Wave functions and scalar products in the Bethe ansatz

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Thèse de doctorat

Résumé

La physique a pour but de décrire la nature, c'est à dire d'expliquer et prédire des phénomènes produits en laboratoire ou se produisant dans l'univers observable. Cette description repose sur un cadre mathématique, un langage formel, au travers duquel sont formulées nos théories. Dans le cadre de cette thèse, ces théories sont la mécanique quantique et la mécanique stochastique. Dans ces contextes, la description des systèmes physique et de leur dynamique est plutôt simple et transparente.

Considérons un système physique. Un état de ce système sera décrit par un élément

$$|\psi\rangle \in \mathcal{H} = \text{ensemble of configurations}$$

appartenant à un espace vectoriel qui décrit l'ensemble des configurations accessible à ce système.
Un tel état peut être décomposé dans une base de vecteurs générant l'espace des configurations tout entier

$$|\psi\rangle = \sum_{\bar{x}} \psi(\bar{x}) \, |\bar{x}\rangle$$

où les objets $|\bar{x}\rangle$ peuvent par exemple être les états de la base des configurations dans l'espace.
Dans le cadre de la mécanique quantique, la mesure $\psi(\bar{x})$, c'est à dire la fonction d'onde dans l'espace des positions, donne accès à la probabilité $|\psi(\bar{x})|^2$ de mesurer l'état $|\psi\rangle$ dans la configuration $|\bar{x}\rangle$. Dans le cadre de la mécanique stochastique, la mesure $\psi(\bar{x})$ est simplement la probabilité de mesurer le système dans l'état $|\psi\rangle$ dans la configuration $|\bar{x}\rangle$.
La dynamique de notre système sera décrite par un equation d'évolution, l'équation de Scrhödinger dans le cadre de la mécanique quantique, qui prend la forme

$$i\partial_t \, |\psi, t\rangle = H \, |\psi, t\rangle$$

Le but final d'une investigation théorique, telle que sera la notre à travers cette thèse, est d'exprimer, sous forme exploitable, les valeurs moyennes de certaines quantités physique dépendant du temps

$$\langle \mathcal{O} \rangle_t = \langle \psi, t | \, \mathcal{O} \, | \psi, t \rangle$$

La majeur partie des système physiques, donnant potentiellement une description pertinente de la nature, ne sont pas exactement solubles, en cela que les quantité physiques qui y sont rattachées ne peuvent être exprimés de manière exacte, analytique. Cela induit un recours à des méthodes de perturbations, à du calcul numérique etc.
Certains rares systèmes physiques simple sont cependant intégrable, en cela que certaines quantités physiques, telles que le spectre en énergie, des fonctions de correlation, etc peuvent être exprimé analytiquement. Ces système sont pour la très simple, dans leur dynamique et dans leur dimensionnalité (la plupart des systèmes intégrables connus, tels que ceux considérés dans cette thèse, sont uni ou bi-dimensionnels) et ne peuvent ainsi être de bon candidats pour décrire notre monde quadri-dimensionnels. Ils peuvent cependant s'avérer être des "modèles jouets" très intéressants, permettant d'étudier certains comportements apparaissant dans des modèles plus complexes non-intégrables, et trouvent des applications dans de nombreux domaines de la physique théoriques tels que la théorie des cordes et la physique de la matière condensée. Cet intérêt croissant pour les modèles intégrables a mené au développement de nombreuses technologies mathématiques puissantes pour leur étude, et notamment l'ansatz de Bethe, qui constitue le cadre technique de cette thèse.

Accéder à une telle quantité $\langle\mathcal{O}\rangle_t$ requière de résoudre certains problèmes formels.

Premièrement, le problème spectral, en d'autres termes la diagonalisation de le Hamiltonien H, qui se traduit par l'équation aux valeurs propres

$$H\,|\psi(\bar{u})\rangle = \epsilon(\bar{u})\,|\psi(\bar{u})\rangle\,,\ \bar{u}\in\ \{\text{ensemble of solution}\}$$

qui donne directement accès à l'évolution temporel de toute superposition de solution

$$|\psi\rangle \equiv \sum_{\bar{u}} C(\bar{u})\,|\psi(\bar{u})\rangle \xrightarrow{\ t\ } |\psi,t\rangle \equiv \sum_{\bar{u}} C(\bar{u})e^{it\epsilon(\bar{u})}\,|\psi(\bar{u})\rangle$$

Si la l'ensemble des états $|\psi(\bar{u})\rangle\,,\ \bar{u}\in\ \{\text{ensemble of solution}\}$ ainsi obtenu est complet, autrement dit qu'il génère l'espace des configurations tout entier, on sait qu'il est possible d'exprimer n'importe quel état de l'espace des configurations sous forme de superposition des états propres de le Hamiltonian.

Ensuite, il peut être très utile de résoudre le problème fonctionnel inverse, à savoir exprimer les éléments d'une base d'intérêt physique (notamment la base des positions pour les problèmes de quench) dans la base des états propres de l'opérateur d'évolution. Concrètement, ce problème se traduit par la recherche de la mesure μ tel que l'équation

$$|\bar{x}\rangle = \sum_{\bar{u}} \mu(\bar{u}|\bar{x})\,|\psi(\bar{u})\rangle$$

soit satisfaite.

Enfin, on cherchera à exprimer de manière exploitable les éléments de matrices d'un opérateur $\mathcal{O}$ dans la base des états propres

$$\langle\psi(\bar{v})|\,\mathcal{O}\,|\psi(\bar{v})\rangle$$

La solution à ces trois problèmes conduira naturellement à une expression intéressante pour la valeur moyenne de l'opérateur $\mathcal{O}$ au cours du temps, considérant un état initial du système $|\psi, t = 0\rangle$

$$\langle \mathcal{O} \rangle_t \equiv \langle \bar{x}, t| \, \mathcal{O} \, |\bar{x}, t\rangle$$

Le problème spectral fait l'objet des deux premiers chapitres de cette thèse. Le premier chapitre donne les bases techniques de l'ansatz de Bethe, et introduit les concepts qui seront seront nécessaire au développement techniques des chapitres suivants. La chaine de spin de Heisenberg (ou chaine de spin XXX) et le modèle du Zero-Range Chipping Model (ZCM) sont présentés, et la résolution de leur problème spectral par ansatz de Bethe en coordonné est exposée. L'ansatz de Bethe en coordonné consiste à considérer les états propre de l'opérateur d'évolution sous la forme de superposition d'onde plane à M paramètres

$$\psi(\bar{x}|\bar{u}) = \sum_{P \in \pi_M} A_P \prod_{i=1}^{M} z_{Pi}^{x_i}$$

Dans le cas des système fini, les ensemble de paramètre $\bar{u}$ sont contraints par des équation couplées, dites équations de Bethe. Dans la limite des systèmes infinis, ces équations donnent lieu à un continuum d'état. Les concept d'états lié et de complétude sont abordés.

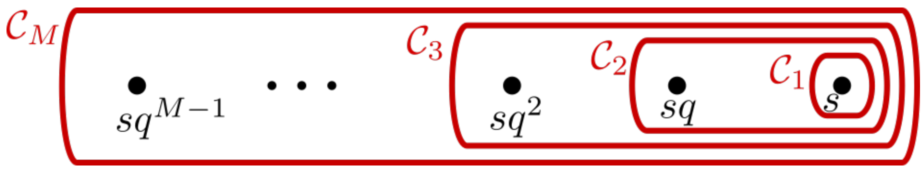
Est ensuite exposé les bases de l'ansatz de Bethe algébrique dans le cadre de la chaine de spin XXX périodique. Dans ce contexte, la diagonalisation du Hamiltonien est effectuée indirectement via diagonalisation de la matrice de transfert du system $t(u)$, qui commute pour différentes valeurs du paramètre spectral $[t(u), t(v)] = 0$, générant les intégrales du mouvement, en particulier le Hamiltonien et le spin total. Les états propres du système sont obtenus par application multiples d'un opérateur de création sur l'état de référence, $B(\bar{u}) \, |\Omega\rangle$, où les paramètres $\bar{u}$ sont contraint par les équations de Bethe. Le problème de la chaine ouverte est introduit, et motive le développement de l'ansatz de Bethe algébrique modifié (MABA), qui est l'objet du second chapitre de cette thèse.

Ce second chapitre est adapté de deux articles écrit en collaboration avec S. Belliard et N. A. Slavnov [23, 24], concernant le MABA pour la chaine de spin XXX twistée, avec spin arbitraire. La principale différence de l'ansatz de Bethe modifié par rapport à son homologue usuel sont que dans ce contexte, la matrice de transfert considérée ne commute plus avec le spin total. Il est donc nécessaire de "remplir" la chaine, c'est à dire de considérer des états propres générés par action multiple de l'opérateur modifié $\tilde{B}$ sur l'état de référence $\tilde{B}(\bar{u}) \, |\Omega\rangle$, où le cardinal $\#\bar{u} = M$ est maintenant contraint par $M = \sum_{i \in \mathcal{L}} 2s_i$, à savoir le nombre maximal de particule dans la chaine, avec s_i la valeur local de la représentation de spin au site i. Diagonaliser la matrice de transfert twistée sur de tel état modifié requière l'étude des actions multiples des opérateur modifiés de la matrice de monodromy twistée, et conduit à des équations de Bethe modifiées, qui contiennent un terme supplémentaire n'apparaissant pas dans les équation de Bethe classiques. On obtient ainsi une nouvelle caractérisation du spectre pour XXX twisté, déjà obtenu et validé dans le cadre de l'ansatz classique, ce qui pourrait permettre d'étudier cette nouvelle caractérisation. Un moyen direct de comparer ces deux caractérisations du spectre serait de comparer les produits scalaires des états de Bethe et de leurs homologues modifiés, qui est l'objet de la seconde partie de ce chapitre, qui se concentre principalement sur l'étude des actions multiples des opérateurs modifiés.

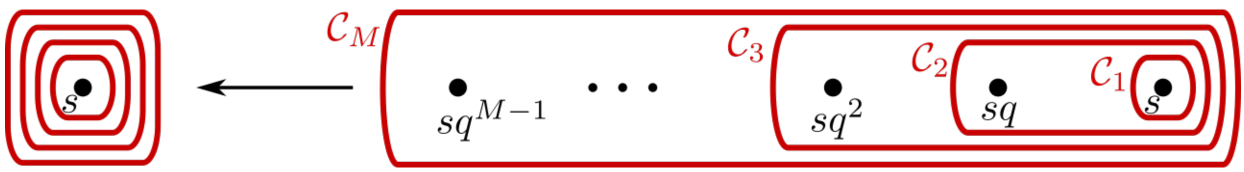
Le troisième chapitre de cette thèse se concentre sur le problème fonctionnel inverse pour les problèmes de quench dans le zero-range chipping model with factorized steady state, puis sur la résolution de l'identité pour la chaine de spin XXZ avec spin arbitraire. L'idée générale pour ces deux calculs sont relativement similaires. On considère le problème fonctionnel inverse dans un régime de spin pour lequel la solution est simple

$$|\bar{y}, t\rangle = \int_{\mathcal{D}} d\bar{u}\ \mu_t(\bar{u}, \bar{y})\, |\psi(\bar{u})\rangle\,, \quad \mu_t(\bar{u}, \bar{y}) = \Lambda(\bar{u})^t \mu(\bar{u}, \bar{y})$$

faisant intervenir des contour emboités $\mathcal{D} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_M$



Des contrainte sur les domaines d'intégration imposés par nos problèmes (le problème de quench pour le zero-range chipping modèle, et le domaine des racines de Bethe pour la résolution de l'identité dans XXZ) imposent de rétrécirent les contours autour d'un pôle unique $\mathcal{D} \to \tilde{\mathcal{D}}$



Durant cette procédure, les contour vont interagir avec les différents pôles provenant de la partie intégrande, qui donne naissance à des contrainte entre les différents paramètre, s'apparentant à des cordes, correspondant à des états liés. On obtient ainsi une somme combinatoire sur toutes les configurations possibles de cordes.

$$\sum_{\bar{p} \leq n} \prod_{i=1}^{\#\bar{p}} \oint_s du_{a_i}\ Res\left[\mu(\bar{u})\Lambda(\bar{u})^t \psi(\bar{x}|\bar{u})\psi(\bar{y}|\bar{u})\right]_{u_j = qu_{j-1},\ j \notin \{a_i\}}$$

où les $\bar{p}$ sont les configuration de corde. On obtient ensuite les régimes de spin demi-entier par prolongement analytique. Dans le cas du zero-range chipping model, la longueur des cordes obtenues ont une valeur maximal, dépendant de la valeur du spin. Une expression compact pour la probabilité conditionnelle d'observer toutes les particules du système à gauche de l'axe étant donné qu'elles se trouvaient toute à droite de l'axe à $t = 0$ sous la forme d'un déterminant de Fredholm

$$\lambda^N \mathbb{P}(\bar{x} \geq 0 | \bar{y} < 0, t) = \oint_0 \frac{dw}{w^{N+1} q^{N^2/2}} \det \left[\mathbb{I} + \mathbb{K}^{[t]}_{\lambda w} \right]$$

est prouvée.

Le quatrième et dernière chapitre de cette thèse, divisé en deux parties, concerne les représentations en déterminant dans l'ansatz de Bethe. La première partie concerne une preuve [25] d'une représentation en déterminant pour les éléments de matrice de l'opérateur nombre de particule dans le gas de Bose avec interaction delta, conjecturée par Véronique Terras

$$\begin{aligned} &\langle \psi(\bar{u}) | \mathcal{O}_\kappa(\bar{\mathbf{x}}) | \psi(\bar{v}) \rangle \\ =&\det{}^{-1}_{i,j} [\frac{ic}{u_i - v_j + ic}] \\ &\times \det{}_{i,j} \left[\frac{(ic)^2}{(v_i - u_j)(v_i - u_j + ic)} \frac{v_i - \bar{u} + ic}{v_i - \bar{u} - ic} \frac{v_i - \bar{v} - ic}{v_i - \bar{v} + ic} + \kappa \frac{(ic)^2}{(v_i - u_j)(v_i - u_j - ic)} \right] \end{aligned}$$

Qui n'est autre que le determinant de Slavnov.

Cette démonstration repose sur la décomposition des vecteurs de Bethe sur l'axe entier sous la forme d'une somme sur des états produit d'états de Bethe sur les demi axes

$$|\psi(\bar{v})\rangle = \sum_N \sum_{\bar{v} \to \{\bar{v}_I, \bar{v}_{II}\}} f(\bar{v}_I, \bar{v}_{II}) \quad |\psi^-_N(\bar{v}_I)\rangle \, |\psi^+_{M-N}(\bar{v}_{II})\rangle$$

pour lesquels l'action de l'opérateur nombre de particule agit trivialement

$$\mathcal{O} \, |\psi^-_N(\bar{v}_I)\rangle \, |\psi^+_{M-N}(\bar{v}_{II})\rangle = \kappa^N$$

et dont les produits scalaires respectif sont connu.

La seconde partie de ce chapitre se concentre sur l'investigation d'une représentation intégrale pour le determinant d'Izergin-Korepin, du déterminant de Slavnov et du déterminant de Gaudin, tissant ainsi un lien formel direct entre ces objets.

Cette thèse aborde les trois problématiques principales de l'étude théorique des valeurs moyennes dépendantes du temps des opérateurs physique, à savoir le problème spectral, le problème fonctionnel inverse, et les élément de matrices de ces opérateurs. Ces investigations ont cependant été menée dans des domaine très différents, quantique ou stochastique, discret ou continue, mais apportent des réponses techniques potentiellement exploitable dans un contexte unique.

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Many thanks to my colleague my family and my friends for their support and presence during those three years.

At last, many thanks Kemal, please don't send me to gulag, I'm innocent.

Contents

About Integrable Systems

What makes the house to stand.

One observes nature, and in an attempt to recognize her a tendency, one writes a law. Extracting and synthesizing the essence of an underlying structure governing our world in such laws and principles requires a formalism, a frame, in which these laws can be formulated and exploited - a scene, in which the play can flow. In this frame, which in the case of physics is fundamentally mathematical, live objects, formal representations of our world, interacting with each other according to our rules - our laws. Frame and laws together form our theory. A theory should obviously ”explain” the already observed phenomena, on which should relies its construction, but also predict a new collection of phenomena, namely be predictive. This question of relevance will however not be of our concern in this thesis. We will dive in the formalism, sometimes almost forgetting about its physical relevance. Indeed, when one considers a physical problem, the theory only brings a formal question - the equation of motion. Solving this equation now deals with mathematics, and can sometimes require subtle efforts. This will be our matter.

Consider two massive bodies floating in the space vacuum with a defined relative motion. One can wonder what will be their trajectories. In the frame of Newton’s gravitation, this simple question can take the form of a simple differential equation, and be solved with a bit of mathematical agility. A simple answer for a simple question.
Now add a third body in our experiment. The new equation of motion remains at the same order of simplicity, but this one can not be solved. This new problem, albeit at first sight very close to the first one, gives rise to a chaotic motion: the tiniest change in the initial conditions would have unpredictably large consequences on the trajectories [1]. Hence, one could only consider approximations of these trajectories requiring numerical computations, only relevant in a small period of time. A very delicate and partial answer for a still very simple question. The first of this example will be said to be solvable, while the second, in opposition, exhibits chaos.

The notion of integrability is strongly linked with the concept of symmetry and with it, through Noether’s theorem[2], the existence of conserved quantities: ”whenever there is a symmetry in nature, there is a conservation law”. More formally, this will be translated in mathematical terms as ”whenever a physical system is invariant under a group of symmetry, there is a conserved charge”. If a system possesses as much conserved quantities as degrees of freedom, it is kept free from any chaotic behavior. In other terms, it is exactly integrable. This thesis focuses on such systems.
In the frame of classical Hamiltonian mechanics, the definition for integrability is rather

simple[2]: a system will be said integrable if it possesses a complete set of action angle variables - these canonical coordinates being associated with a set of generalized momenta, reveling a complete set of conserved quantities of our system. This notion easily extends to the quantum scene, in which integrability relies on the existence of a complete set of independent conserved charges. In a more general manner, integrable systems are systems for which we are able to compute exactly some physical quantities - the most fundamental of these being the energy spectrum - without any use of perturbation theory. However, as in the previously evoked systems of two celestial bodies in mutual interaction, the conserved charges are often not obviously unveiled. Actually, in the approaches we are going to consider, the conserved charges of our system remains formally hidden, albeit still absolutely fundamental in this context.

Among the collection of all possible systems, the exactly solvable models are of null measure, and actually most of the systems solvable by modern techniques, like those we are to consider in this thesis, are $1+1$-dimensional quantum systems, while our world is $3+1$-dimensional. Then, at first sight, solvable models can not be of any help to describe nature. But the architect is not looking for drawing the house, but only what prevents it from falling. What matters is the structure.

Integrability undubiously provides powerful insights in a large spectrum of theoretical physics problems. It found direct application in condensed matter physics - in the Kondo problem and the Hubbard model for instance [3] - and more recently in supersymmetric gauge field theories and in stings theories, for which integrable structures seems to be readable in terms of integrable quantum spin chains[4]. These spin chains so revealed as an essential tool for studying the AdS/CFT correspondence [5], an intriguing correspondence holding between some Super Yang-Mills theories and some string theories. It also appears as a fundamental feature for studying some stochastic problems, which opens a very nice theoretical window on problems at the frontier of physics. Sometimes, the very simple integrable models such as those we are to consider in this thesis can turn out to be relevant toy models for studying more physical problems. Also, schematically, a physical model can sometimes be seen as a small perturbation around its integrable regime. The most eloquent example of this could be the small shift in the energy spectrum of the hydrogen atom, perturbation around the level of energy one obtains when neglecting relativistic and spin-spin interactions in the problem, completely integrable in this regime [6]. At last but not the least, integrable systems revealed as an incredibly deep and rich subject of research, source of mathematical elegance and refinement, but also an endless playground for anyone who likes to deals with mathematical puzzles. Those reasons altogether justified the huge effort of research put in this field during the last decades. The amateur reader should indeed keep in mind that this thesis accounts for an insignificant contribution to a huge and very active domain of research - a tiny square of pottery in a huge mosaic.

Story of a journey in the vast lands of quantum integrability. Among the different approaches for solving the spectral problem that have been developed, the coordinate and algebraic Bethe ansatz probably incarnate today the two most successful and well known, and will be the matter of Chapter 1. This chapter provides an affordable introduction to some models that are to be considered in further chapter, and accounts for a technical introduction before approaching more demanding problems and does not contains any of my original contribution, except for some rare and tiny ones. First is presented, in Section 1.1, the traditional approach as initially proposed by H. Bethe in 1931, so called "coordinate" Bethe ansatz. This direct approach of the problem is very useful in that it provides us with an explicit expression for the eigenfuctions in the coordinate basis. It relies on assuming the wave-function for the eigenstates to be a so called Bethe superposition of plane waves. The diagonalization of the Hamiltonian for the XXX periodic spin chain is reviewed as a first example to introduce the basic aspect of the questions, in which case, due to periodic condition, arise the Bethe equations constraining our states. The diagonalization of the zero range chipping model with factorized steady state by means of the coordinate Bethe ansatz is then presented as an interesting extension of the approach, and as a matter of consistency considering the calculations to follow, in this same context, in a further chapter. The algebraic approach is presented in Section 1.2. This more recent approach provides us with an incredibly powerful tool. In this context, the algebraic structure of the models are exhibited, which allows to consider these in a really wider fashion, focusing on their algebraic underlying structures more than on their physical content. The particular case of the XXX periodic spin chain is once again considered to illustrate the reflection, and the Bethe equations are again obtained in this very different context. The rest of the thesis is devoted to present my personal or collaborative contribution to the domain of integrability.

The usual algebraic approach of the Bethe ansatz, albeit very powerful, requires quite strong formal constraints on our system. These are for instance not satisfied when one consider the problem of the open spin chain with general boundary conditions. Among the different candidate for a new method that could handle this latter problem, the Modified Algebraic Bethe Ansatz proposes to generalize the usual algebraic approach, without giving up on its convenient formalism. The Modified Algebraic Bethe Ansatz is the matter of Chapter 2, in which it is presented in the context of the periodic XXX chain with arbitrary twist. The diagonalization of the twisted transfer matrix is discussed, and the modified Bethe vector are constructed. A second section is devoted to the investigation of multiple action formulas. In particular, the scalar products of modified Bethe vectors are studied, which are the building block for the calculation of form factors and correlation functions, which are objects of real physical interest.

Chapter 3 concerns the inverse functional problem, i.e. the problem of expressing any state of the spin basis in term of the Bethe basis, which can be of a real help when one comes to compute some physical quantities. This problem is addressed in the context of the zero range chipping model with factorized steady state, which constitutes the first step for approaching quench problems, slightly addressed in this section. Independently, an expression for the resolution of the identity in term of Bethe states for the infinite XXX chain is proved. These two calculations are build from the same simple idea. But although they share some similarities, they exhibit important specificities.

At last, Chapter 4 concerns determinant representations, which are objects of recurrent importance in the Bethe ansatz world. An expression for the « matrix elements of the particle number operator » for the delta-Bose gas in terms of a determinant is proved. Independently, some integral representations for the Izergin-Korepin and Slavnov determinants are investigated, in a second part.

Chapter 1

Elements of Bethe Ansatz

It is not the answer, simply the good question.

In the context of quantum mechanics, the equation of motion simply reads as the famous Schrodinger equation

$$i\partial_t \psi = H\psi$$

where ψ is the wave function of our state, and H the Hamiltonian of the system.
In the case of stochastic systems, the dynamics is governed (in the continuous time limit) by the master equation

$$\partial_t \psi = M\psi$$

where M is the transition matrix of the system and ψ is now the probability amplitude describing our state. Albeit those two equations are fundamentally very different in nature, they are formally very close. In our context, namely the spectral problem, we will consider these two pictures as similar, and treat these without any distinction. We would then simplify our discussion and adopt the quantum vocabulary in both of these context.
Solving such an equation of motion can actually sensibly be reduced to the diagonalization of the Hamiltonian, as an arbitrary superposition of the corresponding eigenvectors shall trivially evolve, after time t, as the superposition of the phase shifted eigenvectors

$$\sum_k \alpha_k \psi_k \longrightarrow \sum_k \alpha_k \psi_k e^{it\epsilon_k}$$

where ϵ_k is the eigenvalue corresponding to the eigenstate ψ_k. This makes the spectral problem a central question when one wants to study a system on the physical level. To this end, lot of techniques has been developed in the context of integrable models, such that the Separation of Variable [9, 8], the Coodinate Bethe Ansatz (CBA) and Algebraic Bethe Ansatz (ABA), those latter two being the subject of this first chapter.

By Bethe ansatz one often refers to an ensemble of techniques and mathematical ”technologies” developed in the context of integrable models.
The ansatz (formally ”assumption” in German) is however fundamentally an hypothesis, a bet one makes on some structural property of a model in order to solve it. Thereby the Bethe ansatz provides us with an open question: It proposes us to investigate eigenstates

of a system in a particular form (Bethe superposition of plane waves in the context of the coordinate ansatz, see Section 1.1, and multiple application of the Bethe operator in the algebraic approach, see Section 1.2). The ansatz does thus not provides us with a formal answer, but simply opens a door that appears to lead to a promising way. It is not the answer, simply a good question. At the end of the story, following the way the ansatz indicated, we end up with an expression for eigenstates of the Hamiltonian parametrized by a set of parameters, so called rapidities or Bethe parameters. For finite systems, these parameters still have to satisfy a set of coupled equations, so called Bethe equations, for the corresponding state to effectively be eigenstate of the Hamiltonian. These equations however are almost unsolvable. The result is not absolute, in that the Bethe states also require the good parameterization, but still is very satisfying: the eigenvalue problem as been translated as a very simple (albeit unsolvable) set of equations, and the structure of the eigenstates so exhibited is by itself a very exploitable object.

1.1 The Coordinate Approach

The Bethe Ansatz machinery has been deeply developped from its historical birth with H. Bethe in 1931 [16], who introduced this method to solve the spectral problem of the anti-ferromagnetic Heisenberg - or XXX - spin chain. This first approach for diagonalizing integrable quantum systems, now often refereed to as the Coordinate Bethe Ansatz (CBA) by contrast with alternative approaches (such as the Algebraic Bethe Ansatz, ABA), opened a new field of research. It appears today as a well developed and successful approach in a wide range of physical problems [28, 12, 7, 19, 18].

The Bethe ansatz aim to solve the spectral problem

$$H\,|\psi\rangle = \Lambda\,|\psi\rangle$$

where H is an integrable Hamiltonian. In this context, the ansatz remarkably expresses in a very similar way for systems which are very different in essence. It consists on assuming the wave function for M particles to be a simple superposition of plane waves

$$\psi(\{x_i\}) = \sum_{P\in\pi_M} A_P \prod_i e^{ik_{P_i}x_i}$$

where π_M is the set of permutation of M elements, the set $\{k_i\}$ parameterize our state, and the coefficients A_P is model dependent. This assumption is actually very strong and far from trivial, as it actually corresponds, conceptually speaking, to consider the momenta carried by our particles, the k_i's, to simply be exchanged as two particles scatter, and with it a phase shift given by the diffusion coefficient A.

In the context of the Coordinate Bethe Ansatz (CBA), the integrability relies on the factorization of the M-particle problem as many two-particle problems. Formally here, the diffusion coefficient for the whole particle will factorize over the two-particle diffusion phases $A = \prod_{ij} A_{ij}$.

While the ansatz can be formulated with similar words in different contexts, the answers to come are very different. Indeed, as we will see in this Section, solving the ansatz in different contexts can be an exercise requiring very variable level of trickiness.

In this section, I will first give an overview of the diagonalization by means of the CBA of the periodic XXX spin chain, which constitutes a sensible first approach from a historical and pedagogical point of view. The simplicity of the model retains the mathematics on a very affordable level, which allows us a simple discussion on the different aspect of the problem.

Then is presented the dagonalization of the Zero Range Chipping Model with Factorized Steady State by means of the CBA, which exhibits very interesting (and challenging) aspects, on a bit more technical but still very affordable level. This latter also is introduced as a matter of consistency, as some technical manipulations on these Bethe states will later be ran, see Section 3.1, in the context of the inverse functional problem.

1.1.1 The XXX spin chain

Introduced in the early twentieth century to describe the magnetic behavior of metals [16], the spin 1/2 XXX (or isotropic Heisenberg) spin chain also constitutes one of the simplest integrable quantum system one can consider: a periodic chain (or one-dimensional lattice) of L identical atoms with two levels of energy, interacting with their nearest neighbor. Those two level of energy will be those of the two configuration of spin, $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

The dynamics of the system is governed by an isotropic nearest neighbor interaction Hamiltonian

$$H = \frac{J}{2}\sum_{l=1}^{L} \vec{\sigma}_l \vec{\sigma}_{l+1} \tag{1.1}$$

with $J < 0$ the (ferromagnetic) coupling constant, σ_l^i the Pauli matrices[1] acting on site l, and the sum running over the L sites l of the chain. Fixing the coupling constant to $J = 1$, this Hamiltonian rewrites

$$H = \sum_{l=1}^{L}(1 - P_{l,l+1}) \tag{1.2}$$

with $P_{l,l+1}$ exchanging spins of site l and $l+1$, $P_{l,l+1} = \frac{1}{2}(\vec{\sigma} + 1)$. The periodicity is ensured by the condition $\vec{S}_{l+L} = \vec{S}_l$. In the ferromagnetic case, the spins tend to align, and the ground state of (1.2) is $|\Omega\rangle = |\uparrow\uparrow\uparrow\cdots\uparrow\rangle$, corresponding to the eigenvalue $\Lambda = 0$ (note that this energy state is degenerated[2]). This choice for the ground state is however a matter of convention, as the dual vacuum $|\bar{\Omega}\rangle = |\downarrow\downarrow\downarrow\cdots\downarrow\rangle$ is also a perfect candidate. By acting with M ladder operators $\sigma_l^- = \frac{\sigma_l^x \pm i\sigma_l^y}{2}$ on the ground state $|\Omega\rangle$ one obtains a state of spin $s = \frac{L}{2} - M$, so called M-magnon state, M the number of flipped spins. We write the spin basis vectors $\sigma_{x_1}^- \cdots \sigma_{x_N}^- |\Omega\rangle = |x_1 \cdots x_N\rangle = |\bar{x}\rangle$, $\bar{x} = \{x_1, \cdots, x_N\}$. A

[1] $\sigma_l^x = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}_l$, $\sigma_l^y = \begin{pmatrix} & -i\\ i & \end{pmatrix}_l$, $\sigma_l^y = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}_l$, $[\sigma_l^a, \sigma_k^b] = 2i\delta_{l,k}\epsilon_{abc}\sigma_l^c$

[2] Since $[S^- = \sum_n \sigma_n^-, H] = 0$, $|\Omega\rangle$ is at least degenerate L times: $H|\Omega\rangle = HS^{-l}|\Omega\rangle$, $l < L$

magnon can here be interpreted as a particle, that will propagate and eventually interact with others.

The coordinate Bethe ansatz. I will here briefly review the rather intuitive method, leading to the spectrum of (1.2), so called coordinate Bethe ansatz. A more subtle appraoch, the Algebraic Bethe Ansatz (ABA), will be reviewed in Chapter 1.2. I shall here only explicit the mains arguments and the most important steps of the construction, without detailing all the calculations. For a more precise description of the method, see for instance [28], from which the construction presented in this subsection is drawn.
One can check that the total spin operator $S_z = \sum_l \sigma_l^z$ commutes with the Hamiltonian (1.2), hence these operators being diagonalizable in a common basis. In other words, the number of particle is conserved. One can thus diagonalize (1.2) in a finite spin subsector, in other words to consider states with definite number of magnons M. We thus consider eigenstates of H of the form

$$|\psi\rangle_M = \sum_{\bar{x} \in \mathcal{D}_M^L} a(\bar{x}) \, |\bar{x}\rangle \tag{1.3}$$

$$H \, |\psi\rangle_M = \Lambda \, |\psi\rangle_M \tag{1.4}$$

where $\mathcal{D}_M^L$ is the physical domain for M-particles on the chain of size L, $\mathcal{D}_M^L = \{\bar{x} \in \mathbb{Z}^M, 1 \leq x_1 < \cdots < x_M \leq L\}$ (the inequalities being strict since $\sigma_l^- \sigma_l^- = 0$, i.e. at most one magnon can lie in each site). The ordering here ensures that the expression of ψ_M in terms of the a's is unique.
The equations (1.2), (1.3) and (1.4) define the eigenproblem

$$\sum_{\bar{x}'} (a(\bar{x}) - a(\bar{x}')) = \Lambda a(\bar{x}) \tag{1.5}$$

for $\bar{x} \in \mathcal{D}_M^L$, where the $\bar{x}'$ are obtained from $\bar{x}$ by exchange of two neighbor spins, i.e. $\exists i, x_i' = x_i \pm 1$, with the other x_j unchanged, provided $\bar{x}' \in \mathcal{D}_M^L$.
Let us now extend the definition of the a's over the extended domain $\tilde{\mathcal{D}}_M^L = \{\bar{x} \in \mathbb{Z}^M, 1 \leq x_1 \leq \cdots \leq x_M \leq L\}$, and suppose equation (1.5) remains valid over this domain, i.e.

$$\sum_i (a(\cdots x_i + 1 \cdots) + a(\cdots x_i - 1 \cdots) - 2a(\cdots x_i \cdots)) = \Lambda a(\cdots x_i \cdots) \tag{1.6}$$

Some additional terms will appear in this latter, namely terms containing $a(\cdots x, x \cdots)$, arising from $x_i' = x_i + 1 = x_{i+1}$ and $x_{i+1}' = x_{i+1} - 1 = x_i$.
Since both (1.5) and (1.6) have to be simultaneously satisfied on $\bar{x} \in \mathcal{D}_M^L$, these additional terms in the latter equation have to compensate, which translates as the so called boundary condition

$$a(\cdots x_i + 1, x_i + 1, \cdots) + a(\cdots x_i, x_i, \cdots) - 2a(\cdots x_i, x_i + 1, \cdots) = 0, \quad x_i < x_{i+1} \; \forall i \tag{1.7}$$

The "free particle" equation (1.6), alongside the boundary conditions (1.7), fully define the eigenproblem.
First of all, any plane wave $e^{i(k_1 x_1 + \cdots + k_M x_M)}$ is solutions of (1.6) with corresponding energy

$\Lambda = \sum_{i=1}^{M}(\cos k_i - 1)$. The idea now is to obtain solutions of (1.7) by summing over states of same energy, namely to consider the ansatz

$$a(\bar{x}) = \sum_{P\in\pi_M} A(P) e^{i\sum_{i=1}^{M} k_{Pi}x_i}$$

with π_M the group of permutation of order M, which, from (1.7), leads to

$$A(P) = \prod_{i<j} a(Pi, Pj)$$
$$\frac{a(i,j)}{a(j,i)} = -\frac{e^{i(k_i+k_j)} - 2e^{ik_i} + 1}{e^{i(k_i+k_j)} - 2e^{ik_j} + 1} \equiv S(i,j)$$

which fix the factor $A(P)$ up to global normalization.
The periodicity conditions, here written $a(n_1, n_2 \cdots n_M) \equiv a(n_2 \cdots n_M, n_1{+}L)$ i.e. $A(P) = A(PC)e^{ik_{P1}L}$, with C the circular permutation $Ci = i+1$, imposes

$$e^{ik_iL} = -\prod_j S(i,j)\ \forall i \tag{1.8}$$

the so called Bethe equation. As a matter of convenience, we define $e^{ik_i} \equiv \frac{u_i+i/2}{u_i-i/2} \equiv z_i$. Hence, to summarize, a state

$$|\psi(\bar{u})\rangle = \sum_{\bar{x}\in\mathcal{D}_M^L} \sum_{P\in\pi_M} \prod_{i<j} \frac{u_{Pi} - u_{Pj} + i}{u_{Pi} - u_{Pj}} \prod_k z_{Pk}^{n_k} \sigma_{n_k}^- |\Omega\rangle \tag{1.9}$$

will be eigenstate of the Hamiltonian (1.2) associated to the eigenvalue

$$\Lambda(\bar{u}) = -\frac{1}{2}\sum_{j=1}^{M} \frac{1}{(u_j + i/2)(u_j - i/2)}$$

provided the set of parameter $\bar{u}$ satisfies the Bethe equations

$$\left(\frac{u_i + i/2}{u_i - i/2}\right)^L = -\prod_{j=1}^{M} \frac{u_i - u_j + i}{u_i - u_j - i} \tag{1.10}$$

The states of the form (1.9) satisfying (1.10) will be called *on-shell* Bethe states, *off-shell* otherwise. These states are symmetric under exchange of particle, which reflect the indistinguishable nature of the particles involved in here. As the size of the systems will tend to infinity, the Bethe roots, namely the set of solution for this equations, will turn into a continuum of solutions. Hence, in this case, any state of the form (1.9) will effectively be an eigenstate of (1.1), whatever are its parameters.

Magnons and particles. There is a nice point of view of physical interest to adopt here, particularly favored in the context of the coordinate Bathe ansatz. Considering an M-magnons Bethe state as a state of M particles (excitations) propagating in the chain, the $S(i,j)$ now play the role of diffusion matrices. These S-matrices describe totally elastic scattering implying two particles, simply exchanging their momemta when scattering. Those processes, whose phase are given by the S's, ensure the conservation of all the momenta carried by all the particles. These constants of motion reflect the integrability of the model.

Bound states and strings. Equation (1.10) also provides an interesting insight into the nature of the different Bethe states. The first class of solutions, which we would eventually call free states by contrast with the latter, are solutions parametrized by real rapidities u_i. To consider the other class of solution, namely the bound states, it is more convenient to consider the large L limit (i.e. an infinite chain). In this limit, we can see that for complex u in (1.10) (or equivalently complex k in (1.8)), the right hand side of the equation may diverge or vanish. This implies a relation between two rapidities of the form

$$u_i - u_j = i + \mathcal{O}(L^{-\infty})$$

Such a relation will in the following be denoted $u_i \bowtie u_j$, or equivalently $i \bowtie j$.
If $u_i + u_j = u \in \mathbb{R}$, we are back to the situation of having an excitation freely propagating, described by the real rapidity of the bound state u (and real momentum k)[3]. Otherwise, we continue this grouping of rapidities u_i until obtaining a relation $u_{i_1} = u_{i_2} + i = \cdots = u_{i_N}$, $\sum_n u_{i_n} \in \mathbb{R}$, forming a so called N-string, characterized by a real [4] parameter u (the rapidity for the center of mass of the bound state), and the sequence $\{i_n\}_{n=1\cdots N}$. Such a state will in the following be denoted $\{u_{i_1} \bowtie u_{i_2} \bowtie \cdots \bowtie u_{i_N}\}$, or equivalently $\{i_1 \bowtie i_2 \bowtie \cdots \bowtie i_N\}$.
These bound state are an important part of the Bethe spectrum, as they contribute to its completeness, as wee will see in Section 3.2.

String configurations. One can associate a graph to every Bethe state, by identifying every label with a point of the graph, and link between every points i and j such that $i \bowtie j$. States corresponding to the same graph will be said to belong to the same (class of) string configuration. A string configuration is given by the numbers N_k of strings of length k.
It is also known that a bound state can't have relation of the type $u_i = u_j$, which will be assumed in our development.

Completeness. We constructed here the so called Bethe states, eigenstates of our Hamiltonian, but did not adressed the question of completeness. It is known that these

[3]This can be seen by looking at the wave (3.6), in which will then appear the phase $e^{i(k_i x_i + k_j x_j)} = e^{i/2[(k_i+k_j)(x_i+x_j)+(k_i-k_j)(x_i-x_j)]} = e^{-1/2\alpha(x_i-x_j)}e^{ik_{ij}(x_i+x_j)/2}$, where k_{ij} is the freely propagating bound state momentum. This amplitude decay with $x_i - x_j$, in other words with the distance between the two particles, hence the name bound state.

[4]if $e^{i(k_{i_1}+\cdots k_{i_N})L}$ is not finite, the considered string is not complete, hence the procedure has to be pursued

states form a complete set, i.e. that we obtained a complete spectrum of our Hamiltonian [52, 54, 96, 97]. The set of Bethe state may thus be referred to as the Bethe basis.
I proposed in Section 3.2 an expression for the identity in $\mathcal{H}$ in term of *on-shell* Bethe states in the infinite XXZ spin chain, a deformation of the XXX chain. Doing so, the completeness of this set of sates is proved, and so is the completeness of the spectrum obtained by the coordinate Bethe ansatz for the infinite XXX spin chain.

1.1.2 The Zero-range Chipping Models with Factorized Steady State

I here expose a pedagogical review for the diagonalization of Zero-range Chipping Models with factorized steady state (ZCM) by means of the Coordinate Bethe Ansatz, strongly based on the work of A.M. Povolotsky [18]. Those models, as considered, contain a large family of integrable stochastic particle models as limiting cases. It constitutes a nice and affordable extension of the concepts developed in the previous section in the frame of the XXX spin chain. In this new context, however, we will consider an infinite lattice, hence no Bethe equation in play, but the main idea of the ansatz remains quite unchanged.

A word about stochastic dynamics. In this section we abandon the quantum world to focus on what is called Markov process. In such model, the system evolve at each (discrete) time step according to the state at the previous time step only, without regards to its past trajectory.
Considering an ensemble of configuration χ, the states of the system will be described through the probability for it to be found in a particular configuration, $P(x)$, $x \in \chi$. Its evolution is then govern by the master equation

$$P_{t+1}(x) = \sum_{y \in \chi} M_{x,y} P_t(y)$$

Given the probabilistic interpretation of the object P (the probability amplitude), this latter has to satisfy the two properties of reality, $P \in \mathbb{R}_+$, and of normalization $\sum_x P_t(x) = 1\ \forall t$. This last constraint implies $\sum_x M_{x,y} P_t(y) = 1$, i.e. the vector with all components equal to one is left eigenvector of eigenvalue $\Lambda = 1$.

Among those models are the Assymetric Simple Exclusion Processes (ASEP), introduced in 1970 by F. Spitzer [26]. These are very simple markov processes, in which a particle on a discrete lattice will, at each discrete time step, jump to the right with probability p or to the right with probability $1-p$, provided in each cases those sites are unoccupied. These models has been deeply studied since then, which led to a large variety of results [19, 7]. These systems, albeit very simple considering their dynamics, are very interesting integrable systems, giving raise to complex mathematical structures when one comes to considering dynamical problems such as quenches problems [19].

In the models we will consider now, the Zero-range Chipping Models with factorized steady state (ZCM), particles on a discrete lattice jump to the site at their right with a probability depending on the number of particles initially occupying their site. The coordinate approach of the Bethe ansatz is usually employed to diagonalize a Hamiltonian. This appraoch here surprizingly applies to the diagonalization of a more delicate object,

the transition matrix, based on an elegant formulation of the problem by A. M. Povolotsky [18]. As we will see, imposing the system to be integrable by means of the coordinate Bethe anstaz constraints the jumping probability in a particular form, so that the M-particle interaction actually factorizes over 2-particle interactions. We also impose the steady state measure, namely the measure for the state toward which the system will tend after infinite time, to factorizes over the sites of the lattice. Those two constraints completely fix the system, up to three free complex parameters.
The basic idea for the ansatz remains the same as for the case of the XXX spin chain in the previous section. We first obtain a simple equation for the probability amplitudes for non-interacting particles on the lattice. After that, extending the domain of the definition for the probability amplitude to the whole physical domain, we obtain the so called boundary conditions. In our case however, several particles can lie in a single site, while only one could in the XXX spin chain. The condition for the system to be integrable rely on the factorization of these M-particle boudary conditions over 2-particle boundary conditions, i.e. such that the problem reduces to the simple case of XXX. This impose constraints on the transition probabilities defining the system.

In a certain sector of the model, where two parameters satisfy a simple relation, the obtained eigenstates collapse to the Bethe states of the ASEP. This link is intriguing since in our model the particle only jump on the right, while they jump in both direction in ASEP. This link, quickly discussed, will be here left as an open question, albeit it seems to already be well understood [18, 95]. The case of the more general half-integer spin value in however not addressed in this latter paper, and is introduced here as a anticipation on the inverse functional problem of Chapter 3.This link is however a good example enlightening the formal background that different physical systems can share.

The System. Let us consider a one dimensional system of N particles living on an infinite lattice $\mathcal{L}$. We describe the configuration of these particles by their position

$$\bar{x} = \{x_i\}_{i=1,\cdots,N}, \ x_i \in \mathcal{L},$$

or equivalently by the occupation configuration

$$\bar{n}[\bar{x}] = \{\#\{x \in \bar{x}, x = i\}\}_{i\in\mathcal{L}}.$$

In our description, they is no restriction on the number of particle occupying a site at any time. The dynamical behaviour of the system is described by the probability $\varphi(m|n)$ for $m = 0, \cdots, n$ particles at a site occupied by n particles to jump on the next site at its right, at each time step, satisfying the normalization condition

$$\sum_{m=0}^{n} \varphi(m|n) = 1 \tag{1.11}$$

We also consider a parallel evolution at each site, i.e. all sites evolve simultaneously.
For an initial condition characterized by the probability distribution $P_0(\bar{n})$, the system will be characterized by probability distribution $P_t(\bar{n})$ to find the system in a configuration $\bar{n}$ at time t, this distribution evolving according to the master equation

$$P_{t+1}(\bar{n}') = \sum_{\bar{n}} \mathbf{M}_{\bar{n}',\bar{n}} P_t(\bar{n})$$

where the elements of the transition matrix $\mathbf{M}$ are given by

$$\mathbf{M}_{\bar{n}',\bar{n}} = \sum_{\{m_k \in \mathbb{N}\}_{k\in\mathcal{L}}} \prod_{i\in\mathcal{L}} L_{n_i',n_i}^{m_{i-1},m_i} \tag{1.12}$$

with

$$L_{n_i',n_i}^{m_{i-1},m_i} = \delta_{(n_i'-n_i),(m_{i-1}-m_i)} \varphi(m_i|n_i)$$

and we define $\varphi(m|n) = 0$ for $m > n$, i.e. a site can not be "more than empty".

Remark 1.1.1. *Formally we adopt a lowest weight representation for the space of configuration, but a priori not highest weight, as we can have as many particle as we want on each site. As we will see intuitively, highest weight representation will corresponds to particular relations of the parameters of our system. This discussion would however be favorized by an algebraic approach.*

The term $\delta_{(n_i'-n_i),(m_{i-1}-m_i)}$ here insures the conservation of the total number of particle, $\sum_{i\in\mathcal{L}} n_i = N$. The element $\mathbf{M}_{\bar{n}',\bar{n}}$ corresponds to the weight for a transition from a configuration $\bar{n}$ to a configuration $\bar{n}'$. The existence of the stationary state, the right eigenvector of $\mathbf{M}$ with eigenvalue $\Lambda = 1$, is ensured by the condition (1.11).
A very important point in our reasoning is that the stationary state measure will be a product measure, i.e. factorizes over all the sites of the lattice,

$$P_{st}(\bar{n}) = \prod_{i\in\mathcal{L}} f(n_i)$$

if and only if there exist two functions v and w such that [20]

$$\varphi(m|n) = \frac{v(m)w(n-m)}{\sum_{k=0}^{n} v(k)w(n-k)}$$

in which case the one-site weight for the stationary measure writes

$$f(n) = \sum_{k=0}^{n} v(k)w(n-k) \tag{1.13}$$

due to the normalization condition (1.11). We will assume this in what follows.
We can notice that fixing $\bar{n}$ and $\bar{n}'$ in (1.12) constrains the value of $\bar{m}$, the numbers of particles jumping to the right, at each site, during the process, due to the conservation of the total number of particle.
The matrix elements of the transition matrix $\mathbf{M}$ can then be written

$$\begin{aligned} \mathbf{M}_{\bar{n}',\bar{n}} =& P_{st}(\bar{n})^{-1} \prod_{i\in\mathcal{L}} v(m_i)w(n_i - m_i) \\ n_i' - n_i =& m_{i-1} - m_i \quad \forall\, i \in \mathcal{L} \end{aligned}$$

We are now to consider an infinite lattice, namely $\mathcal{L} = \mathbb{Z}$. To make the Bethe Ansatz more workable, it is actually sensible to consider the conjugated matrix

$$\mathbf{M^0} = \mathbf{SMS}^{-1} \tag{1.14}$$

where $\mathbf{S}^{-1}$ is the diagonal matrix of elements

$$\mathbf{S}_{\bar{n}',\bar{n}} = \delta_{\bar{n}',\bar{n}}/P_{st}(\bar{n})$$

such that the conjugation (1.14) replaces in $\mathbf{M}$ the terms $P_{st}(\bar{n})^{-1}$ by $P_{st}(\bar{n}')^{-1}$.

$$\begin{aligned}\mathbf{M^0}_{\bar{n}',\bar{n}} =& P_{st}(\bar{n}')^{-1} \prod_{i\in\mathcal{L}} v(m_{i-1}) w(n'_i - m_{i-1}) \\ n'_i - n_i =& m_{i-1} - m_i \quad \forall\ i \in \mathcal{L}\end{aligned}$$

The reason we performed the similarity transformation on the transition matrix is that the matrix element $\mathbf{M^0}_{\bar{n}',\bar{n}}$ does only explicitly depend on the final configuration $\bar{n}'$, independently of the initial configuration $\bar{n}$, which will be of great help. This will become clearer as we will progress in the reasoning.
Once a solution ψ^0 to this problem is found, one easily obtains the corresponding right eigenvector of $\mathbf{M}$ by composition $\psi = \mathbf{S}^{-1}\psi^0$.

Integrability of the model. We want to construct solutions of the equation

$$\Lambda\psi^0(\bar{n}') = \sum_{\bar{n}} \mathbf{M^0}_{\bar{n}',\bar{n}} \psi^0(\bar{n})$$

An equivalent description for the configuration of the system can be given through the coordinates of the particles on the lattice. We will now adopt this description, and specify a M-particle configuration by their ordered coordinates $\bar{x} \in \mathcal{D}_M = \{\{x_1, \cdots, x_M\} \in \mathcal{L}^M,\ x_{i+1} \geq x_i\}$.

For one particle. In this case the eigenproblem simply reads

$$\Lambda\psi^0(x) = p\psi^0(x-1) + (1-p)\psi^0(x)$$

where $p = \frac{v(1)}{v(1)+w(1)}$. The parameter p is the probability for the particle to jump to the right at each time iteration. As we will see, this will also be in the more general M particle case in a sector in which the particle does not interact with others, i.e. the free particle sector.

For two particles. This case is slightly more delicate, as interaction between particles will appear. Two cases thus are to be considered . First, the analog of the one particle case, for which the particles does not interact, i.e. for $x_1 < x_2$, in which case the eigenproblem reads

$$\begin{aligned}\Lambda\psi^0(x_1, x_2) =& (1-p)[p\psi^0(x_1 - 1, x_2) + (1-p)\psi^0(x_1, x_2)] \\ &+ p[p\psi^0(x_1 - 1, x_2 - 1) + (1-p)\psi^0(x_1, x_2 - 1)]\end{aligned} \tag{1.15}$$

which is obviously the combination of two one-particle problems for x_1 and x_2 independently. This equation is however not valid on all the physical domain $\mathcal{D}$, as the eigenproblem will take another form in the case of $x_1 = x_2 = x$, which in turn reads

$$\Lambda\psi^0(x,x) = f^{-1}(2)[w(2)\psi^0(x,x) + v(1)w(1)\psi^0(x-1,x) + v(2)\psi^0(x-1,x-1)] \quad (1.16)$$

where, once again, we see that the terms $v(m)$ appears when m particles jump to the right, and $w(n)$ for n particles staying on the site, independently of the initial number of particle present on the site.

The idea now is to extend the validity of the free equation (1.15) to the whole physical domain, giving rise to additional boundary conditions. To obtain this latter, let us consider the free eigenproblem (1.15) in the particular case $x_1 = x_2 = x$, which reads

$$\begin{aligned}\Lambda\psi^0(x,x) =&(1-p)[p\psi^0(x-1,x) + (1-p)\psi^0(x,x)] \\ &+ p[p\psi^0(x-1,x-1) + (1-p)\psi^0(x,x-1)]\end{aligned} \quad (1.17)$$

We see here a so called forbidden term appearing, i.e. a term for which the ordering of the coordinate is violated: $\psi^0(x,x-1)$. Combining (1.16) and (1.17), we obtain the condition on the forbidden term, namely the boundary condition,

$$\psi^0(x,x-1) = \alpha\psi^0(x-1,x-1) + \beta\psi^0(x-1,x) + \gamma\psi^0(x,x) \quad (1.18)$$

where

$$\alpha = \frac{v(2)/f(2) - p^2}{p(1-p)}, \; \beta = \frac{v(1)w(1)/f(2)}{p(1-p)}, \; \gamma = \frac{w(2)/f(2) - (1-p)^2}{p(1-p)} \quad (1.19)$$

We can thus completely define the two-particle eigenproblem through the free eigenproblem (1.15) on the entire physical domain, alongside the boundary condition (1.18).

For M particles. This case once again requires a bit more of attention.
The non interacting equation here writes, for $x_{i+1} > x_i$,

$$\Lambda\psi^0(\bar{x}) = \sum_{\bar{k}\in\{0,1\}^{\otimes M}} p^{||\bar{k}||}(1-p)^{M-||\bar{k}||}\psi^0(\bar{x}-\bar{k}) \quad (1.20)$$

where we adopted the notations $||\bar{k}|| = \sum_{i=1}^M k_i$, and $\bar{x} - \bar{k} = \{x_i - k_i\}_{i=1,\cdots,M}$. The choice of conjugation we performed on the transition matrix now takes all its sense, as its coefficients factorize over single sites, only depending on the number of incoming particles, and number of particles after the transition.

Therefore, in turn, the equation with the term $\Lambda\psi^0(\cdots, x^n, \cdots)$ in the l.h.s. will contain the sum

$$\sum_{k=0}^{n} \varphi(k|n)\psi^0(\cdots,(x-1)^k, x^{n-k},\cdots) \quad (1.21)$$

where x^n represents a string $x_i, \cdots, x_{i+n-1}$ of value x, i.e. for n particles at the same site x. The corresponding term from the non-interacting equation, (which contains forbidden terms of the form $\psi^0(\cdots, x, x-1, \cdots)$), is given by

$$\sum_{\bar{k}\in\{0,1\}^{\otimes n}} p^{||\bar{k}||}(1-p)^{n-||\bar{k}||}\psi^0(\cdots, x-k_1, \cdots, x-k_n, \cdots) \tag{1.22}$$

The condition for this system to be integrable by means of the coordinate Bethe ansatz can now be clearly enlightened: the boundary conditions for the M-particle problem can be obtained by successive iterations of the generalized two-particle boundary condition

$$\begin{aligned}&\psi^0(\cdots, x, x-1, \cdots)\\ &= \alpha\psi^0(\cdots, x-1, x-1, \cdots) + \beta\psi^0(\cdots, x-1, x, \cdots) + \gamma\psi^0(\cdots, x, x, \cdots)\end{aligned}$$

Indeed, if that is the case, the complete spectral problem will be specified by the free particle equation (1.20) alongside the above boundary conditions, which are altogether solvable by CBA, as we will see.

The following reasoning relies on a very simple but sharp idea due to A.M. Povolotsky[18], which is to translate the condition for the factorization of the boundary condition, i.e. the condition for integrability, into a very simple algebraic problem: Defining the algebra generated by two elements A and B obeying the relation

$$BA = \alpha AA + \beta AB + \gamma BB \tag{1.23}$$

the transition coefficients φ will have to satisfy

$$(pA + (1-p)B)^n = \sum_{m=0}^{n} \varphi(m|n) A^m B^{n-m} \tag{1.24}$$

Philosophically, the l.h.s. here identifies with equation (1.22), and the r.h.s. with equation (1.21). Albeit the problem is now expressed in a very simple form, its solution is not trivial, which can be summarized in the following proposition.

Proposition 1.1.1. *[18] Consider an associative algebra over complex numbers, generated by A and B, satisfying the relation (1.23), where α, β, γ are arbitrary complex numbers constrained by $\alpha+\beta+\gamma = 1$, and p a complex number. The coefficients $\varphi(m|n)$ in (1.24) are then given by*

$$\varphi(m|n) = \mu^m \frac{(\nu/\mu; q)_m (\mu; q)_{n-m}}{(\nu; q)_n} \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}} \tag{1.25}$$

and the function f (1.13) reads

$$f(n) = \frac{(\nu; q)_n}{(q; q)_n} \tag{1.26}$$

where ν, μ, q parametrize α, β, γ and p as

$$\alpha = \frac{\nu(1-q)}{1-q\nu}, \quad \beta = \frac{q-\nu}{1-q\nu}, \quad \gamma = \frac{1-q}{1-q\nu} \tag{1.27}$$

$$\mu = p + \nu(1-p) \tag{1.28}$$

and we suppose that $\nexists k \in \mathbb{N}$, $\nu = q^{-k}$.

The constraint we imposed on the system, in order to guaranty its integrability by means of CBA, completely fixed the system up to three free parameters μ, $\nu\, q$. As we will see, the latter constraint $\nexists k \in \mathbb{N},\ \nu = q^{-k}$ excludes the half-integer spin values, the spin value s corresponding to the relation $\nu q^{2s} = 1$. This particular class of cases will require a bit more of attention, which is to be exposed at the end of this section.
Note that the weight for the stationary state $f(n)$ does not depends on the spectral parameter μ. This property relies on the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{1.29}$$

from which we deduce, via a discrete Fourier transform, that

$$f(n) \equiv \sum_{m=0}^{n} \mu^m \frac{(\nu/\mu;q)_m}{(q;q)_m} \frac{(\mu;q)_{n-m}}{(q;q)_{n-m}} = \frac{(\nu;q)_n}{(q;q)_n} \tag{1.30}$$

In addition, the three parameters α, β, γ do not depend on the spectral parameter μ. This will imply, in turn, the Bethe states, whose form are constrained by the generalized two-particle boundary conditions, to also be independent of the spectral parameter.

The Ansatz. All the ingredients are now available to make the Bethe ansatz work for diagonalizing $\mathbf{M^0}$. The eigenproblem is now reformulated as the non-interacting equations, valid on the whole physical domain $\mathcal{D}$,

$$\Lambda \psi^0(\bar{x}) = \sum_{\bar{k} \in \{0,1\}^{\otimes M}} p^{||\bar{k}||}(1-p)^{M-||\bar{k}||} \psi^0(\bar{x} - \bar{k}) \tag{1.31}$$

alongside the boundary conditions

$$\begin{aligned} &\psi^0(\cdots, x, x-1, \cdots) \\ =&\alpha\psi^0(\cdots, x-1, x-1, \cdots) + \beta\psi^0(\cdots, x-1, x, \cdots) + \gamma\psi^0(\cdots, x, x, \cdots) \end{aligned} \tag{1.32}$$

We are looking for solutions of the form

$$\psi^0(\bar{x}|\bar{z}) = \sum_P A_P(\bar{z}) \prod_i z_{Pi}^{x_i} \tag{1.33}$$

where the z_i are the quantum numbers parametrizing the states. On this assumption on the form of the amplitudes consists the ansatz. Plugging the ansatz (1.33) in the non-interacting equation (1.31) provides us with an expression for the eigenvalue

$$\Lambda(\bar{z}) = \prod_i (1 - p + p/z_i) \tag{1.34}$$

associated with the amplitude $\psi^0(\bar{x}|\bar{z})$. On the other hand, the boundary conditions (1.32) are fulfilled if the A coefficients satisfy

$$\frac{A_P(\bar{z})}{A_{P(jj+1)}(\bar{z})} = -\frac{\alpha + \beta z_{Pj} + \gamma z_{Pj} z_{P(j+1)} - z_{P(j+1)}}{\alpha + \beta z_{P(j+1)} + \gamma z_{Pj} z_{P(j+1)} - z_{Pj}} \tag{1.35}$$

This is solved for

$$A_P(\bar{z}) = \sigma(P) \prod_{i<j} \frac{\alpha + \beta z_{Pi} + \gamma z_{Pi} z_{Pj} - z_{Pj}}{\alpha + \beta z_j + \gamma z_i z_j - z_j} \tag{1.36}$$

where we chose to fix the global multiplicative factor such that $A_{id} = 1$.

Example. *For 2 particles, the ansatz writes*

$$\psi^0(x_1, x_2|z_1, z_2) = A(z_1, z_2) z_1^{x_1} z_2^{x_2} + A(z_2, z_1) z_2^{x_1} z_1^{x_2} \tag{1.37}$$

so that the condition (1.32) *translates to*

$$\begin{aligned} A(z_1, z_2)/z_2 + A(z_2, z_1)/z_1 =& \alpha[A(z_1, z_2) + A(z_2, z_1)]/(z_1 z_2) \qquad (1.38) \\ &+ \beta[A(z_1, z_2)/z_1 + A(z_2, z_1)/z_2] + \gamma[A(z_1, z_2) + A(z_2, z_1)] \\ \Longleftrightarrow & \frac{A(z_1, z_2)}{A(z_2, z_1)} = -\frac{\alpha + \beta z_1 + \gamma z_1 z_2 - z_2}{\alpha + \beta z_2 + \gamma z_1 z_2 - z_1} \qquad (1.39) \end{aligned}$$

which is obviously solved for

$$A(z_{P1}, z_{P2}) = \sigma(P) \frac{\alpha + \beta z_{P1} + \gamma z_{P1} z_{P2} - z_{P2}}{\alpha + \beta z_1 + \gamma z_1 z_2 - z_2} \tag{1.40}$$

The M-particle case is slightly trickier, but relies on the same scheme. In this case, we consider the boundary condition (1.32) *in which the two fixed coordinates are those of the particles i and $i+1$, i.e. $x_i = x$, $x_{i+1} = x - 1$ in the l.h.s. of* (1.32)*. If, among all the permutations involved in the equation, we only consider P and $P(ii+1)$, $P \in \pi_M$, as we did for two particles, we indeed obtain the condition* (1.35)*. This condition is then sufficient, albeit a priori not necessary.*

We can from now on introduce a convenient parameterization of our states through the change of variable

$$z_i = \frac{u_i - \nu^{1/2}}{u_i - \nu^{-1/2}} \tag{1.41}$$

This change of variable slightly differ from the one adopted by Povolotsky in its paper, but will be more convenient for our forthcoming development of Chaper 3.

Now, we can obtain the right eigenvectors of $\mathbf{M}$ by rotation $\psi = \mathbf{S}^{-1}\psi^0$. Then, we obtain the expression for the Bethe vector, eigenvector of (1.12), and associated eigenvalue, which read as

$$\psi(\bar{x}|\bar{u}) = \prod_{i \in \mathcal{L}} \frac{(\nu; q)_{n_i[\bar{x}]}}{(q; q)_{n_i[\bar{x}]}} \sum_P \sigma(P) \prod_{i<j} \frac{q u_{Pi} - u_{Pj}}{q u_i - u_j} \prod_i \left(\frac{u_{Pi} - \nu^{1/2}}{u_{Pi} - \nu^{-1/2}} \right)^{x_i} \tag{1.42}$$

$$\Lambda(\bar{u}) = \frac{u - \mu\nu^{-1/2}}{u - \nu^{1/2}} \tag{1.43}$$

where n_i is the number of particle at site i, $n_i[\bar{x}] = \#x \in \bar{x}, x = i$. This family of states has been shown to be complete [95], hence fully diagonalizing the model.

Remark 1.1.2. *It is important here to mention that the Bethe states are a priori non physical, in that no probabilistic interpretation can be made about their amplitude, which can be complex or not normalized. These states remains of a real interest, as the time evolution of a physical state will be straightforwardly provided provided we know how this physical state decompose over the Bethe basis. Such a decomposition, for initial state which are simple elements of the spin basis, which in turn is physical, is performed in Section 3.1.*

Remark 1.1.3. *As in the frame of the algebraic Bethe ansatz, to be exposed in the forthcoming Chapter 1.2, we here considered the transition matrix, analogous to the quantum transfer matrix, as the central object of our description. This operator, depending on an external parameter (the spectral parameter), acts non-locally on the system of particle. Although this question has not been addressed here, we expect our transfer matrix to commute for different values of the spectral parameter. A first clue is that the obtained Bethe states (1.42) does not depend on the spectral parameter μ, which would constitute a sufficient condition if we knew the obtained spectrum to be complete. At least, we know that the Bethe states would be eigenstates of any operator generated by the transfer matrix. We can actually obtain an operator, generated by the transfer matrix, which has a local action on our system of particle, i.e. a pseudo Hamiltonian. Given that such a computation doesn't seem to appear in the literature, I give it here for the curious reader. We proceed by analogy with the quantum Hamiltonian for the XXX spin chain (See Section 1.2.3).*

As in this latter case, we can see that our transfer matrix reduces to a product of local permutation at some particular value of the spectral parameter, namely $\mu = 1$ here ($(\mathbf{M}|_{\mu=1})_{\bar{n},\bar{m}} = \prod_{j\in\mathcal{L}} \delta_{n_j,m_{j-1}}$, given that $\varphi(m|n)|_{\mu=1} = \delta_{n,m}$), and similarly define the pseudo Hamiltonian $\mathcal{H} = \partial_\mu \ln \mathbf{M}|_{\mu=1}$.

Its elements are given by

$$\mathcal{H}_{\bar{n},\bar{n}'} = \sum_{j\in\mathcal{L}} \left(\prod_{i\neq j-1,j} \delta_{n_i,n'_i} \right) h(n_j|n'_j)\delta_{n'_{j-1}+n'_j,n_{j-1}+n_j} \tag{1.44}$$

where $h(n|m) = \partial_\mu \varphi(n|m)|_{\mu=1}$.

A Bethe state $\psi(\bar{x}|\bar{u})$ (1.42) is by construction eigenstate of $\mathcal{H}$ associated to the eigenvalue $\epsilon(u) = \frac{1}{1-u\nu^{1/2}}$. This Hamiltonian is a sum of local Hamiltonian $\mathcal{H}^j$, which act trivially on the whole chain except on the sites j and $j-1$, i.e. describe nearest neighbor interaction.

Proof. To prove this, we just need a few ingredients, which can easily be computed:

$$\begin{aligned}
\varphi(m|n)|_{\mu=1} &= \delta_{n,m} \\
\Rightarrow L_{n,n'}^{m,m'}|_{\mu=1} &= \delta_{n,m'}\delta_{n',m} \\
\Rightarrow (\mathbf{M}|_{\mu=1})_{\bar{n},\bar{m}} &= \prod_{j\in\mathcal{L}} \delta_{n_j,m_{j-1}} \\
\Rightarrow \left(\mathbf{M}^{-1}|_{\mu=1}\right)_{\bar{n},\bar{m}} &= \prod_{j\in\mathcal{L}} \delta_{n_{j-1},m_j} \\
&\textit{and} \\
(\partial_\mu \mathbf{M}|_{\mu=1})_{\bar{n},\bar{m}} &= \sum_{j\in\mathcal{L}} \left(\prod_{i\neq j,j+1} \delta_{n_i,m_{i-1}} \right) \delta_{m_{j-1}+m_j,n_j+n_{j+1}} h(n_{j+1}|m_j)
\end{aligned}$$

Compiling these expression, we obtain the expression for the Hamiltonian as defined before. □

We see that this pseudo Hamiltonian acts locally on the lattice, as expected. Also, particles evolving with respect to this Hamiltonian can only jump to the left.

The half-integer spin case. This particular case will require to truncate the transfer matrix. Indeed, for $k \in \mathbb{N}$, the function $\varphi(m|n)$ will diverge for $n > k$ as νq^k tends to 1, and so will some elements of $\mathbf{M}$. On the other hand, the amplitude $\psi(\bar{x}|\bar{z})$ will vanish if there exists i such that $n_i[\bar{x}] > k$. Then, if we restrict the entries $\mathbf{M^0}_{\bar{n}',\bar{n}}$ to the values $n_i', n_i \leq k \ \forall i$, we can without trouble take the limit $\nu q^k \to 1$, for which the amplitude $\psi(\bar{x}|\bar{z})$ will still describe an eigenstate. We can for instance evoke the spin 1/2 case, for which the Bethe states coincides with Bethe states for the ASEP. Indeed, setting the new parameterization $\xi_i = \frac{u_i - \nu^{1/2}}{u_i - \nu^{-1/2}}$, we have

$$\psi(\bar{x}|\bar{z}) = \left(\frac{1-\nu}{1-q}\right)^M \sum_P \sigma(P) \prod_{i<j} \frac{(1-p)\xi_{Pi}\xi_{Pj} + p - \xi_{Pi}}{(1-p)\xi_i\xi_j + p - \xi_i} \prod_i \xi_{Pi}^{x_i} \tag{1.45}$$

with $p = \frac{1}{1+q}$, which is obviously the Bethe function for ASEP [19]. We thus know that the Hamiltonian for ASEP commutes with the (truncated) spin 1/2 transition matrix of Povolotsky, at least in the subspace spanned by the Bethe states. Although the dynamics described by our transition matrix or the generated pseudo Hamiltonian is fundamentally different from its ASEP cousin, as in our case particles are to jump only in one direction, the Bethe state for ZCM may be seen as deformation of the ASEP Bethe state in some context, as for instance the transition amplitude treated in Section 3.1. In this context, we will see that the analytic structure of the Bethe states play a central role in the calculation, while the attached dynamics (i.e. the time dependence) remains, at least formally, trivial and silent. The fact that the particles evolve in one direction only may however lead to some simplifications in the calculation of some physical quantities involving time evolution, as for instance the probability to find the m^{th} particle at position x, after time evolution t from step initial condition. Indeed, we hope in this case the dynamics of a particle to only be influenced by its partners located at its right. This question will be approach in Chapter 3.1.

1.2 The Algebraic Approach

In this section are introduced the main ingredients on which relies the Algebraic Bethe Ansatz (ABA). Developed in the eighties by the Leningrad group [15], this sophisticated machinery provides a powerful alternative approach to the coordinate ansatz for the diagonalization of integrable systems, although it tends to keep the physical aspects of the considered system hidden behind its elegant formalism.
This approach requires the consideration of the so called auxiliary space, a non-physical space that closely interact with its physical counterpart. This level of formalism actually provides a precious window opened on the deeper mathematical aspect of the considered systems. The different model are treated on the algebraic level, considering the so called corresponding quantum groups, without necessary regards to their particular representation. The machinery thus handle a class of systems that share a common algebraic structure, that may have very different physical interpretations.

The ABA is intended for the diagonalization of an object more fundamental than the Hamiltonian, so called transfer matrix, which is the generator for the integral of motions of the system. The ansatz here consists on generating eigenstates of the transfer matrix from application of a particular operator on the physical vacuum, that depends on external parameters

$$B(u_1)\cdots B(u_M)\,|\Omega\rangle$$

Similarly as in the coordinate approach, these parameters will have to satisfy the Bethe equation for the considered state to be eigenstate of the transfer matrix. The ABA thus provides a very different although equivalent characterization of the energy spectrum. This new formulation of the result, in the form of multiple application of proper operators on the vacuum, opens a new way to compute physical quantities, such as scalar products and correlation functions. In practice, these imply using the exchange relations of the different operators in play. These relations actually define (or depend on) the so called quantum group that define our system on the algebraic level, level on which many important physical properties can thus already be established [12].
The ABA is now very well developped in the litterature, and numerous reviews exists on the subject, see e.g. [13, 12, 14] from which this Chapter is drawn.

1.2.1 Quantum Inverse Scattering Method

Let us introduce the auxiliary and physical spaces, $\mathcal{A}$ and $\mathcal{H}$ respectively, and keep these undefined for now. We suppose the existence of parametric matrices $R(u) \in End(\mathcal{A}\otimes\mathcal{A})$ and so called Lax Operator $L(u) \in End(\mathcal{A}\otimes\mathcal{H})$, $u \in \mathbb{C}$, such that holds the identity

$$R_{ab}(u-v)L_a(u)L_b(v) = L_b(v)L_a(u)R_{ab}(u-v) \tag{1.46}$$

usually referred to as the RTT relation, where the subscripts a and b specifiy the copy of the auxiliary space in which the operator acts non trivially.
The choice of the R-matrix in a sense projects the problem on the algebraic level, namely specifies the quantum group of the model, while the choice of the L-matrix specifies its

representation. Two models with different physical interpretation can share the same algebraic background, which makes the algebraic approach much deeper than the coordinate one.

Remark 1.2.1. *Using (1.46), we have*

$$\begin{aligned} L_a L_b L_c =& R_{ab}^{-1} R_{ac}^{-1} R_{bc}^{-1} L_c L_b L_a R_{bc} R_{ac} R_{ab} \\ =& R_{bc}^{-1} R_{ac}^{-1} R_{ab}^{-1} L_c L_b L_a R_{ab} R_{ac} R_{bc} \end{aligned}$$

where we used the notation $L_i(u_i) = L_i$, $R_{ij}(u_i - u_j) = R_{ij}$, which, except from pathological cases, imply the famous Yang-Baxter equation

$$R_{ab}(u-v)R_{ac}(u-w)R_{bc}(v-w) = R_{bc}(v-w)R_{ac}(u-w)R_{ab}(u-v) \tag{1.47}$$

In turn, if an R-matrix $R(u)$ satisfies the Yang-Baxter equation, it provides a trivial associated L-matrix satisfying the RTT relation (1.46). Indeed, considering $\mathcal{H} = \mathcal{A}$, and defining $L_i(u) = R_{ic}(u-w)$, $i = a, b$, then (1.46) and (1.47) coincide. In practice, we will always require our R-matrix to satisfy the Yang-Baxter equation.

Let us now consider a broader collection of physical spaces $\{\mathcal{H}_i\}_{i\in\mathcal{L}}$, where $\mathcal{L}$ is for instance a one dimensional lattice, the subscript i then referring to a particular site of the lattice, and associated local Lax matrices $L^i(u) \in End(\mathcal{A}\otimes\mathcal{H}_i)$, such as hold the relations

$$R_{ab}(u-v)L_a^i(u-w)L_b^i(v-w) = L_b^i(v-w)L_a^i(u-w)R_{ab}(u-v), \;\; i \in \mathcal{L} \tag{1.48}$$

where the indexes a, b specify the auxiliary space, and i the quantum space in which these operator act non trivially.
One can check that if such $\{L_i(u)\}_{i\in\mathcal{L}}$ satisfies the previous relation, then the (ordered) product of the Lax matrices, so called monodromy matrix,

$$T(u) \equiv \overrightarrow{\prod}_{i\in\mathcal{L}} L_i(u) \tag{1.49}$$

[5] also does:

$$R_{ab}(u-v)T_a(u-w)T_b(v-w) = T_b(v-w)T_a(u-w)R_{ab}(u-v) \tag{1.50}$$

This identity encodes the relations between the matrix elements (projected in the auxiliary space) of the monodromy matrix T. It is interesting to see that the Lax operator $L(u)$ here plays the role of a local monodromy matrix as they play the same role in regard to our scheme of reasoning. They will however obviously differ from one to the other when

[5] the product here has to be ordered, since the matrices doesn't commute a priori. This ordering is however arbitrary.

specifying the representation for these algebras. While the Lax operator will correspond to a representation for the algebra describing the physical space at a particular site of the lattice, the monodromy matrix will be a representation for the algebra on the the entire lattice. In a more general manner, for two local monodromy matrices $T_1(u)$ and $T_2(u)$ satisfying (1.50), i.e. acting in two different physical spaces (e.g. two halves of a spin chain), the product $T_1(u)T_2(u)$ will also satisfy (1.50) (and then describe the two halves of the spin chain chain glued together).
We now arrive to a very central point of our reasoning, as the trace of the monodromy matrix over the auxiliary space $t(u) \equiv tr_a[T_a(u)]$, the so called transfer matrix, defines a continuous family of commuting operators

$$[t(u), t(v)] = 0 \tag{1.51}$$

This identity is obtained by taking the trace in equation (1.50). The monodromy then generates a family of commuting operators, which can be the first clue for integrability (which would effectively feature the system if this family is complete). Indeed, the more linearly independent commuting operator we have, the more conserved quantity there are. And if the number of conserved quantity fits the number of degrees of freedom, the system is integrable.
If the system is integrable, then the Hamiltonian, generated by the transfer matrix, commutes with this latter. Hence, solving the spectrum problem here translates into finding the eigenvalues and associated eigenstates of the transfer matrix, which is the role of ABA. The idea is to construct eigenvectors of the transfer matrix by application of elements of the monodromy matrix on a proper vacuum. From the exchange relation of these elements, obtained from the RTT relation (1.50), one obtains conditions for our parameters, so called Bethe equation, for the obtained vectors to indeed be eigenstates of $t(u)$.

1.2.2 The Rational 6-vertex Model

We will now specify the objects introduced above, and in particular the R-matrix, such that describing the so called rational 6-vertex model [14]. Its diagonalization by means of the ABA will require some assumption on the Lax matrices in play, but their very representation will be kept unspecified in this section.
The diagonalization of the rational 6-vertex model is pedagogically interesting in that it constitutes one of the simplest realization of ABA for quantum integrable systems. It is a also a necessary step for approaching the problem of the Modified Algebraic Bethe Ansatz for the closed XXX spin chain with arbitrary twist, that will be the subject of Section 2.

We here consider the auxiliary space $\mathcal{A} = \mathbb{C}^2$, and one of the simplest solution of the Yang-Baxter equation (1.47), which is the rational R-matrix for the 6-vertex model

$$R_{ab}(u,v) = \frac{u-v}{c} + P_{ab} = \begin{pmatrix} \frac{u-v+c}{c} & 0 & 0 & 0 \\ 0 & \frac{u-v}{c} & 1 & 0 \\ 0 & 1 & \frac{u-v}{c} & 0 \\ 0 & 0 & 0 & \frac{u-v+c}{c} \end{pmatrix}_{ab} \tag{1.52}$$

$$= \frac{u-v+c/2}{c} + \frac{\sum_\alpha \sigma_a^\alpha \sigma_b^\alpha}{2} \tag{1.53}$$

$$= \frac{u-v+c/2}{c} + \frac{\sigma_a^z \sigma_b^z}{2} + \sigma_a^+ \sigma_b^- + \sigma_a^- \sigma_b^+ \tag{1.54}$$

where P is the permutation operator, $P = \sum_{i,j=1,2} E_{ij} \otimes E_{ji}$, $(E_{ij})_{kl} = \delta_{i,k}\delta_{j,l}$, i.e. P acts on a basis vector of $\mathcal{A} \otimes \mathcal{A}$ as $P_{ab} e_1 \otimes e_2 = e_2 \otimes e_1$. Using the identities $P_{ij}P_{jk}P_{ij} = P_{ik}$, $P^2 = P$, $P_{ij} = P_{ji}$, one easily checks that this matrix indeed satisfies the Yang-Baxter equation (1.47).

As we will see, it is not necessary to specify the representation of the algebra (i.e. to specify the associated monodromy matrix), neither even the quantum space in consideration, to perfom the ABA. We will however need to suppose the elements of the monodromy matrix to properly act on a reference state (ideally the physical vacuum).

Let us in the following consider a monodromy matrix $T(u)$ such that holds the RTT relation (1.50). We write it in the auxiliary basis as

$$T_a(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{pmatrix}_a \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \tag{1.55}$$

where the t_{ij} act on the physical space $\mathcal{H}$, which can still be kept unspecified.

To perform the ABA, we have to also suppose the existence of a vacuum $|0\rangle$ such that holds the action

$$T_a(u)\,|0\rangle = \begin{pmatrix} \lambda_1(u) & B(u) \\ 0 & \lambda_2(u) \end{pmatrix}_a |0\rangle \tag{1.56}$$

what we shall refer to as a proper action, on in turn a proper vacuum. While C anihilates the vaccum, the two diagonal operators of the transfer matrix A and D act linearly on the vacuum, with coefficient $\lambda_{1,2}$. In turn B, which acts non trivially on the vacuum, will be understood as a creation operator, that will generate eigenvalues of the transfer matrix (the so called Bethe states). We now gathered all the ingredients we need to perform ABA.

Proposition 1.2.1. *[14] Let consider an ensemble of Lax operators*

$$L_{ai}(u) = \begin{pmatrix} \alpha_i(u) & \beta_i(u) \\ \gamma_i(u) & \delta_i(u) \end{pmatrix}_a \tag{1.57}$$

acting in $\mathbb{C}^2 \otimes \mathcal{H}_i$, *and assume corresponding vacua* $|0\rangle_i \in \mathcal{H}_i$ *such that*

$$L_{ai}(u)\,|0\rangle_i = \begin{pmatrix} \lambda_{1i}(u) & \beta_i(u) \\ 0 & \lambda_{2i}(u) \end{pmatrix} |0\rangle_i \tag{1.58}$$

where $\lambda_{i1,2}$ are complex functions, then the monodromy matrix

$$T_a(u) = L_{aN}(u) \cdots L_{a1}(u)$$

satisfies (1.56) *for the vacuum* $|0\rangle = |0\rangle_N \otimes \cdots \otimes |0\rangle_1$.

In this case, the vacuum eigenvalues of the monodromy matrix are given by the product of the local vacuum eigenvalues of the Lax operator:

$$\lambda_k(u) = \prod_{i=1}^{N} \lambda_{ki}(u), \;\; k = 1, 2 \tag{1.59}$$

These properties can be proved by induction over N, simply multiplying matrices in the auxiliary basis, and using the fact that the different elements of the Lax matrices act on different spaces and thus commute.
Once again, the Lax operators can be seen as local monodromy matrix in our frame of reasoning. More generally, if two monodromy matrices T_1 and T_2 acting in $\mathbb{C}^2 \otimes \mathcal{H}_1$ and $\mathbb{C}^2 \otimes \mathcal{H}_2$ respectively (e.g. describing the two halves of a chain) satisfy the relations (1.56), so will the product $T(u) = T_1(u)T_2(u)$ and so on and so forth.

We are now going to construct eigenstates of $t(u)$ by successive application of operators B on the vacuum, i.e. states of the form $B(\bar{u})\,|0\rangle \equiv \prod_i B(u_i)\,|0\rangle$ for a generic set $\bar{u} = \{u_1, \cdots, u_M\}$, what we call Bethe states.
The RTT relations (1.50) are in our case (1.52) equivalent to the exchange relations

$$\begin{aligned} t_{ij}(v)t_{ij}(u) =& t_{ij}(u)t_{ij}(v) && (1.60)\\ t_{ij}(v)t_{ik}(u) =& f(u,v)t_{ik}(u)t_{ij}(v) + g(v,u)t_{ik}(v)t_{ij}(u) && (1.61)\\ t_{ij}(v)t_{kj}(u) =& f(v,u)t_{kj}(u)t_{ij}(v) + g(u,v)t_{kj}(v)t_{ij}(u) && (1.62)\\ [t_{ij}(u), t_{\bar{i}\bar{j}}(v)] =& g(u,v)\{t_{\bar{i}j}(u)t_{i\bar{j}}(v) - t_{\bar{i}j}(v)t_{i\bar{j}}(u)\} && (1.63) \end{aligned}$$

where $\bar{1} = 2,\ \bar{2} = 1,\ g(u,v) = \frac{c}{u-v}$ and $f(u,v) = 1 + g(u,v)$. From these relations, we deduce the identities,

$$\begin{aligned} t_{ij}(v)t_{ik}(\bar{u}) =& f(\bar{u},v)t_{ik}(\bar{u})t_{ij}(v) && (1.64)\\ &+ \sum_{i=1}^{M} g(v,u_i)f(\bar{u}_i,u_i)t_{ik}(\bar{u}_i)t_{ik}(v)t_{ij}(u_i) && (1.65)\\ t_{jk}(v)t_{ik}(\bar{u}) =& f(v,\bar{u})t_{ik}(\bar{u})t_{jk}(v) && (1.66)\\ &+ \sum_{i=1}^{M} g(u_i,v)f(u_i,\bar{u}_i)t_{ik}(\bar{u}_i)t_{ik}(v)t_{jk}(u_i) && (1.67) \end{aligned}$$

where we used the notation, for a function or an operator depending on a unique variable $\mathcal{F}(u)$ and a set $\bar{u}$, $\mathcal{F}(\bar{u}) = \prod_i \mathcal{F}(u_i)$ (assuming this operator commute for different parameter), and $\bar{u}_i = \bar{u} \setminus \{u_i\}$.

Proof. Let prove the particular expression

$$A(v)B(\bar{u}) = f(\bar{u}, v)B(\bar{u})A(v) \tag{1.68}$$

$$+ \sum_{i=1}^{M} g(v, u_i)f(\bar{u}_i, u_i)B(\bar{u}_i)B(v)A(u_i) \tag{1.69}$$

We can first remark that given the exchange relations for A and B from (1.61), we can translate A through the Bs. At each step, the parameter carried by A can or remain attached to it, bringing a factor f, either be exchanged with the parameter carried by B, then bringing a factor g.

Considering the relation of exchange, and the symmetry over permutation of the u_i (given that $[t_{ij}(u), t_{ij}(v)] = 0$), we now we can write

$$A(v)B(\bar{u}) = c(v, \bar{u})B(\bar{u})A(v) \tag{1.70}$$

$$+ \sum_{i=1}^{M} c_i(v, \bar{u})B(\bar{u}_i)B(v)A(u_i) \tag{1.71}$$

The first coefficient c corresponds to the term for which the A keeps its parameter at each step, i.e. $c(v, \bar{u}) = \prod_i f(u_i, v)$.
The second coefficient c_i corresponds to terms for which at least one parameter has been exchanged. Since the t_{ij} permute for different parameter, i.e. $[t_{ij}(u), t_{ij}(v)] = 0$, we can write the product $A(v)B(\bar{u}) = A(v)B(u_i)B(\bar{u}_i)$. Exchanging the first to operators, we obtain $f(u_i, v)B(u_i)A(v)B(\bar{u}_i) + g(u_i, v)B(v)A(u_i)B(\bar{u}_i)$. The first term see u_i at the left side of A, so that it can not give rise to terms with $A(u_i)$ after making this operator go to the right. The second therm, in turn, contains $A(u_i)$. Among all the term that are to appear when making this operator going through the right ones, will be the term for which no parameter are exchanged, hence with the coefficient $f(\bar{u}_i, u_i)$. In all the other term, in which the parameter is exchanged, u_i will be attached with a B. Hence $c_i = g(v, u_i)f(\bar{u}_i, u_i)$. The other coefficients are obtained by symmetry. □

Now, by application of equations (1.64) and (1.67) on the vacuum, we obtain

$$t(v)B(\bar{u})|0\rangle = \Lambda(v, \bar{u})B(\bar{u})|0\rangle \tag{1.72}$$

$$+ \Lambda_i(v, \bar{u})B(\bar{u}_i)B(v)|0\rangle \tag{1.73}$$

where $\Lambda(v, \bar{u}) = g(v, u_i)\lambda_1(v)f(\bar{u}, v) + \lambda_2(v)f(v, \bar{u})$,
and $\Lambda(v, \bar{u}) = g(v, u_i)(\lambda_1(u_i)f(\bar{u}_i, u_i) - \lambda_2(u_i)f(u_i, \bar{u}_i))$. A Bethe vector $B(\bar{u})|0\rangle$ will then be eigenstate of the transfer matrix $t(u)$ if and only if $\Lambda_i(v, \bar{u}) = 0$, $\forall i = 1, \cdots, M$, i.e. if the rapidities u_i satisfy the so called Bethe equations

$$\frac{\lambda_1(u_i)}{\lambda_2(u_i)} = \frac{f(u_i, \bar{u}_i)}{f(\bar{u}_i, u_i)} \tag{1.74}$$

If so, the corresponding eigenvalue reads

$$\Lambda(v, \bar{u}) = g(v, u_i)(\lambda_1(u_i) f(\bar{u}_i, u_i) - \lambda_2(u_i) f(u_i, \bar{u}_i))$$

.

Remark 1.2.2. *We saw in 1.2.1 that if a set of Lax operators $L_i(u)$ satisfiy the RTT relation (1.50), so does the monodromy matrix $T(u) = \prod_{i \in \mathcal{L}} L_i(u)$. Furthermore, we saw in proposition 1.2.1 that if the Lax operators satisfy the properties (1.58), also would the monodromy matrix. The aim will then be to find a family of local L operators satisfying those two properties, from which we can construct a non local operator, the monodromy matrix, whose trace can then be diagonalized by means of the ABA.*

1.2.3 The XXX Spin Chain

We are now going to concretely specify the Lax operator involved in the last Section, i.e. to specify the representation of the algebra (1.50).

We can already build the simplest L-matrix directly from the R-matrix (1.52), considering the local quantum space $\mathcal{H}_i = \mathbb{C}^2$, defined as $L_i(u) = R_{ai}(u)$. In turn, we can construct the monodromy matrix $T(u) = L_1(u - \theta_1) \cdots L_L(u - \theta_L)$ [6], that satisfies, according to the previous section, to the RTT-relation

$$R_{ab}(u - v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u - v). \tag{1.75}$$

We can actually consider the generalized representation, by defining the Lax matrices

$$L_{ai}(u) = \begin{pmatrix} \frac{u}{c} + \frac{1}{2} + S_i^z & S_i^- \\ S_i^+ & \frac{u}{c} + \frac{1}{2} - S_i^z \end{pmatrix}_a = \frac{u - \theta_i}{c} + \sigma_a^z S_i^z + \sigma_a^+ S_i^- + \sigma_a^- S_i^+ \tag{1.76}$$

where we define the generators of the gl_2 algebra $\{S^{\pm}, S^z, I\}$ that satisfy the commutation relations

$$[S^+, S^-] = 2S^z, \quad [S^z, S^{\pm}] = \pm S^{\pm}, \quad [I, S^a] = 0 \tag{1.77}$$

We will consider several representation of this algebra, $\{S_i^{\pm}, S_i^z, I_i\}_{i \in \mathcal{L}}$, acting on different vector spaces (the local physical spaces, labeled by i, which can for instance be interpreted as the sites of a spin chain). Although these representations don't need to be specified further, neither to be equivalent at different sites, we will have to consider these to be highest weight representations, i.e. we assume a highest weight vector $|0\rangle_i$ such that $S_i^+ |0\rangle = 0$. The local physical space $\mathcal{H}_i$ is then generated by action of S_i^- on the vacuum, $\mathcal{H}_i = Vec\{(S_i^-)^n |0\rangle_i\}_{n \in \mathbb{N}}$ (one can easily check, using the commutation relations above, that this family is closed under action of the generators).
We will also consider, as a matter of simplicity, each representation to be finite dimensional. More precisely, we consider finite dimensional representations with an arbitrary

[6] We here introduced some homogeneities $\{\theta_i\}_{i \in \mathcal{L}}$, as this shall not imply any changes in the ansatz, and could be useful for further reasonings.

positive half-integer spin s_i for each site of the chain, which are higher and lowest weight representation of the gl_2 Lie algebra. Let us define the highest weight vector of spin s representation by $|s\rangle$. The action of the gl_2 generators are given by

$$S^z|s\rangle = s|s\rangle, \quad S^+|s\rangle = 0, \quad \left(S^-\right)^{2s+1}|s\rangle = 0, \quad I|s\rangle = |s\rangle. \tag{1.78}$$

The action of the gl_2 generators on the lowest weight vector $|-s\rangle$ of spin s representation is

$$S^z|-s\rangle = -s|-s\rangle, \quad S^-|-s\rangle = 0, \quad \left(S^+\right)^{2s+1}|-s\rangle = 0, \quad I|-s\rangle = |-s\rangle. \tag{1.79}$$

The lowest and the highest vector are related by the formula

$$|-s\rangle = \frac{\left(S^-\right)^{2s}}{\prod_{j=1}^{2s}\sqrt{j(2s-(j-1))}}|s\rangle. \tag{1.80}$$

We take the highest (lowest) weight vector as a tensor product of gl_2 highest (lowest) weight vectors with different spins. They are given by

$$|0\rangle = \bigotimes_{i=1}^{N}|s_i\rangle, \quad |\hat{0}\rangle = \bigotimes_{i=1}^{N}|-s_i\rangle. \tag{1.81}$$

The monodromy matrix for N arbitrary positive (half)-integer spins $\bar{s} = \{s_1, \ldots, s_N\}$ is then given by

$$T_a(u) = L_{a1}(u-\theta_1)\ldots L_{aN}(u-\theta_N), \tag{1.82}$$

where $\{\theta_1, \ldots, \theta_N\}$ are inhomogeneity parameters, introduced as a matter of generality. These have no impact on our reasoning.

On one hand it is quite simple to check, using the commutation relation (1.77), that for any site $i \in \mathcal{L}$ the Lax matrix L_{ai} actually satisfy the RTT relations (1.75), and so does the monodromy matrix $T(u)$. On the other hand, we directly observe that L_{ai} acts as required on the vacuum, and so does $T_a(u)$ according to Proposition 1.2.1.

The actions of the monodromy matrix on the two vacua are given by

$$T_a(u)\,|0\rangle = \begin{pmatrix} \lambda_1(u) & t_{12}(u) \\ 0 & \lambda_2(u) \end{pmatrix}_a |0\rangle\,, \quad T_a(u)\,|\hat{0}\rangle = \begin{pmatrix} \lambda_2(u) & 0 \\ t_{21}(u) & \lambda_1(u) \end{pmatrix}_a |\hat{0}\rangle \tag{1.83}$$

We can then apply the ABA as described before and obtain a family of eigenvectors for the transfer matrix. The results of the previous section then directly apply, with the associated weight functions

$$\lambda_1(u) = \prod_{i=1}^{N}\frac{u-\theta_i+c(s_i+\frac{1}{2})}{c}, \quad \lambda_2(u) = \prod_{i=1}^{N}\frac{u-\theta_i-c(s_i-\frac{1}{2})}{c}. \tag{1.84}$$

Remark 1.2.3. *Note that here, given the particular expression of the R-matrix, holds the property* $[R_{ab}(u), K_a K_b] = 0$, *for any* $K \in End(\mathbb{C}^2)$, *so called* gl_2 *invariance, so that the twisted monodromy matrix* $KT(u)$ *also satisfies* (1.75). *We thus constructed a family of commuting operators, generated by the twisted transfer matrix* $t_K(u) = tr_a[K_a T_a(u)]$, *i.e. such that* $[t_K(u), t_K(v)] = 0$ *according the previous section. As we shall see now, the ABA will here be workable only for a twist* K *which is diagonalizable and invertible.*

The monodromy matrix contains the gl_2 Lie algebra generators $\{S^+, S^-, S^z, I\}$ realized as $S^\alpha = \sum_{i=1}^N S_i^\alpha$,

$$T(u) = \left(\frac{u}{c}\right)^N + \left(\begin{pmatrix} \frac{I+S^z}{2} & S^- \\ S^+ & \frac{I-S^z}{2} \end{pmatrix} - \sum_{i=1}^N \theta_i\right)\left(\frac{u}{c}\right)^{N-1} + ... \tag{1.85}$$

Using the RTT relation we can extract the relation between the gl_2 Lie algebra generators and the $t_{ij}(u)$, in particular we have:

$$[S^z, t_{ii}(u)] = 0, \quad [S^z, t_{12}(u)] = t_{12}(u), \quad [S^z, t_{21}(u)] = -t_{21}(u). \tag{1.86}$$

We say that the transfer matrix has a $U(1)$ symmetry if $[S^z, t(u)] = 0$, that is, in the case of a diagonal twist K. In fact, if $\mathrm{Det}(K) \neq 0$, then due to the global gl_2 invariance we can always restore the $U(1)$ symmetry, by diagonalizing the twist matrix and doing a proper similarity transformation to perform usual ABA, as we are to see in the next Section 1.2.4.

Starting form the highest or the lowest weight vector we can construct two possible basis, that we call Bethe vectors, to span the Hilbert space of the model $\bigotimes_{i=1}^N \mathbb{C}^{2s_i+1}$. Let $\bar{u}$ denote a set of arbitrary complex parameters $\{u_1, ..., u_M\}$ with cardinality $\#\bar{u} = M$, and let $S = \sum_{i=1}^N 2s_i$. The Bethe state states are then constructed as

$$|\psi^M(\bar{u})\rangle = \prod_{i=1}^M t_{12}(u_i)|0\rangle \tag{1.87}$$

with $\#\bar{u} = M$ and $M = 0, 1, ..., S$. Or equivalently, by

$$|\hat{\psi}^M(\bar{v})\rangle = \prod_{i=1}^{\hat{M}} t_{21}(v_i)|\hat{0}\rangle \tag{1.88}$$

with $\#\bar{v} = \hat{M}$ and $\hat{M} = 0, 1, ..., S$.

These two basis are related by a morphism of the Yangian. Let us consider the mappings

$$\psi_1(T(u)) = T(-u), \quad \psi_2(T(u)) = T^t(u), \tag{1.89}$$

where t is the usual transposition in the auxiliary space, $A_{ij}^t = A_{ji}$. They define anti-morphism of the RTT relation (see *e.g.* [49]). Taking the composition of these mappings we obtain an automorphism of the Yangian

$$\phi(T(u)) = \psi_1 \circ \psi_2(T(u)) = T^t(-u). \tag{1.90}$$

This mapping relates the two different bases of the representation space of the Yangian. Indeed we have

$$\phi(|\psi^M(\bar{u})\rangle) = |\hat{\psi}^M(-\bar{u})\rangle\,, \tag{1.91}$$

where we have to apply the rule $\phi(\lambda_i(u)) = \lambda_{3-i}(-u)$ and $\phi(|0\rangle) = |\hat{0}\rangle$ to preserve the structure of the action. Thus, ϕ relates the highest weight and the lowest weight representations. We remark also that $\phi(t(u)) = t(-u)$, and therefore the action of the transfer matrix on one of the bases defines the action on the other.

The Hamiltonian for the Heisenberg Spin Chain from the 6-vertex Model For an invertible twist K, we can here define a Hamiltonian as the logarithmic derivative of the homogeneous transfer matrix (i.e. without inhomogeneities) around the point for which $R = P$:

$$H \equiv \partial_u ln[t_K(u)]|_{u=0} = t_K^{-1}(0)\,\partial_u t_K(u)|_{u=0} \tag{1.92}$$

$$= \sum_{k=1}^{l-1} P_{ii+1} + K_l P_{1l} K_l^{-1} \tag{1.93}$$

$$= \sum_{k=1}^{l} \frac{1}{2}(\overrightarrow{\sigma}_i \cdot \overrightarrow{\sigma}_{i+1} + 1) \tag{1.94}$$

along with the periodicity condition $\sigma^a_{l+1} = K_1^{-1}\sigma^a_1 K_1$. We then recovered the Hamiltonian for XXX as explicitly defined on the spin basis in (1.1).

Remark 1.2.4. *We just diagonalized the transfer matrix for the XXX spin chain by means of the ABA, which as we saw generate the Hamiltonian for the XXX spin chain in the case of the spin $1/2$ representation. In the frame of the ABA, the eigenstates are generated by multiple application of the t_{12} operator on the vacuum, provided their arguments actually satisfy the Bethe equation for the model. These equation corresponds to the ones we obtained in the case of the coordinate Bethe ansatz in Section 1.1.1. This strongly suggest that these two basis actually are the same, namely that $|\psi(\bar{u})\rangle \propto B(\bar{u})\,|0\rangle$, where $|\psi(\bar{u})\rangle$ is the Bethe state obtain by coordinate Bethe ansatz in Section 1.1.1. This link is actually very delicate to establish, in that the two characterization are fundamentally different.*

1.2.4 The Twisted Chain

In this section we will consider the twisted transfer matrix $T_K(u) = tr_a[K_a T(u)]$. The important point here is that the twisted elements will satisfy the same exchange relation that those of the untwisted matrix, due to the symmetry of the system. We would however have to consider a twisted vacuum, in order to preserve proper action of the monodromy matrix, which is the second and last necessary ingredient to perform the ABA.

Let us consider a matrix $M_a \in \mathbb{C}^2$. If it is invertible, we can write it in the exponentiated form [21]

$$M_a = e^{\sum_\alpha m_\alpha \frac{1}{2}\sigma_a^\alpha}$$

We can straightforwardly generalize this object to the other representations as

$$M_i = e^{\sum_\alpha m_\alpha S_i^\alpha}$$

what we shall name the i^{th} counterpart of M. Then holds the local gl_2 invariance

$$[L_{ai}(u), M_a M_i] = 0 \tag{1.95}$$

where we considered the representation (1.76) for the L matrix.

Proof. The result here relies on the identity

$$[L_{ai}(u), S_i^\beta + \frac{1}{2}\sigma^\beta] = 0$$

which is obtained using the definition of $L_{ai}(u)$ and the commutation relations for the S^α and the σ^α. Then

$$[L_{ai}(u), \sum_\beta m_\beta (S_i^\beta + \frac{1}{2}\sigma^\beta)] = 0$$

and hence the invariance. □

We construct the quantum counterpart of M as $\mathcal{M} = \prod_{i\in\mathcal{L}} M_i$. From the definition of $T(u)$ and property (1.95), we deduce the global gl_2 invariance of the chain

$$[T_a(u), M_a\mathcal{M}] = 0 \tag{1.96}$$

On this invariance relies the simple reasonning, to be exposed thereafter, on which we diagonalize the twisted transfer matrix.

Consider a twist matrix $K \in End(\mathbb{C}^2)$. For diagonalizing the corresponding transfer matrix $T_K(u) = tr_a[K_a T(u)]$ by means of the usual ABA as exposed before, this matrix needs to be invertible and diaonalizable. We then define an invertible matrix M and a diagonal matrix $D = diag(\tilde{\kappa}, \kappa)$, such that $K = MDM^{-1}$. Given the cyclicity of the trace, we can consider the similar twisted monodromy matrix $\tilde{T}(u) = DM^{-1}T(u)M$, i.e. such that $t_K(u) = tr_a[\tilde{T}(u)]$. Using the global $gl(2)$ invariance (1.96), we get $\tilde{T}(u) = \mathcal{M}DT(u)\mathcal{M}^{-1}$.

The proper vacuum to be considered here is obtained by similarity transformation on the old vacuum, $|\tilde{0}\rangle = \mathcal{M}\,|0\rangle$. We can then notice that

$$\tilde{T}(u)\,|\tilde{0}\rangle = \begin{pmatrix} \tilde{\kappa}\lambda_1(u) & \tilde{t}_{12}(u) \\ 0 & \kappa\lambda_2(u) \end{pmatrix} |\tilde{0}\rangle \tag{1.97}$$

and then exploit the results obtained for the non diagonal case. The vector

$$\tilde{t}_{12}(u)\,|\tilde{0}\rangle \tag{1.98}$$

will be eigenstate of $t_K(u)$ with the corresponding eigenvalue

$$\Lambda_K(u,\bar{u}) = \tilde{\kappa}\lambda_1(v)f(\bar{u},v) + \kappa\lambda_2(v)f(v,\bar{u}) \tag{1.99}$$

if and only if the Bethe parameters $\bar{u}$ satisfy the Bethe equations

$$\frac{\tilde{\kappa}\lambda_1(u_i)}{\kappa\lambda_2(u_i)} = \frac{f(u_i,\bar{u}_i)}{f(\bar{u}_i,u_i)}. \tag{1.100}$$

Equivalently, from the lowest weight representation, defining $|\tilde{\hat{0}}\rangle = \mathcal{M}\,|\hat{0}\rangle$, the vector

$$\tilde{t}_{21}(v)\,|\tilde{\hat{0}}\rangle \tag{1.101}$$

will be eigenstate of $t_K(u)$ with corresponding eigenvalue

$$\Lambda_K(u,\bar{u}) = \kappa\lambda_1(v)f(\bar{u},v) + \tilde{\kappa}\lambda_2(v)f(v,\bar{u}) \tag{1.102}$$

if and only if the Bethe parameters $\bar{u}$ satisfy the Bethe equations

$$\frac{\kappa\lambda_1(u_i)}{\tilde{\kappa}\lambda_2(u_i)} = \frac{f(u_i,\bar{u}_i)}{f(\bar{u}_i,u_i)}. \tag{1.103}$$

We here see that the characterization of the spectrum, namely the Bethe ansatz, only depends on the diagonal component of the twist, without regards to its rotation. This symmetry is however linked to our particular model, given that it emerged from the gl_2 symmetry in play here.

1.2.5 Quantum Wronskian Equation

We can rewrite the two results of the previous section in terms of Baxter Q-polynomials. Let us define the polynomials

$$q_+^M(z) = \prod_{i=1}^{M}\frac{z-u_i}{c} = \big(g(z,\bar{u})\big)^{-1} \tag{1.104}$$

and

$$q_-^{\hat{M}}(z) = \prod_{i=1}^{\hat{M}}\frac{z-v_i}{c} = \big(g(z,\bar{v})\big)^{-1}. \tag{1.105}$$

This allows us to rewrite the spectral problem (1.100) and (1.103) in terms of Baxter T-Q equations

$$\Lambda_d(z)q_+^M(z) = \tilde{\kappa}\lambda_1(z)q_+^M(z-c) + \kappa\lambda_2(z)q_+^M(z+c), \tag{1.106}$$

and

$$\Lambda_d(z)q_-^{\hat{M}}(z) = \kappa\lambda_1(z)q_-^{\hat{M}}(z-c) + \tilde{\kappa}\lambda_2(z)q_-^{\hat{M}}(z+c). \tag{1.107}$$

The two characterizations of the spectrum by Baxter T-Q equation satisfy a quantum Wronskian condition.

Theorem 1.2.1. *The two Baxter Q-polynomials $q_+^M(z)$ and $q_-^{\hat{M}}(z)$ are related by the quantum Wronskian equation [55]*

$$W_d(z) - W_d(z+c) = 0, \tag{1.108}$$

with

$$W_d(z) = \frac{\tilde{\kappa} q_+^{\hat{M}}(z-c) q_-^M(z) - \kappa q_+^{\hat{M}}(z) q_-^M(z-c)}{\lambda(z)}, \tag{1.109}$$

and

$$\frac{\lambda_1(z)}{\lambda_2(z)} = \frac{\lambda(z+c)}{\lambda(z)}, \quad \lambda(z) = \prod_{i=1}^{N} \prod_{k=1}^{2s_i} \frac{z - \theta_i + c(s_i - k + \frac{1}{2})}{c}. \tag{1.110}$$

This implies $\hat{M} + M = S = \sum_{i=1}^N 2s_i$ and $W_d(z) = (\tilde{\kappa} - \kappa)$.

Proof. For a given eigenvalue $\Lambda_d(z)$ of the transfer matrix, the quantum Wronskian equation follows from the difference between (1.106) times $q_-^{\hat{M}}(z)$ and (1.107) times $q_+^M(z)$, and from the fact that $\frac{\lambda_1(z)}{\lambda_2(z)} = \frac{\lambda(z+c)}{\lambda(z)}$. Then, due to (1.108) we conclude that $W_d(z)$ is a constant, up to a periodic function. Taking the limit $z \to \infty$ we find that $\hat{M} + M = S$ and $W_d(z) = (\tilde{\kappa} - \kappa)$. □

This quantum Wronskian equation, equivalent anlogous of the Bethe equations, eventually becomes interesting when one wants to investigate Bethe solutions [55]. These will find generalization in Chapter 2 with the quantum Wronskian equation for the modified Bethe states.

1.2.6 The Open Chain

The trace of a product of operators is invariant under cyclic permutation of these objects. This results in the periodicity of the system described by the transfer matrix in the previous chapter, which is the trace of the product of local operator from first to last site of the lattice: the consideration of the transfer matrix in previous sections allowed us to consider the diagonalization of the periodic XXX spin chain. This section, strongly based on the work of Slyanin [22] on the open XXZ spin chain, aims for a pedestrian approach of the open chain problem, which require its own investigation.

The description of non-periodic systems in the frame of QISM require the consideration of two other algebras $\mathcal{T}^-$ and $\mathcal{T}^+$, the reflection algebras, respectively defined by the relations

$$R_{12}(u_1 - u_2)\mathcal{T}_1^-(u_1)R_{12}(u_1 + u_2)\mathcal{T}_2^-(u_2) = \tag{1.111}$$
$$\mathcal{T}_2^-(u_2)R_{12}(u_1 + u_2)\mathcal{T}_1^-(u_1)R_{12}(u_1 - u_2) \tag{1.112}$$

and

$$R_{12}(-u_1 + u_2)(\mathcal{T}_1^+)^{t_1}(u_1)R_{12}(-u_1 - u_2 - 2\eta)(\mathcal{T}_2^+)^{t_2}(u_2) = \tag{1.113}$$
$$(\mathcal{T}_2^+)^{t_2}(u_2)R_{12}(-u_1 - u_2 - 2\eta)(\mathcal{T}_1^+)^{t_1}(u_1)R_{12}(-u_1 + u_2) \tag{1.114}$$

where the matrix R is solution of the Yang-Baxter equation (1.47). We also impose the R-matrix to satisfy the properties of

1. symmetry: $P_{12}R_{12}(u)P_{12} = R_{12}(u)$ and $R_{12}^{t_1}(u) = R_{12}^{t_2}(u)$
2. unitarity: $R_{12}(u)R_{12}(-u) = \rho(u)$
3. crossing unitarity: $R_{12}^{t_1}(u)R_{12}^{t_1}(-u-2\eta) = \tilde{\rho}(u)$

where t_i accounts for the transposition in the space i, $\tilde{\rho}$ and ρ are complex functions, and η is a complex constant which characterize R. We will in the following assume these conditions for the R-matrix.

Remark 1.2.5. *The rational R-matrix for the 6-vertex model (1.52), considered in the case of the XXX spin chain, satisfies these conditions with $\eta = c$, $\rho(u) = \frac{1-u^2}{c}$, and $\tilde{\rho}(u) = 1 - \frac{(u+c)^2}{c^2}$.*

Theorem 1.2.1. *For $\mathcal{T}^-(u)$ and $\mathcal{T}^+(u)$ representations of the algebras $\mathcal{T}^-$ and $\mathcal{T}^+$ respectively, the quantity*

$$t(u) = tr[\mathcal{T}^+(u)\mathcal{T}^-(u)] \tag{1.115}$$

forms a continuous family of commuting operators [22], i.e.

$$[t(u), t(v)] = 0, \;\; \forall u, v \tag{1.116}$$

One can thus consider $t(u)$ as a generating function of conserved quantities, as was the transfer matrix for the periodic chain. But this is just the beginning of the story, as we still have to make this relevant for describing open spin chains.

Let us also consider a representation of the T algebra (1.46) that satisfies the crossing symmetry

$$\{T^a(u)\}^a = \theta(u)T(u-2\eta) \tag{1.117}$$

where θ is a complex function and we define the morphism $\{T_1\}^a(u) = \{T_1^{-1}(u)\}^{t_1}$.

Remark 1.2.6. *It is easy to show that the trivial representation of T defined as $T_0(u) = R_{0N}(u)\cdots R_{01}(u)$ satisfies this latter property with $\theta(u) = (\rho(u)\tilde{\rho}^{-1}(-u))^N$, using the properties of symmetry and crossing symmetry of the R matrix.*

This leads us to an important property, that links this algebra and the description of the open chain:

Proposition 1.2.2. *Let $\tilde{\mathcal{T}}^{\pm}(u)$ be representations of the $\mathcal{T}^{\pm}$ algebras in $\tilde{W}^{\pm}$, and $T^{\pm}(u)$ representation of the T algebra in $W^{\pm}$. Then [22] $\mathcal{T}^{\pm}(u)$ defined as*

$$\mathcal{T}^-(u) = T^-(u)\tilde{\mathcal{T}}^-(u)\{T^-\}^{-1}(-u) \tag{1.118}$$

and

$$\{\mathcal{T}^+\}^t(u) = \{T^+\}^t(u)\{\tilde{\mathcal{T}}^+\}^t(u)\{T^+\}^a(-u) \tag{1.119}$$

are representations of the $\mathcal{T}^{\pm}$ algebras in $\tilde{W}^{\pm} \otimes W^{\pm}$.

For applications to the open spin chain, we can choose the particular representations

$$T^-(u) = L_{0M}(u) \cdots L_{01}(u) \tag{1.120}$$
$$T^+(u) = L_{0N}(u) \cdots L_{0M+1}(u) \tag{1.121}$$
$$\tilde{\mathcal{T}}^{\pm}(u) = K^{\pm}(u) \tag{1.122}$$

with L_{0i} some simple representation of the T algebra (1.46), and $K^{\pm}(u)$ are representations of the $\mathcal{T}^{\pm}$ in $\mathbb{C}$, i.e. complex matrices in $End(V)$. This leads us to the main result of QISM for the open spin chain.

Proposition 1.2.3. *For representations $\mathcal{T}^{\pm}(u)$ of the $T^{\pm}$ algebras defined in* (1.118)-(1.122)*, the transfer matrix $t(u)$* (1.115) *reads*

$$t(u) = tr[\mathcal{T}^+(u)\mathcal{T}^-(u)] = tr[K^+(u)T(u)K^-(u)T^{-1}(-u)] \tag{1.123}$$

where $T(u) = T^+(u)T^-(u) = L_{0N}(u) \cdots L_{01}(u)$.

Note that this result is absolutely independent on the factorization of the matrix $T(u)$ into $T^+(u)$ and $T^-(u)$. The considered transfer matrix thus generates, according to (1.116), integrals of motions of the system. As an example, let us consider the R-matrix for the 6-vertex model defined in (1.52), and the corresponding trivial representation for the T algebra (1.46) $T_0(u) = R_{0N}(u) \cdots R_{01}(u)$. It is shown in [22] that the Hamiltonian

$$H = \sum_{n=1}^{N-1} P_{nn+1} + \frac{1}{2}\partial_u K_1^-(u)|_{u=0} + \frac{K_N^+(0)}{tr[K^+(0)]} \tag{1.124}$$

is generated by the transfer matrix according to

$$H = \frac{\partial_u t(u)|_{u=0} - tr[\partial_u K^+(u)|_{u=0}]}{2tr[K^+(0)]}. \tag{1.125}$$

Hence this Hamiltonian commutes with $t(u)$ according to (1.116).

We can now understand why the considered transfer matrix is said to describe the open spin chain, by looking at this Hamiltonian. We see a first term, describing the nearest neighbor interaction in the spin chain, as for the closed XXX-spin chain, and the two last terms, describing interaction of the first and the last site with the boundaries of the system. Note that we are far to be able to consider the general boundary condition for the open chain, as these are defined from the $K^{\pm}$ matrices, which are representation of the $\mathcal{T}^{\pm}$ respectively.

The Algebraic Bethe Ansatz for the Open XXX-Spin Chain

We will now sketch the main ideas of the diagonalization of the transfer matrix of the type (1.123) by means of the ABA, in the case of the rational 6-vertex model (1.52). We consider a representation $T(u)$ of the T algebra (1.46), and suppose the existence of a vacuum $|0\rangle$ such that condition (1.56) holds, for instance the representation (1.76).

For the sake of pedagogy, we will only consider the case of diagonal K matrices (1.122), $K^{\pm}(u) = D^{\pm}(u) = diag(\alpha^{\pm}(u), \beta^{\pm}(u))$, representations of the $\mathcal{T}^{\pm}$ algebras.

We are going to construct eigenstates of the transfer matrix $t(u) = tr[D^{+}(u)\tilde{T}(u)]$, with $\tilde{T}(u) = T(u)D^{-}(u)T^{-1}(-u)$.

This latter matrix actually satisfies the properties (1.56) with the same vacuum. Ideed, using the notation t_{ij} and $\tilde{t}_{ij}$ (1.55) for the matrix elements of $T(u)$ and $\tilde{T}(u)$ respectively, we obtain

$$\tilde{T}(u)\,|0\rangle = \begin{pmatrix} \tilde{\lambda}_1(u) & \tilde{t}_{12}(u) \\ 0 & \tilde{\lambda}_2(u) \end{pmatrix} |0\rangle \tag{1.126}$$

where

$$\begin{aligned}
\tilde{t}_{12}(u) =& \alpha_{+}(u)\alpha_{-}(u)t_{11}(u)t_{12}(-u) + \alpha_{+}(u)\beta_{-}(u)t_{12}(u)t_{22}(-u) \\
\tilde{\lambda}_1(u) =& \alpha_{+}(u)\alpha_{-}(u)\lambda_1(u)\lambda_1(-u) \\
\tilde{\lambda}_2(u) =& \beta_{+}(u)\alpha_{-}(u)\frac{c}{2u}\{\lambda_1(u)\lambda_2(-u) - \lambda_1(-u)\lambda_2(u)\} + \beta_{+}(u)\beta_{-}(u)\lambda_2(u)\lambda_2(-u)
\end{aligned}$$

and where we used the commutation relation (1.63) to obtain the expression for $\tilde{\lambda}_2(u)$. We can then here see the operator $\tilde{t}_{12}(u)$ as a creation operator, that will generate Bethe states.

We would still need a last ingredient to perform the ABA, which is the commutation relations for the operators $\tilde{t}_{ij}(u)$, using the relation (1.111). Using these, one could diagonalize the transfer matrix similarly to what is described for the closed spin chain in Section 1.2.2, although the exchange relations for the $\tilde{t}_{ij}(u)$ operators slightly differ from the usual ones. This last procedure can be found in [22] in the (more general) case of the open XXZ spin chain. An important point to notice here is that the dynamics of the spin chain at its boundaries is described through elements, for instance the $K^{\pm}$ matrices for the Hamiltonian (1.124), that are representations of the $\mathcal{T}^{\pm}$ algebras. This requirement for the boundary condition is quite constraining, in that it is a necessary condition for the system to be solvable by means of the usual ABA. Solving the more general case is still challenging at this day, but some approach are investigated. Among these, the so called Modified Algebraic Bethe Ansatz, which is the subject of the following Chapter 2.

1.3 Conclusion

In this chapter has been exposed two very different approaches for the diagonalization of integrable quantum and stochastic systems. On one hand, the Coordinate Bethe Ansatz (CBA) provides a rather direct approach, that lead to an expression for eigenstates of the Hamiltonian, the so called Bethe states, expressed in the spin basis, in both contexts of the Heisenberg spin chain and the Zero-range Chipping Model with factorized steady state (ZCM). The diagonalization of the transition matrix for the ZCM turned out to be more challenging, in that it requires to consider multi-particle occupation. The transition probability amplitudes has been shown to be constrained in a particular form, parameterized by three free parameters, so that the system effectively is integrable by means of

CBA. This constrain corresponds to a multi-particle dynamics factorizing to two-particle dynamics. Having an explicit expression for the Bethe states in the spin basis, such that obtained with CBA, can turn out to be very convenient and in particular when approaching quench problem, for which one need to symmetrically express the spin basis in terms of Bethe states. Such a gymnastic is performed in Section 3.1 in the context of the ZCM, which appears as a very non trivial exercise. On the other hand, the Algebraic Bethe Ansatz (ABA) provided us with a set of eigenvectors one obtains by multiple application of suitable operators on the physical vacuum. In this approach, the ingredient ensuring integrability are much more easy to identify. An L matrix and a corresponding R matrix, and a good vacuum. This formalism is very favorable for approaching calculations of scalar product, keystones for the computation of form factor and correlation functions. This main ingredients for those, the multiple actions of the matrix elements of the monodromy matrix, will be studied in the context of the Modified Algebraic Bethe ansatz in Chapter 2.

In the case of the XXX spin chain, both the CBA and the ABA has been used to diagonalize the Hamiltonian, and we saw the Bethe equations raising, in a perfectly similar form. As argued, the eigenvectors obtained in each context, parameterized by the Bethe numbers, should correspond. This link is however delicate to formally draw. These two approaches, although very different, led to the same answer formulated with different words. We also mentioned the existence of Bound states, corresponding to strings of Bethe parameters, which are actual part of the spectrum. These bound states has been shown to effectively exhibit a "bound particle" behavior in the CBA, favored by the explicit form of the Bethe states.
The case of the open spin chain, have also been explored. In this context, the constraints on the boundary conditions are very strong. Away from these, we are not able to find a good vacuum. This motivated the developed the so called Modified Algebraic Bethe Ansatz, which is the subject of Chapter 2.

Chapter 2

Modified Algebraic Bethe Ansatz: the Twisted XXX Case

We approached in Chapter 1 the Algebraic Bethe Ansatz (ABA) machinery and employed it in the context of the XXX spin chain. This approach proved to be successful in the context of the periodic chain with invertible periodic conditions, which is rather satisfying. In the case of the open chain, in turn, the ansatz will be successful provided the boundary conditions satisfy a very strong constraint, namely to be representations of the reflection algebras. This significant failure of ABA, due to the fact that no proper vacuum can be considered in these contexts, motivated the development of new approaches of the problem. This chapter present one of these, the Modified Algebraic Bethe Ansatz (MABA), in the context of the twisted XXX spin chain. It is an adapted version of two papers [23, 24] elaborated in collaboration with S. Belliard and N. Slavnov. My contribution to this research project mainly consisted on technical efforts on the formal questions addressed here, namely the maths behind the philosophical considerations. We are here going to consider the periodic XXX spin chain, which can already be handled by the usual ABA, but considering a reference state which is not a "suitable vacuum" from the usual ABA point of view. This would eventually allows one to compare the two approaches in a sector in which they can both be developed. If this new approach appears to be reliable, it could then hopefully be extended to the open spin chain with general boundary conditions for which, also, no good vacuum can be found.

The question of the spectral problem of the lattice quantum integrable models without $U(1)$ symmetry, as the XXZ Heisenberg spin chain on the segment with the most general integrable boundary condition [22], has led to the development of new techniques to perform the Bethe ansatz. Among the proposed methods such as the *off diagonal Bethe ansatz* [33], the *separation of the variables* [21, 34], the *modified algebraic Bethe ansatz* (MABA) proposes to address this problem from the algebraic Bethe ansatz point of view. In this section we study this method and determine what changes are needed with respect to the case with $U(1)$ symmetry.

The new features in the Bethe ansatz for models without $U(1)$ symmetry are as follows. Firstly, the Baxter T-Q equation has a new term [35, 36]. Secondly, the Bethe vectors are linear combinations of all Bethe vectors of the associated model where the $U(1)$ symmetry is restored, and these vectors are factorized in terms of a modified creation operator [37, 38, 39], that was proved in [40, 41] by means of the Baxter T-Q equation[1]. Thirdly, the number of Bethe roots is fixed and depends on the model under consideration [42, 36].

The MABA also can be applied to models with quasi-periodic boundary conditions such as the XXX spin chain with an arbitrary twist. In particular, one can use this approach for studying XXX spin-$\frac{1}{2}$ chain

$$H = \sum_{k=1}^{N} \Big(\sigma_k^x \otimes \sigma_{k+1}^x + \sigma_k^y \otimes \sigma_{k+1}^y + \sigma_k^z \otimes \sigma_{k+1}^z \Big), \tag{2.1}$$

subject to the following non-diagonal boundary conditions:

$$\gamma\sigma_{N+1}^x = \frac{\tilde{\kappa}^2 + \kappa^2 - \kappa_+^2 - \kappa_-^2}{2}\sigma_1^x + i\frac{\kappa^2 - \tilde{\kappa}^2 - \kappa_+^2 + \kappa_-^2}{2}\sigma_1^y + (\kappa\kappa_- - \tilde{\kappa}\kappa_+)\sigma_1^z, \tag{2.2}$$

$$\gamma\sigma_{N+1}^y = i\frac{\tilde{\kappa}^2 - \kappa^2 - \kappa_+^2 + \kappa_-^2}{2}\sigma_1^x + \frac{\tilde{\kappa}^2 + \kappa^2 + \kappa_+^2 + \kappa_-^2}{2}\sigma_1^y - i(\tilde{\kappa}\kappa_+ + \kappa\kappa_-)\sigma_1^z, \tag{2.3}$$

$$\gamma\sigma_{N+1}^z = (\kappa\kappa_+ - \tilde{\kappa}\kappa_-)\sigma_1^x + i(\tilde{\kappa}\kappa_- + \kappa\kappa_+)\sigma_1^y + (\tilde{\kappa}\kappa + \kappa_+\kappa_-)\sigma_1^z. \tag{2.4}$$

The twist parameters $\{\kappa, \tilde{\kappa}, \kappa_+, \kappa_-\} \in \mathbb{C}^4$ are generic and $\gamma = \tilde{\kappa}\kappa - \kappa_+\kappa_-$. The Pauli matrices[2] σ_k^α with $\alpha = x, y, z$ act non-trivially on the k^{th} component of the quantum space $\mathcal{H} = \otimes_{k=1}^N V_k$ with $V_k = \mathbb{C}^2$. This model provides a simple example where the new properties of the method can be studied. In particular, in the case of the Hamiltonian (2.1), several conjectures about the MABA were formulated in [43], where a special transformation of the twist matrix was proposed. One more conjecture of [43], formulated on the base of direct calculations for the chains of small length, concerns a special off-shell action of the modified creation operator, which generates a new term of the Baxter T-Q equation.

In this chapter we consider the most general case of the closed XXX Heisenberg spin chain with an arbitrary twist and arbitrary positive (half)-integer spins transfer matrix[3]. We prove the conjecture about the multiple action of the modified creation operator on the pseudo-vacuum state given in [43]. In fact, we prove this property independently of the action on the pseudo-vacuum state and go beyond the proofs done for other models [44, 45, 46]. Moreover, we consider two different bases for solving the spectral problem of this family of models and we relate the two solutions by a modified quantum Wronskian equation.

In the second Section we study multiple actions of the modified monodromy matrix entries on the modified Bethe vectors. The obtained formulas of the multiple actions allow us to calculate the scalar products of the modified Bethe vectors. We find an analog of Izergin–Korepin formula for the scalar products. This formula involves modified Izergin determinants and can be expressed as sums over partitions of the Bethe parameters.

[1] Contrary to this method, the MABA provides a constructive way to build Bethe vectors and clarifies the algebraic origin of the new term in the Baxter T-Q equation.

[2] $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^x = \sigma^+ + \sigma^-, \quad \sigma^y = i(\sigma^- - \sigma^+)$.

[3] Explicit Hamiltonian for this case is not known. It can be formulated in the more restrictive case of L_0-regular spin chains [47], where L_0 arbitrary positive (half)-integer spins are periodically repeated. The case $L_0 = 1$ with spin $\frac{1}{2}$ corresponds to the Hamiltonian (2.1).

2.1 Caracterization of the Spectral Problem and Quantum Wronskian Equation

This section is devoted to the case with non-diagonal twist ($\kappa^+ \neq 0$ and $\kappa^- \neq 0$) and to the study of the MABA scheme. Here we give two basis for the Bethe vectors as well as their corresponding spectrum, which allows us to obtain the modified quantum Wronskian equation.

We consider the XXX spin chain with arbitrary spin defined in Section 1.2.3.

For a non-diagonal twist, the transfer matrix

$$t(z) = \operatorname*{Tr}_a \left(K_a T_a(z)\right) = \tilde{\kappa} t_{11}(z) + \kappa t_{22}(z) + \kappa^+ t_{21}(z) + \kappa^- t_{12}(z) \tag{2.5}$$

with

$$K = \begin{pmatrix} \tilde{\kappa} & \kappa^+ \\ \kappa^- & \kappa \end{pmatrix}, \tag{2.6}$$

does not commute with the operator S^z. We have $[S^z, t(z)] = \kappa^+ t_{21}(z) - \kappa^- t_{12}(z)$. As a consequence, the highest and lowest weight vectors are not anymore eigenvectors of the transfer matrix. The action of the twisted transfer matrix on the highest weight vector (1.81), which is not anymore eigenstate unless $\kappa^- = 0$, is given by

$$t(z)|0\rangle = (\tilde{\kappa}\lambda_1(z) + \kappa\lambda_2(z))|0\rangle + \kappa^- t_{12}(z)|0\rangle. \tag{2.7}$$

Similarly the lowest weight vector (1.81) is not an eigenstate either (unless $\kappa^+ = 0$), and the action of the twisted transfer matrix on it is

$$t(z)|\hat{0}\rangle = (\tilde{\kappa}\lambda_2(z) + \kappa\lambda_1(z))|\hat{0}\rangle + \kappa^+ t_{21}(z)|\hat{0}\rangle. \tag{2.8}$$

The MABA allows us to construct the modified Bethe vector keeping the highest or lowest weight vectors as a starting point. The idea of the modified algebraic Bethe ansatz [37, 38, 39, 45, 43] relies in the construction of modified operators that preserve the operator algebra structure[4]. The transformation of the monodromy matrix

$$\bar{T}(z) = AT(z)B = \begin{pmatrix} \nu_{11}(z) & \nu_{12}(z) \\ \nu_{21}(z) & \nu_{22}(z) \end{pmatrix}, \tag{2.9}$$

where A and B are two arbitrary two by two matrices, is an automorphism of the Yangian of $\mathfrak{gl}_2$, i.e. new operators satisfy the same commutation relations as the operators $t_{ij}(z)$ (see Appendix 2.3.4). We remark that the modified operators can be seen as a transfer matrix for a model with a twist that has null determinant:

$$\nu_{ij}(z) = \operatorname*{Tr}_a \left((V_{ji})_a T_a(z)\right) \tag{2.10}$$

with

$$V_{ji} = BE_{ji}A, \tag{2.11}$$

[4] For XXZ spin chain the underling operator algebra structure is not preserved but mapped in a more complex one (see [38, 39, 45] and references therein).

that satisfies $\mathrm{Det}(V_{ji}) = 0$. We call such operators *null twisted transfer matrices*. The Yangian generators $t_{ij}(u)$, the modified operators [37, 43], as well as 'good' operators [56] are such objects.

We look for a deformation of the action of the transfer matrix on the highest weight vector in terms of the modified creation operator $\nu_{12}(u)$ of the following form:

$$t(z)|0\rangle = \Big((\tilde{\kappa} - \rho_1)\lambda_1(z) + (\kappa - \rho_2)\lambda_2(z)\Big)|0\rangle + \eta\, \nu_{12}(z)|0\rangle. \tag{2.12}$$

Here ρ_i and η are some scalars that can be seen as deformation parameters that go to zero when we restore the $U(1)$ symmetry. The matrix V_{21} is uniquely determined in terms of $\{\kappa, \tilde{\kappa}, \kappa^-, \kappa^+, \rho_1, \rho_2, \eta\}$.

We do the same for the action on the lowest weight vector and look for the modified creation operator $\nu_{21}(z)$ such that

$$t(z)|\hat{0}\rangle = \Big((\tilde{\kappa} - \rho_1)\lambda_2(z) + (\kappa - \rho_2)\lambda_1(z)\Big)|\hat{0}\rangle + \hat{\eta}\, \nu_{21}(z)|\hat{0}\rangle. \tag{2.13}$$

The matrix V_{12} is uniquely determined in terms of $\{\kappa, \tilde{\kappa}, \kappa^-, \kappa^+, \rho_1, \rho_2, \hat{\eta}\}$. We fix also the factorisation of the twist

$$K = BDA, \tag{2.14}$$

with

$$D = \begin{pmatrix} \tilde{\kappa} - \rho_1 & 0 \\ 0 & \kappa - \rho_2 \end{pmatrix}. \tag{2.15}$$

Thus, we have the diagonal modified transfer matrix

$$t(z) = \mathrm{Tr}(D\bar{T}(z)) = (\tilde{\kappa} - \rho_1)\nu_{11}(z) + (\kappa - \rho_2)\nu_{22}(z).$$

Then, it is easy to see that A and B are given by[5],

$$A = \sqrt{\mu}\begin{pmatrix} 1 & \frac{\rho_2}{\kappa^-} \\ \frac{\rho_1}{\kappa^+} & 1 \end{pmatrix}, \quad B = \sqrt{\mu}\begin{pmatrix} 1 & \frac{\rho_1}{\kappa^-} \\ \frac{\rho_2}{\kappa^+} & 1 \end{pmatrix}, \quad \mu = \frac{1}{1 - \frac{\rho_1\rho_2}{\kappa^+\kappa^-}}, \tag{2.16}$$

provided ρ_1 and ρ_2 satisfy the following relation:

$$\kappa^+\kappa^- - (\rho_1\kappa + \rho_2\tilde{\kappa}) + \rho_1\rho_2 = 0. \tag{2.17}$$

We can now express the modified operators as linear combination of the original operators $t_{ij}(z)$

$$\nu_{11}(z) = \mu\Big(t_{11}(z) + \frac{\rho_2}{\kappa^+}t_{12}(z) + \frac{\rho_2}{\kappa^-}t_{21}(z) + \frac{\rho_2^2}{\kappa^-\kappa^+}t_{22}(z)\Big), \tag{2.18}$$

$$\nu_{22}(z) = \mu\Big(t_{22}(z) + \frac{\rho_1}{\kappa^+}t_{12}(z) + \frac{\rho_1}{\kappa^-}t_{21}(z) + \frac{\rho_1^2}{\kappa^-\kappa^+}t_{11}(z)\Big), \tag{2.19}$$

$$\nu_{12}(z) = \mu\Big(t_{12}(z) + \frac{\rho_1}{\kappa^-}t_{11}(z) + \frac{\rho_2}{\kappa^-}t_{22}(z) + \frac{\rho_1\rho_2}{(\kappa^-)^2}t_{21}(z)\Big), \tag{2.20}$$

$$\nu_{21}(z) = \mu\Big(t_{21}(z) + \frac{\rho_1}{\kappa^+}t_{11}(z) + \frac{\rho_2}{\kappa^+}t_{22}(z) + \frac{\rho_1\rho_2}{(\kappa^+)^2}t_{12}(z)\Big), \tag{2.21}$$

[5] We still have some freedom that follows from the transformation $A \to C^{-1}A$ and $B \to BC$ with $[C, D] = 0$ that leaves (2.14) invariant.

with

$$\mu = \frac{1}{1 - \frac{\rho_1\rho_2}{\kappa^-\kappa^+}}. \tag{2.22}$$

Remark 2.1.1. *In the case $\rho_1 = \rho_2 = \rho$ we recover the decomposition of the twist matrix given in [43] with $A = B = L$.*

Remark 2.1.2. *With the constraint $\kappa^+ = \kappa^- = 0$ we recover the diagonal twist and the action formulas (2.28)–(2.35) take the usual ABA form. From the point of view of the decomposition of the twist matrix $K = BDA$ we can recover the diagonal twist in two ways. On the one hand, taking $\rho_i = 0$ and then $\kappa^+ = \kappa^- = 0$ we send the modified operators $\nu_{ij}(z)$ to the initial operators $t_{ij}(z)$. On the other hand, taking first $\kappa^+ = \kappa^-$, redefining $\rho_i = \kappa^+\bar{\rho}_i$ with $\bar{\rho}_i \neq 0$ and then specifying $\kappa^+ = 0$ we keep the modified operators $\nu_{ij}(z)$ provided $\bar{\rho}_1\kappa + \bar{\rho}_2\tilde{\kappa} = 0$. The B^{good} operator in gl_2 case proposed in [56] belongs to this special case of the modified operators.*

Using (2.18)–(2.19) we can calculate the actions of the modified operators on the highest weight and the lowest weight vectors.

Proposition 2.1.1. *If we consider $\nu_{12}(z)$ as a modified creation operator, then the actions of the modified operators $\{\nu_{11}(z), \nu_{22}(z), \nu_{21}(z)\}$ on the highest weight vector (1.81) are given by*

$$\begin{aligned}
\nu_{11}(z)|0\rangle &= \lambda_1(z)|0\rangle + \frac{\rho_2}{\kappa^+}\nu_{12}(z)|0\rangle, \\
\nu_{22}(z)|0\rangle &= \lambda_2(z)|0\rangle + \frac{\rho_1}{\kappa^+}\nu_{12}(z)|0\rangle, \\
\nu_{21}(z)|0\rangle &= \Big(\frac{\rho_1}{\kappa^+}\lambda_1(z) + \frac{\rho_2}{\kappa^+}\lambda_2(z)\Big)|0\rangle + \frac{\rho_1\rho_2}{(\kappa^+)^2}\nu_{12}(z)|0\rangle.
\end{aligned} \tag{2.23}$$

On the other hand, if we choose $\nu_{21}(z)$ as a modified creation operator, then the actions of the modified operators $\{\nu_{11}(z), \nu_{22}(z), \nu_{12}(z)\}$ on the lowest weight vector (1.81) are given by

$$\begin{aligned}
\nu_{11}(z)|\hat{0}\rangle &= \lambda_2(z)|\hat{0}\rangle + \frac{\rho_2}{\kappa^-}\nu_{21}(z)|\hat{0}\rangle, \\
\nu_{22}(z)|\hat{0}\rangle &= \lambda_1(z)|\hat{0}\rangle + \frac{\rho_1}{\kappa^-}\nu_{21}(z)|\hat{0}\rangle, \\
\nu_{12}(z)|\hat{0}\rangle &= \Big(\frac{\rho_2}{\kappa^-}\lambda_1(z) + \frac{\rho_1}{\kappa^-}\lambda_2(z)\Big)|\hat{0}\rangle + \frac{\rho_1\rho_2}{(\kappa^-)^2}\nu_{21}(z)|\hat{0}\rangle.
\end{aligned} \tag{2.24}$$

Proof. The proposition follows from the explicit form of the modified operators in terms of the original operators $t_{kl}(z)$ given by (2.18)–(2.21) and the action on the weight vectors (1.83). □

Remark 2.1.3. *For a given weight vector $|0\rangle$ or $|\hat{0}\rangle$ the role of $\nu_{12}(z)$ and $\nu_{21}(z)$ can be inverted. Indeed, as an example, from the last equation of (2.23) we can express $\nu_{12}(z)|0\rangle$ in term of $\nu_{21}(z)|0\rangle$ and $|0\rangle$ and then consider $\nu_{21}(z)$ as a creation operator on $|0\rangle$.*

It follows from the action of the modified operators that the actions of the transfer matrix on the weight vectors are given by

$$t(z)|0\rangle = \Big((\tilde{\kappa} - \rho_1)\lambda_1(z) + (\kappa - \rho_2)\lambda_2(z)\Big)|0\rangle + \frac{\kappa^-}{\mu}\nu_{12}(z)|0\rangle, \tag{2.25}$$

and

$$t(z)|\hat{0}\rangle = \Big((\tilde{\kappa}-\rho_1)\lambda_2(z)+(\kappa-\rho_2)\lambda_1(z)\Big)|\hat{0}\rangle + \frac{\kappa^+}{\mu}\nu_{21}(z)|\hat{0}\rangle. \tag{2.26}$$

Let us define the shorthand notation for the product of functions or of commuting operators,

$$\nu_{12}(\bar{u}) = \prod_{i=1}^{M}\nu_{12}(u_i), \qquad \nu_{21}(\bar{v}) = \prod_{i=1}^{\hat{M}}\nu_{21}(v_i). \tag{2.27}$$

Then the action of $t(z)$ on the modified Bethe vector $\nu_{12}(\bar{u})|0\rangle$ with $\#\bar{u} = M$ has been derived in [43] and is given by

$$t(z)\nu_{12}(\bar{u})|0\rangle = \frac{\kappa^-}{\mu}\nu_{12}(z)\nu_{12}(\bar{u})|0\rangle + \Lambda_1^M(z,\bar{u})\nu_{12}(\bar{u})|0\rangle \tag{2.28}$$

$$+\sum_{i=1}^{M} g(u_i,z)E_1^M(u_i,\bar{u}_i)\nu_{12}(z)\nu_{12}(\bar{u}_i)|0\rangle, \tag{2.29}$$

where

$$\Lambda_1^M(z,\bar{u}) = (\tilde{\kappa}-\rho_1)\lambda_1(z)f(\bar{u},z)+(\kappa-\rho_2)\lambda_2(z)f(z,\bar{u}), \tag{2.30}$$

$$E_1^M(u_i,\bar{u}_i) = (\kappa-\rho_2)\lambda_2(u_i)f(u_i,\bar{u}_i)-(\tilde{\kappa}-\rho_1)\lambda_1(u_i)f(\bar{u}_i,u_i). \tag{2.31}$$

On the other hand, the action of $t(z)$ on the second family of Bethe vectors given by $\nu_{21}(\bar{v})|\hat{0}\rangle$ with $\#\bar{v} = \hat{M}$ reads

$$t(z)\nu_{21}(\bar{v})|\hat{0}\rangle = \frac{\kappa^+}{\mu}\nu_{21}(z)\nu_{21}(\bar{v})|\hat{0}\rangle + \hat{\Lambda}_1^{\hat{M}}(z,\bar{v})\nu_{12}(\bar{v})|\hat{0}\rangle \tag{2.32}$$

$$+\sum_{i=1}^{\hat{M}} g(v_i,z)\hat{E}_1^{\hat{M}}(v_i,\bar{v}_i)\nu_{21}(z)\nu_{21}(\bar{v}_i)|\hat{0}\rangle, \tag{2.33}$$

where

$$\hat{\Lambda}_1^{\hat{M}}(z,\bar{v}) = (\kappa-\rho_2)\lambda_1(z)f(\bar{v},z)+(\tilde{\kappa}-\rho_1)\lambda_2(z)f(z,\bar{v}), \tag{2.34}$$

$$\hat{E}_1^{\hat{M}}(v_i,\bar{v}_i) = (\tilde{\kappa}-\rho_1)\lambda_2(v_i)f(v_i,\bar{v}_i)-(\kappa-\rho_2)\lambda_1(v_i)f(\bar{v}_i,v_i). \tag{2.35}$$

For the finite dimensional representation given by the monodromy matrix (1.82), the multiple product of a null transfer matrix $\nu(z) = \mathrm{Tr}(VT(z))$ with $\mathrm{Det}(V) = 0$ (such as the modified creation and annihilation operators) satisfies the following theorem.

Theorem 2.1.1. *Let $\bar{u}$ be a set of arbitrary parameters of cardinality $\#\bar{u} = \sum_{i=1}^{N} 2s_i = S$. For a finite dimensional representation of the monodromy matrix given by (1.82), the following operator identity holds*

$$\nu(z)\nu(\bar{u}) = \mathrm{Tr}(V)\Big(F(z)g(z,\bar{u})\nu(\bar{u}) + \sum_{i=1}^{S} g(u_i,z)F(u_i)g(u_i,\bar{u}_i)\nu(z)\nu(\bar{u}_i)\Big), \tag{2.36}$$

where $\nu(\bar{u}_i) = \prod_{j=1,\, j\neq i}^{S} \nu(u_j)$ *and*

$$F(z) = \prod_{i=1}^{N} \prod_{k=0}^{2s_i} \frac{z - \theta_i + c(s_i - k + \frac{1}{2})}{c}. \tag{2.37}$$

The l.h.s of (2.36) involves the product of $S + 1$ *operators* ν *and each term of the r.h.s involves the product of* S *operators* ν*.*

Proof. It is proved in the Appendix 2.3.2 that for $\mathrm{Tr}(V) \neq 0$ and for arbitrary inhomogeneity parameters $\bar{\theta} = \{\theta_1, \ldots, \theta_N\}$, the operator $\nu(z)$ has simple spectrum. Moreover, its inverse multiplied by the function $F(z)$ has polynomial eigenvalues of degree $S = \sum_{i=1}^{N} 2s_i$ with the leading term given by

$$F(z)\nu_{12}^{-1}(z) = \mathrm{Tr}(V)^{-1} \left(\frac{z}{c}\right)^{S} + ..., \qquad z \to \infty. \tag{2.38}$$

Let $\#\bar{u} = S + 1$. Consider a product of operators

$$F(z)\nu_{12}^{-1}(z)\nu(\bar{u})g(z, \bar{u}). \tag{2.39}$$

It has simples poles at the points u_i and behaves as z^{-1} at infinity. Then, taking the sum of all residues we find that

$$\nu(\bar{u}) = \mathrm{Tr}(V) \sum_{i=1}^{S+1} F(u_i)g(u_i, \bar{u}_i)\nu(\bar{u}_i). \tag{2.40}$$

Setting $u_{S+1} = z$ we complete the proof. □

Corollary 2.1.1. *We can specify theorem 2.1.1 by taking* $V = BE_{21}A$ *to find*

$$\frac{\kappa^-}{\mu}\nu_{12}(z)\nu_{12}(\bar{u}) = (\rho_1 + \rho_2)\Big(F(z)g(z, \bar{u})\nu_{12}(\bar{u}) \tag{2.41}$$

$$+ \sum_{i=1}^{S} g(u_i, z)F(u_i)g(u_i, \bar{u}_i)\nu_{12}(z)\nu_{12}(\bar{u}_i)\Big), \tag{2.42}$$

and by taking $V = BE_{12}A$ *to find*

$$\frac{\kappa^+}{\mu}\nu_{21}(z)\nu_{21}(\bar{v}) = (\rho_1 + \rho_2)\Big(F(z)g(z, \bar{v})\nu_{21}(\bar{v}) \tag{2.43}$$

$$+ \sum_{i=1}^{S} g(v_i, z)F(v_i)g(v_i, \bar{v}_i)\nu_{21}(z)\nu_{21}(\bar{v}_i)\Big). \tag{2.44}$$

These examples show the algebraic origin of the inhomogeneous term of the modified Baxter T-Q equation [35, 36, 33]. It was conjectured and then proved in [38, 39, 45, 44, 43] for models on a segment. Here we go beyond and prove this property independently of the action on the highest weight vector in the case the twisted XXX spin chain with arbitrary positive (half)-integer spins.

Remark 2.1.4. *For the fundamental representation, $s_i = \frac{1}{2}$, we have $F(z) = \lambda_1(z)\lambda_2(z)$. This proves the conjecture of [43].*

Remark 2.1.5. *The entries of the monodromy matrix can be treated as null transfer matrices $t_{ij}(z) = \mathrm{Tr}(E_{ji}T(z))$. Then we have from theorem 2.1.1*

$$t_{ii}(z)t_{ii}(\bar{u}) = F(z)g(z,\bar{u})t_{ii}(z)t_{ii}(\bar{u}) + \sum_{j=1}^{S} F(u_j)g(u_j,z)g(u_j,\bar{u}_j)t_{ii}(z)t_{ii}(\bar{u}_j), \tag{2.45}$$

$$t_{ij}(z)t_{ij}(\bar{u}) = 0, \tag{2.46}$$

for $i \neq j$, $\#\bar{u} = S$, and the function $F(z)$ given by (2.37).

Theorem 2.1.2. *For finite dimensional representation of the monodromy matrix given by (1.82), the action of the transfer matrix on the Bethe vector $\nu_{12}(\bar{u})|0\rangle$ with $\#\bar{u} = \sum_{i=1}^{N} 2s_i = S$ is given by*

$$t(z)\nu_{12}(\bar{u})|0\rangle = \Lambda(z,\bar{u})\nu_{12}(\bar{u})|0\rangle + \sum_{i=1}^{S} g(u_i,z)E(u_i,\bar{u}_i)\nu_{12}(z)\nu_{12}(\bar{u}_i)|0\rangle, \tag{2.47}$$

where

$$\Lambda(z,\bar{u}) = \quad (\tilde{\kappa}-\rho_1)\lambda_1(z)f(\bar{u},z) \tag{2.48}$$

$$+(\kappa-\rho_2)\lambda_2(z)f(z,\bar{u}) + (\rho_1+\rho_2)F(z)g(z,\bar{u}), \tag{2.49}$$

$$E(u_i,\bar{u}_i) = \quad (\kappa-\rho_2)\lambda_2(u_i)f(u_i,\bar{u}_i) \tag{2.50}$$

$$-(\tilde{\kappa}-\rho_1)\lambda_1(u_i)f(\bar{u}_i,u_i) + (\rho_1+\rho_2)F(u_i)g(u_i,\bar{u}_i). \tag{2.51}$$

The action of the transfer matrix on the Bethe vector $\nu_{21}(\bar{v})|\hat{0}\rangle$ with $\bar{v} = \sum_{i=1}^{N} 2s_i = S$ is given by

$$t(z)\nu_{21}(\bar{v})|\hat{0}\rangle = \hat{\Lambda}(z,\bar{v})\nu_{21}(\bar{v})|\hat{0}\rangle + \sum_{i=1}^{S} g(v_i,z)\hat{E}(v_i,\bar{v}_i)\nu_{21}(z)\nu_{21}(\bar{v}_i)|\hat{0}\rangle, \tag{2.52}$$

where

$$\hat{\Lambda}(z,\bar{v}) = (\kappa-\rho_2)\lambda_1(z)f(\bar{v},z) \tag{2.53}$$

$$+(\tilde{\kappa}-\rho_1)\lambda_2(z)f(z,\bar{v}) + (\rho_1+\rho_2)F(z)g(z,\bar{v}), \tag{2.54}$$

$$\hat{E}(v_i,\bar{v}_i) = (\tilde{\kappa}-\rho_1)\lambda_2(v_i)f(v_i,\bar{v}_i) \tag{2.55}$$

$$-(\kappa-\rho_2)\lambda_1(v_i)f(\bar{v}_i,v_i) + (\rho_1+\rho_2)F(v_i)g(v_i,\bar{v}_i). \tag{2.56}$$

Thus, when the inhomogeneous Bethe equations are satisfied, i.e. $E(u_i,\bar{u}_i) = 0$ and $\hat{E}(v_i,\bar{v}_i) = 0$ for $i = 1,\dots,S$, the vectors $\nu_{12}(\bar{u})|0\rangle$ and $\nu_{21}(\bar{v})|\hat{0}\rangle$ are eigenvectors of the transfer matrix.

Proof. The theorem follows from the actions (2.28)–(2.32) and corollary 2.1.1. □

Then we can rewrite the two eigenvalues by two inhomogeneous Baxter T-Q functional equation. We define the functional Q-operators $Q^+(z) = (g(z,\bar{u}))^{-1}$ and $Q^-(z) = (g(z,\bar{v}))^{-1}$, that are two polynomials in z of degree S, to find that

$$\Lambda(z)Q^+(z) = (\tilde{\kappa}-\rho_1)\lambda_1(z)Q^+(z-c) + (\kappa-\rho_2)\lambda_2(z)Q^+(z+c) + (\rho_1+\rho_2)F(z) \quad (2.57)$$

and

$$\Lambda(z)Q^-(z) = (\kappa-\rho_2)\lambda_1(z)Q^-(z-c) + (\tilde{\kappa}-\rho_1)\lambda_2(z)Q^-(z+c) + (\rho_1+\rho_2)F(z). \quad (2.58)$$

Then we can construct quantum Wronskian equations [55].

Theorem 2.1.3. *The two functional Baxter Q-operators $Q^\pm(z)$ are related by the modified quantum Wronskian equation given as*

$$\begin{aligned} W(z) - W(z+c) &= (\rho_1+\rho_2)(Q^+(z) - Q^-(z)), \\ W(z) &= \frac{(\tilde{\kappa}-\rho_1)Q^+(z-c)Q^-(z) - (\kappa-\rho_2)Q^+(z)Q^-(z-c)}{\lambda(z)}. \end{aligned} \quad (2.59)$$

Proof. For a given eigenvalue $\Lambda(z)$ of the transfer matrix, one should multiply (2.57) and (2.58) respectively by $Q^-(z)$ and $Q^+(z)$, and consider the difference of the resulting expressions. Then the use of $\frac{\lambda_1(z)}{\lambda_2(z)} = \frac{\lambda(z+c)}{\lambda(z)}$ and $\lambda_1(z)\lambda(z) = F(z)$ directly leads to the modified quantum Wronskian equation. □

Remark 2.1.6. *For an invertible twist, the transfer matrix (2.5) can also be characterized from the usual Baxter T-Q equations [57]. Defining $\alpha^\pm$ to be eigenvalues of the twist matrix K that satisfy $\alpha^+ + \alpha^- = \tilde{\kappa} + \kappa$ and $\alpha^+\alpha^- = \operatorname{Det} K$. Then we have*

$$\Lambda(z)q^M(z) = \alpha^+\lambda_1(z)q^M(z-c) + \alpha^-\lambda_2(z)q^M(z+c). \quad (2.60)$$

We can construct Wronskian type equation between this parametrization and the inhomogeneous one (2.57)

$$W(z) - W(z+c) = \left(\frac{\alpha^-}{\tilde{\kappa}-\rho_1}\right)^{z/c} (\rho_1+\rho_2)q(z), \quad (2.61)$$

with

$$W(z) = \left(\frac{\alpha^-}{\tilde{\kappa}-\rho_1}\right)^{z/c} \frac{\alpha^+ q(z-c)Q^+(z) - (\tilde{\kappa}-\rho_1)q(z)Q^+(z-c)}{\lambda(z)}. \quad (2.62)$$

Here we used the identity $\alpha^+\alpha^- = (\tilde{\kappa}-\rho_1)(\kappa-\rho_2)$.

2.1.1 Conclusion

In this section has been presented several proofs of the new steps needed to perform the algebraic Bethe ansatz for the models without $U(1)$ symmetry. We showed that the appearance of the new term in the Baxter T-Q equation follows from the analysis of the product of the modified creation operators and that it is a general property of the null twisted transfer matrix. It is not necessary to consider the action on a weight vector.

Then the Bethe ansatz characterization of the spectral problem of the XXX Heisenberg spin chain with an arbitrary twist and arbitrary positive (half)-integer spin at each site of the chain is fully understood by means of the MABA. We also derived, for arbitrary positive (half)-integer spins, a modified quantum Wronskian equation that relates two different characterizations of the spectral problem. It should be of interest to relate the modified quantum Wronskian equation with Hirota equation [58]. Moreover finding the numerical solutions of the Bethe equations remains a challenging open problem (see recent development in [52, 53, 54]), and modified quantum Wronskian equation can be used to address it.

The new actions of the modified operators on the weight vectors deserve to be studied in details. They also appear in the context of the separation of variable by introduction of the B^{good} operators [56] (see also recent work of two of the authors [59] that considers the $\mathfrak{gl}_2$ case from the point of view of the ABA). These actions provide formulas for the development of the modified Bethe vector in terms of the original $t_{12}(u)$ creation operator. The multiple action of the modified operators should lead to generalization of the results of [51]. These multiple action formulas are very useful for the calculation of Bethe vector's scalar products, form factors, and correlation functions and are investigated in the next section.

An important step should be to achieve the MABA for the twisted XXX sl_3 spin chain and to other types of boundary conditions or other higher rank algebras. For a lower block twist, there are no new difficulties, as the first step of the nested Bethe ansatz preserves the $U(1)$ symmetry. However, for the upper block twist some new tools have to be found.

2.2 Scalar product of modified Bethe vectors

We consider in the previous section the development of the algebraic Bethe ansatz (ABA) for the models without $U(1)$ symmetry[6], the so called *modified algebraic Bethe ansatz* (MABA), that gives access to the spectrum and associated eigenstates of the twisted XXX spin chain. A further natural task is to calculate the correlation functions within the framework of this method. Development in this direction would allow to adapt the technique of the usual ABA for the study of correlation functions for models with $U(1)$ symmetry [12] to models without $U(1)$ symmetry. In turn, this would allow to obtain exact solutions in a wide range of fields, such as statistical physics, condensed matter physics, high energy physics, mathematical physics, and so on.

In the study of correlation functions within the framework of the ABA, the scalar products of Bethe vectors play an important role [61, 12, 27, 62, 63, 64, 65]. Whereas the scalar products are known, one can compute the form factors of local operators [66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76]. In turn, knowing the form factors, it is possible to calculate the correlation functions by means of their form factor expansion [77, 78, 79, 80, 81, 82, 83, 84, 85].

The calculation of the scalar products is based on the formulas for the multiple action [7] of the monodromy matrix entries on the Bethe vectors [86, 51, 87, 88]. The specificity of the MABA is that the action of the elements of the monodromy matrix on the highest weight vector is nonstandard. Usually, this vector is an eigenvector of the diagonal elements and it is annihilated by the lower-triangular part of the monodromy matrix. However, the monodromy matrix of MABA is obtained from the usual one by means of a non-diagonal twist transformation. This transformation does not affect the commutation relations between the matrix elements, but changes their actions on the highest weight vector[8]. In particular, the latter is no longer an eigenvector of the diagonal entries of the monodromy matrix. As a result, the multiple actions formulas change significantly.

In this section we consider gl_2-invariant integrable models, as the XXX spin-$\frac{1}{2}$ chain with non diagonal twist (2.1) considered in the previous section.

Our consideration is not restricted to the Hamiltonian (2.1). Actually, we consider a more general case with arbitrary highest weight representation and arbitrary non-diagonal twist transformation of the monodromy matrix. We find the multiple actions of the *modified operators* on the *modified Bethe vectors*. This corresponds to the repeated action of the same operator which depends, in general, on different parameters. This allows us to find a closed expression for the scalar product of two modified Bethe vectors. Multiple action formulas of the usual ABA are expressed in terms of a partition function of the six-vertex model with domain wall boundary condition [61]. This latter has an explicit representation in terms of the Izergin determinant [90]. Within the framework of the MABA one deals with certain deformation of the Izergin determinant that we call *a modified Izergin determinant*. It depends on the parameters of the modified Bethe vectors,

[6] Namely models for which the total spin operator or Cartan operator do not commute with the transfer matrix and the Hamiltonian.

[7] By the multiple action of an operator, we refer to the application of a product of operators of its kind.

[8] A similar transformation occurs on the framework of the so-called B^{good} operator, see [56, 59, 89].

but also on the twist parameters. Remarkably, the multiple action formulas and the scalar products of Bethe vectors, being written in terms of the modified Izergin determinant, have almost the same form as their analogs in the usual ABA. Recent progress in this direction by S. Belliard and N. Slavnov confirm the hope for a compact expression for the scalar product of modified Bethe vectors, as they managed to synthesize the scalar product of modified Bethe vectors through a determiinent representation [10], hence generalizing what has been obtained in the framework of the usual algebraic Bethe ansatz[12, 14, 29].

In section 2.2.1 are introduced some notations and the modified Izergin determinant. We recall multiple actions and a scalar product formula within the standard framework of the ABA in section 2.2.2. In section 2.2.3 we introduce the modified operators and consider their multiple actions on the modified Bethe vectors. Section 2.2.4 is devoted to the calculation of the scalar product of modified Bethe vectors. Auxiliary formulas are gathered in appendices. In appendix 2.3.3 we list some properties of the modified Izergin determinant. In appendix 2.3.4 we give simple and multiple commutation relations of the monodromy matrix entries within the standard framework of the ABA. Appendix 2.3.5 contains a description of a special automorphism of the Yangian of gl_2.

2.2.1 Notation and Modified Izergin Determinant

Let us define the rational functions

$$g(u,v) = \frac{c}{u-v}, \quad f(u,v) = 1 + g(u,v) = \frac{u-v+c}{u-v}, \tag{2.63}$$

$$h(u,v) = \frac{f(u,v)}{g(u,v)} = \frac{u-v+c}{c}, \tag{2.64}$$

where c is the constant entering the R-matrix (1.52). Actually, all these functions depend on the difference of their arguments. However we do not stress this dependence. This will in particular allow us to use a special shorthand notation (see (2.67)). It is easy to see that the functions introduced above possess the following properties:

$$\chi(u,v)\Big|_{c\to -c} = \chi(v,u), \qquad \chi(-u,-v) = \chi(v,u), \qquad \chi(u-c,v) = \chi(u,v+c), \tag{2.65}$$

where χ is any of the three functions. One can also convinces one that

$$g(u,v-c) = \frac{1}{h(u,v)}, \quad h(u,v+c) = \frac{1}{g(u,v)}, \quad f(u,v+c) = \frac{1}{f(v,u)}. \tag{2.66}$$

Below we consider sets of complex parameters and denote them by a bar . For example, $\bar{u} = \{u_1, \dots, u_n\}$. The notation $\bar{u} \pm c$ means that $\pm c$ is added to all the elements of the set $\bar{u}$. We agree upon that the notation $\bar{u}_k$ refers to the set that is complementary in $\bar{u}$ to the element u_k, that is, $\bar{u}_k = \bar{u} \setminus u_k$.

To make the formulas more compact, we use a shorthand notation for the products of functions or operators. Namely, if the function (operator) is fed with a set of variables, then we express the product with respect to the corresponding set. For example,

$$t_{kl}(\bar{u}) = \prod_{j=1}^{n} t_{kl}(u_j), \quad \lambda_i(\bar{u}) = \prod_{j=1}^{n} \lambda_i(u_j), \quad f(z,\bar{u}) = \prod_{j=1}^{n} f(z,u_j), \quad f(\bar{u}_k,u_k) = \prod_{\substack{j=1\\ j\neq k}}^{n} f(u_j,u_k), \tag{2.67}$$

and so on. Note that due to commutativity of the t_{kl}-operators the first product in (2.67) is well defined. Notation $f(\bar{u},\bar{v})$ means the double product over the sets $\bar{u}$ and $\bar{v}$. By definition any product over the empty set is equal to 1. A double product is equal to 1 if at least one of the sets is empty.

Later we will extend this convention to the products of matrix elements of the twisted monodromy matrix.

Modified Izergin determinant

In many formulas of the ABA the Izergin determinant appears [61, 90]. Within the framework of the MABA we have to deal with a deformation of this object that we call a modified Izergin determinant.

Definition 2.2.1. *Let $\bar{u} = \{u_1, \dots, u_n\}$, $\bar{v} = \{v_1, \dots, v_m\}$ and z be a complex number. Then the modified Izergin determinant $K^{(z)}_{n,m}(\bar{u}|\bar{v})$ is defined by*

$$K^{(z)}_{n,m}(\bar{u}|\bar{v}) = \det_m \left(-z\delta_{jk} + \frac{f(\bar{u}, v_j) f(v_j, \bar{v}_j)}{h(v_j, v_k)} \right). \tag{2.68}$$

Alternatively the modified Izergin determinant can be presented as

$$K^{(z)}_{n,m}(\bar{u}|\bar{v}) = (1-z)^{m-n} \det_n \left(\delta_{jk} f(u_j, \bar{v}) - z \frac{f(u_j, \bar{u}_j)}{h(u_j, u_k)} \right). \tag{2.69}$$

The proof of the equivalence of representations (2.68) and (2.69) can be found in proposition 4.1 of [91]. It is based on the recursive property (2.180). The modified Izergin determinant is related to the partial domain wall partition functions [92]. Other correspondences will be discussed elsewhere.

It is also convenient to introduce a conjugated modified Izergin determinant as

$$\overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v}) = K^{(z)}_{n,m}(\bar{u}|\bar{v})\Big|_{c\to -c} = \det_m \left(-z\delta_{jk} + \frac{f(v_j, \bar{u}) f(\bar{v}_j, v_j)}{h(v_k, v_j)} \right), \tag{2.70}$$

or equivalently

$$\overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v}) = (1-z)^{m-n} \det_n \left(\delta_{jk} f(\bar{v}, u_j) - z \frac{f(\bar{u}_j, u_j)}{h(u_k, u_j)} \right). \tag{2.71}$$

In the particular case $z = 1$ and $\#\bar{u} = \#\bar{v} = n$ the modified Izergin determinant turns into the ordinary Izergin determinant, that we traditionally denote by $K_n(\bar{u}|\bar{v})$:

$$K^{(1)}_{n,n}(\bar{u}|\bar{v}) = K_n(\bar{u}|\bar{v}). \tag{2.72}$$

This property can be seen from the recursion (2.180) and the initial condition (2.165). It also follows from (2.69) that

$$K^{(1)}_{n,m}(\bar{u}|\bar{v}) = 0, \qquad \text{for} \qquad n < m. \tag{2.73}$$

Other properties of the modified Izergin determinant are collected in Appendix 2.3.3.

2.2.2 Multiple Actions

Actions of the operators $t_{ij}(u)$ on the Bethe vectors (1.87) were computed in [15] (see also [12]). To study the problem of the scalar products one should calculate multiple actions of the form

$$t_{ij}(\bar u)t_{12}(\bar v)|0\rangle. \tag{2.74}$$

Here, according to the convention on the shorthand notation (2.67) $t_{ij}(\bar u)$ is the product of the operators t_{ij} over the set $\bar u = \{u_1, \dots, u_n\}$.

Multiple action formulas are given in terms of sums over partitions of the set $\bar w = \{\bar u, \bar v\}$ into subsets. Here and below we mostly denote the subsets by Roman subscripts (except for some special cases). Notation $\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}\}$ (and similar ones) means that the set $\bar w$ is divided into subsets $\bar w_{\rm I}$ and $\bar w_{\rm II}$ such that $\bar w_{\rm I} \cup \bar w_{\rm II} = \bar w$ and $\bar w_{\rm I} \cap \bar w_{\rm II} = \emptyset$.

Proposition 2.2.1. *[51] Let $\#\bar u = n$, $\#\bar v = m$, $\bar w = \{\bar u, \bar v\}$, and K_n be the Izergin determinant (2.72). Then*

$$t_{12}(\bar u)t_{12}(\bar v)|0\rangle = t_{12}(\bar w)|0\rangle. \tag{2.75}$$

The actions of the diagonal elements t_{ii} are given by

$$t_{11}(\bar u)t_{12}(\bar v)|0\rangle = (-1)^n \sum_{\substack{\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}\} \\ \#\bar w_{\rm I} = n}} \lambda_1(\bar w_{\rm I}) \overline{K}_n(\bar u|\bar w_{\rm I} - c) f(\bar w_{\rm II}, \bar w_{\rm I}) t_{12}(\bar w_{\rm II})|0\rangle, \tag{2.76}$$

$$t_{22}(\bar u)t_{12}(\bar v)|0\rangle = (-1)^n \sum_{\substack{\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}\} \\ \#\bar w_{\rm I} = n}} \lambda_2(\bar w_{\rm I}) K_n(\bar u|\bar w_{\rm I} + c) f(\bar w_{\rm I}, \bar w_{\rm II}) t_{12}(\bar w_{\rm II})|0\rangle, \tag{2.77}$$

where the sums are taken over partitions $\{\bar u, \bar v\} = \bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}\}$ such that $\#\bar w_{\rm I} = n$. The action of the elements t_{21} reads

$$\begin{aligned} t_{21}(\bar u)t_{12}(\bar v)|0\rangle = \sum_{\substack{\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}, \bar w_{\rm III}\} \\ \#\bar w_{\rm I} = \#\bar w_{\rm II} = n}} & \lambda_2(\bar w_{\rm I})\lambda_1(\bar w_{\rm II}) K_n(\bar u|\bar w_{\rm I} + c) \overline{K}_n(\bar u|\bar w_{\rm II} - c) \\ & \times f(\bar w_{\rm I}, \bar w_{\rm II}) f(\bar w_{\rm I}, \bar w_{\rm III}) f(\bar w_{\rm III}, \bar w_{\rm II}) t_{12}(\bar w_{\rm III})|0\rangle, \end{aligned} \tag{2.78}$$

where the sum is taken over partitions $\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}, \bar w_{\rm III}\}$ such that $\#\bar w_{\rm I} = \#\bar w_{\rm II} = n$.

Note that the action formulas (2.76) are the direct consequence of the commutation relations (2.202).

Equation (2.78) gives immediate access to the scalar product of Bethe vectors defined by

$$S_t^n(\bar u, \bar v) = \langle 0|t_{21}(\bar u)t_{12}(\bar v)|0\rangle, \tag{2.79}$$

where $\#\bar u = \#\bar v = n$.

Theorem 2.2.1. *Let $\#\bar u = \#\bar v = n$. Then the scalar product of two Bethe vectors is given by*

$$S_t^n(\bar u, \bar v) = \sum_{\substack{\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}\} \\ \#\bar w_{\rm I} = \#\bar w_{\rm II} = n}} \lambda_2(\bar w_{\rm I})\lambda_1(\bar w_{\rm II}) K_n(\bar u|\bar w_{\rm I} + c) \overline{K}_n(\bar u|\bar w_{\rm II} - c) f(\bar w_{\rm I}, \bar w_{\rm II}). \tag{2.80}$$

where the sum is taken over partitions $\bar w \Rightarrow \{\bar w_{\rm I}, \bar w_{\rm II}\}$ such that $\#\bar w_{\rm I} = \#\bar w_{\rm II} = n$.

The sum (2.80) can also be written in the form of the sum over independent partitions of the sets $\bar{u}$ and $\bar{v}$. Then it corresponds to the Izergin–Korepin formula [12].

Corollary 2.2.1. *Let $\#\bar{u} = \#\bar{v} = n$. Then the scalar product of two Bethe vectors is given by*

$$S_t^n(\bar{u},\bar{v}) = \sum_{\substack{\bar{u}\Rightarrow\{\bar{u}_{\rm I},\bar{u}_{\rm II}\}\\ \bar{v}\Rightarrow\{\bar{v}_{\rm I},\bar{v}_{\rm II}\}\\ \#\bar{u}_{\rm I}=\#\bar{v}_{\rm I}}} \lambda_2(\bar{u}_{\rm I})\lambda_2(\bar{v}_{\rm II})\lambda_1(\bar{u}_{\rm II})\lambda_1(\bar{v}_{\rm I})K_{n_2}(\bar{v}_{\rm II}|\bar{u}_{\rm II})\overline{K}_{n_1}(\bar{v}_{\rm I}|\bar{u}_{\rm I})f(\bar{u}_{\rm I},\bar{u}_{\rm II})f(\bar{v}_{\rm II},\bar{v}_{\rm I}), \tag{2.81}$$

where the sum is taken over partitions $\bar{u} \Rightarrow \{\bar{u}_{\rm I},\bar{u}_{\rm II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\bar{v}_{\rm II}\}$ such that $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = n_1$, $\#\bar{u}_{\rm II} = \#\bar{v}_{\rm II} = n_2$, where $n_1 = 0,1,\dots,n$ and $n = n_1+n_2$.

Proof. We set in (2.80) $\bar{w}_{\rm I} \Rightarrow \{\bar{u}_{\rm I},\bar{v}_{\rm II}\}$ and $\bar{w}_{\rm II} \Rightarrow \{\bar{u}_{\rm II},\bar{v}_{\rm I}\}$. Let $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = n_1$, $\#\bar{u}_{\rm II} = \#\bar{v}_{\rm II} = n_2$, where $n_1 = 0,1,\dots,n$ and $n = n_1+n_2$. Using (2.166) and (2.167) we obtain

$$S_t^n(\bar{u},\bar{v}) = (-1)^n \sum_{\substack{\bar{u}\Rightarrow\{\bar{u}_{\rm I},\bar{u}_{\rm II}\}\\ \bar{v}\Rightarrow\{\bar{v}_{\rm I},\bar{v}_{\rm II}\}\\ \#\bar{u}_{\rm I}=\#\bar{v}_{\rm I}}} \lambda_2(\bar{u}_{\rm I})\lambda_2(\bar{v}_{\rm II})\lambda_1(\bar{u}_{\rm II})\lambda_1(\bar{v}_{\rm I})K_{n_2}(\bar{u}_{\rm II}|\bar{v}_{\rm II}+c)\overline{K}_{n_1}(\bar{u}_{\rm I}|\bar{v}_{\rm I}-c)$$
$$\times f(\bar{u}_{\rm I},\bar{u}_{\rm II})f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm I},\bar{v}_{\rm I})f(\bar{v}_{\rm II},\bar{u}_{\rm II}). \tag{2.82}$$

Then the use of (2.176) and (2.177) immediately leads us to (2.81). □

2.2.3 Multiple Actions of Modified Operators on Bethe Vectors

A monodromy matrix of MABA is constructed as a twist transformation of the original monodromy matrix (1.55). In [43, 23] we discussed the factorisation of the twist matrix $K = BDA$ (where D is a diagonal matrix), which allows us to use the MABA. It includes some freedom by the transformation $A \to SA$ and $B \to BS^{-1}$ for any invertible diagonal matrix S. Let us consider the following parametrization of the two matrices A and B:

$$A = \sqrt{\mu}\begin{pmatrix} 1 & \frac{\rho_2}{\kappa^-} \\ \frac{\rho_1}{\kappa^+} & 1 \end{pmatrix}, \quad B = \sqrt{\mu}\begin{pmatrix} 1 & \frac{\rho_1}{\kappa^-} \\ \frac{\rho_2}{\kappa^+} & 1 \end{pmatrix}, \quad \mu = \frac{1}{1-\frac{\rho_1\rho_2}{\kappa^+\kappa^-}}. \tag{2.83}$$

Here ρ_i and $\kappa^\pm$ are generic parameters. Due to the gl_2 invariance (1.95), the transformation of the monodromy matrix

$$\bar{T}(u) = AT(u)B = \begin{pmatrix} \nu_{11}(u) & \nu_{12}(u) \\ \nu_{21}(u) & \nu_{22}(u) \end{pmatrix} \tag{2.84}$$

is an automorphism of the Yangian of gl_2, i.e. new operators ν_{ij} satisfy the same commutation relations as the $t_{ij}(z)$, given in Appendix 2.3.4. However, the actions of the modified operators $\{\nu_{ii}(u),\nu_{21}(u)\}$ on the highest weight vector (1.81) change. It is easy to see that now they are given by

$$\nu_{11}(u)|0\rangle = \lambda_1(u)|0\rangle + \beta_2\nu_{12}(u)|0\rangle, \tag{2.85}$$
$$\nu_{22}(u)|0\rangle = \lambda_2(u)|0\rangle + \beta_1\nu_{12}(u)|0\rangle, \tag{2.86}$$
$$\nu_{21}(u)|0\rangle = \Big(\beta_1\lambda_1(u) + \beta_2\lambda_2(u)\Big)|0\rangle + \beta_1\beta_2\nu_{12}(u)|0\rangle, \tag{2.87}$$

where $\beta_i = \frac{\rho_i}{\kappa^+}$.

The modified Bethe vectors are given by

$$\nu_{12}(\bar{v})|0\rangle = \prod_{i=1}^{m} \nu_{12}(v_i)|0\rangle \tag{2.88}$$

with $m = 0, 1, \ldots$. Here we extended the convention on the shorthand notation (2.67) to the products of the operators ν_{ij}. Since the commutation relations of ν_{ij} are the same as the ones of t_{ij}, we have, in particular, $[\nu_{ij}(u), \nu_{ij}(v)] = 0$. Thus, the products $\nu_{ij}(\bar{v})$ are well defined.

Multiple actions of the modified diagonal operators

It is clear that changing the action on the highest weight vector leads to a modification of the multiple action formulas.

Proposition 2.2.2. *The multiple actions of the products of the diagonal modified operators $\nu_{ii}(\bar{u})$, with $\bar{u} = \{u_1, \ldots, u_n\}$, on the modified Bethe vector $\nu_{12}(\bar{v})|0\rangle$, with $\bar{v} = \{v_1, \ldots, v_m\}$, are given by*

$$\nu_{11}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_2^n \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}} (-\beta_2)^{-l} \lambda_1(\bar{w}_{\rm I}) \overline{K}^{(1)}_{n,l}(\bar{u}|\bar{w}_{\rm I} - c) f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) \nu_{12}(\bar{w}_{\rm II})|0\rangle, \tag{2.89}$$

$$\nu_{22}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_1^n \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}} (-\beta_1)^{-l} \lambda_2(\bar{w}_{\rm I}) K^{(1)}_{n,l}(\bar{u}|\bar{w}_{\rm I} + c) f(\bar{w}_{\rm I}, \bar{w}_{\rm II}) \nu_{12}(\bar{w}_{\rm II})|0\rangle. \tag{2.90}$$

Here $l = \#\bar{w}_{\rm I}$. The sum is taken over all partitions $\{\bar{u}, \bar{v}\} = \bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}$. There is no restrictions on the cardinalities of the subsets. The function $K^{(1)}_{n,l}$ and $\overline{K}^{(1)}_{n,l}$ respectively are the modified Izergin determinants (2.68) and (2.70) at $z = 1$.

Remark 2.2.1. *The main difference between modified action formulas and equations (2.76) is the replacement of the ordinary Izergin determinants with the modified Izergin determinants. This leads to the fact that there is no restriction on the cardinalities of the subsets in formulas (2.89), (2.90). However, due to the property $K^{(1)}_{n,l}(\bar{u}|\bar{v}) = \overline{K}^{(1)}_{n,l}(\bar{u}|\bar{v}) = 0$ for $n < l$, the summation in (2.89), (2.90) is carried out only over those partitions for which $l \leq n$.*

Proof. We give a detailed proof of formula (2.89). The proof of formula (2.90) is completely analogous. It also follows form (2.89) due to the symmetry of the Yangian described in appendix 2.3.5.

We first consider the case $n = \#\bar{u} = 1$. In fact, in this case, equation (2.89) was first conjectured in [43] and then proved in [59]. Therefore, we consider this case for the sake of completeness only.

Since the operators ν_{ij} possess the same commutation relations as t_{ij}, we can use (2.202) for $n = 1$:

$$\nu_{11}(u)\nu_{12}(\bar{v}) = -\sum_{\substack{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}\\ \#\bar{w}_{\rm I}=1}} \overline{K}_1(u|\bar{w}_{\rm I} - c) f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) \nu_{12}(\bar{w}_{\rm II}) \nu_{11}(\bar{w}_{\rm I}). \tag{2.91}$$

Here $\bar{w} = \{u, \bar{v}\}$. The sum is taken over partitions $\bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}$ such that $\#\bar{w}_{\rm I} = 1$. Applying this equation to $|0\rangle$ and using (2.85) we obtain

$$\nu_{11}(u)\nu_{12}(\bar{v})|0\rangle = -\sum_{\substack{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}\\ \#\bar{w}_{\rm I}=1}} \overline{K}_1(u|\bar{w}_{\rm I} - c)f(\bar{w}_{\rm II}, \bar{w}_{\rm I})\nu_{12}(\bar{w}_{\rm II})\Big(\lambda_1(\bar{w}_{\rm I}) + \beta_2\nu_{12}(\bar{w}_{\rm I})\Big)|0\rangle. \tag{2.92}$$

The sum over partitions in the term proportional to β_2 can be computed explicitly. Indeed, we have

$$-\overline{K}_1(u|\bar{w}_{\rm I} - c) = \frac{c}{u - \bar{w}_{\rm I} + c} = \frac{1}{h(u, \bar{w}_{\rm I})}.$$

Then

$$-\beta_2 \sum_{\substack{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}\\ \#\bar{w}_{\rm I}=1}} \overline{K}_1(u|\bar{w}_{\rm I} - c)f(\bar{w}_{\rm II}, \bar{w}_{\rm I})\nu_{12}(\bar{w}_{\rm II})\nu_{12}(\bar{w}_{\rm I})|0\rangle = \beta_2\nu_{12}(\bar{w})|0\rangle\, G, \tag{2.93}$$

where

$$G = \sum_{\substack{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}\\ \#\bar{w}_{\rm I}=1}} \frac{f(\bar{w}_{\rm II}, \bar{w}_{\rm I})}{h(u, \bar{w}_{\rm I})}. \tag{2.94}$$

To calculate the sum over partitions (2.94) it is enough to present it as a contour integral

$$G = \frac{-1}{2\pi i c}\oint_{|z|=R\to\infty} \frac{f(\bar{w}, z)}{h(u, z)}\, dz. \tag{2.95}$$

Taking the residue at infinity we obtain[9] $G = 1$. Thus,

$$\nu_{11}(u)\nu_{12}(\bar{v})|0\rangle = \beta_2\nu_{12}(\bar{w})|0\rangle - \sum_{\substack{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}\\ \#\bar{w}_{\rm I}=1}} \lambda_1(\bar{w}_{\rm I})\overline{K}_1(u|\bar{w}_{\rm I} - c)f(\bar{w}_{\rm II}, \bar{w}_{\rm I})\nu_{12}(\bar{w}_{\rm II})|0\rangle. \tag{2.96}$$

It remains to compare the result obtained with equation (2.89) for $n = 1$. In this case either $l = 0$ or $l = 1$. It is easy to see that the first term in (2.96) corresponds to the case $l = 0$, while the second term gives the sum over partitions for $l = 1$. Thus, the action (2.89) is proved for $n = 1$.

To proceed further we use induction over n. Assume that (2.89) holds for some $n-1$. Then the action of $\nu_{11}(\bar{u})$ on the modified Bethe vector $\nu_{12}(\bar{v})|0\rangle$ can be computed as the successive action of $\nu_{11}(\bar{u}_n)$ and $\nu_{11}(u_n)$ (recall that $\bar{u}_n = \bar{u}\setminus u_n$). At the first step we have

$$\nu_{11}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \nu_{11}(u_n)\beta_2^{n-1}\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} (-\beta_2)^{-l_{\rm I}}\lambda_1(\bar{\xi}_{\rm I})\overline{K}^{(1)}_{n-1,l}(\bar{u}_n|\bar{\xi}_{\rm I} - c)f(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I})\nu_{12}(\bar{\xi}_{\rm II})|0\rangle. \tag{2.97}$$

Here $\bar{\xi} = \{\bar{u}_n, \bar{v}\}$. The sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}\}$, and $l_{\rm I} = \#\bar{\xi}_{\rm I}$. Acting with $\nu_{11}(u_n)$ on the vector $\nu_{12}(\bar{\xi}_{\rm II})|0\rangle$ we obtain

$$\begin{aligned}\nu_{11}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_2^{n}&\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} (-\beta_2)^{-l_{\rm I}}\lambda_1(\bar{\xi}_{\rm I})\overline{K}^{(1)}_{n-1,l_{\rm I}}(\bar{u}_n|\bar{\xi}_{\rm I} - c)f(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I})\\ &\times\sum_{\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}} (-\beta_2)^{-k_{\rm I}}\lambda_1(\bar{\eta}_{\rm I})\overline{K}^{(1)}_{1,k_{\rm I}}(u_n|\bar{\eta}_{\rm I} - c)f(\bar{\eta}_{\rm II}, \bar{\eta}_{\rm I})\nu_{12}(\bar{\eta}_{\rm II})|0\rangle.\end{aligned} \tag{2.98}$$

[9] Recall that $u \in \bar{w}$, and thus, there is no pole at $z = u + c$.

Here we have one more sum over partitions of the set $\bar{\eta} = \{u_n, \bar{\xi}_{\rm II}\} \Rightarrow \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}\}$, and $k_{\rm I} = \#\bar{\eta}_{\rm I}$.

Thus, in (2.98), the set $\{\bar{u}, \bar{v}\}$ eventually is divided into three subsets $\bar{\xi}_{\rm I}$, $\bar{\eta}_{\rm I}$, and $\bar{\eta}_{\rm II}$. The subset $\bar{\xi}_{\rm II}$ plays an intermediate role and should be understood as $\bar{\xi}_{\rm II} = \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}\} \setminus \{u_n\}$. The only restriction on these partitions is that $u_n \notin \bar{\xi}_{\rm I}$.

Let $\bar{w} = \{\bar{u}, \bar{v}\}$. Denote $\bar{\xi}_{\rm I} = \bar{w}_{\rm I}$, $\bar{\eta}_{\rm I} = \bar{w}_{\rm II}$, and $\bar{\eta}_{\rm II} = \bar{w}_{\rm III}$. Then $\bar{\xi}_{\rm II} = \{\bar{w}_{\rm II}, \bar{w}_{\rm III}\} \setminus \{u_n\}$ and

$$f(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I}) = \frac{f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) f(\bar{w}_{\rm III}, \bar{w}_{\rm I})}{f(u_n, \bar{w}_{\rm I})}. \tag{2.99}$$

Notice that the right hand side of (2.99) vanishes as soon as $u_n \in \bar{w}_{\rm I}$. Thus, the condition $u_n \notin \bar{\xi}_{\rm I}$ holds automatically. Equation (2.98) then takes the following form:

$$\begin{aligned}\nu_{11}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_2^n \sum_{\bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}, \bar{w}_{\rm III}\}} (-\beta_2)^{-r_{\rm I} - r_{\rm II}} \lambda_1(\bar{w}_{\rm I}) \lambda_1(\bar{w}_{\rm II}) \\ \times \overline{K}^{(1)}_{n-1, r_{\rm I}}(\bar{u}_n | \bar{w}_{\rm I} - c) \overline{K}^{(1)}_{1, r_{\rm II}}(u_n | \bar{w}_{\rm II} - c) \frac{f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) f(\bar{w}_{\rm III}, \bar{w}_{\rm I}) f(\bar{w}_{\rm III}, \bar{w}_{\rm II})}{f(u_n, \bar{w}_{\rm I})} \nu_{12}(\bar{w}_{\rm III})|0\rangle. \end{aligned} \tag{2.100}$$

Here $r_{\rm I} = \#\bar{w}_{\rm I}$ and $r_{\rm II} = \#\bar{w}_{\rm II}$. Let $\{\bar{w}_{\rm I}, \bar{w}_{\rm II}\} = \bar{w}_0$ and $r_0 = \#\bar{w}_0$. Then, we recast (2.100) as follows:

$$\begin{aligned}\nu_{11}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_2^n \sum_{\bar{w} \Rightarrow \{\bar{w}_0, \bar{w}_{\rm III}\}} (-\beta_2)^{-r_0} \lambda_1(\bar{w}_0) f(\bar{w}_{\rm III}, \bar{w}_0) \nu_{12}(\bar{w}_{\rm III})|0\rangle \\ \times \sum_{\bar{w}_0 \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}} \overline{K}^{(1)}_{n-1, r_{\rm I}}(\bar{u}_n | \bar{w}_{\rm I} - c) \overline{K}^{(1)}_{1, r_{\rm II}}(u_n | \bar{w}_{\rm II} - c) \frac{f(\bar{w}_{\rm II}, \bar{w}_{\rm I})}{f(u_n, \bar{w}_{\rm I})}. \end{aligned} \tag{2.101}$$

The sum over partitions is now organized in two steps. First, the set $\bar{w}$ is divided into two subsets $\bar{w}_0 \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}$. Then the subset $\bar{w}_0$ is divided once more as $\bar{w}_0 \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}$. It is easy to see that the sum over partitions in the second line of (2.101) reduces to the modified Izergin determinant due to (2.190):

$$\sum_{\bar{w}_0 \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}} \overline{K}^{(1)}_{n-1, r_{\rm I}}(\bar{u}_n | \bar{w}_{\rm I} - c) \overline{K}^{(1)}_{1, r_{\rm II}}(u_n | \bar{w}_{\rm II} - c) \frac{f(\bar{w}_{\rm II}, \bar{w}_{\rm I})}{f(u_n, \bar{w}_{\rm I})} = \overline{K}^{(1)}_{n, r_0}(\bar{u} | \bar{w}_0 - c). \tag{2.102}$$

Thus, we arrive at

$$\nu_{11}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_2^n \sum_{\bar{w} \Rightarrow \{\bar{w}_0, \bar{w}_{\rm III}\}} (-\beta_2)^{-r_0} \lambda_1(\bar{w}_0) \overline{K}^{(1)}_{n, r_0}(\bar{u} | \bar{w}_0 - c) f(\bar{w}_{\rm III}, \bar{w}_0) \nu_{12}(\bar{w}_{\rm III})|0\rangle. \tag{2.103}$$

This equation coincides with (2.89) for $\#\bar{u} = n$ up to the labels of the subsets. □

Multiple action of the modified operator ν_{21}

Proposition 2.2.3. *The multiple action of the product of modified operators $\nu_{21}(\bar{u})$, with $\bar{u} = \{u_1, ..., u_n\}$, on the modified Bethe vector $\nu_{12}(\bar{v})|0\rangle$, with $\bar{v} = \{v_1, ..., v_m\}$, is given by*

$$\begin{aligned}\nu_{21}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \sum_{\bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}, \bar{w}_{\rm III}\}} (-\beta_1)^{n - l_{\rm I}} (-\beta_2)^{n - l_{\rm II}} \lambda_2(\bar{w}_{\rm I}) \lambda_1(\bar{w}_{\rm II}) \\ \times K^{(1)}_{n, l_{\rm I}}(\bar{u} | \bar{w}_{\rm I} + c) \overline{K}^{(1)}_{n, l_{\rm II}}(\bar{u} | \bar{w}_{\rm II} - c) f(\bar{w}_{\rm I}, \bar{w}_{\rm II}) f(\bar{w}_{\rm I}, \bar{w}_{\rm III}) f(\bar{w}_{\rm III}, \bar{w}_{\rm II}) \nu_{12}(\bar{w}_{\rm III})|0\rangle. \end{aligned} \tag{2.104}$$

Here $l_{\rm I} = \bar{w}_{\rm I}$ and $l_{\rm II} = \bar{w}_{\rm II}$. The sum is taken over all partitions $\{\bar{u}, \bar{v}\} = \bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}, \bar{w}_{\rm III}\}$. The function $K^{(1)}_{n,l_{\rm I}}$ and $\overline{K}^{(1)}_{n,l_{\rm II}}$ respectively are the modified Izergin determinants (2.68) and (2.70) at $z = 1$.

Proof. To prove (2.104) we first use induction over $m = \#\bar{v}$ and then over $n = \#\bar{u}$.

Let $n = 1$ and, hence, $\bar{u} = u$. Note that in spite of the sum in (2.104) is taken over all possible partitions of the set $\bar{w} = \{u, \bar{v}\}$, in fact, it is restricted by the condition $l_i \le n$ ($i = {\rm I}, {\rm II}$), because otherwise the modified Izergin determinants vanish. Thus, for $n = 1$ the cardinalities of the subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ are either 0 or 1. Then, it is easy to see that for $n = 1$ and $m = 0$, equation (2.104) coincides with the action formula (2.87).

Assume that (2.104) holds for some $m - 1$, where $m > 0$. Using commutation relation (2.201) we obtain

$$\nu_{21}(u)\nu_{12}(\bar{v})|0\rangle = \big[\nu_{12}(v_m)\nu_{21}(u) + g(u, v_m)\big(\nu_{11}(v_m)\nu_{22}(u) - \nu_{11}(u)\nu_{22}(v_m)\big)\big]\nu_{12}(\bar{v}_m)|0\rangle. \tag{2.105}$$

Let us first consider the contribution of the term $\nu_{12}(v_m)\nu_{21}(u)$. Due to the induction assumption we have

$$\nu_{12}(v_m)\nu_{21}(u)\nu_{12}(\bar{v}_m)|0\rangle = \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}} (-\beta_1)^{1-l_{\rm I}}(-\beta_2)^{1-l_{\rm II}}\, f(\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}) f(\bar{\xi}_{\rm I}, \bar{\xi}_{\rm III}) f(\bar{\xi}_{\rm III}, \bar{\xi}_{\rm II})$$
$$\times \lambda_2(\bar{\xi}_{\rm I})\lambda_1(\bar{\xi}_{\rm II}) K^{(1)}_{1,l_{\rm I}}(u|\bar{\xi}_{\rm I} + c)\overline{K}^{(1)}_{1,l_{\rm II}}(u|\bar{\xi}_{\rm II} - c)\nu_{12}(\{v_m, \bar{\xi}_{\rm III}\})|0\rangle, \tag{2.106}$$

where $\bar{\xi} = \{u, \bar{v}_m\}$. Let $\bar{w} = \{u, \bar{v}\}$. Then equation (2.106) is equivalent to

$$\nu_{12}(v_m)\nu_{21}(u)\nu_{12}(\bar{v}_m)|0\rangle = \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II},\bar{w}_{\rm III}\}} (-\beta_1)^{1-l_{\rm I}}(-\beta_2)^{1-l_{\rm II}}\, \frac{f(\bar{w}_{\rm I}, \bar{w}_{\rm II}) f(\bar{w}_{\rm I}, \bar{w}_{\rm III}) f(\bar{w}_{\rm III}, \bar{w}_{\rm II})}{f(\bar{w}_{\rm I}, v_m) f(v_m, \bar{w}_{\rm II})}$$
$$\times \lambda_2(\bar{w}_{\rm I})\lambda_1(\bar{w}_{\rm II}) K^{(1)}_{1,l_{\rm I}}(u|\bar{w}_{\rm I} + c)\overline{K}^{(1)}_{1,l_{\rm II}}(u|\bar{w}_{\rm II} - c)\nu_{12}(\bar{w}_{\rm III})|0\rangle. \tag{2.107}$$

Indeed, due to the factor $\big(f(\bar{w}_{\rm I}, v_m) f(v_m, \bar{w}_{\rm II})\big)^{-1}$ we have $v_m \notin \bar{w}_{\rm I}$ and $v_m \notin \bar{w}_{\rm II}$, because otherwise the corresponding contribution vanishes. Thus, $v_m \in \bar{w}_{\rm III}$. Setting $\bar{w}_{\rm I} = \bar{\xi}_{\rm I}$, $\bar{w}_{\rm II} = \bar{\xi}_{\rm II}$, and $\bar{w}_{\rm III} = \{v_m, \bar{\xi}_{\rm III}\}$ in (2.107) we immediately arrive at (2.106).

The action of the terms $\nu_{11}(v_m)\nu_{22}(u)$ and $\nu_{11}(u)\nu_{22}(v_m)$ in (2.105) can be computed using Proposition 2.2.2. We omit simple but rather cumbersome intermediate calculations and give the final result:

$$g(v_m, u)\big(\nu_{11}(u)\nu_{22}(v_m) - \nu_{11}(v_m)\nu_{22}(u)\big)\nu(\bar{v}_m)|0\rangle = \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II},\bar{w}_{\rm III}\}} (-\beta_1)^{1-l_{\rm I}}(-\beta_2)^{1-l_{\rm II}}$$
$$\times f(\bar{w}_{\rm I}, \bar{w}_{\rm II}) f(\bar{w}_{\rm I}, \bar{w}_{\rm III}) f(\bar{w}_{\rm III}, \bar{w}_{\rm II}) K^{(1)}_{1,l_{\rm I}}(u|\bar{w}_{\rm I} + c)\overline{K}^{(1)}_{1,l_{\rm II}}(u|\bar{w}_{\rm II} - c)$$
$$\times \left(1 - \frac{1}{f(\bar{w}_{\rm I}, v_m) f(v_m, \bar{w}_{\rm II})}\right)\lambda_2(\bar{w}_{\rm I})\lambda_1(\bar{w}_{\rm II})\nu_{12}(\bar{w}_{\rm III})|0\rangle. \tag{2.108}$$

Combining equations (2.107) and (2.108) we obtain (2.104) for $\#\bar{v} = m$. Thus, the first step of induction is completed.

Let now assume that (2.104) holds for some $n-1$. We prove that then it holds for $\#\bar{u}=n$. The proof is very similar to the one of proposition 2.2.2, however, it is more bulky.

We act successively as $\nu_{21}(\bar{u})=\nu_{21}(u_n)\nu_{21}(\bar{u}_n)$. Then

$$\nu_{21}(\bar{u})\nu_{12}(\bar{v})|0\rangle=\nu_{21}(u_n)\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}}(-\beta_1)^{n-1-l_{\rm I}}(-\beta_2)^{n-1-l_{\rm II}}\lambda_2(\bar{\xi}_{\rm I})\lambda_1(\bar{\xi}_{\rm II})$$
$$\times f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm III})f(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})K^{(1)}_{n-1,l_{\rm I}}(\bar{u}_n|\bar{\xi}_{\rm I}+c)\overline{K}^{(1)}_{n-1,l_{\rm II}}(\bar{u}_n|\bar{\xi}_{\rm II}-c)\nu_{12}(\bar{\xi}_{\rm III})|0\rangle. \quad (2.109)$$

Here $\bar{\xi}=\{\bar{u}_n,\bar{v}\}$, $l_{\rm I}=\#\bar{\xi}_{\rm I}$, and $l_{\rm II}=\#\bar{\xi}_{\rm II}$. The action of $\nu_{21}(u_n)$ gives us an additional sum over partitions

$$\nu_{21}(\bar{u})\nu_{12}(\bar{v})|0\rangle=\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}}(-\beta_1)^{n-l_{\rm I}}(-\beta_2)^{n-l_{\rm II}}\lambda_2(\bar{\xi}_{\rm I})\lambda_1(\bar{\xi}_{\rm II})\ f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm III})f(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})$$
$$\times K^{(1)}_{n-1,l_{\rm I}}(\bar{u}_n|\bar{\xi}_{\rm I}+c)\overline{K}^{(1)}_{n-1,l_{\rm II}}(\bar{u}_n|\bar{\xi}_{\rm II}-c)\sum_{\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}}(-\beta_1)^{-k_{\rm I}}(-\beta_2)^{-k_{\rm II}}\lambda_2(\bar{\eta}_{\rm I})\lambda_1(\bar{\eta}_{\rm II})$$
$$\times\ f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm III})f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})K^{(1)}_{1,k_{\rm I}}(u_n|\bar{\eta}_{\rm I}+c)\overline{K}^{(1)}_{1,k_{\rm II}}(u_n|\bar{\eta}_{\rm II}-c)\nu_{12}(\bar{\eta}_{\rm III})|0\rangle, \quad (2.110)$$

where $\bar{\eta}=\{\bar{\xi}_{\rm III},v_n\}$, $k_{\rm I}=\#\bar{\eta}_{\rm I}$, and $k_{\rm II}=\#\bar{\eta}_{\rm II}$. Thus, eventually the sum is taken over partitions of the set $\{\bar{u},\bar{v}\}$ into five subsets $\bar{\xi}_{\rm I}$, $\bar{\xi}_{\rm II}$, $\bar{\eta}_{\rm I}$, $\bar{\eta}_{\rm II}$, and $\bar{\eta}_{\rm III}$ such that $u_n\notin\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$. The subset $\bar{\xi}_{\rm III}$ should be understood as $\bar{\xi}_{\rm III}=\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}\setminus\{u_n\}$.

Let $\bar{w}=\{\bar{u},\bar{v}\}$. We denote $\bar{\xi}_{\rm I}=\bar{w}_{\rm I}$, $\bar{\xi}_{\rm II}=\bar{w}_{\rm II}$, $\bar{\eta}_{\rm I}=\bar{w}_{\rm i}$, $\bar{\eta}_{\rm ii}=\bar{w}_{\rm ii}$, and $\bar{\eta}_{\rm III}=\bar{w}_{\rm iii}$. Respectively, the cardinalities of the subsets are denoted by $r_{\rm I}=\#\bar{w}_{\rm I}$, $r_{\rm II}=\#\bar{w}_{\rm II}$, $r_{\rm i}=\#\bar{w}_{\rm i}$, $r_{\rm ii}=\#\bar{w}_{\rm ii}$. Then equation (2.110) takes the form

$$\nu_{21}(\bar{u})\nu_{12}(\bar{v})|0\rangle=\sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II},\bar{w}_{\rm i},\bar{w}_{\rm ii},\bar{w}_{\rm iii}\}}(-\beta_1)^{n-r_{\rm I}-r_{\rm i}}(-\beta_2)^{n-r_{\rm II}-r_{\rm ii}}\lambda_2(\bar{w}_{\rm i})\lambda_2(\bar{w}_{\rm I})\lambda_1(\bar{w}_{\rm II})\lambda_1(\bar{w}_{\rm ii})$$
$$\times\frac{f(\bar{w}_{\rm I},\bar{w}_{\rm II})f(\bar{w}_{\rm I},\bar{w}_{\rm i})f(\bar{w}_{\rm I},\bar{w}_{\rm ii})f(\bar{w}_{\rm I},\bar{w}_{\rm iii})f(\bar{w}_{\rm i},\bar{w}_{\rm II})f(\bar{w}_{\rm ii},\bar{w}_{\rm II})f(\bar{w}_{\rm iii},\bar{w}_{\rm II})}{f(\bar{w}_{\rm I},u_n)f(u_n,\bar{w}_{\rm II})}$$
$$\times f(\bar{w}_{\rm i},\bar{w}_{\rm ii})f(\bar{w}_{\rm i},\bar{w}_{\rm iii})f(\bar{w}_{\rm iii},\bar{w}_{\rm ii})$$
$$\times K^{(1)}_{n-1,r_{\rm I}}(\bar{u}_n|\bar{w}_{\rm I}+c)K^{(1)}_{1,r_{\rm i}}(u_n|\bar{w}_{\rm i}+c)\overline{K}^{(1)}_{n-1,r_{\rm II}}(\bar{u}_n|\bar{w}_{\rm II}-c)\overline{K}^{(1)}_{1,r_{\rm ii}}(u_n|\bar{w}_{\rm ii}-c)\nu_{12}(\bar{w}_{\rm iii})|0\rangle. \quad (2.111)$$

Observe that the restriction $u_n\notin\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}$ holds automatically, because $\big(f(\bar{w}_{\rm I},u_n)f(u_n,\bar{w}_{\rm II})\big)^{-1}=0$ for $u_n\in\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}$. Setting $\{\bar{w}_{\rm II},\bar{w}_{\rm ii}\}=\bar{w}_0$, $\{\bar{w}_{\rm I},\bar{w}_{\rm i}\}=\bar{w}_{0'}$, and $\bar{w}_{\rm iii}=\bar{w}_{\rm III}$ we recast (2.111) as follows:

$$\nu_{21}(\bar{u})\nu_{12}(\bar{v})|0\rangle=\sum_{\bar{w}=\{\bar{w}_{0'},\bar{w}_0,\bar{w}_{\rm III}\}}(-\beta_1)^{n-r_{0'}}(-\beta_2)^{n-r_0}\lambda_2(\bar{w}_{0'})\lambda_1(\bar{w}_0)\nu_{12}(\bar{w}_{\rm III})|0\rangle$$
$$\times f(\bar{w}_{0'},\bar{w}_0)f(\bar{w}_{0'},\bar{w}_{\rm III})f(\bar{w}_{\rm III},\bar{w}_0)\ W\overline{W}, \quad (2.112)$$

where $r_{0'}=\#\bar{w}_{0'}$, $r_0=\#\bar{w}_0$, and

$$W=\sum_{\bar{w}_{0'}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm i}\}}\frac{f(\bar{w}_{\rm I},\bar{w}_{\rm i})}{f(\bar{w}_{\rm I},u_n)}K^{(1)}_{n-1,r_{\rm I}}(\bar{u}_n|\bar{w}_{\rm I}+c)K^{(1)}_{1,r_{\rm i}}(u_n|\bar{w}_{\rm i}+c), \quad (2.113)$$

$$\overline{W}=\sum_{\bar w_0\Rightarrow\{\bar w_{\rm II},\bar w_{\rm ii}\}}\frac{f(\bar w_{\rm ii},\bar w_{\rm II})}{f(u_n,\bar w_{\rm II})}\overline{K}^{(1)}_{n-1,r_{\rm II}}(\bar u_n|\bar w_{\rm II}-c)\overline{K}^{(1)}_{1,r_{\rm ii}}(u_n|\bar w_{\rm ii}-c). \tag{2.114}$$

Observe that the sums over partitions of the subsets $\bar w_{0'}$ and $\bar w_0$ can be obtained one from another via the replacement $c\to -c$. Moreover, the sum (2.114) was computed in (2.102). Thus,

$$W=K^{(1)}_{n,r_{0'}}(\bar u|\bar w_{0'}+c),\qquad \overline{W}=\overline{K}^{(1)}_{n,r_0}(\bar u|\bar w_0-c). \tag{2.115}$$

Substituting this into (2.112) we obtain

$$\begin{aligned}\nu_{21}(\bar u)\nu_{12}(\bar v)|0\rangle=&\sum_{\bar w=\{\bar w_{0'},\bar w_0,\bar w_{\rm III}\}}(-\beta_1)^{n-r_{0'}}(-\beta_2)^{n-r_0}\lambda_2(\bar w_{0'})\lambda_1(\bar w_0)\nu_{12}(\bar w_{\rm III})|0\rangle\\ &\times f(\bar w_{0'},\bar w_0)f(\bar w_{0'},\bar w_{\rm III})f(\bar w_{\rm III},\bar w_0)\;K^{(1)}_{n,r_{0'}}(\bar u|\bar w_{0'}+c)\overline{K}^{(1)}_{n,r_0}(\bar u|\bar w_0-c),\end{aligned} \tag{2.116}$$

which coincides with (2.104) up to the labels of the subsets. This ends the proof. □

Multiple action of the modified operator ν_{12}

Up to now all the multiple action formulas were valid for an arbitrary highest wight representation of the Yangian of $\mathfrak{gl}_2$. The following proposition is valid for finite dimensional representations only.

Proposition 2.2.4. *Let $\#u=n$ and $\#v=m$. Consider an irreducible finite dimensional representation of the Yangian. Then there exists an integer S and a function $F(u)$ such that for all n and m such that $m+n\ge S$ the following multiple action holds:*

$$\begin{aligned}\nu_{12}(\bar u)\nu_{12}(\bar v)|0\rangle=\nu_{12}(\bar w)|0\rangle=&\Big(\frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big)^{m+n-S}\\ &\times\sum_{\bar w\Rightarrow\{\bar w_{\rm I},\bar w_{\rm II}\}}F(\bar w_{\rm I})g(\bar w_{\rm I},\bar w_{\rm II})\nu_{12}(\bar w_{\rm II})|0\rangle.\end{aligned} \tag{2.117}$$

The sum is taken over partitions $\{\bar v,\bar u\}=\bar w\Rightarrow\{\bar w_{\rm I},\bar w_{\rm II}\}$ such that $\#\bar w_{\rm I}=m+n-S$, $\#\bar w_{\rm II}=S$. The constant μ is defined in (2.83).

Remark 2.2.2. *The value of S and the explicit form of the function $F(u)$ depend on the concrete representation [23]. In particular, for the case of the fundamental representation of the inhomogeneous XXX spin-1/2 chain with N sites one has $S=N$ and*

$$F(u)=\prod_{i=1}^{N}\frac{h(u,\theta_i)}{g(u,\theta_i)}, \tag{2.118}$$

where θ_i are inhomogeneity parameters.

Remark 2.2.3. *Equation (2.117) shows that if the number of the operators ν_{12} exceeds S, then their successive action on $|0\rangle$ reduces to the action of exactly S such operators. This property is a peculiarity of finite-dimensional representations, and it is this property that is key for implementation of the MABA. In particular, the function $F(u)$ gives rise to the inhomogeneous term introduced in the context of the off-diagonal Bethe ansatz [35, 33].*

Proof. To prove proposition 2.2.4 we use induction over $n = \#\bar{u}$ with $n + m \geq S$. The case $n = 1$ was first conjectured in [43] for the fundamental representation. Then, it was proved in [23] that for any irreducible finite dimension representation there exists an integer S and a function $F(u)$ such that

$$\nu_{12}(u)\nu_{12}(\bar{v}) = \frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big(F(u)g(u,\bar{v})\nu_{12}(\bar{v}) + \sum_{i=1}^{S} g(v_i,u)F(v_i)g(v_i,\bar{v}_i)\nu_{12}(u)\nu_{12}(\bar{v}_i)\Big). \quad (2.119)$$

The reader can find the explicit form of $F(u)$ and the corresponding S in [23]. It is easy to see that the term in the first line of (2.119) corresponds to the partition $\bar{w}_{\rm I} = u$, $\bar{w}_{\rm II} = \bar{v}$ in (2.117). The terms in the second line of (2.119) correspond to the partitions $\bar{w}_{\rm I} = v_i$, $\bar{w}_{\rm II} = \{u, \bar{v}_i\}$ $(i = 1, \dots, m)$ in (2.117). Thus, (2.117) coincides with (2.119) for $n = 1$.

Let (2.117) be valid for $n - 1 = \#\bar{u}_n$ such that $n - 1 + m > S$. Consider the action of $\nu_{12}(\bar{u})$ with $n = \#\bar{u}$. We can act successively, firstly by $\nu_{12}(\bar{u}_n)$ and secondly by $\nu_{12}(u_n)$. Due to the induction assumption we obtain at the first step

$$\nu_{12}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \Big(\frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big)^{m+n-S-1} \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} F(\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})\nu_{12}(u_n)\nu_{12}(\bar{\xi}_{\rm II})|0\rangle, \quad (2.120)$$

where the sum is taken over partitions $\bar{\xi} = \{\bar{u}_n, \bar{v}\} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}\}$ such that $\#\bar{\xi}_{\rm I} = n - 1$, $\#\bar{\xi}_{\rm II} = m$. Acting with $\nu_{12}(u_n)$ on $\nu_{12}(\bar{\xi}_{\rm II})|0\rangle$ via (2.119) we find

$$\nu_{12}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \Big(\frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big)^{m+n-S} \times \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} \sum_{\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}} F(\bar{\xi}_{\rm I})F(\bar{\eta}_{\rm I})g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})g(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})\nu_{12}(\bar{\eta}_{\rm II})|0\rangle, \quad (2.121)$$

where we have additional partitions $\bar{\eta} = \{u_n, \bar{\xi}_{\rm II}\} \Rightarrow \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}\}$ such that $\#\bar{\eta}_{\rm I} = 1$ and $\#\bar{\eta}_{\rm II} = m$. Thus, eventually we deal with the partitions of the set $\bar{w} = \{\bar{u}, \bar{v}\}$ into three subsets: $\bar{\xi}_{\rm I}$, $\bar{\eta}_{\rm I}$, and $\bar{\eta}_{\rm II}$. The subset $\bar{\xi}_{\rm II}$ should be understood as $\bar{\xi}_{\rm II} = \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}\} \setminus \{u_n\}$. Besides the restrictions on the cardinalities of the subsets we have the additional restriction $u_n \notin \bar{\xi}_{\rm I}$.

Let $\bar{\xi}_{\rm I} = \bar{w}_{\rm I}$, $\bar{\eta}_{\rm I} = \bar{w}_{\rm II}$, and $\bar{\eta}_{\rm II} = \bar{w}_{\rm III}$. Then $\bar{\xi}_{\rm II} = \{\bar{w}_{\rm II}, \bar{w}_{\rm III}\} \setminus \{u_n\}$, and equation (2.121) takes the form

$$\nu_{12}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \Big(\frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big)^{m+n-S} \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II},\bar{w}_{\rm III}\}} F(\bar{w}_{\rm I})F(\bar{w}_{\rm II})\nu_{12}(\bar{w}_{\rm III})|0\rangle \times \frac{g(\bar{w}_{\rm I},\bar{w}_{\rm II})g(\bar{w}_{\rm I},\bar{w}_{\rm III})}{g(\bar{w}_{\rm I},u_n)} g(\bar{w}_{\rm II},\bar{w}_{\rm III}). \quad (2.122)$$

Observe that the condition $u_n \notin \bar{w}_{\rm I}$ is valid automatically due to the factor $\Big(g(\bar{w}_{\rm I}, u_n)\Big)^{-1}$

that vanishes if $u_n \in \bar{w}_{\text{I}}$. Setting $\bar{w}_0 = \{\bar{w}_{\text{I}}, \bar{w}_{\text{II}}\}$ we recast (2.122) as follows:

$$\nu_{12}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \Big(\frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big)^{m+n-S} \sum_{\bar{w}\Rightarrow\{\bar{w}_0,\bar{w}_{\text{III}}\}} F(\bar{w}_0)g(\bar{w}_0,\bar{w}_{\text{III}})\nu_{12}(\bar{w}_{\text{III}})|0\rangle \\ \times \sum_{\bar{w}_0\Rightarrow\{\bar{w}_{\text{I}},\bar{w}_{\text{II}}\}} \frac{g(\bar{w}_{\text{I}},\bar{w}_{\text{II}})}{g(\bar{w}_{\text{I}},u_n)}. \tag{2.123}$$

The sum over partitions is now taken in two steps. First, the set $\bar{w} = \{\bar{u}, \bar{v}\}$ is divided into subsets $\{\bar{w}_0, \bar{w}_{\text{III}}\}$ such that $\#\bar{w}_0 = n$ and $\#\bar{w}_{\text{III}} = m$. Then the subset $\bar{w}_0$ is divided into subsets $\{\bar{w}_{\text{I}}, \bar{w}_{\text{II}}\}$ such that $\#\bar{w}_{\text{I}} = n-1$ and $\#\bar{w}_{\text{II}} = 1$. Let us prove that the latter sum is equal to 1. We have

$$\sum_{\bar{w}_0\Rightarrow\{\bar{w}_{\text{I}},\bar{w}_{\text{II}}\}} \frac{g(\bar{w}_{\text{I}},\bar{w}_{\text{II}})}{g(\bar{w}_{\text{I}},u_n)} = \lim_{x\to u_n} \frac{1}{g(\bar{w}_0,x)} \sum_{\bar{w}_0\Rightarrow\{\bar{w}_{\text{I}},\bar{w}_{\text{II}}\}} g(\bar{w}_{\text{I}},\bar{w}_{\text{II}})g(\bar{w}_{\text{II}},x). \tag{2.124}$$

Here we have replaced u_n by x in order to avoid possible singularity at $\bar{w}_{\text{II}} = u_n$. Recall that $\#\bar{w}_{\text{II}} = 1$. Thus, the sum over partitions in the right hand side of (2.124) is given by a contour integral

$$\sum_{\bar{w}_0\Rightarrow\{\bar{w}_{\text{I}},\bar{w}_{\text{II}}\}} g(\bar{w}_{\text{I}},\bar{w}_{\text{II}})g(\bar{w}_{\text{II}},x) = \frac{-1}{2\pi ic}\oint_{\Gamma(\bar{w}_0)} g(\bar{w}_0,z)g(z,x)\,dz, \tag{2.125}$$

where the anticlockwise oriented contour $\Gamma(\bar{w}_0)$ surrounds the points $\bar{w}_0$ and does not contain any other singularities of the integrand. Taking the integral by the residue outside the integration contour (that is, at $z = x$) we immediately obtain

$$\sum_{\bar{w}_0\Rightarrow\{\bar{w}_{\text{I}},\bar{w}_{\text{II}}\}} g(\bar{w}_{\text{I}},\bar{w}_{\text{II}})g(\bar{w}_{\text{II}},x) = g(\bar{w}_0,x), \tag{2.126}$$

leading to

$$\sum_{\bar{w}_0\Rightarrow\{\bar{w}_{\text{I}},\bar{w}_{\text{II}}\}} \frac{g(\bar{w}_{\text{I}},\bar{w}_{\text{II}})}{g(\bar{w}_{\text{I}},u_n)} = 1. \tag{2.127}$$

Substituting this into (2.123) we arrive at

$$\nu_{12}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \Big(\frac{(\mu-1)(\beta_1+\beta_2)}{\beta_1\beta_2}\Big)^{m+n-S} \sum_{\bar{w}\Rightarrow\{\bar{w}_0,\bar{w}_{\text{III}}\}} F(\bar{w}_0)g(\bar{w}_0,\bar{w}_{\text{III}})\nu_{12}(\bar{w}_{\text{III}})|0\rangle, \tag{2.128}$$

which coincides with (2.117) up to the labels of the subsets. Thus, the proof is completed. □

2.2.4 Modified Scalar Product

We can now consider the scalar product of the modified Bethe vectors.

Theorem 2.2.2. *Let $\#\bar{u}=n$ and $\#\bar{v}=m$, and define the dual highest weight $|0\rangle$ by*

$$\langle 0|\, t_{ii}(u)=\lambda_i(u)\,\langle 0|\,,\quad \langle 0|\, t_{12}(u)=0,\quad \langle 0|0\rangle=1 \tag{2.129}$$

Then the scalar product of two modified Bethe vectors

$$S_\nu^{n,m}(\bar{u},\bar{v})=\langle 0|\nu_{21}(\bar{u})\nu_{12}(\bar{v})|0\rangle \tag{2.130}$$

is given by

$$S_\nu^{n,m}(\bar{u},\bar{v})=\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}}(-\beta_1)^{n-l_{\rm I}}(-\beta_2)^{n-l_{\rm II}}\lambda_2(\bar{\xi}_{\rm I})\lambda_1(\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}) \\ \times K^{(\mu)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}+c)\overline{K}^{(\mu)}_{n,l_{\rm II}}(\bar{u}|\bar{\xi}_{\rm II}-c). \tag{2.131}$$

Here $\bar{\xi}=\{\bar{u},\bar{v}\}$, $l_{\rm I}=\bar{\xi}_{\rm I}$, and $l_{\rm II}=\bar{\xi}_{\rm II}$. The sum is taken over all partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$. There is no restriction on the cardinalities of the subsets. The functions $K^{(\mu)}_{n,l_{\rm I}}$ and $\overline{K}^{(\mu)}_{n,l_{\rm II}}$ respectively are the modified Izergin determinants (2.68) and (2.70) at $z=\mu$.

Proof. Acting with the dual highest weight vector (2.129) onto (2.104) we find

$$S_\nu^{n,m}(\bar{u},\bar{v})=\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}}(-\beta_1)^{n-l_{\rm I}}(-\beta_2)^{n-l_{\rm II}}\lambda_2(\bar{\xi}_{\rm I})\lambda_1(\bar{\xi}_{\rm II})\, f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm III})f(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II}) \\ \times\, K^{(1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}+c)\overline{K}^{(1)}_{n,l_{\rm II}}(\bar{u}|\bar{\xi}_{\rm II}-c)\langle 0|\nu_{12}(\bar{\xi}_{\rm III})|0\rangle. \tag{2.132}$$

Recall that here $\bar{\xi}=\{\bar{u},\bar{v}\}$, $\#\bar{\xi}_{\rm I}=l_{\rm I}$, and $\#\bar{\xi}_{\rm II}=l_{\rm II}$. The sum is taken over all partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}$.

The vacuum average $\langle 0|\nu(\bar{\xi}_{\rm III})|0\rangle$ was computed in [59]:

$$\langle 0|\nu_{12}(\bar{w})|0\rangle=(1-\mu)^p\sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}}(-\beta_2)^{-\#\bar{w}_{\rm II}}(-\beta_1)^{-\#\bar{w}_{\rm I}}\lambda_2(\bar{w}_{\rm I})\lambda_1(\bar{w}_{\rm II})\, f(\bar{w}_{\rm I},\bar{w}_{\rm II}), \tag{2.133}$$

where $\#\bar{w}=p$ and the sum is taken over all partitions $\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}$. Substituting (2.133) into (2.132) and decomposing $\bar{\xi}_{\rm III}=\{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ we find

$$S_\nu^{n,m}(\bar{u},\bar{v})=\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}}(1-\mu)^{l_{\rm i}+l_{\rm ii}}(-\beta_1)^{n-l_{\rm I}-l_{\rm i}}(-\beta_2)^{n-l_{\rm II}-l_{\rm ii}}\lambda_2(\bar{\xi}_{\rm I})\lambda_1(\bar{\xi}_{\rm II})\lambda_2(\bar{\xi}_{\rm i})\lambda_1(\bar{\xi}_{\rm ii}) \\ \times\; f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm i})f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm ii})f(\bar{\xi}_{\rm i},\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm II})f(\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii})K^{(1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}+c)\overline{K}^{(1)}_{n,l_{\rm II}}(\bar{u}|\bar{\xi}_{\rm II}-c). \tag{2.134}$$

Here the sum is taken over partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$. The cardinalities of the subsets are denoted by l with the corresponding subscript.

Now we set $\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\}=\bar{\xi}_0$, $\{\bar{\xi}_{\rm II},\bar{\xi}_{\rm ii}\}=\bar{\xi}_{0'}$. Then we arrive at

$$S_\nu^{n,m}(\bar{u},\bar{v})=\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_0,\bar{\xi}_{0'}\}}(-\beta_1)^{n-l_0}(-\beta_2)^{n-l_{0'}}\lambda_2(\bar{\xi}_0)\lambda_1(\bar{\xi}_{0'})f(\bar{\xi}_0,\bar{\xi}_{0'})\mathcal{L}(\bar{\xi}_0)\overline{\mathcal{L}}(\bar{\xi}_{0'}), \tag{2.135}$$

where

$$\mathcal{L}(\bar{\xi}_0)=\sum_{\bar{\xi}_0\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\}}(1-\mu)^{l_{\rm i}}K^{(1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}+c)f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}) \tag{2.136}$$

and
$$\overline{\mathcal{L}}(\bar\xi_{0'}) = \sum_{\bar\xi_{0'}\Rightarrow\{\bar\xi_{\rm II},\bar\xi_{\rm ii}\}} (1-\mu)^{l_{\rm ii}}\overline{K}^{(1)}_{n,l_{\rm II}}(\bar u|\bar\xi_{\rm II}-c)f(\bar\xi_{\rm ii},\bar\xi_{\rm II}). \tag{2.137}$$

The sums (2.136) and (2.137) are computed in proposition 2.3.9:
$$\mathcal{L}(\bar\xi_0) = K^{(\mu)}_{n,l_0}(\bar u|\bar\xi_0+c), \qquad \overline{\mathcal{L}}(\bar\xi_{0'}) = \overline{K}^{(\mu)}_{n,l_{0'}}(\bar u|\bar\xi_{0'}-c). \tag{2.138}$$

Then equation (2.135) coincides with (2.131) up to the labels of the subsets.

□

Remarkably, this formula has exactly the same form as representation (2.80) for the scalar product in the usual ABA (for $m=n$). However, instead of the ordinary Izergin determinants we have now modified Izergin determinants. Furthermore, we have no restrictions on the cardinalities of the subsets.

Consider the case $\mu=1$ and $n=m$. Then, due to (2.73) a non-vanishing contribution occurs if and only if $n\ge \#\bar\xi_{\rm I}$ and $n\ge\#\bar\xi_{\rm II}$. Since $\#\bar\xi_{\rm I}+\#\bar\xi_{\rm II}=2n$, we conclude that $n=\#\bar\xi_{\rm I}$ and $n=\#\bar\xi_{\rm II}$. This leads us to
$$S^{n,n}_\nu(\bar u,\bar v)\Big|_{\mu=1} = \sum_{\substack{\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}\\ \#\bar\xi_{\rm I}=\#\bar\xi_{\rm II}=n}} \lambda_2(\bar\xi_{\rm I})\lambda_1(\bar\xi_{\rm II})f(\bar\xi_{\rm I},\bar\xi_{\rm II})K_n(\bar u|\bar\xi_{\rm I}+c)\overline{K}_n(\bar u|\bar\xi_{\rm II}-c), \tag{2.139}$$

and we reproduce the usual ABA scalar product S^n_t given by theorem 2.2.1.

Similarly to (2.81) the sum (2.131) can be written in the form of the sum over independent partitions of the sets $\bar u$ and $\bar v$ (*modified Izergin–Korepin formula*).

Corollary 2.2.2. *Let $\#\bar u=n$ and $\#\bar v=m$. Then the modified scalar product of two Bethe vectors is given by*
$$S^{n,m}_\nu(\bar u,\bar v) = \mu^{2n}(1-\mu)^{m-n}\sum(-\beta_1)^{n_2-m_2}(-\beta_2)^{n_1-m_1}\lambda_2(\bar u_{\rm I})\lambda_2(\bar v_{\rm II})\lambda_1(\bar u_{\rm II})\lambda_1(\bar v_{\rm I})$$
$$\times f(\bar u_{\rm I},\bar u_{\rm II})f(\bar v_{\rm II},\bar v_{\rm I})K^{(1/\mu)}_{m_2,n_2}(\bar v_{\rm II}|\bar u_{\rm II})\overline{K}^{(1/\mu)}_{m_1,n_1}(\bar v_{\rm I}|\bar u_{\rm I}), \tag{2.140}$$

where the sum is taken over all partitions $\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}$ and $\bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}$ such that $\#\bar v_{\rm I}=m_1$, $\#\bar v_{\rm II}=m_2$ and $\#\bar u_{\rm I}=n_1$, $\#\bar u_{\rm II}=n_2$, where $n_1=0,1,\dots,n$ and $m_1=0,1,\dots,m$.

Proof. We set $\bar w_{\rm I}\Rightarrow\{\bar u_{\rm I},\bar v_{\rm II}\}$ and $\bar w_{\rm II}\Rightarrow\{\bar u_{\rm II},\bar v_{\rm I}\}$ with $\#\bar u_{\rm I}=n_1$, $\#\bar v_{\rm I}=m_1$, $\#\bar u_{\rm II}=n_2$, $\#\bar v_{\rm II}=m_2$ and $n=n_1+n_2$, $m=m_1+m_2$ in (2.131). Using (2.166) and (2.167) we obtain:
$$S^{n,m}_\nu(\bar u,\bar v) = (-\mu)^n\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}}(-\beta_1)^{n-n_1-m_2}(-\beta_2)^{n-n_2-m_1}\lambda_2(\bar u_{\rm I})\lambda_2(\bar v_{\rm II})\lambda_1(\bar u_{\rm II})\lambda_1(\bar v_{\rm I})$$
$$\times K^{(\mu)}_{n_2,m_2}(\bar u_{\rm II}|\bar v_{\rm II}+c)\overline{K}^{(\mu)}_{n_1,m_1}(\bar u_{\rm I}|\bar v_{\rm I}-c)f(\bar u_{\rm I},\bar u_{\rm II})f(\bar v_{\rm II},\bar v_{\rm I})f(\bar u_{\rm I},\bar v_{\rm I})f(\bar v_{\rm II},\bar u_{\rm II}). \tag{2.141}$$

Then the use of (2.176) and (2.177) for the modified Izergin determinants immediately gives (2.140). □

2.2.5 Conclusion

We considered multiple actions of the modified monodromy matrix entries on the modified Bethe vectors within the framework of the MABA. We shown that they look very similar to the standard multiple actions obtained for the ordinary ABA in [51]. The main difference is that the ordinary Izergin determinant [90] is modified according to (2.68) and (2.70), and the sum over partitions of the Bethe parameters should be taken without restrictions on the cardinalities of the subsets. The same changes apply to the formula for the scalar product of the modified Bethe vectors. It would be interesting to compare this result with those that follow from the separation of variable approach [93].

Further development of the method, as the one proposed in this paper, can be carried out in several directions. It is quite possible that the multiple action formulas admit a deformation to the XXZ model. In this case, however, the property (1.95) is no longer valid for arbitrary twist matrices, i.e. the system is no longer gl_2 invariant. Therefore, one should consider a more sophisticated face-vertex transformation of the twist (see *e.g.* [41, 39] and references therein).

It is also interesting to consider models with higher rank algebra. The main open problem in this direction is to construct the Bethe vectors in the twisted periodic case.

Finally, a very attractive way for further development is to consider particular cases of the scalar products of the modified Bethe vectors. It is well known from the ABA that if one of the vectors is an eigenvector of the transfer matrix (on-shell Bethe vector), then the scalar product admit a compact determinant representation, which involves the Jacobian of the transfer matrix eigenvalue [27]. It was conjectured in [43] that a similar representation also exists in the case of the scalar products involving the modified on-shell Bethe vectors. We will provide a proof of this conjecture in our forthcoming publication.

2.3 Appendix

2.3.1 Commutation Relations of the $t_{ij}(u)$ and Multiple Actions on the Bethe Vector

The RTT relation (1.50) yields the following commutation relations:

$$t_{ij}(v)t_{ij}(u) = t_{ij}(u)t_{ij}(v), \quad (2.142)$$

$$t_{ij}(v)t_{ik}(u) = f(u,v)t_{ik}(u)t_{ij}(v) + g(v,u)t_{ik}(v)t_{ij}(u), \quad (2.143)$$

$$t_{ij}(v)t_{kj}(u) = f(v,u)t_{kj}(u)t_{ij}(v) + g(u,v)t_{kj}(v)t_{ij}(u). \quad (2.144)$$

They imply the following actions on the products of M operators:

$$t_{ij}(v)t_{ik}(\bar{u}) = f(\bar{u},v)t_{ik}(\bar{u})t_{ij}(v) + \sum_{i=1}^{M} g(v,u_i)f(\bar{u}_i,u_i)t_{ik}(v)t_{ik}(\bar{u}_i)t_{ij}(u_i), \quad (2.145)$$

$$t_{ij}(v)t_{kj}(\bar{u}) = f(v,\bar{u})t_{kj}(\bar{u})t_{ij}(v) + \sum_{i=1}^{M} g(u_i,v)f(u_i,\bar{u}_i)t_{kj}(v)t_{kj}(\bar{u}_i)t_{ij}(u_i) \quad (2.146)$$

The same commutation relations are valid for the modified operators $\nu_{ij}(u)$.

2.3.2 SoV Basis for the Modified Creation Operator $\nu_{12}(z)$

Let us construct the Separation of variables basis [60] of the modified creation operator $\nu_{12}(z)$. We assume that the inhomogeneity parameters $\bar{\theta}$ are generic complex numbers and that the product AB has non zero entries.

Let us introduce a vector

$$|y\rangle = \mathcal{A}_1^{-1}\mathcal{A}_2^{-1}\dots\mathcal{A}_N^{-1}|\hat{0}\rangle, \quad (2.147)$$

where

$$\mathcal{A}_i = a_0 \ \exp\Big(a^+S_i^+ + a^-S_i^- + 2a_3S_i^3\Big), \quad (2.148)$$

and the parameters $\{a_0, a^+, a^-, a_3\}$ are fixed by the equality

$$A = \sqrt{\mu}\begin{pmatrix} 1 & \frac{\rho_2}{\kappa^-} \\ \frac{\rho_1}{\kappa^+} & 1 \end{pmatrix} = a_0 \ \exp\Big(a^+\sigma^+ + a^-\sigma^- + a_3\sigma^3\Big). \quad (2.149)$$

Using $\mathfrak{gl}_2$ invariance we can show that

$$\nu_{12}(z)|y\rangle = \frac{\mu}{\kappa^-}(\rho_1+\rho_2)\lambda_2(z)|y\rangle. \quad (2.150)$$

Indeed, we have

$$\nu_{12}(z)|y\rangle = \mathrm{Tr}(E_{21}AT(z)B)\mathcal{A}_1^{-1}\mathcal{A}_2^{-1}\dots\mathcal{A}_N^{-1}|\hat{0}\rangle \quad (2.151)$$

$$= \mathcal{A}_1^{-1}\mathcal{A}_2^{-1}\dots\mathcal{A}_N^{-1}\,\mathrm{Tr}(E_{21}T(z)AB)|\hat{0}\rangle \quad (2.152)$$

$$= (AB)_{12}\lambda_2(z)|y\rangle. \quad (2.153)$$

Consider a set of vectors

$$|Y(\bar{k})\rangle = \prod_{i=1}^{N}\prod_{j=1}^{k_i}\nu_{22}(\theta_i + c(s_i + \frac{1}{2} - j))|y\rangle, \tag{2.154}$$

where $\bar{k}$ is a set of N integers such that $k_i \in \{0, 1, \ldots, 2s_i\}$ for $i = 1, ..., N$. Then it follows from the commutation relation (2.146) that

$$\nu_{12}(z)\prod_{j=1}^{n}\nu_{22}(u - c\,j) = \Big(\prod_{j=1}^{n} f(z, u - c\,j)\Big)\Big(\prod_{j=1}^{n}\nu_{22}(u - c\,j)\Big)\nu_{12}(z)$$

$$+g(u - c, z)\Big(\prod_{j=2}^{n} f(u - c, u - c\,j)\Big) \tag{2.155}$$

$$\times\nu_{22}(z)\Big(\prod_{j=2}^{n}\nu_{22}(u - c\,j)\Big)\nu_{12}(u - c). \tag{2.156}$$

Here we have used $f(x - c, x) = 0$ and $\lambda_2(\theta_i + c(s_i - \frac{1}{2})) = 0$. Hence, the action of $\nu_{12}(z)$ on $|Y(\bar{k})\rangle$ reads

$$\nu_{12}(z)|Y(\bar{k})\rangle = \frac{\mu}{\kappa^-}(\rho_1 + \rho_2)\frac{F(z)}{\Lambda_{12}(z, \bar{k})}|Y(\bar{k})\rangle, \tag{2.157}$$

where $F(z)$ given by (2.37) and

$$\Lambda_{12}(z, \bar{k}) = \prod_{i=1}^{N}\Big(\prod_{j=1}^{k_i} h(z - c, \theta_i + c(s_i + \frac{1}{2} - j))\prod_{j=k_i+1}^{2s_i} h(z, \theta_i + c(s_i + \frac{1}{2} - j))\Big), \tag{2.158}$$

with $h(u, v) = f(u, v)/g(u, v)$.

Remark 2.3.1. *Using $\mathfrak{gl}_2$ invariance we can show that*

$$\langle\hat{0}|\mathcal{A}_1^{-1}\mathcal{A}_2^{-1}\ldots\mathcal{A}_N^{-1}|Y(\bar{k})\rangle = \prod_{i=1}^{N}\prod_{j=1}^{k_i}(AB)_{22}\lambda_1(\theta_i + c(s_i + \frac{1}{2} - j)) \neq 0. \tag{2.159}$$

Thus, all the vectors under consideration are non zero.

Thus, we have constructed $\prod_{i=1}^{N}(2s_i+1)$ different eigenvectors of the modified creation operator with different eigenvalues. As the vectors are independent and the representation is finite (being of dimension $\prod_{i=1}^{N}(2s_i+1)$), this implies that the modified creation operator has simple spectrum and it is invertible. Thus, we have

$$F(z)(\nu_{12}(z))^{-1}|Y(\bar{k})\rangle = \frac{\kappa^-}{\mu(\rho_1 + \rho_2)}\Lambda_{12}(z, \bar{k})|Y(\bar{k})\rangle. \tag{2.160}$$

Therefore, $F(z)(\nu_{12}(z))^{-1}$ is a polynomial in z of degree z^S.

Observe that the modified creation operator

$$\nu_{12}(z) = \mathrm{Tr}(V_{12}T(z)), \qquad V_{12} = \mu\begin{pmatrix}\frac{\rho_1}{\kappa^-} & \frac{\rho_1\rho_2}{(\kappa^-)^2}\\ 1 & \frac{\rho_2}{\kappa^-}\end{pmatrix} \tag{2.161}$$

is a null twisted transfer matrix, because $\mathrm{Det}(V_{12}) = 0$. Since we can consider ρ_i, μ, and κ^- as free parameters in our construction, the result applies to any null twisted transfer matrix.

2.3.3 Properties of Modified Izergin Determinant

In this section we give a list of properties of the modified Izergin determinant introduced in section 2.2.1. In all the propositions listed below $\bar{u}$ and $\bar{v}$ are two sets of arbitrary complex numbers with cardinalities $\#\bar{u} = n$ and $\#\bar{v} = m$.

Basic properties

Proposition 2.3.1.

$$\begin{aligned} K^{(z)}_{n,m}(\bar{u} - c|\bar{v}) &= K^{(z)}_{n,m}(\bar{u}|\bar{v} + c), \\ \overline{K}^{(z)}_{n,m}(\bar{u} - c|\bar{v}) &= \overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v} + c). \end{aligned} \tag{2.162}$$

$$K^{(z)}_{n,m}(-\bar{u}| - \bar{v}) = \overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v}). \tag{2.163}$$

Proof. These formulas directly follow from (2.68)–(2.71) and the definition of the rational functions (2.63). □

Proposition 2.3.2.

$$K^{(z)}_{n,0}(\bar{u}|\emptyset) = \overline{K}^{(z)}_{n,0}(\bar{u}|\emptyset) = 1, \qquad K^{(z)}_{0,n}(\emptyset|\bar{v}) = \overline{K}^{(z)}_{0,n}(\emptyset|\bar{v}) = (1 - z)^n, \tag{2.164}$$

$$\begin{aligned} K^{(z)}_{1,m}(u|\bar{v}) &= (1 - z)^{m-1}\big(f(u, \bar{v}) - z\big), \\ K^{(z)}_{n,1}(\bar{u}|v) &= f(\bar{u}, v) - z, \\ \overline{K}^{(z)}_{1,m}(u|\bar{v}) &= (1 - z)^{m-1}\big(f(\bar{v}, u) - z\big), \\ \overline{K}^{(z)}_{n,1}(\bar{u}|v) &= f(v, \bar{u}) - z. \end{aligned} \tag{2.165}$$

Proof. These formulas directly follow from (2.68)–(2.71). □

Proposition 2.3.3.

$$K^{(z)}_{n+1,m+1}(\{\bar{u}, w - c\}|\{\bar{v}, w\}) = -zK^{(z)}_{n,m}(\bar{u}|\bar{v}). \tag{2.166}$$

$$\overline{K}^{(z)}_{n+1,m+1}(\{\bar{u}, w + c\}|\{\bar{v}, w\}) = -z\overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v}). \tag{2.167}$$

Proof. We use representation (2.68). We see that only the term $-z\delta_{m+1,k}$ survives in the last row of the determinant due to $f(w - c, w) = 0$. Then we obtain

$$\begin{aligned} K^{(z)}_{n+1,m+1}(\{\bar{u}, w - c\}|\{\bar{v}, w\}) &= -z \det_m \left(-z\delta_{jk} + \frac{f(\bar{u}, v_j)f(w - c, v_j)f(v_j, \bar{v}_j)f(v_j, w)}{h(v_j, v_k)}\right) \\ &= -z \det_m \left(-z\delta_{jk} + \frac{f(\bar{u}, v_j)f(v_j, \bar{v}_j)}{h(v_j, v_k)}\right) = -zK^{(z)}_{n,m}(\bar{u}|\bar{v}), \end{aligned} \tag{2.168}$$

because $f(w - c, v_j)f(v_j, w) = 1$ due to (2.66). Equation (2.167) then follows from the replacement $c \to -c$. □

Proposition 2.3.4.

$$\begin{aligned} K^{(z)}_{n,m}(\bar u|\bar v) &= \sum_{\bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}} (-z)^{\#\bar v_{\rm II}} f(\bar u,\bar v_{\rm I})f(\bar v_{\rm I},\bar v_{\rm II}),\\ \overline{K}^{(z)}_{n,m}(\bar u|\bar v) &= \sum_{\bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}} (-z)^{\#\bar v_{\rm II}} f(\bar v_{\rm I},\bar u)f(\bar v_{\rm II},\bar v_{\rm I}). \end{aligned} \tag{2.169}$$

Here the sum is taken over all partitions $\bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}$.

$$\begin{aligned} K^{(z)}_{n,m}(\bar u|\bar v) &= (1-z)^{m-n}\sum_{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}} (-z)^{\#\bar u_{\rm I}} f(\bar u_{\rm II},\bar v)f(\bar u_{\rm I},\bar u_{\rm II}),\\ \overline{K}^{(z)}_{n,m}(\bar u|\bar v) &= (1-z)^{m-n}\sum_{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}} (-z)^{\#\bar u_{\rm I}} f(\bar v,\bar u_{\rm II})f(\bar u_{\rm II},\bar u_{\rm I}). \end{aligned} \tag{2.170}$$

Here the sum is taken over all partitions $\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}$.

Proof. Expanding the determinant (2.68) over diagonal minors we find

$$\begin{aligned} &\det_n\left(\frac{f(v_j,\bar v_j)f(\bar u,v_j)}{h(v_j,v_k)}-z\delta_{jk}\right)\\ &\quad=(-z)^n+\sum_{s=1}^n(-z)^{n-s}\sum_{1\le j_1<\dots<j_s\le n}\left(\prod_{p=1}^s f(v_{j_p},\bar v_{j_p})f(\bar u,v_{j_p})\right)\det_s\frac{1}{h(v_{j_i},v_{j_k})}. \end{aligned} \tag{2.171}$$

The determinant in the right hand side is the Cauchy determinant, hence,

$$\det_s\frac{1}{h(v_{j_i},v_{j_k})}=\prod_{\substack{p,q=1\\p\ne q}}^s\frac{1}{f(v_{j_p},v_{j_q})}. \tag{2.172}$$

Thus, we obtain

$$\begin{aligned} &\det_n\left(\frac{f(v_j,\bar v_j)f(\bar u,v_j)}{h(v_j,v_k)}-z\delta_{jk}\right)\\ &\quad=(-z)^n+\sum_{s=1}^n(-z)^{n-s}\sum_{1\le j_1<\dots<j_s\le n}\left(\prod_{p=1}^s f(v_{j_p},\bar v_{j_p})f(\bar u,v_{j_p})\right)\prod_{\substack{p,q=1\\p\ne q}}^s\frac{1}{f(v_{j_p},v_{j_q})}. \end{aligned} \tag{2.173}$$

This is exactly the sum over partitions given by the first equation (2.169). The second equation (2.169) then follows by means of the replacement $c\to-c$. Equations (2.170) can be proved exactly in the same manner starting from the representation (2.69). □

Proposition 2.3.5.

$$\overline{K}^{(z)}_{n,m}(\bar u|\bar v)=(1-z)^{m-n}K^{(z)}_{m,n}(\bar v|\bar u) \tag{2.174}$$

Proof. Replacing $\bar{u} \leftrightarrow \bar{v}$ and $n \leftrightarrow m$ in (2.170) we obtain

$$(1-z)^{m-n} K^{(z)}_{m,n}(\bar{v}|\bar{u}) = \sum_{\bar{v}\Rightarrow\{\bar{v}_{\rm I},\bar{v}_{\rm II}\}} (-z)^{\#\bar{v}_{\rm I}} f(\bar{v}_{\rm II},\bar{u}) f(\bar{v}_{\rm I},\bar{v}_{\rm II}). \tag{2.175}$$

Comparing this expansion with the second equation (2.169) we see that they coincide up to the labels of the subsets. □

Proposition 2.3.6.

$$K^{(z)}_{n,m}(\bar{u}|\bar{v}+c) = \frac{(-z)^n(1-z)^{m-n}}{f(\bar{v},\bar{u})} K^{(1/z)}_{m,n}(\bar{v}|\bar{u}). \tag{2.176}$$

$$\overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v}-c) = \frac{(-z)^n(1-z)^{m-n}}{f(\bar{u},\bar{v})} \overline{K}^{(1/z)}_{m,n}(\bar{v}|\bar{u}). \tag{2.177}$$

Proof. Using (2.68) we obtain

$$K^{(z)}_{n,m}(\bar{u}|\bar{v}+c) = \det_m\left(-z\delta_{jk} + \frac{f(v_j,\bar{v}_j)}{f(v_j,\bar{u})h(v_j,v_k)}\right) = \frac{(-z)^m}{f(\bar{v},\bar{u})}\det_m\left(\delta_{jk}f(v_j,\bar{u}) - \frac{1}{z}\frac{f(v_j,\bar{v}_j)}{h(v_j,v_k)}\right). \tag{2.178}$$

On the other hand, using (2.69) for $K^{(1/z)}_{m,n}(\bar{v}|\bar{u})$ we obtain

$$K^{(1/z)}_{m,n}(\bar{u}|\bar{v}) = \left(1-\tfrac{1}{z}\right)^{n-m}\det_m\left(\delta_{jk}f(v_j,\bar{u}) - \frac{1}{z}\frac{f(v_j,\bar{v}_j)}{h(v_j,v_k)}\right). \tag{2.179}$$

Comparing (2.178) and (2.179) we arrive at (2.176). Equation (2.177) follows from the replacement $c \to -c$. □

Proposition 2.3.7. *The function $K^{(z)}_{n,m}(\bar{u}|\bar{v})$ has poles at $u_j = v_k$. The residue at $u_n = v_m$ is given by*

$$\begin{aligned} K^{(z)}_{n,m}(\bar{u}|\bar{v})\Big|_{u_n\to v_m} &= g(u_n,v_m)f(\bar{u}_n,u_n)f(v_m,\bar{v}_m)K^{(z)}_{n-1,m-1}(\bar{u}_n|\bar{v}_m) + reg,\\ \overline{K}^{(z)}_{n,m}(\bar{u}|\bar{v})\Big|_{u_n\to v_m} &= g(v_m,u_n)f(u_n,\bar{u}_n)f(\bar{v}_m,v_m)\overline{K}^{(z)}_{n-1,m-1}(\bar{u}_n|\bar{v}_m) + reg, \end{aligned} \tag{2.180}$$

where reg means regular part.

Proof. It is clear that the two equations (2.180) are related by the replacement $c \to -c$. To prove the first equation we use (2.69). Then for $u_n = v_m$ the pole occurs only in the matrix element $\delta_{nk}f(u_n,\bar{v})$. The determinant reduces to the product of this element and the corresponding minor:

$$\begin{aligned} K^{(z)}_{n,m}(\bar{u}|\bar{v})\Big|_{u_n\to v_m} &= (1-z)^{m-n}g(u_n,v_m)f(v_m,\bar{v}_m)\\ &\times \det_{n-1}\left(\delta_{jk}f(u_j,\bar{v}_m)f(u_j,v_m) - z\frac{f(u_j,\bar{u}_{j,n})f(u_j,u_n)}{h(u_j,u_k)}\right) + reg, \end{aligned} \tag{2.181}$$

where $\bar{u}_{j,n} = \bar{u} \setminus \{u_j, u_n\}$. We see that for $u_n = v_m$ we can extract the factor $f(u_j, u_n)$ form the j-th row of the matrix. Thus,

$$K^{(z)}_{n,m}(\bar{u}|\bar{v})\Big|_{u_n\to v_m} = (1-z)^{m-n} g(u_n, v_m) f(v_m, \bar{v}_m) f(\bar{u}_n, u_n) \\ \times \det_{n-1}\left(\delta_{jk} f(u_j, \bar{v}_m) - z\frac{f(u_j, \bar{u}_{j,n})}{h(u_j, u_k)}\right) + reg, \quad (2.182)$$

which ends the proof. □

Summation formulas

Proposition 2.3.8. *Let $\bar{\xi}$, $\bar{u}$, and $\bar{v}$ be sets of arbitrary complex numbers such that $\#\bar{\xi} = l$, $\#\bar{u} = n$, and $\#\bar{v} = m$. Then*

$$\begin{aligned} \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} z_2^{l_{\rm I}} K^{(z_1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}) K^{(z_2)}_{m,l_{\rm II}}(\bar{v}|\bar{\xi}_{\rm II}) f(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I}) f(\bar{u}, \bar{\xi}_{\rm II}) &= K^{(z_1 z_2)}_{n+m,l}(\{\bar{u}, \bar{v}\}|\bar{\xi}),\\ \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} z_2^{l_{\rm I}} \overline{K}^{(z_1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}) \overline{K}^{(z_2)}_{m,l_{\rm II}}(\bar{v}|\bar{\xi}_{\rm II}) f(\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}) f(\bar{\xi}_{\rm II}, \bar{u}) &= \overline{K}^{(z_1 z_2)}_{n+m,l}(\{\bar{u}, \bar{v}\}|\bar{\xi}). \end{aligned} \quad (2.183)$$

Here $l_{\rm I} = \#\bar{\xi}_{\rm I}$ and $l_{\rm II} = \#\bar{\xi}_{\rm II}$. The sums are taken with respect to all partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}\}$. There is no restriction on the cardinalities of the subsets.

Proof. It is clear that the two equations (2.183) are related by the replacement $c \to -c$. To prove the first equation we use (2.169):

$$\begin{aligned} K^{(z_1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}) &= \sum_{\bar{\xi}_{\rm I}\Rightarrow\{\bar{\xi}_1,\bar{\xi}_2\}} (-z_1)^{l_2} f(\bar{u}, \bar{\xi}_1) f(\bar{\xi}_1, \bar{\xi}_2),\\ K^{(z_2)}_{m,l_{\rm II}}(\bar{v}|\bar{\xi}_{\rm II}) &= \sum_{\bar{\xi}_{\rm II}\Rightarrow\{\bar{\xi}_3,\bar{\xi}_4\}} (-z_2)^{l_4} f(\bar{v}, \bar{\xi}_3) f(\bar{\xi}_3, \bar{\xi}_4). \end{aligned} \quad (2.184)$$

Here we use Arabic numbers for numeration the subsets. The corresponding cardinalities are $l_i = \#\bar{\xi}_i$, $i = 1, 2, 3, 4$. Thus, $l_{\rm I} = l_1 + l_2$ and $l_{\rm II} = l_3 + l_4$. Denoting the left hand side of the first equation (2.183) by Λ we obtain

$$\Lambda = \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3,\bar{\xi}_4\}} (-1)^{l_4} z_2^{l_1+l_2+l_4} (-z_1)^{l_2} f(\bar{u}, \bar{\xi}_1) f(\bar{v}, \bar{\xi}_3) f(\bar{\xi}_1, \bar{\xi}_2) f(\bar{\xi}_3, \bar{\xi}_4) \\ \times f(\bar{u}, \bar{\xi}_3) f(\bar{u}, \bar{\xi}_4) f(\bar{\xi}_3, \bar{\xi}_1) f(\bar{\xi}_3, \bar{\xi}_2) f(\bar{\xi}_4, \bar{\xi}_1) f(\bar{\xi}_4, \bar{\xi}_2). \quad (2.185)$$

Setting $\{\bar{\xi}_4, \bar{\xi}_1\} = \bar{\xi}_0$ we find

$$\Lambda = f(\bar{u}, \bar{\xi}) \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_0,\bar{\xi}_2,\bar{\xi}_3\}} z_2^{l_0+l_2} (-z_1)^{l_2} \frac{f(\bar{v}, \bar{\xi}_3)}{f(\bar{u}, \bar{\xi}_2)} f(\bar{\xi}_0, \bar{\xi}_2) f(\bar{\xi}_3, \bar{\xi}_0) f(\bar{\xi}_3, \bar{\xi}_2) \\ \times \sum_{\bar{\xi}_0\Rightarrow\{\bar{\xi}_1,\bar{\xi}_4\}} (-1)^{l_4} f(\bar{\xi}_4, \bar{\xi}_1). \quad (2.186)$$

It was proved in [59] that for any set of variables $\bar{x}$ such that $\#\bar{x} = p$ the following identity holds:

$$\sum_{\substack{\bar{x}\Rightarrow\{\bar{x}_{\rm I},\bar{x}_{\rm II}\}\\ \#\bar{x}_{\rm I}=k}} f(\bar{x}_{\rm II},\bar{x}_{\rm I}) = \sum_{\substack{\bar{x}\Rightarrow\{\bar{x}_{\rm I},\bar{x}_{\rm II}\}\\ \#\bar{x}_{\rm I}=k}} f(\bar{x}_{\rm I},\bar{x}_{\rm II}) = \binom{p}{k}. \tag{2.187}$$

Here the sum is taken over partitions $\bar{x} \Rightarrow \{\bar{x}_{\rm I},\bar{x}_{\rm II}\}$ such that the cardinality of the subset $\bar{x}_{\rm I}$ is fixed by $\#\bar{x}_{\rm I} = k$, $k \le p$. Applying this result to the sum over partitions $\bar{\xi}_0 \Rightarrow \{\bar{\xi}_1,\bar{\xi}_4\}$ we see that this sum vanishes if $\bar{\xi}_0 \neq \emptyset$:

$$\sum_{\bar{\xi}_0\Rightarrow\{\bar{\xi}_1,\bar{\xi}_4\}} (-1)^{l_4} f(\bar{\xi}_4,\bar{\xi}_1) = \sum_{l_4=0}^{l_0}(-1)^{l_4}\binom{l_0}{l_4} = (1-1)^{l_0}. \tag{2.188}$$

Thus, we obtain

$$\Lambda = \sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_2,\bar{\xi}_3\}} (-z_1z_2)^{l_2} f(\bar{v},\bar{\xi}_3)f(\bar{u},\bar{\xi}_3)f(\bar{\xi}_3,\bar{\xi}_2) = K^{(z_1z_2)}_{n+m,l}(\{\bar{u},\bar{v}\}|\bar{\xi}), \tag{2.189}$$

due to (2.169).

Replacing $\bar{\xi}$ by $\bar{\xi} \pm c$ and setting $z_1 = z_2 = 1$ in (2.183) we obtain

$$\begin{aligned} &\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} K^{(1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}+c)K^{(1)}_{m,l_{\rm II}}(\bar{v}|\bar{\xi}_{\rm II}+c)\frac{f(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{f(\bar{\xi}_{\rm II},\bar{u})} = K^{(1)}_{n+m,k}(\{\bar{u},\bar{v}\}|\bar{\xi}+c),\\ &\sum_{\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}} \overline{K}^{(1)}_{n,l_{\rm I}}(\bar{u}|\bar{\xi}_{\rm I}-c)\overline{K}^{(1)}_{m,l_{\rm II}}(\bar{v}|\bar{\xi}_{\rm II}-c)\frac{f(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})}{f(\bar{u},\bar{\xi}_{\rm II})} = \overline{K}^{(1)}_{n+m,k}(\{\bar{u},\bar{v}\}|\bar{\xi}-c). \end{aligned} \tag{2.190}$$

These formulas were used in sections 2.2.3 and 2.2.3.

□

Proposition 2.3.9. *Let $\bar{u}$ and $\bar{v}$ be sets of arbitrary complex numbers such that $\#\bar{u} = n$ and $\#\bar{v} = m$. Then*

$$\begin{aligned} &\sum_{\bar{v}\Rightarrow\{\bar{v}_{\rm I},\bar{v}_{\rm II}\}} z_1^{l_{\rm II}} K^{(z_2)}_{n,l_{\rm I}}(\bar{u}|\bar{v}_{\rm I})f(\bar{v}_{\rm I},\bar{v}_{\rm II}) = K^{(z_2-z_1)}_{n,m}(\bar{u}|\bar{v}),\\ &\sum_{\bar{v}\Rightarrow\{\bar{v}_{\rm I},\bar{v}_{\rm II}\}} z_1^{l_{\rm II}} \overline{K}^{(z_2)}_{n,l_{\rm I}}(\bar{u}|\bar{v}_{\rm I})f(\bar{v}_{\rm II},\bar{v}_{\rm I}) = \overline{K}^{(z_2-z_1)}_{n,m}(\bar{u}|\bar{v}). \end{aligned} \tag{2.191}$$

Here $l_{\rm II} = \#\bar{v}_{\rm II}$. The sums are taken with respect to all partitions $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\bar{v}_{\rm II}\}$. There is no restriction on the cardinalities of the subsets.

Proof. Obviously, the two equations (2.191) are related by the replacement $c \to -c$, therefore, we prove only the first equation. Let

$$\mathcal{L} = \sum_{\bar{v}\Rightarrow\{\bar{v}_{\rm I},\bar{v}_{\rm II}\}} z_1^{l_{\rm II}} K^{(z_2)}_{n,l_{\rm I}}(\bar{u}|\bar{v}_{\rm I})f(\bar{v}_{\rm I},\bar{v}_{\rm II}). \tag{2.192}$$

Using (2.169) we obtain

$$\mathcal{L}=\sum_{\bar v\Rightarrow\{\bar v_{\rm i},\bar v_{\rm ii},\bar v_{\rm II}\}} z_1^{l_{\rm II}}(-z_2)^{l_{\rm ii}}f(\bar u,\bar v_{\rm i})f(\bar v_{\rm i},\bar v_{\rm ii})f(\bar v_{\rm i},\bar v_{\rm II})f(\bar v_{\rm ii},\bar v_{\rm II}). \tag{2.193}$$

Here $l_{\rm II}=\#\bar v_{\rm II}$, $l_{\rm ii}=\#\bar v_{\rm ii}$, and the sum is taken with respect to all partitions $\bar v\Rightarrow\{\bar v_{\rm i},\bar v_{\rm ii},\bar v_{\rm II}\}$. Setting $\bar v_0=\{\bar v_{\rm ii},\bar v_{\rm II}\}$ and $l_0=\#\bar v_0$ we find

$$\mathcal{L}=\sum_{\bar v\Rightarrow\{\bar v_{\rm i},\bar v_0\}} z_1^{l_0}f(\bar u,\bar v_{\rm i})f(\bar v_{\rm i},\bar v_0)\sum_{\bar v_0\Rightarrow\{\bar v_{\rm ii},\bar v_{\rm II}\}}\Big(-\tfrac{z_2}{z_1}\Big)^{l_{\rm ii}}f(\bar v_{\rm ii},\bar v_{\rm II}). \tag{2.194}$$

Here we first have the sum over partitions $\bar v\Rightarrow\{\bar v_{\rm i},\bar v_0\}$ and then the subset $\bar v_0$ is divided once more as $\bar v_0\Rightarrow\{\bar v_{\rm ii},\bar v_{\rm II}\}$. Using (2.187) we find

$$\sum_{\bar v_0\Rightarrow\{\bar v_{\rm ii},\bar v_{\rm II}\}}\Big(-\tfrac{z_2}{z_1}\Big)^{l_{\rm ii}}f(\bar v_{\rm ii},\bar v_{\rm II})=\sum_{l_{\rm ii}=0}^{l_0}\Big(-\tfrac{z_2}{z_1}\Big)^{l_{\rm ii}}\binom{l_0}{l_{\rm ii}}=\Big(1-\tfrac{z_2}{z_1}\Big)^{l_0}. \tag{2.195}$$

Substituting this result into (2.194) we immediately arrive at

$$\mathcal{L}=\sum_{\bar v\Rightarrow\{\bar v_{\rm i},\bar v_0\}}(z_1-z_2)^{l_0}f(\bar u,\bar v_{\rm i})f(\bar v_{\rm i},\bar v_0)=K_{n,m}^{(z_2-z_1)}(\bar u|\bar v), \tag{2.196}$$

due to (2.169). □

2.3.4 Commutation Relations of the $t_{ij}(u)$ and $\nu_{ij}(u)$

The RTT relation (1.50) yields to the following commutation relations:

$$[t_{ij}(u),t_{kl}(v)]=g(u,v)\big(t_{kj}(v)t_{il}(u)-t_{kj}(u)t_{il}(v)\big). \tag{2.197}$$

In particular,

$$t_{ij}(u)t_{ij}(v)=t_{ij}(v)t_{ij}(u),\qquad \forall i,j, \tag{2.198}$$
$$t_{11}(u)t_{12}(v)=f(v,u)t_{12}(v)t_{11}(u)+g(u,v)t_{12}(u)t_{11}(v), \tag{2.199}$$
$$t_{22}(u)t_{12}(v)=f(u,v)t_{12}(v)t_{22}(u)+g(v,u)t_{12}(u)t_{22}(v), \tag{2.200}$$
$$[t_{21}(u),t_{12}(v)]=g(u,v)\big(t_{11}(v)t_{22}(u)-t_{11}(u)t_{22}(v)\big). \tag{2.201}$$

In turn, commutation relations (2.199) and (2.200) imply the following multiple commutation relations [51]:

$$\begin{aligned}t_{11}(\bar u)t_{12}(\bar v)&=(-1)^n\sum_{\#\bar w_{\rm I}=n}\overline{K}_n(\bar u|\bar w_{\rm I}+c)f(\bar w_{\rm II},\bar w_{\rm I})t_{12}(\bar w_{\rm II})t_{11}(\bar w_{\rm I}),\\ t_{22}(\bar u)t_{12}(\bar v)&=(-1)^n\sum_{\#\bar w_{\rm I}=n}K_n(\bar u|\bar w_{\rm I}+c)f(\bar w_{\rm I},\bar w_{\rm II})t_{12}(\bar w_{\rm II})t_{22}(\bar w_{\rm I}).\end{aligned} \tag{2.202}$$

Here $\#\bar u=n$, $\#\bar v=m$, $\bar w=\{\bar u,\bar v\}$, and K_n is the Izergin determinant. The sums are taken over partitions $\bar w\Rightarrow\{\bar w_{\rm I},\bar w_{\rm II}\}$ such that $\#\bar w_{\rm I}=n$.

The same commutation relations are valid for the modified operators $\nu_{ij}(u)$.

2.3.5 Symmetries of the Yangian

Consider a mapping

$$\phi(T(u)) = T^{\tau}(-u), \tag{2.203}$$

where τ is the diagonal transposition $A^{\tau} = \sigma^1 A^t \sigma^1$ and $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. It defines an automorphism of the Yangian of $\mathfrak{gl}_2$ [94]. This automorphism allows us to find the action of $\nu_{22}(u)$ on the modified Bethe vector knowing those for $\nu_{11}(u)$

$$\phi(\nu_{11}(u)\nu_{12}(\bar{v})|0\rangle) = \nu_{22}(-u)\nu_{12}(-\bar{v})|0\rangle. \tag{2.204}$$

Here we have to apply the following prescriptions: $\phi(\lambda_i(u)) = \lambda_{3-i}(-u)$ and $\phi(\beta_i) = \beta_{3-i}$.

Let us consider as an example the action (2.89). Applying the mapping ϕ to this equation we obtain

$$\nu_{22}(-\bar{u})\nu_{12}(-\bar{v})|0\rangle = \beta_1^n \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}} (-\beta_1)^{-l}\lambda_2(-\bar{w}_{\rm I})\overline{K}^{(1)}_{n,l}(\bar{u}|\bar{w}_{\rm I}-c)f(\bar{w}_{\rm II},\bar{w}_{\rm I})\nu_{12}(-\bar{w}_{\rm II})|0\rangle. \tag{2.205}$$

Changing $\bar{u} \to -\bar{u}$ and $\bar{v} \to -\bar{v}$ we arrive at

$$\nu_{22}(\bar{u})\nu_{12}(\bar{v})|0\rangle = \beta_1^n \sum_{\bar{w}\Rightarrow\{\bar{w}_{\rm I},\bar{w}_{\rm II}\}} (-\beta_1)^{-l}\lambda_2(\bar{w}_{\rm I})\overline{K}^{(1)}_{n,l}(-\bar{u}|-\bar{w}_{\rm I}-c)f(\bar{w}_{\rm I},\bar{w}_{\rm II})\nu_{12}(\bar{w}_{\rm II})|0\rangle. \tag{2.206}$$

Finally, using (2.163) we reproduce equation (2.90).

Chapter 3

The Inverse Functional Problem

The two previous chapters, in different contexts and approaches, have mainly been devoted to solving the spectral problem, namely constructing the Bethe states of different systems. When approaching dynamical problems, the Bethe basis will play a central role, as its elements evolve trivially through time (in the continuous time limit)

$$\sum_{\bar{u}} \alpha_{\bar{u}} |\psi(\bar{u})\rangle \longrightarrow \sum_{\bar{u}} \alpha_{\bar{u}} |\psi(\bar{u})\rangle e^{it\epsilon(\bar{u})} \tag{3.1}$$

where the sum runs over the Bethe roots $\bar{u}$, and $\epsilon(\bar{u})$ is the eigenvalue corresponding to the state $|\psi(\bar{u})\rangle$. In the case of a discrete time evolution (as for the ZCM in Section 3.1), the time dependence is slightly different but still trivially equivalent at that stage. Where the spectral problem has been addressed, one can approach dynamical problems, and in particular quench problems. This is the subject of this chapter.

Let us consider a system whose dynamics (in the discrete time limit) is governed by an operator H - roughly speaking, an Hamiltonian. Mathematically speaking, a quench schematically consists on considering a particular state of the system which is not eigenstate of our H (e.g. filling the left part of a chain with particles and leaving the right part empty), and then see how this state evolves with time. In more "experimentalist" words, one consider the system prepared in a initial state, the steady state of an initial Hamiltonian H_0 (obtained by letting the system relax with asymptotic time), brutally tune the system such as described by a new Hamiltonian H, and then observe how the system evolves with time. This sudden tuning, keeping the system away from any adiabatic evolution, can for instance consists on brutally switching off a magnetic field initially constraining some atoms in particular regions of space. Studying quench is thus a matter of the vast domain of non equilibrium physics, which is a very active and prolific field of research in experimental and theoretical physics.

In this context, we will consider initial states that trivially expresses in the spin basis (or position basis), what we shall call a "spin state" without regards to the physical content of our system. Knowing the decomposition of any spin state in the Bethe basis, i.e. knowing the coefficient $\alpha_{\bar{u}}$ in (3.1), one trivially obtains the expression of the state after time t.

The spin basis flavor will be obviously favored in the context of the coordinate Bethe ansatz, where we obtained explicit expressions for the Bethe states in the spin basis (see (3.6) for XXX and (1.42) for the ZCM),

$$|\psi(\bar{u})\rangle = \sum_{\bar{x}} \psi(\bar{x}|\bar{u})\,|\bar{x}\rangle \tag{3.2}$$

where the set $\bar{x}$ are the set of positions (positions of the magnons for XXX and positions of the particles in the ZCM), and if we want to consider initial states that are defined in the position basis, $|\psi_0\rangle = \sum_{\bar{x}} C_{\bar{x}}\,|\bar{x}\rangle$, which is for instance the case when considering to fill with particles only a half of a lattice, we would then naturally need to express the position basis states in term of Bethe states

$$|\bar{x}\rangle = \sum_{\bar{u}} \mu(\bar{x}|\bar{u})\,|\psi(\bar{u})\rangle \tag{3.3}$$

where the $\bar{k}$ again are Bethe roots, such that we could eventually express our initial state in terms of Bethe states as $|\psi_0\rangle = \sum_{\bar{k}} \alpha(\bar{k})\,|\psi(\bar{k})\rangle$.
It is clear now that considering the decomposition of Bethe states (3.2), the problem of finding the coefficients $\mu(\bar{x})$ in (3.3) would reduce to what we shall here call the Inverse Functional Problem, namely find the measure μ such that holds the equation

$$\sum_{\bar{u}} \mu(\bar{y}|\bar{u})\psi(\bar{x}|\bar{u}) = \delta_{\bar{x},\bar{y}} \tag{3.4}$$

The existence of such a measure would be guarantied if the considered set of Bethe states is complete. The completeness problem can in turn be related with a very close but significantly nonequivalent problem, which is to express the so called resolution of the identity, expressed

$$\sum_{\bar{u}} \tilde{\mu}(\bar{u})\psi^{\star}(\bar{y}|\bar{u})\psi(\bar{x}|\bar{u}) = \delta_{\bar{x},\bar{y}} \tag{3.5}$$

where $\psi^{\star}$ is the wave function for the dual vector, i.e. the left eigenstate of the Hamiltonian.
Obviously, the knowledge of $\tilde{\mu}$ implies the knowledge of μ, but also implies that the set of Bethe states is complete. Indeed, as indicates its name, the resolution of the identity (3.5) is equivalent to the expression $\mathbb{I}_M = \sum_{\bar{u}} \tilde{\mu}(\bar{u})\,|\psi(\bar{u})\rangle\,\langle\psi(\bar{u})|$ for the identity in the M particles sector, hence the completeness. Looking at this latter expression, one may be tempted to think the problem to be trivially solved by $\tilde{\mu}(\bar{u}) = \langle\psi(\bar{u})|\psi(\bar{u})\rangle^{-1}$. In the case of a system of finite volume would raise the Bethe equation, and hence a discrete set of Bethe roots, this would effectively be the case. The problem in this case is that Bethe equations are very complicated to solve, which makes the sum over Bethe roots highly hypothetical. In the case of a system of infinite volume, the Bethe equation vanish, and the sum would turn into an integral, and with it the emergence of a measure which can be very far from trivial. Most of all, the resolution of the identity can turn to be a very welcome tool in problems of scalar products. Schematically, a scalar product of two

arbitrary states $\langle\phi_1|\phi_2\rangle$ can be rewritten $\sum_{\bar{u}} \tilde{\mu}(\bar{u}) \langle\phi_1|\psi(\bar{u})\rangle \langle\psi(\bar{u})|\phi_2\rangle$. The scalar product in the first expression may be very tricky, but in the second each term of the sum involves the scalar product with an on-shell Bethe state, for which many tools are available, i.e. one can reduce a very complicated problem to many simpler problem.

This chapter is organized as follows. Section 3.1 is devoted to the study of quench problem in the zero-range chipping Model with factorized steady state on the infinite lattice, with an initial state where particles are homogeneously distributed in on one side of the lattice. We first obtain in Section 3.1.1 the first building block required to approach the problem, which is the Inverse Functional Problem. Then is proved in Section 3.1.2 a very compact expression for the leftmost particle, i.e. the sum of the probability amplitudes of a Bethe state over one half of the lattice. Those two ingredients, put together, provide an expression for the transition probability from a state initially homogeneously spread on the right side of the lattice, after a time t, toward a position basis state, given in Section 3.1.3. Finally, in Section 3.1.3, we discuss the transition probability from step initial condition to its symmetrical step final condition, which is proved to express as a Fredholm determinant. This problem however, albeit interesting, is absolutely non-physical.
In Section 3.2, we prove an expression for the resolution of the identity in the XXZ infinite chain with continuous spin. In Section 3.2.4 is considered the half-integer spin case, and then in 3.2.5 the result is obtain for the XXX spin chain as a limit.
As we will see, in both these context, the half integer regime (reached by analytic continuation) will produce a combinatorial sum over terms corresponding to bound states of the system. Very intriguingly, in the context of IFP for the ZCM with half-integer spin value, the length of the corresponding string are constrained by the value of the spin.

Both those calculations are inspired by the calculation for the propagator for the δ-Bose gas [30] of S. Prolhac and H. Spohn. The idea borrowed to the authors is to express the IFP in a spin regime in which the expression is almost trivial (typically the negative spin regime, in which no bound state survive). The half-integer spin regime is then reached by analytic continuation. During this procedure, the contours are shrunk, grabbing poles, and producing strings of constrained parameter. These correspond to bound states of the system. Albeit this scheme apply to the three computations, they all require a particular treatment, as they all have their own particularities.

3.1 Toward Quench Problems in Zero-range Chipping Model with Factorized Steady States

The development exposed in this Section aims the obtaining, in the context of the Zero-range Chipping Model with factorized steady state (ZCM) with half integer spin, see Section 1.1.2, of a compact expression for the transition probability from a state initially homogeneously distributed on one side of the lattice, after a time t, to a position basis state. The problem of quench from step pinitial condition has been adressed by B. Derrida and A. Gerschenfeld [7] for the Symmetric Simple Exclusion Process (SSEP). This problem seems in this case to be greatly simplified by the symmetry exhibited by the

dynamics in this context. Later on, C. A. Tracy and H. Widom in [19] approached the problem for the Asymmetric Simple Exclusion Process (ASEP), where is demonstrated an expression for the probability transition for step initial condition.
This section, motivated by a research project still running with V. Pasquier on quench problems in ZCM, is the result of my personal work, but didn't led to a publication yet. We followed here the approach adopted by C. A. Tracy and H. Widom for ASEP, and aim to express the transition probability for the m^{th} particle to be found above the origin given that all the particles where initially homogeneously spread below the origin in the form of a Fredholm determinant. However, the problem significantly complexifies in the ZCM, as bound states are to appear during our manipulations. Hence made more painful to reached in our context, the results exposed in this thesis unfortenatly didn't reached the m^{th} particle case yet, but are limited to the case of the 1^{st} particle. The starting point of our reflection, namely the IFP for the non half-integer spin regime, can already be found in the literature [95].

This Section is organized as follows. In Section 3.1.1 is given the first ingredient of the reflection, which is an expression for the probability amplitude for finding the system at $\bar{x}$ at a time t, given it was prepared at $\bar{y}$ at $t = 0$. Since time evolution is governed by the monodromy matrix, we will need to express this amplitude as a linear combination of its eignenstates (the Bethe states), that is to say to solve the inverse functional problem

$$\mathbb{P}(\bar{x}|\bar{y}, t) = \int d\bar{u} F(\bar{u}, \bar{y}, t)\psi(\bar{x}|\bar{u})$$

where the measure $F(\bar{u}, \bar{y}, t)$, to be found, here also carrying the time dependence and imposes the initial condition $\mathbb{P}(\bar{x}|\bar{y}, t = 0) = \delta_{\bar{x},\bar{y}}$. Since the time dependence is trivial, $F(\bar{u}, \bar{y}, t) = F(\bar{u}, \bar{y}, 0)\Lambda(\bar{u})^t$, we see that this problem, setting $t = 0$, actually simply reduces to the Inverse Functional Problem (3.4), here in integral form.
First is given in Lemma 3.1.1 a simple expression for the IFP in the regime away from half integer spin. Then, we reach by analytic continuation the half integer spin regime. This continuation procedure, which produces new terms corresponding to the contribution of bound states, leads to theorem 3.1.4, which gives an expression (3.18) for $\mathbb{P}(\bar{x}|\bar{y}, t)$ as a sum over integrals of bound states. This theorem clearly exhibit the symmetry between the initial and the final state for the IFP, which is rather obviously physically understandable by a simple gedanken experiment of reversion of the time flow. As evoked before, the main idea of analytic continuation procedure is inspired by the computation of the propagator for the δ-Bose gas [30], but some additional complications will have to be handle here. Then in Section 3.1.2 is proven a very compact expression for the leftmost particle, what we call the summation Lemma, which the second ingredient of our computations, namely the probability for a Bethe state to be found above a certain point. These to results are trivially combined in Section 3.1.3 to obtained the transition probability from step initial and final condition. This latter expression is in turn carefully transformed in Section 3.1.3 into a Fredholm determinant, dressed with an integral over an additional parameter.

A quick recap about the objects in play here. We consider the Bethe states described in Section 1.1.2 in the context of the zero-range chipping models with factorized steady states,

$$\psi(\bar{x}|\bar{u}) = C_{\bar{n}[\bar{x}]} \sum_{P \in S_N} A_P(\bar{u}) \prod_i z(u_{Pi})^{x_i} \tag{3.6}$$

where we define the Bethe amplitudes and phases

$$A_P(\bar{u}) = \prod_{1 \leq i < j \leq N} \frac{q u_{Pi} - u_{Pj}}{u_{Pi} - u_{Pj}}, \quad z(u) = \frac{u - s}{u - s^{-1}} \tag{3.7}$$

and the occupation number coefficient

$$C_{\bar{n}[\bar{x}]} = \prod_{i \in \mathcal{L}} \frac{(s^2; q)_{n_i[\bar{x}]} (1-q)^{n_i[\bar{x}]}}{(q;q)_{n_i[\bar{x}]}}, \quad n_i[\bar{x}] = \#\{x \in \bar{x}, x = i\}, \ i \in \mathcal{L} \tag{3.8}$$

with the usual q-factorial

$$(a;q)_k = \begin{cases} \prod_{i=0}^{k-1} (1 - aq^i) & \text{for } k > 0 \\ 1 & \text{for } k = 0 \end{cases} \tag{3.9}$$

The amplitudes (3.6) describe eigenstates of the monodromy matrix (1.12), without restriction over the set of parameter given that we consider an infinite system (i.e. no Bethe equations involved). These are associated to the eigenvalue

$$\Lambda(\bar{u}) = \prod_{i=1}^{M} \left(\frac{u_i - \mu s^{-1}}{u_i - s} \right) \tag{3.10}$$

The half integer spin $n/2$, $n \in \mathbb{N}$, here corresponds to the relation $s^2 q^n = 1$, for which at most n particles can stay in one site.

3.1.1 The Initial Spin Condition: Inverse Functional Problem

The aim here is to solve the IFP for the ZCM with half integer spin. As we will see, this problem is far from trivial and require careful analytic manipulations. In the main text are only given the different Lemmas, the main steps of the technical reasoning eventually leading us to the main Theorem 3.1.4, and with these some keys to understand what is the idea behind the maths. The proofs of these are given in the appendix.

We first solve the IFP in the generic non-half integer spin regime. This problem, summarized in the following Lemma, has already been addressed in the literature [95]. As this was not to my knowledge when I wrote this thesis, I provide here my personal proof. In this case, we can consider adequately nested contours so that the integral reads very simply. These contour prevent the different variables to interact with each others. Such a simple expression can be straightforwardly reached in the negative-half integer spin regime, i.e. for $s^2 q^{-n} = 1$, but this case won't be of any interest in our reflection. The following lemma will be the starting point toward the initial spin condition in the half integer spin case, which is the matter of this subsection.

Lemma 3.1.1. *For $s^2(qe^{i\epsilon})^n = 1$, $\epsilon \neq 0$, and $\bar{x}, \bar{y} \in \mathbb{Z}^N$ two sets of N ordered integers, $x_{i+1} \geq x_i$, $y_{i+1} \geq y_i$, we have*

$$\prod_{i=1}^{N} \oint_{\mathcal{C}_i} \frac{du_i}{2i\pi} \mu(u_i)\psi(\bar{x}|\bar{u})\mathcal{M}(\bar{y}|\bar{u}) = \delta_{\bar{x},\bar{y}} \tag{3.11}$$

where

$$\mathcal{M}(\bar{y}|\bar{u}) = A_{id}^{-1}(\bar{u}) \prod_i z(u_i)^{-y_i} \tag{3.12}$$

$$\mu(u_i) = \frac{1}{s(u_i - s)(u_i - s^{-1})} \tag{3.13}$$

and $\mathcal{C}_i$ encircle s and $q\mathcal{C}_i \subset \mathcal{C}_{i+1}$, these contours being as shrunk as necessary such that $s^{-1} \not\subset \mathcal{C}_i$.

This Lemma is proved in the Appendix, see Section 3.1.4.

The next step toward the half integer spin regime is to shrink all the contours of our integrals around a single point. During this subtle procedure, that has to be run with caution, some poles linking the different variables will be grabbed, then producing strings of particles. These correspond to constraints of the form $u_j = qu_i$, $i < j$.

Lemma 3.1.2. *For $s^2(qe^{i\epsilon})^n = 1$, $\epsilon \neq 0$, and $\bar{x}, \bar{y} \in \mathbb{Z}^N$ two sets of N ordered integers, $x_{i+1} \geq x_i$, $y_{i+1} \geq y_i$, we have*

$$\delta_{\bar{x},\bar{y}} = \sum_{N \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}^<} \prod_{i=1}^{k} \oint_s \frac{du_{\alpha_{i,1}}}{2i\pi} \tag{3.14}$$

$$\times \left[\mu(u_i)\psi(\bar{x}|\bar{u})\mathcal{M}(\bar{y}|\bar{u}) \prod_{i=1}^{k} \prod_{j=1}^{\#\bar{\alpha}_i - 1} (u_{\alpha_{i,j+1}} - qu_{\alpha_{i,j}}) \right]_{u_{\alpha_{i,j+1}} = qu_{\alpha_{i,j}}} \tag{3.15}$$

where $N \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}^<$ are the ordered partitions of $\{1, \cdots, N\}$: $\bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset$, $\cup_i \bar{\alpha}_i = \{1, \cdots, N\}$, $\alpha_{i,1} < \alpha_{i+1,1}$ and $\alpha_{i,j} < \alpha_{i,j+1}$.

This Lemma is proved in the Appendix, see Section 3.1.4.
Note that so far, the spin regime has been kept unchanged, i.e. it remained in the non-half integer spin regime. We only proceed to manipulate the contours of our integrals, that made the bound states appear. Also notice that the length of these strings are not constrained.

We proceed to the analytic continuation to reach the regime $s^2q^n = 1$.

Lemma 3.1.3. *For $s^2q^n = 1$, and $\bar{x}, \bar{y} \in \mathbb{Z}^N$ two sets of N ordered integers, $x_{i+1} \geq x_i$, $y_{i+1} \geq y_i$,*

$$\delta_{\bar{x},\bar{y}} = \sum_{N \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}_n^<} \prod_{i=1}^{k} \oint_s \frac{du_{\alpha_{i,1}}}{2i\pi} \tag{3.16}$$

$$\times \left[\mu(u_i)\psi(\bar{x}|\bar{u})\mathcal{M}(\bar{y}|\bar{u}) \prod_{i=1}^{k} \prod_{j=1}^{\#\bar{\alpha}_{i-1}} (u_{\alpha_{i,j+1}} - qu_{\alpha_{i,j}}) \right]_{u_{\alpha_{i,j+1}} = qu_{\alpha_{i,j}}} \tag{3.17}$$

where $N \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}_n^<$ are the ordered partitions of $\{1, \cdots, N\}$ of module lower than n: $\bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset$, $\cup_i \bar{\alpha}_i = \{1, \cdots, N\}$, $\alpha_{i,1} < \alpha_{i+1,1}$ and $\alpha_{i,j} < \alpha_{i,j+1}$, $\#\bar{\alpha}_i \leq n$ $\forall i$.

This Lemma is proved in the Appendix, see Section 3.1.4.
As we can see in this Lemma, the analytic continuation to the half-integer spin regime $s^2q^n = 1$ leads to an upper bound for the length of the strings involved in the IFP. Indeed, as we can see in the proof, the contribution for the strings of length greater than n will systematically vanishes, whatever the other string configuration are. Here, as in the previous Lemmas, the sum over bound states is made through a sum over the (now bound) partitions of the ensemble of variables.

We are now going to re-express the IFP obtained in the previous Lemma, such that providing us with a convenient expression for the reflection to come latter (namely step initial and final conditions). The aim of this last tricky procedure for the IFP is to re-expression the sum over bound states as a sum simply over integer, rather than a sum over partition themselves. This imply to "symmetrise" the integrand of Lemma 3.1.3.

Theorem 3.1.4. *For $\bar{x}, \bar{y} \in \mathbb{Z}^N$ two sets of N ordered integers, $x_{i+1} \geq x_i$, $y_{i+1} \geq y_i$, $t \in \mathbb{N}$ and $s^2q^n = 1$, the probability to find the system in the configuration $\bar{x}$ at time t given it was initially at $\bar{y}$ is given by*

$$\mathbb{P}(\bar{x}|\bar{y}, t) = \sum_{\bar{p}} \frac{1}{(\#\bar{p})!} \prod_{i=1}^{\#\bar{p}} \oint_s du_{a_i} \tag{3.18}$$

$$\times \left[\mu(\bar{u}) A_{id}^{-1}(\bar{u}) A_{\Pi}^{-1}(\bar{u}) \Lambda(\bar{u})^t \psi(\bar{x}|\bar{u}) \psi(\bar{w}|\bar{u}) \prod_{j \notin \{a_i\}}^{k} (u_j - qu_{j-1}) \right]_{u_j = qu_{j-1},\ j \notin \{a_i\}}$$

where the ensembles $\bar{p}$ are such that $1 \leq p_i \leq n$, $\sum_i p_i = N$, $a_i = 1 + \sum_{j=1}^{i-1} p_i$, and we accordingly define $w_i \equiv -y_{\Pi i}$.

This Theorem is proved in the Appendix, see Section 3.1.4.
Schematically speaking, the sum over the partition will be translated as a sum over the string length combined with a sum over permutations of the particles. This latter one can be inserted into the integrand. This new sum over permutation allows us to make appear another wave function in the integrand, depending on the initial state only, the absolutely symmetric partner of the first introduced wave function depending on the final

state only. As we will see in Section 3.1.3, this expression will appear as very convenient for the problem of initial and/or final condition. Still miss, however, a last ingredient, the leftmost particle, which is the matter of the next section.

3.1.2 The Leftmost Particle: the Summation Lemma

We now want to express the probability, considering a Bethe state, for the leftmost particle to be found at the right of some point m.

Lemma 3.1.5. *For $s, q \in \mathbb{C}\backslash\{1\}$, $\bar{v} \in \mathbb{C}^N$ such that $|z(v_i)| < 1$ $\forall i$, and $m \in \mathbb{Z}$, we have*

$$\sum_{x_{i+1} \geq x_i \geq m} \psi(\bar{x}|\bar{v}) = \prod_{i=1}^{N}(1 - sv_i) \prod_{i=1}^{N} z(v_i)^m \tag{3.19}$$

This Theorem is proved in the Appendix, see Section 3.1.4, which is obtained by taking the limit $\bar{u} \to \infty$ of equation (3.44) in Lemma 3.1.10. This limit however has to be taking with caution.

It is obvious that this result and the IFP obtained in the previous section can be straightforwardly combined to obtain the probability transition from step initial condition.

3.1.3 The Probability Transition for Step Initial and Final Condition as a Fredholm Determinant

We are now going to combine the IFP expressed in Theorem 3.1.4 and the summation lemma 3.1.5. As we evoqued before, the expression we obtained for the IFP exhibit a trivial symmetry between the initial and the final state, as two wave functions, each depending on one of these, explicitly emerged. We can thus apply the summation Lemma twice, on the initial and final state, and obtain the probability transition, at time t from step initial condition to step final condition, namely the probability for all the particle to be found above a certain point given they all where below the origin at $t = 0$.

Remark 3.1.1. *We are going now to consider the double sum*

$$\sum_{x_{i+1} \geq x_i \geq a} \sum_{y_i \leq y_{i+1} \leq b} \mathbb{P}(\bar{x}|\bar{y}, t)$$

The first of these sum, which corresponds to consider the probability to find the first particle of the system above the site a, has a sensible physical meaning. In turn, the second sum can not be interpreted from the probabilistic point of view, hence loosing its physical relevance.
Indeed, in our context, a state defined by its amplitudes $\varphi(\bar{y})$ will be considered as physical if the two conditions $\varphi(\bar{y}) \in \mathbb{R}^+$ $\forall \bar{y}$ and $\sum_{y_{i+1} \geq y_i} \varphi(\bar{y}) = 1$ are satisfied. This latter condition, the normalization condition, can also be expressed $\langle \Omega | \varphi \rangle = 1$, where $\langle \Omega |$, the left eigenvector of the transition matrix of eigenvalue 1, has all its components in the position basis equal to 1. This guaranty the conservation of probability through time,

$\partial_t \langle\Omega|\varphi_t\rangle = \partial_t \langle\Omega| \mathbf{M}^t |\varphi\rangle = 0$. *In other words, the family of physical states is closed under the action of the transition matrix.*
Let us now come back to our problem. By the sum $\sum_{y_i \le y_{i+1} \le b} \mathbb{P}(\bar{x}|\bar{y}, t)$ *we consider the initial state* $|\varphi\rangle = \sum_{y_i \le y_{i+1} \le b} C_{\bar{n}[\bar{y}]} |\bar{y}\rangle$, *which can in no way be renormalized such that satisfying the normalization condition* $\langle\Omega|\varphi\rangle = \sum_{y_i \le y_{i+1} \le b} C_{\bar{n}[\bar{y}]} = 1$. *This initial state has thus no physical relevance.*
The following results are however given as their demonstration could be technically helpful for further developments.

Lemma 3.1.6. *For* $q > 1$, $s^2 q^n = 1$,$\frac{q(1+s^2)}{2} > 1$ *and* $t \in \mathbb{N}$, *we have*

$$\tilde{\mathbb{P}}(\bar{x} \ge a|\bar{y} \le (a-b-1), t) = s^{-2N} \sum_{\bar{p}} \frac{1}{(\#\bar{p})!} \prod_{i=1}^{\#\bar{p}} \oint_s du_{a_i} \tag{3.20}$$

$$\times \left[A_{id}^{-1}(\bar{u}) A_{\Pi}^{-1}(\bar{u}) \Lambda(\bar{u})^t z(\bar{u})^b \prod_{j \notin \{a_i\}} (u_j - q u_{j-1}) \right]_{u_j = q u_{j-1},\ j \notin \{a_i\}} \tag{3.21}$$

where we defined

$$\tilde{\mathbb{P}}(\bar{x} \ge a|\bar{y} \le b, t) = \sum_{x_{i+1} \ge x_i \ge a} \sum_{y_i \le y_{i+1} \le b} \mathbb{P}(\bar{x}|\bar{y}, t)$$

and where Π *is the mirror permutation* $\Pi i = N - i + 1$.

Proof. First of all, one can check that for $\frac{q(1+s^2)}{2} > 1$ and u close enough to s, we have $\left|\frac{uq^k - s}{uq^k - s^{-1}}\right| < 1$ for $k = 0, \cdots, n-1$. We can insert the sum over $\bar{x}$ and $\bar{y}$ inside the brackets in (3.18), remarking that $\sum_{y_i \le y_{i+1} \le m} = \sum_{w_{i+1} \ge w_i \ge -m}$. Now, we straightforwardly apply, twice in a row, lemma 3.1.5 to the definition of $\psi(\bar{x}|\bar{u})$ and $\psi(\bar{w}|\bar{u})$, hence the Lemma. □

This expression, after careful manipulation, can be re-expressed as a dressed Fredholm determinant. These object are appreciated since they exhibited very nice behavior when considering some limits of the problem, as for instance the asymptotic time limit [19].

Theorem 3.1.7. *For* $q > 1$, $s^2 q^n = 1$,$\frac{q(1+s^2)}{2} > 1$ *and* $t \in \mathbb{N}$, *we have*

$$\lambda^N \tilde{\mathbb{P}}_N(\bar{x} \ge a|\bar{y} \le a-b-1, t) = \oint_0 \frac{dw}{w^{N+1} q^{N^2/2}} \det\left[\mathbb{I} + \mathbb{K}_{\lambda w}^{[t,b]}\right] \tag{3.22}$$

where $\mathbb{K}_x^{[t,m]}$ *is defined by its kernel*

$$K_x^{[t,m]}(u, v) = \sum_{k=1}^{n} \frac{u(1-q^k)}{u - vq^k} \Lambda_k^t(u) z_k^m(u) \frac{q^{k^2/2-k+1}}{(q;q)_k} x^k \tag{3.23}$$

$$\Lambda_k^t(u) = \prod_{i=0}^{k-1} \Lambda(uq^i)^t \tag{3.24}$$

$$z_k^m(u) = \prod_{i=0}^{k-1} z(uq^i)^m \tag{3.25}$$

This theorem is proved in the appendix, see Section 3.1.4.
For the spin value $1/2$, this dressing (namely the integral) wouldn't be needed here, and one could easily recover the formula for ASEP [19].

3.1.4 Appendix

Proof of Lemma 3.1.1.

Proof. For each permutation $P \in S_N$, we define a graph $\Gamma(P)$ as the ensemble of labeled points $i = 1, \cdots, N$, where two points i and j will be linked if and only if $\{i,j\}_P$[1]. If two ensemble of points are disconnected, the two sets of integral corresponding to these points will decouple (since a coupling between u_a and u_b appears from $A^{-1}_{id}(\bar{u})A_{P^{-1}}(\bar{u}) = \prod_{\{i,j\}_P} \frac{u_i - qu_j}{qu_i - u_j}$ if and only if $\{a,b\}_P$ or $\{b,a\}_P$). More precisely, a connected sub-graph of $\Gamma(P)$ will correspond to an ensemble of points G which is stable by permutation P, i.e. such that $P : G \mapsto G$.
During the following development, for a fixed permutation, we may identify an ensemble of points with its corresponding sub-graph, as a matter of simplicity.

• We are going first to prove that for any permutation P^{-1}, the corresponding contribution in the lhs of (3.11) won't vanish only for $x_{P^{-1}i} = y_i \; \forall i$.

Consider a permutation P and a connected sub-graph of $\Gamma(P)$ (as the integrals factorize over connected sub-graphs of $\Gamma(P)$, we treat each of them independently).
If it consists on a unique point i (which implies $P^{-1}i = i$), the corresponding term is proportional to

$$\oint_{\mathcal{C}_i} \frac{du_i}{(u_i - s)(su_i - 1)} \left(\frac{u_i - s}{u_i - s^{-1}}\right)^{x_i - y_i}$$

which is non zero only for $x_i = y_i$.
Otherwise, if the connected sub-graph contains more than one point, we define j_{min} a minimal point of this sub-graph (i.e. there is no j such that $\{j, j_{min}\}_P$), and j_{max} a maximal link of j_{min} (i.e. $\{j_{min}, j_{max}\}_P$, and there is no j such that $\{j_{max}, j\}_P$). Note that these points may not be unique, but exist given that the sub-graph contains more that one point. If $x_{P^{-1}j_{min}} - y_{j_{min}} > 0$, the only poles in $u_{j_{min}}$ are at s^{-1} and possibly at $q^{-1}u_j$, $j > j_{min}$, i.e. outside the integration circle (given the definition of the contours) and so the integral vanishes. Similarly, if $x_{P^{-1}j_{max}} - y_{j_{max}} < 0$, the only poles for $z_{j_{max}}$ are inside the integration circle (note that there is no pole at infinity) and so the integral vanishes as well. Then, since $x_{P^{-1}j_{min}} - y_{j_{min}} \geq x_{P^{-1}j_{max}} - y_{j_{max}}$ (given the ordering of $\bar{x}$ and $\bar{y}$), the integral won't vanish only for $x_{P^{-1}j_{min}} - y_{j_{min}} = x_{P^{-1}j_{max}} - y_{j_{max}} = 0$.
Since for any j_{min} there is at least one linked j_{max} (and the other way round), we know that for any extremal point j_{ext}, $x_{P^{-1}j_{ext}} - y_{j_{ext}} = 0$.
Now if j is not an extremal point, we know by definition that there exist j_{min} and j_{max} such that $\{j_{min}, j\}_P$ and $\{j, j_{max}\}_P$, but then $x_{P^{-1}j_{min}} - y_{j_{min}} \geq x_{P^{-1}j} - y_j \geq x_{P^{-1}j_{max}} - y_{j_{max}}$,

[1] We define this internal binary relation in $\{1, \cdots, N\}^{\otimes 2}$ as $\{i,j\}_P \Leftrightarrow i < j,\; P^{-1}i > P^{-1}j$. Remark that this relation is transitive, i.e. $\{i,j\}_P$ *and* $\{j,k\}_P \Rightarrow \{i,k\}_P$.

and so $x_{P^{-1}j_{min}} - y_{j_{min}} = 0 = x_{P^{-1}j_{max}} - y_{j_{max}} \Rightarrow x_{P^{-1}j} - y_j = 0$.

We thus proved that for any permutation P, the corresponding contribution does not vanish only for $x_{P^{-1}i} = y_i, \ \forall i$.

• We now prove that a non zero contribution implies $x_i = y_i, \forall i$.
First remember that P factorizes over connected sub-graphs: for G the points corresponding to a connected sub-graph of $\Gamma(P)$, i.e. $P : G \mapsto G$, we just proved that $\forall i \in G, \ x_{P^{-1}i} = y_i$. Then, for $i, j \in G$ such that $\{i, j\}_P$, given the ordering of $\bar{x}$ and $\bar{y}$, we have $x_{P^{-1}i} \geq x_{P^{-1}j} = y_j \geq y_i = x_{P^{-1}i}$, i.e. $x_{P^{-1}i} = x_{P^{-1}j} = y_i = y_j \ \forall i, j$. The subgraph G being connected and stable by P^{-1}, we have $x_i = y_j, \ \forall i, j \in G$, hence this bullet point proved.

• At last, we assume $x_i = y_i \ \forall i$, and compute the corresponding value of the integral in the lhs of (3.11). As we just saw, only permutations P such that $P^{-1} : G \mapsto G$, with $x_i = y_j \ \forall i, j \in G$, will contribute. Then, the integral will factorizes over these G. We describe our spin configuration by the set $\bar{n}[\bar{x}] = \{\#\{x \in \bar{x}, x = i\}\}_{\ i \in \mathcal{L}}$. The (non zero) elements of $\bar{n}[\bar{x}]$ then also are the cardinal of the sets G described above.
The resulting term can be written

$$C_{\bar{n}[\bar{x}]} \prod_{l \in \mathcal{L}} \left[\prod_{i=1}^{n_l[\bar{x}]} \oint_{\mathcal{C}_{(i,l)}} \frac{du_i}{2i\pi} \mu(u_i) A_{id}^{-1}(\bar{u}) \sum_{P \in S_{k_l}} A_P(\bar{u}) \right] \tag{3.26}$$

where the contours here have a slightly different labeling, but still the same nesting properties: $(i, l) = \sum_{j=1}^{l-1} n_j[\bar{x}] + i$ (i.e. $q\mathcal{C}_{(i,l)} \subset \mathcal{C}_{(i+1,l)}, \ s \in \mathcal{C}_{(i,l)}$ and $s^{-1} \notin \mathcal{C}_{(i,l)}$).
So, using Lemma 3.1.9, we get

$$C_{\bar{n}[\bar{x}]} \prod_{l \in \mathcal{L}} \left[\frac{(q;q)_{k_l}}{(1-q)^{k_l}} \prod_{i=1}^{n_l[\bar{x}]} \oint_{\mathcal{C}_{(i,l)}} \frac{du_i}{2i\pi(u_i - s)(su_i - 1)} \prod_{i<j} \frac{u_i - u_j}{qu_i - u_j} \right] \tag{3.27}$$

For each $l \in \mathcal{L}$, we integrated over u_i, beginning from $i = 1$ up to $i = k_l$, grabbing for each variables the poles inside the integration contour, i.e. at $u_i = s$. After integrating p times, we obtain

$$\prod_{i=1}^{n_l[\bar{x}]} \oint_{\mathcal{C}_{(i,l)}} \frac{du_i}{2i\pi(u_i - s)(su_i - 1)} \prod_{i<j} \frac{u_i - u_j}{qu_i - u_j} = \tag{3.28}$$

$$\frac{1}{(s^2;q)_p} \prod_{i=p+1}^{n_l[\bar{x}]} \oint_{\mathcal{C}_{(i,l)}} \frac{du_i}{2i\pi(u_i - s)(sq^p u_i - 1)} \prod_{p<i<j} \frac{u_i - u_j}{qu_i - u_j} \tag{3.29}$$

which gives, for $p = n_l[\bar{x}] \ \forall l \in \mathcal{L}$,

$$C_{\bar{n}[\bar{x}]} \prod_{l\in\mathcal{L}} \frac{(q;q)_{k_l}}{(s^2;q)_{k_l}(1-q)^{k_l}} = 1 \tag{3.30}$$

which ends the proof of the Lemma. □

The following lemma will be useful in the demonstration of Lemma 3.1.2.

Lemma 3.1.8.

$$A_{id}^{-1}(\bar{u})\psi(\bar{x}|\bar{u}) = \prod_{i<j}(u_i - u_j)G(\bar{x}|\bar{u}) \tag{3.31}$$

where $G(\bar{x}|\bar{u})$ is a rational function regular at $u_i = u_j$ $\forall i, j$.

Proof. We write

$$A_{id}^{-1}(\bar{u})\psi(\bar{x}|\bar{u}) = \prod_{i<j} \frac{1}{qu_i - u_j} \sum_P \sigma(P) \prod_{i<j}(qu_{Pi} - u_{Pj}) \prod_i z(u_{Pi})^{x_i} \tag{3.32}$$

It is obvious that the sum over permutation is anti-symmetric in the u_i, and hence proportional to $\prod_{i<j}(u_i - u_j)$, and so is $A_{id}^{-1}(\bar{u})\psi(\bar{x}|\bar{u})$. □

Proof of Lemma 3.1.2.

Proof. The idea to prove this lemma is to shrink the contours $\mathcal{C}_j$ around s from $j = 1$ up to $j = N$. Doing so, the variable u_j will grab some poles at $u_j = qu_i$, $i < j$. We will then obtain a sum over configuration of links of the form $u_{i_m} = qu_{i_{m-1}} = \cdots = q^m u_{i_0}$, $i_j < i_{j+1}$, what we call a string configuration. The final result is obtain by recurrence over l:

$$\delta_{\bar{x},\bar{y}} = \sum_{l\vdash\{\bar{\alpha}_1,\cdots,\bar{\alpha}_k\}^<} \prod_{i=1}^{k} \oint_s \frac{du_{\alpha_{i,1}}}{2i\pi} \prod_{j=l+1}^{N} \oint_{\mathcal{C}_j} \frac{du_j}{2i\pi} \tag{3.33}$$

$$\times \left[\mu(u_i)\psi(\bar{x}|\bar{u})\mathcal{M}(\bar{y}|\bar{u}) \prod_{i=1}^{k} \prod_{j=1}^{\#\bar{\alpha}_{i-1}} (u_{\alpha_{i,j+1}} - qu_{\alpha_{i,j}})\right]_{u_{\alpha_{i,j}}=qu_{\alpha_{i,j}}} \tag{3.34}$$

where $l \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}^<$ are the ordered partitions of $\{1, \cdots, l\}$: $\bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset$, $\cup_i \bar{\alpha}_i = \{1, \cdots, N\}$, $\alpha_{i,1} < \alpha_{i+1,1}$ and $\alpha_{i,j} < \alpha_{i,j+1}$.

Assume the previous expression true for $l - 1$. Let us see what happens when we shrink the contour for u_l as $\mathcal{C}_l \to s$. Doing so we will cross all the poles inside $\mathcal{C}_l$ except the pole at s.

Consider a fixed spin configuration $(l-1) \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}$. The dependence in u_l is given by

$$\frac{1}{(u_l - s)(su_l - 1)} \sum_P \prod_{\{i,l\}_P} \frac{qu_l - u_i}{u_l - qu_i} \prod_{\{l,j\}_P} \frac{qu_j - u_l}{u_j - qu_l} \left(\frac{u_l - s}{u_l - s^{-1}} \right)^{x_{P^{-1}l} - y_l} \Bigg|_{u_{\alpha_{i,j+1}} = qu_{\alpha_{i,j}}} \tag{3.35}$$

Given the definition of the contours, $q\mathcal{C}_k \subset \mathcal{C}_j,\ j > k$, we know there will be no interaction with the u_j for $j > k$ during the contour shrinking. We are thus interested in all the (potential) poles at $u_k = qu_i,\ i < k$, with $i \in \bar{\alpha}_m = \{\alpha_1, \cdots, \alpha_p\}$ for some m. If $i \neq \alpha_p$, we already have the link $u_i = q^{-1}u_j,\ j \in \bar{\alpha}_m,\ j > i$, and the corresponding pole for u_l is given by $\frac{1}{z_l - z_j}$. From lemma 3.1.8, we know that this pole is actually suppressed. Then there only remains the pole at $z_l = tz_{\alpha_p}$. In other words, z_l will possibly only get linked with the end of previously constructed strings. The obtained sum corresponds to the case l of (3.33), and the lemma is obtained for $l = N$. □

Proof of Lemma 3.1.3.

Proof. First of all, if there are more than n particle per site (i.e. there is some i for which $n_i[\bar{x}] > n$), the occupation coefficient will tend to zero with the analytic continuation, in which case the lemma is straightforwardly obtained from the previous one. We then now assume that there are less than n particle per site.

As we will show, only the partitions $N \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}$ with $\#\bar{\alpha}_i \leq n\ \forall i$ will survive from (3.1.2) after the analytic continuation in t to its final real value: $s^2(qe^{i\epsilon})^n = 1 \to s^2q^n = 1$.
Note that during all the procedure, the modulus of t remains greater than one, so that no poles from terms of the form $\frac{1}{u - qu'}$ would cross the contours. We are thus only interested in terms of the form $\frac{1}{s^{\pm 1} + q^k u}$, where u is integrated over around s.

Let's consider a fixed partition $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}$. It is important to mention first that only permutations P such that P^{-1} inverts the ordering inside $\bar{\alpha}_i$ will survive. Indeed, if $P^{-1}\alpha_{a,b} > P^{-1}\alpha_{a,b+1}$ (remember that $\alpha_{a,b} < \alpha_{a,b+1}$ by definition of the ordered partitions), the corresponding term appears in $\prod_{\{i,j\}_P} \frac{qu_j - u_i}{u_j - qu_i}$, and so the residue at $u_{\alpha_{a,b+1}} = qu_{\alpha_{a,b}}$ is non zero. Otherwise there is no pole at that point, so the corresponding residue is zero. We thus assume $P\alpha_{i,j} > P\alpha_{i,j+1}\ \forall i,\ j, j+1 \in \bar{\alpha}_i$ from now on. The other terms will be of the form $\frac{1}{q^b u_{\alpha_{a,1}} - s^{\pm 1}}$, with $b = 0, \cdots, \#\bar{\alpha}_a - 1$, with $u_{\alpha_{i,1}}$ closely integrated around s. Before the analytic continuation (i.e. for $s^2(qe^{i\epsilon})^n = 1$), only the poles at $u_{\alpha_{a,1}} = s$ are inside integration contour, while after (i.e. for $s^2q^n = 1$), the poles at $u_{\alpha_{a,1}} = s^{-1}q^n$ also are (which would only appear for $\#\bar{\alpha}_a > n$). Then no pole will cross the contours during the analytic continuation if $\#\bar{\alpha}_a \leq n\ \forall a$, in which case the lemma trivially follows.

We thus now assume that there exists at least one a such that $\#\bar{\alpha}_a \equiv p > n$. To simplify a bit our expressions, we define $\xi_{j-1} = x_{Pj} - y_j,\ j = 1, \cdots, p$. Note that $\xi_{j+1} \leq \xi_j$, and that in particular $\xi_0 > \xi_n + 1$, given we assumed the number of particle per site to be $\leq n$, and that P invert the order inside $\bar{\alpha}_a$. We also rename $z_{\alpha_{a,1}} = z$. In the integrated part the following term appears

$$\prod_{j=0}^{p-1} \frac{1}{(q^j u - s)s(q^j u - s^{-1})} \left(\frac{q^j u - s}{q^j z - s^{-1}} \right)^{\xi_j} \tag{3.36}$$

which in particular contains

$$(u - s)^{\xi_0 - 1}(q^n u - s^{-1})^{-(\xi_n+1)}. \tag{3.37}$$

We now integrate over u, picking the poles inside contour, i.e. at $u_{\alpha_{i,1}} = s$. For $\xi_0 > 0$, there is no pole at s, i.e. the residue is zero, and so the lemma trivially follows. Otherwise, for $\xi_0 \leq 0$, the residue is given by

$$\mathcal{R} = \frac{1}{(1-\xi_0)!} \partial_u^{-\xi_0} \left[(q^n u - s^{-1})^{-(\xi_n+1)} \mathcal{F}(\bar{u}) \right] \Big|_{u=s} \tag{3.38}$$

where $\mathcal{F}(\bar{z})$ is regular in $z = s$ (before and after the continuation). Then, given that $0 < -\xi_0 < -(\xi_n + 1)$ (since we assumed less than n particle per site), we know that at least one $(q^n u - s^{-1})$ will survive to the successive derivatives. We then end up with a term proportional to $(q^n u - s^{-1})|_{u=s}$, which tends to zero when analytically continuing q as $s^2(qe^{i\epsilon})^n = 1 \to s^2 q^n = 1$. This ends the proof of the lemma. □

Proof of Theorem 3.1.4.

Proof. Let us first introduce the time dependence that has to be considered here: Defining $\mathcal{M}^t(\bar{u}) = \mathcal{M}(\bar{u})\Lambda(\bar{u})^t$, carrying the time dependence for a Bethe state: $\psi(\bar{x}|\bar{u}, t) = \psi(\bar{x}|\bar{u})\Lambda(\bar{u})^t$, we know that for $\bar{x}, \bar{y} \in \mathbb{Z}^N$ two sets of N ordered integers, $x_{i+1} \geq x_i$, $y_{i+1} \geq y_i$, $t \in \mathbb{N}$ and $s^2 q^n = 1$, the probability to find the system in the configuration $\bar{x}$ at time t given it was initially at $\bar{y}$ is given by

$$\mathbb{P}(\bar{x}|\bar{y}, t) = C_{\bar{n}[\bar{y}]} \sum_{N \vdash \{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}_n^<} \frac{1}{k!} \prod_{i=1}^{k} \oint_s du_{\alpha_{i,1}} \tag{3.39}$$
$$\times \left[\mu(\bar{u})\psi(\bar{x}|\bar{u})\mathcal{M}_t(\bar{y}|\bar{u}) \prod_{i=1}^{k} \prod_{j=1}^{\#\bar{\alpha}_{i-1}} (u_{\alpha_{i,j+1}} - q u_{\alpha_{i,j}}) \right]_{u_{\alpha_{i,j+1}} = q u_{\alpha_{i,j}}}$$

such that at $t = 0$ we indeed recover Lemma 3.1.3 from (3.18), since $\mathbb{P}(\bar{x}|\bar{y}) = C_{\bar{n}[\bar{y}]}\delta_{\bar{x},\bar{y}}$.

We can now reparametrize the sum over string configurations as a sum over what we call *fundamental strings*, alongside a sum over permutations. In lemma 3.1.3, we considered two ordering conditions for the string configuration we will now get rid of:
• inside a string: we can see that if inside a string two successive terms are unordered, $\alpha_{i,j} > \alpha_{i,j+1}$, then the term $(u_{\alpha_{i,j+1}} - q u_{\alpha_{i,j}})$ won't be compensated from $\psi(\bar{x}|\bar{u})\mathcal{M}_t(\bar{y}|\bar{u})$. We can then ignore this first condition.
• The second ordering concerns the first elements of each strings, $\alpha_{i,1} < \alpha_{i+1,1}$. But given

that the $u_{\alpha_{i,1}}$ are integrated on the same contour, they can actually be exchanged. Summing over all of these new configuration introduce a redundancy of magnitude $\frac{1}{k!}$, with k the number of strings (i.e. the number of integrated particles).
• Now, remark that the sum over string configurations in (3.39) can be replaced by a sum over fundamental string configurations and a sum over permutation: $\sum_{N\vdash\{\bar{\alpha}_1,\cdots,\bar{\alpha}_k\}_n} \to \sum_{\bar{p}}\sum_P$, with $\sum_i p_i = N$, with the fundamental string configurations given by $\{\{1,\cdots,p_1\},\{p_1+1,\cdots,p_1+p_2\},\cdots\}$, and the one to one link with the general configuration made through

$$\begin{aligned}&\{\{\alpha_{1,1},\cdots,\alpha_{1,p_1}\},\{\alpha_{2,1},\cdots,\alpha_{2,p_2}\},\cdots\}\\=&\{\{P1,\cdots,Pp_1\},\{P(p_1+1),\cdots,P(p_1+p_2)\},\cdots\}.\end{aligned}$$

Under this parametrization (3.39) reads

$$\begin{aligned}&\mathbb{P}(\bar{x}|\bar{y},t) = \\&\sum_{\bar{p}}\sum_P \frac{1}{(\#\bar{p})!}\prod_{i=1}^{\#\bar{p}}\oint_s du_{Pa_i}\Bigg[C_{\bar{n}[\bar{y}]}A_\Pi(\bar{u})\prod_i z(u_i)^{-y_i} \\&\mu(\bar{u})A_{id}^{-1}(\bar{u})A_\Pi^{-1}(\bar{u})\Lambda^t(\bar{u})\psi(\bar{x}|\bar{u})\prod_{j\notin\{a_i\}}^{k}(u_{Pj}-qu_{P(j-1)})\Bigg]_{u_{Pj}=qu_{P(j-1)},\ j\notin\{a_i\}}\end{aligned} \tag{3.40}$$

Now, given the symmetry of the integration contours, we can proceed, for each P inside the sum over permutations, to the relabeling $u_i \to u_{P^{-1}i}$, and then re-parametrize the sum as $P^{-1} \to P\Pi$. Defining the ensemble $\bar{w}$ as $w_i = -y_{\Pi i}$, and noticing that the term $\mu(\bar{u})A_{id}^{-1}(\bar{u})A_\Pi^{-1}(\bar{u})\Lambda^t(\bar{u})\psi(\bar{x}|\bar{u})$ is symmetric under permutation over the u_i, we obtain

$$\begin{aligned}&\mathbb{P}(\bar{x}|\bar{y},t)\\=&\sum_{\bar{p}}\frac{1}{(\#\bar{p})!}\prod_{i=1}^{\#\bar{p}}\oint_s du_{a_i}\Bigg[C_{\bar{n}[\bar{y}]}\sum_P A_P(\bar{u})\prod_i z(u_{Pi})^{w_i} \\&\mu(\bar{u})A_{id}^{-1}(\bar{u})A_\Pi^{-1}(\bar{u})\Lambda^t(\bar{u})\psi(\bar{x}|\bar{u})\prod_{j\notin\{a_i\}}^{k}(u_j-qu_{j-1})\Bigg]_{u_j=qu_{j-1},\ j\notin\{a_i\}}\end{aligned} \tag{3.41}$$

where we recognize the expression for the Bethe amplitude

$$\psi(\bar{w}|\bar{u}) = C_{\bar{n}\sum_P[\bar{y}]}A_P(\bar{u})\prod_i z(u_{Pi})^{w_i}$$

which concludes the proof.

□

The two following lemmas will be useful for the forthcoming proof of Lemma 3.1.5.

Lemma 3.1.9. *For $\bar{u}$ a disjoint set of N complex numbers, and q a complex parameter, we have*

$$\sum_{P\in S_N} A_P(\bar{u}) = \frac{(q;q)_N}{(1-q)^N} \tag{3.42}$$

Proof. First of all, we can rewrite

$$X_N \equiv \sum_{P\in S_N} A_P(\bar{u}) = \frac{1}{\prod_{1\le i<j\le N}(u_i-u_j)} \sum_{P\in S_N} \sigma(P) \prod_{1\le i<j\le N} (qu_{Pi}-u_{Pj})$$

with $\sigma(P)$ the signature of P.

The sum over permutation being obviously anti-symmetric in $\bar{u}$, and considering its order in u_i, we know it is proportional to $\prod_{1\le i<j\le N}(u_i-u_j)$, and the whole expression is then a constant (i.e. independent on $\bar{u}$).

We can then consider the particular case $u_i = uq^i$. After relabeling $P^{-1} \to P\Pi$ (Π the mirror permutation: $\Pi i = N-i+1$), one get

$$X_N = \sum_{P\in S_N} \prod_{1\le Pi<Pj\le N} \frac{q^i - q^{j+1}}{q^i - q^j}$$

A permutation P will here contribute if and only if $P(j+1) > Pj$ $\forall j$, i.e. $P = id$. We can thus write

$$\begin{aligned} X_N = & \prod_{1\le i<j\le N} \frac{q^i - q^{j+1}}{q^i-q^j} \\ =& X_{N-1} q^{N-1} \prod_{n=1}^{N-1} \frac{q^{i-1}-q^N}{q^i - q^N} \\ =& X_{N-1} q^{N-1} \frac{1-q^N}{q^{N-1}-q^N} \\ =& X_{N-1} \frac{1-q^N}{1-q} \end{aligned} \tag{3.43}$$

A trivial recurrence from $X_1 = 1$ leads to $X_N = \frac{(q;q)_N}{(1-q)^N}$, hence the Lemma proved. □

Lemma 3.1.10. *For $(s,q) \in \mathbb{R}\times\mathbb{C}$ and $\bar{u}, \bar{v}$ two self-disjoint sets of complex parameter of cardinal N, such that $|\tilde{z}(u_i)\tilde{z}(v_j)| < 1$ $\forall i,j$, we have*

$$\prod_{i=1}^{N} \frac{1}{(su_i-1)(sv_i-1)} \sum_{x_i\ge x_{i+1}\ge 0} C_{\bar{n}[\bar{x}]} \tilde{\psi}(\bar{x}|\bar{v}) \tilde{\psi}(\bar{x}|\bar{u}) = q^{N(N-1)/2} \frac{D_N(\bar{u},\bar{v})}{\delta_N(\bar{u},\bar{v})} \tag{3.44}$$

where

$$\tilde{\psi}(\bar{x}|\bar{u}) \equiv \sum_{P\in S_N} A_{P\Pi}(\bar{u}) \prod_i \tilde{z}(u_{Pi})^{x_i} \tag{3.45}$$

$$\tilde{z}(u) \equiv \frac{u-s}{su-1} \tag{3.46}$$

$$\delta_N(\bar{u},\bar{v}) \equiv \det\left[\frac{1}{1-qu_iv_j}\right] \tag{3.47}$$

$$D_N(\bar{u},\bar{v}) \equiv \det\left[\frac{1}{(1-u_iv_j)(1-qu_iv_j)}\right] \tag{3.48}$$

Proof. This is proved in [17] (equation (157)). □

Proof of Lemma 3.1.5.

Proof. This lemma is obtained by taking the limit $\bar{u} \to \infty$ of equation (3.44) in Lemma 3.1.10. Note that the hypothesis $\left|\left(\frac{v_i-s}{sv_i-1}\right)\left(\frac{u_j-s}{su_j-1}\right)\right| < 1\ \forall i,j$ required in Lemma 3.1.10 turns into $\left|\frac{v_i-s}{s(sv_i-1)}\right| < 1\ \forall i$ as $u_i \to \infty$.

From Lemma 3.1.9 we obviously obtain

$$\tilde{\psi}(\bar{x}|\bar{u})\Big|_{\bar{u}\to\infty} = \frac{(q;q)_N}{(1-q)^N} \prod_i \left(\frac{1}{s}\right)^{x_i} \tag{3.49}$$

i.e.

$$\tilde{\psi}(\bar{x}|\bar{u})\tilde{\psi}(\bar{x}|\bar{v})\Big|_{\bar{u}\to\infty} = \frac{(q;q)_N}{(1-q)^N}\bar{\psi}(\bar{x}|\bar{v})$$

where

$$\bar{\psi}(\bar{x}|\bar{v}) = \sum_{P\in S_N} A_{P\Pi}(\bar{v}) \prod_i \left(\frac{v_{Pi}-s}{s(sv_{Pi}-1)}\right)^{x_i}$$

so that equation (3.44) rewrites

$$\frac{1}{\prod_{i=1}^N (sv_i-1)} \sum_{x_i \geq x_{i+1}\geq 0} \frac{(s^2;q)_{\bar{n}[\bar{x}]}}{(q;q)_{\bar{n}[\bar{x}]}}\bar{\psi}(\bar{x}|\bar{v}) = \prod_{i=1}^N (su_i-1)\frac{q^{N(N-1)/2}}{(q;q)_N}\frac{D_N(\bar{u},\bar{v})}{\delta_N(\bar{u},\bar{v})}\Bigg|_{\bar{u}\to\infty} \tag{3.50}$$

Now, using the factorized expression for the Cauchy determinant, we have

$$\delta_N(\bar{u},\bar{v}) = q^{N(N-1)/2}\frac{\prod_{i<j}(u_i-u_j)(v_i-v_j)}{\prod_{i,j}(1-qu_iv_j)} \tag{3.51}$$

$$\begin{aligned} D_N(\bar{u},\bar{v}) &= \det\left[\frac{1}{(1-u_iv_j)(1-qu_iv_j)}\right] \\ &= \frac{1}{(q-1)^N\prod_i u_iv_i}\det\left[\frac{1}{1-qu_iv_j} - \frac{1}{1-u_iv_j}\right] \\ &= \frac{1}{(q-1)^N\prod_i u_iv_i}\prod_i (1-\mathcal{D}_{v_i}^{-1})\delta_N(\bar{u},\bar{v}) \end{aligned} \tag{3.52}$$

where $\mathcal{D}_v$ is the shift operator defined by $\mathcal{D}_v f(v) = f(qv)$ (i.e. $\mathcal{D}_v = e^{ln(q)\partial_{ln(v)}}$). Using the identity $\prod_{i=1}^N (1-\mathcal{D}_{v_i}^{-1}) = \sum_{N\vdash\{I,II\}} (-)^{\#I} \prod_{i\in I} \mathcal{D}_{v_i}^{-1}$, where $N \vdash \{I,II\}$ are the 2-partitions of $\{1,\cdots,N\}$, we can write

$$\frac{\delta_N(\bar{u},\bar{v})}{D_N(\bar{u},\bar{v})} = \frac{1}{(q-1)^N \prod_i u_i v_i} \frac{\prod_{i,j}(1-qu_iv_j)}{\prod_{i<j}(u_i-u_j)(v_i-v_j)} \tag{3.53}$$

$$\times \sum_{N\vdash\{I,II\}} (-)^{\#I} \prod_{i\in I} \mathcal{D}_{v_i}^{-1} \frac{\prod_{i<j}(u_i-u_j)(v_i-v_j)}{\prod_{i,j}(1-qu_iv_j)}$$

$$= \frac{1}{(q-1)^N \prod_i u_i v_i} \sum_{N\vdash\{I,II\}} (-)^{\#I} q^{-\#I(\#I-1)/2} \tag{3.54}$$

$$\times \prod_{i\in\{I,II\}} \prod_{j\in I} \frac{1-qu_iv_j}{1-u_iv_j} \prod_{i\in I} \prod_{j\in II} \frac{q^{-1}v_i - v_j}{v_i - v_j}$$

and so

$$\prod_i (su_i - 1) \frac{\delta_N(\bar{u},\bar{v})}{D_N(\bar{u},\bar{v})} \Bigg|_{\bar{u}\to\infty}$$

$$= \frac{s^N}{(q-1)^N \prod_j v_j} \sum_{N\vdash\{I,II\}} (-)^{\#I} q^{-\#I(\#I-1)/2} \tag{3.55}$$

$$\times q^{N\#I} \prod_{i\in I} \prod_{j\in II} \frac{q^{-1}v_i - v_j}{v_i - v_j}$$

$$= \frac{s^N}{(q-1)^N \prod_j v_j} \sum_{N\vdash\{I,II\}} (-q^N)^{\#I} \tag{3.56}$$

$$\times \prod_{i<j} \frac{1}{v_i - v_j} \prod_{i\in I} \mathcal{D}_{v_i}$$

$$= \frac{s^N}{(q-1)^N \prod_j v_j} \prod_{i<j} \frac{1}{v_i - v_j} \prod_i (1 - q^N \mathcal{D}_{v_i}^{-1}) \prod_{i<j} (v_i - v_j) \tag{3.57}$$

Now, using the factorized expression for the Vandermonde determinant, $\det[v_i^{j-1}] = \prod_{i<j}(v_i - v_j)$, we can write

$$\prod_{i<j} \frac{1}{v_i - v_j} \prod_i (1 - q^N \mathcal{D}_{v_i}^{-1}) \prod_{i<j} (v_i - v_j) \tag{3.58}$$

$$= \frac{1}{det[v_i^{j-1}]} \det\left[(1-q^{N-j+1}) v_i^{j-1}\right]$$

$$= \frac{1}{det[v_i^{j-1}]} \det\left[(1-q^{N-j+1})\right] \det\left[v_i^{j-1}\right]$$

$$= (q;q)_N$$

So we finally obtain

$$\prod_{i=1}^{N} \frac{1}{(sv_i - 1)} \sum_{x_i \ge x_{i+1} \ge 0} \frac{(s^2;q)_{\bar{n}[\bar{x}]}}{(q;q)_{\bar{n}[\bar{x}]}} \bar{\psi}(\bar{x}|\bar{v}) = \frac{s^N}{(q-1)^N \prod_j v_j} q^{N(N-1)/2} \tag{3.59}$$

i.e.

$$\sum_{x_i \ge x_{i+1} \ge 0} C_{\bar{n}[\bar{x}]} \bar{\psi}(\bar{x}|\bar{v}) = \prod_{i=1}^{N} s \frac{sv_i - 1}{v_i} (-)^N q^{N(N-1)/2} \tag{3.60}$$

Eventually, we make the change of variable $\{s, q, v_i\} \to \{s^{-1}, q^{-1}, v_i^{-1}\}$, and redefine $\{v_i, x_i\} \to \{v_{\Pi i}, x_{\Pi i}\}$. Under these transformations, the objects under consideration transform as

$$\left(\frac{v_i - s}{s(sv_i - 1)}\right) \to z(v_i) \tag{3.61}$$

$$C_{\bar{n}[\bar{x}]} \to \left(\frac{-1}{s^2}\right)^N C_{\bar{n}[\bar{x}]} \tag{3.62}$$

$$A_{P\Pi}(\bar{v}) \to \frac{1}{q^{N(N-1)/2}} A_P(\bar{v}) \tag{3.63}$$

i.e.

$$C_{\bar{n}[\bar{x}]} \bar{\psi}(\bar{x}|\bar{v}) \to \left(\frac{-1}{s^2}\right)^N \frac{1}{q^{N(N-1)/2}} \psi(\bar{x}|\bar{v}) \tag{3.64}$$

so that, taking the transformation into account in the RHS of (3.60), we obtain the Lemma for $m = 0$. The case $m \in \mathbb{Z}$ is straightforwardly obtained by shift of the probability amplitude: $\psi(\bar{x} + m|\bar{v}) = \psi(\bar{x}|\bar{v}) \prod_{i=1}^{N} z(v_i)^m$ $(C_{\bar{n}[\bar{x}]} = C_{\bar{n}[\bar{x}+m]})$.

□

Proof of Theorem 3.1.7.

Proof. One can easily show that for a fundamental string configuration $\{p_1, \cdots, p_k\}$, we have

$$\begin{aligned} &\lambda^N \left[A_{id}^{-1}(\bar{u}) A_{\Pi}^{-1}(\bar{u}) \Lambda(\bar{u})^t z(\bar{u})^m \prod_{j \notin \{a_i\}} (u_j - q u_{j-1}) \right]_{u_j = q u_{j-1},\ j \notin \{a_i\}} \\ &= \prod_{i=1}^{k} \lambda^{p_i} \frac{1}{q^{p_i - 1}(q;q)_{p_i}} \Lambda_{p_j}^t(u_{a_j}) z_{p_j}(u_{a_j})^m \prod_{1 \le i < j}^{k} \frac{1}{q^{k_i k_j}} \frac{(u_{a_i} - u_{a_j})(q^{p_i} u_{a_i} - q^{p_j} u_{a_j})}{(u_{a_i} - q^{p_j} u_{a_j})(q^{p_i} u_{a_i} - u_{a_j})} \\ &= \frac{1}{q^{(\sum_{i=1}^{k} p_i)^2/2}} \prod_{i=1}^{k} \lambda^{p_i} \frac{q^{p_i^2/2 - p_i + 1}}{(q;q)_{p_i}} \Lambda_{p_j}^t(u_{a_j}) z_{p_j}^m(u_{a_j}) \det_{1 \le i,j \le k} \left[\frac{u_{a_i}(1 - q^{p_j})}{u_{a_i} - u_{a_j} q^{p_j}} \right] \end{aligned} \tag{3.65}$$

so that (3.20) reads, after relabeling $u_{a_i} \to u_i$, as

$$\lambda^N \mathbb{P}(\bar{x} \geq a | \bar{y} \leq a - b - 1, t) = \sum_{\bar{p}_n^N} \frac{1}{(\#\bar{p})!} \prod_{i=1}^{\#\bar{p}} \oint_s du_i \tag{3.66}$$

$$\times \frac{1}{q^{(\sum_{i=1}^k p_i)^2/2}} \det_{1 \leq i,j \leq k} \left[\lambda^{p_j} \frac{q^{p_j^2/2 - p_j + 1}}{(q;q)_{p_j}} \Lambda^t_{p_j}(u_j) z^b_{p_j}(u_j) \frac{u_i(1-q^j)}{u_i - u_j q^{p_j}} \right]$$

We see that the only reason we cannot distribute the sum over the p_i in each column of the determinant is the constraint $\sum_i p_i = N$. We overcome this by imposing this constraint through an additional integral:

$$\lambda^N \mathbb{P}(\bar{x} \geq a | \bar{y} \leq a - b - 1, t) \tag{3.67}$$

$$= \sum_{k=1}^N \frac{1}{k!} \sum_{p_1=1}^n \cdots \sum_{p_k=1}^n \oint_0 \frac{dw}{w^{N - \sum_{j=1}^k p_j + 1} q^{N^2/2}} \tag{3.68}$$

$$\times \prod_{i=1}^k \oint_s du_i \det_{1 \leq i,j \leq k} \left[\lambda^{p_j} \frac{q^{p_j^2/2 - p_j + 1}}{(q;q)_{p_j}} \Lambda^t_{p_j}(u_j) z^b_{p_j}(u_j) \frac{u_i(1-q^j)}{u_i - u_j q^{p_j}} \right]$$

$$= \oint_0 \frac{dw}{w^{N+1} q^{N^2/2}} \sum_{k \in \mathbb{N}} \frac{1}{k!} \prod_{i=1}^k \oint_s du_i \tag{3.69}$$

$$\times \det_{1 \leq i,j \leq k} \left[\sum_{p_j=1}^m w^{p_j} \lambda^{p_j} \frac{q^{p_j^2/2 - p_j + 1}}{(q;q)_{p_j}} \Lambda^t_{p_j}(u_j) z^b_{p_j}(u_j) \frac{u_i(1-q^j)}{u_i - u_j q^{p_j}} \right]$$

$$= \oint_0 \frac{dz}{z^{N+1} q^{N^2/2}} \sum_{k \in \mathbb{N}} \frac{1}{k!} \prod_{i=1}^k \oint_s du_i \det_{1 \leq i,j \leq k} \left[K^{[t,b]}_{\lambda z}(u_i, u_j) \right] \tag{3.70}$$

Note that for $N < 0$ or $k > N$, this expression simply vanishes, given that $p_j > 0$. We here recognize the expression for the Fredholm determinant, hence the theorem proved. □

3.2 The Resolution of Identity for the Infinite XXZ Spin Chain

The resolution of the identity in term of on-shell Bethe states can appear to be a very practical tool in the computation of physical quantities, as it provides an interesting alternative path for the computation of scalar products. Indeed, inserting it in a scalar product

$$\langle \psi_1 | \mathcal{O} | \psi_2 \rangle = \sum \langle \psi_1 | \mathcal{O} | n \rangle \langle n | \psi_2 \rangle$$

would result in addition scalar product involving Bethe states $|n\rangle$, which are often well behaved and for which has been developed numerous identities.
This section is devoted to the derivation the resolution of the identity identity in terms of the Bethe states, in the infinite spin XXZ chain with generalized half integer spin. It is the results of a technical exercise set up by Didina Serban as I was beginning my PhD, but did unfortunately not resulted in a publication, as my researched efforts have been deflected before I could reach the result for the case of the finite volume system, which was the first motivation. As in the case of the IFP for ZCM in Section 3.1, my calculations to follow here has been inspired by similar calculations for the propagator in of the infinite δ-Bose gas [30], but despite schematic similarities (the main idea of analytic continuation), additional difficulties has to be treated here. The results presented here already appeared in the literature in the case of the XXZ 1/2-spin chain [96, 97], but to my knowledge, the result presented here (namely in the generalized half integer spin case) can't be found in the literature and is stated here as a theorem.
We will proceed similarly as for the IFP in Section 3.1, again inspired by[30]. Starting from the almost trivial expression in the negative spin regime, the positive half-spin case is reached by analytic continuation. During this procedure, the contours are shrunk so that links between particles are to appear, corresponding the contribution of bound states. The question for the XXX spin chain is then straightforwardly answered by limit of the XXZ system.

In Section 3.2.1 the final Theorem, namely the resolution of the identity for the infinite XXZ half integer spin chain, is presented. The next sections are devoted to proving this theorem. The reader should enter these with great care, as the technical development exposed here has been managed by brute force. It is the result of the first work of a young PhD student, that would deserve a bit of clarification.

The keys of the reasoning are however given on a intuitive level all along the development.

We consider the periodic continuous spin XXZ chain [2], whose eigenstates of the corresponding Hamiltonian are in the form [17]

$$\begin{aligned} |\varphi(\mathbf{u}), N\rangle \equiv & \sum_{\mathbf{p}} \sum_{\tau \in S_N} \prod_{\substack{j,k=1 \\ j<k \\ \tau j > \tau k}}^{N} \frac{t u_{\tau j} - u_{\tau k}}{u_{\tau j} - t u_{\tau k}} \prod_{j=1}^{N} \left(\frac{u_{\tau j} + s}{1 + s u_{\tau j}} \right)^{p_j} |\mathbf{p}\rangle \\ = & \prod_{j<k} \frac{u_j - t u_k}{t u_j - u_k} \sum_{\mathbf{p}} \sum_{\tau \in S_N} \prod_{\substack{j,k=1 \\ j<k}}^{N} \frac{t u_{\tau j} - u_{\tau k}}{u_{\tau j} - t u_{\tau k}} \prod_{j=1}^{N} \left(\frac{u_{\tau j} + s}{1 + s u_{\tau j}} \right)^{p_j} |\mathbf{p}\rangle \end{aligned} \tag{3.71}$$

where S_N is the group of permutation of N elements, $\mathbf{p} = \{p_1, \cdots, p_N\}$ $1 \le p_1 < \cdots < p_N \le L$, L the length of the chain, and $|\mathbf{p}\rangle = \sigma_{p_1}^{-1} \cdots \sigma_{p_N}^{-1} |\Omega\rangle$ with $|\Omega\rangle$ the ground state

[2]The system under consideration is parametrized by two complex variables s and t. Considering the L matrix in [17], the spin k is defined by the relation $s^2 t^k = 1$. The particular case $s^2 t = 1$ leads as expected to the spin 1/2 XXZ, and so do the states (3.71) in this limit.

$|\uparrow\cdots\uparrow\rangle$. In the spin 1/2 case, i.e. for $s^2t = 1$, these states are eigenvectors of the Hamiltonian

$$H = \sum_i \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z$$

an anisotropic Δ-deformation of the XXX Hamiltonian (1.1). Indeed, as we will see in Section 3.2.5, the Bethe states for XXZ will collapse into their XXX cousins in the $s \to 1$ limit.

The states (3.71) will effectively be eigenstate of the master operator, then referred to as *on-shell* Bethe states, provided the rapidities u_j satisfy the Bethe equation

$$\left(\frac{u_j + s}{1 + u_j s}\right)^L = \prod_{k\neq j} \frac{tu_j - u_k}{u_j - tu_k} \quad \forall j \tag{3.72}$$

Remark 3.2.1. *For $s^2t = 1$, the equation (3.72) corresponds to the spin value $1/2$, with an anisotropy parameter given by $\Delta = \frac{1}{2}(s + 1/s)$. Note that for $s \in \mathbb{R}^*$, as we are to consider in the final theorem, we have $|\Delta| \geq 1$. In this context, the resolution for the identity has already been obtained, see for instance [96, 97] for the case $|\Delta| = 1$ and $|\Delta| > 0$ respectively. More generally, the constraint $s^2t^n = 1$, $s \in \mathbb{R}^*$, which will be assumed in the following theorem, corresponds to the highest weight representation of spin $n/2$, hence describing a system with at most n magnons per site.*

3.2.1 Results

To begin with, we derive the expression for the resolution of the identity in the negative spin regime, i.e. for $(|t|-1)(|s|-1) > 0$, which can be expressed

$$\mathbb{I} = \frac{1}{N!}\sum_{N=1}^{L}\sum_{\sigma\in S_N}\prod_{j=1}^{N}\left(\oint_{u_j\in S_2} \frac{du_j}{2i\pi}\frac{1-s^2}{(u_j+s)(1+su_j)}\right)|\varphi(\mathbf{u}),N\rangle\langle\varphi(\mathbf{u}),N| \tag{3.73}$$

For positive spin, i.e. $(|t|-1)(|s|-1) < 0$, things become more complex. In this regime, the Bethe equations (3.72) admit non unitary solutions in the large L limit, with relations of the type $u_j = tu_k$, the so called bound states. These take part in the completeness of Bethe states, hence appearing in the resolution of the identity. Note that such states are not realized in the negative spin chain regime.

Remark 3.2.2. *These two different regimes of spin force a rapprochement with the concepts of repulsive and attractive systems, for which bound state will or not appear depending on the sign of the coupling constant. Doing so we could refer to negative spin systems as repulsive and positive spin systems as attractive. In addition to this similarity, some aspects of our calculations are actually shows similarity with what can be encountered in the case of the Bose gas with δ interaction [30].*

A bound state will be expressed

$$|\varphi(\mathbf{u},\mathbf{n},M)\rangle \equiv \sum_{\mathbf{p}} \sum_{\tau\in S''_n(\mathbf{n})} \prod_{\substack{j,k=1\\ j<k}}^{M} \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k\\ \tau^{-1}(a)>\tau^{-1}(b)}} \frac{u_{j,a}-tu_{k,b}}{tu_{j,a}-u_{k,b}} \prod_{j=1}^{M}\prod_{a\in\Omega_j}\left(\frac{u_{j,a}+s_j}{1+s_j u_{j,a}}\right)^{p_{\tau^{-1}a}} |\mathbf{p}\rangle \tag{3.74}$$

which is actually a state (3.71) with constraints over rapidities $\mathbf{u}$, up to re-normalization.

Let us briefly explain these notations and their relevance at the current stage of the discussion.
Linked rapidities $u_{j_1} = tu_{j_2} = t^2u_{j_3} = \cdots$ will correspond to a single-like particle, a bound state of excitations, so called string, described by what one could call a cluster $\Omega_j = \{j_1, j_2, j_3, \cdots\}$. Here N is the number of magnons, and M is the number of independent rapidities, or number of string. A bound state will be described by its string configuration $\mathbf{n} = \{n_1, \cdots, n_m\}$, $n_j = |\Omega_j|$, and its strings rapidities u_j, $u_{j,a}$ being the rapidity of the magnon $a \in \Omega_j$ so that all the $u_{j,a}$ are dependent of the string rapidity.
Finally, $S''_N(\mathbf{n})$ is the set of elements of S_N such that exchanging the order of all elements of a cluster: $a, b \in \Omega_j$, $a < b \Rightarrow S''_N(\mathbf{n})a > S''_N(\mathbf{n})b$. This constraint over the summed permutation in (3.74) is an important manifestation of the clustering mechanism at work in the positive spin regime.
All these objects will be precisely defined later.

The main part of this chapter focuses on the proof for the following theorem.

Theorem 3.2.1. *For $s^2t^n = 1$, $s \in \mathbb{R}$, the resolution of the identity in term of Bethe states writes*

$$\begin{aligned}\mathbb{I} = &\sum_{N=1}^{L}\sum_{M=1}^{N}\frac{1}{M!}\sum_{\mathbf{n}\in D_{N,M}}\prod_{j=1}^{M}\left(\oint_{u_j\in S_2}\frac{du_j}{2i\pi u_j}\right)\prod_{r=1}^{n_j-1}\frac{(t^r-t^{-1})(t^r-1)}{t^r(1-t^{-1})}\\ &\times\prod_{j=1}^{M}\prod_{a\in\Omega_j}\frac{(1-s^2)u_{j,a}}{(s+u_{j,a})(1+su_{j,a})}\\ &\times|\varphi(\mathbf{u},M,\mathbf{n},\mathbf{s})\rangle\langle\varphi(\mathbf{u},M,\mathbf{n},\mathbf{s})|\end{aligned} \tag{3.75}$$

The most interesting aspect of this formula undoubtedly concerns its combinatorics, which is the matter of the first Subsection 3.2.2. To provide the reader with a rapid insight to the reflection, an informal approach is proposed and the three magnons case is sketched, providing us with all the schematic material involved in the rigorous proof that follows.
The second Section 3.2.3 is devoted to the study of the analyticity of the integral. It is shown that while barycentering the rapidities of our string[3], which is necessary to obtained

[3]For a bound state to effectively be a physical state, the string parameter needs to belong to a particular region of the complex plane. By barycentering the string we here refer to deforming the integral contours into this particular region. The obtained integral then corresponds to summing over eigenstates of the Hamiltonian

on-shell bound Bethe states, no additional term would appear, as the integrated part is analytic in a large enough region.
In Section 3.2.4, we eventually adjust the s parameter such that reaching the constraint $s^2 t^n = 1$, $s \in \mathbb{R}^*$, corresponding to the $n/2$ spin regime. We show that additional terms appearing during this analytic continuation procedure will eventually vanish for these particular values of s. This would achieve the proof for Theorem 3.2.1.
Finally, in Section 3.2.5, a similar formula is obtained for the $XXX_{n/2}$ spin chain as a limit of the XXZ case.

3.2.2 Clustering

We first suppose $(|s|-1)(|t|-1) > 0$, and we choose, without loss of generality, $|s|, |t| < 1$. This first assumption corresponds to a repulsive-like interaction, for which there are no bound states.
Indeed, setting $\mathbf{p} = \{p_j\}_{j=1,..,N}$ and $\mathbf{q} = \{q_j\}_{j=1,..,N}$ two sets of ordered integers in $\{1, \cdots, L\}$, i.e. such that $1 \le p_1 < \cdots < p_N \le L$ and same for q, we have

$$I_\delta(\mathbf{p}, \mathbf{q}) \tag{3.76}$$

$$\equiv \sum_{\sigma \in S_N} \prod_{j=1}^{N} \oint_{u_j \in S_2} \frac{du_j}{2i\pi} \frac{1 - s^2}{(u_j + s)(1 + su_j)} \prod_{\substack{j<k \\ \sigma^{-1}(j) > \sigma^{-1}(k)}} \frac{tu_j - u_k}{u_j - tu_k} \prod_{j=1}^{N} \left(\frac{u_j + s}{1 + su_j} \right)^{\xi_{\sigma,j}} \tag{3.77}$$

$$= \prod_{j=1}^{N} \delta_{p_j, q_j} \tag{3.78}$$

with $\xi_{\sigma,j} \equiv p_j - q_{\sigma^{-1}(j)}$.

Proof. Let us define $\Gamma(\sigma^{-1})$ the graph of N points of labels $j = 1, \cdots, N$ with a link between j and k for $j < k$ and $\sigma^{-1}(j) > \sigma^{-1}(k)$. Such a relation would be denoted $\{j, k\}_{\sigma^{-1}}$.

- For $\sigma \neq Id$, $\exists\ \gamma$ at least one connected subgraphs of $\Gamma(\sigma)$ not only containing a unique, i.e. isolated, point. We then define j_{min} and j_{max} the minimum and maximum of this graph respectively.
  We know by definition that $\nexists\ j > j_{max}$, $\sigma^{-1}(j_{max}) > \sigma^{-1}(j)$ (otherwise j_{max} is not the maximum), and similarly $\nexists\ j < j_{max}$, $\sigma^{-1}(j) > \sigma^{-1}(j_{max})$.
  So, if $\xi_{j_{max}} > 0$, all the poles for u_k in (3.76) are outside the integration circle and the integral vanishes.
  Similarly, if $\xi_{j_{min}} < 0$, all the poles for u_k are inside the integration circle and the integral vanishes as well.
  In addition, by definition, $\xi_{j_{min}} < \xi_{j_{max}}$, so at least one of the two conditions above is fulfilled.

- Now if $\sigma = Id$, none of the points are linked, and the integral vanishes except for $\mathbf{p} = \mathbf{q}$, in which case the integral is equal to 1.

□

All the rapidities being integrated on the same contour, it is symmetric under the relabeling $u_j \to u_{\tau j}$ in (3.76) and we can then sum over $\tau \in S_N$ with an additional combinatorics factor $\frac{1}{N!}$. Then, noticing the factorization

$$\begin{aligned}&\prod_{\substack{\tau^{-1}(j)<\tau^{-1}(k)\\ \sigma^{-1}(j)>\sigma^{-1}(k)}} \frac{tu_j - u_k}{u_j - tu_k} \\ &= \prod_{\substack{j<k\\ \sigma^{-1}(j)>\sigma^{-1}(k)}} \frac{tu_j - u_k}{u_j - tu_k} \times \prod_{\substack{j<k\\ \tau^{-1}(j)>\tau^{-1}(k)}} \frac{u_j - tu_k}{tu_j - u_k}\end{aligned} \tag{3.79}$$

(detailed in equation (3.92)), one obtains

$$\mathbb{I} = \frac{1}{N!} \sum_{N=1}^{L} \sum_{\sigma \in S_N} \prod_{j=1}^{N} \left(\oint_{u_j \in S_2} \frac{du_j}{2i\pi} \frac{1 - s^2}{(u_j + s)(1 + su_j)} \right) |\varphi(\mathbf{u}), N\rangle \langle \varphi(\mathbf{u}), N| \tag{3.80}$$

where the Bethe states $|\varphi(\mathbf{u}), N\rangle$ are defined in (3.71).

The attractive regime. We are now going to split the integration contour outside the unit circle, close enough so that no poles are grabbed (i.e. not interacting with the pole depending on s, which will be the matter of a last treatment). This will allows us to move freely the parameter t from $|t| < 1$ to $|t| > 1$. Then, shifting back the integration contour to the unit circle in a later procedure, these previously moved poles will be crossed, giving rise to the clusterings of our particles. The first two of these steps are the matter of the following lemma.

Albeit the whole following development does not seem to depend on the phase of t and s, but the proofs are formally developed for s and t real.
Let us define the parameterization $t = e^x$, $s = e^{-y}$, $\rho = e^r$, and $b(i)$ the *anti-normal-order* of $i \in \{1, \cdots, N\}$: $b(j) = N - j$.

Lemma 3.2.2. *For* $0 < x < r < \frac{y}{N}$, *we have*

$$I_\delta(\mathbf{p}, \mathbf{q}) = \sum_{\sigma \in S_N} \prod_{j=1}^{N} \oint_{|u_j| = e^{rb(j)}} \frac{du_j}{2i\pi} \frac{1 - s^2}{(u_j + s)(1 + su_j)} \prod_{\substack{j<k\\ \sigma^{-1}(j)>\sigma^{-1}(k)}} \frac{tu_j - u_k}{u_j - tu_k} \prod_{j=1}^{N} \left(\frac{u_j + s}{1 + su_j} \right)^{\xi_{\sigma,j}} \tag{3.81}$$

Proof. We first consider $x < 0$. For $r = 0$ we come back to (3.76).
We now are now going to shift the integral contour in (3.76) according to $|u_j| = e^{rb(j)}$, $r = 0 \to r > 0$.

- By definition we have $\{i,j\}_{\sigma^{-1}} \Rightarrow b(i) \geq b(j)+1 \Rightarrow rb(i) > rb(j)+x$, since $r > x$, so the integration contour in (3.76) won't cross any pole from the term $\prod_{\substack{j<k \\ \sigma^{-1}(j)>\sigma^{-1}(k)}} \frac{tu_j-u_k}{u_j-tu_k}$ while shifting $r=0 \to r>0$.

- Now, since $-y < 0 \leq rb(j) < rN < y$, the integration contour in (3.76) won't cross any pole from the term $\left(\frac{u_j+s}{1+su_j}\right)^{\xi_{\sigma,j}} \frac{1}{(u_j+s)(1+su_j)}$ while shifting $r=0 \to 0<r<y/N$.

- In (3.81), the poles depending on $t=e^x$ are from the $\frac{1}{u_j-tu_k}$ for $\{j,k\}_{\sigma^{-1}}$, with $|u_j| = e^{b(j)r}$, $|tu_k| = e^{b(k)r+x}$. Now (3.81) is obviously analytic in t, hence remains unchanged while shifting $t=e^x$ as long as $x<r$, i.e. if the corresponding poles don't cross any contour.

□

From now on we assume $0 < x < r < \frac{y}{N}$.

This condition over y is conceptually equivalent to sending $1/s$ to infinity, so that we don't care about these poles for now.

Grabbing poles, constructing trees: combinatorics. We will now deal with the combinatorics that arises with the clustering of magnons while shifting our contour back. The clustering is summed up in the following Lemma.

Let's first introduce some new objects:

- $\mathbb{I}_{\{c\}} = 1$ if the condition c is true, 0 otherwise.

- $D_{N,M} = \{\mathbf{n} = \{n_1, \cdots, n_M\}, \sum_{i=1}^M n_i = N, n_i \geq 1\}$, the set of clustering, or string configuration.

- $P_N(\mathbf{n}) = \{\mathbf{A} = (A_1, \cdots, A_M),\ |A_j| = n_j,\ A_j \cap A_{k\neq j} = \emptyset,\ A_j \subset \{1, \cdots, N\}\}$ the set of *ordered* M-partitions of $\{1, \cdots, N\}$, i.e. $\cup_{j=1}^M A_j = \{1, \cdots, N\}$, $A_j = \{\cdots, j, k, \cdots\} \Rightarrow j<k$.

The A_j are conceptually equivalent to the Ω_j (evoked in the paragraph after (3.74), and defined later) in that they describe the clustering of our rapidities. Their difference is technical, these two sets of objects being just related by a particular set of permutation, as we will see later while re-expressing our sum to make the Behte states appear.

For now we are to identify $\mathbf{A}$ with its class of equivalence under permutation of its elements, i.e. $\forall R \in S_M$, $\{A_i\}$ and $\{A_{Ri}\}$ are considered to be the same element of $P(\mathbf{n})$.

- For $a \in A_j$, $d(a)$ the *anti-normal* order of a in A_j, such that $d(min[A_j]) = |A_j|-1$, $d(max[A_j]) = 0$.

Lemma 3.2.3. *For $0 < x < r < y/N$, let ϵ_j, $j = 1, \cdots N$ be distinct arbitrary numbers such that $0 < \epsilon_j < r - x < y/N$. Then, for $l = 0, \cdots, N$, one has*

$$\begin{aligned} I_\delta(\mathbf{p},\mathbf{q}) = I_l \equiv & \sum_{M=1}^{N} \sum_{\mathbf{n}\in D_{N,M}} \prod_{j=1}^{l} \left(\oint_{|u_j|=e^{rb(j)}} du_j \right) \prod_{j=l+1}^{M} \left(\oint_{|u_j|=e^{\epsilon_j}} du_j \right) \\ & \times \sum_{\sigma\in S_N} \sum_{\mathbf{A}\in P(\mathbf{n})} \prod_{j=1}^{l} \mathbb{I}_{\{A_j=\{j\}\}} \prod_{j=1}^{M} \prod_{\substack{a,b\in A_j \\ a<b}} \mathbb{I}_{\{\sigma^{-1}a>\sigma^{-1}b\}} \\ & \times \frac{(1-s^2)^N}{(2i\pi)^M} \prod_{j=1}^{M} \prod_{r=1}^{n_j-1} \frac{(t^r - t^{-1})(t^r-1)}{t^r(1-t^{-1})} \prod_{j=1}^{M} \frac{1}{u_j} \prod_{a\in A_j} u_j t^{d(a)} \\ & \times \prod_{\substack{j,k=1 \\ j\neq k}}^{M} \prod_{\substack{a\in A_j \\ b\in A_k \\ a<b \\ \sigma^{-1}(a)>\sigma^{-1}(b)}} \frac{u_j t^{d(a)+1} - u_k t^{d(b)}}{u_j t^{d(a)} - u_k t^{d(b)+1}} \prod_{j=1}^{M} \prod_{a\in A_j} \frac{(u_j t^{d(a)} + s)^{\xi_{\sigma,a}-1}}{(1+su_j t^{d(a)})^{\xi_{\sigma,a}+1}} \end{aligned} \tag{3.82}$$

where $I_\delta(\mathbf{p},\mathbf{q})$, as defined before, does not carry any dependence in l.

Before going for the formal proof. Let's have a look to a more intuitive and informal reasoning to approach the mechanism around the clustering in (3.82).

For a given l, the variables u_j, $j = 1, \cdots, l$ are untouched. Their integration contour has not been shifted, and they belong to trivial clusters, given the condition $\prod_{j=1}^{l} \mathbb{I}_{\{A_j=\{j\}\}}$. These can be seen as 1-strings, and we have $d(j) = 0$: $a \in A_j \Rightarrow u_j t^{d(a)} = u_j$.

For $l = N$, we are back to (3.81).

Now, we shift the contour for u_l to the unit circle. As we will see in the proof, this shift will grab all the poles at $u_j = u_m t^{n_m}$ provided $\sigma^{-1}(l) > \sigma^{-1}(a)\ \forall a \in A_m$ (the other either not being grabbed, either canceling while summing over permutation), where u_m is an already shifted rapidity. This is an important result, since it constrains the clusters to branchless trees: in A_m, the rapidities are $u_m t^{n_m-1}, u_m t^{n_m-2}, \cdots, u_m$, so the new link is made with the end of the string, $u_j = u_m t^{n_m}$. Now considering $j \in A_m$ (and $\mathbf{n}$ corresponding with this new string configuration, i.e. $\{j\}, A_m \to \{j, A_m\}$), we see that in the formula for $l-1$ all clustering with such a link (and only these) will appear, given we have the constraint $\prod_{\substack{a,b\in A_m \\ a<b}} \mathbb{I}_{\{\sigma^{-1}a>\sigma^{-1}b\}}$. Of course, in addition to all these term, we have to consider the term with u_j still integrated, i.e. not connected with another string. Thus, while for $l = N$ only the term $M = N$ was contributing, at each shift the term M would give a term $M - 1$ (the shifted variable linked to a string) and a term M (the shifted variable still integrated), so for $l = 0$ all the values of M will contribute.

Example: the three magnons case. We come back to our representation of a permutation σ^{-1} as the graph $\Gamma_{\sigma^{-1}}$ in which points j and k are linked iff $j < k$, $\sigma^{-1}(j) > \sigma^{-1}(k)$.

We can thus consider $id \in S_3$ as the graph $\Gamma_{id} = \{\{1\},\{2\},\{3\}\}$, or (13) (exchanging 1 and 3) as $\Gamma_{(13)} = \{\{1,2,3\}\}$. We will consider this latter to illustrate our construction.

To do so, we have to follow the three following rules concerning created (and not canceling) links, as described before:
-the first shift (here of u_3) is trivial.
-a shifted variable will link to already shifted variables belonging to the same graph.
-a link will be made at the end of a string.

Thus: The shift of u_3 won't produce any link. We describe this configuration by the cluster $P_N = \{\{1\},\{2\},\{3\}\}$. The shift of u_2 will produce the two terms $P_N = \{\{1\},\{2,3\}\}$ and $P_N = \{\{1\},\{2\},\{3\}\}$. The shift of u_1 will produce, from this first term, $P_N = \{\{1,2,3\}\}$ and $P_N = \{\{1\},\{2,3\}\}$, and from the second one $P_N = \{\{1,2\},\{3\}\}$, $P_N = \{\{1,3\},\{2\}\}$ and $P_N = \{\{1\},\{2\},\{3\}\}$.

The formal proof.

Proof. • For $l = N-1$,

$A_j = \{j\}\ \forall j = 1,\cdots,N-1$ so we are back to (3.81), except from a shift $|u_N| = 1 \to e^{\epsilon_N}$, which does not cross any pole given that the pole in u_N are at $z = t^{-1}u_j,\ j = 1,\cdots,N-1,\ |z| = e^{rb(j)-x} > e^{\epsilon_N} > 1$, and at $z = -s^{\pm 1},\ -y < 0 < \epsilon_N < y$.

• Let us assume (3.82) is true for $l \leq N-1$. We are now going to prove that $I_l = I_{l-1}$.

We move in I_l the contour of integration over u_l: $|u_l| = e^{rb(l)} \to e^{\epsilon_l}$. Crossed poles will come from

$$\prod_{\substack{j,k=1\\ j\neq k}}^{M} \prod_{\substack{a\in A_j\\ b\in A_k\\ a<b\\ \sigma^-1(a)>\sigma^-1(b)}} \frac{u_j t^{d(a)+1} - u_k t^{d(b)}}{u_j t^{d(a)} - u_k t^{d(b)+1}}$$

(the poles $z = s$ and $z = 1/s$ will not be crossed given that we imposed $0 < \epsilon_j < y$).

⋆ For $j = 1,\cdots,l-1$, we may have $\frac{1}{u_j - tu_l}$, but since $e^{\epsilon_l} < e^{rb(l)} < e^{rb(j)+x}$, none of these poles are crossed.

⋆ For $c \in A_k,\ k = l+1,\cdots,M$, we have $\frac{1}{u_l - u_k t^{d(c)+1}}$ for $\sigma^{-1}(l) > \sigma^{-1}(c)$. Here we have $e^{rb(l)} > e^{\epsilon_k + x(d(c)+1)} > e^{\epsilon_l}$ since $d(c)+1 \leq |A_j| \leq N-l$ and $\epsilon_k < x$, so all the poles from these terms are crossed.

Thus shifting the integration contour of u_l will produce several terms. A first term in which u_l is still integrated, corresponding to the term $M' = M$ in I_{l-1}, and another term with $u_l = u_m t^{d(c)+1}$ for every c such that $c \in \{l+1,\cdots,N\},\ c \in A_m$, provided $\sigma^{-1}(l) > \sigma^{-1}(c)$. The corresponding residue is:

$$
\begin{aligned}
&\frac{(1-s^2)^N}{(2i\pi)^M}\prod_{j=1}^{M}\prod_{r=1}^{n_j-1}\frac{(t^r-t^{-1})(t^r-1)}{t^{2r-1}(t-1)}\prod_{j=1}^{l-1}\left(\oint_{|u_j|=e^{rb(j)}}du_j\right)\prod_{j=l+1}^{M}\left(\oint_{|u_j|=e^{\epsilon_j}}du_j\right)\\
&\times\sum_{\sigma\in S_N}\sum_{\mathbf{A}\in P(\mathbf{n})}\prod_{j=1}^{l}\mathbb{I}_{\{A_j=\{j\}\}}\prod_{j}^{M}\prod_{\substack{a,b\in A_j\\a<b}}\mathbb{I}_{\{\sigma^{-1}a>\sigma^{-1}b\}}\\
&\times\frac{(u_mt^{d(c)+1}+s)^{\xi_{\sigma,a}-1}}{(1+su_mt^{d(c)+1})^{\xi_{\sigma,a}+1}}\times\prod_{\substack{j\in\Gamma_{l,M}\\j\neq l}}\prod_{a\in A_j}\frac{(u_jt^{d(a)}+s)^{\xi_{\sigma,a}-1}}{(1+su_jt^{d(a)})^{\xi_{\sigma,a}+1}}\\
&\times\prod_{\substack{j,k=1\\j\neq k\\j,k\neq l}}^{M}\prod_{\substack{a\in A_j\\b\in A_k\\a<b\\\sigma^{-1}(a)>\sigma^{-1}(b)}}\frac{u_jt^{d(a)+1}-u_kt^{d(b)}}{u_jt^{d(a)}-u_kt^{d(b)+1}}\prod_{\substack{j=1\\j\neq l,m}}^{M}\prod_{\substack{a\in A_j\\a<l\\\sigma^{-1}(a)>\sigma^{-1}(l)}}\frac{u_jt^{d(a)+1}-u_mt^{d(c)+1}}{u_jt^{d(a)}-u_mt^{d(c)+2}}\\
&\times\prod_{\substack{k=1\\k\neq l,m}}^{M}\prod_{\substack{b\in A_k\\l<b\\\sigma^{-1}(l)>\sigma^{-1}(b)}}\frac{u_mt^{d(c)+2}-u_kt^{d(b)}}{u_mt^{d(c)+1}-u_kt^{d(b)+1}}\prod_{\substack{a\in A_m\\a\neq c\\\sigma^{-1}(l)>\sigma^{-1}(a)}}\frac{u_mt^{d(c)+2}-u_mt^{d(a)}}{u_mt^{d(c)+1}-u_mt^{d(a)+1}}\\
&\times(2i\pi)t(u_mt^{d(c)+1}-u_mt^{d(c)-1})
\end{aligned}
\tag{3.83}
$$

The last line result in a factor $(2i\pi)u_m(t-t^{-1})t^{d(c)+1}$, and the line before can write as

$$
\prod_{\substack{a\in A_m\\a\neq c\\\sigma^{-1}(a)>\sigma^{-1}(l)}}\frac{t^{d(c)+2}-t^{d(a)}}{t^{d(c)+1}-t^{d(a)+1}}
\tag{3.84}
$$

Branchless trees:

⋆ If $c=min(A_m)$ ($\Rightarrow d(c)=n_m-1$), $\sigma^{-1}(c)>\sigma^{-1}(a)\ \forall a\in A_m$ so all the terms in (3.84) contribute since by hypothesis $\sigma^{-1}(l)>\sigma^{-1}(c)$, and (3.84) rewrite

$$
\prod_{r=1}^{n_m-1}\frac{t^2-t^{-r}}{t-t^{-r+1}}=\frac{(t^{n_m}-t^{-1})(t^{n_m}-1)}{t^{n_m}(1-t^{-1})(t-t^{-1})}
\tag{3.85}
$$

⋆ We are now going to prove that for $c\neq min(A_m)$, residues cancel each other. We know that $d(b)$ and $\sigma^{-1}(b)$ are ordered in the same way in A_m, so $\exists!f\in A_m/\ for\ b\in A_m,\ b\geq f\Rightarrow\sigma^{-1}l>\sigma^{-1}b$ and $b<f\Rightarrow\sigma^{-1}b>\sigma^{-1}l$. Then (3.84) reads

$$
\prod_{\substack{a\in A_m\\a\neq c\\a\geq f}}\frac{t^{d(c)+2}-t^{d(a)}}{t^{d(c)}+1-t^{d(a)+1}}
\tag{3.86}
$$

Now if $d(f) \geq d(c)+2$, $\exists b \in A_m / \ \bar{d}(b) = \bar{d}(c)+2$ and (3.86) is equal to zero. Now either $\bar{d}(f) = \bar{d}(c)$ $(f = c)$, and (3.86) rewrite

$$\prod_{\substack{a\in A_m\\ a>c}} \frac{t^{d(c)+2} - t^{d(a)}}{t^{d(c)+1} - t^{d(a)+1}} \tag{3.87}$$

either $d(f) = d(c)+1$ and (3.86) rewrite

$$\frac{t^{d(c)+2} - t^{d(c)+1}}{t^{d(c)+1} - t^{d(c)+2}} \prod_{\substack{a\in A_m\\ a>c}} \frac{t^{d(c)+2} - t^{d(a)}}{t^{d(c)+1} - t^{d(a)+1}} = -\prod_{\substack{a\in A_m\\ a>c}} \frac{t^{d(c)+2} - t^{d(a)}}{t^{d(c)+1} - t^{d(a)+1}} \tag{3.88}$$

Consider now to replace σ^{-1} by $\sigma^{-1} \circ \theta_{l,c'}$, with c' such that $\bar{d}(c') = \bar{d}(c)+1$, i.e. replacing $f = c$ by $f = c'$, or the other way round. This is equivalent to exchange (3.87) and (3.88), and thus these term will cancel while summing over σ.

Summary. There will be a new term (3.83) for all $c = min(A_j)$, $j = l+1, \cdots, M$ provided $\sigma^{-1}(l) > \sigma^{-1}(c)$. In other words, u_l will be linked to any string provided l is related to any element of this string, the link being made at the end of the string. Let's call $\mathbf{B}$ the clustering obtained by merging A_m with $\{l\}$: $B_m = \{l, A_m\}$ (in this new clustering, $l \in B_m$ and $d(l) = |B_m| - 1 = |A_m| = n_m = \bar{d}(c)+1$, $n'_m = |B_m| = n_m + 1$), and to relabel $j \to j-1$, $j = l+1, \cdots, M$. These correspond to all the terms appearing in I_{l-1} for $M' = M-1$, i.e. all clustering $\mathbf{A}$ such that $\prod_{j=1}^{l-1} \mathbb{I}_{\{A_j=\{j\}\}} \prod_{j=1}^{M} \prod_{\substack{a,b\in A_j\\ a<b}} \mathbb{I}_{\{\sigma^{-1}a>\sigma^{-1}b\}} \neq 0$.

Remark 3.2.3. *We may here be tempted to add a combinatorial term* $\frac{1}{M!}$ *for all the permutation* R *in* S_M*, but remember that we here identify the cluster* $\{A_j\}$ *with* $\{A_{R(j)}\}$*, namely we sum over the class of equivalence of* $P_N(\mathbf{n})$*.*

This completes the proof of Lemma 3.2.3.

□

We then obtain, from $l = 0$ in Lemma 3.2.3,

$$
\begin{aligned}
I_\delta(\mathbf{p},\mathbf{q}) = &\sum_{M=1}^{N} \sum_{\mathbf{n}\in D_{N,M}} \prod_{j=1}^{M} \left(\oint_{|u_j|=e^{\epsilon_j}} du_j \right) \\
&\times \sum_{\sigma\in S_N} \sum_{\mathbf{A}\in P(\mathbf{n})} \prod_{j=1}^{M} \prod_{\substack{a,b\in A_j\\ a<b}} \mathbb{I}_{\{\sigma^{-1}a>\sigma^{-1}b\}} \\
&\times \frac{(1-s^2)^N}{(2i\pi)^M} \prod_{j=1}^{M} \prod_{r=1}^{n_j-1} \frac{(t^r-t^{-1})(t^r-1)}{t^r(1-t^{-1})} \prod_{j=1}^{M} \frac{1}{u_j} \prod_{a\in A_j} u_j t^{d(a)} \\
&\times \prod_{\substack{j,k=1\\ j\neq k}}^{M} \prod_{\substack{a\in A_j\\ b\in A_k\\ a<b\\ \sigma^{-1}1(a)>\sigma^{-1}1(b)}} \frac{u_j t^{d(a)+1} - u_k t^{d(b)}}{u_j t^{d(a)} - u_k t^{d(b)+1}} \prod_{j=1}^{M} \prod_{a\in A_j} \frac{(u_j t^{d(a)}+s)^{\xi_{\sigma,a}-1}}{(1+s u_j t^{d(a)})^{\xi_{\sigma,a}+1}}
\end{aligned}
\tag{3.89}
$$

This expression does not depend on $\rho = e^r$, hence it is valid for any value of $0 < x < y/N$.

3.2.3 Analyticity: Barycentring the Strings

In this section, we first rewrite the integrated object so that it exhibits analyticity in a region large enough to proceed to barycenter the strings. Indeed, from consideration on the Bethe equation and on bound states, we know that the string j has to be centered around a unitary variable $u_j \in S_2$: $u_{j_a} = u_j t^{-\frac{n_j+1}{2}+a}$, $a = 1, \cdots, n_j$. We will then be able to perform the corresponding shift.

Let us introduce some new objects:

• $\Omega_j(\mathbf{n}) \equiv \Omega_j = \{n_1 + \cdots + n_{j-1} + 1, \cdots, n_1 + \cdots + n_j\}$

• $r_{\mathbf{n}}(a) \equiv r(a) = s$ for $a = n_1 + \cdots + n_{j-1} + s \in \Omega_j$, the normal order of a in Ω_j.

• $S'_N(\mathbf{n}) = \{\sigma \in S_N |\ a, b \in \Omega_j,\ a < b, \sigma^{-1}(a) < \sigma^{-1}(b)\}$
and $S''_N(\mathbf{n}) = \{\tau \in S_N |\ a, b \in \Omega_j,\ a < b, \tau^{-1}(a) > \tau^{-1}(b)\}$
The ϵ_j being arbitrary, we can sum over permutations of the ϵ_j, equivalently over the u_j, or the A_j. This latter choice is equivalent to consider the $\{A_j\}$ as ordered, i.e. to consider $\{A_j\}$ and $\{A_{Rj}\}$, $R \in S_M \setminus Id$, as distinct elements of $P(\mathbf{n})$. These ordered sets can be bijectively associated to the elements of $S''_n(\mathbf{n})$: $A_j = \{\tau^{-1}(a),\ a \in \Omega_j(\mathbf{n})\} = \tau^{-1}(\Omega_j)$, given that the $\tau^{-1}(a)$, $a \in \Omega_j$, are ordered. Such a sum would be introduced with a combinatorial factor $\frac{1}{M!}$. Doing so in (3.89) we can also relabel $a \to \tau^{-1}a$, $\sigma \to \sigma \circ \tau^{-1}$, so that would appear in the obtained sum the term

$$
\prod_{j=1}^{M} \prod_{\substack{a,b\in\Omega_j\\ \tau^{-1}a<\tau^{-1}b}} \mathbb{I}_{\{\sigma^{-1}a>\sigma^{-1}b\}}
\tag{3.90}
$$

which is equivalent to $\sigma \in S'_n(\mathbf{n})$.
Noticing that for $\tau \in S''_N(\mathbf{n})$ $d(\tau^{-1}a) = r(a) - 1$, we can at last rewrite (3.89) as

$$\begin{aligned} I_\delta(\mathbf{p},\mathbf{q}) &= \sum_{M=1}^{N} \frac{1}{M!} \sum_{\mathbf{n}\in D_{N,M}} \prod_{j=1}^{M} \left(\oint_{|u_j|=e^{\epsilon_j}} \frac{du_j}{u_j} \right) \sum_{\sigma\in S'_N(\mathbf{n})} \sum_{\tau\in S''_N(\mathbf{n})} \\ &\times \frac{(1-s^2)^N}{(2i\pi)^M} \prod_{j=1}^{M} \prod_{r=1}^{n_j-1} \frac{(t^r-t^{-1})(t^r-1)}{t^r(1-t^{-1})} \prod_{j=1}^{M} \prod_{a\in\Omega_j} u_j t^{r(a)-1} \\ &\times \prod_{\substack{j,k=1 \\ j\neq k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \tau^{-1}(a)<\tau^{-1}(b) \\ \sigma^-1(a)>\sigma^-1(b)}} \frac{u_j t^{r(a)+1} - u_k t^{r(b)}}{u_j t^{r(a)} - u_k t^{r(b)+1}} \prod_{j=1}^{M} \prod_{a\in\Omega_j} \frac{(u_j t^{r(a)-1}+s)^{\xi_{\sigma,\tau,a}-1}}{(1+s u_j t^{r(a)-1})^{\xi_{\sigma,\tau,a}+1}} \end{aligned} \tag{3.91}$$

with $\xi_{\sigma,\tau,a} \equiv p_{\tau^{-1}(a)} - q_{\sigma^{-1}(a)}$.
One can now factorize

$$\begin{aligned} &\prod_{\substack{j,k=1 \\ j\neq k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \tau^{-1}(a)<\tau^{-1}(b) \\ \sigma^-1(a)>\sigma^-1(b)}} \frac{u_j t^{r(a)+1} - u_k t^{r(b)}}{u_j t^{r(a)} - u_k t^{r(b)+1}} \\ &= \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \tau^{-1}(a)<\tau^{-1}(b) \\ \sigma^-1(a)>\sigma^-1(b)}} \frac{u_j t^{r(a)+1} - u_k t^{r(b)}}{u_j t^{r(a)} - u_k t^{r(b)+1}} \times \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \tau^{-1}(a)>\tau^{-1}(b) \\ \sigma^-1(a)<\sigma^-1(b)}} \frac{u_j t^{r(a)} - u_k t^{r(b)+1}}{u_j t^{r(a)+1} - u_k t^{r(b)}} \\ &= \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \sigma^-1(a)>\sigma^-1(b)}} \frac{u_j t^{r(a)+1} - u_k t^{r(b)}}{u_j t^{r(a)} - u_k t^{r(b)+1}} \times \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \tau^{-1}(a)>\tau^{-1}(b)}} \frac{u_j t^{r(a)} - u_k t^{r(b)+1}}{u_j t^{r(a)+1} - u_k t^{r(b)}} \end{aligned} \tag{3.92}$$

Then, defining

$$\chi(\mathbf{u},M,\mathbf{n},\mathbf{p}) \equiv \sum_{\tau\in S''_n(\mathbf{n})} \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \tau^{-1}(a)>\tau^{-1}(b)}} \frac{u_j t^{r(a)} - u_k t^{r(b)+1}}{u_j t^{r(a)+1} - u_k t^{r(b)}} \prod_{j=1}^{M} \prod_{a\in\Omega_j} \frac{(u_j t^{r(a)-1}+s)^{p_{\tau^{-1}a}}}{(1+s u_j t^{r(a)-1})^{p_{\tau^{-1}a}+1}} \tag{3.93}$$

and

$$\tilde{\chi}(\mathbf{u},M,\mathbf{n},\mathbf{q}) \equiv \sum_{\sigma\in S'_n(\mathbf{n})} \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a\in\Omega_j \\ b\in\Omega_k \\ \sigma^-1(a)>\sigma^-1(b)}} \frac{u_j t^{r(a)+1} - u_k t^{r(b)}}{u_j t^{r(a)} - u_k t^{r(b)+1}} \prod_{j=1}^{M} \prod_{a\in\Omega_j} \frac{(u_j t^{r(a)-1}+s)^{-q_{\sigma^{-1}a}-1}}{(1+s u_j t^{r(a)-1})^{-q_{\sigma^{-1}a}}} \tag{3.94}$$

equation (3.91) rewrites as

$$I_\delta(\mathbf{p},\mathbf{q}) = \sum_{M=1}^{N} \frac{1}{M!} \sum_{\mathbf{n}\in D_{N,M}} \prod_{j=1}^{M} \left(\oint_{|u_j|=e^{\epsilon_j}} \frac{du_j}{u_j} \right) \times \frac{(1-s^2)^N}{(2i\pi)^M} \prod_{j=1}^{M} \prod_{r=1}^{n_j-1} \frac{(t^r-t^{-1})(t^r-1)}{t^r(1-t^{-1})} \prod_{j=1}^{M} \prod_{a\in\Omega_j} u_j t^{r(a)-1} \times \chi(\mathbf{u},M,\mathbf{n},\mathbf{p})\tilde{\chi}(\mathbf{u},M,\mathbf{n},\mathbf{q}) \tag{3.95}$$

Analyticity. Let's now state the following analytical property:

Lemma 3.2.4.

$$\tilde{g}(\mathbf{u},M,\mathbf{n},\mathbf{q}) \equiv \tilde{\chi}(\mathbf{u},M,\mathbf{n},\mathbf{q}) \times \prod_{\substack{j<k\\ j\neq k}}^{M} \prod_{r=1}^{n_j} \prod_{s=1}^{n_k} \frac{u_j t^r - u_k t^{s+1}}{u_j t^r - u_k t^s} \tag{3.96}$$

and

$$g(\mathbf{u},M,\mathbf{n},\mathbf{p}) \equiv \chi(\mathbf{u},M,\mathbf{n},\mathbf{p}) \times \prod_{\substack{j<k\\ j\neq k}}^{M} \prod_{r=1}^{n_j} \prod_{s=1}^{n_k} \frac{u_j t^{n_j-r+1} - u_k t^{n_k-s}}{u_j t^{n_j-r} - u_k t^{n_k-s}} \tag{3.97}$$

are analytic in u_j *for* $u_j \in \mathcal{D}(s) \equiv \{u \in \mathbb{C} \setminus \{-st^{-r(a)+1}, -s^{-1}t^{-r(a)+1}\}\}$

Proof. Let's first prove the first part of the Lemma.

- On one hand, we can write

$$\prod_{\substack{j,k=1\\ j<k}}^{M} \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k\\ \sigma^{-1}(a)>\sigma^{-1}(b)}} \frac{u_j t^{r(a)+1} - u_k t^{r(b)}}{u_j t^{r(a)} - u_k t^{r(b)+1}} \times \prod_{\substack{j,k=1\\ j<k}}^{M} \prod_{r=1}^{n_j} \prod_{s=1}^{n_k} \frac{u_j t^r - u_k t^{s+1}}{u_j t^{r+1} - u_k t^s} = \prod_{\substack{j,k=1\\ j<k}}^{M} \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k}} \frac{u_j t^{r(a)} - u_k t^{r(b)-sign(\sigma^{-1}(a)-\sigma^{-1}(b))}}{u_j t^{r(a)} - u_k t^{r(b)}} t^{(sign(\sigma^{-1}(a)-\sigma^{-1}(b))+1)/2} \tag{3.98}$$

So, noticing that

$$\prod_{\substack{a,b\in\Omega_j\\ a<b}} \frac{u_j t^{r(a)} - u_j t^{r(b)-sign(\sigma^{-1}(a)-\sigma^{-1}(b))}}{u_j t^{r(a)} - u_j t^{r(b)}} = \begin{cases} 0 & \sigma \notin S'_n(\mathbf{n}) \\ \prod_{s,r=1}^{n_j} \frac{t^s-t^{r+1}}{t^s-t^r} \equiv \frac{1}{f(n_j)} & \sigma \in S'_n(\mathbf{n}) \end{cases} \tag{3.99}$$

we can instead sum over all $\sigma \in S_n$, and define $sign(\sigma,a,b) \equiv sign(\sigma^{-1}(a)-\sigma^{-1}(b))$, so (3.96) rewrite

$$\prod_j f(n_j) \sum_{\sigma\in S_n} \prod_{\substack{a,b=1\\ a<b}}^{N} t^{1/2} \frac{u_j t^{r(a)+sign(\sigma,a,b)/2} - u_k t^{r(b)-sign(\sigma,a,b)/2}}{u_j t^{r(a)} - u_k t^{r(b)}} \prod_{j=1}^{M} \prod_{a\in\Omega_j} \frac{(u_j t^{r(a)-1}+s)^{-q_{\sigma^{-1}a}-1}}{(1+su_j t^{r(a)-1})^{-q_{\sigma^{-1}a}}} \tag{3.100}$$

Let's define, for $a \in \Omega_j$, $\lambda_a = u_j t^{r(a)-1}$, and

$$
\begin{aligned}
&\phi(\lambda, M, \mathbf{n}, q) \\
&\equiv \prod_j f(n_j) \sum_{\sigma \in S_n} \prod_{\substack{a,b=1 \\ a<b}}^{N} t^{1/2} \frac{\lambda_a t^{sign(\sigma,a,b)/2} - \lambda_b t^{-sign(\sigma,a,b)/2}}{\lambda_a - \lambda_b} \prod_{j=1}^{M} \prod_{a \in \Omega_j} \frac{(\lambda_a + s)^{-q_{\sigma^{-1}a}-1}}{(1+s\lambda_a)^{-q_{\sigma^{-1}a}}}
\end{aligned} \tag{3.101}
$$

Poles at $\lambda_c = \lambda_d$ are simple poles for $\{\lambda_a\}_{a=1,\cdots,N} \in \mathcal{D}_\lambda(\mathbf{n})$, and the corresponding residues are (for $c < d$):

$$
\begin{aligned}
&\prod_j f(n_j) \sum_{\sigma \in S_n} \prod_{a<c} t^{1/2} \frac{\lambda_a t^{sign(\sigma,a,c)/2} - \lambda_d t^{-sign(\sigma,a,c)/2}}{\lambda_a - \lambda_d} \prod_{\substack{c<b \\ b \neq d}} t^{1/2} \frac{\lambda_d t^{sign(\sigma,c,b)/2} - \lambda_b t^{-sign(\sigma,c,b)/2}}{\lambda_d - \lambda_b} \\
&\times \prod_{\substack{a<d \\ a \neq c}} t^{1/2} \frac{\lambda_a t^{sign(\sigma,a,d)/2} - \lambda_d t^{-sign(\sigma,a,d)/2}}{\lambda_a - \lambda_d} \prod_{d<b} t^{1/2} \frac{\lambda_d t^{sign(\sigma,d,b)/2} - \lambda_b t^{-sign(\sigma,d,b)/2}}{\lambda_d - \lambda_b} \\
&\times \prod_{a \neq c,d} \left(\frac{\lambda_a t^{-1} + s}{1 + s\lambda_a t^{-1}} \right)^{-q_{\sigma^{-1}a}} \times \left(\frac{\lambda_d t^{-1} + s}{1 + s\lambda_d t^{-1}} \right)^{-q_{\sigma^{-1}c} - q_{\sigma^{-1}d}} \\
&\times \lambda_d t^{1/2} (t^{sign(\sigma,c,d)/2} - t^{-sign(\sigma,c,d)/2}) \\
&= \sum_{\sigma \in S_n} \prod_{a<c} t^{1/2} \prod_{a \neq c,d} \frac{\lambda_a t^{sign(\sigma,a,c)/2} - \lambda_d t^{-sign(\sigma,a,c)/2}}{\lambda_a - \lambda_d} \times \prod_{a \neq c,d} \frac{\lambda_a t^{sign(\sigma,a,d)/2} - \lambda_d t^{-sign(\sigma,a,d)/2}}{\lambda_a - \lambda_d} \\
&\times \prod_{a \neq c,d} \left(\frac{\lambda_a t^{-1} + s}{1 + s\lambda_a t^{-1}} \right)^{-q_{\sigma^{-1}a}} \times \left(\frac{\lambda_d t^{-1} + s}{1 + s\lambda_d t^{-1}} \right)^{-q_{\sigma^{-1}c} - q_{\sigma^{-1}d}} \\
&\times \lambda_d t^{1/2} (t^{sign(\sigma,c,d)/2} - t^{-sign(\sigma,c,d)/2})
\end{aligned} \tag{3.102}
$$

This term will just pick a minus sign for exchanging $\sigma^{-1}(c)$ and $\sigma^{-1}(d)$, so the residue cancel each other while summing over σ. Thus ϕ has no poles except at $\lambda_a = -s^{\pm 1}$ and so (3.96) is analytic in $u_j \in \mathcal{D}(s)$.

This completes the proof of the first part of Lemma 3.2.4.

• On the other hand, in equation (3.97), defining $R_{\mathbf{n}} \in S_n$ such as acting as the mirror permutation inside Ω_j $\forall j$, $R_{\mathbf{n}}^2 = Id$, we have

$$
\begin{aligned}
&\sum_{\tau \in S_n''} \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a \in \Omega_j \\ b \in \Omega_k \\ \tau^{-1}(a) > \tau^{-1}(b)}} \frac{u_j t^{r(a)} - u_k t^{r(b)+1}}{u_j t^{r(a)+1} - u_k t^{r(b)}} \\
&= \sum_{R_{\mathbf{n}} \circ \tau \in S_n''} \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a \in \Omega_j \\ b \in \Omega_k \\ \tau^{-1}(a) > \tau^{-1}(b)}} \frac{u_j t^{r(R_{\mathbf{n}}(a))} - u_k t^{r(R_{\mathbf{n}}(b))+1}}{u_j t^{r(R_{\mathbf{n}}(a))+1} - u_k t^{r(R_{\mathbf{n}}(b))}}
\end{aligned} \tag{3.103}
$$

Now we can notice that $r(R_{\mathbf{n}}a) = n_j - r(a) + 1$ and that $R_{\mathbf{n}} \circ \tau \in S''_n(\mathbf{n}) \Leftrightarrow \tau \in S'_n(\mathbf{n})$, so we end up with the term

$$\sum_{\tau \in S'_n} \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{\substack{a \in \Omega_j \\ b \in \Omega_k \\ \tau^{-1}(a) > \tau^{-1}(b)}} \frac{u_j t^{n_j - r(a)} - u_k t^{n_k - r(b)+1}}{u_j t^{n_j - r(a)+1} - u_k t^{n_k - r(b)}} \tag{3.104}$$

and the second part of Lemma 3.2.4 can be proved similarly to the first one.

□

Using the previous Lemmas, we can prove the following one, which is the last piece needed to proceed to the barycentering our our strings.

Lemma 3.2.5.

$$g(u, M, \mathbf{n}, p)\tilde{g}(u, M, \mathbf{n}, q) \prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{r=1}^{n_j} \prod_{s=1}^{n_k} \frac{u_j t^r - u_k t^s}{u_j t^r - u_k t^{s+1}} \frac{u_j t^{n_j - r} - u_k t^{n_k - s}}{u_j t^{n_j - r+1} - u_k t^{n_k - s}} \tag{3.105}$$

is analytic for $u_j \in \mathcal{D}(s)$ and $t^{1/2-n_i} < |u_i| < t^{1/2}$ $\forall i$, with g and $\tilde{g}$ defined in Lemma 3.2.4.

Proof. It is easy to show that

$$\prod_{\substack{j,k=1 \\ j<k}}^{M} \prod_{r=1}^{n_j} \prod_{s=1}^{n_k} \frac{u_j t^r - u_k t^s}{u_j t^r - u_k t^{s+1}} \frac{u_j t^{n_j - r} - u_k t^{n_k - s}}{u_j t^{n_j - r+1} - u_k t^{n_k - s}} = \prod_{\substack{j,k=1 \\ j<k}}^{M} \frac{(u_j - u_k)(u_j t^{-n_k} - u_k t^{-n_j})}{(u_j t^{n_j} - u_k)(u_j - u_k t^{n_k})} \tag{3.106}$$

We know from Lemma 3.2.4 that g and $\tilde{g}$ are analytic in $\mathcal{D}(s)$. Thus for $\mathbf{u} \in \mathcal{D}(s)$ the only poles in (3.105) are for $u_j = u_k t^{-n_j}$ and $u_j = u_k t^{n_k}$.

By definition, in $\mathcal{D}(\mathbf{n})$, $t^{1/2-n_i} < |u_i| < t^{1/2}$ $\forall i$, so $|u_k t^{-n_j}| < t^{1/2-n_j} < |u_j|$ and $|u_k t^{n_k}| > t^{1/2} > |u_j|$.

Thus the Lemma is proved.

□

Note that (3.95) can be written

$$\begin{aligned} I_\delta(\mathbf{p}, \mathbf{q}) = \sum_{M=1}^{N} \frac{1}{M!} \sum_{\mathbf{n} \in D_{N,M}} \prod_{j=1}^{M} \left(\oint_{|u_j| = e^{\epsilon_j}} \frac{du_j}{u_j} \right) \\ \times \frac{(1-s^2)^N}{(2i\pi)^M} \prod_{j=1}^{M} \prod_{r=1}^{n_j - 1} \frac{(t^r - t^{-1})(t^r - 1)}{t^r(1 - t^{-1})} \prod_{j=1}^{M} \prod_{a \in \Omega_j} u_j t^{r(a)-1} \\ \times g(u, M, \mathbf{n}, p)\tilde{g}(u, M, \mathbf{n}, q) \prod_{\substack{j,k=1 \\ j<k}}^{M} \frac{(u_j - u_k)(u_j t^{-n_k} - u_k t^{-n_j})}{(u_j t^{n_j} - u_k)(u_j - u_k t^{n_k})} \end{aligned} \tag{3.107}$$

Barycentering the integrals. Let us now define

$$
\begin{aligned}
|\varphi(\mathbf{u},\mathbf{n},M,\mathbf{s})\rangle \equiv & \sum_{1\le p_1<\cdots<p_N\le L}\ \sum_{\tau\in S''_n(\mathbf{n})}\ \prod_{\substack{j,k=1\\ j<k}}^{M}\ \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k\\ \tau^{-1}(a)>\tau^{-1}(b)}} \frac{u_{j,a}-tu_{k,b}}{tu_{j,a}-u_{k,b}} \prod_{j=1}^{M}\prod_{a\in\Omega_j}\left(\frac{u_{j,a}+s_j}{1+s_ju_{j,a}}\right)^{p_{\tau^{-1}a}} |\mathbf{p}\rangle \\
\equiv & \sum_{1\le p_1<\cdots<p_N\le L} \varphi(\mathbf{u},\mathbf{n},M,\mathbf{p},\mathbf{s})\,|\mathbf{p}\rangle
\end{aligned}
\tag{3.108}
$$

with $u_{j,a}\equiv u_jt^{r(a)-(n_j+1)/2}$, and where the s_j are initially set to s and introduced for a later procedure. For $s_j=s$ these states are the Bethe states (3.74), which are nothing else that re-normalized states (3.71) with constraints on the rapidities (some links, related to clustering), as one can notice as rewriting the sum over S_N as a sum over $S''_N(\mathbf{n})$, using calculation similar to what has been seen in equation (3.99).
We can notice that for $\mathbf{u}\in S_2^M$ ($u_j^\star=u_j^{-1}$)

$$
\begin{aligned}
&\sum_{\sigma\in S'_n(\mathbf{n})}\prod_{\substack{j,k=1\\ j<k}}^{M}\ \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k\\ \sigma^{-1}(a)>\sigma^{-1}(b)}} \frac{u_jt^{r(a)-(n_j+1)/2+1}-u_kt^{r(b)-(n_k+1)/2}}{u_jt^{r(a)-(n_j+1)/2}-u_kt^{r(b)-(n_k+1)/2+1}} \\
&\times\prod_{j=1}^{M}\prod_{a\in\Omega_j}\left(\frac{u_jt^{r(a)-(n_j+1)/2}+s_j}{1+s_ju_jt^{r(a)-(n_j+1)/2}}\right)^{-q_{\sigma^{-1}a}} \\
=&\sum_{\sigma\in S''_n(\mathbf{n})}\prod_{\substack{j,k=1\\ j<k}}^{M}\ \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k\\ \sigma^{-1}(a)>\sigma^{-1}(b)}} \frac{u_jt^{-r(a)+(n_j+1)/2+1}-u_kt^{-r(b)+(n_k+1)/2}}{u_jt^{-r(a)+(n_j+1)/2}-u_kt^{-r(b)+(n_k+1)/2+1}} \\
&\times\prod_{j=1}^{M}\prod_{a\in\Omega_j}\left(\frac{u_jt^{-r(a)+(n_j+1)/2}+s_j}{1+s_ju_jt^{-r(a)+(n_j+1)/2}}\right)^{-q_{\sigma^{-1}a}} \\
=&\overline{\varphi(\mathbf{u},\mathbf{n},M,\mathbf{q},\mathbf{s})}
\end{aligned}
\tag{3.109}
$$

the second line being obtained summing over $R_\mathbf{n}\circ\sigma\in S'_N$, i.e. $\sigma\in S''_N$ and relabeling $a\to R_\mathbf{n}(a)$ ($r(R_\mathbf{n}(a))=n_j-r(a)+1$). We can thus write, after a shift $|u_j|=e^{\epsilon_j}\to e^{x(-n_j+1)/2}$ in (3.95) (which has no impact, after Lemma 3.2.5, still supposing $Nx<y$) and the change of variable $u_j\to u_jt^{(-n_j+1)/2}$:

$$
\begin{aligned}
I_\delta(\mathbf{p},\mathbf{q}) =& \sum_{M=1}^{N}\frac{1}{M!}\sum_{\mathbf{n}\in D_{N,M}}\prod_{j=1}^{M}\left(\oint_{u_j\in S_2}\frac{du_j}{2i\pi u_j}\right)\prod_{r=1}^{n_j-1}\frac{(t^r-t^{-1})(t^r-1)}{t^r(1-t^{-1})} \\
&\times\prod_{j=1}^{M}\prod_{a\in\Omega_j}\frac{(1-s_j^2)u_{j,a}}{(s_j+u_{j,a})(1+s_ju_{j,a})} \\
&\times\varphi(\mathbf{u},M,\mathbf{n},p,\mathbf{s})\overline{\varphi(\mathbf{u},M,\mathbf{n},q,\mathbf{s})}
\end{aligned}
\tag{3.110}
$$

Or equivalently

$$
\begin{aligned}
\mathbb{I} = & \sum_{N=1}^{L} \sum_{M=1}^{N} \frac{1}{M!} \sum_{\mathbf{n} \in D_{N,M}} \prod_{j=1}^{M} \left(\oint_{u_j \in S_2} \frac{du_j}{2i\pi u_j} \right) \prod_{r=1}^{n_j - 1} \frac{(t^r - t^{-1})(t^r - 1)}{t^r(1 - t^{-1})} \\
& \times \prod_{j=1}^{M} \prod_{a \in \Omega_j} \frac{(1 - s_j^2) u_{j,a}}{(s_j + u_{j,a})(1 + s_j u_{j,a})} \\
& \times |\varphi(\mathbf{u}, M, \mathbf{n}, \mathbf{s})\rangle \langle \varphi(\mathbf{u}, M, \mathbf{n}, \mathbf{s})|
\end{aligned} \tag{3.111}
$$

3.2.4 The $XXZ_{n/2}$ spin chain

We are almost done with our proof. The last point to be treated concerns the variable s. Until now we considered it small ($1/s$ large). In the spin chain of finite size L, we can chose s such that for any number of magnon N, with of course $N \leq L$, the condition $0 < x < y/N$ is always true. In the large size limit however, as the number of magnons is unbounded, this condition cannot hold anymore and has to be relaxed. In addition, for finite size spin chains, the relation between $t = e^x$ and $s = e^{-y}$ is physical, i.e. inherent to our system, and is not related to the size of our chain.

We are thus now going to shift s_j in (3.110): $s_j > Nx \to s_j = t^{-n/2} \equiv t^{-\frac{n_j+1}{2}+m}$, $n \in \mathbb{N}^\star$ ($m \leq n_j/2$), and show that this shift does not produce any additional terms, namely that the residues of the grabbed poles shall vanish. During this shift some of the poles from

$$
\prod_{a \in \Omega_j} \frac{(s_j + t^{-\frac{-n_j+1}{2}+r(a)} u_j)^{\xi_{\sigma,\tau,a}-1}}{(1 + s_j t^{-\frac{-n_j+1}{2}+r(a)} u_j)^{\xi_{\sigma,\tau,a}+1}} \tag{3.112}
$$

will cross the integration contour, and we will show that all the corresponding residues will vanish while s_j goes to $t^{-\frac{n_j+1}{2}+m}$.

If $m \leq 0$, none of these poles are grabbed and s_j can be directly shifted. In the following we assume $m \geq 1$. Equation (3.112) can be rewritten

$$
\prod_{k=1}^{n_j} \frac{(s_j + t^{-\frac{-n_j+1}{2}+k} u_j)^{\tilde{p}_k - 1}}{(1 + s_j t^{-\frac{-n_j+1}{2}+k} u_j)^{\tilde{p}_k + 1}} \tag{3.113}
$$

with $\tilde{p}_k \equiv \xi_{\sigma,\tau,a}$, $r(a) = k$. Following from the definition of ξ, $\sigma \in S'_N(\mathbf{n})$ and $\tau \in S''_N(\mathbf{n})$, we have $j < k \Rightarrow \tilde{p}_j \geq \tilde{p}_k + 2$. Poles in (3.113) crossed during the shift $s_j > Nx \to t^{-\frac{n_j+1}{2}+m}$, $m \leq n_j/2$, will be in

$$
z = -st^{(n_j+1)/2-k} \text{ from } (s + t^{-(n_j+1)/2+k} u_j)^{\tilde{p}_{n_j-k+1}-1}, \; k \leq m \leq n_j/2, \; \tilde{p}_{n_j-k+1} \leq 0 \tag{3.114}
$$

and

$$z = -s^{-1}t^{-(n_j+1)/2+k} \text{ from } (1+st^{(n_j+1)/2-k}u_j)^{-\tilde{p}_k-1},\ k \leq m \leq n_j/2,\ \tilde{p}_k \geq 0 \quad (3.115)$$

The residue at (3.114) can be written, setting $l = n_j + 1 - 2m + k$ ($\Rightarrow n_j \geq l \geq k+1$)

$$\partial_{u_j}^{-\tilde{p}_{n_j-k+1}} \left(\cdots (1+st^{-(n_j+1)/2+l}u_j)^{-\tilde{p}_{n_j-l+1}-1} \cdots \right)\Big|_{u_j=-st^{(n_j+1)/2-k}} \quad (3.116)$$

but since $l \geq k+1$, we have $0 \leq -\tilde{p}_{n_j-k+1} < -\tilde{p}_{n_j-l+1} - 1$, so (3.116) is equal to a sum in which each term contains a $(1+st^{-(n_j+1)/2+l}u_j)^\alpha\big|_{u_j=-st^{(n_j+1)/2-k}}$ with $\alpha > 0$, which goes to zero with $s \to t^{-(n_j+1)/2+m}$.

On the other hand the residue at (3.115) can be written ($n_j \geq l \geq k+1$)

$$\partial_{u_j}^{\tilde{p}_k} \left(\cdots (s+t^{-(n_j+1)/2+k}u_j)^{\tilde{p}_{n_j-k+1}-1} \cdots \right)\Big|_{u_j=-s^{-1}t^{-(n_j+1)/2+k}} \quad (3.117)$$

but since $k < n_j - k + 1$, we have $\tilde{p}_k < \tilde{p}_{n_j-k+1} - 1$, so (3.117) is equal to a sum in which each term contain $(s+t^{-(n_j+1)/2+k}u_j)^\alpha\big|_{u_j=-s^{-1}t^{-(n_j+1)/2+k}}$ with $\alpha > 0$, which goes to zero with $s_j \to t^{-(n_j+1)/2+m}$. We can in this manner shift all the s_j to $s = t^{-(n_j+1)/2+m}$. Thus (3.110) remains valid for $s = t^{-n/2} = n_j\ \forall j,\ n \in \mathbb{N}^\star$, and Theorem 3.2.1 is proved.

3.2.5 The $XXX_{n/2}$ spin chain

The $XXX_{n/2}$ spin chain sector would be reached by taking the limit $s \to 1$, keeping $s^2t^n = 1$. To see that, let us proceed the change change of variable $u_j = \frac{x_j+\alpha in/2}{x_j-\alpha in/2}$, $\alpha = \frac{1+s}{1-s}$.

Remark 3.2.4. *One should be careful with the direction of integration here:* $du = \frac{-in\alpha dx}{(x-n\alpha i/2)^2}$ *but* $\oint du = \int_{-\infty}^{+\infty} \frac{+in\alpha dx}{(x-n\alpha i/2)^2}$.

Then we directly obtain, by setting $s = 1 - \epsilon n/2$, $t = 1+\epsilon$, and letting ϵ go to zero,

$$\begin{aligned}
&\frac{u_{j,a}+s}{1+su_{j,a}} \to \frac{x_{j,a}+in/2}{x_{j,a}-in/2} \\
&\frac{u_{j,a}-tu_{k,b}}{tu_{j,a}-u_{k,b}} \to \frac{x_{j,a}-x_{k,b}-i}{x_{j,a}-x_{k,b}+i} \\
&\oint_{u_j\in S_2} \frac{du_j}{u_j} \prod_{r=1}^{n_j-1} \frac{(t^r-t^{-1})(t^r-1)}{t^r(1-t^{-1})} \prod_{a\in\Omega_j} \frac{(1-s^2)u_{j,a}}{(s+u_{j,a})(1+su_{j,a})} \cdots \\
&\to \int_{\mathbb{R}} idx_j n^{n_j}(n_j-1)!n_j! \prod_{a\in\Omega_j}^{n_j} \frac{1}{(x_{j,a}+i\frac{n}{2})(x_{j,a}-i\frac{n}{2})} \cdots
\end{aligned} \quad (3.118)$$

where $x_{j,a} \equiv x + i(r(a) - (n_j+1/2)/2)$.

So defining the Bethe states for $XXX_{n/2}$

$$
\begin{aligned}
&|\Phi(\mathbf{x}, M, \mathbf{n})\rangle \equiv \\
&\sum_{1\le p_1<\cdots<p_N\le L} \sum_{\tau\in S''_n(\mathbf{n})} \prod_{\substack{j,k=1\\ j<k}}^{M} \prod_{\substack{a\in\Omega_j\\ b\in\Omega_k\\ \tau^{-1}(a)>\tau^{-1}(b)}} \frac{x_{j,a}-x_{k,b}-i}{x_{j,a}-x_{k,b}+i} \prod_{j=1}^{M}\prod_{a\in\Omega_j}\left(\frac{x_{j,a}+in/2}{x_{j,a}-in/2}\right)^{p_{\tau^{-1}a}} |\mathbf{p}\rangle \\
\equiv &\sum_{1\le p_1<\cdots<p_N\le L} \Phi(\mathbf{x}, M, \mathbf{n}, \mathbf{p})\,|\mathbf{p}\rangle
\end{aligned}
\tag{3.119}
$$

we can write

$$
\begin{aligned}
I_\delta(\mathbf{p},\mathbf{q}) = &\sum_{M=1}^{N}\frac{n^N}{M!}\sum_{\mathbf{n}\in D_{N,M}} \\
&\times \prod_{j=1}^{M}\left(\int_{x_j\in\mathbb{R}}\frac{idx_j}{2i\pi}\right) n_j!(n_j-1)!\prod_{a\in\Omega_j}\frac{1}{(x_{j,a}+in/2)(x_{j,a}-in/2)} \\
&\times \Phi(\mathbf{x}, M, \mathbf{n}, p)\overline{\Phi(\mathbf{x}, M, \mathbf{n}, q)}
\end{aligned}
\tag{3.120}
$$

or equivalently

$$
\begin{aligned}
\mathbb{I} = &\sum_{N=1}^{L}\frac{n^N}{M!}\sum_{M=1}^{N}\sum_{\mathbf{n}\in D_{N,M}} \\
&\times \prod_{j=1}^{M}\left(\int_{x_j\in\mathbb{R}}\frac{idx_j}{2i\pi}\right) n_j!(n_j-1)!\prod_{a\in\Omega_j}\frac{1}{(x_{j,a}+ni/2)(x_{j,a}-ni/2)} \\
&\times |\Phi(\mathbf{x}, M, \mathbf{n}, p)\rangle\,\langle\Phi(\mathbf{x}, M, \mathbf{n}, q)|
\end{aligned}
\tag{3.121}
$$

Remark 3.2.5. *To be expressed in the more familiar Bethe language, one can equivalently rewrite* (3.119)

$$
\begin{aligned}
\Phi(\mathbf{x}, M, \mathbf{n}, p) = &\sum_{\tau\in S''_n(\mathbf{n})}\prod_{\substack{a\in\tau^{-1}(\Omega_j)\\ b\in\tau^{-1}(\Omega_k)\\ j\neq k\\ \{a,b\}_\tau}}\frac{x_{j,\tau(a)}-x_{k,\tau(b)}+i}{x_{j,\tau(a)}-x_{k,\tau(b)}-i} \\
&\times\prod_{a=1}^{N}\left(\frac{x_{j,\tau(a)}+ni/2}{x_{j,\tau(a)}-ni/2}\right)^{p_a}
\end{aligned}
\tag{3.122}
$$

The Bethe states φ and Φ defined before are, except from the free state $M = N$, *on-shell* only in the $L \to \infty$ limit. Otherwise they are (at least slightly) *off-shell*, since $x_j = x_k + i$ cannot be solution of the Bethe equation for a finite L. In the general case, one thus need to consider finite size corrections, which has still to be investigated.

3.3 Conclusion

The two previous discussions, although taking place in two different contexts, involves similar calculations, as the Inverse Functional Problem and the Resolution of the Identity are two very close concepts. In both cases, we begun from a particularly favorable spin regime, and reached a more flavored one through analytic continuation. This implied a procedure of contour shrinking, during which poles are grabbed, linking rapidities one to another, namely generating strings of bound states. This results in a combinatorial sum over all string configurations. Surprisingly, in the case of half integer positive spin value, these string are constrained in length in the Zero-range Chipping Model with factorized steady state (ZCM), which is not the case for the resolution of the identity in XXZ. This is quite surprising given the Bethe states in these two context expresses in a very similar way, and that the concerned quantities we handled in both context (the resolution of the identity and the inverse functional problem) are relatively equivalent. This quantitative difference between those two conceptually close problems would deserve a deeper investigation. So far no physical interpretation seems to clearly emerge.

As evoked in Section 1.1.2, the Bethe states for ZCM collapses to the ASEP Bethe states in the $s^2q = 1$ (i.e. spin 1/2). This formally corresponds to the trivial case for the IFP for ZCM, as in this case the lengths of the strings are constrained to 1, i.e. simply don't appear.

In both cases, the calculation have been ran in the context of infinite volume systems, hence no Bethe equations in play. This feature is very simplifying the problem, as we can run our summation (namely our integrals) over a continuous set of rapidity. The finite volume case would in turn require a very different treatment. In such a case, a sensible choice wouldn't be to explicitly solve the Bethe equation for eventually summing over the Bethe roots, but rather to implement the Bethe equation inside the integrals (via poles depending on Bethe equations, not on their roots themselves), keeping those silent and hidden. Schematically, the integral

$$\oint dz \frac{1}{f(z)} \sim \sum_{z, f(z)=0}$$

accounts for the sum over Bethe equation $f(z) = 0$, without need of explicitly solving these. This interesting technical question could be a matter of future research efforts, but haven't been addressed during my thesis.

Chapter 4

Determinant Representations

The good guy.

During my journey in the vast world of integrable quantum models, the determinant representations proved to be unavoidable and reliable fellows. In particular, these structures appears when considering scalar products of Bethe states, as we saw in the context of the modified algebraic Bethe ansatz in Chapter 2, in the form of Izergin-Korepin and Slavnov determinant [12, 14, 28]. These objects thus are of particular interest given that they provide the first building blocks for the computation of form factors, and in turn correlation functions, namely opens the door on physical investigation.
These objects are very appreciate for two reasons. First of all, they provide a very compact and still explicit expression for objects that usually emerge from very complicated structures. Furthermore, they proved to be very reliable and flexible structures. For instance, we will see how the norm of Bethe states as a determinant, namely the Gaudin's determinant [28], can straightforwardly be obtained as the limit of the determinant for the on-shell/off-shell scalar products, namely the Slavnov determinant [29]. They also nicely behave in the large size limit, and a lot of analytic tools are available for studying these as they have independently been objects of interest for mathematicians. In other words, the determinant is the good guy. Everyone is happy when he shows up.

Now, inserting an operator inside the scalar product[1] may spoil the party, as the structure of Bethe states could then be altered. In some cases however, the operator in play preserve part of the structure of Bethe states. That is for instance the case for the Particle Number Operator (PNO) for the δ-Bose gas, which is the subject of the first part of this chapter, Section 4.1, which is adapted from my paper [25], in which is demonstrated a conjecture due to V. Terras [31]. As we will see, the action of the PNO on Bethe states can be roughly understood as cleaving the Bethe states living on the infinite lattice into two Bethe states living each in one half of the lattice, preserving the Bethe structures in each of these states. This is far more complicated in reality, but this results in a decomposition matrix elements of the PNO in the Bethe basis as a sum of scalar products of two half-Bethe states, each of these readable in terms of an Izergin-Korepin determinant. What is very interesting and intriguing is that this decomposition, when properly written,

[1] i.e. considering objects of the form $\langle\psi|\,\mathcal{O}\,|\psi'\rangle$

can be re-summed as a unique determinant, formally similar to a Slavnov's determinant. This expression for the matrix elements of the PNO, initially conjectured by V. Terras [31], could provide a very useful theoretical tool for probing the current behavior of the system.
In the second section 4.2 I explore a nice integral representation for the Izergin-Korepin determinant introduced by E. Bettelheim and I. Kostov [11]. Once supplied, this representation seems rather obvious, and the proof for its equivalence with the Izergin-Korepin determinant is straightforward. I then provide a proof for its equivalence with the Slavnov determinant, which is in turn more delicate. The link is also drawn with the Gaudin's determinant, leading to calculations of technical interest.

4.1 δ-Bose Gas: the Matrix Elements of the Particle Number Operator as a Determinant

This Section in adapted from my article [25]. In an introductory Section 4.1.1 I first expose a review of the Coordinate Bethe Ansatz for the δ-Bose gas on an infinite axis, which is a well know application of the coordinate Bethe ansatz [28]. The scalar product of Bethe states [28], also briefly reviewed, provides the keystone to compute a compact expression for the Matrix Elements of the Particle Number Operator as conjectured by V. Terras, the main result of the reflection. This question is motivated by potential applications of the result on quench problems, for which such a physical quantity constitutes an interesting tool, for instance in providing a useful probe for studying the current at the origin of the lattice.

4.1.1 Introduction

The Coordinate Bethe Ansatz for the δ-Bose gas on an infinite axis, an excessively simple integrable model: the coordinate Bethe ansatz for the δ-Bose gas incarnates the most primitive realization of the Bethe Ansatz machinery, while the absence of boundary or cyclic conditions in this system makes the Bethe equations to vanish. While the ansatz is friendly, the arising computations have to be ran with caution. To this end I put some efforts, in this introductory part, in fixing some conventions and notations that will be necessary for the forthcoming reasonings. These notations may seem heavy, but will turn out to be very convenient for our technical reasonings. Then, the spectral problem of the δ-Bose gas is presented, and solved by means of the Coordinate Bethe Ansatz, and an expression for the scalar product in terms of the Izergin-Korepin determinant is reached. These two first steps are very well developed in the literature, see e.g. [28].
The expression for the scalar product as a determinant provides the keystone to compute a compact expression for the Matrix Elements of the Particle Number Operator as conjectured by V. Terras. This result is stated in Section 4.1.2 as a theorem, alongside some necessary definitions of the objects taking part to the reflection.
Then is exposed the proof in Section 4.1.3. While the main steps of the reflection and intermediate lemmas are provided in this section, the interested reader will be send to the Appendix for detailed achievement of the technical steps.

Convention and Notations

The calculations to come will involve numerous manipulations over sets of variables, as well as products of functions over their elements. It will then be very convenient to introduce some notations to keep our expressions as sober and intelligible as possible.

• For $\bar{\alpha}$ a set of parameters, its j^{th} element will be written $(\bar{\alpha})_j$, or simply α_j when no confusion is possible, and its cardinal $\#\bar{\alpha}$.

The ordering of such a set will play an essential role in our development, as sums over these ordering are to appear.

The different ordering will be generated by action of permutations, which will act on a set by reordering its elements: for $P \in \pi_n$ a permutation and a set $\bar{\alpha} = \{\alpha_1, \cdots, \alpha_n\}$, we define $\bar{P}\bar{\alpha} = \{\alpha_{P^{-1}1}, \cdots, \alpha_{P^{-1}n}\}$. In other words we define the i^{th} element of $\bar{\alpha}$ to be the Pi^{th} element of $\bar{P}\bar{\alpha}$: a permutation act on the ordering of the elements of a set, not fundamentally on their labels.

This definition, which may seem unnatural, ensures the associativity of this action: $\overline{Q}(\overline{P}\overline{\alpha}) = \overline{QP}\overline{\alpha}$.

Example. Consider a set of three elements $\bar{\alpha} = \{a, b, c\}$ and the cyclic permutation $P = (123) = (12)(23)$ (where we used the notation (ij) for the transposition of elements i and j, $(ij)j = i$).

Thus we have the permuted sets $\overline{P}\bar{\alpha} = \{c, a, b\}$, $\overline{P^2}\bar{\alpha} = \{b, c, a\}$ and $\overline{P^3}\bar{\alpha} = \{a, b, c\} = \bar{\alpha}$.

• It is now natural to define the class of equivalence $\mathcal{C}(\bar{\alpha})$ of a set $\bar{\alpha}$ by the ensemble of sets related by permutations containing $\bar{\alpha}$: $\mathcal{C}(\bar{\alpha}) = \{\bar{P}\bar{\alpha}, P \in S_n\}$. A class of equivalence only contains information about the content of its attached sets, without regards to their ordering.

As sums over classes of equivalence of partitions will appear in the following, it will be convenient to define the representative element of a class $\mathcal{C}(\bar{\alpha})$, referred to as the *normal ordering* of $\bar{\alpha}$, denoted $: \bar{\alpha} :$. By normal ordering we thus consider the choice, arbitrary and unspecified, of a particular element of any class of equivalence.

• By feeding a function[2] with a set of arguments we express the product of functions: $f(\bar{\alpha}, \bar{\beta}) = \prod_{i,j} f(\alpha_i, \beta_j)$.

It will as well sometimes be necessary to operate these product over ordered indexes: For $\sim$ a relation of order, $f^{\sim}(\bar{\alpha}, \bar{\beta}) \equiv \prod_{i \sim j} f(\alpha_i, \beta_j)$.

• We define the scalar product of two sets $\bar{u}$ and $\bar{x}$ of lenghts M by $(\bar{u}, \bar{x}) = \sum_{i=1}^{M} u_i x_i$. Note that for any permutation $P \in S_M$ we have $(\bar{u}, \bar{x}) = (\bar{P}\bar{u}, \bar{P}\bar{x})$.

[2]for functions initially defined for single arguments, i.e. not fundamentally depending on sets of variables.

The Ansatz

In this Section is exposed a review of the δ-Bose gas problem and a compact expression for the scalar product in the infinite size case is reached [28].

Our system consists on M indistinguishable particle on an infinite axis, with hard core interaction of intensity c. Its dynamics is governed by the so called non-linear Schrodinger Hamiltonian:

$$H = -\sum_{j=1}^{M} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{i<j} \delta(x_i - x_j) \quad c \in \mathbb{R} \tag{4.1}$$

This system can actually be seen as the limit of the XXX spin chain, see for instance [28] Chapter 6, but this link is not obvious. It will however manifest by a great similarity of the objects appearing when solving the spectral problem, namely defining the Bethe states. This reflect the common algebraic background shared by these two model, albeit they have different physical interpretations. The first step here is to solve the eigenproblem

$$H\psi = \Lambda\psi \tag{4.2}$$

i.e. to find eigenstates ψ of H, by means of the Behte ansatz.

For P a permutation of S_N and $M \in \mathbb{N}$, we define the *elementary domain* $D_P = \{\bar{x} \in \mathbb{R}^M, x_{Pi} < x_{P(i+1)}\}$. Note that $\bar{x} \in D \Leftrightarrow \bar{P}\bar{x} \in D_P$ (where $D = D_{id}$). In the D_P sector, the non-linear Schrodinger equation (4.1) describes the dynamics of M free particles:

$$\left(\sum_{j=1}^{M} \frac{\partial^2 \psi}{\partial x_j^2} + \Lambda)\psi\right|_{D_P} = 0. \tag{4.3}$$

The full dynamical problem is specified by moreover imposing the boundary conditions:

$$\left.\frac{\partial \psi}{\partial x_{i+1}} - \frac{\partial \psi}{\partial x_i}\right|_{x_{i+1}-x_i=0^+} = c\psi|_{x_{i+1}=x_i} \tag{4.4}$$

obtained from (4.1) by integrating x_i on an infinitesimal domain slightly surrounding x_{i+1}, and exploiting the complete symmetry of the wave-function under exchange of particles. The ansatz here consists on assuming the fundamental solution of (4.1) in the domain D to be a Bethe superposition of plane waves

$$\psi_{\bar{u}}(\bar{x})|_D = \sum_{P \in S_M} A(\bar{P}\bar{u}) e^{i(\bar{x}, \bar{P}\bar{u})} \tag{4.5}$$

where A is a set of amplitudes to be determined and $\bar{u} \in \mathbb{C}^M$ the so called set ofrapidities, obviously satisfying (4.3) with $\Lambda = \sum_i u_i^2$. Note that our convention for the dependence on the permutation P here is the opposite of what has been chosen in Chapter 1. The current convention, although it makes our expressions a bit heavy at that stage, will turn out to be very practical in the next Section, in which resides all the technical subtlety.

Remark 4.1.1. *We here assume these parameter to be complex in order to handle the case of bound states that would occur for $c < 0$ (see Section 1.1.1). These bound states are characterized by strings of particle of rapidities linked by relations*

$$u_{i+1} - u_i + ic,\ \ c \in \mathbb{R},\ \sum_j u_j \in \mathbb{R}. \tag{4.6}$$

The existence of bound states will however be ignored in the following, given it has no impact on our reasoning. We will consider our rapidities as being real in our manipulations, as a matter of clarity, and treat the bound state case on a more informal level.

The boundary conditions (4.4) in turn reads

$$A(\overline{P_{jj+1}P}\bar{u}) = \frac{u_{P^{-1}j} - u_{P^{-1}(j+1)} - ic}{u_{P^{-1}j} - u_{P^{-1}(j+1)} + ic} A(\bar{P}\bar{u}), \quad \forall P \in S_N,\ \forall j \tag{4.7}$$

leading to the unique[3] solution

$$A(\bar{u}) = F(\bar{u}) \equiv \prod_{i<j} \left(1 + \frac{ic}{u_i - u_j}\right) \tag{4.8}$$

Remark 4.1.2. *For $c \in \mathbb{R}$ we have $F^\star(\bar{u}) = F(\bar{T}\bar{u})$, where we define the mirror permutation $Ti = \#\bar{u} - i + 1,\ i \in \{1, \cdots, M\}$.*

We here obtained an expression for the wave function restricted to the fundamental domain D. Its expression in any other fundamental domain can straightforwardly be obtained by exploiting the complete symmetry of our wave-function under exchange of particle:

$$\psi_{\bar{u}}(\bar{Q}\bar{x})\big|_{D_Q} = \psi_{\bar{u}}(\bar{x})|_D \Longrightarrow \psi_{\bar{u}}(\bar{x})|_{D_Q} = \sum_P F(\overline{Q^{-1}P}\bar{u}) e^{i(\bar{x},\bar{P}\bar{u})} \tag{4.9}$$

We can now write the Bethe states, eigenvectors of (4.1),

$$|\psi(\bar{u})\rangle \equiv c^{M/2} \int_D \psi_{\bar{u}}(\bar{x})|_D\, |\bar{x}\rangle \tag{4.10}$$

$$= c^{M/2} \int_{-\infty}^{\infty} dx_M \int_{-\infty}^{x_M} dx_{M-1} \cdots \int_{-\infty}^{x_2} dx_1 \sum_{P \in S_M} F(\bar{P}\bar{u}) e^{i(\bar{x},\bar{P}\bar{u})}\, |\bar{x}\rangle \tag{4.11}$$

associated with the eigenvalue

$$\lambda = \sum_i u_i^2 \tag{4.12}$$

where the state $|\bar{x}\rangle$ is the quantum state of M particles described by positions $\bar{x} \in D$. The integration here only runs over D, which is consistent given the symmetry of our

[3] up to global normalization

state. The global normalization factor $c^{M/2}$ is introduced by anticipation, as a matter of compactness. Given that the considered Hamiltonian is hermitian [4], the left eigenvectors can straightforwardly be written

$$\langle\psi(\bar{u})| \equiv c^{M/2} \int_D \psi^\star_{\bar{u}}(\bar{x})|_D \langle\bar{x}| \tag{4.13}$$

where $\psi^\star$ is the complex conjugated of ψ, and the dual basis is defined as $\langle\bar{x}|\bar{y}\rangle = \delta_{\bar{x},\bar{y}}$.

Scalar product and the Izergin-Korepin determinant

Let us now briefly review the computation of the scalar product of two Bethe states, following Gaudin (see [28] Chapter 4).

The expression (4.11) for the Bethe functions is unfriendly in that it involves integrals of parameters over domains depending explicitly on the other integrated ones. In this regard, we will shift the integrals in (4.11) such that the domains of integration become independent.
Let us define, for two sets $\bar{x}$ and $\bar{u}$ of cardinal M and $n \leq M$ an integer, the shifted sets $\{\bar{x}]^n$ and $[\bar{u}\}^n$ by their (i^{th}) elements:

$$\{\bar{x}]^n_i \equiv \begin{cases} \sum_{j=i}^n x_j & i \leq n \\ \sum_{j=n+1}^i x_j & i > n \end{cases} \qquad [\bar{u}\}^n_i \equiv \begin{cases} \sum_{j=1}^i u_j & i \leq n \\ \sum_{j=i}^{\#\bar{u}} u_j & i > n \end{cases} \tag{4.14}$$

We can easily check that $(\{\bar{x}]^n, \bar{u}) = (\bar{x}, [\bar{u}\}^n)$, and that $[\bar{u}\}^n + [\bar{u}'\}^n = [\bar{u} + \bar{u}'\}^n$ (and same for $\{\bar{x}]$). Note however that in general $\bar{P}[\bar{u}\}^n \neq [\bar{P}\bar{u}\}^n$ (and same for the action of a permutation over $\{\bar{x}]$).

We now proceed in (4.11) to the change of variables $\bar{x} \to \{\bar{x}]^M$, and rewrite our Bethe states, for $\#\bar{u} = M$:

$$|\psi(\bar{u})\rangle = c^{M/2} \int_{-\infty}^{\infty} dx_M \int_{-\infty}^{0} dx_{M-1} \cdots \int_{-\infty}^{0} dx_1 \sum_{P\in S_M} F(\bar{P}\bar{u}) e^{i(\bar{x},[\bar{P}\bar{u}\}^M)} |\{\bar{x}]^M\rangle \tag{4.15}$$

where we used the identity $(\{\bar{x}]^M, \bar{P}\bar{u}) = (\bar{x}, [\bar{P}\bar{u}\}^M)$. The compact form for the shifted exponents make this expression a bit ugly, and may seems absolutely unjustified at that stage. This will however turn out to be a very practical choice as the reflection will complexify in the following.

Using (4.15) and the identity $\langle\{\bar{x}]^M|\{\bar{y}]^M\rangle = \langle\bar{x}|\bar{y}\rangle = \prod_i \delta(x_i - y_i)$, we have

$$\langle\psi(\bar{u})|\psi(\bar{v})\rangle = c^M \sum_{P,Q\in S_M} F^\star(\bar{P}\bar{u}) F(\bar{Q}\bar{v}) \int_{-\infty}^{\infty} dx_M \int_{-\infty}^{0} dx_{M-1} \cdots \int_{-\infty}^{0} dx_1 e^{-i(\bar{x},[\bar{P}\bar{u}-\bar{Q}\bar{v}\}^M)} \tag{4.16}$$

[4] $\int \psi_1^\star H \psi_2 = \int (H\psi_1)^\star \psi_2$

Integrating this expression, we obtain:

$$\begin{aligned}\langle\psi(\bar u)|\psi(\bar v)\rangle =& c^M \sum_{P,Q\in S_M} F^\star(\bar P\bar u)F(\bar Q\bar v)2\pi\delta\left([\bar P\bar u-\bar Q\bar v\}_M^M\right)\frac{i^{M-1}}{\prod_{i=1}^{M-1}[\bar P\bar u-\bar Q\bar v - i0\}_i^M}\\ =&\left(\sum_i(u_i-v_i)\right)\frac{2\pi}{i}\delta\left(\sum_i(u_i-v_i)\right)\sum_{P,Q\in S_M}F^\star(\bar P\bar u)F(\bar Q\bar v)\frac{(ic)^M}{\prod_{i=1}^{M}[\bar P\bar u-\bar Q\bar v\}_i^M}\end{aligned} \tag{4.17}$$

the last line being obtained noticing that $[\bar P\bar u-\bar Q\bar v\}_M^M=\sum_{i=1}^M(u_i-v_i)$, independently of P and Q, and taking the limit $i0=0$ legally.

Remark 4.1.3. *The case of bound states wouldn't here require any particular treatment (Since the rapidities are no longer real, we have to consider $\bar u\to\bar u^\star$ in the right hand side of (4.16)). Indeed, for $P\in S_M$, one can see from (4.6) that $\mathfrak{Im}[\bar P\bar u^\star\}_i^M>0\Longrightarrow F(\bar P\bar u^\star)=0$, and same for the Q part, so the corresponding integral in (4.16) would actually vanish. The same argument will hold for the integration of the matrix elements of the particle number operator.*

One can now make use of Lemma 4.1.2 (See Appendix 4.1.5), due to Gaudin, and finally obtain

$$\langle\psi(\bar u)|\psi(\bar v)\rangle=\frac{2\pi}{i}\left(\sum_i(u_i-v_i)\right)\delta\left(\sum_i(u_i-v_i)\right)\mathcal{K}_M(\bar u|\bar v) \tag{4.18}$$

where we defined, for two sets of parameter $\bar u$ and $\bar v$ of equal length, $\#\bar u=\#\bar v=n$, the Izergin-Korepin determinant

$$\mathcal{K}_n(\bar u|\bar v)\equiv g^<(\bar u,\bar u)\ g^>(\bar v,\bar v)\ h(\bar u,\bar v)\ \det_{i,j}[t(u_i,v_j)] \tag{4.19}$$

$$=\det_{i,j}[h^{-1}(u_i,v_j)]\det_{i,j}[t(u_i,v_j)] \tag{4.20}$$

where we used the well know factorized expression for the Cauchy determinant in the last line, and

$$f(u,v)=1+g(u,v)\equiv 1+\frac{ic}{u-v} \tag{4.21}$$

$$h(u,v)=\frac{f(u,v)}{g(u,v)}=\frac{u-v+ic}{ic} \tag{4.22}$$

$$t(u,v)=\frac{g(u,v)}{h(u,v)}=\frac{(ic)^2}{(u-v)(u-v+ic)}=\frac{ic}{u-v}-\frac{ic}{u-v-ic} \tag{4.23}$$

Note that $g(u,v)=-g(v,u)$. This anti-symmetry, alongside the anti-symmetry of the determinant under transposition of lines or columns, makes this object totally symmetric under permutation inside each family of parameters:

$$\mathcal{K}_n(\bar u|\bar v)=\mathcal{K}_n(\bar P\bar u|\bar Q\bar v)\quad\forall P,Q\in S_n \tag{4.24}$$

As expected, expression (4.18) for the scalar product exhibits the orthogonality of Bethe states (note that $\mathcal{K}_M(\bar{u}|\bar{v}) \propto \frac{1}{\sum_i (u_i - v_i)}$).

Remark 4.1.4. *It is really tempting to evoke the strong similarity that this object shares with its XXX cousin.*
Indeed, the scalar product of two M-magnons states in an XXX spin chain of length $2M$ can also be expressed in term of the Izergin-Korepin determinant:
For a periodic XXX spin chain of inhomogeneities $\bar{\theta}$ and two sets of rapidities $\bar{v}_I$ and $\bar{v}_{II}$ (one of these satisfying the Bethe equations), with $\#\bar{\theta} = 2\#\bar{v}_I = 2\#\bar{v}_{II}$, the scalar product of the two corresponding Bethe states can be expressed as [29]

$$\langle \psi(\bar{v}_I) | \psi(\bar{v}_{II}) \rangle = \mathcal{K}_{2M}(\bar{\theta} - ic/2 | \{\bar{v}_I, \bar{v}_{II}\}). \tag{4.25}$$

While this formal similarity appearing between the scalar products in XXX and in the δ-Bose gas is indubitable, the role played by our parameters are in each cases very (and intriguingly) different: the formal map can be expressed as $\{\bar{u}, \bar{v}\} \to \{\bar{\theta} - ic/2, \{\bar{v}_I, \bar{v}_{II}\}\}$, i.e. one set of rapidities in the δ-Bose gas plays the role of the inhomogeneities in XXX.

Remark 4.1.5. *The question of completeness of the Bethe states is not adressed here. Although very strong, the Bethe hypothesis, on which relies the ansatz, actually leads to a complete set of states. This highly non trivial fundamental feature, reflect of integrability, can be appraoched through different strategies. One can for instence cite [30] where is obtained an expression for the identity as a sum over Bethe states projectors, hence demonstrating the completeness of the Bethe states.*

In this section has been reviewed the diagonalization of the Hamiltonian for the δ-Bose gas on an infinite axis (4.1) by means of the coordinate Bethe ansatz, which led us to an expression for the Bethe States (4.11).

In the next section, I propse in this context a reasoning and computations that led me to a compact expression for the Matrix Elements (in the Bethe basis) of the Particle Number Operator (MEPNO) in term of a determinant, as conjectured by V. Terras [31]. This Section is adapted from my paper [25].

4.1.2 Definition and Result

The Paticle Number Operator is defined as follows.
Let κ be a complex parameter. We define the particle number operator $\mathcal{O}_\kappa(\bar{\mathbf{x}})$, with $\mathbf{x}$ the coordinate operator, as counting the number of particles with negative coordinate: for $\mathcal{C}(\bar{x}) \in \mathcal{C}(\mathbb{R}_-^n \otimes \mathbb{R}_+^{M-n})$, $\mathcal{O}_\kappa(\bar{\mathbf{x}}) |\bar{x}\rangle = \kappa^n |\bar{x}\rangle$:

$$\begin{aligned} \mathcal{O}_\kappa(\bar{\mathbf{x}}) &\equiv \prod_{i=1}^{M} \mathcal{O}_\kappa(\mathbf{x}_i) \\ \mathcal{O}_\kappa(\mathbf{x}_i) &\equiv \kappa^{\theta(-\mathbf{x}_i)} \\ \mathbf{x}_i |\bar{x}\rangle &\equiv x_i |\bar{x}\rangle \\ \theta(x) &= \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \end{aligned} \tag{4.26}$$

Note that the elements of $|\mathcal{C}(\mathbb{R}_-^n \times \mathbb{R}_+^{M-n})\rangle$ for $n \in \{0, \cdots, M\}$ are the maximal eigen-domains of $\mathcal{O}_\kappa(\bar{x})$, i.e. any eigen-domain of $\mathcal{O}_\kappa$ is contained in one these.

Remark 4.1.6. *It is clear that the "particle number operator" as defined above is not actually returning the number of particle (on the left side of the axis) as an eigenvalue, but is a generator of this latter operator, which can straightforwardly be defined as* $\mathcal{N}(\bar{\mathbf{x}}) = \frac{\partial \mathcal{O}_\kappa(\bar{\mathbf{x}})}{\partial \kappa}\Big|_{\kappa=1}$.

The resulting expression for the MEPNO is given by the following theorem.

Theorem 4.1.1. *For two real sets of rapidities $\bar{u}, \bar{v} \in \mathbb{R}^M$, defining the Bethe states,*

$$
\begin{aligned}
|\psi(\bar{u})\rangle \equiv & \int_D \psi_{\bar{u}}(\bar{x})|_D \, |\bar{x}\rangle \\
= & c^{M/2} \int_{-\infty}^{\infty} dx_M \int_{-\infty}^{x_M} dx_{M-1} \cdots \int_{-\infty}^{x_2} dx_1 \sum_{P \in S_M} F(\bar{P}\bar{u}) e^{i(\bar{P}\bar{u}, \bar{x})} \, |\bar{x}\rangle
\end{aligned}
$$

F defined in (4.8), the MEPNO is expressed as

$$
\langle \psi(\bar{u}) | \, \mathcal{O}_\kappa(\bar{\mathbf{x}}) \, | \psi(\bar{v}) \rangle \equiv \mathcal{S}_\kappa^M(\bar{u}|\bar{v}) \tag{4.27}
$$

$$
= det_{i,j}^{-1}[h^{-1}(u_i, v_j)] det_{i,j} \left[t(v_i, u_j) \frac{h(v_i, \bar{u})}{h(\bar{u}, v_i)} \frac{h(\bar{v}, v_i)}{h(v_i, \bar{v})} + \kappa t(u_j, v_i) \right] \tag{4.28}
$$

$$
\begin{aligned}
= & det_{i,j}^{-1}\left[\frac{ic}{u_i - v_j + ic}\right] \\
& \times det_{i,j} \left[\frac{(ic)^2}{(v_i - u_j)(v_i - u_j + ic)} \frac{v_i - \bar{u} + ic}{v_i - \bar{u} - ic} \frac{v_i - \bar{v} - ic}{v_i - \bar{v} + ic} + \kappa \frac{(ic)^2}{(v_i - u_j)(v_i - u_j - ic)} \right]
\end{aligned}
$$

Remark 4.1.7. *We can here be tempted to compare this expression for the MEPNO in the δ-Bose gas with the expression for the scalar product in XXX, as obtained by N. A. Slavnov. For $|\psi(\bar{u})\rangle_\kappa$ a Bethe state for the κ-twisted*[5] *XXX spin chain and $|\psi(\bar{v})\rangle$ a Bethe vector for the non-twisted chain, the scalar product can be expressed, see [27], in term of the so called Slavnov determinant as*

$$
\langle \psi(\bar{v}) | \psi(\bar{u}) \rangle_\kappa = det_{i,j}^{-1}[h^{-1}(u_i, v_j)] det_{i,j} \left[\kappa t(u_j, v_i) + t(v_i, u_j) \frac{h(v_i, \bar{u})}{h(\bar{u}, v_i)} \frac{h(\bar{v}, v_i)}{h(v_i, \bar{v})} \right]
$$

which exactly corresponds to (4.27).

While the link between the scalar product in the δ-Bose gas and in XXX was formally established via a one-to-one map of the spectral and inhomogeneity parameters (see Remark 4.1.4), the link here seems more direct.

[5] We refer here to a diagonal twist $\begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}$ of the monodromy matrix in ABA.

4.1.3 Proof

As for the computation of the scalar product, it will be convenient to rewrite the Bethe state (4.11) as integrals over domains independent on the integrated parameters. In this case it is appropriate to write a Bethe states as a sum over eigen-domains of the particle number operator, namely domains made of product of $\mathbb{R}^{\pm}$. Such a writing is performed in Appendix 4.1.5, so that we can rewrite the MEPNO (4.27) as

$$\mathcal{S}_{\kappa}^{M}(\bar{u}|\bar{v}) = c^M \sum_{n=0}^{M} \kappa^n \prod_{i=1}^{n} \int_{\mathbb{R}_-} dx_i \prod_{i=n+1}^{M} \int_{\mathbb{R}_+} dx_i \sum_{P,Q\in S_M} F^{\star}(\bar{P}\bar{u})F(\bar{Q}\bar{v})e^{-i(\bar{x},[\bar{P}\bar{u}-\bar{Q}\bar{v}\}^n)} \quad (4.29)$$

After integrating $\bar{x}$, one obtains

$$\mathcal{S}_{\kappa}^{M}(\bar{u}|\bar{v}) = \sum_{n=0}^{M} \kappa^n (-)^{M-n} \sum_{P,Q\in S_M} \frac{(ic)^M F^{\star}(\bar{P}\bar{u})F(\bar{Q}\bar{v})}{\prod_{i=1}^{M}[\bar{P}\bar{u}-\bar{Q}\bar{v}\}_i^n} \quad (4.30)$$

We are now going to re-express the summation in (4.30).
On one hand we notice that the two terms $\prod_{i=1}^{n}[\bar{P}\bar{u}-\bar{Q}\bar{v}\}_i^n$ and $\prod_{i=n+1}^{M}[\bar{P}\bar{u}-\bar{Q}\bar{v}\}_i^n$ depend on two disjoint sets of rapidity, $\{(\bar{P}\bar{u}-\bar{Q}\bar{v})_i\}_{i=1,\cdots,n}$ and $\{(\bar{P}\bar{u}-\bar{Q}\bar{v})_i\}_{i=n+1,\cdots,M}$ respectively.
On the other hand one can see that the term $F^{\star}(\bar{P}\bar{u})F(\bar{Q}\bar{v})$ can be factorized in two terms depending separately on these two disjoint sets and a third crossed term depending on the normal ordered version of these sets (see Appendix 4.1.5).
We are thus going to transform the summation in (4.30) as a sum over the normally ordered partitions $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ with $\#\bar{u}_I = \#\bar{v}_I$, and a sum over permutations acting inside each partitioned subset. The re-summation is rigorously performed in Appendix 4.1.5, and it leads us to a re-expression of (4.30) as

$$\begin{aligned}\mathcal{S}_{\kappa}^{M}(\bar{u}|\bar{v}) = &\sum \kappa^{\#I} f(\bar{u}_{II},\bar{u}_I) f(\bar{v}_I,\bar{v}_{II}) \\ &\times \sum_{P_I,Q_I\in S_{\#I}} \frac{(ic)^{\#I} F^{\star}(\bar{P}_I\bar{u}_I)F(\bar{Q}_I\bar{v}_I)}{\prod_{i=1}^{\#I}[\bar{P}_I\bar{u}_I-\bar{Q}_I\bar{v}_I\}_i^{\#I}} \\ &\times \left(\sum_{P_{II},Q_{II}\in S_{\#II}} \frac{(ic)^{\#II} F^{\star}(\bar{P}_{II}\bar{u}_{II})F(\bar{Q}_{II}\bar{v}_{II})}{\prod_{i=1}^{\#II}[\bar{P}_{II}\bar{u}_{II}-\bar{Q}_{II}\bar{v}_{II}\}_i^{\#II}} \right)^{\star}\end{aligned} \quad (4.31)$$

where the first sum runs over the normal ordered partitions $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ with $\#\bar{u}_I = \#\bar{v}_I$, and is made use of the notation $\#I = \#\bar{u}_I = \#\bar{v}_I$ (and same for $\#II$).

Remark 4.1.8. *This expression can be intuitively read as a sum over the number of particle on one side of the axis, n, of the product of two scalar products of states each living on one side of the axis, weighted by κ^n. This is not so surprising given that we were previously able to rewrite our states as decomposed on the operator $\mathcal{O}$'s subsectors, as in* (4.41).

We can now obviously make use of Lemma 4.1.2 for the two summations over permutations acting on the subsets I and II separately, and rewrite (4.31) as

$$\mathcal{S}_{\kappa}^{M}(\bar{u}|\bar{v}) = \sum \kappa^{\#\bar{v}_I} f(\bar{v}_I, \bar{v}_{II}) \sum \mathcal{K}_{\#\bar{u}_I}(\bar{u}_I, \bar{v}_I)\mathcal{K}_{\#\bar{u}_{II}}(\bar{v}_{II}, \bar{u}_{II}) f(\bar{u}_{II}, \bar{u}_I) \tag{4.32}$$

where we used the identity $\mathcal{K}_n^{\star}(\bar{u}|\bar{v}) = \mathcal{K}_n(\bar{v}|\bar{u})$, and the sum over partitions split into a first sum over normally ordered partitions $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ and a second one over $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$, with $\#\bar{u}_I = \#\bar{v}_I$.

We can now in turn straightforwardly apply Lemma 4.1.3 to the second sum in (4.32) and we obtain

$$\mathcal{S}_{\kappa}^{M}(\bar{u}|\bar{v}) = \sum_{\bar{v}\Rightarrow\{\bar{v}_I,\bar{v}_{II}\}} (-\kappa)^{\#\bar{v}_I} f(\bar{v}_I, \bar{v}_{II}) f(\bar{u}, \bar{v}_I)\mathcal{K}_M(:\{\bar{v}_I - c, \bar{v}_{II}\}:|\bar{u}) \tag{4.33}$$

We are here allowed to choose, as a matter of simplicity, the normal ordering for the sets in $\mathcal{K}$, given the convenient symmetry of this latter (we here simply considered $:\bar{u}:=\bar{u}$ and $:\bar{v}:=\bar{v}$).

We need for the following to define the shift operator: $D_{\bar{w}}^{-1} \equiv \prod_{i=1}^{\#\bar{w}} D_{(\bar{w})_i}^{-1}$ with $D_w f(w) \equiv f(w+c)$, i.e. $D_w = e^{c\partial_w}$.

Remark 4.1.9. *The normal ordering of a shifted set can be properly defined as the shifted normal ordered set. For $\bar{v} = \{\bar{v}_I, \bar{v}_{II}\}$ and $:\bar{v}:=\bar{v}$, $:\{\bar{v}_I - c, \bar{v}_{II}\}: \equiv D_{\bar{v}_I}^{-1}\bar{v}$, and so $\mathcal{K}_M(:\{\bar{v}_I - c, \bar{v}_{II}\}:|\bar{u}) = D_{\bar{v}_I}^{-1}\mathcal{K}_M(\bar{v}|\bar{u})$ (we here consider $:\bar{v}:=\bar{v}$).*

After developing the Izergin-Korepin determinant in (4.33) according to (4.20), making use of the property of the different functions at play and taking care of the product over ordered indexes, we can simplify and clean a bit our expression (see Appendix 4.1.5), after what we can eventually rewrite (4.33)

$$\begin{aligned}\mathcal{S}_{\kappa}^{M}(\bar{u}|\bar{v}) =& g^{<}(\bar{u}, \bar{u}) g^{>}(\bar{v}, \bar{v}) h^{-1}(\bar{v}, \bar{v}) \\ &\times \sum_{\bar{v}\Rightarrow\{\bar{v}_I,\bar{v}_{II}\}} \kappa^{\#I} h(\bar{u}, \bar{v}_I) h(\bar{v}_I, \bar{v}) h(\bar{v}_{II}, \bar{u}) h(\bar{v}, \bar{v}_{II}) D_{\bar{v}_I}^{-1} \\ &\times \det_{i,j}\left[t(v_i, u_j)\right]\end{aligned} \tag{4.34}$$

Now, on one hand the element in the last line, namely the determinant, does not depends on the summed partitions in the line above. On the other hand, the whole term in the second line can be factorized as a product of operator, using the development $\prod_{i\in\alpha}(x_i + y_i) = \sum_{\beta\subseteq\alpha}\left[\prod_{i\in\beta} x_i \prod_{j\in\alpha\setminus\beta} y_j\right]$ (provided that $[x_i, y_j] = 0\ \forall i, j$). Doing so we get from (4.34)

$$\mathcal{S}_\kappa^M(\bar{u}|\bar{v}) = \lim_{\bar{w}\to\bar{v}} g^<(\bar{u},\bar{u})g^>(\bar{v},\bar{v})h^{-1}(\bar{v},\bar{v})$$
$$\times \prod_{i=1}^{M}(h(v_i,\bar{u})h(\bar{w},v_i) + \kappa h(\bar{u},v_i)h(v_i,\bar{w})D_{v_i}^{-1})\mathrm{det}_{i,j}\left[t(v_i,u_j)\right] \quad (4.35)$$

where we simply introduce the limit $\bar{w} \to \bar{v}$ to prevent our shift operators from interacting with the other terms of the product (i.e. forcing their commutativity).

Now the last step is straightforward: we distribute the different terms of the product of operators over the different lines of the matrix, on which each act independently, and eventually obtain

$$\mathcal{S}_\kappa^M(\bar{u}|\bar{v}) = g^<(\bar{u},\bar{u})g^>(\bar{v},\bar{v})h^{-1}(\bar{v},\bar{v})$$
$$\times \mathrm{det}_{i,j}\left[h(v_i,\bar{u})h(\bar{v},v_i)t(v_i,u_j) + \kappa h(\bar{u},v_i)h(v_i,\bar{v})t(u_j,v_i)\right]$$
$$= \mathrm{det}_{i,j}^{-1}[h^{-1}(u_i,v_j)]\mathrm{det}_{i,j}\left[\kappa t(u_j,v_i) + t(v_i,u_j)\frac{h(v_i,\bar{u})}{h(\bar{u},v_i)}\frac{h(\bar{v},v_i)}{h(v_i,\bar{v})}\right] \quad (4.36)$$

where we used the identity for the Cauchy determinant for the first term, and developed terms of the form $\prod_{i=1}^{M} f(x,u_i)$ over lines or column of the matrix in the last determinant.

Hence Theorem 4.1.1 is proved.

4.1.4 Conclusion

The brute force approach we followed here led to the quantitative aspect of interest: a nice and compact expression for the Matrix Elements of the Particle Number Operator, which involves a determinant.
The link between the MEPNO in the δ-Bose gas and the scalar product in the XXX spin chain is direct, albeit tricky. Indeed, although we may agree that diagonally twisting the XXX spin chain and weighting the Particle Number are of the same nature, conceptually speaking, the lack of consistency for the concept of "number of particles on one side of a periodic spin chain" makes this connection dubious. We also previously drew a formal link between the scalar products of Bethe states in the δ-Bose gas and in XXX through a mapping between rapidities of the first system and inhomogeneities of the latter, see Remark 4.1.4. These two links seem to reflect the common algebraic background shared by the XXX spin chain and the δ-Bose gas. Note that the MEPNO on a segement in the periodic XXX spin chain has actually already been obtained through ABA, see for instance [32] Section 7.2, or [27]. This result however appears in a very different form than for the MEPNO in the δ-Bose gas or the scalar product in the spin chain. This important formal inadequacy simply translate the deep difference in nature between these two objects, which are the number of particle 'at the right side', and 'on a segment'.

The δ-Bose gas is well known as a limit of its XXX cousin [28]. An approach through ABA would thus provides us with a more promising development, as it may enables us to treat the problem on the more fundamental algebraic level.

Our result may moreover entail a practical interest, in that the MEPNO can, obviously, be used to count the number of particles lying on one side of the axis, and by extension (namely its time derivative) provides us with a concrete probe for the current at the origin of the axis, and its fluctuations, e.g. following a quench of the system, on the analytic level. This latter problem however brings its fistful of trouble, as the most natural approach would suggest to express our evolving states in the Bethe basis, which seems, at least at first sight, to be a very non-trivial task, see Chapter 3.

4.1.5 Appendix

Two Useful Lemmas

Lemma 4.1.2. *Let $\bar{u}$ and $\bar{v}$, with $\#\bar{u} = \#\bar{v} = M$, be two sets of real parameters. Then*

$$\sum_{P,Q\in S_M} F^\star(\bar{P}\bar{u})F(\bar{Q}\bar{v})\frac{(ic)^M}{\prod_{i=1}^M [P\bar{u} - Q\bar{v}\}_i^M} = \mathcal{K}_M(\bar{u}|\bar{v}) \tag{4.37}$$

The proof of this Lemma can be found in [28], Appendix B.

Lemma 4.1.3. *Let $\bar{\gamma}$, $\bar{\alpha}$ and $\bar{\beta}$ be sets of complex parameters with $\#\bar{\alpha} = m_1$, $\#\bar{\beta} = m_2$ and $\#\bar{\gamma} = m_1 + m_2$. Then*

$$\sum \mathcal{K}_{m_1}(\bar{\gamma}_I|\bar{\alpha})\mathcal{K}_{m_2}(\bar{\beta}|\bar{\gamma}_{II})f(\bar{\gamma}_{II},\bar{\gamma}_I) = (-)^{m_1} f(\bar{\gamma},\bar{\alpha})\mathcal{K}_{m_1+m_2}(\{\bar{\alpha}-c,\bar{\beta}\}|\bar{\gamma}) \tag{4.38}$$

where the summation is taken over the normal ordered partitions $\bar{\gamma} \Rightarrow \{\bar{\gamma}_I, \bar{\gamma}_{II}\}$, with $\#\bar{\gamma}_I = \#\bar{\alpha}$.

The proof for this lemma can be found in [63].

Projecting the integrals over *Particle Number Operator* maximal eigen-domains.

Let's define the domain $D^n \subset D$ as the domain of (strictly) ordered positions with n of these being negatives:

$$D^n = \{\bar{x} \in \mathbb{R}^M | x_1 < \cdots < x_n < 0 < x_{n+1} < \cdots < x_M\} \tag{4.39}$$

These domains are actually the ordered *particle number operator*'s eigen-domains: $\mathcal{O}_\kappa(\mathbf{\bar{x}})\,|D^n\rangle = \kappa^n\,|D^n\rangle$.

Then the Bethe functions (4.11) can be rewritten as a sum over integrated domain D^n

$$\begin{aligned}
|\psi(\bar{u})\rangle =& c^{M/2}\sum_{n=0}^{M}\int_{D^n} d\bar{x} \sum_{P\in S_M} F(\bar{P}\bar{u})e^{-i(\bar{P}\bar{u},\bar{x})}\,|\bar{x}\rangle \\
=& c^{M/2}\sum_{n=0}^{M}\int_{-\infty}^{0} dx_n \int_{-\infty}^{x_n} dx_{n-1}\cdots\int_{-\infty}^{x_2} dx_1 \int_0^\infty dx_{n+1}\int_{x_{n+1}}^\infty dx_{n+2}\cdots\int_{x_{M-1}}^\infty dx_M \\
&\times \sum_{P\in S_M} F(\bar{P}\bar{u})e^{-i(\bar{P}\bar{u},\bar{x})}\,|\bar{x}\rangle
\end{aligned} \tag{4.40}$$

It will appear important, for the computations of the desired quantity, to shift the integration domains to domains independent of the integrated positions. Thus, shifting the integration boundaries in (4.40) to $\mathbb{R}^{\pm}$, one can write the Bethe function

$$
\begin{aligned}
|\psi(\bar{u})\rangle =& c^{M/2}\sum_{n=0}^{M}\prod_{i=1}^{n}\int_{\mathbb{R}^-} dx_i \prod_{i=n+1}^{M}\int_{\mathbb{R}^+} dx_i \sum_{P\in S_M} F(\bar{P}\bar{u})e^{-i(\{\bar{x}]^n,\bar{P}\bar{u})}\,|\{\bar{x}]^n\rangle \\
=& c^{M/2}\sum_{n=0}^{M}\prod_{i=1}^{n}\int_{\mathbb{R}^-} dx_i \prod_{i=n+1}^{M}\int_{\mathbb{R}^+} dx_i \sum_{P\in S_M} F(\bar{P}\bar{u})e^{-i(\bar{x},[\bar{P}\bar{u}\}^n)}\,|\{\bar{x}]^n\rangle
\end{aligned}
\tag{4.41}
$$

We here used the identity $(\{\bar{x}]^n, \bar{P}\bar{u}) = (\bar{x}, [\bar{P}\bar{u}\}^n)$.

Resummation over 2-partitions

For any P and Q elements of S_M and $n \leq M$ an integer, we uniquely define

$$
\begin{aligned}
\bar{P}\bar{u} =& \{\bar{P}_I\bar{u}_I, \bar{P}_{II}\bar{u}_{II}\} \\
\bar{u}_I =& : \bar{P}_I\bar{u}_I : \\
\bar{u}_{II} =& : \bar{P}_{II}\bar{u}_{II} : \\
\\
\#\bar{u}_I =& n = \#\bar{v}_I \\
\\
\bar{Q}\bar{v} =& \{\bar{Q}_I\bar{v}_I, \bar{Q}_{II}\bar{v}_{II}\} \\
\bar{v}_I =& : \bar{Q}_I\bar{u}_I : \\
\bar{v}_{II} =& : \bar{Q}_{II}\bar{v}_{II} :
\end{aligned}
\tag{4.42}
$$

Now, one can easily show that for $\bar{w} = \{\bar{w}_I, \bar{w}_{II}\}$, with $\#\bar{w} = M$ and $\#\bar{w}_I = n$, we have

$$
\prod_{i=1}^{n}[\bar{w}\}_i^n = \prod_{i=1}^{n}[\bar{w}_I\}_i^n \tag{4.43}
$$

$$
\prod_{i=n+1}^{M}[\bar{w}\}_i^n = \prod_{i=1}^{M-n}[\bar{w}_{II}\}_i^0 \tag{4.44}
$$

and that

$$
F(\bar{w}) = F(\bar{w}_I)F(\bar{w}_{II})f(\bar{w}_I, \bar{w}_{II}) \tag{4.45}
$$

$$
f(\bar{w}_I, \bar{w}_{II}) = f(\bar{P}\bar{w}_I, \bar{Q}\bar{w}_{II}) \quad \forall\ P, Q. \tag{4.46}
$$

Combining the preceding matchings and properties, one can effectively rewrite (4.30) as

$$\mathcal{S}^M_\kappa(\bar u,\bar v) = \sum\sum \kappa^{\#\bar u_I}(-)^{\#\bar u_{II}} f^\star(\bar u_I,\bar u_{II}) f(\bar v_I,\bar v_{II})$$
$$\times \sum_{P_I,Q_I} \frac{(ic)^{\#\bar u_I} F^\star(\bar{P_I}\bar u_I) F(\bar{Q_I}\bar v_I)}{\prod_{i=1}^{\#\bar u_I}[\bar{P_I}\bar u_I - \bar{Q_I}\bar v_I\}_i^{\#\bar u_I}} \times \sum_{P_{II},Q_{II}} \frac{(ic)^{\#\bar u_{II}} F^\star(\bar{P_{II}}\bar u_{II}) F(\bar{Q_{II}}\bar v_{II})}{\prod_{i=1}^{\#\bar u_{II}}[\bar{P_{II}}\bar u_{II} - \bar{Q_{II}}\bar v_{II}\}_i^{0}} \tag{4.47}$$

where the first pair of sum run over the normal ordered partitions $\bar u \Rightarrow \{\bar u_I,\bar u_{II}\}$ and $\bar v \Rightarrow \{\bar v_I,\bar v_{II}\}$, with $\#\bar u_I = \#\bar v_I$, and P_I, P_{II}, Q_I and Q_{II} acts on $\bar u_I, \bar u_{II}, \bar v_I$ and $\bar v_{II}$ respectively.

Now one last bit of rewriting: using

$$f^\star(u,v) = f(v,u) \tag{4.48}$$
$$\Rightarrow F^\star(\bar w) = F(\bar T\bar w) \tag{4.49}$$
$$\prod_{i=1}^{\#\bar w}[\bar w\}_i^0 = \prod_{i=1}^{\#\bar w}[\bar T\bar w\}_i^{\#\bar w} \tag{4.50}$$

where T is defined as the "mirror" permutation: $Ti = n - i + 1$, with $n = \#\bar w$, one can rewrite (4.47) as

$$\begin{aligned}\mathcal{S}^M_\kappa(\bar u,\bar v) =& \sum\sum \kappa^{\#\bar u_I} f(\bar u_{II},\bar u_I) f(\bar v_I,\bar v_{II})\\ &\times \sum_{P_I,Q_I} \frac{(ic)^{\#\bar u_I} F^\star(\bar{P_I}\bar u_I) F(\bar{Q_I}\bar v_I)}{\prod_{i=1}^{\#\bar u_I}[\bar{P_I}\bar u_I - \bar{Q_I}\bar v_I\}_i^{\#\bar u_I}} \left(\sum_{P_{II},Q_{II}} \frac{(ic)^{\#\bar u_{II}} F^\star(\bar T\bar{P_{II}}\bar u_{II}) F(\bar T\bar{Q_{II}}\bar v_{II})}{\prod_{i=1}^{\#\bar u_{II}}[\bar T(\bar{P_{II}}\bar u_{II} - \bar{Q_{II}}\bar v_{II})\}_i^{\#\bar u_{II}}}\right)^\star\\ =& \sum\sum \kappa^{\#\bar u_I} f(\bar u_{II},\bar u_I) f(\bar v_I,\bar v_{II})\\ &\times \sum_{P_I,Q_I} \frac{(ic)^{\#\bar u_I} F^\star(\bar{P_I}\bar u_I) F(\bar{Q_I}\bar v_I)}{\prod_{i=1}^{\#\bar u_I}[\bar{P_I}\bar u_I - \bar{Q_I}\bar v_I\}_i^{\#\bar u_I}} \left(\sum_{P_{II},Q_{II}} \frac{(ic)^{\#\bar u_{II}} F^\star(\bar{P_{II}}\bar u_{II}) F(\bar{Q_{II}}\bar v_{II})}{\prod_{i=1}^{\#\bar u_{II}}[\bar{P_{II}}\bar u_{II} - \bar{Q_{II}}\bar v_{II}\}_i^{\#\bar u_{II}}}\right)^\star\end{aligned} \tag{4.51}$$

Simplifying and cleaning

Using the definition (4.20) of $\mathcal{K}_n(\bar x|\bar y)$ we can develop the Izergin-Korepin determinent

$$f(\bar v_I,\bar v_{II}) f(\bar u,\bar v_I) D^{-1}_{\bar v_I}\mathcal{K}_M(\bar v|\bar u)$$
$$= f(\bar v_I,\bar v_{II}) f(\bar u,\bar v_I) \tag{4.52}$$
$$\times g^<(\bar v_I,\bar v_I) g^<(\bar v_{II},\bar v_{II}) g^<(\bar v_I - c,\bar v_{II}) g^<(\bar v_{II},\bar v_I - c) g^>(\bar u,\bar u)$$
$$\times h(\bar v_I - c,\bar u) h(\bar v_{II},\bar u) D^{-1}_{\bar v_I} det_{i,j}\left[t(v_i,u_j)\right] \tag{4.53}$$

Now using the properties (from (4.21)-(4.23))

$$f(x,y) = h(x,y)g(x,y) \tag{4.54}$$
$$g(x,y) = -g(y,x) \tag{4.55}$$
$$g(x,y-c) = h^{-1}(x,y) \Rightarrow g^<(\bar v_{II}, \bar v_I - c) = \left(h^<(\bar v_{II}, \bar v_I)\right)^{-1} \tag{4.56}$$
$$g(x-c,y) = -h^{-1}(y,x) \Rightarrow g^<(\bar v_I - c, \bar v_{II}) = \left(h^>(\bar v_{II}, \bar v_I)\right)^{-1}(-)^{\xi(\bar v_I, \bar v_{II})} \tag{4.57}$$
$$h(x-c,y) = g^{-1}(x,y) \tag{4.58}$$

where we defined $(-)^{\xi(\bar v_I, \bar v_{II})} \equiv \prod_{\substack{v_i \in \bar v_I \\ v_j \in \bar v_{II} \\ i<j}} (-1)$, we can write (4.53) as

$$\begin{aligned} f(\bar v_I, \bar v_{II}) f(\bar u, \bar v_I) D_{\bar v_I}^{-1} \bar v \mathcal{K}_M(\bar v|\bar u) = & h(\bar v_I, \bar v_{II}) g(\bar v_I, \bar v_{II}) h(\bar u, \bar v_I) g(\bar u, \bar v_I) \\ & \times g^<(\bar v_I, \bar v_I) g^<(\bar v_{II}, \bar v_{II}) \left(h^>(\bar v_{II}, \bar v_I)\right)^{-1} \\ & \times (-)^{\xi(\bar v_I, \bar v_{II})} \left(h^<(\bar v_{II}, \bar v_I)\right)^{-1} \\ & \times g^>(\bar u, \bar u) \\ & \times (-)^{\# \bar v_I M} g^{-1}(\bar u, \bar v_I) h(\bar v_{II}, \bar u) \\ & \times D_{\bar v_I}^{-1} det_{i,j} \left[t(v_i, u_j)\right]. \end{aligned} \tag{4.59}$$

Now notice that

$$g(\bar v_I, \bar v_{II}) g^<(\bar v_I, \bar v_I) g^<(\bar v_{II}, \bar v_{II}) (-)^{\xi(\bar v_I, \bar v_{II})} = g^<(\bar v, \bar v)(-)^{\#I\#II}$$

$(\xi(\bar v_{II}, \bar v_I) + \xi(\bar v_I, \bar v_{II}) = \#I\#II)$ and that

$$\frac{h(\bar v_I, \bar v_{II})}{h(\bar v_{II}, \bar v_I)} = \frac{h(\bar v_I, \bar v)}{h(\bar v, \bar v_I)} = \frac{h(\bar v, \bar v_{II}) h(\bar v_I, \bar v)}{h(\bar v, \bar v)}$$

so (4.53) eventually rewrites

$$\begin{aligned} f(\bar v_I, \bar v_{II}) f(\bar u, \bar v_I) D_{\bar v_I}^{-1} \bar v \mathcal{K}_M(\bar v|\bar u) = & g^<(\bar u, \bar u) g^>(\bar v, \bar v) h^{-1}(\bar v, \bar v) \\ & \times h(\bar u, \bar v_I) h(\bar v_I, \bar v) h(\bar v_{II}, \bar u) h(\bar v, \bar v_{II}) (-)^{\#I\#II} (-)^{M\#I} \\ & \times D_{\bar v_I}^{-1} det\left[t(v_i, u_j)\right] \end{aligned} \tag{4.60}$$

Just notice that $(-)^{\#I(M-\#II)} = (-)^{\#I^2} = (-)^{\#I}$.

4.2 Integral Representation for the Gaudin's, Izergin-Korepin's and Slavnov's Determinants

It is not about the goal, but the way.

On one hand the Izergin-Korepin determinant, initially found to be the partition function for the 6-vertex model[12, 14], is now well known to provide a compact form for the on-shell/off-shell[6] scalar product in the XXX spin chain , for states of number of magnon M equal to half of the chain's length [29]. For two Bethe states of M magnons $|\psi(\bar{u})\rangle_M$ and $|\psi(\bar{v})\rangle_M$ (i.e. $\#\bar{u} = \#\bar{v} = M$) in a chain of size $L = 2M$, it takes the form of the determinant of a matrix of size $2M$, as one of its entries corresponds to the set of the union of the rapidities of the two states $\{\bar{u}, \bar{v}\}$, of cardinal $2M$, while the other corresponds to the set of inhomogeneity of the chain $\bar{\theta}$, of cardinal $L = 2M$:

$$_M \langle\psi(\bar{u})|\psi(\bar{v})\rangle_M = det_{2M}K(\{\bar{u},\bar{v}\}|\bar{\theta}), \quad L = \#\bar{\theta} = 2M \tag{4.61}$$

All these object will be properly defined latter.
On the other hand the on-shell/off-shell scalar product of two states of M magnons in a chain of length L can be written as a Slavnov determinant of size M, where the size of the chain is now unconstrained except by $M \leq L$ [14]

$$_M \langle\psi(\bar{u})|\psi(\bar{v})\rangle_M = det_M S_{\bar{\theta}}(\bar{u}|\bar{v}), \quad M \leq L \tag{4.62}$$

Note that while the Izergin-Korepin and Slavnov determinants are define for a wide family of parameters, these compact forms would only account for the scalar product for restricted family of parameter, i.e. if one of its subsets satisfies the Bethe equations.
More recently, an expression for the Izergin-Korepin determinant in an integral form has been proposed [11]. I will here propose to draw a link between the different determinant through the integral representation. The understanding of these integral form is motivated by form factor computation in integrable gauge theories, since these integral expression are expected to behave nicely in the large size limit.

The reflection is organized as follows. At the end of this introduction are defined the notations and objects that are going to be used. After are exposed and proved the identities holding between integral representation and the different determinants of interest:
As a warm up we first re-establish the integral representation for the Izergin-Korepin determinant in Section 4.2.1, initially established by E. Bettelheim and I. Kostov [11],

$$det_N K(\bar{u}|\bar{\theta}) = \mathcal{A}_{\bar{\theta}}(\bar{u}), \quad L = \#\bar{\theta} = N \tag{4.63}$$

and proceed symmetrically for the Slavnov determinant in Section 4.2.1

$$det_M S(\bar{u}|\bar{v}) = \mathcal{A}(\{\bar{u},\bar{v}\}) \tag{4.64}$$

The latter will require a slightly trickier but still similar treatment. Through this integral representation, we drew a formal link between these determinants, hence proving they equivalence in an original fashion.
The on-shell/off-shell scalar product, with some of off-shell state rapidities coinciding with the on-shell ones:

$$_M \langle\psi(\bar{u})|\psi(\{\bar{u}_I,\bar{v}_{II}\})\rangle_M = \lim_{\bar{v}_I\to\bar{u}_I} {}_M \langle\psi(\bar{u})|\psi(\bar{v})\rangle_M\,, \quad \bar{u}_I \subseteq \bar{u} \tag{4.65}$$

[6] by on-shell we refer to a state of rapidities satisfying the Bethe equations of the system

An expression for this scalar product can straightforwardly be obtained as a limit of the Slavnov determinant

$$_M \langle \psi(\bar{u}) | \psi(\{\bar{u}_I, \bar{v}_{II}\}) \rangle_M = \lim_{\bar{v}_I \to \bar{u}_I} det_M S(\bar{u} | \bar{v}_I \cup \bar{v}_{II}) \tag{4.66}$$

which is performed in Section 4.2.1 for the special case of $II = \emptyset$, i.e. $\bar{v} \to \bar{u}$.
I also propose here to directly obtain it from the integral representation in Section 4.2.2, where we expose a compact form for

$$_M \langle \psi(\bar{u}) | \psi(\{\bar{u}_I, \bar{v}_{II}\}) \rangle_M = \mathcal{A}(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) \tag{4.67}$$

the corresponding proof being of potential interest.
As expected, taking $\bar{u}_I = \emptyset$ leads to the Slavnov expression, while for $\bar{u}_I = \bar{u}$ we recover the Gaudin's norm

$$_M \langle \psi(\bar{u}) | \psi(\bar{u}) \rangle_M = \mathcal{N}(\bar{u}) \tag{4.68}$$

Some notations and definitions. For $\bar{v} = \{v_1, \cdots, v_N\}$ a set of complex parameter, we write $\bar{v}_i \equiv \bar{v} \setminus \{v_i\}$ and $\#\bar{v} = N$ its cardinal.
When setting a subset of these "full" sets, we will implicitly define a subset of the "full" set of indexes: $\bar{u}_I \subseteq \bar{u} \iff I \subseteq \{1, \cdots, N\}$, $\bar{u}_I = \{u_i\}_{i \in I}$. As a matter of simplicity, we may use the same notation for a set and its cardinal, $\#I \sim I$, as this couldn't lead to confusion.
When a set of parameters appears in an analytic expression, it will refer to a product over these parameters, e.g.

$$\frac{x - \bar{u} + c}{x - \bar{u}} \equiv \prod_i \frac{x - u_i + c}{x - u_i} \tag{4.69}$$

$$\frac{\bar{x} - \bar{u} + c}{\bar{x} - \bar{u}} \equiv \prod_a \prod_i \frac{x_a - u_i + c}{x_a - u_i} \tag{4.70}$$

When appearing as an argument of a function or as an index, such a set wouldn't refer to a product but only to a general dependence of this object, e.g. in $\mathcal{J}_{\bar{u}}(x)$ (defined hereafter).

Definition 4.2.1. *For $\bar{u}$ a set of variables and $Z : \mathbb{C} \mapsto \mathbb{C}$ a complex function regular at u_k $\forall k$, we define the Bethe quantifier, or counting function, as*

$$\mathcal{J}^Z_{\bar{v}}(x) \equiv Z(x) \frac{x - \bar{v} + c}{x - \bar{v} - c}. \tag{4.71}$$

A set $\bar{v}$ will be said to satisfy the Bethe equations, and then the corresponding Bethe state to be on-shell, if and only if $\mathcal{J}^Z_{\bar{v}}(v_i) = -1$ $\forall i$. Note that in this case $\mathcal{J}^Z_{\bar{v}_i}(v_i) = 1$ $\forall i$.

Remark. *In the case of the XXX spin chain, the Bethe equation is specified by*

$$Z^{XXX}_{\bar{\theta}}(x) = \frac{\lambda_2(x)}{\lambda_1(x)} \equiv \frac{x - \bar{\theta} - c/2}{x - \bar{\theta} + c/2}$$

with $\#\bar{\theta} = L$ the length of the spin chain.

However, some of our results holding for a wider family of function Z, we will consider this function in its most general definition, and will implicitly consider $Z_{\bar{\theta}}^{XXX}$ when referring to the scalar product.

Definition 4.2.2. *For $\bar{u}$ and $\bar{v}$ two sets of complex variables, we define the Cauchy determinant*

$$\delta(\bar{u}, \bar{v}) = det_{ij}\left[\frac{c}{u_i - v_j + c}\right] = c^{2\#\bar{u}}\frac{\prod_{i<j}(u_i - u_j)(v_i - v_j)}{\prod_{i,j}(u_i - v_j + c)}, \tag{4.72}$$

Definition 4.2.3. *For $\bar{v}$ and $\bar{\theta}$ two sets of complex variables of cardinal N, and c a complex parameter, we define the Izergin-Korepin determinant*

$$\mathcal{K}_N(\bar{v}|\bar{\theta}) \equiv det_{i,j}^{-1}\left[\frac{c}{v_i - \theta_j - c/2}\right] det_{i,j}\left[\frac{c}{v_i - \theta_j - c/2} - \frac{c}{v_i - \theta_j + c/2}\right] \tag{4.73}$$

Definition 4.2.4. *For $\bar{u}$ and $\bar{w}$ two sets of complex variables of same cardinal, $Z : \mathbb{C} \mapsto \mathbb{C}$ a complex function regular at u_k and w_k $\forall k$, and c a complex parameter, we define the Slavnov determinant*

$$\mathcal{S}_Z(\bar{u}|\bar{w}) \equiv det_{i,j}^{-1}\left[\frac{c}{u_i - w_j + c}\right] det_{i,j}\left[c^2\frac{\mathcal{J}_{\bar{u}_i}^{Z}(w_j) - 1}{(u_i - w_j)(u_i - w_j + c)}\right] \tag{4.74}$$

Definition 4.2.5. *Let $\bar{v}$ be a set of complex variables, c a complex parameter and Z : $\mathbb{C} \mapsto \mathbb{C}$ a complex function regular at v_k $\forall k$.*
We define the integral representation

$$\mathcal{A}_Z(\bar{v}) = \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\prod_{a=1}^{n}\left(\oint_{\mathcal{C}_{\bar{v}}}\frac{dx_a}{2i\pi c}Z(x_j)\frac{x_a - \bar{v} + c}{x_a - \bar{v}}\right)\prod_{a\neq b}^{n}\frac{x_a - x_b}{x_a - x_b + c} \tag{4.75}$$

where $\mathcal{C}_{\bar{v}}$ is a contour slightly encircling the v_k.

Definition 4.2.6. *We define the shift operator*

$$D_x f(x) = f(x + c) \tag{4.76}$$

$$D_x = e^{c\frac{\partial}{\partial x}} \tag{4.77}$$

4.2.1 The Izergin-Korepin and the Slavnov determinants

In this section I re-establish the link [11] between the integral representation and the Izergin-Korepin determinants. The Slavnov determinant is also obtained in a similar, though a bit more hairy manner.
We also show how we straightforwardly obtain the Gaundin determinant for the norm as a limit of the Slavnov determinant, by contrast with its direct obtaining from the integral representation, which is the matter of a next section.

Izergin-Korepin and its Integral Representation

As a first step toward more demanding results, we prove the following Theorem.

Theorem 4.2.7. *For $\bar{v}$ and $\bar{\theta}$ two sets of disjoint complex variables with $\#\bar{v} = \#\bar{\theta} = N$, and c a complex parameter, we have the integral representation for the Izergin-Korepin determinant*

$$\mathcal{K}_N(\bar{v}|\bar{\theta}) \equiv det^{-1}_{i,j}\left[\frac{c}{v_i - \theta_j - c/2}\right] det_{i,j}\left[\frac{c}{v_i - \theta_j - c/2} - \frac{c}{v_i - \theta_j + c/2}\right] \tag{4.78}$$

$$= \mathcal{A}_{Z_{\bar{\theta}}}(\bar{v}) \equiv \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \prod_{a=1}^{n} \left(\oint_{\mathcal{C}_{\bar{v}}} \frac{dx_a}{2i\pi c} \frac{x_a - \bar{\theta} - c/2}{x_a - \bar{\theta} + c/2} \frac{x_a - \bar{v} + c}{x_a - \bar{v}} \right) \prod_{a \neq b}^{n} \frac{x_a - x_b}{x_a - x_b + c} \tag{4.79}$$

where $Z_{\bar{\theta}}(x) \equiv \frac{x - \bar{\theta} - c/2}{x - \bar{\theta} + c/2}$.

Proof. See Appendix 4.2.3. ☐

This first approach to the integral representation is a good exercise to get used to the main objects and techniques under consideration.
As mentioned, we know that for $\bar{v} = \{\bar{u}, \bar{w}\}$, with $\#\bar{u} = \#\bar{w} = M$ and $\bar{u}$ satisfying the Bethe equations for XXX, the Izergin-Korepin determinant accounts for the scalar product ${}_M\langle\psi(\bar{u})|\psi(\bar{w})\rangle_M$, and so does the integral representation:

$${}_M\langle\psi(\bar{u})|\psi(\bar{w})\rangle_M = \mathcal{K}_N(\{\bar{u},\bar{w}\}|\bar{\theta}) = \mathcal{A}_{Z_{\bar{\theta}}}(\{\bar{u},\bar{w}\}), \quad \mathcal{J}_{\bar{u}}^{Z_{\bar{\theta}}^{XXX}}(u_i) = -1 \ \forall i \tag{4.80}$$

For any number of magnons M, this on-shell/off-shell scalar product can be expressed as a determinant of size M, so called Slavnov determinant.

Slavnov determinant and its integral representation: Theorem

We previously presented an integral representation for the Izergin-Korepin determinant. Now we are a bit more familiar with the mechanisms in play in these integrals, we can carry this integral representation to the other side of the story, that is to say the Slavnov determinant side.

Theorem 4.2.8. *For $\bar{v} = \{\bar{u}, \bar{w}\}$, $\#\bar{u} = \#\bar{w} = M$, a set of disjoint complex parameters with $\bar{u}$ satisfying the Bethe equations $\mathcal{J}^Z_{\bar{u}_i}(u_i) = 1 \ \forall i$, and $Z : \mathbb{C} \longmapsto \mathbb{C}$ any complex function regular on $\bar{u}$, holds the equality between the integral representation and the Slavnov determinant*

$$\mathcal{S}_Z(\bar{u}|\bar{w}) \equiv det^{-1}\left[\frac{c}{u_i - w_j + c}\right] det\left[c^2 \frac{\mathcal{J}^Z_{\bar{u}_i}(w_j) - \mathcal{J}^Z_{\bar{u}_i}(u_i)}{(u_i - w_j)(u_i - w_j + c)}\right] \tag{4.81}$$

$$= \mathcal{A}_Z(\{\bar{u},\bar{w}\}) \equiv \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \prod_{a=1}^{n} \left(\oint_{\mathcal{C}_{\{\bar{u},\bar{w}\}}} \frac{dx_a}{2i\pi c} Z(x) \frac{x_a - \bar{u} + c}{x_a - \bar{u}} \frac{x_a - \bar{w} + c}{x_a - \bar{w}} \right) \prod_{a \neq b}^{n} \frac{x_a - x_b}{x_a - x_b + c} \tag{4.82}$$

where $\mathcal{C}_{\{\bar{u},\bar{w}\}}$ closely encircle $\{\bar{u},\bar{w}\}$, $\mathcal{J}^Z_{\bar{u}}(x) = Z(x)\frac{x - \bar{u} + c}{x - \bar{u} - c}$ and $\bar{u}_i \equiv \bar{u} \setminus u_i$.

Proof. See Appendix 4.2.3 □

Proving this theorem, alongside theorem 4.2.7, obviously establish the link between the Izergin-Korepin determinant and the Slavnov determinant:

$$K_{2M}(\{\bar{u},\bar{w}\}|\bar{\theta}) = \mathcal{S}_{Z_{\bar{\theta}}}(\bar{u}|\bar{w}), \quad \mathcal{J}_{\bar{u}}^{Z_{\bar{\theta}}^{XXX}}(u_i) = -1\ \forall i.$$

As mentioned, the Slavnov determinant provides a compact form for the on-shell/off-shell scalar product, for a chain of unconstrained length $L = \#\bar{\theta}$, and so does the integral representation:

$$_M\langle\psi(\bar{u})|\psi(\bar{w})\rangle_M = \mathcal{S}_{Z_{\bar{\theta}}^{XXX}}(\bar{u}|\bar{w}) = \mathcal{A}_{Z_{\bar{\theta}}^{XXX}}(\{\bar{u},\bar{w}\}), \quad \mathcal{J}_{\bar{u}}^{Z_{\bar{\theta}}^{XXX}}(u_i) = -1\ \forall i.$$

Gaudin's norm as a limit of the Slavnov determinant

The expression for the scalar product given in Theorem 4.2.8 provides a simple way to obtain the Gaudin's norm, as we are to demonstrate now.
By definition of the norm as a scalar product, the norm of an on-shell state of rapidity $\bar{u}$ reads as

$$\langle\psi(\bar{u})|\psi(\bar{u})\rangle \equiv \mathcal{N}(\bar{u}) = \mathcal{A}_Z(\{\bar{u},\bar{u}\}) = \lim_{\bar{v}\to\bar{u}} \mathcal{S}_Z(\{\bar{u},\bar{v}\}), \quad \mathcal{J}_{\bar{u}_i}^{Z}(u_i) = 1\ \forall i \tag{4.83}$$

As suggested by the equalities above, two options are obviously available.
• One option is to take the limit $\bar{v} \to \bar{u}$ inside integrals of $\mathcal{A}$, then obtaining double poles inside integration contours. This would require a different treatment and will be the matter of the next section.
• A second option, we are to propose here first, the most simple and direct, is to take this limit of the Slavnov determinant $\mathcal{S}$: $\lim_{\bar{v}\to\bar{u}} \mathcal{S}_Z(\{\bar{u},\bar{v}\})$
Let's consider the square matrix S, the Slavnov matrix, defined by its matrix elements

$$S_{ij}(\bar{u},\bar{v}) \equiv c^2 \frac{\mathcal{J}_{\bar{u}_i}^{Z}(w_j) - 1}{(u_i - w_j)(u_i - w_j + c)} \tag{4.84}$$

$$\text{i.e.} \quad \mathcal{S}_Z(\{\bar{u},\bar{w}\}) = det_{i,j}^{-1}\left[\frac{c}{u_i - w_j + c}\right] det[S(\bar{u},\bar{v})] \tag{4.85}$$

The norm can then be written

$$\mathcal{N}(\bar{u}) = det_{i,j}^{-1}\left[\frac{c}{u_i - u_j + c}\right] \lim_{\bar{w}\to\bar{u}} det[S(\bar{u},\bar{w})], \quad \mathcal{J}_{\bar{u}_i}^{Z}(u_i) = 1, \forall i$$

When taking this limit, one will obtain an expression for the norm as a determinant, so called Gaudin determinant, according to the following theorem:

Theorem 4.2.9. *For $\bar{u}$ satisfying the Bethe equations $\mathcal{J}_{\bar{u}_i}^{Z}(u_i) = 1\ \forall i$ with $Z : \mathbb{C} \longmapsto \mathbb{C}$ any complex function regular on $\bar{u}$, we have*

$$\mathcal{N}(\bar{u}) = det^{-1}_{i,j}\left[\frac{c}{u_i - u_j + c}\right] \lim_{\bar{w}\to\bar{u}} det_{i,j}[S_{ij}(\bar{u},\bar{w})] \tag{4.86}$$

$$= det^{-1}_{i,j}\left[\frac{c}{u_i - u_j + c}\right] det[G(\bar{u})] \tag{4.87}$$

where we define the Gaudin matrix G by its matrix elements

$$G_{ij}(\bar{u}) \equiv c\partial_{u_j}\mathcal{J}^Z_{\bar{u}}(u_i) \tag{4.88}$$

Proof. The proof is direct.

For $i \neq j$, the Bethe equations being satisfied we have $\mathcal{J}^Z_{\bar{u}_i}(u_i) = 1\ \forall i$, and so $\mathcal{J}^Z_{\bar{u}_i}(u_j) = -\frac{u_i-u_j+c}{u_i-u_j-c}$. Then we can write

$$\lim_{\bar{w}\to\bar{u}} S_{ij}(\bar{u},\bar{w}) = c^2\frac{-\frac{u_i-u_j+c}{u_i-u_j-c} - 1}{(u_i-u_j)(u_i-u_j+c)} = \frac{-2c^2}{(u_i-u_j)^2 - c^2} = -c\frac{\partial\mathcal{J}^Z_{\bar{u}_i}(u_i)}{\partial u_j} = c\frac{\partial\mathcal{J}^Z_{\bar{u}}(u_i)}{\partial u_j} \tag{4.89}$$

$$\lim_{\bar{w}\to\bar{u}} S_{ii}(\bar{u},\bar{w}) = \lim_{w_i\to u_i} c\frac{\mathcal{J}^Z_{\bar{u}_i}(w_i) - 1}{u_i - w_i} = \lim_{\epsilon\to 0} c\frac{\mathcal{J}^Z_{\bar{u}_i}(u_i+\epsilon) - \mathcal{J}^Z_{\bar{u}_i}(u_i)}{-\epsilon} = -c\frac{\partial\mathcal{J}^Z_{\bar{u}_i}(u_i)}{\partial u_i} = c\frac{\partial\mathcal{J}^Z_{\bar{u}}(u_i)}{\partial u_i} \tag{4.90}$$

Note that here we of course consider the partial derivative not only act on the argument of the quantifier, but on the full object, also depending on the variables in indexes.

□

4.2.2 Between Slavnov and Gaudin determinants

We saw before the case of the scalar product between two Bethe states of rapidity $\bar{u}$ and $\bar{v}$. For Z regular at $\bar{u}\cup\bar{v}$ and $\bar{u}$ satisfying the Bethe equations $\mathcal{J}^Z_{\bar{u}_i}(u_i) = 1\ \forall i$, we proved the integral representation $\mathcal{A}_Z(\{\bar{u},\bar{v}\})$ to be equal to the Slavnov determinant $\mathcal{S}_Z(\bar{u},\bar{v})$. When taking the limit $\bar{v}\to\bar{u}$ of the Slavnov determinant, we obtained the Gaudin's expression for the norm. We are now to obtain this last result directly from the integral representation, i.e. to consider

$$\langle\psi(\bar{u})|\psi(\bar{u})\rangle \equiv \mathcal{N}(\bar{u}) = \mathcal{A}_Z(\{\bar{u},\bar{u}\}), \quad \mathcal{J}^Z_{\bar{u}_i}(u_i) = 1\ \forall i \tag{4.91}$$

We are actually to consider a bit more general limit, in which only a subset of $\bar{v}$ tends to a mirror subset of $\bar{u}$:
$\bar{v} = \{\bar{v}_I, \bar{v}_{II}\} \to \{\bar{u}_I, \bar{v}_{II}\}$

$$\langle\psi(\bar{u})|\psi(\{\bar{u}_I,\bar{v}_{II}\})\rangle = \mathcal{A}_Z(\{\bar{u},\bar{u}_I,\bar{v}_{II}\}), \quad \mathcal{J}^Z_{\bar{u}_i}(u_i) = 1\ \forall i \tag{4.92}$$

obviously accounting for the case of the Gaudin's norm for $II = \emptyset$. This result can be summed up in the following theorem.

Theorem 4.2.10. *Let $\{I, II\}$ be a partition of $\{1, \cdots, M\}$. We consider two disjoint sets of rapidity $\bar{u} \equiv \{\bar{u}_I, \bar{u}_{II}\}$ and $\bar{v}_{II}$, such that $\bar{u}$ satisfy the Bethe equations $\mathcal{J}^Z_{\bar{u}}(u_i) = -1\ \forall i$, with Z a complex function regular on $\bar{u} \cup \bar{v}_{II}$.*

Then we have

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) \tag{4.93}$$

$$\equiv \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \prod_{a=1}^{n} \left(\oint_{\mathcal{C}} \frac{dx_a}{2i\pi c} Z(x_j) \left(\frac{x_a - \bar{u}_I + c}{x_a - \bar{u}_I} \right)^2 \frac{x_a - \bar{u}_{II} - c}{x_a - \bar{u}_{II}} \frac{x_a - \bar{v}_{II} + c}{x_a - \bar{v}_{II}} \right) \prod_{a \neq b}^{n} \frac{x_a - x_b}{x_a - x_b + c} \tag{4.94}$$

$$= \delta^{-1}(\bar{u}, \{\bar{u}_I, \bar{v}_{II}\}) \det \begin{pmatrix} G_{I,I} & G_{I,II} \\ S_{II,I} & S_{II,II} \end{pmatrix} \tag{4.95}$$

where δ is the Cauchy determinant defined in the introduction, the Gaudin's matrix elements for two sets α and β is defined as

$$(G_{\alpha\beta})_{ij} = G_{\alpha_i \beta_j} = \left(c\partial_{u_{\alpha_j}} \mathcal{J}^Z_{\bar{u}}(u_{\alpha_i}) \right), \tag{4.96}$$

the Slavnov's matrix elements defined as

$$(S_{\beta\alpha})_{ij} = S_{\alpha_i \beta_j} = c \frac{\mathcal{J}^Z_{\bar{u}_{\alpha_i}}(v_{\beta_j}) - 1}{(u_{\alpha_i} - v_{\beta_j})(u_{\alpha_i} - v_{\beta_j} + c)}, \tag{4.97}$$

and where the contour $\mathcal{C} = \mathcal{C}_{\bar{u} \cup \bar{v}_{II}}$ closely encircles the whole set of rapidity, $\bar{u} \cup \bar{v}_{II}$.

Remark. *We consider in this theorem the two sets $\bar{u}_{II}$ and $\bar{v}_{II}$ to be labeled by the same set of indices. This is made without loss of generality given that the considered objects, the integral representation and the Gaudin and Slavnov determinants (and in between) are invariant under permutations of their arguments: while the determinant is antisymmetric, so is its prefactor. Thus, taking the limit $\bar{P}\bar{v} \to \bar{P}\{\bar{u}_I, \bar{v}_{II}\}$, where $\bar{P}$ permutes the elements of the sets it applies to, would lead to the exact same result. The choice $\bar{P} = \bar{id}$ is a sensible choice of simplicity.*

Remark. *It's here clear that this result would be straightforwardly obtained from the Slavnov determinant 4.81, as had been the particular case of the Gaudin's norm. Considering the entries of the Slavnov matrix on columns for the u_i's and and lines for the v_i's, we see that taking $v_i \to u_i$ turn the corresponding line into its Gaudin's form, while it remains in the Slavnov form otherwise.*

Proof. Before moving to the formal proof of the theorem, let us sketch our reasoning on an intuitive level. Doing so, we draw the global structure of the proof, which may provide a useful support for the reader, to keep track of the reflection while progressing in the rigorous development.

Sketching the proof

We consider the integral representation for two intersecting sets of variables. Then, some of the poles inside the integral contours are now double poles (for $\{\bar{u}, \bar{v}\} \to \{\bar{u}, \bar{u}_I, \bar{v}_{II}\} = \{\bar{u}_I, \bar{u}_I, \bar{u}_{II}, \bar{v}_{II}\}$, the double poles are at $\bar{u}_I$, the others corresponding to single poles).

When grabbed, these poles would produce a derivative of the integrated core (the residue of this second order pole), and in particular the derivative of the quantifier (or counting function) $\mathcal{J}$.

Remark. *This mechanism can of course be compared with what happened while taking the limit $v_i \to u_i$ in the Slavnov determinant, see section 4.2.1, according to the following picture:*

$$\lim_{\epsilon\to 0} \oint \frac{dx}{2i\pi} \frac{f(x)}{(x-z)(x-z-\epsilon)} = \lim_{\epsilon\to 0} \left(\frac{f(z)}{-\epsilon} + \frac{f(z+\epsilon)}{\epsilon} \right) \tag{4.98}$$

$$= \partial_x f(x)|_z \tag{4.99}$$

$$= \oint \frac{dx}{2i\pi} \frac{f(x)}{(x-z)^2} \tag{4.100}$$

These two different approaches, that is to say taking the limit after integrating or inside the integrated core, lead to the same result. While we followed the former in section 4.2.1, we now are to follow the latter.

We will proceed step by step:

• In Section 4.2.2 we split our contours as contours around simple poles and others around double poles. Schematically seeing the integral representation as a sum over the number of integral, it would then be written as a double sum over the number of integrals around each contours, (see equation (4.114)):

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) \text{``}=\text{''} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\oint_{\mathcal{C}} dx \mathcal{J}(x) \right)^n \tag{4.101}$$

$$\text{``}=\text{''} \sum_{p=0}^{\infty} \frac{(-)^p}{p!} \left(\oint_{\mathcal{C}_I} dx \mathcal{J}(x) \right)^p \sum_{q=0}^{\infty} \frac{(-)^q}{q!} \left(\oint_{\mathcal{C}_{II}} dx \mathcal{J}(x) \right)^q \tag{4.102}$$

where $\mathcal{C}_I$ and $\mathcal{C}_{II}$ closely encircle double and single poles, respectively.

• Then in Section 4.2.2 we extract the quantifier for variables integrated around double poles. To do so we first integrate around double poles, giving rise to derivative of the quantifier and derivative of the rest of the core. We then step back for the second term, according to the following scheme:

$$\oint \frac{dx}{2i\pi} \frac{\mathcal{J}(x) f(x)}{\prod_i (x-u_i)^2} = \sum_i \left(\partial_x \mathcal{J}(x)|_{u_i} f(u_i) + \mathcal{J}(u_i)\, \partial_x f(x)|_{u_i} \right) \frac{1}{\prod_{j\neq i} (u_i - u_j)^2} \tag{4.103}$$

$$= \sum_i \left(\partial_x \mathcal{J}(x)|_{u_i} f(u_i) \frac{1}{\prod_{j\neq i}(u_i-u_j)^2} \right) + \mathcal{J} \oint \frac{dx}{2i\pi} \frac{f(x)}{\prod_i (x-u_i)^2}$$

or in our case (see Lemma 4.2.11), schematically:

$$\left(\oint_{\mathcal{C}_I} dx \mathcal{J}(x) \right) = \sum_k \partial_x \mathcal{J}(x)|_{u_k} + \mathcal{J} \left(\oint_{\mathcal{C}_I} dx \right) \tag{4.104}$$

where we assumed the Bethe equation to be satisfied, i.e. $\mathcal{J}(u_i) = 1\ \forall i$. This assumption is here fundamental, since it allows the terms $\mathcal{J}(u_i)$ to be factorized in the second term of the first line, and then to go back to integrated form in the last line.

We see in (4.103) that one can, step by step, extract the quantifier $\mathcal{J}$, i.e. making it disappear from the integrated core for double pole variables.
We then repeat this step until all the quantifier are either derivated at u_i, either integrated for single pole variables, i.e. until all the quantifier for double pole variables have been extracted. Once a variable has been grabbed during integration, its corresponding pole would disappear from the other integrals. We then obtain a sum over subsets α of "double pole variables" at which the quantifier are derivated.
Such subset would be absent from the "still to be integrated" part (see Lemma 4.2.12):

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) \\ \text{“} = \text{”} \sum_{I \Rightarrow \{\alpha,\beta\}} \prod_{i\in\alpha} (\partial_x \mathcal{J}(x)|_{u_i}) \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \left(\oint_{\mathcal{C}_{II}} dx\right)^n \left(\oint_{\mathcal{C}_I} dx \mathcal{J}(x)\right)^q F(\bar{u}_\beta, \bar{u}_{II}, \bar{v}_{II}) \tag{4.105}$$

F the "still to be integrated" part.

• In Section 4.2.2 we consider the "still to be integrated" part for a particular subset β of I. Bringing back this expression to what has been considered for the integral representation in the Slavnov case, we apply Theorem 4.2.8 and obtain a determinant on our subset.

$$\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \left(\oint_{\mathcal{C}_{II}} dx\right)^n \left(\oint_{\mathcal{C}_I} dx \mathcal{J}(x)\right)^q F(\bar{u}_\beta) = det_\beta(\tilde{M}) \tag{4.106}$$

• At last, in Section 4.2.2, we interpret our expression as a sum over subsets of the product of two minor determinants, on the said subsets and its complementary.

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \sum_{I \Rightarrow \{\alpha,\beta\}} \prod_{i\in\alpha} (\partial_x \mathcal{J}(x)|_{u_i}) det_\beta(\tilde{M}) \tag{4.107}$$

$$= \sum_{I \Rightarrow \{\alpha,\beta\}} det_\alpha(J) det_\beta(\tilde{M}) \tag{4.108}$$

J the diagonal matrix of element $\partial_x \mathcal{J}(x)|_{u_i}$, $i \in \alpha$.

We can then use the Laplace identity to rewrite this a the determinant of the sum of two matrices, which are straightforwardly shown to correspond to the desired matrix in (4.95),

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = det(J + \tilde{M}) \tag{4.109}$$

$$= det(M) \tag{4.110}$$

hence the theorem proved.
Let's proceed!

Splitting the contours into single and double pole contours

Using the definition of $\mathcal{J}^Z_{\bar{u}}(x)$, we can rewrite (4.93) as

$$\mathcal{A}_Z(\{\bar{u},\bar{u}_I,\bar{v}_{II}\}) \tag{4.111}$$
$$= \sum_{n=0}^{2M} \frac{(-)^n}{n!} \prod_{a=1}^{n} \oint_{\mathcal{C}} \frac{dx_a}{2i\pi c} \mathcal{J}^Z_{\bar{u}}(x_a) \frac{(x_a-\bar{u}_I)^2-c^2}{(x_a-\bar{u}_I)^2} \frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}} \frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}} \prod_{a\neq b}^{n} \frac{x_a-x_b}{x_a-x_b+c}$$

The pole in $\bar{u}_I$ are here double poles, and will thus require a particular treatment. We are now to split our contours around double and single poles, such that our integral representation express as a sum over single pole and double pole contours.

The expression (4.111) is symmetric with respect to permutations of the integrated variables x_a, and since the only encircled poles are $\bar{u}\cup\bar{v}_{II}$, we can write $\mathcal{C}\equiv\mathcal{C}_{\bar{u}\cup\bar{v}_{II}} \simeq \mathcal{C}_{\bar{u}_I}+\mathcal{C}_{\bar{u}_{II}\cup\bar{v}_{II}} \equiv \mathcal{C}_I+\mathcal{C}_{II}$, i.e. $\mathcal{C}_I$ and $\mathcal{C}_{II}$ respectively encircle double and single poles.

We thus have

$$\frac{(-)^n}{n!}\left(\oint_{\mathcal{C}} dx\right)^n = \frac{(-)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\oint_{\mathcal{C}_I} dx\right)^k\left(\oint_{\mathcal{C}_{II}} dx\right)^{n-k} \tag{4.112}$$
$$= \sum_{p+q=n} \frac{(-)^p}{p!}\left(\oint_{\mathcal{C}_I} dx\right)^p \frac{(-)^q}{q!}\left(\oint_{\mathcal{C}_{II}} dx\right)^q \tag{4.113}$$

and (4.111) rewrites

$$\mathcal{A}_Z(\{\bar{u},\bar{u}_I,\bar{v}_{II}\}) = \sum_{p=0}^{\infty}\frac{(-)^p}{p!}\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^p \sum_{q=0}^{\infty}\frac{(-)^q}{q!}\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\right)^q \tag{4.114}$$
$$\times \prod_{a=1}^{p+q} \mathcal{J}^Z_{\bar{u}}(x_a)\frac{(x_a-\bar{u}_I)^2-c^2}{(x_a-\bar{u}_I)^2}\frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}}\frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}}\prod_{a\neq b}^{p+q}\frac{x_a-x_b}{x_a-x_b+c} \tag{4.115}$$

Extracting the quantifier: sum over subsets

We are now to extract the quantifier from double pole integrals, i.e. extract $\mathcal{J}(x)$ with x integrated over $\mathcal{C}_I$, step by step, until none of these remains.

To do so let us define, for the partition $I\Rightarrow\{\alpha,\beta\}$, the object

$$\mathcal{I}_q(m,n,\alpha) = \prod_{i\in\alpha}\left(c\partial_x\mathcal{J}^Z_{\bar{u}}(x)\big|_{u_i}\right)\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^m\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q \tag{4.116}$$
$$\times\prod_a\frac{(x_a-\bar{u}_\beta)^2-c^2}{(x_a-\bar{u}_\beta)^2}\frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}}\frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}}\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c} \tag{4.117}$$
$$\times\prod_{k<l\in\alpha}\frac{(u_k-u_l)^2-c^2}{(u_k-u_l)^2}\frac{(\bar{u}_\alpha-\bar{u}_\beta)^2-c^2}{(\bar{u}_\alpha-\bar{u}_\beta)^2}\frac{\bar{u}_\alpha-\bar{u}_{II}-c}{\bar{u}_\alpha-\bar{u}_{II}}\frac{\bar{u}_\alpha-\bar{v}_{II}+c}{\bar{u}_\alpha-\bar{v}_{II}} \tag{4.118}$$

Accordingly, we can write (4.114) as

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \sum_{p=0}^{\infty} \frac{(-)^p}{p!} \sum_{q=0}^{\infty} \frac{(-)^q}{q!} \mathcal{I}_q(p, 0, \emptyset) \tag{4.119}$$

This expression will take interest from the two following lemmas.

Lemma 4.2.11. *For $\alpha \subseteq I$, $n \in \mathbb{N}$, $m \in \mathbb{N}^+$, and the partition $I \Rightarrow \{\alpha, \beta\}$, we have*

$$\mathcal{I}_q(m, n, \alpha) = -\sum_{k\in\beta} \mathcal{I}_q(m-1, n, \alpha \cup \{k\}) - \mathcal{I}_q(m-1, n+1, \alpha) \tag{4.120}$$

Proof. See Appendix 4.2.3. □

The idea behind this Lemma is what has been drawn in the sketched proof, see equations (4.103).
We are able, step by step, to extract the quantifier integrated around double pole contours $\mathcal{C}_I$: For each iteration of the procedure depicted in Lemma 4.2.11, two terms with one less integral on $\mathcal{C}_I$ containing $\mathcal{J}(x)$ (i.e. $m \to m-1$) are produced: one term with $\alpha \to \alpha \cup \{k\}$ (i.e. an increasing the set of derivative of $\mathcal{J}$) and another term with $n \to n+1$ (i.e. one more integral on $\mathcal{C}_I$ not containing $\mathcal{J}(x)$). Each step then reduces the number of integrated quantifiers on $\mathcal{C}_I$, and conserve the quantity $m + n + \#\alpha$.

Remark. *The set α can here be completed, at each step, by an element of its complementary in I. This is so given that once a pole has been grabbed, the corresponding pole disappears for the other integrated variables (considering the term with derivative of the quantifier $\mathcal{J}$ only) and can then not be grabbed again.*

This first Lemma can be extended to the following one:

Lemma 4.2.12. *For $r \leq m \leq 2I$*

$$\mathcal{I}_q(m, 0, \emptyset) = (-)^r \sum_{n=0}^{r} \sum_{\substack{\alpha\subseteq I \\ \#\alpha = r-n}} \mathcal{I}_q(m-r, n, \alpha) \frac{r!}{n!} \tag{4.121}$$

Proof. See Appendix 4.2.3 □

Here we repeatedly applied the procedure of Lemma 4.2.11 r times, i.e. proceed $m \to m - r$. It logically produces terms with α subset of "double pole variables" I, of cardinal at most r, since each step adds to this subset (at most) one element, randomly picked in $\bar{u}_I$.

Using Lemma 4.2.12 for $r = m$ (i.e. extracting all the quantifier on $\mathcal{C}_I$) and the expression (4.119), we can write

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{n=0}^{p} \sum_{\substack{\alpha\subseteq I \\ \#\alpha=p-n}} \mathcal{I}_q(0,n,\alpha) \frac{p!}{n!} \tag{4.122}$$

$$= \sum_{\alpha\subseteq I} \sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{I}_q(0,n,\alpha) \tag{4.123}$$

Completing the integration: to a determinant in the complementary subset

We now want to factorize $\mathcal{I}$ as a part still to be integrated, and another part that contains the partial derivatives of the quantifier (and a plethora of decorating terms). To this end we define

$$\mathcal{K}(\alpha) \equiv \prod_{i\in\alpha} \left(c\partial_x \mathcal{J}_{\bar{u}}^Z(x)\big|_{u_i} \right) \tag{4.124}$$

$$\times \prod_{k<l\in\alpha} \frac{(u_k-u_l)^2-c^2}{(u_k-u_l)^2} \frac{(\bar{u}_\alpha-\bar{u}_{I\setminus\alpha})^2-c^2}{(\bar{u}_\alpha-\bar{u}_{I\setminus\alpha})^2} \frac{\bar{u}_\alpha-\bar{u}_{II}-c}{\bar{u}_\alpha-\bar{u}_{II}} \frac{\bar{u}_\alpha-\bar{v}_{II}+c}{\bar{u}_\alpha-\bar{v}_{II}} \tag{4.125}$$

and

$$\tilde{\mathcal{K}}(n,q,\alpha) \equiv \left(\oint_{\mathcal{C}_I} \frac{dx}{2i\pi c} \right)^n \left(\oint_{\mathcal{C}_{II}} \frac{dx}{2i\pi c} \mathcal{J}_{\bar{u}}^Z(x) \right)^q \tag{4.126}$$

$$\times \prod_a \frac{(x_a-\bar{u}_\alpha)^2-c^2}{(x_a-\bar{u}_\alpha)^2} \frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}} \frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}} \prod_{a\neq b} \frac{x_a-x_b}{x_a-x_b+c} \tag{4.127}$$

i.e. $\mathcal{I}_q(0,n,\alpha) = \mathcal{K}(\alpha)\tilde{\mathcal{K}}(n,q,\beta), \quad \beta = I\setminus\alpha.$

Now the integral representation (4.122) rewrite

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \sum_{I\Rightarrow\{\alpha,\beta\}} \mathcal{K}(\alpha) \sum_{q=1}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\mathcal{K}}(n,q,\beta) \tag{4.128}$$

Lemma 4.2.13. *For $\beta \subseteq I$, we have*

$$\sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\mathcal{K}}(n,q,\beta) \equiv \sum_{q=q}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\oint_{\mathcal{C}_I} \frac{dx}{2i\pi c} \right)^n \left(\oint_{\mathcal{C}_{II}} \frac{dx}{2i\pi c} \mathcal{J}_{\bar{u}}^Z(x) \right)^q \tag{4.129}$$

$$\times \prod_a \frac{(x_a-\bar{u}_\beta)^2-c^2}{(x_a-\bar{u}_\beta)^2} \frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}} \frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}} \prod_{a\neq b} \frac{x_a-x_b}{x_a-x_b+c} \tag{4.130}$$

$$= \delta^{-1}(\{\bar{u}_\beta, \bar{u}_{II}\}, \{\bar{u}_\beta, \bar{v}_{II}\}) \det \begin{pmatrix} \tilde{G}_{\beta,\beta} & \tilde{G}_{\beta,II} \\ S_{II,\beta} & S_{II,II} \end{pmatrix} \tag{4.131}$$

where

$$\left(\tilde{G}_{\delta\gamma}\right)_{ij} \equiv \frac{-2c^2}{(u_{\delta_i} - u_{\gamma_j})^2 - c^2}$$

from which we can rewrite the "still to be integrated" term as a determinant.

Proof. See Appendix 4.2.3 □

Laplace identity: ressumation as a single determinant

According to Lemma 4.2.13 equation (4.128) rewrites

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \delta^{-1}(\bar{u}, \{\bar{u}_I, \bar{v}_{II}\}) \sum_{I \Rightarrow \{\alpha,\beta\}} \prod_{i\in\alpha} \left((-c)\partial_x \mathcal{J}^Z_{\bar{u}}(x)\big|_{u_i}\right) \begin{pmatrix} \tilde{G}_{\beta,\beta} & \tilde{G}_{\beta,II} \\ S_{II,\beta} & S_{II,II} \end{pmatrix} \tag{4.132}$$

where δ is the Cauchy determinant, defined in the introduction.

We can interpret this a a sum of minor determinants, by defining the diagonal matrix J_β, β a set of indexes, of matrix elements

$$(J_\beta)_{ij} \equiv \delta_{ij}\, (-c)\partial_x \mathcal{J}^Z_{\bar{u}}(x)\big|_{u_{\beta_i}} \tag{4.133}$$

such that 4.132 rewrites

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \delta^{-1}(\bar{u}, \{\bar{u}_I, \bar{v}_{II}\}) \sum_{\alpha \subseteq I \Rightarrow \{\alpha,\beta\}} \det J_\alpha \det \begin{pmatrix} \tilde{G}_{\beta,\beta} & \tilde{G}_{\beta,II} \\ S_{II,\beta} & S_{II,II} \end{pmatrix} \tag{4.134}$$

Given that J is diagonal, we can now use the Laplace identity for the determinant of the sum of two matrices on the set E,

$$det_E\,(A+B) = \sum_{\substack{E \Rightarrow \{\alpha_I, \alpha_{II}\} \\ E \Rightarrow \{\beta_I, \beta_{II}\}}} \det(A_{\alpha_I,\beta_I}) \det(B_{\alpha_{II},\beta_{II}}) \tag{4.135}$$

or, for A diagonal,

$$det_E\,(A+B) = \sum_{E \Rightarrow \{\alpha_I, \alpha_{II}\}} \det(A_{\alpha_I,\alpha_I}) \det(B_{\alpha_{II},\alpha_{II}}) \tag{4.136}$$

We then obtain, from (4.134),

$$\mathcal{A}_Z(\{\bar{u}, \bar{u}_I, \bar{v}_{II}\}) = \delta^{-1}(\bar{u}, \{\bar{u}_I, \bar{v}_{II}\}) \det\left[\begin{pmatrix} J_I & 0_{I,II} \\ 0_{II,I} & 0_{II,II} \end{pmatrix} + \begin{pmatrix} \tilde{G}_{I,I} & \tilde{G}_{I,II} \\ S_{II,I} & S_{II,II} \end{pmatrix}\right] \tag{4.137}$$

Finally, one can see that the non diagonal terms of $\tilde{G}$ are those of G, while for the diagonal terms we have: $\tilde{G}_{ii} + J_i = -2 + c\,\partial_x \mathcal{J}^Z_{\bar{u}}(x)\big|_{u_i} = -2 + \partial_{u_i}\mathcal{J}^Z_{\bar{u}}(u_i) + 2 = G_{ii}$.

Hence the theorem is proved.

□

Conclusion

Through this subsection we learned a bit more to understand the integral representation, the mechanics at work in these and how they do behave under relevant limits, such that we obtained a coherent mapping between the different determinants for the norm and the scalar product, and the integral representation. Our reflection however limits to a very particular (and very simple) case of spin chain, and it would be interesting to see how the integral representation could be generalized or adapted to more exotic cases.

4.2.3 Appendix

Proof of Theorem 4.2.7

Proof. • On one side we have

$$\begin{aligned}\mathcal{K}_N(\bar{v}|\bar{\theta}) \equiv & det^{-1}\left[\frac{1}{v_i-\theta_j-c/2}\right] det\left[\frac{1}{v_i-\theta_j-c/2}-\frac{1}{v_i-\theta_j+c/2}\right] \\ = & \frac{\prod_{i,j}(v_i-\theta_j-c/2)}{\prod_{i<j}(v_i-v_j)\prod_{i<j}(\theta_i-\theta_j)} det\left[(1-D_{v_i})(\frac{1}{v_i-\theta_j-c/2})\right] \\ = & \frac{\prod_{i,j}(v_i-\theta_j-c/2)}{\prod_{i<j}(v_i-v_j)\prod_{i<j}(\theta_i-\theta_j)} \prod_{i=1}^{\#\bar{v}}(1-D_{v_i}) det\left[\frac{1}{v_i-\theta_j-c/2}\right]\end{aligned} \tag{4.138}$$

where we made use of the shift operator D, and factorize the operators $1-D$ as a product, since each of these act separately on a line of the matrix inside determinant.

This product can now be developed as a sum over 2-partitions of the set of rapidities $\bar{v} \Rightarrow \{\bar{v}_\alpha, \bar{v}_{\bar{\alpha}}\}$

$$\begin{aligned}\mathcal{K}_N(\bar{v}|\bar{\theta}) = & \frac{\prod_{i,j}(v_i-\theta_j-c/2)}{\prod_{i<j}(v_i-v_j)\prod_{i<j}(\theta_i-\theta_j)}\left(\sum_{\bar{v}\Rightarrow\{\bar{v}_\alpha,\bar{v}_{\bar{\alpha}}\}}(-)^{\#u_\alpha}D_{v_\alpha}\right)\frac{\prod_{i<j}(v_i-v_j)\prod_{i<j}(\theta_i-\theta_j)}{\prod_{i,j}(v_i-\theta_j-c/2)} \\ = & \sum_{\bar{v}\Rightarrow\{\bar{v}_\alpha,\bar{v}_{\bar{\alpha}}\}}(-)^{\#u_\alpha}\frac{\prod_{i,j}(v_i-\theta_j-c/2)}{\prod_{i<j}(v_i-v_j)\prod_{i<j}(\theta_i-\theta_j)}D_{v_\alpha}\frac{\prod_{i<j}(v_i-v_j)\prod_{i<j}(\theta_i-\theta_j)}{\prod_{i,j}(v_i-\theta_j-c/2)}\end{aligned} \tag{4.139}$$

we can now make the shift operators act on the developed form of the Cauchy determinant, and obtain

$$\mathcal{K}_N(\bar{v}|\bar{\theta}) = \sum_{\bar{v}\Rightarrow\{\bar{v}_\alpha,\bar{v}_{\bar{\alpha}}\}}(-)^{\#u_\alpha}\frac{\bar{v}_\alpha-\bar{\theta}-c/2}{\bar{v}_\alpha-\bar{\theta}+c/2}\frac{\bar{v}_\alpha-\bar{v}_{\bar{\alpha}}+c}{\bar{v}_\alpha-\bar{v}_{\bar{\alpha}}} \tag{4.140}$$

• Now on the other side of the story we have

$$\mathcal{A}_{Z_{\bar{\theta}}}(\bar{v}) = \sum_{n=0}^{N}\frac{(-)^n}{n!}\prod_{a=1}^{n}\left(\oint_{\mathcal{C}_{\mathbf{u}}}\frac{dx_a}{2i\pi c}\frac{x_a-\bar{\theta}-c/2}{x_a-\bar{\theta}+c/2}\frac{x_a-\bar{v}+c}{x_a-\bar{v}}\right)\prod_{a\neq b}^{n}\frac{x_a-x_b}{x_a-x_b+c} \tag{4.141}$$

where $\mathcal{C}_{\bar{v}}$ closely encircle the rapidities $\bar{v}$, such that the only poles are at $x_a = v_j$.

When integrating a variable x_c, it will grab some pole at $x_c = v_j$, such that the term $\prod_{a\neq b}^{n} \frac{x_a - x_b}{x_a - x_b + c}$ will produce $\prod_{a\neq c} \frac{(x_a - v_j)^2}{(x_a - v_j)^2 + c^2}$. Then, the pole in $x_b = v_j$ disappears for all b. In other words a parameter v_j can be grabbed only once.

We thus obtain

$$\mathcal{A}_{Z_{\bar{\theta}}}(\bar{v}) = \sum_{n=1}^{N} (-)^n \sum_{\bar{v}_\alpha \subseteq \bar{v},\ \#\bar{v}=n} \frac{\bar{v}_\alpha - \bar{\theta} - c/2}{\bar{v}_\alpha - \bar{\theta} + c/2} \frac{\bar{v}_\alpha - \bar{v} + c}{\bar{v}_\alpha - \bar{v}} \prod_{i\neq j\ i,j\in\alpha}^{n} \frac{v_i - v_j}{v_i - v_j + c} \tag{4.142}$$

$$= \sum_{\bar{v}_\alpha \subseteq \bar{v}} (-)^{\#\bar{v}_\alpha} \frac{\bar{v}_\alpha - \bar{\theta} - c/2}{\bar{v}_\alpha - \bar{\theta} + c/2} \frac{\bar{v}_\alpha - \bar{v}_{\bar{\alpha}} + c}{\bar{v}_\alpha - \bar{v}_{\bar{\alpha}}} \tag{4.143}$$

Linking these two sides, the theorem is proved.

□

Proof of Theorem 4.2.8

For $\bar{v} = \{\bar{u}, \bar{w}\}$, $\#\bar{u} = \#\bar{w} = M$ a set of disjoint complex parameters with $\bar{u}$ satisfying the Bethe equations $\mathcal{J}^{Z}_{\bar{u}_i}(u_i) = 1\ \forall i$, and $Z : \mathbb{C} \longmapsto \mathbb{C}$ any complex function regular at $u_j\ \forall j$, holds the equality between the integral representation and the Slavnov determinant

$$\mathcal{A}_Z(\{\bar{u}, \bar{w}\}) = det^{-1}\left[\frac{c}{u_i - w_j + c}\right] det\left[c^2 \frac{\mathcal{J}^{Z}_{\bar{u}_i}(w_j) - 1}{(u_i - w_j)(u_i - w_j + c)}\right] \equiv \mathcal{S}_Z(\bar{u}|\bar{w}) \tag{4.144}$$

Proof. In the following we consider $\bar{v} = \{\bar{u}, \bar{w}\}$, $\#\bar{u} = \#\bar{w} = M$ to be a set of disjoint complex parameter, and $Z : \mathbb{C} \longmapsto \mathbb{C}$ any complex function regular on $\bar{v}$.

$$\begin{aligned}\mathcal{A}_Z(\{\bar{u}, \bar{w}\}) &= \sum_{n=0}^{N} \frac{(-)^n}{n!} \prod_{a=1}^{n} \left(\oint_{\mathcal{C}_{\bar{v}}} \frac{dx_a}{2i\pi c} Z(x_j) \frac{x_a - \bar{u} + c}{x_a - \bar{u}} \frac{x_a - \bar{w} + c}{x_a - \bar{w}} \right) \prod_{a\neq b}^{n} \frac{x_a - x_b}{x_a - x_b + c} \\ &= \sum_{n=0}^{N} \frac{(-)^n}{n!} \prod_{a=1}^{n} \left(\oint_{\mathcal{C}_{\bar{v}}} \frac{dx_a}{2i\pi c} \mathcal{J}^{Z}_{\bar{u}}(x_j) \frac{x_a - \bar{u} - c}{x_a - \bar{u}} \frac{x_a - \bar{w} + c}{x_a - \bar{w}} \right) \prod_{a\neq b}^{n} \frac{x_a - x_b}{x_a - x_b + c}\end{aligned} \tag{4.145}$$

with $\mathcal{C}_{\bar{v}}$ closely encircling $\{\bar{u}, \bar{w}\}$.

As before, we know that a pole $x_a = v_j$ can be grabbed only once while integrating. We thus obtain

$$\begin{aligned}\mathcal{A}_Z(\{\bar{u}, \bar{w}\}) = &\sum_{n=0}^{2M} \left(\frac{-1}{c}\right)^n \sum_{\substack{\bar{u}_\alpha \subseteq \bar{u} \\ \bar{w}_\beta \subseteq \bar{w} \\ \#\bar{u}_\alpha + \#\bar{w}_\beta = n}} \mathcal{J}^{Z}_{\bar{u}}(\bar{u}_\alpha) \mathcal{J}^{Z}_{\bar{u}}(\bar{w}_\beta) (-c)^{\#\bar{u}_\alpha} c^{\#\bar{w}_\beta} \\ &\times \frac{\bar{u}_\alpha - \bar{u} - c}{\bar{u}_\alpha - \bar{u}} \frac{\bar{u}_\alpha - \bar{w} + c}{\bar{u}_\alpha - \bar{w}} \frac{\bar{w}_\beta - \bar{u} - c}{\bar{w}_\beta - \bar{u}} \frac{\bar{w}_\beta - \bar{w} + c}{\bar{w}_\beta - \bar{w}} \\ &\times \frac{\bar{u}_\alpha - \bar{u}_\alpha}{\bar{u}_\alpha - \bar{u}_\alpha - c} \frac{\bar{w}_\beta - \bar{w}_\beta}{\bar{w}_\beta - \bar{w}_\beta + c} \frac{\bar{u}_\alpha - \bar{w}_\beta}{\bar{u}_\alpha - \bar{w}_\beta + c} \frac{\bar{w}_\beta - \bar{u}_\alpha}{\bar{w}_\beta - \bar{u}_\alpha + c}\end{aligned} \tag{4.146}$$

here we sum over the sets $\bar{u}_\alpha$ and $\bar{w}_\beta$ corresponding to the grabbed subset of $\bar{u}$ and $\bar{w}$ respectively.

Simplifying a bit, we get

$$\mathcal{A}_Z(\{\bar{u},\bar{w}\}) = \sum_{\bar{u}_\alpha\subseteq\bar{u},\bar{w}_\beta\subseteq\bar{w}} (-)^{\#\bar{w}_\beta}\mathcal{J}^Z_{\bar{u}}(\bar{w}_\beta)\mathcal{J}^Z_{\bar{u}}(\bar{u}_\alpha) \tag{4.147}$$

$$\times\frac{\bar{u}_\alpha-\bar{u}_{\bar{\alpha}}-c}{\bar{u}_\alpha-\bar{u}_{\bar{\alpha}}}\frac{\bar{u}_\alpha-\bar{w}_{\bar{\beta}}+c}{\bar{u}_\alpha-\bar{w}_{\bar{\beta}}}\frac{\bar{u}_\alpha-\bar{w}_\beta+c}{\bar{u}_\alpha-\bar{w}_\beta-c}\frac{\bar{u}_{\bar{\alpha}}-\bar{w}_\beta+c}{\bar{u}_{\bar{\alpha}}-\bar{w}_\beta}\frac{\bar{w}_\beta-\bar{w}_{\bar{\beta}}+c}{\bar{w}_\beta-\bar{w}_{\bar{\beta}}} \tag{4.148}$$

We can now make use of the shift operator, and write

$$\mathcal{A}_Z(\{\bar{u},\bar{w}\}) = \left[\frac{\prod_{i<j}(u_i-u_j)(w_i-w_j)}{\prod_{i,j}(u_i-w_j+c)}\right]^{-1}\prod_{i,j=1}^{M}(1+\mathcal{J}^Z_{\bar{u}}(u_i)D^{-1}_{u_i})(1-\mathcal{J}^Z_{\bar{u}}(w_j)D_{w_j}) \tag{4.149}$$

$$\times\frac{\prod_{i<j}(u_i-u_j)(w_i-w_j)}{\prod_{i,j}(u_i-w_j+c)}$$

$$=det^{-1}\left[\frac{1}{u_i-w_j+c}\right]\prod_{i,j=1}^{M}(1+\mathcal{J}^Z_{\bar{u}}(u_i)D^{-1}_{u_i})(1-\mathcal{J}^Z_{\bar{u}}(w_j)D_{w_j})det\left[\frac{1}{u_i-w_j+c}\right] \tag{4.150}$$

with the shift operator $D_x f(x) = f(x+c)$.

Distributing these product of independent operator over lines and column of the Cauchy matrix, we obtain (add intermediate step here, with the product developed as a sum over partitions)

$$\mathcal{A}_Z(\{\bar{u},\bar{w}\}) =det^{-1}\left[\frac{1}{u_i-w_j+c}\right]det\left[\frac{(1-\mathcal{J}^Z_{\bar{u}_i}(w_j))(1-\mathcal{J}^Z_{\bar{u}_i}(u_i))}{(u_i-w_j+c)}+c\frac{\mathcal{J}^Z_{\bar{u}_i}(w_j)-\mathcal{J}^Z_{\bar{u}_i}(u_i)}{(u_i-w_j)(u_i-w_j+c)}\right] \tag{4.151}$$

where we used $\mathcal{J}^Z_{\bar{u}}(w_j) = \mathcal{J}^Z_{\bar{u}_i}(w_j)\frac{u_i-w_j-c}{u_i-w_j+c}$, and $\mathcal{J}^Z_{\bar{u}}(u_i) = -\mathcal{J}^Z_{\bar{u}_i}(u_i)$.

Hence, for $\bar{u}$ satisfying the Bethe equations, i.e. $\mathcal{J}^Z_{\bar{u}_i}(u_i) = 1\forall i$, we obtain the theorem. hence the theorem proved. □

Proof of Lemma 4.2.11

For $\alpha\subseteq\lambda_I$ and $n\in\mathbb{N}$ and $m\in\mathbb{N}^*$,

$$\mathcal{I}_q(m,n,\alpha) = \sum_{k\in\lambda_I\backslash\alpha}\mathcal{I}_q(m-1,n,\alpha\cup\{k\}) - \mathcal{I}_q(m-1,n+1,\alpha) \tag{4.152}$$

Proof. We have

$$\mathcal{I}_q(m,n,\alpha) = \prod_{i\in\alpha}\left(c\partial_x \mathcal{J}^Z_{\bar{u}}(x)\big|_{u_i}\right)\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^m\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q \tag{4.153}$$

$$\times\prod_a \frac{(x_a-\bar{u}_{\lambda_I\setminus\alpha})^2-c^2}{(x_a-\bar{u}_{\lambda_I\setminus\alpha})^2}\ \frac{x_a-\bar{u}_{\lambda_{II}}-c}{x_a-\bar{u}_{\lambda_{II}}}\ \frac{x_a-\bar{v}_{\lambda_{II}}+c}{x_a-\bar{v}_{\lambda_{II}}}\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c} \tag{4.154}$$

$$\times\prod_{k<l\in\alpha}\frac{(u_k-u_l)^2-c^2}{(u_k-u_l)^2}\prod_{k\in\alpha,\ l\in\lambda_I\setminus\alpha}\frac{(u_k-u_l)^2-c^2}{(u_k-u_l)^2} \tag{4.155}$$

We are to look at the term

$$X = \left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^m\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q \tag{4.156}$$

$$\times\prod_a \frac{(x_a-\bar{u}_{\lambda_I\setminus\alpha})^2-c^2}{(x_a-\bar{u}_{\lambda_I\setminus\alpha})^2}\ \frac{x_a-\bar{u}_{\lambda_{II}}-c}{x_a-\bar{u}_{\lambda_{II}}}\ \frac{x_a-\bar{v}_{\lambda_{II}}+c}{x_a-\bar{v}_{\lambda_{II}}}\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c} \tag{4.157}$$

and integrate it in $x = y$, for y a variable integrated in $\mathcal{C}_I$ and such that appears $\mathcal{J}^Z_{\bar{u}}(y)$.

For such a contour, uncounted poles will be doubles, hence appearance of a derivative:

$$\begin{aligned}
X = &\sum_{k\in\lambda_I\setminus\alpha}\left(-c\partial_x \mathcal{J}^Z_{\bar{u}}(x)\big|_{u_k}\right)\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^{m-1}\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q\\
&\times\prod_a \frac{(x_a-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2-c^2}{(x_a-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2}\ \frac{x_a-\bar{u}_{\lambda_{II}}-c}{x_a-\bar{u}_{\lambda_{II}}}\ \frac{x_a-\bar{v}_{\lambda_{II}}+c}{x_a-\bar{v}_{\lambda_{II}}}\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c}\\
&\times\frac{(u_k-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2-c^2}{(u_k-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2}\ \frac{u_k-\bar{u}_{\lambda_{II}}-c}{u_k-\bar{u}_{\lambda_{II}}}\ \frac{u_k-\bar{v}_{\lambda_{II}}+c}{u_k-\bar{v}_{\lambda_{II}}}\\
&-\sum_{k\in\lambda_I\setminus\alpha}\frac{1}{c}\partial_y\left(((y-u_k)^2-c^2)\frac{(y-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2-c^2}{(y-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2}\ \frac{y-\bar{u}_{\lambda_{II}}-c}{y-\bar{u}_{\lambda_{II}}}\ \frac{y-\bar{v}_{\lambda_{II}}+c}{y-\bar{v}_{\lambda_{II}}}\right)\Bigg|_{y=u_k}\\
&\times\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^{m-1}\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q\\
&\times\prod_a \frac{(x_a-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2-c^2}{(x_a-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2}\ \frac{x_a-\bar{u}_{\lambda_{II}}-c}{x_a-\bar{u}_{\lambda_{II}}}\ \frac{x_a-\bar{v}_{\lambda_{II}}+c}{x_a-\bar{v}_{\lambda_{II}}}\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c}\\
&\times\frac{(u_k-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2-c^2}{(u_k-\bar{u}_{\lambda_I\setminus(\alpha\cup\{k\})})^2}\ \frac{u_k-\bar{u}_{\lambda_{II}}-c}{u_k-\bar{u}_{\lambda_{II}}}\ \frac{u_k-\bar{v}_{\lambda_{II}}+c}{u_k-\bar{v}_{\lambda_{II}}}
\end{aligned} \tag{4.158}$$

where the minus sign in front of the second term comes from $\mathcal{J}^Z_{\bar{u}}(u_k) = -1\ \forall k$. Since this value is independent of k, and hence factorize the second term, wan can rewrite it as integration of y of the same term, without $\mathcal{J}^Z_{\bar{u}}(y)$. The second term then rewrites

$$
-\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^{m-1}\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^{n+1}\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^{q} \tag{4.159}
$$
$$
\times\prod_a\frac{(x_a-\bar{u}_{\lambda_I\backslash\alpha})^2-c^2}{(x_a-\bar{u}_{\lambda_I\backslash\alpha})^2}\frac{x_a-\bar{u}_{\lambda_{II}}-c}{x_a-\bar{u}_{\lambda_{II}}}\frac{x_a-\bar{v}_{\lambda_{II}}+c}{x_a-\bar{v}_{\lambda_{II}}}\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c}
$$

Using the definition of $\mathcal{I}_q(m,n,\alpha)$, the theorem follows. □

Proof of Lemma 4.2.12

For $r\leq m\leq 2\#\lambda_I$

$$
\mathcal{I}_q(m,0,\alpha)=\sum_{n=0}^{r}\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=r-n}\mathcal{I}_q(m-r,n,\alpha)(-)^n\frac{r!}{n!} \tag{4.160}
$$

Proof. We are going to proceed by recursion.

The initialization is insured by Lemma 4.2.11. Let us assume (4.160) valid for r. We are to prove it remains valid for $r'=r+1$.

We have

$$
\mathcal{I}_q(m,0,\alpha)=\sum_{n=0}^{r}\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=r-n}\mathcal{I}_q(m-r,n,\alpha)(-)^n\frac{r!}{n!}. \tag{4.161}
$$

Using Lemma 4.2.11, it rewrites

$$
\begin{aligned}
\mathcal{I}_q(m,0,\alpha)=&\sum_{n=0}^{r}\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=r-n}\sum_{k\in I\backslash\alpha}\mathcal{I}_q(m-(r+1),n,\alpha\cup\{k\})(-)^n\frac{r!}{n!}\\
&-\sum_{n=0}^{r}\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=r-n}\mathcal{I}_q(m-(r+1),n+1,\alpha)(-)^n\frac{r!}{n!}.
\end{aligned} \tag{4.162}
$$

We can rewrite the sum in the first term $\sum\limits_{\substack{\alpha\subseteq\lambda_I\\ \#\alpha=r-n}}\sum_{k\in I\backslash\alpha}=(r+1-n)\sum\limits_{\substack{\alpha\subseteq\lambda_I\\ \#\alpha=(r+1)-n}}$, and shifting $n\to n+1$ in the second term, we get

$$
\begin{aligned}
\mathcal{I}_q(m,0,\alpha)=&\sum_{n=0}^{r}(r+1-n)\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=(r+1)-n}\mathcal{I}_q(m-(r+1),n,\alpha)(-)^n\frac{r!}{n!}\\
&+\sum_{n=1}^{r+1}\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=(r+1)-n}\mathcal{I}_q(m-(r+1),n,\alpha)(-)^n\frac{r!}{(n-1)!}
\end{aligned} \tag{4.163}
$$

we can now separate these sums as a term with $n=0$ from the first line, a sum of terms for $1\leq n\leq r$ from the two lines, and a term with $n=r+1$ from the second line:

$$\mathcal{I}_q(m,0,\alpha) = \sum_{\alpha\subseteq\lambda_I,\ \#\alpha=r+1} (r+1)\mathcal{I}_q(m-(r+1),0,\alpha)(-)^n r! \tag{4.164}$$

$$+\sum_{n=1}^{r} \sum_{\alpha\subseteq\lambda_I,\ \#\alpha=(r+1)-n} \mathcal{I}_q(m-(r+1),n,\alpha)(-)^n r!\left(\frac{(r+1)-n}{n!}+\frac{1}{(n-1)!}\right)$$

$$+\sum_{\alpha\subseteq\lambda_I,\ \#\alpha=0} \mathcal{I}_q(m-(r+1),r+1,\alpha)(-)^{r+1}\frac{r!}{((r+1)-1)!}$$

$$=\sum_{n=0}^{r+1} \sum_{\alpha\subseteq\lambda_I,\ \#\alpha=(r+1)-n} \mathcal{I}_q(m-(r+1),n,\alpha)(-)^n\frac{(r+1)!}{n!} \tag{4.165}$$

where we obtained the last line by assembling the three terms of the former ones. Hence the theorem is proved. □

Proof of Lemma 4.2.13

$$\sum_{q=0}^{\infty}\frac{(-)^q}{q!}\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\mathcal{K}}(n,q,\alpha) \equiv \tag{4.166}$$

$$\sum_{q=0}^{\infty}\frac{(-)^q}{q!}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q$$

$$\times\prod_a \frac{(x_a-\bar{u}_\alpha)^2-c^2}{(x_a-\bar{u}_\alpha)^2}\frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}}\frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}}\prod_{a\neq b}^{p+q}\frac{x_a-x_b}{x_a-x_b+c}$$

$$=\delta^{-1}(\{\bar{u}_\alpha,\bar{u}_{II}\},\{\bar{u}_\alpha,\bar{v}_{II}\})\det\begin{pmatrix}\tilde{G}_{\alpha,\alpha} & \tilde{G}_{\alpha,II}\\ S_{II,\alpha} & S_{II,II}\end{pmatrix} \tag{4.167}$$

where

$$\left(\tilde{G}_{\beta\gamma}\right)_{ij}\equiv\frac{-2c^2}{(u_{\beta_i}-u_{\gamma_j})^2-c^2}$$

Proof. First, we consider (4.166) as the limit

$$\sum_{q=0}^{\infty}\frac{(-)^q}{q!}\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\mathcal{K}}(n,q,\alpha)$$

$$=\lim_{\bar{v}_\alpha\to\bar{u}_\alpha}\sum_{q=0}^{\infty}\frac{(-)^q}{q!}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\oint_{\mathcal{C}_I}\frac{dx}{2i\pi c}\right)^n\left(\oint_{\mathcal{C}_{II}}\frac{dx}{2i\pi c}\mathcal{J}^Z_{\bar{u}}(x)\right)^q \tag{4.168}$$

$$\times\prod_a\frac{x_a-\bar{u}_\alpha-c}{x_a-\bar{u}_\alpha}\frac{x_a-\bar{v}_\alpha+c}{x_a-\bar{v}_\alpha}\frac{x_a-\bar{u}_{II}-c}{x_a-\bar{u}_{II}}\frac{x_a-\bar{v}_{II}+c}{x_a-\bar{v}_{II}}$$

$$\times\prod_{a\neq b}\frac{x_a-x_b}{x_a-x_b+c}$$

We define $\tilde{\mathcal{J}}^Z_{\bar{u}}$, the modified $\mathcal{J}^Z_{\bar{u}}$, a regular function on $\bar{u} \cup \bar{v}_{II}$, such that

$$\tilde{\mathcal{J}}^Z_{\bar{u}}(y) = \begin{cases} \mathcal{J}^Z_{\bar{u}} & y \text{ in the vicinity of } \mathcal{C}_{II} \cup \bar{u}_{II} \cup \bar{v}_{II} \\ -1 & y \in \text{ in the vicinity of } \mathcal{C}_I \cup \bar{u}_\alpha \cup \bar{v}_\alpha \end{cases}$$

Doing so, 4.168 rewrites

$$\begin{aligned} &\sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\mathcal{K}}(n,q,\alpha) \\ &= \lim_{\bar{v}_\alpha \to \bar{u}_\alpha} \sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\oint_{\mathcal{C}_I} \frac{dx}{2i\pi c} \tilde{\mathcal{J}}^Z_{\bar{u}}(x) \right)^n \left(\oint_{\mathcal{C}_{II}} \frac{dx}{2i\pi c} \tilde{\mathcal{J}}^Z_{\bar{u}}(x) \right)^q \\ &\times \prod_a \frac{x_a - \bar{u}_\alpha - c}{x_a - \bar{u}_\alpha} \frac{x_a - \bar{v}_\alpha + c}{x_a - \bar{v}_\alpha} \frac{x_a - \bar{u}_{II} - c}{x_a - \bar{u}_{II}} \frac{x_a - \bar{v}_{II} + c}{x_a - \bar{v}_{II}} \\ &\times \prod_{a \neq b} \frac{x_a - x_b}{x_a - x_b + c} \end{aligned} \tag{4.169}$$

We can recombine the contours according to (4.2.2), and so obtain

$$\begin{aligned} &\sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\mathcal{K}}(n,q,\alpha) \\ &= \lim_{\bar{v}_\alpha \to \bar{u}_\alpha} \sum_{p=0}^{\infty} \frac{(-)^p}{p!} \left(\oint_{\mathcal{C}_I + \mathcal{C}_{II}} \frac{dx}{2i\pi c} \tilde{\mathcal{J}}^Z_{\bar{u}}(x) \right)^p \\ &\times \prod_a \frac{x_a - \bar{u}_\alpha - c}{x_a - \bar{u}_\alpha} \frac{x_a - \bar{v}_\alpha + c}{x_a - \bar{v}_\alpha} \frac{x_a - \bar{u}_{II} - c}{x_a - \bar{u}_{II}} \frac{x_a - \bar{v}_{II} + c}{x_a - \bar{v}_{II}} \\ &\times \prod_{a \neq b} \frac{x_a - x_b}{x_a - x_b + c} \end{aligned} \tag{4.170}$$

We can now make use of Lemma 4.2.12, and we get

$$\begin{aligned} &\sum_{q=0}^{\infty} \frac{(-)^q}{q!} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\mathcal{K}}(n,q,\alpha) \\ &= \lim_{\bar{v}_\alpha \to \bar{u}_\alpha} \det^{-1} \left[\frac{c}{u_i - v_j + c} \right] \det \left[c^2 \frac{\tilde{\mathcal{J}}^Z_{\bar{u}}(v_j)}{(u_i - v_j)(u_i - v_j - c)} + \frac{\tilde{\mathcal{J}}^Z_{\bar{u}}(u_i)}{(u_i - v_j)(u_i - v_j + c)} \right] \end{aligned} \tag{4.171}$$

The Lemma straightforwardly follows. □

The skeptical reader could as well directly prove it following a similar procedure as had been used to prove Lemma 4.2.12.

4.3 Conclusion

A new ship for new lands.

This chapter offered us to get acquainted with some of the most simple and useful determinant representation that can be encountered in the frame of quantum integrability. In the frame of the δ-Bose gas such a determinant, the Izergin-Korepin determinant, naturally appearing when considering the Matrix Element of the Particle Number Operator (MEPNO), nicely re-sums an a unique Slavnov determinant. It would be interesting to apply this compact result to the study of quench problems. To approach such a problem, one would naturally need to obtain nice expressions for interesting physical state in terms of Bethe states, as has been done in Chapter 3 in the context of the zero-range chipping model with factorized steady state, as such an expression would trivially provide us with the time dependence of the considered state. One could then combine the to results to express the evolution of the particle number at the origin of the lattice after a quench.

We also explored the nice integral representation for these determinants, introduced by E. Bettelheim and I. Kostov. Using this integral representation, we drew a formal link between these determinants, hence proving they equivalence in an original fashion. These integral representation are very simple, and let hope for really nice behaviors in various limits. Indeed, albeit the determinant and the integral representation are technically equivalent, their fundamentally different formulations may provide very nicely behaved objects, in totally different contexts. And so has been built a new ship, made of an other wood. A new ship, for new lands.

It would for instance be interesting to express the MEPNO through this integral representation and see if it has something more to tell us in this particular form. It could also be very interesting, albeit far more technically demanding, to extend this integral representation to the modified Izergin determinant introduced in the frame of the modified algebraic Bethe ansatz in Chapter 2.

General Conclusion

Walking in the dark, the weakest light is welcome.

So ends my journey in the wild lands of quantum Integrability, and now comes the time to make a step back to look at what we collected on the way.
In a first chapter has been reviewed the two well known machineries of the Coordinate Bethe Ansatz (CBA) and the Algebraic Bethe Ansatz (ABA), both developed on the purpose of solving the spectral problem of a wide range of quantum integrable models. This led us to the diagonalization of some interesting problem, but although allowed us to draw a formal link between systems of fundamentally different nature, physically disconnected. In the context of CBA we obtained solution of the spectral problem, the Bethe states, in the context of the XXX spin chain and the Zero-range Chipping Model with factorized steady state (ZCM). In both these contexts, integrability seemed to express in a very similar way, and the answers that the CBA brought are remarkably similar. The case of the ZCM war particularly interesting in that the approach we adopted actually leads to reaching the integrable regime of a class of systems, the essence of integrability manifesting in this case as the factorization of the M-particle dynamics to the 2-particle dynamics.

When one gets interest for integrable systems, one is inevitably keen on exploring the underlying structures of the many available models. In this regard, the ABA provides a particularly suitable alternative machinery in the quest of solutions for the spectral problem. The formalism developed in this context tends to make these structures, namely the algebraic background of our models, to emerge to the light of the day. We reviewed the main aspects of this ansatz, once more in the context of the XXX spin chain. This led us to run into a famous limit of the power of ABA the, which is the open spin chain. This failing relies on the absence of a suitable vacuum. To circumvent this problem has been developed a new approach, so called Modified Algebraic Bethe Ansatz (MABA), extending the usual ABA, which relies on the same mathematical lexicon. A new path, only slightly diverging from the main road. This modified ansatz brought its fistful of subtlety, and we eventually obtained a characterization of the spectrum through modified Bethe equations. We also studied multiple action of modified operators which, as one could have expected, can read as deformations of the usual ones, such as the modified Izergin determinant emerging from the scalar product of modified Bethe states. This new interesting approach however comes with critical questionings, and in particular the question of completeness of the modified Bethe states. The different attempt one could make for solving the completeness problem may hopefully be very favored here by the fact that solutions for the considered system, the twisted XXX spin chain, are already provided by the usual ABA. Walking in the dark, the weakest light is welcome.

A very important dissimilarity between the coordinate and the algebraic approaches of the Bethe ansatz concerns the forms in which is formulated their answers. These two are equivalent, and both are very suitable in different contexts. While the ABA proposes to generate the Bethe states by application of suitable operators, which is particularly favorable for the investigation of scalar products and form factors, the CBA provides us with an explicit expression for the Bethe states in the spin basis. This forme showed up to be particularly adapted in the frame of the Inverse Functional Problem (IFP), which we solved in the context of the ZCM and of the XXZ spin chain. We followed in both cases a simple idea that proved to be reliable in the context of IFP for the δ-Bose gas. As one could have expected, given that these problems obviously share important structural similarities, they developments naturally exhibited important similarities, to begin with the raise of bound states. More surprisingly, they also exhibited profound differences, which brings an interesting formal question around the meaning of the vanishing of longest strings in the case of IFP for the ZCM, while all string seem to survive in the XXZ case. If this if effectively the case, this inadequacy may account for a deeper structural difference between these two contexts.
Despite this interesting questioning, our development led us to satisfying results. In particular, the IFP for ZCM provided us with the first ingredient for approaching quench problems, which have only barely been touched in my thesis. The resolution of the identity in turn, obtained in a similar manner (evidencing on the way the completeness of Bethe states), couldn't find proper application in this thesis. It may however open the way for an extension of the result to the finite chain, which constitutes a very challenging problem. We also had a look on some of the most common determinant representations recurrently appearing in the context of the Bethe ansatz. We saw such representation appearing while considering the Matrix Elements of the Particle Number Operator (MEPNO) in the δ-Bose gas. The structure of Bethe states here played a very important role, alongside its stability under the action of the particle number operator. A formal link between such representations has been established through an interesting integral representation. It would be very interesting to look for a generalization of these integral representations to the case of the modified determinant representation obtained in the frame of the MABA.

While the two concepts of IFP and MEPNO seems to point toward the same direction, namely dynamical problems, these constitutes two very different aspect of the question. On one hand, the IFP gives direct access the dynamics of our physical states, through their decomposition on the Bethe basis. On the other hand, the MEPNO provides a probe for some physical quantity, namely here the number of particle. Supposing these two results to be available in the same context, a next natural step would now be to gather these two building blocks in order to obtain express the evolution of the current following a quench.

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ÉCOLE DOCTORALE
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Titre : Fonctions d'Onde et Produits Scalaires dans l'Ansatz de Bethe

Mots clés : Fonctions d'onde, ansatz de Bethe, produits scalaires

Résumé : Les modèles intégrables sont des modèles physiques pour lesquels certaines quantités peuvent être calculées de manière exacte, sans recours aux méthodes de perturbations. Ces modèles très particuliers suscitent un intérêt croissant en physique théorique. Les applications directes en physique de la matière condensée et les liens subtils plus récemment mis en évidence avec certaines théories de jauge supersymétriques ont motivé depuis des décennies l'élaboration d'outils mathématiques complexes. Parmi eux, l'ansatz de Bethe a joué un rôle central, et permis la diagonalisation de nombreux modèles de natures très différentes. Le premier chapitre de cette thèse est consacré à une introduction aux deux approches de l'ansatz de Bethe, dites "en coordonnée" et "algébrique", dans le cadre de la chaîne de spin de Heisenberg et d'un modèle stochastique généralisant à un spin continu le modèle du Totally Asymmetric Simple Exclusion Process.
Le deuxième chapitre de cette thèse présente l'ansatz algébrique modifié pour la chaîne XXX périodique. Cet ansatz modifié est proposé pour résoudre le cas de la chaîne ouverte, pour laquelle l'ansatz classique n'est plus efficace. Le produit scalaire des états de Bethe modifiés ainsi obtenus est étudié.
Le troisième chapitre concerne la résolution de l'identité, et le problème fonctionnel inverse. Une expression pour les états de spin en terme des états de Bethe est présentée pour le q-TASEP, et une expression de la résolution de l'identité en terme des états de Bethe pour la chaîne de spin XXZ infinie est démontrée, faisant intervenir dans les deux cas la contribution des états liés.
Enfin, le quatrième chapitre concerne les représentions en déterminant dans l'ansatz de Bethe. Une expression pour les éléments de matrice de l'opérateur Nombre de Particule pour le gaz de Bose avec interaction delta en terme d'un déterminent est démontrée, et des représentations intégrales pour les déterminants d'Izergin-Korepin et de Slavnov sont investiguées, établissant ainsi un nouveau lien formel direct entre ces deux représentations en déterminant.

Title : Wave Functions and Scalar Product in the Bethe Ansatz

Keywords : Bethe ansatz, Wave function, scalar product

Abstract : Integrable models are physical models for which some quantities can be exactly obtained, without use of perturbation theory. Those very special models are source of an increasing interest in theoretical physics. The direct applications in condensed matter physics and the subtle links evidenced more recently with some supersymmetric gauges theories motivated the development of complex mathematical tools. Among these, Behte ansatz played an important role, and provides an efficient approach for diagonalizing a lot of models of various nature.
The first chapter of this thesis is devoted to the introduction to the two approaches of the Bethe anstaz, said ≪ coordinate ≫ and ≪ algebraic ≫, in the context of the XXX Heisenberg spin chain and a continuous spin generalization of the Totally Asymmetric Simple Exclusion Process, the so called Zero-range Chipping model with factorized steady state (ZCM).
The second chapter is devoted to the Modified Algebraic Bethe Ansatz in the context of the periodic XXX chain. This modified ansatz is proposed for solving the spectral problem of the open spin chain, for which the usual ansatz fails. The scalar product of the obtained modified Bethe states is studied.
The third chapter concerns the resolution of the identity and the inverse functional problem. An expression for the spin states in terms of Bethe states est presented for the ZCM, and an expression for the resolution of the identity in term of Bethe states for the infinite XXZ chain is proved, involving in both cases the contribution of bound states.
At last, the fourth chapter concerns determinant representations in the Bethe ansatz. An expression for the ≪ matrix elements of the particle number operator ≫ for the delta-Bose gas in terms of a determinant is proved, and some integral representations for the Izergin-Korepin and Slavnov determinants are investigated, then establishing a new formal link between these two determinant representations.

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