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Lorentz Invariance in Relativistic Particle Mechanics

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Abstract: The notion of invariance under the Lorentz transformation is fundamental to special relativity and its continuation beyond the speed of light. Theories and solutions with this characteristic are stronger and more powerful than conventional theories or conventional solutions because the Lorentz-invariant approach automatically embodies the conventional approach. We propose a Lorentz-invariant extension of Newton's second law, which includes both special relativistic mechanics and Schrödinger's quantum wave theory. Here, we determine new general expressions for energy–momentum, which are Lorentz-invariant. We also examine the Lorentz-invariant power-law energy–momentum expressions, which include Einstein's energy relation as a particular case.

Keywords: special relativity; relativistic mechanics; energy–momentum relations; unified physical theories

1. Introduction

In this paper, we determine the general solutions to the Lorentz-invariant theory proposed in [1] that are also Lorentz-invariant. The formal solutions to partial differential equations by means of one-parameter groups of transformations that leave the equation invariants have been well established (see [2,3]). However, all solutions to any theories remaining invariant under a one-parameter group of transformations are not necessarily all invariant. A familiar example is the one-dimensional heat equation, $\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2}$, for temperature for $\Theta = \Theta(x, t)$. This equation remains unchanged by the stretching transformations $x^* = e^{\epsilon} x$, $t^* = e^{2\epsilon} t$ and $\Theta^* = e^{m\epsilon}\Theta$, where ϵ denotes the one-parameter group and m denotes an arbitrary constant. This one-parameter stretching group shows that there are similar solutions to the partial differential equation with the functional form $\Theta(x, t) = t^{m/2}\Phi(x/t^{1/2})$ for a function, Φ, and the partial differential equation may be reduced to a second-order ordinary differential equation. It does not imply that every solution to the heat equation has this structure, and there are solutions that are not similar. A corresponding situation applies to the Lorentz-invariant theory proposed in [1,4,5], which admits solutions and embodies consequences that are not fully Lorentz-invariant. Here, we determine the general one-dimensional family of energy–momentum solutions that are Lorentz-invariant.

The requirement that the governing equations be invariant under a Lorentz transformation of coordinates for any well-formed physical model has become an important guiding principle that should apply to any properly formulated physical theory, and the notion of Lorentz invariance has emerged as an indispensable tool. Invariance under the Lorentz group of transformations, in particular, and more generally under other symmetry groups, has proved to be significant in determining both the fundamental chemical structure and physical theory (see Refs. [6,7] for work on the chemical structure and Refs. [8,9] for work on physical theory). Various notions of Lorentz invariance have been widely invoked throughout theoretical physics, including high-precision testing and violations of Lorentz invariance from both terrestrial and astrophysical experiments [10–12]; quantum gravity, quantum entanglement, and Lorentz invariance [13–15]; Lorentz-invariant theories of gravity and general relativity [16,17]; well-formed numerical and computational methods [18]; and many others. The theory formulated in [1,4,5] is invariant under both

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the Lorentz group of coordinate transformations and the associated energy–momentum relations. For the one spatial dimension, these are specified in (6)–(9)).

Although Lorentz invariance and its consequences are well established in special relativity (see [19–22]), it seems to have gone unnoticed that—for special relativistic motions—this can be exploited to determine certain general solutions involving arbitrary functions. For example, the requirement that the velocity equations $\mathbf{dx}/dt = \mathbf{u}(\mathbf{x}, t)$ be Lorentz-invariant implies that the velocity field $\mathbf{u}(\mathbf{x}, t)$ satisfies certain first-order partial differential equations, which admit general solutions involving arbitrary functions. In three papers [23–25], for one, two, and three dimensions, the authors examined each of the three distinct geometries to determine Lorentz-invariant velocity fields for certain special relativistic motions. In [23], for a single Cartesian spatial dimension x, the requirement that the velocity equation $dx/dt = u(x, t)$ remains invariant under a Lorentz transformation implies that the velocity $u(x, t)$ satisfies the first-order partial differential equation, as follows:

$$t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t} = 1 - \left(\frac{u}{c}\right)^2, \tag{1}$$

and a general solution to the velocity, $u(x, t)$, involving a single arbitrary function $\phi(\delta)$ is given by the following:

$$\frac{u(x,t)}{c} = \frac{[(ct+x)\phi(\delta)]^2 - 1}{[(ct+x)\phi(\delta)]^2 + 1}, \tag{2}$$

where $\delta = [(ct)^2 - x^2]^{1/2}$.

For a single spatial dimension, x, the general Lorentz-invariant solutions to the energy $e(x, t) = mc^2$ and momentum $p(x, t) = mu$ of the forms $e = F(x, t)$ and $p = G(x, t)$ satisfy the first-order coupled partial differential equations, as follows:

$$t\frac{\partial e}{\partial x} + \frac{x}{c^2}\frac{\partial e}{\partial t} = p, \qquad t\frac{\partial p}{\partial x} + \frac{x}{c^2}\frac{\partial p}{\partial t} = \frac{e}{c^2}, \tag{3}$$

where $m(x, t)$ denotes the mass and $u(x, t) = dx/dt$ denotes the velocity. These equations are cited in [23] but are not solved. Here, we present their general solutions in terms of two arbitrary functions, namely the following:

$$e(x,t) = \Phi(\delta)x + \Psi(\delta)ct, \qquad cp(x,t) = \Phi(\delta)ct + \Psi(\delta)x, \tag{4}$$

where $\Phi(\delta)$ and $\Psi(\delta)$ denote arbitrary functions of $\delta = [(ct)^2 - x^2]^{1/2}$. The results given here for (3) involve two arbitrary functions and are inclusive of those obtained in [23] for the partial differential Equation (1) involving only a single arbitrary function. With $u/c = pc/e$, the velocity from (4) agrees with the general solution (2), with the following identification:

$$\delta\phi(\delta) = \left(\frac{1+\chi(\delta)}{1-\chi(\delta)}\right)^{1/2}, \qquad \chi(\delta) = \frac{[\delta\phi(\delta)]^2 - 1}{[\delta\phi(\delta)]^2 + 1},$$

where $\chi(\delta) = \Phi(\delta)/\Psi(\delta)$. Generally, with $u/c = pc/e$, (3) constitutes necessary conditions for the validity of Equation (1), but not sufficient. In other words, if Equation (3) are satisfied, then with $u/c = pc/e$, (1) is satisfied. However, if (1) is satisfied, then generally, (3) will not hold.

Formally, we may extend the general solution (4) by the implicit relations, as follows:

$$e(x,t) = \Phi(\delta,\Delta,\zeta)x + \Psi(\delta,\Delta,\zeta)ct, \quad cp(x,t) = \Phi(\delta,\Delta,\zeta)ct + \Psi(\delta,\Delta,\zeta)x, \tag{5}$$

where $\Phi(\delta, \Delta, \zeta)$ and $\Psi(\delta, \Delta, \zeta)$ denote two arbitrary functions of the three variables (δ, Δ, ζ); $\Delta = [e^2 - (cp)^2]^{1/2}$; and ζ is a further Lorentz-invariant defined by (12) below. These details are relegated to Appendix A.

In the following section, for a single space dimension x, we state the two basic sets of space–time Lorentz transformations and energy–momentum Lorentz transformations, which underpin this work. Following that, we use the infinitesimal versions of these Lorentz transformations to derive first-order partial differential equations for Lorentz-invariant quantities and we solve the partial differential equations using Lagrange's characteristic method to deduce general solutions to the Lorentz-invariant quantities. In the subsequent section, we connect the coupled partial differential equation (3) for particle energy, e, and particle momentum, p, with the Lorentz-invariant theory proposed in [1] and Equation (21). Together, these relations constitute four equations to determine the four partial derivatives, $\partial e/\partial x$, $\partial e/\partial t$, $\partial p/\partial x$, and $\partial p/\partial t$; moreover, the two compatibility conditions (see Equation (25)) confirm that the full Equation (21) are only invariant if the two applied forces, $f(x,t)$ and $cg(x,t)$, are both invariant. In the penultimate section of the paper, as an illustrative example, we examine the Lorentz-invariant power-law energy–momentum relations originally proposed in [26].

2. Lorentz Transformations

With reference to Figure 1, capital letters refer to the fixed (X,T) reference frame and lowercase letters refer to the moving (x,t) reference frame. In the notation of Figure 1, if v denotes the constant relative frame velocity, then the Lorentz transformations $(0 \leqslant v < c)$ are given by the following:

$$x = \frac{X - vT}{[1-(v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{6}$$

with the following inverse transformation:

$$X = \frac{x + vt}{[1-(v/c)^2]^{1/2}}, \quad T = \frac{t + vx/c^2}{[1-(v/c)^2]^{1/2}}, \tag{7}$$

and the identity transformations $x = X$ and $t = T$ are characterised by $v = 0$. Various derivations of (6) and (7) can be found in texts, such as the work by Feynmann et al. [27] and Landau and Lifshitz [28]. Other derivations of Lorentz transformations are given by Lee and Kalotas [29] and Levy-Leblond [30].

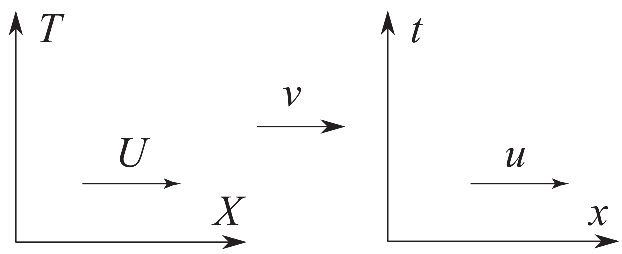

Figure 1. Two inertial frames moving along the x-axis with relative velocity, v.

For $0 \leqslant v < c$ the standard Lorentz-invariant energy–momentum relations (see [1], page 40) are given by the following:

$$e = \frac{E - Pv}{[1-(v/c)^2]^{1/2}}, \quad p = \frac{P - Ev/c^2}{[1-(v/c)^2]^{1/2}}, \tag{8}$$

with inverse relations given by the following:

$$E = \frac{e + pv}{[1-(v/c)^2]^{1/2}}, \quad P = \frac{p + ev/c^2}{[1-(v/c)^2]^{1/2}}, \tag{9}$$

Together, these equations are referred to as the Lorentz-invariant energy–momentum relations, and we obtain the following:

$$\begin{aligned} ct+x &= \left(\tfrac{1-v/c}{1+v/c}\right)^{1/2}(cT+X), \quad ct-x = \left(\tfrac{1+v/c}{1-v/c}\right)^{1/2}(cT-X), \\ e+pc &= \left(\tfrac{1-v/c}{1+v/c}\right)^{1/2}(E+Pc), \quad e-pc = \left(\tfrac{1+v/c}{1-v/c}\right)^{1/2}(E-Pc). \end{aligned} \tag{10}$$

From these relations, the Lorentz invariants $(ct)^2 - x^2 = (cT)^2 - X^2$ and $e^2 - (pc)^2 = E^2 - (Pc)^2$ are apparent, and we have the following:

$$\left(\frac{ct+x}{ct-x}\right) = \left(\frac{1-v/c}{1+v/c}\right)\left(\frac{cT+X}{cT-X}\right), \quad \left(\frac{e+pc}{e-pc}\right) = \left(\frac{1-v/c}{1+v/c}\right)\left(\frac{E+Pc}{E-Pc}\right), \tag{11}$$

from which, a further Lorentz-invariant is also apparent:

$$\zeta = \left(\frac{e+pc}{e-pc}\right)\left(\frac{ct-x}{ct+x}\right) = \left(\frac{E+Pc}{E-Pc}\right)\left(\frac{cT-X}{cT+X}\right), \tag{12}$$

Finally, in this section, we remark that for those problems involving both partial differential equations and boundary or initial conditions, the invariance of the equation and associated conditions under a one-parameter Lie group of transformations usually produces a simplified problem (see [31]). Thus, any solutions to (3) will produce solutions to the special relativistic problem, provided the accompanying conditions are also invariant under the Lorentz transformation. For one spatial dimension, x, this means that any associated boundaries or initial data must involve the three invariants $\delta = [(ct)^2 - x^2]^{1/2}$, $\Delta = [e^2 - (cp)^2]^{1/2}$ or the above invariant (12); therefore, the accompanying data must have the following general form:

$$H\left\{[(ct)^2 - x^2]^{1/2}, [e^2 - (cp)^2]^{1/2}, \left(\frac{e+pc}{e-pc}\right)\left(\frac{ct-x}{ct+x}\right)\right\} = constant,$$

for some function $H(\delta, \Delta, \zeta)$.

3. Partial Differential Equations Arising from Lorentz Invariance

In this section, we make use of the infinitesimal versions of (6) and (8), thus, $x \approx X - vT$, $t \approx T - vX/c^2$, $e \approx E - Pv$, and $p \approx P - Ev/c^2$ to generate partial differential equations for Lorentz-invariant quantities. For example, consider a function that remains invariant under one of these transformations, such as $F(x,t) = F(X,T)$ or $G(e,p) = G(E,P)$. By expanding using Taylor's series, collecting first-order terms in v, and then reverting to lowercase variables, we can deduce the following first-order partial differential equations:

$$t\frac{\partial F}{\partial x} + \frac{x}{c^2}\frac{\partial F}{\partial t} = 0, \qquad p\frac{\partial G}{\partial e} + \frac{e}{c^2}\frac{\partial G}{\partial p} = 0,$$

and Lagrange's characteristic method gives the general solutions $F(x,t) = \Phi([(ct)^2 - x^2]^{1/2})$ and $G(e,p) = \Psi([e^2 - (pc)^2]^{1/2})$, where Φ and Ψ denote arbitrary functions of the indicated arguments.

Similarly, for Lorentz-invariant energy–momentum solutions to the forms $e = F(x,t)$ and $p = G(x,t)$, we suppose these relations remain invariant under both Lorentz transformations (6) and (8), and we may use the same procedure to deduce the two coupled partial differential equations (3). Upon introducing variables $\rho = e + pc$ and $\sigma = e - pc$, these uncouple to give the following:

$$t\frac{\partial \rho}{\partial x} + \frac{x}{c^2}\frac{\partial \rho}{\partial t} = \frac{\rho}{c}, \qquad t\frac{\partial \sigma}{\partial x} + \frac{x}{c^2}\frac{\partial \sigma}{\partial t} = -\frac{\sigma}{c}, \tag{13}$$

which are both readily solved using Lagrange's characteristic method. We introduce a characteristic parameter, s, such that we have the following:

$$\frac{dx}{ds} = t, \qquad \frac{dt}{ds} = \frac{x}{c^2}, \qquad \frac{d\rho}{ds} = \frac{\rho}{c}, \qquad \frac{d\sigma}{ds} = -\frac{\sigma}{c}. \tag{14}$$

From the first two, we obtain $\delta = [(ct)^2 - x^2]^{1/2} = constant$, and from the third and fourth, using $ct = (x^2 + \delta^2)^{1/2}$, we have the following:

$$\frac{d\rho}{dx} = \frac{\rho}{ct} = \frac{\rho}{(x^2+\delta^2)^{1/2}}, \qquad \frac{d\sigma}{dx} = -\frac{\sigma}{ct} = -\frac{\sigma}{(x^2+\delta^2)^{1/2}}.$$

For these integrations, δ can be regarded as a constant, so both may be integrated with the substitution $x = \delta \sinh\theta$, thus, we have the following:

$$\frac{d\rho}{\rho} = \frac{dx}{(x^2+\delta^2)^{1/2}} = d\theta, \qquad \frac{d\sigma}{\sigma} = -\frac{dx}{(x^2+\delta^2)^{1/2}} = -d\theta,$$

and upon using $\sinh^{-1}(z) = \log(z + (z^2+1)^{1/2})$, we obtain the following:

$$\rho = C_1\left(\frac{ct+x}{ct-x}\right)^{1/2}, \qquad \sigma = C_2\left(\frac{ct-x}{ct+x}\right)^{1/2}, \tag{15}$$

where C_1 and C_2 denote arbitrary constants. Thus, the general solutions to the coupled partial differential equations (13) are determined from the following:

$$\rho = e + pc = I(\delta)\left(\frac{ct+x}{ct-x}\right)^{1/2}, \qquad \sigma = e - pc = J(\delta)\left(\frac{ct-x}{ct+x}\right)^{1/2}, \tag{16}$$

where I and J denote arbitrary functions of $\delta = [(ct)^2 - x^2]^{1/2}$.

Specifically, the general Lorentz-invariant energy–momentum solutions to the forms $e = F(x,t)$ and $p = G(x,t)$ are given explicitly by the following:

$$\begin{aligned} e(x,t) &= \tfrac{1}{2}\left\{I(\delta)\left(\tfrac{ct+x}{ct-x}\right)^{1/2} + J(\delta)\left(\tfrac{ct-x}{ct+x}\right)^{1/2}\right\}, \\ cp(x,t) &= \tfrac{1}{2}\left\{I(\delta)\left(\tfrac{ct+x}{ct-x}\right)^{1/2} - J(\delta)\left(\tfrac{ct-x}{ct+x}\right)^{1/2}\right\}, \end{aligned} \tag{17}$$

where I and J denote arbitrary functions of $\delta = [(ct)^2 - x^2]^{1/2}$. From these equations, the particle velocity $u = u(x,t)$ is given by the following:

$$\frac{u(x,t)}{c} = \frac{cp(x,t)}{e(x,t)} = \frac{\frac{I(\delta)}{J(\delta)}\left(\frac{ct+x}{ct-x}\right) - 1}{\frac{I(\delta)}{J(\delta)}\left(\frac{ct+x}{ct-x}\right) + 1}, \tag{18}$$

upon noting that $\delta = [(ct)^2 - x^2]^{1/2}$, this expression is entirely in accord with the general solution (2).

We observe that these general solutions can be alternatively expressed as follows:

$$\begin{aligned} e(x,t) &= \tfrac{1}{2\delta}\{[I(\delta) + J(\delta)]ct + [I(\delta) - J(\delta)]x\}, \\ cp(x,t) &= \tfrac{1}{2\delta}\{[I(\delta) - J(\delta)]ct + [I(\delta) + I\delta)]x\}, \end{aligned}$$

so that in terms of redefined arbitrary functions $\Phi(\delta)$ and $\Psi(\delta)$, we obtain (4), and these relations provide the precise structures of the general solutions to (5) for energy–momentum.

The particle velocity $u = u(x,t)$ is given by the following:

$$\frac{u(x,t)}{c} = \frac{cp(x,t)}{e(x,t)} = \frac{I(\delta)(ct+x) - J(\delta)(ct-x)}{I(\delta)(ct+x) + J(\delta)(ct-x)} = \frac{x + ct\chi(\delta)}{ct + x\chi(\delta)},$$

where $\chi(\delta)$ denotes the single arbitrary function of δ, which is defined by $\chi(\delta) = [I(\delta) - J(\delta)]/[I(\delta) + J(\delta)] = \Phi(\delta)/\Psi(\delta)$. Upon noting that $u = dx/dt$, we observe that the particle paths may be obtained by integrating the following:

$$\frac{d\delta}{dt} = \frac{1}{\delta}(c^2t - xu) = \frac{1}{\delta}\left\{c^2t - xc\left(\frac{x + ct\chi(\delta)}{ct + x\chi(\delta)}\right)\right\} = \frac{c\delta}{ct + x\chi(\delta)}, \tag{19}$$

so, from (19) with $x = ((ct)^2 - \delta^2)^{1/2}$, we need to perform the following integration:

$$\frac{d\delta}{dt} = \frac{c\delta}{ct + ((ct)^2 - \delta^2)^{1/2}\chi(\delta)}.$$

Upon making the substitution $ct = \delta\cosh\phi$, this equation is as follows:

$$d\delta = \frac{\cosh\phi d\delta + \delta\sinh\phi d\phi}{\cosh\phi + \chi(\delta)\sinh\phi},$$

which formally integrates trivially to give $\phi = \int \chi(\delta)d\delta/\delta + constant$, and the particle paths may be formally obtained from $ct = \delta\cosh(\int \chi(\delta)d\delta/\delta + C)$, where $\delta = [(ct)^2 - x^2]^{1/2}$, $\chi(\delta)$ is related to the previously introduced arbitrary functions Φ and Ψ by $\chi(\delta) = \Phi(\delta)/\Psi(\delta)$ and C denotes an arbitrary constant.

4. Lorentz-Invariant Extension of Newton's Second Law

In special relativistic mechanics with one spatial dimension x, Newton's second law posits that the applied force f equals the total time rate of change of momentum. This is expressed as $f = \frac{d}{dt}\left(m\frac{dx}{dt}\right)$ for a particle of mass m. Accordingly, on face value, space x and time t play fundamentally different roles. In other words, in conventional special relativistic mechanics, Newton's second law states that the applied force is equal to the rate of change of momentum, expressed as $f = dp/dt$. Using the differential formula (22), the total time differential operator, d/dt, under a Lorentz transformation transforms according to the following:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = \frac{(1-(v/c)^2)^{1/2}}{(1 - Uv/c^2)}\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right),$$

and is, therefore, not Lorentz-invariant. Further, under a Lorentz transformation momentum, p alone is not Lorentz-invariant but satisfies the coupled Lorentz-invariant energy–momentum relations (8) and (9). In order to develop a Lorentz-invariant version of Newton's second law and a mechanical model in which space and time enjoy identical roles, the author of [1,4,5] proposed three-dimensional vectorial equations (20), so that the right-hand sides of equations, such as (21), are automatically invariant under the Lorentz transformation.

We first provide a synopsis of the Lorentz-invariant theory proposed in [1,4,5]. This model assumes that all quantities are position ($\mathbf{x}$) and time (t)-dependent, with the momentum vector $\mathbf{p} = m\mathbf{u}$ and particle energy $e = mc^2$, where $m = m_0(1 - (u/c)^2)^{-1/2}$ denotes the relativistic mass, m_0 denotes the constant rest mass, and u denotes the magnitude of the particle velocity. The following three-dimensional extension of Newton's second law is proposed, such that the force, $\mathbf{f}$, and energy–mass production, g, are given by the following:

$$\mathbf{f} = \frac{\partial\mathbf{p}}{\partial t} + \nabla e, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla\cdot\mathbf{p}, \tag{20}$$

and all space and time derivatives in (20) are partial. The right-hand sides are invariants under the Lorentz transformation, and if the forces $(\mathbf{f}, g)$ are generated from a potential $V(\mathbf{x}, t)$, such that we have the following:

$$\mathbf{f} = -\nabla V, \qquad gc^2 = -\frac{\partial V}{\partial t},$$

then there exists a conservation of energy principle $e + \mathscr{E} + V =$ constant. If the potential $V(\mathbf{x}, t)$ satisfies the classical wave equation, then the left-hand side of (20) is also invariant under the Lorentz transformation, so that in this circumstance, the full Equation (20) is invariant under the Lorentz transformation. Thus, if $(\mathbf{f}, gc)$ are generated as external forces from a scalar potential $V(\mathbf{x}, t)$, then the mechanical system is conservative in the sense that a conservation of energy principle applies $e + \mathscr{E} + V =$ constant, which we interpret as $Particle\ Energy + Wave\ Energy + Potential\ Energy = constant$.

The proposed model [1,4,5] aims to include both Newtonian mechanics and quantum mechanics in the form of Schrödinger's quantum wave theory in a single theory, and there are many situations for which Equation (20) coincides with conventional Newtonian mechanics. The model encompasses Newtonian mechanics with $g = 0$, and Schrödinger's quantum wave theory with $\mathbf{f} = \mathbf{0}$. It is a mechanical theory for which space and time are on equal footing but it is only invariant under the Lorentz transformations. It, therefore, shares some of the features of general relativity, such as the field determination that may emerge from the solution to the problem itself. The general theory of relativity is precisely what the name suggests, which is a general approach to mechanics that is invariant under arbitrary coordinate transformations. Accordingly, there is a complexity, such that it is sometimes difficult to judge what is or is not included in the theory.

For a single space dimension, x, the Lorentz-invariant equations for energy–momentum $e = F(x, t)$ and $p = G(x, t)$ respectively, are as follows:

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{21}$$

where $f(x, t)$ and $cg(x, t)$ denote the applied external forces in the space and time directions, respectively. From the Lorentz transformations (6), we may deduce the differential formulae as follows:

$$\frac{\partial}{\partial x} = \frac{1}{(1 - (v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \quad \frac{\partial}{\partial t} = \frac{1}{(1 - (v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\}, \tag{22}$$

which, together with the Lorentz transformation (8), we may show that (21) remains invariant; in other words, we have the following:

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = \frac{\partial P}{\partial T} + \frac{\partial E}{\partial X}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial E}{\partial T} + \frac{\partial P}{\partial X}.$$

Accordingly, the left-hand sides of (21) are automatically invariant under the combined Lorentz transformations (6) and (8). However, the full equations are invariant only if the applied forces are such that $f(x, t) = f(X, T)$ and $g(x, t) = g(X, T)$, which we have noted implies that $f(x, t)$ and $g(x, t)$ satisfy the partial differential equations, as follows:

$$t\frac{\partial f}{\partial x} + \frac{x}{c^2}\frac{\partial f}{\partial t} = 0, \qquad t\frac{\partial g}{\partial x} + \frac{x}{c^2}\frac{\partial g}{\partial t} = 0, \tag{23}$$

and, therefore, we have general solutions $f(x, t) = f([(ct)^2 - x^2]^{1/2}) = f(\delta)$ and $g(x, t) = g([(ct)^2 - x^2]^{1/2}) = g(\delta)$, where $\delta = [(ct)^2 - x^2]^{1/2}$ and f and g now denote arbitrary functions of δ. This expected outcome is also obtained alternatively, as follows.

Assuming the Lorentz-invariant theory [1], and for prescribed applied external forces $f(x, t)$ and $cg(x, t)$, we may view the four equations (3) and (21) as determining equations for the four partial derivatives $\partial e/\partial x$, $\partial e/\partial t$, $\partial p/\partial x$, and $\partial p/\partial t$, which upon solving yields the following:

$$\begin{aligned} \frac{\partial e}{\partial x} &= \frac{\{(ex+c^2tp)-Ax\}}{\delta^2}, \qquad \frac{\partial e}{\partial t} = \frac{c^2\{At-(et+xp)\}}{\delta^2}, \\ \frac{\partial p}{\partial x} &= \frac{\{(px+et)-Bx\}}{\delta^2}, \qquad \frac{\partial p}{\partial t} = \frac{\{c^2Bt-(ex+c^2tp)\}}{\delta^2}, \end{aligned} \tag{24}$$

where $\delta = [(ct)^2 - x^2]^{1/2}$, $A(x,t) = xf(x,t) + c^2tg(x,t)$ and $B(x,t) = tf(x,t) + xg(x,t)$. Now, when performing the various partial differentiations and using Equation (24) in the final evaluation of the two compatibility equations, we have the following:

$$\frac{\partial}{\partial t}\left(\frac{\partial e}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial e}{\partial t}\right), \qquad \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right), \tag{25}$$

we may confirm that these two compatibility relations together with (24) ultimately yield the following two equations:

$$t\frac{\partial A}{\partial x} + \frac{x}{c^2}\frac{\partial A}{\partial t} = B, \qquad t\frac{\partial B}{\partial x} + \frac{x}{c^2}\frac{\partial B}{\partial t} = \frac{A}{c^2}.$$

which have general solutions as given by (17). It is clear that these general solutions are entirely consistent with the definitions of A and B, namely, $A = xf + c^2tg$ and $B = tf + xg$, and the general solutions (17) and those of Equation (23), namely $f = f(\delta)$ and $g = g(\delta)$; here, both f and g now denote arbitrary functions. Thus, the two compatibility conditions (25) together with the four relations (24) are formally equivalent to the invariance of the applied external forces f and g (namely $f(x,t) = f(X,T)$ and $g(x,t) = g(X,T)$) under the Lorentz transformation (6). Further, the functions $f(\delta)$ and $g(\delta)$ are related to the arbitrary functions $\Phi(\delta)$ and $\Psi(\delta)$ appearing in (4) through the following relations:

$$f(\delta) = 2\Phi(\delta) + \delta\frac{d\Phi(\delta)}{d\delta}, \qquad cg(\delta) = 2\Psi(\delta) + \delta\frac{d\Psi(\delta)}{d\delta}. \tag{26}$$

In the following section, we examine how the Lorentz-invariant power-law energies first obtained in [26] fit into the general formulation.

Finally, in this section, as an example, one of the major physical characteristics of the theory [1] is the possibility of sustaining motion in the absence of any applied forces since the particle energy alone can act as an applied potential generating the motion. For the general solution (4), it is apparent that such a possibility arises if the arbitrary functions $\Phi(\delta)$ and $\Psi(\delta)$ are chosen, such that $f = g = 0$, namely $\Phi(\delta) = C_1/\delta^2$ and $\Psi(\delta) = C_2/\delta^2$, where C_1 and C_2 denote arbitrary constants; then, we have the following:

$$e(x,t) = \frac{C_1x + C_2ct}{(ct)^2 - x^2}, \qquad cp(x,t) = \frac{C_1ct + C_2x}{(ct)^2 - x^2},$$

which can be alternatively expressed in terms of singularities at $\pm c$; thus, we have the following:

$$e(x,t) = \frac{1}{2}\frac{(C_1 + C_2)}{ct - x} - \frac{1}{2}\frac{(C_1 - C_2)}{ct + x}, \quad cp(x,t) = \frac{1}{2}\frac{(C_1 + C_2)}{ct - x} + \frac{1}{2}\frac{(C_1 - C_2)}{ct + x},$$

and the velocity distribution is $u(x,t) = c(C_1ct + C_2x)/(C_1x + C_2ct)$.

5. Power-Law Lorentz-Invariant Energy–Momentum Relations

In this section, we examine the Lorentz-invariant power-law energy–momentum relations (28) stated below and originally derived in [26] (see also [1] (page 49)). We note that Einstein's energy expression, $e^2 - (pc)^2 = e_0^2$, where e_0 denotes the constant rest energy, follows the conventional notion for the accrual of particle energy, e, defined as the accumulated work done due to an applied force f arising from $de = fdx$, and $f = dp/dt$ gives $de = udp$ or $de/dp = u$, as may be confirmed by a straightforward differentiation of $e^2 - (pc)^2 = e_0^2$; thus, $de/dp = c^2p/e = u$. For the power-law energy expressions (28) given below, the equation $de = f dx$ no longer applies, and $de/dp \neq u$. This means that the accrual of particle energy takes place through mechanisms that are additional to the applied external force. In other areas of applied mathematics, the use of power-law expressions as approximations for fitting experimental data has been extremely effective.

The power-law energy–momentum relations given below are based on the Lorentz-invariant energy–momentum equation, that is,

$$\frac{de}{dp} = c\left(\frac{\kappa + u/c}{1 + \kappa u/c}\right), \tag{27}$$

involving the arbitrary constant κ, for which the particle and wave velocities arise as two special cases corresponding, respectively, to the values $\kappa = 0$ ($de/dp = u$) and $\kappa = \pm\infty$ ($de/dp = c^2/u$). In [1,26], it is shown that (27) admits the following energy–momentum expressions:

$$\begin{aligned} e(u) &= \tfrac{e_0}{(1-(u/c)^2)^{1/2}}\left(\tfrac{1+u/c}{1-u/c}\right)^{\kappa/2}, \\ cp(u) &= \tfrac{e_0(u/c)}{(1-(u/c)^2)^{1/2}}\left(\tfrac{1+u/c}{1-u/c}\right)^{\kappa/2}, \end{aligned} \tag{28}$$

where e_0 denotes the rest energy, and the following relation applies:

$$e(u)^2 - (cp(u))^2 = e_0^2\left(\frac{1+u/c}{1-u/c}\right)^{\kappa} = e_0^2\left(\frac{e+cp}{e-cp}\right)^{\kappa}, \tag{29}$$

upon using $u/c = cp/e$. This equation in turn is as follows:

$$(e+cp)^{1-\kappa}(e-cp)^{1+\kappa} = e_0^2, \tag{30}$$

which agrees with Einstein's relation $e^2 - (cp)^2 = e_0^2$ for $\kappa = 0$.

While Equation (27) is fully Lorentz-invariant under the energy–momentum transformations (8), for the entire formulation given by (28) and (29) to be invariant, the assumed constant rest energies e_0 and E_0 must transform according to the following relation:

$$E_0 = e_0\left(\frac{1-v/c}{1+v/c}\right)^{\kappa/2}. \tag{31}$$

Along with $e(-u) \neq e(u)$, this means that Expression (28) for $\kappa \neq 0$ exhibits both a directional and frame dependence. A directional dependence has always been inherent in the combined Planck–de Broglie relations $e = \pm pc$ (see [26] and [1] (page 52) for further details). Quite independently, both [32,33] have suggested that the universe may have a directional dependence in the sense that the properties of the universe may not be isotropic and a preferred direction may exist. The frame dependence (31) indicates that the universe may also be inhomogeneous as well as anisotropic.

In [26] and [1] (page 49), the invariance of (27) under the Lorentz transformation (6) is established using Einstein's addition of velocities law. We may alternatively demonstrate this invariance using the variables $\rho = e + pc$ and $\sigma = e - pc$ and the Lorentz transformations (8). In terms of these new variables, Equation (27) simply becomes as follows:

$$\frac{d\rho}{d\sigma} = -\frac{1}{\lambda}\frac{\rho}{\sigma}, \tag{32}$$

where $\lambda = (1-\kappa)/(1+\kappa)$ and from (10) or (11), the following is clear:

$$\frac{\rho}{\sigma} = \frac{e+pc}{e-pc} = \left(\frac{1-v/c}{1+v/c}\right)\left(\frac{E+Pc}{E-Pc}\right), \tag{33}$$

and Equation (32) is unchanged by (33) and integrates immediately to give $\rho^{\lambda}\sigma = constant$ or $\rho^{1-\kappa}\sigma^{1+\kappa} = constant$ in agreement with (30).

If in addition to assuming Lorentz-invariant energy $e = e(x,t)$ and momentum $p = p(x,t)$ solutions and the Lorentz-invariant theory [1], we impose the further condition of the power-law energy–momentum relation (29), then from (16) we have the following:

$$e(u)^2 - (cp(u))^2 = I(\delta)J(\delta) = e_0^2\left(\frac{e+cp}{e-cp}\right)^{\kappa} = e_0^2\left(\frac{I(\delta)}{J(\delta)}\left(\frac{ct+x}{ct-x}\right)\right)^{\kappa},$$

so that for $e_0 \neq 0$ and $\kappa \neq 0$, we have the following:

$$\frac{I(\delta)}{J(\delta)}\left(\frac{ct+x}{ct-x}\right) = \left(\frac{I(\delta)J(\delta)}{e_0^2}\right)^{1/\kappa},$$

and from Expression (18) for the particle velocity $u = u(x,t)$, there is the immediate implication that $u = u(\delta)$. However, from (1), it is clear that the only particle velocity for which this is possible is the speed of light, namely $u = \pm c$. Accordingly, the imposition of Lorentz-invariant energy–momentum solutions to a Lorentz-invariant theory, together with the assumption of the power-law energy relations, restricts the particle motion to the constant speed of light.

6. Summary and Conclusions

Any assumption of spatial invariance means that similarity solutions to partial differential equations necessarily only apply to isolated single-particle systems that are dominated either by a much larger particle or by some symmetrically applied external force. However, they often correspond to the most fundamental problems in the discipline. This is certainly the case in topics such as heat conduction and fluid mechanics and will be shown to also be the case for the fundamental problems of mathematical physics. Conventional special relativistic mechanics is a “space” dominated model, whereas Schrödinger’s quantum wave theory is a “time” dominated model. Here, the areas of application in mind are those intermediate situations where both space and time contribute. It is natural to first consider the problems in one spatial dimension.

The vectorial equations (20) apply to two and three spatial dimensions and exhibit numerous similarity energy–momentum profiles. In particular, for both axially and centrally symmetric particle motions (see [1] (page 233)), following [24,25], there will be interesting results for higher dimensional problems (although clearly far more complicated). For multi-particle systems, the general vectorial equations (20) still apply for each component particle, provided that the appropriate particle–particle interactions are properly incorporated in the applied external forces $(\mathbf{f}, gc)$ applied to each individual particle.

In this paper, we investigated some of the implications of the assumption of Lorentz invariance both with regard to mechanical theories and the formal solutions to such theories. Lorentz-invariant theories or solutions with this characteristic are stronger and more powerful than conventional Newtonian theories or conventional solutions because the Lorentz-invariant approach automatically embodies the conventional approach. This is the case for the Lorentz-invariant power-law energy–momentum relations (28) involving the arbitrary parameter, κ, which are inclusive of Einstein’s energy relations arising from the special case, $\kappa = 0$. It is also the case for the Lorentz-invariant extension of Newton’s second law proposed in [1], which included special relativistic mechanics and Schrödinger’s quantum wave theory.

For any Lorentz-invariant mechanical theory, not all solutions will be invariant, and here we have focussed solely on determining Lorentz-invariant solutions to a particular Lorentz-invariant theory [1]. For Theory [1], and for a single spatial dimension, x, the governing equations for energy $e(x,t) = mc^2$ and momentum $p(x,t) = mu$ are given by (21), where $m(x,t)$ is mass, $u(x,t) = dx/dt$ is velocity, and $f(x,t)$ and $cg(x,t)$ are the applied forces in the space and time directions, respectively. We have shown that the most general Lorentz-invariant solutions to (3) and (21), of the forms $e = F(x,t)$ and $p = G(x,t)$, are given by (4), where $\Phi(\delta)$ and $\Psi(\delta)$ are arbitrary functions of $\delta = [(ct)^2 - x^2]^{1/2}$, provided that both f and g are both Lorentz-invariant functions, implying that $f = f(\delta)$ and $g = g(\delta)$, and are related to the arbitrary functions $\Phi(\delta)$ and $\Psi(\delta)$ appearing in (4) through (26).

With regard to the formal solutions to (3) and (21), it is clear that further incorporation of the other two Lorentz invariants $\Delta = [e^2 - (cp)^2]^{1/2}$ and $\zeta = [(e+pc)(ct-x)]/[(e-pc)(ct+x)]$ will not greatly affect these general solutions; in Appendix A, we show that (4) can be generalised by (5). This is possible since, ultimately, both Δ and ζ must reduce to functions of δ only, namely, $\Delta = \Delta(\delta)$ and $\zeta = \zeta(\delta)$, along with the assumption that the applied forces have the structures $f = f(\delta, \Delta, \zeta)$ and $g = g(\delta, \Delta, \zeta)$, and are related to the functions $\Phi(\delta, \Delta, \zeta)$ and $\Psi(\delta, \Delta, \zeta)$ appearing in (5) through Relation (A5), in overall agreement with (26). In other words, for prescribed applied forces $f = f(\delta, \Delta, \zeta)$ and $g = g(\delta, \Delta, \zeta)$, (5) provides an exact solution to the four equations (3) and (21), provided that the functions $\Phi(\delta, \Delta, \zeta)$ and $\Psi(\delta, \Delta, \zeta)$ satisfy (A5).

In this paper, we have determined energy–momentum profiles $e = F(x,t)$ and $p = G(x,t)$ given by (4) corresponding to the general Lorentz-invariant solutions to (3) and (21). While these solutions strictly only apply to a single space dimension x, they are nevertheless general solutions involving two arbitrary functions. As such, they may be applicable for particle paths well approximated by straight lines or for approximating particle paths with several straight segments. This approach can be useful in testing Lorentz invariance violations, such as with atmospheric neutrino experiments [11] or in testing Lorentz symmetry for photons travelling from the outer universe, revealing the early cosmic expansion history of the universe [12].

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Appendix A. Extending the General Solutions (4)

In this appendix, we present the formal details establishing that the general solutions to (3) and (21), namely (4), may be generalised by expressions of the form (5), where $\Phi(\delta, \Delta, \zeta)$ and $\Psi(\delta, \Delta, \zeta)$ denote two arbitrary functions of the three Lorentz invariants (δ, Δ, ζ), which are defined by the following:

$$\delta = [(ct)^2 - x^2]^{1/2}, \qquad \Delta = [e^2 - (cp)^2]^{1/2}, \qquad \zeta = \left(\frac{e+pc}{e-pc}\right)\left(\frac{ct-x}{ct+x}\right),$$

and by making use of (5), we have the following identities:

$$\Delta^2 = (\Psi^2 - \Phi^2)\delta^2, \qquad \zeta = \frac{\Psi + \Phi}{\Psi - \Phi},$$

since $\rho = e + cp = (\Psi + \Phi)(ct + x)$ and $\sigma = e - cp = (\Psi - \Phi)(ct - x)$.

Clearly, the implicit expression (5) is not as useful as the explicit general Solution (4). However, their existence provides considerable insight into the formal structures of solutions and the vast complexities of possible solution behaviours. In the following calculations, we assume either basic Equations (3) and (21) or Expression (24) for the four partial derivatives, $\partial e/\partial x$, $\partial e/\partial t$, $\partial p/\partial x$, and $\partial p/\partial t$. In other words, we make use of one form of the determining Equations (3) and (21). In the following, we calculate the partial derivatives, $\partial/\partial x$, and $\partial/\partial t$, based on Expression (5) using the chain rule for partial differentiation. The final results for Equations (3) and (21) are as follows:

$$
\begin{aligned}
t\tfrac{\partial e}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial e}{\partial t} &= t\Phi+\tfrac{x}{c}\Psi+\left(x\tfrac{\partial\Phi}{\partial\delta}+ct\tfrac{\partial\Psi}{\partial\delta}\right)\left(t\tfrac{\partial\delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\delta}{\partial t}\right)\\
&+\left(x\tfrac{\partial\Phi}{\partial\Delta}+ct\tfrac{\partial\Psi}{\partial\Delta}\right)\left(t\tfrac{\partial\Delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\Delta}{\partial t}\right)+\left(x\tfrac{\partial\Phi}{\partial\zeta}+ct\tfrac{\partial\Psi}{\partial\zeta}\right)\left(t\tfrac{\partial\zeta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\zeta}{\partial t}\right),\\
t\tfrac{\partial p}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial p}{\partial t} &= \tfrac{t}{c}\Psi+\tfrac{x}{c^2}\Phi+\left(t\tfrac{\partial\Phi}{\partial\delta}+\tfrac{x}{c}\tfrac{\partial\Psi}{\partial\delta}\right)\left(t\tfrac{\partial\delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\delta}{\partial t}\right)\\
&+\left(t\tfrac{\partial\Phi}{\partial\Delta}+\tfrac{x}{c}\tfrac{\partial\Psi}{\partial\Delta}\right)\left(t\tfrac{\partial\Delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\Delta}{\partial t}\right)+\left(t\tfrac{\partial\Phi}{\partial\zeta}+\tfrac{x}{c}\tfrac{\partial\Psi}{\partial\zeta}\right)\left(t\tfrac{\partial\zeta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\zeta}{\partial t}\right),\\
\tfrac{\partial p}{\partial t}+\tfrac{\partial e}{\partial x} &= \tfrac{\partial\Phi}{\partial\delta}\left(x\tfrac{\partial\delta}{\partial x}+t\tfrac{\partial\delta}{\partial t}\right)+\tfrac{\partial\Phi}{\partial\Delta}\left(x\tfrac{\partial\Delta}{\partial x}+t\tfrac{\partial\Delta}{\partial t}\right)+\tfrac{\partial\Phi}{\partial\zeta}\left(x\tfrac{\partial\zeta}{\partial x}+t\tfrac{\partial\zeta}{\partial t}\right)\\
&+2\Phi+c\tfrac{\partial\Psi}{\partial\delta}\left(t\tfrac{\partial\delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\delta}{\partial t}\right)+c\tfrac{\partial\Psi}{\partial\Delta}\left(t\tfrac{\partial\Delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\Delta}{\partial t}\right)+c\tfrac{\partial\Psi}{\partial\zeta}\left(t\tfrac{\partial\zeta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\zeta}{\partial t}\right), \qquad \text{(A1)}\\
\tfrac{1}{c^2}\tfrac{\partial e}{\partial t}+\tfrac{\partial p}{\partial x} &= \tfrac{\partial\Phi}{\partial\delta}\left(t\tfrac{\partial\delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\delta}{\partial t}\right)+\tfrac{\partial\Phi}{\partial\Delta}\left(t\tfrac{\partial\Delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\Delta}{\partial t}\right)+\tfrac{\partial\Phi}{\partial\zeta}\left(t\tfrac{\partial\zeta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\zeta}{\partial t}\right)\\
&+\tfrac{2}{c}\Psi+\tfrac{1}{c}\tfrac{\partial\Psi}{\partial\delta}\left(x\tfrac{\partial\delta}{\partial x}+t\tfrac{\partial\delta}{\partial t}\right)+\tfrac{1}{c}\tfrac{\partial\Psi}{\partial\Delta}\left(x\tfrac{\partial\Delta}{\partial x}+t\tfrac{\partial\Delta}{\partial t}\right)+\tfrac{1}{c}\tfrac{\partial\Psi}{\partial\zeta}\left(x\tfrac{\partial\zeta}{\partial x}+t\tfrac{\partial\zeta}{\partial t}\right). \qquad \text{(A2)}
\end{aligned}
$$

Now, each term vanishes, namely the following:

$$t\frac{\partial\delta}{\partial x}+\frac{x}{c^2}\frac{\partial\delta}{\partial t}=t\frac{\partial\Delta}{\partial x}+\frac{x}{c^2}\frac{\partial\Delta}{\partial t}=t\frac{\partial\zeta}{\partial x}+\frac{x}{c^2}\frac{\partial\zeta}{\partial t}=0,$$

due to Lorentz invariance and the fact that δ, Δ, and ζ are invariants. The invariant character of these results is most easily established using the variables $\rho = e + pc$, $\sigma = e - pc$ and the characteristic equations defined by (14), namely the following:

$$\frac{dx}{ds}=t, \qquad \frac{dt}{ds}=\frac{x}{c^2}, \qquad \frac{d\rho}{ds}=\frac{\rho}{c}, \qquad \frac{d\sigma}{ds}=-\frac{\sigma}{c}. \qquad \text{(A3)}$$

When using (A3) with $\Delta^2 = \rho\sigma$ and $\zeta = \rho(ct - x)/\sigma(ct + x)$, we have the following:

$$
\begin{aligned}
2\Delta\tfrac{d\Delta}{ds} &= \rho\tfrac{d\sigma}{ds}+\sigma\tfrac{d\rho}{ds}=-\tfrac{\rho\sigma}{c}+\tfrac{\rho\sigma}{c}=0,\\
\tfrac{d\zeta}{ds} &= \left(\tfrac{\rho}{\sigma}\right)\tfrac{d}{ds}\left(\tfrac{ct-x}{ct+x}\right)+\left(\tfrac{ct-x}{ct+x}\right)\tfrac{d}{ds}\left(\tfrac{\rho}{\sigma}\right)\\
&= -\tfrac{2\rho}{c\sigma}\left(\tfrac{ct-x}{ct+x}\right)+\tfrac{2\rho}{c\sigma}\left(\tfrac{ct-x}{ct+x}\right)=0.
\end{aligned}
$$

Below, we provide formal proofs. The fact that $\delta = [(ct)^2 - x^2]^{1/2}$ satisfies this equation is immediate upon evaluating the two partial derivatives, $\partial\delta/\partial x = -x/\delta$ and $\partial\delta/\partial t = c^2 t/\delta$. For $\Delta = [e^2 - (cp)^2]^{1/2}$ and the identity $t\partial\Delta/\partial x + x/c^2\partial\Delta/\partial t = 0$, we can either use the partial differential relations (3) or the explicit formulae (24). In the former event, we have the following:

$$
\begin{aligned}
t\tfrac{\partial\Delta}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial\Delta}{\partial t} &= \tfrac{t}{\Delta}\left(e\tfrac{\partial e}{\partial x}-c^2p\tfrac{\partial p}{\partial x}\right)+\tfrac{x}{\Delta c^2}\left(e\tfrac{\partial e}{\partial t}-c^2p\tfrac{\partial p}{\partial t}\right)\\
&= \tfrac{e}{\Delta}\left(t\tfrac{\partial e}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial e}{\partial t}\right)-\tfrac{c^2p}{\Delta}\left(t\tfrac{\partial p}{\partial x}+\tfrac{x}{c^2}\tfrac{\partial p}{\partial t}\right)=0.
\end{aligned}
$$

In establishing the identity $t\partial\zeta/\partial x + x/c^2\partial\zeta/\partial t = 0$, we need the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$, which are employed throughout [1] and are first introduced on page 41. In evaluating the partial derivatives $\partial\zeta/\partial x$ and $\partial\zeta/\partial t$, we obtain the following:

$$
\begin{aligned}
\tfrac{\partial\zeta}{\partial x} &= -\left(\tfrac{e+pc}{e-pc}\right)\tfrac{2ct}{(ct+x)^2}-\left(\tfrac{ct-x}{ct+x}\right)\tfrac{2c(e\partial p/\partial x-p\partial e/\partial x)}{(e-pc)^2},\\
\tfrac{\partial\zeta}{\partial t} &= \left(\tfrac{e+pc}{e-pc}\right)\tfrac{2cx}{(ct+x)^2}+\left(\tfrac{ct-x}{ct+x}\right)\tfrac{2c(e\partial p/\partial t-p\partial e/\partial t)}{(e-pc)^2},
\end{aligned}
$$

and upon using Expression (24) for the partial derivatives, we obtain the following:

$$\frac{\partial\zeta}{\partial x} = -\left(\frac{e+pc}{e-pc}\right)\frac{2ct}{(ct+x)^2} + \frac{2c}{\delta^2}\left(\frac{ct-x}{ct+x}\right)\frac{(t\Delta^2+x(f\eta-g\xi))}{(e-pc)^2},$$
$$\frac{\partial\zeta}{\partial t} = \left(\frac{e+pc}{e-pc}\right)\frac{2cx}{(ct+x)^2} - \frac{2c}{\delta^2}\left(\frac{ct-x}{ct+x}\right)\frac{(x\Delta^2+c^2t(f\eta-g\xi))}{(e-pc)^2},$$

and these expressions further simplify, giving the following:

$$\frac{\partial\zeta}{\partial x} = \frac{2cx}{\delta^2}\left(\frac{ct-x}{ct+x}\right)\frac{(f\eta-g\xi)}{(e-pc)^2} = \frac{2cx\zeta(f\eta-g\xi)}{\delta^2\Delta^2},$$
$$\frac{\partial\zeta}{\partial t} = -\frac{2c^3t}{\delta^2}\left(\frac{ct-x}{ct+x}\right)\frac{(f\eta-g\xi)}{(e-pc)^2} = -\frac{2c^3t\zeta(f\eta-g\xi)}{\delta^2\Delta^2},$$

from which, evidently, we have the following:

$$t\frac{\partial\zeta}{\partial x} + \frac{x}{c^2}\frac{\partial\zeta}{\partial t} = t\frac{2cx\zeta(f\eta-g\xi)}{\delta^2\Delta^2} - \frac{x}{c^2}\frac{2c^3t\zeta(f\eta-g\xi)}{\delta^2\Delta^2} = 0,$$

as required.

We might similarly establish the following identities:

$$x\frac{\partial\delta}{\partial x} + t\frac{\partial\delta}{\partial t} = \delta, \qquad x\frac{\partial\Delta}{\partial x} + t\frac{\partial\Delta}{\partial t} = \frac{1}{\Delta}(f\xi - c^2g\eta - \Delta^2),$$
$$x\frac{\partial\zeta}{\partial x} + t\frac{\partial\zeta}{\partial t} = -\frac{2c\zeta}{\Delta^2}(f\eta - g\xi).$$

The first and third are immediately apparent using the given expressions for the partial derivatives. For the second result, we have the following:

$$\begin{aligned} f\xi - c^2g\eta &= \left(\frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}\right)(ex - c^2pt) - \left(\frac{\partial e}{\partial t} + c^2\frac{\partial p}{\partial x}\right)(px - et) \\ &= x\left(e\frac{\partial e}{\partial x} - c^2p\frac{\partial p}{\partial x}\right) + t\left(e\frac{\partial e}{\partial t} - c^2p\frac{\partial p}{\partial t}\right) \\ &\quad + c^2e\left(t\frac{\partial p}{\partial x} + \frac{x}{c^2}\frac{\partial p}{\partial t}\right) - c^2p\left(t\frac{\partial e}{\partial x} + \frac{x}{c^2}\frac{\partial e}{\partial t}\right) \\ &= \Delta\left(x\frac{\partial\Delta}{\partial x} + t\frac{\partial\Delta}{\partial t}\right) + e^2 - (cp)^2, \end{aligned}$$

upon using Relation (3), from which the desired result follows.

Thus, from (21), (A1), and (A2), we obtain the following expressions for the applied f and g forces:

$$f = 2\Phi + \frac{\partial\Phi}{\partial\delta}\left(x\frac{\partial\delta}{\partial x} + t\frac{\partial\delta}{\partial t}\right) + \frac{\partial\Phi}{\partial\Delta}\left(x\frac{\partial\Delta}{\partial x} + t\frac{\partial\Delta}{\partial t}\right) + \frac{\partial\Phi}{\partial\zeta}\left(x\frac{\partial\zeta}{\partial x} + t\frac{\partial\zeta}{\partial t}\right),$$

$$gc = 2\Psi + \frac{\partial\Psi}{\partial\delta}\left(x\frac{\partial\delta}{\partial x} + t\frac{\partial\delta}{\partial t}\right) + \frac{\partial\Psi}{\partial\Delta}\left(x\frac{\partial\Delta}{\partial x} + t\frac{\partial\Delta}{\partial t}\right) + \frac{\partial\Psi}{\partial\zeta}\left(x\frac{\partial\zeta}{\partial x} + t\frac{\partial\zeta}{\partial t}\right),$$

which, upon using the above results, simplifies to yield the following:

$$f = 2\Phi + \delta\frac{\partial\Phi}{\partial\delta} + \frac{1}{\Delta}\frac{\partial\Phi}{\partial\Delta}(f\xi - c^2g\eta - \Delta^2) - \frac{2c\zeta}{\Delta^2}\frac{\partial\Phi}{\partial\zeta}(f\eta - g\xi),$$

$$gc = 2\Psi + \delta\frac{\partial\Psi}{\partial\delta} + \frac{1}{\Delta}\frac{\partial\Psi}{\partial\Delta}(f\xi - c^2g\eta - \Delta^2) - \frac{2c\zeta}{\Delta^2}\frac{\partial\Psi}{\partial\zeta}(f\eta - g\xi).$$

Now, when making use of the above-given expressions for the various partial derivatives, either from the vanishing of the three Jacobians, that is,

$$\frac{\partial(\delta,\Delta)}{\partial(x,t)} = \frac{\partial(\delta,\zeta)}{\partial(x,t)} = \frac{\partial(\Delta,\zeta)}{\partial(x,t)} = 0,$$

or the two equations, that is, $t\partial\Delta/\partial x + x/c^2\partial\Delta/\partial t = 0$ and $t\partial\zeta/\partial x + x/c^2\partial\zeta/\partial t = 0$, this implies that, ultimately, both Δ and ζ must reduce to functions of δ only, namely $\Delta = \Delta(\delta)$ and $\zeta = \zeta(\delta)$ with the following implications:

$$f\xi - c^2 g\eta = \Delta\left(\Delta + \delta\frac{d\Delta}{d\delta}\right), \qquad c(f\eta - g\xi) = -\frac{\delta\Delta^2}{2\zeta}\frac{d\zeta}{d\delta}, \tag{A4}$$

assuming that the applied forces have the structures $f = f(\delta, \Delta, \zeta)$, and $g = g(\delta, \Delta, \zeta)$, and are related to the functions $\Phi(\delta, \Delta, \zeta)$ and $\Psi(\delta, \Delta, \zeta)$

$$\begin{aligned} f &= 2\Phi + \delta\tfrac{\partial\Phi}{\partial\delta} + \delta\tfrac{\partial\Phi}{\partial\Delta}\tfrac{d\Delta}{d\delta} + \delta\tfrac{\partial\Phi}{\partial\zeta}\tfrac{d\zeta}{d\delta} = 2\Phi + \delta\left(\tfrac{\partial\Phi}{\partial\delta} + \tfrac{\partial\Phi}{\partial\Delta}\tfrac{d\Delta}{d\delta} + \tfrac{\partial\Phi}{\partial\zeta}\tfrac{d\zeta}{d\delta}\right) \\ &= 2\Phi + \delta\tfrac{d\Phi}{d\delta}, \\ gc &= 2\Psi + \delta\tfrac{\partial\Psi}{\partial\delta} + \delta\tfrac{\partial\Psi}{\partial\Delta}\tfrac{d\Delta}{d\delta} + \delta\tfrac{\partial\Psi}{\partial\zeta}\tfrac{d\zeta}{d\delta} = 2\Psi + \delta\left(\tfrac{\partial\Psi}{\partial\delta} + \tfrac{\partial\Psi}{\partial\Delta}\tfrac{d\Delta}{d\delta} + \tfrac{\partial\Psi}{\partial\zeta}\tfrac{d\zeta}{d\delta}\right) \\ &= 2\Psi + \delta\tfrac{d\Psi}{d\delta}. \end{aligned} \tag{A5}$$

This coincides with (26). Overall, for the given applied forces $f = f(\delta, \Delta, \zeta)$ and $g = g(\delta, \Delta, \zeta)$, along with Relations (A4) and (A5) as two determining equations for the Lorentz invariants, ξ and η imply that ξ and η must be functions of δ only, as might be expected for Lorentz-invariant quantities.

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