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Quantum mechanics with a minimal length and time scale

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Abstract. Many theories of quantum gravity suggest that there should exist a minimal scale with which it is possible to measure distances and time. We present a formalism of quantum mechanics exchibiting a minimal length and time scale. The work is based on the Page-Wootters formalism which gives a possible solution to the problem of time in quantum mechanics. This formalism is an example of a relational approach to quantum dynamics. It is based on the idea that reference frames with respect to which motion is described are themselves physical systems interacting with the degrees of freedom they wish to describe. Quantum dynamics is described in terms of quantum correlations between a clock and a system. In our work we modify the commutation relations between the time and frequency operators leading to a minimal uncertainty of time measurement. This results in a modified version of the Schrödinger equation. A minimal time scale also allows us to introduce a discrete Schrödinger equation describing time evolution on a lattice. We show that both descriptions of time evolution are equivalent. We further investigate the effects of such modification on a couple simple quantum systems.

1 Introduction

Various theories of quantum gravity (such as string theory) predict the existence of a minimum measurable length scale, usually on the order of the Planck length [1, 2]. Black hole physics also suggests the existence of a fundamental limit with which we can measure distances [3, 4].

Motivated by these results, we investigate a quantum theory that incorporates a minimum measurable length and time scale. First, we consider quantum mechanics exhibiting a minimal length scale.

Next, we consider quantum mechanics with only a minimal time scale. More details on this part can be found in [5]. Our starting point is the Page-Wootters formalism of time evolution in quantum mechanics, which is based on the idea that temporal dynamics emerge from correlations between a "clock" and the rest of the system.

To introduce the minimal time scale into the framework, we modify the commutation relations between the time operator $\hat{T}$ and its conjugate, the frequency operator $\hat{\Omega}$. This modification implies that the time observable can no longer be represented by a self-adjoint operator, but only by a symmetric one. This outcome is natural, since it becomes impossible to consider system states at a precisely defined instant of time.

Instead, we employ "clock" states that are maximally localized around specific moments to construct a continuous-time representation of the system, leading to a modified Schrödinger equation governing its evolution. The existence of a minimal time scale also enables us to formulate a discrete-time representation and a corresponding discrete Schrödinger equation describing evolution on a temporal lattice. Remarkably, the continuous and discrete representations turn out to be equivalent, both capturing the same physical time evolution of the system.

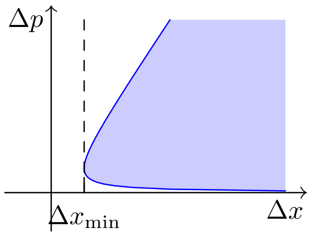

Figure 1: Plot of the uncertainty principle (2) showing that the uncertainty in position cannot be smaller than $\Delta x_{\min}$.

This paper is structured in the following way. Section 2 contains the description of quantum mechanics exhibiting a minimal length scale. In Section 3 we review the Page-Wootters formalism on which the developed theory will be based. In Section 4, we introduce a minimal time scale into the theory by modifying the commutation relations between the time and frequency operators. Sections 5-7 contain the analysis of the time evolution of a couple simple quantum systems when a minimal time scale is present. The conclusions and a discussion of the received results is given in Section 8.

2 Minimal length scale

A way of achieving a minimal length and time scale is through modification of canonical commutation relations. Such modified commutation relations will then lead to modified uncertainty relations which are referred in the literature as Generalized Uncertainty Principles. The simplest modification of commutation relations between position and momentum operators in a one-dimensional case and nonrelativistic regime was proposed in [6]

$$[\hat{x}, \hat{p}] = i\hbar\left(\hat{1} + \beta\hat{p}^2\right), \tag{1}$$

where β is some positive constant. This commutation relation leads to the following uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta(\Delta p)^2 + \beta\langle\hat{p}\rangle^2\right) \tag{2}$$

from which we can infer that the uncertainty in position cannot be smaller than

$$\Delta x_{\min} = \hbar\sqrt{\beta}\sqrt{1 + \beta\langle\hat{p}\rangle^2} \tag{3}$$

see Fig. 1. Every physical state cannot have the uncertainty in position smaller than this value. For states for which the expectation value of the momentum operator $\hat{p}$ is equal zero, we receive the absolutely smallest uncertainty in position equaling $\Delta x_0 = \hbar\sqrt{\beta}$. The nonzero value of Δx_0 is a manifestation of the quantization of space. There exists a lower bound to the possible resolution with which we can measure distances.

There are many possible generalizations of the commutation relations (1) to n dimensions. The most general commutation relations for which the right hand side depends only on $\hat{p}$ and the momentum operators commute are of the form

$$[\hat{x}_j, \hat{p}_k] = i\hbar\Theta_{jk}(\hat{\boldsymbol{p}}), \quad [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{x}_k] = i\hbar\{\hat{x}_l, \Theta^{-1}_{lr}(\hat{\boldsymbol{p}})\Theta_{s[j}(\hat{\boldsymbol{p}})\Theta_{k]r,s}(\hat{\boldsymbol{p}})\}, \tag{4}$$

where Θ_{jk} is some general function and $\{\,\cdot\,,\,\cdot\,\}$ stands for the anti-commutator. The last equation in (4) follows from the first two and the Jacobi identity. The noncommutativity of position operators indicate that the position space is described by a noncommutative geometry. Particular cases of (4):

$$[\hat{x}_j, \hat{p}_k] = i\hbar(\delta_{jk}\hat{1} + \beta\hat{p}_j\hat{p}_k), \quad [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{x}_k] = i\hbar\beta(\hat{p}_j\hat{x}_k - \hat{p}_k\hat{x}_j), \tag{5}$$

$$[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}(\hat{1} + \beta\hat{\boldsymbol{p}}^2), \quad [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{x}_k] = 2i\hbar\beta(\hat{p}_j\hat{x}_k - \hat{p}_k\hat{x}_j), \tag{6}$$

$$[\hat{x}_j, \hat{p}_k] = i\hbar\left(\frac{\beta\hat{\boldsymbol{p}}^2}{\sqrt{1 + 2\beta\hat{\boldsymbol{p}}^2} - 1}\delta_{jk} + \beta\hat{p}_j\hat{p}_k\right), \quad [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{x}_k] = 0, \tag{7}$$

$$[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}(\hat{1} + \beta\hat{p}_j\hat{p}_k), \quad [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{x}_k] = 0. \tag{8}$$

The first three commutation relations have the property of being rotation invariant.

3 Page-Wootters formalism

The same approach used to introduce a minimal length scale in quantum theory can be applied to introduce a minimal time scale. For this we need a suitable formalism of quantum dynamics. The formalism we will focus on was introduced by D.N. Page and W.K. Wootters [7].

The Page-Wootters formalism is a relational approach to quantum dynamics. It is based on the idea that reference frames with respect to which motion is described are themselves physical systems interacting with the degrees of freedom they wish to describe. Quantum dynamics is described in terms of quantum correlations between a clock and a system.

The clock is described by the Hilbert space $\mathcal{H}_T$ and the system undergoing time evolution is described by the Hilbert space $\mathcal{H}_S$. The joint Hilbert space of the clock and the system is $\mathcal{H} = \mathcal{H}_T \otimes \mathcal{H}_S$. To receive an ordinary time evolution we take $\mathcal{H}_T$ isomorphic to the Hilbert space of a particle on a line $L^2(\mathbb{R})$. On $\mathcal{H}_T$ we introduce the time operator $\hat{T}$ corresponding to the measurements of time and frequency operator $\hat{\Omega}$ conjugate to $\hat{T}$: $[\hat{T}, \hat{\Omega}] = i\hat{1}$.

On $\mathcal{H}$, we introduce the constraint operator of the model

$$\hat{J} = \hbar\hat{\Omega} \otimes \hat{1}_S + \hat{1}_T \otimes \hat{H}_S, \tag{9}$$

with $\hat{H}_S$ the system Hamiltonian. Such constraint describes the simplest case of the "clock" non-interacting with the system and where the system Hamiltonian is time independent. It is, however, possible to consider more general cases. Vectors $|\Psi\rangle\rangle$ in $\mathcal{H}$ which are (generalized) eigenvectors of the operator $\hat{J}$ associated with the null eigenvalue: $\hat{J}|\Psi\rangle\rangle = 0$ are considered as physical states of the model.

By taking a generalized eigenvector $|t\rangle_T$ of the time operator $\hat{T}$ corresponding to an eigenvalue t: $\hat{T}|t\rangle_T = t|t\rangle_T$ and projecting it with $|\Psi\rangle\rangle$ we receive a conventional state $|\psi(t)\rangle_S = {}_T\langle t|\Psi\rangle\rangle$ of the system S at time t.

By writing the constraint equation in the time representation in $\mathcal{H}_T$: ${}_T\langle t|\hat{J}|\Psi\rangle\rangle = 0$ and using the fact that the frequency operator $\hat{\Omega}$ in the time representation takes the form of a differentiation operator: ${}_T\langle t|\hat{\Omega}|\psi\rangle_T = -i\frac{\partial}{\partial t}{}_T\langle t|\psi\rangle_T$ we obtain the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}_S|\psi(t)\rangle_S. \tag{10}$$

A similar considerations with $|t\rangle_T$ replaced by eigenvectors $|\omega\rangle_T$ of $\hat{\Omega}$ lead to a frequency representation in $\mathcal{H}_T$ and the constraint equation in the form of the eigenvector equation of the operator $\hat{H}_S$:

$$\hat{H}_S|\psi(\omega)\rangle_S = -\hbar\omega|\psi(\omega)\rangle_S. \tag{11}$$

4 Incorporation of a minimal time scale

Similarly as was done with a minimal length scale, cf. (1), by modifying the commutation relations for the operators of time and frequency, $\hat{T}$ and $\hat{\Omega}$ we can incorporate a minimal time scale into the formalism:

$$[\hat{T}, \hat{\Omega}] = i\left(\hat{1} + \kappa\hat{\Omega}^2\right). \tag{12}$$

Here κ is a positive constant describing the smallest possible resolution with which we can measure time $\Delta t_0 = \sqrt{\kappa}$.

The operators $\hat{T}$ and $\hat{\Omega}$ on a Hilbert space $\mathcal{H}_T$ can be defined as symmetric operators with common domain D of physical states, i.e. states for which the uncertainty of $\hat{T}$ is not smaller than Δt_0. However, only the operator $\hat{\Omega}$ will be essentially self-adjoint. This implies that the eigenvectors of $\hat{T}$ are not physical states and there are no sequences of physical states which would approximate point localisation. It is no longer possible to consider states of the system S at particular instances of time.

However, we can use $\hat{\Omega}$ to construct a frequency representation of the Hilbert space $\mathcal{H}_T$. The Hilbert space $\mathcal{H}_T$ is represented as a space $L^2(\mathbb{R}, \mathrm{d}\mu)$ of functions defined on $\mathbb{R}$ and square integrable with respect to the measure $\mathrm{d}\mu(\omega) = \dfrac{\mathrm{d}\omega}{1+\kappa\omega^2}$. Such functions can be formally constructed by projecting states $|\psi\rangle_T$ onto eigenvectors $|\omega\rangle_T$ of $\hat{\Omega}$: $\psi(\omega) = {}_T\langle\omega|\psi\rangle_T$.

The scalar product in the Hilbert space $L^2(\mathbb{R}, \mathrm{d}\mu)$ is given by the formula

$${}_T\langle\varphi|\psi\rangle_T = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{1+\kappa\omega^2}\overline{\varphi(\omega)}\psi(\omega), \tag{13}$$

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and the frequency and time operators $\hat{\Omega}$, $\hat{T}$ are represented as appropriate multiplication and differentiation operators:

$$\hat{\Omega}\psi(\omega) = \omega\psi(\omega), \quad \hat{T}\psi(\omega) = i(1+\kappa\omega^2)\partial_\omega\psi(\omega). \tag{14}$$

Although eigenvectors of the time operator $\hat{T}$ are not physical states, it is possible to define states which are physical and closely resemble eigenstates of $\hat{T}$. These are states $|\varphi_\tau^{ML}\rangle_T$ for which

$$_T\langle\varphi_\tau^{ML}|\hat{T}|\varphi_\tau^{ML}\rangle_T = \tau, \quad (\Delta t)_{|\varphi_\tau^{ML}\rangle_T} = \Delta t_0 \tag{15}$$

meaning that they will be maximally localized around instances of time τ.

The conditions in (15) uniquely determine the state $|\varphi_\tau^{ML}\rangle_T$, which in the frequency representation is given by the formula

$$\varphi_\tau^{ML}(\omega) = \sqrt{\frac{2\sqrt{\kappa}}{\pi}}(1+\kappa\omega^2)^{-1/2}\exp\left(-i\tau\frac{\arctan(\sqrt{\kappa}\omega)}{\sqrt{\kappa}}\right). \tag{16}$$

Using the states of maximal localization we can construct a continuous time representation of the Hilbert space $\mathcal{H}_T$: $\psi(\tau) = {}_T\langle\varphi_\tau^{ML}|\psi\rangle_T$. The received wave functions $\psi(\tau)$ describe the probability amplitude for the system being maximally localized around the instance of time τ.

The operators $\hat{\Omega}$ and $\hat{T}$ in the continuous time representation take the form

$$\hat{\Omega}\psi(\tau) = \frac{\tan(-i\sqrt{\kappa}\partial_\tau)}{\sqrt{\kappa}}\psi(\tau), \quad \hat{T}\psi(\tau) = \left(\tau + i\kappa\frac{\tan(-i\sqrt{\kappa}\partial_\tau)}{\sqrt{\kappa}}\right)\psi(\tau). \tag{17}$$

The family of states $\{|\varphi_\tau^{ML}\rangle_T\}$ for $\tau \in \mathbb{R}$ forms an overcomplete set of vectors. For fixed $\lambda \in [0,1)$ we can choose from this set a smaller countable set forming a basis in the Hilbert space $\mathcal{H}_T$:

$$\{|\varphi_{2\sqrt{\kappa}(\lambda+n)}^{ML}\rangle_T \mid n \in \mathbb{Z}\}. \tag{18}$$

Using this basis, we can construct a discrete time representation. These bases can be viewed as lattices with spacing $2\sqrt{\kappa} = 2\Delta t_0$ shifted by $2\sqrt{\kappa}\lambda$.

The basis vectors $|\varphi_{2\sqrt{\kappa}(\lambda+n)}^{ML}\rangle_T$ are orthonormal except for neighboring vectors:

$$_T\langle\varphi_{2\sqrt{\kappa}(\lambda+m)}^{ML}|\varphi_{2\sqrt{\kappa}(\lambda+n)}^{ML}\rangle_T = \begin{cases} 1 & \text{for } m=n \\ \frac{1}{2} & \text{for } m = n\pm 1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Using this basis, we can construct a discrete time representation of the Hilbert space $\mathcal{H}_T$: $\psi_n = {}_T\langle\varphi_{2\sqrt{\kappa}(\lambda+n)}^{ML}|\psi\rangle_T$. Note, that because $|\varphi_{2\sqrt{\kappa}(\lambda+n)}^{ML}\rangle_T$ are not orthonormal this representation differs from representing a state $|\psi\rangle_T$ as a sequence of coefficients in the expansion of the state $|\psi\rangle_T$ in the base $\{|\varphi_{2\sqrt{\kappa}(\lambda+n)}^{ML}\rangle_T\}_{n\in\mathbb{Z}}$ but such representation will have a direct physical interpretation.

The operators $\hat{\Omega}$ and $\hat{T}$ in the discrete time representation take the form

$$\hat{\Omega}\psi_n = \frac{f(-i\sqrt{\kappa}D_n)}{\sqrt{\kappa}}\psi_n, \quad \hat{T}\psi_n = \left(2\sqrt{\kappa}(\lambda+n) + i\kappa\frac{f(-i\sqrt{\kappa}D_n)}{\sqrt{\kappa}}\right)\psi_n, \tag{20}$$

where $f(x) = \dfrac{2x}{1+\sqrt{1-4x^2}}$ and $D_n\psi_n = \dfrac{\psi_{n+1}-\psi_{n-1}}{4\Delta t_0}$ is a discrete derivative.

The continuous and discrete time representations are equivalent. The transformation of a state's wave function in the continuous time representation into its discrete time representation takes the form

$$\psi_n = \psi\left(2\sqrt{\kappa}(\lambda+n)\right) \tag{21}$$

with an inverse transform equal

$$\psi(\tau) = \sum_{n=-\infty}^{\infty}\psi_n \operatorname{sinc}\left(\frac{\tau - 2\sqrt{\kappa}(\lambda+n)}{2\sqrt{\kappa}}\right), \tag{22}$$

where $\operatorname{sinc} x = \dfrac{\sin(\pi x)}{\pi x}$.

It might seem surprising that the function $\psi(\tau)$ can be reproduced from the knowledge of its values at a countable set of points. This is, however, nothing strange because $\psi(\tau)$ is a function which extends to an entire function of exponential type and entire functions are completely determined by their values at a countable set of points.

The state $|\psi(\tau)\rangle_S$ of the system S maximally localized around an instance of time τ can be obtained via projection of the physical state $|\Psi\rangle\rangle$ with the maximal localization state $|\varphi^{ML}_\tau\rangle_T$: $|\psi(\tau)\rangle_S = {}_T\langle\varphi^{ML}_\tau|\Psi\rangle\rangle$.

The constraint equation $\hat{J}|\Phi\rangle\rangle = 0$ in the continuous time representation in $\mathcal{H}_T$ takes the form of a modified Schrödinger equation

$$i\frac{\partial}{\partial\tau}|\psi(\tau)\rangle_S = \frac{1}{\sqrt{\kappa}}\arctan\left(\sqrt{\kappa}\,\hat{H}_S/\hbar\right)|\psi(\tau)\rangle_S. \tag{23}$$

The discrete time representation of the physical state $|\Psi\rangle\rangle$ can be received by projecting it with the maximal localization states $|\varphi^{ML}_{2\sqrt{\kappa}(\lambda+n)}\rangle_T$: $|\psi_n\rangle_S = {}_T\langle\varphi^{ML}_{2\sqrt{\kappa}(\lambda+n)}|\Psi\rangle\rangle$.

The constraint equation $\hat{J}|\Phi\rangle\rangle = 0$ in the discrete time representation in $\mathcal{H}_T$ takes the form of a discrete Schrödinger equation

$$i\hbar D_n|\psi_n\rangle_S = \hat{H}_S\left(\hat{1} + \kappa(\hat{H}_S/\hbar)^2\right)^{-1}|\psi_n\rangle_S. \tag{24}$$

5 Example: Free particle

The Hilbert space of the system is $\mathcal{H}_S = L^2(\mathbb{R}, \mathrm{d}x)$ and the Hamiltonian of the system is $\hat{H}_S = \dfrac{1}{2m}\hat{p}^2$, where m is the mass of a particle. Let us assume that the system is in an arbitrary initial state

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(p)e^{\frac{i}{\hbar}px}\,\mathrm{d}p, \tag{25}$$

written as a Fourier transform of an arbitrary function f such that ψ will be in the domains of the position and momentum operators $\hat{x}$, $\hat{p}$ and their squares $\hat{x}^2$, $\hat{p}^2$.

The modified time evolution of the state $\psi(x)$ takes the form

$$\psi(\tau, x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(p)e^{\frac{i}{\hbar}(px-E(p)\tau)}\,\mathrm{d}p, \tag{26}$$

where $E(p) = \dfrac{\hbar}{\sqrt{\kappa}}\arctan\left(\dfrac{\sqrt{\kappa}p^2}{2m\hbar}\right)$.

In accordance with (23), the effective Hamiltonian describing the time evolution of the system is

$$\hat{H}_{\text{eff}} = \frac{\hbar}{\sqrt{\kappa}}\arctan\left(\sqrt{\kappa}\,\hat{H}_S/\hbar\right) = \frac{\hbar}{\sqrt{\kappa}}\arctan\left(\frac{\sqrt{\kappa}\hat{p}^2}{2m\hbar}\right). \tag{27}$$

In classical Hamiltonian mechanics velocity is defined as a derivative of the Hamilton function H with respect to momentum p in accordance with Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{28}$$

By formally differentiating (27) with respect to $\hat{p}$ we can define a velocity operator by the formula

$$\hat{v} = \frac{\hat{p}}{m}\left(\hat{1} + \frac{\kappa\hat{p}^4}{4m^2\hbar^2}\right)^{-1}. \tag{29}$$

We can calculate the expectation values of position, momentum, and velocity operators:

$$\begin{aligned}
&\langle\hat{x}\rangle_{\psi(\tau)} = \langle\hat{x}\rangle_{\psi(0)} + \tau\langle\hat{v}\rangle_{\psi(0)},\\
&\langle\hat{p}\rangle_{\psi(\tau)} = \langle\hat{p}\rangle_{\psi(0)},\\
&\langle\hat{v}\rangle_{\psi(\tau)} = \langle\hat{v}\rangle_{\psi(0)} = \int_{-\infty}^{+\infty}|f(p)|^2\frac{p/m}{1+\kappa p^4/4m^2\hbar^2}\,\mathrm{d}p \le v_{\max},
\end{aligned} \tag{30}$$

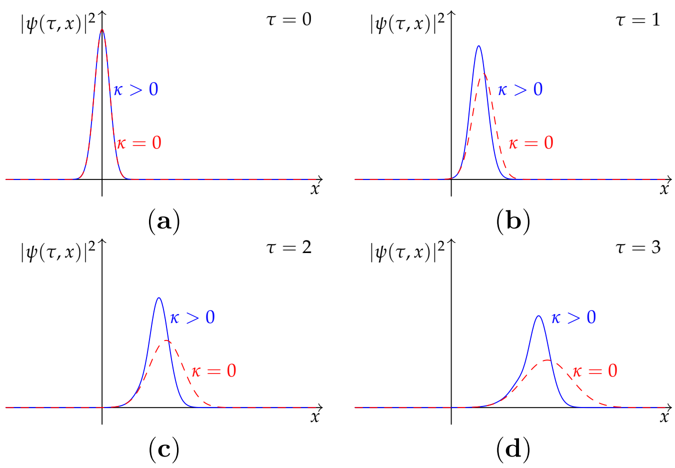

Figure 2: (**a**–**d**) Plots of the probability density of the Gaussian wave packet $\psi(\tau, x)$ with $\Delta p = 1/\sqrt{2}$ and $p_0 = 3$ for particular instances of time τ. Mass $m = 1$, $\hbar = 1$, $\kappa = 0.005$ (blue plots).

where $v_{\max} = \sqrt{\dfrac{3\sqrt{3}\hbar}{8m\sqrt{\kappa}}}$ is the upper limit on the speed of propagation of wave packets. Notice that $v_{\max}$ does not depend on the initial state $\psi(x)$, but it depends on the mass m of the particle. The heavier the particle, the smaller this speed limit is. For example if $\Delta t_0 = t_p$ (the Planck's time) we get

$$\begin{aligned}
&v_{\max} \approx 8.72 \cdot 10^{17}\,\frac{\mathrm{m}}{\mathrm{s}} && \text{for } m = \text{mass of the proton},\\
&v_{\max} = \sqrt{\frac{3\sqrt{3}}{8}}c \approx 0.8c && \text{for } m = m_p \text{ (the Planck's mass)},\\
&v_{\max} = 0.05c && \text{for } m = 150\sqrt{3}m_p \approx 5.65\,\mathrm{mg}.
\end{aligned}$$

In Figure 2, is presented the time evolution of the probability density of the initial state (25) in the form of a Gaussian wave packet with

$$f(p) = \frac{1}{(2\pi)^{1/4}(\Delta p)^{1/2}} \exp\left(-\frac{(p-p_0)^2}{4(\Delta p)^2}\right). \tag{31}$$

From these plots, we can see that the velocity and spreading of a Gaussian wave packet are smaller for the $\kappa > 0$ case.

6 Example: Harmonic oscillator

The Hilbert space of the system is $\mathcal{H}_S = L^2(\mathbb{R}, \mathrm{d}x)$ and the Hamiltonian of the system is

$$\hat{H}_S = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2, \tag{32}$$

where m is the mass of a particle and ω is a frequency of oscillations. Let us assume that the system is initially in a coherent state

$$\psi_\alpha(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x-x_0)^2} e^{\frac{i}{\hbar}(x-\frac{1}{2}x_0)p_0}. \tag{33}$$

In Figure 3, is presented the time evolution of the probability density of this state received by numerical calculations. These plots illustrate how the coherence of the state is destroyed during time evolution as a result of the existence of the fundamental limit for the precision with which we can measure time.

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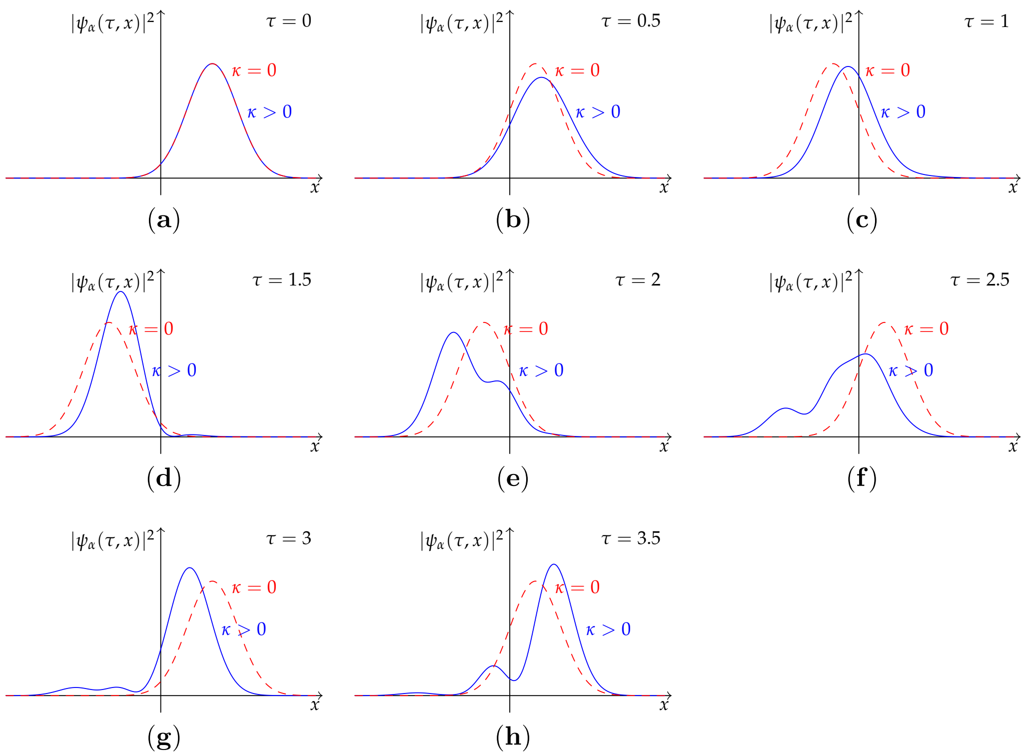

Figure 3: (**a**–**h**) Plots of the probability density of the time-evolved coherent state $\psi_\alpha(\tau, x)$ with $x_0 = 1$ and $p_0 = 0$ for particular instances of time τ. Mass $m = 1$, frequency $\omega = 2\pi/3$, $\hbar = 1$, $\kappa = 0.01$ (blue plots).

7 Example: System of n non-interacting spins-1/2 in a magnetic field

The Hilbert space of the system is $\mathcal{H}_S = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2n}$ and the Hamiltonian of the system is

$$\begin{aligned}\hat{H}_S &= -\gamma B_0(\hat{S}_z \otimes \hat{1} \otimes \cdots \otimes \hat{1} + \hat{1} \otimes \hat{S}_z \otimes \cdots \otimes \hat{1} + \cdots + \hat{1} \otimes \hat{1} \otimes \cdots \otimes \hat{S}_z) \\ &= \frac{\hbar\omega_0}{2}(\sigma_z \otimes I \otimes \cdots \otimes I + I \otimes \sigma_z \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes \sigma_z),\end{aligned} \tag{34}$$

where γ is the gyromagnetic ratio; B_0 is the strength of the magnetic field, which we assume is directed along the z-axis; $\omega_0 = -\gamma B_0$; and $\hat{S}_z$ is the z-component of the spin operator expressed by the Pauli matrix σ_z. Let us assume that the system is initially in an arbitrary non-entangled state

$$|\psi\rangle_S = |\theta_1, \varphi_1\rangle \otimes |\theta_2, \varphi_2\rangle \otimes \cdots \otimes |\theta_n, \varphi_n\rangle, \tag{35}$$

where

$$|\theta, \varphi\rangle = \cos(\theta/2)e^{-i\varphi/2}|\uparrow\rangle + \sin(\theta/2)e^{i\varphi/2}|\downarrow\rangle, \tag{36}$$

$\theta, \varphi \in \mathbb{R}$ and $|\uparrow\rangle_S$, $|\downarrow\rangle_S$ are spin-up and spin-down eigenstates of the spin operator $\hat{S}_z$.

In the case of one spin ($n = 1$) the modified time evolution of the state $|\psi\rangle_S$ takes the form

$$|\psi(\tau)\rangle_S = |\theta, \varphi + \omega_\kappa \tau\rangle, \tag{37}$$

where $\omega_\kappa = \dfrac{2}{\sqrt{\kappa}} \arctan(\sqrt{\kappa}\omega_0/2)$. The spin is precessing around the z-axis with the frequency ω_κ. Since arctan is a bounded function, there is an upper limit for the frequency with which a spin can precess. The frequency of precession is necessarily smaller than $\pi/\sqrt{\kappa}$.

In the case of two spins ($n = 2$) the modified time evolution of the state $|\psi\rangle_S$ takes the form

$$|\psi(\tau)\rangle_S = |\theta_1, \varphi_1 + \omega_\kappa \tau\rangle \otimes |\theta_2, \varphi_2 + \omega_\kappa \tau\rangle, \tag{38}$$

where $\omega_\kappa = \dfrac{1}{\sqrt{\kappa}} \arctan(\sqrt{\kappa}\omega_0)$. The spins are precessing around the z-axis with the frequency ω_κ. The frequency of precession is smaller than in the case of only one spin.

In the case of three spins ($n = 3$) the modified time evolution of the state $|\psi\rangle_S$ takes the form

$$\begin{aligned} |\psi(\tau)\rangle_S = \cos(\lambda_\kappa\tau)&|\theta_1, \varphi_1 + \omega_\kappa\tau\rangle \otimes |\theta_2, \varphi_2 + \omega_\kappa\tau\rangle \otimes |\theta_3, \varphi_3 + \omega_\kappa\tau\rangle \\ - i\sin(\lambda_\kappa\tau)&|-\theta_1, \varphi_1 + \omega_\kappa\tau\rangle \otimes |-\theta_2, \varphi_2 + \omega_\kappa\tau\rangle \otimes |-\theta_3, \varphi_3 + \omega_\kappa\tau\rangle, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \omega_\kappa &= \frac{1}{4\sqrt{\kappa}} \left(\arctan\left(\frac{3\sqrt{\kappa}\omega_0}{2}\right) + \arctan\left(\frac{\sqrt{\kappa}\omega_0}{2}\right) \right), \\ \lambda_\kappa &= \frac{1}{4\sqrt{\kappa}} \left(\arctan\left(\frac{3\sqrt{\kappa}\omega_0}{2}\right) - 3\arctan\left(\frac{\sqrt{\kappa}\omega_0}{2}\right) \right). \end{aligned} \tag{40}$$

The spins are precessing around the z-axis with the frequency ω_κ. The spins initially non-entangled get entangled during time evolution.

8 Conclusions

The modification of the Page-Wootters formalism was presented which led to a minimal time scale. Physical consequences of such modification were discussed. It is possible to extend the developed formalism into relativistic quantum mechanics exhibiting minimal time and length scale by imposing generalized commutation relations between space-time variables which will be Lorentz covariant. For example the following generalizations of the commutation relations (5) and (6) can be used

$$\begin{aligned} &[\hat{x}^\mu, \hat{p}^\nu] = i\hbar(g^{\mu\nu}\hat{1} + \beta\hat{p}^\mu\hat{p}^\nu) \\ &[\hat{p}^\mu, \hat{p}^\nu] = 0 \\ &[\hat{x}^\mu, \hat{x}^\nu] = i\hbar\beta(\hat{p}^\mu\hat{x}^\nu - \hat{p}^\nu\hat{x}^\mu) \end{aligned} \qquad \text{or} \qquad \begin{aligned} &[\hat{x}^\mu, \hat{p}^\nu] = i\hbar g^{\mu\nu}(\hat{1} + \beta\hat{p}_\rho\hat{p}^\rho) \\ &[\hat{p}^\mu, \hat{p}^\nu] = 0 \\ &[\hat{x}^\mu, \hat{x}^\nu] = 2i\hbar\beta(\hat{p}^\mu\hat{x}^\nu - \hat{p}^\nu\hat{x}^\mu) \end{aligned}$$

where $g^{\mu\nu}$ is a space-time metric. The representation of the operators $\hat{x}^\mu$ and $\hat{p}^\nu$ on an appropriate Hilbert space, the states of maximal localization, and modified evolution equations can be found also in such cases.

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