A Toy-Model Resolving the Black Hole Information Paradox

by

Savin Shayok Dipro
19111001

A thesis submitted to the Department of Mathematics and Natural Sciences
in partial fulfillment of the requirements for the degree of
B.Sc. in Physics

Department of Mathematics and Natural Sciences
Brac University
December 2024

Declaration

I hereby declare that my thesis, ”A Toy-Model Resolving the Black Hole Information Paradox,” which I submitted to the Department of Mathematics and Natural Sciences to partially fulfill the requirements for a Bachelor of Science in Physics degree, is entirely original with properly cited references and has not been submitted anywhere else for publication or degree requirements.

Candidate,

Savin Shayok Dipro
ID: 19111001

Approval

The thesis titled “A Toy-Model Resolving the Black Hole Information Paradox” submitted by

1. Savin Shayok Dipro (19111001)

Of Fall, 2024 has been accepted as satisfactory in partial fulfillment of the requirement for the degree of B.Sc. in Physics on December 30, 2024.

Examining Committee:

Supervisor:
(Member)

____

Dr. Tibra Ali

Professor,
Department of Mathematics and Natural Sciences
Associate Dean, Research,
School of Data & Sciences
Brac University

Co-Supervisor:
(Member)

____

Ahmed Rakin Kamal

Lecturer(On Leave),
Department of Mathematics and Natural Sciences
Brac University

Program Coordinator:
(Member)

____

Muhammad Lutfor Rahman

Assistant Professor,
Department of Mathematics and Natural Sciences
Brac University

Head of Department:
(Chair)

____

Md. Firoze H. Haque

Associate Professor,
Department of Mathematics and Natural Sciences
Brac University

Abstract

This dissertation focuses on how considering a toy model that allows non-local gravitational interaction among the Hawking radiated pairs along with the chosen bell pair state gets closer to resolving the black hole information paradox. We begin by presenting some of the required background on Hawking radiation and the small correction to the leading order Hawking state before focusing on the non-local gravitational interaction prescription of the toy model for the entanglement entropy. We show how introducing two new parameters, which depend on mass depletion rate of a Black Hole and interactions between Hawking radiated pairs leads to the decrease of the leading order Hawking state entanglement entropy resulting in the recovery of information.

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Chapter 1

Introduction

When Einstein first established the general theory of relativity 10 years after the publication of his paper on special relativity, it was not apparent that the general theory of relativity had much more implications in physics other than gravity. His field equations had much deeper implications than it ought to be. One of which led to the discoveries of gravitational waves and black holes. Although gravitational waves were predicted by scientists almost a century ago, the first direct observation of gravitational waves was made on 14 September 2015 and was announced by the LIGO and Virgo collaborations on 11 February 2016 [43], [47]. On the other hand, black holes are relatively not that old theory. The discovery team, the North American Nanohertz Observatory for Gravitational Waves, or NANOGrav, collaboration, strongly suspects the ripples in space-time were created from merging supermassive black holes, each a billion times more massive than our sun [56]. Despite these observations, black holes are still ongoing research in physics. In 1974 Hawking showed using quantum field theory in a curved spacetime that black holes emit radiation as a perfect black body [8]. Despite being a well predicted theory Hawking radiation is yet to be observed. Furthermore, this groundbreaking result is perhaps the first step towards a quantum black hole. Hawking's result further imply that radiating black holes slowly evaporate. This shows a concern about the information that has been thrown into the black hole. If a book is thrown into the black hole where does the information contained in the book go after the black hole evaporates? Naively, this is the black hole information paradox.

To resolve the information paradox there have been many given competing ideas. Although a majority of theorists agree that the information can be recovered, a minority of them agree that the information is indeed lost . Some theories, such as the fuzzball solution [28] suggests that the geometry of the black hole is non-trivial and that we can recover the lost information through a complicated geometry of spacetime that does not allow the information to be lost. However, most theorists believe that to understand where the information resides, we need a full description of quantum gravity. String theory is by far the most explored theory of quantum gravity. In string theory the horizon of the black hole is of great importance. One aspect of string theory suggests the holographic principle as a resolution to the black hole information paradox [23]. The AdS/CFT correspondence, which gives a better understanding of the black hole entropy is an example of the holographic principle. This approach shows how the unitarity of black hole is saved. [25].

In this dissertation, we start with an introduction to the concept of black holes by discussing the Schwarzschild metric and introducing other coordinate systems. We then discuss the mechanics of black holes and how they are related to thermodynamics. Next, we give a brief qualitative discussion on the Hawking radiation and the information paradox. Moreover, we will see how small correction to the leading order Hawking state do not decrease the entanglement entropy of the radiated pairs, formulated by Mathur [30]. Lastly, we present a toy model that considers non-local gravitational interaction and bypasses the entropy bounds.

Chapter 2

Background

2.1 The Schwarzschild Solution

Assuming that the spherical mass's electric charge, angular momentum, and universal cosmological constant are all zero, the Schwarzschild Solution[1] is an exact solution to the Einstein field equations that characterize the gravitational field outside a spherical mass. To Einsteins surprise, Karl Schwarzschild was the first one to find the exact solutions to Einsteins field equations in 1916 just after a year of Einsteins paper on general relativity.
Some of the main features of the Schwarzschild metric are,

1. It is a static metric. A static metric has two requirements; all the metric components are independent of time co-ordinate and the spacetime geometry is unchanged by parity meaning that the sign reversal of the time co-ordinate do not change the geometry of the metric in space time.

2. It is an isotropic metric meaning that the geometry of the metric remains the same in every direction from a particular point.

3. It represents the spacetime geometry outside a spherically symmetric mass.

4. Birkhoff's theorem[57] establishes that the Schwarzschild metric is the only spherically symmetric, asymptotically flat solution to Einstein's vacuum field equations.

With the above features the Schwarzschild metric is given as,

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.1}$$

which is also written as,

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.2}$$

where, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
The Schwarzschild geometry is the geometry of vacuum space-time outside a spherical mass such as a star. There are some important things to notice about the Schwarzschild metric. Firstly, the Schwarzschild metric is continuous in the region $\infty < r < \frac{2GM}{c^2}$. Furthermore, If we notice carefully we will see that the metric has two singularities; one at $r = \frac{2GM}{c^2}$ mentioned as the co-ordinate singularity and another at $r = 0$ which is the spacetime or curvature singularity. We should bear in mind that the curvature singularity can not be changed from $r = 0$. However, throughout different courses of scientific history people have adopted different methods specifically different choices of co-ordinates to get passed the co-ordinate singularity. We will be talking about two of them namely, Eddington–Finkelstein coordinates[45] and Kruskal–Szekeres coordinates[55] in the next sections.

2.2 The Eddington–Finkelstein coordinates

We start by finding the geodesic equation[37] for particles moving in a Schwarzschild metric. Geodesics are the path taken by free falling particles/objects as they hover through spacetime. The actual path taken by an object in spacetime is its worldline which can be expressed as a four vector where each component is parameterized in terms of some affine parameter λ as $x^\mu = x^\mu(\lambda)$.In case of a particle with mass this affine parameter is the proper time τ. For massless particle the affine parameter is λ as $d\tau = 0$ ($ds^2 = -c^2 d\tau^2$).Furthermore, geodesics parallel transport their own tangent vectors. Particles with mass follow timelike world lines (i.e $U^\mu U_\mu = -c^2$) and massless particles follow a null geodesic (i.e $U^\mu U_\mu = 0$) where U^μ are the components of the four velocity of the particle. The metric $g_{\alpha\beta}$ of Schwarzschild metric is given as follows,

$$g_{\mu\nu} = \begin{pmatrix} -c^2\left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \tag{2.3}$$

Let us consider the simplest geodesic; the radial null geodesics. ”Radial” means that θ and ϕ are constant along the geodesic. Hence, For photons following a radially null geodesic we have $d\theta$=0 and $d\phi$=0. Furthermore, Form the null geodesic condition for photons $g_{\mu\nu}U^{\mu}U^{\nu}=0$ we get,

$$
\begin{aligned}
&g_{\mu\nu}U^{\mu}U^{\nu}=0\\
&g_{tt}\frac{dt}{d\lambda}\frac{dt}{d\lambda}+g_{rr}\frac{dr}{d\lambda}\frac{dr}{d\lambda}+g_{\theta\theta}\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda}+g_{\phi\phi}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}=0\\
&g_{tt}\left(\frac{dt}{d\lambda}\right)^2+g_{rr}\left(\frac{dr}{d\lambda}\right)^2=0\\
&-c^2\left(1-\frac{2GM}{c^2r}\right)dt^2+\left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2=0\\
&c^2\left(1-\frac{2GM}{c^2r}\right)dt^2=\left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2\\
&c^2\left(1-\frac{2GM}{c^2r}\right)^2dt^2=dr^2\\
&\left(\frac{dr}{dt}\right)^2=c^2\left(1-\frac{2GM}{c^2r}\right)^2\\
&\frac{dr}{dt}=\pm c\left(1-\frac{2GM}{c^2r}\right)\\
&\int\frac{dr}{\left(1-\frac{2GM}{c^2r}\right)}=\pm\int cdt
\end{aligned}
$$

The first solution of the integral for an outgoing photon is given as,

$$ct=r+\frac{2GM}{c^2}\ln(c^2r-2GM)+Constant \tag{2.4}$$

On the other hand, the second solution for an ingoing photons we replace $t\rightarrow -t$ and get,

$$ct=-r-\frac{2GM}{c^2}\ln(c^2r-2GM)+Constant \tag{2.5}$$

Now the idea is to use these null incoming and outgoing geodesics of photons as our new coordinate. First we use the ingoing null geodesic. Let us use the integrating constant as our new coordinate 'u' which is known as the advanced time parameter[45],

$$u=r+\frac{2GM}{c^2}\ln(c^2r-2GM)+ct \tag{2.6}$$

Note that u is a null co-ordinate as, $d(constant) = du = 0 \implies ds^2 = 0$.
Taking the differential of equation (2.6) we get,

$$
\begin{aligned}
du =& dr + \frac{2GM}{c^2}\frac{c^2}{c^2r-2GM}dr + cdt \\
du =& dr + \frac{2GM}{c^2r-2GM}dr + cdt \\
du =& \frac{c^2r}{c^2r-2GM}dr + cdt \\
du =& \frac{1}{1-\frac{2GM}{c^2r}}dr + cdt \\
du =& \left(1-\frac{2GM}{c^2r}\right)^{-1} dr + cdt \\
cdt =& du - \left(1-\frac{2GM}{c^2r}\right)^{-1} dr \\
\implies c^2dt^2 =& du^2 - 2\left(1-\frac{2GM}{c^2r}\right)^{-1} drdu + \left(1-\frac{2GM}{c^2r}\right)^{-2} dr^2
\end{aligned} \tag{2.7}
$$

Substituting equation (2.7) in the Schwarzschild metric equation (2.2) we get,

$$
\begin{aligned}
ds^2 &= -c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2d\Omega^2 \\
ds^2 &= -\left(1-\frac{2GM}{c^2r}\right)\left[du^2 - 2\left(1-\frac{2GM}{c^2r}\right)^{-1} drdu + \left(1-\frac{2GM}{c^2r}\right)^{-2} dr^2\right] + \left(1-\frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2d\Omega^2 \\
\implies ds^2 &= -\left(1-\frac{2GM}{c^2r}\right)du^2 + 2dudr + r^2d\Omega^2
\end{aligned} \tag{2.8}
$$

Notice that the new co-ordinates (u, r, θ, ϕ) has no co-ordinate singularity in the line element. Thus, the line element is now defined in the region, $0 < r < \infty$. In addition, since we have considered radially null geodesic of a photon, surfaces of constant u in this new coordinate are drawn as light cones. Now, we look at the radial motion photons in the equatorial plane of this new coordinate. For which $ds = d\theta = d\phi = 0$. Then the above equation turns out as follows,

$$
\begin{aligned}
0 &= -\left(1-\frac{2GM}{c^2r}\right)du^2 + 2drdu \\
0 &= \left(1-\frac{2GM}{c^2r}\right)du^2 - 2drdu \\
0 &= \left(1-\frac{2GM}{c^2r}\right)\left(\frac{du}{dr}\right)^2 - 2\frac{du}{dr} \\
&\left[\left(1-\frac{2GM}{c^2r}\right)\frac{du}{dr} - 2\right]\frac{du}{dr} = 0
\end{aligned}
$$

The two solutions correspond to ingoing and outgoing photons in the coordinate(u, r, θ, ϕ).The two solutions are,

$$\frac{du}{dr} = 0 \implies u = constant \tag{2.9}$$

and

$$\left(1-\frac{2GM}{c^2r}\right)\frac{du}{dr} - 2 = 0 \implies u = 2r + \frac{4GM}{c^2}\ln|c^2r - 2GM| + constant \tag{2.10}$$

Notice that u $\to \infty$ as r $\to \frac{2GM}{c^2}$ as you would expect from the Schwarzschild solution. Since, u is not single valued it is convenient to choose a more familiar coordinate in which we can draw the spacetime coordinate easily. Let us define a timelike coordinate t' as $ct' = u - r$.Then from equation (2.6) we get,

$$ct' = r + \frac{2GM}{c^2}\ln(c^2r - 2GM) + ct - r \tag{2.11}$$

$$ct = ct' - \frac{2GM}{c^2}\ln(c^2r - 2GM) \tag{2.12}$$

Furthermore, differentiating equation (2.12) we get,

$$cdt = cdt' - \frac{2GM}{c^2}\frac{c^2}{c^2r - 2GM}dr \tag{2.13}$$

$$cdt = cdt' - \frac{2GM}{c^2r}\left(1 - \frac{2GM}{c^2r}\right)^{-1} dr \tag{2.14}$$

$$c^2dt^2 = c^2dt'^2 - \frac{4MG}{cr}\left(1 - \frac{2GM}{c^2r}\right)^{-1} dt'dr + \frac{4G^2M^2}{c^4r^2}\left(1 - \frac{2GM}{c^2r}\right)^{-2} dr^2 \tag{2.15}$$

Substituting equation (2.15) in the Schwarzschild metric equation (2.2) we get,

$$ds^2 = -\left(1 - \tfrac{2GM}{c^2r}\right)\left[c^2dt'^2 - \tfrac{4MG}{cr}\left(1 - \tfrac{2GM}{c^2r}\right)^{-1} dt'dr + \tfrac{4G^2M^2}{c^4r^2}\left(1 - \tfrac{2GM}{c^2r}\right)^{-2} dr^2\right] + \left(1 - \tfrac{2GM}{c^2r}\right)^{-1} dr^2 + r^2d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt'^2 + \frac{4MG}{cr}dt'dr - \frac{4G^2M^2}{c^4r^2}\left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt'^2 + \frac{4MG}{cr}dt'dr + \left[1 - \left(\frac{2GM}{c^2r}\right)^2\right]\left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt'^2 + \frac{4MG}{cr}dt'dr + \left(1 - \frac{2GM}{c^2r}\right)\left(1 + \frac{2GM}{c^2r}\right)\left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt'^2 + \frac{4MG}{cr}dt'dr + \left(1 + \frac{2GM}{c^2r}\right)dr^2 + r^2d\Omega^2 \tag{2.16}$$

The coordinate (t', r, θ, ϕ) is known as the advanced Eddington-Finkelstein coordinates. Notice that the line element in advanced Eddington-Finkelstein coordinates has no coordinate singularity and it is defined in the region $0 < r < \infty$.However, this line element is not invariant under parity. Furthermore, from the solutions of u found previously we see that,
Given $ct' = u - r$,

$$ct' = constant - r$$

Which is the ingoing radial null geodesic of photon. On the other hand,

$$ct' = -r + 2r + \frac{4GM}{c^2}\ln|c^2r - 2GM| + constant$$

$$ct' = r + \frac{4GM}{c^2}\ln|c^2r - 2GM| + constant$$

Which is the outgoing radial null geodesic of photon.
Now, we look at the solution of outgoing photons in the Eddington-Finkelstein coordinate with the new coordinate (v, r, θ, ϕ),

$$v = -r - \frac{2GM}{c^2}\ln(c^2r - 2GM) + ct \tag{2.17}$$

The corresponding metric in this coordinates is then,

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 - 2dvdr + r^2 d\Omega^2 \tag{2.18}$$

Notice that just like coordinate (u, r, θ, ϕ) the new coordinate (v, r, θ, ϕ) has removed the coordinate singularity and it is defined in the region $0 < r < \infty$. Now again, we look at the radial motion photons in the equatorial plane of this new coordinate. For which $ds = d\theta = d\phi = 0$. Then the above equation turns out as follows,

$$\left[\left(1 - \frac{2GM}{c^2 r}\right)\frac{dv}{dr} + 2\right]\frac{dv}{dr} = 0 \tag{2.19}$$

The two solutions correspond to ingoing and outgoing photons in the coordinate(v, r, θ, ϕ).The two solutions are,

$$\frac{dv}{dr} = 0 \implies v = constant \tag{2.20}$$

and

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dv}{dr} + 2 = 0 \implies v = -2r - \frac{4GM}{c^2}\ln|c^2 r - 2GM| + constant \tag{2.21}$$

Now, we consider another coordinate (t^*, r, θ, ϕ) in which $ct^* = v + r$.Then from equation (2.17) we get,

$$\begin{aligned} ct^* &= -r - \frac{2GM}{c^2}\ln(c^2 r - 2GM) + ct + r \\ ct &= ct^* + \frac{2GM}{c^2}\ln(c^2 r - 2GM) \end{aligned} \tag{2.22}$$

differentiating equation (2.22) we arrive at,

$$c^2 dt^2 = c^2 dt^{*2} + \frac{4MG}{cr}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} dt^* dr + \frac{4G^2M^2}{c^4 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-2} dr^2 \tag{2.23}$$

Plugging equation (2.23) in the Schwarzschild metric equation (2.2) we get,

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^{*2} - \frac{4MG}{cr} dt^* dr + \left(1 + \frac{2GM}{c^2 r}\right) dr^2 + r^2 d\Omega^2 \tag{2.24}$$

The coordinate (t^*, r, θ, ϕ) is known as the retarded Eddington-Finkelstein coordinates. We can see that the line element in retarded Eddington-Finkelstein coordinates has no coordinate singularity and it is defined in the region $0 < r < \infty$.However, this line element is also not invariant under parity. Furthermore, from the solutions of v found previously we see that,
Given $ct^* = v + r$,

$$ct^* = constant + r \tag{2.25}$$

Which is the outgoing radial null geodesic of photon. On the other hand,

$$\begin{aligned} ct^* &= -2r - \frac{4GM}{c^2}\ln|c^2 r - 2GM| + r + constant \\ ct^* &= -r - \frac{4GM}{c^2}\ln|c^2 r - 2GM| + constant \end{aligned} \tag{2.26}$$

Which is the ingoing radial null geodesic of photon. There are some subtle differences between the advanced and retarded Eddington-Finkelstein coordinates. While both of them are smooth and continues down to the singularity at $r = 0$, the key difference is in the cross term of the metric. We shall explore some of the differences next.

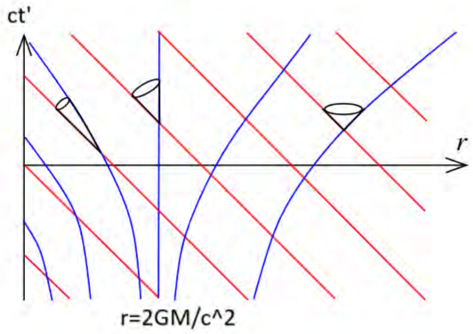

Figure 2.1: The advanced Eddington-Finkelstein diagram.

From the figure 2.1 of the advanced Eddington-Finkelstein coordinate we can see that the ingoing light rays which are shown in red travel at 45 degrees in the (ct', r) plane. Meanwhile, the outgoing light rays shown in blue depends on their position with respect to the horizon. The outgoing geodesics that lie inside the horizon move towards the singularity at $r = 0$ while the outgoing geodesics that lie outside the horizon move outwards. These two regions are bounded by the null geodesics which simply run along the horizon $r = \frac{2GM}{c^2}$.

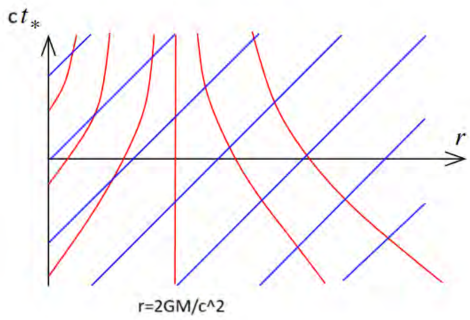

Figure 2.2: Retarded Eddington-Finkelstein diagram

From the figure 2.2 of the retarded Eddington-Finkelstein coordinate we can see that the outgoing light rays which are shown in blue travel at 45 degrees in the (ct', r) plane. For the outgoing light rays in this case, always go out, despite where they start with respect to the horizon. Meanwhile, the ingoing light rays shown in red depends on their position with respect to the horizon. Therefore, we see that the region $r < \frac{2GM}{c^2}$ here describes a different part of spacetime than the region $r < \frac{2GM}{c^2}$ in advanced Eddington-Finkelstein coordinates! The ingoing light rays that are outside the horizon do not reach the singularity. Instead, they move towards the horizon. In the same manner, the ingoing light rays that lie behind the horizon also move towards the horizon, again not being able to cross it. This is clearly not a physics analogous to a black hole. In fact, this solution resembles to a white hole, an object that expels all matter from inside. A white hole therefore is the time reversal of a black hole. If we turn the Figure 2.2 of white hole upside down, we get the Figure 2.1 of black hole. Einstein equations admit the solutions of white holes. Since a solution of Einstein equations lead to black holes from which nothing can escape there had to be a time reversal invariance of the Einsteins equations that permits the solution which nothing can enter. Nevertheless, white holes are not physically relevant as unlike black holes, one cannot form them from gravitational collapse.

2.3 The Kruskal-Szekeres Coordinates

We saw that in the advanced and retarded Eddington-Finkelstein coordinates we were able to remove the coordinate singularity. However, we found discontinuity for the outgoing radial photon null geodesics and the ingoing radial photon null geodesics in the advanced Eddington-Finkelstein and retarded Eddington-Finkelstein coordinates respectively. Now the idea is to find a system where both ingoing and outgoing geodesics are continuous. In other words, we introduce coordinates which cover the entire spacetime , including both black and white holes.

We start with the advanced null coordinate 'u' found in equation (2.6) and retarded null coordinate 'v' found in equation (2.17) from the previous section[1],

$$u = ct + r + \frac{2GM}{c^2}\ln\left|\frac{c^2 r}{2GM} - 1\right| \tag{2.27}$$

and

$$v = ct - r - \frac{2GM}{c^2}\ln\left|\frac{c^2 r}{2GM} - 1\right| \tag{2.28}$$

Upon taking the differential of equation (2.27) and (2.28) we arrive at,

$$du = cdt + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr \tag{2.29}$$

and

$$dv = cdt - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr \tag{2.30}$$

[2]let us now look at the product dudv,

$$dudv = c^2dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-2} dr^2$$

$$\implies dudv = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left[-\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2d\Omega^2\right] + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} r^2 d\Omega^2$$

$$\implies dudv = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} ds^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} r^2 d\Omega^2$$

$$\implies ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dudv + r^2 d\Omega^2 \tag{2.31}$$

In this coordinate the metric is degenerate at $r = \frac{2GM}{c^2}$. To get around this degeneracy, Kruskal introduced the following coordinates,

$$U = exp\left(\frac{c^2 u}{4GM}\right) \tag{2.32}$$

and

$$V = -exp\left(-\frac{c^2 v}{4GM}\right) \tag{2.33}$$

Notice that these are null coordinates. We now figure out the line element as before. Taking the differential of equations (2.32) and (2.33) we get,

$$dU = \frac{c^2}{4GM} exp\left(\frac{c^2 u}{4GM}\right) du$$

$$\implies du = \frac{4GM}{c^2} exp\left(-\frac{c^2 u}{4GM}\right) dU \tag{2.34}$$

[1]Here we have consumed the constant term $\frac{2GM}{c^2}\ln[2GM]$ inside both u and v.

[2]Here we have considered the line element of the Schwarzschild metric.

and

$$
\begin{aligned}
dV =& \frac{c^2}{4GM} exp\left(-\frac{c^2 v}{4GM}\right) dv \\
\implies dv =& \frac{4GM}{c^2} exp\left(\frac{c^2 v}{4GM}\right) dV
\end{aligned}
\tag{2.35}
$$

Furthermore, if we take the product of $dudv$ from equations (2.34) and (2.35) we arrive at,

$$
dudv = \frac{16G^2M^2}{c^4} exp\left(-\frac{c^2(u-v)}{4GM}\right) dUdV \tag{2.36}
$$

Plugging equation (2.36) into the metric equation (2.31) we get the following metric,

$$
ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) \frac{16G^2M^2}{c^4} exp\left(-\frac{c^2(u-v)}{4GM}\right) dUdV + r^2 d\Omega^2 \tag{2.37}
$$

We now introduce the tortoise coordinate[27] ,

$$
\tilde{r} = \frac{1}{2}(u-v) = r + \frac{2GM}{c^2} ln\left|\frac{c^2 r}{2GM} - 1\right| \tag{2.38}
$$

Performing a little algebra on the tortoise coordinate we can show that,

$$
\left(1 - \frac{2GM}{c^2 r}\right) = \frac{2GM}{c^2 r} exp\left(\frac{c^2(u-v)}{4GM}\right) exp\left(-\frac{c^2 r}{2GM}\right) \tag{2.39}
$$

Thus, the line element of equation (2.37) becomes,

$$
\begin{aligned}
ds^2 =& -\frac{2GM}{c^2 r} exp\left(\frac{c^2(u-v)}{4GM}\right) exp\left(-\frac{c^2 r}{2GM}\right) \frac{16G^2M^2}{c^4} exp\left(-\frac{c^2(u-v)}{4GM}\right) dUdV + r^2 d\Omega^2 \\
ds^2 =& -\frac{32}{r}\left(\frac{GM}{c^2}\right)^3 exp\left(-\frac{c^2 r}{2GM}\right) dUdV + r^2 d\Omega^2
\end{aligned}
\tag{2.40}
$$

Furthermore, using the tortoise coordinate and equations (2.32) and (2.33) we get outside the horizon,

$$
\begin{aligned}
UV =& -exp\left(\frac{c^2 u}{4GM}\right) exp\left(-\frac{c^2 v}{4GM}\right) \\
UV =& -exp\left(\frac{c^2(u-v)}{4GM}\right) \\
UV =& -exp\left(\frac{c^2 r}{2GM} + ln\left|\frac{c^2 r}{2GM} - 1\right|\right) \\
UV =& \left(1 - \frac{c^2 r}{2GM}\right) exp\left(\frac{c^2 r}{2GM}\right)
\end{aligned}
\tag{2.41}
$$

Similarly we can show that,

$$
\frac{U}{V} = -exp\left(-\frac{c^2 t}{2GM}\right) \tag{2.42}
$$

The badness that we encountered in the previous line element of equation (2.31) due to degeneracy at $r = \frac{2GM}{c^2}$ has been removed in the preceding line element. The new coordinate (u, v, θ, ϕ) is known as the Kruskal-Szekeres coordinates. We can see that there is no coordinate singularity at $r = \frac{2GM}{c^2}$ in this coordinate. Furthermore, the two coordinates are now combined. In fact, the Kruskal spacetime is the maximal extension of the Schwarzschild solution[50]. To check whether a given spacetime can be further extended we look at where all geodesics end up. If the geodesics can be followed for infinite affine parameter, then they escape to infinity. On the other hand, if we come to an end following the geodesics at some finite affine parameter then either they run into a coordinate singularity, or they run into a curvature singularity. In the later case we can not do anything about it. In the former case however, we can extend the spacetime as we have above. Thus, by definition, a manifold is considered to be maximally extended if every geodesic from any point on the manifold can be extended in either direction to infinity unless it terminates on an intrinsic singularity[32]. In simple words, We have the maximally extended spacetime when any geodesics that come to an abrupt end is due to curvature or spacetime singularities. Moreover, at $r = \frac{2GM}{c^2}$ we see $U = 0$ or $V = 0$ from equation (2.41), Which tells us that the horizon is actually two null surfaces instead of one null surface, intersecting at the point $U = V = 0$.The null surface $U = 0$ represents the horizon of the black hole which is called the future horizon. On the other hand, the null surface $V = 0$ represents the horizon of the white hole which is the past horizon. Now, if we look at the singularity at $r = 0$, we have $UV = 1$ which resembles a hyperbola. Furthermore, for $U,V > 0$ we get the singularity of the black hole. On the other hand, for $U,V < 0$ we get the singularity of the white hole. We can further asses these situation in the following diagram,

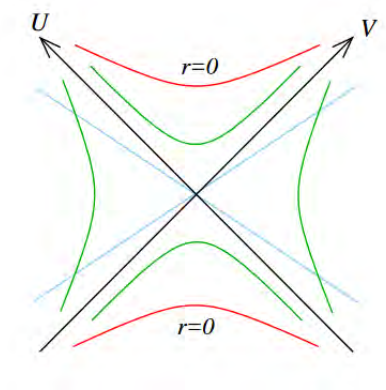

Figure 2.3: Kruskal Diagram

Since U and V are null lines their axes are drawn at 45 degrees. These are the two horizons. In the given diagram, the time $T = \frac{1}{2}(V + U)$ is shown as the vertical direction while $X = \frac{1}{2}(V - U)$ represents the horizontal spatial direction. The red curves resembles the singularities for which $UV = 1$. The diagram further illustrates constant r drawn as green lines and constant t drawn as blue lines. From (2.41), we see that $UV = constant$ gives us lines of constant r. Meanwhile, from (2.42), we see that $\frac{U}{V} = constant$ gives us lines of constant t that are linear. Let us now turn to the quadrants of the Kruskal diagram. The right-hand quadrant is the exterior of the black hole. The exterior of the black hole in this case is the spacetime covered by the original Schwarzschild coordinates. The upper quadrant is the interior of the black hole and the lower quadrant is the interior of the white hole. We are now left with the left-hand quadrant which is a surprise as it is just another copy of the exterior of the black hole, which suggests that there exists a spacelike wormhole between the two regions. this wormhole is known as the Einstein-Rosen bridge[44].

2.4 The Penrose Diagram

Further into the discussion of black hole we come across the Penrose diagrams. The Penrose diagrams incorporate the causal structure of spacetime in a similar fashion as the Kruskal diagram. Nevertheless, Penrose diagrams are wonderful in the sense that you can capture the infinite spacetime within a finite paper sheet with preserving the trajectory of photons at 45 degrees. The procedure to obtain the Penrose diagram for the Schwarzschild metric is essentially a conformal transformation. A conformal transformation is a co-ordinate transformation($x^\mu \rightarrow x'^\mu(x^\mu)$) which transforms the metric as , $g_{\mu\nu}$'$(x) \rightarrow \Omega^2(x)g_{\mu\nu}$[52]. Before embarking on the discussion of the diagrams, we should indulge in some of the feature of the transformation required to proceed to the diagrams,

1. The spacetime in Penrose diagrams is asymptotically flat, meaning that it appears to be like Minkowski spacetime from a far.

2. The angular coordinates are suppressed as the paper is only 2D. However, since conformal transformation preserves the angles, the trajectory of light rays remain the same; at 45 degrees.

3. We include points at infinity by setting the conformal factor $\Omega = \frac{1}{r}$. Therefore, $r = 0$ corresponds to the point at infinity imposing a conformal boundary to the diagram of the space-time itself.

4. The conformal infinity is a disjoint union of timelike, lightlike and spacelike infinity. We will discuss this next.

There are 5 ways to reach infinity. These are defined as follows,

- Future timelike infinity(i^+): Where you can reach by fixing r and taking t to ∞.

- Past timelike infinity(i^-):where you can reach by fixing r and taking t to $-\infty$.

- spacelike infinity(i^0):where you can reach by fixing t and taking r to ∞.

- future null infinity($\mathscr{L}^+$):where photons can reach by fixing u and taking v to ∞.

- past null infinity($\mathscr{L}^-$):where photons can reach by fixing v and taking u to $-\infty$.

The Penrose diagram for Minkowski spacetime is,

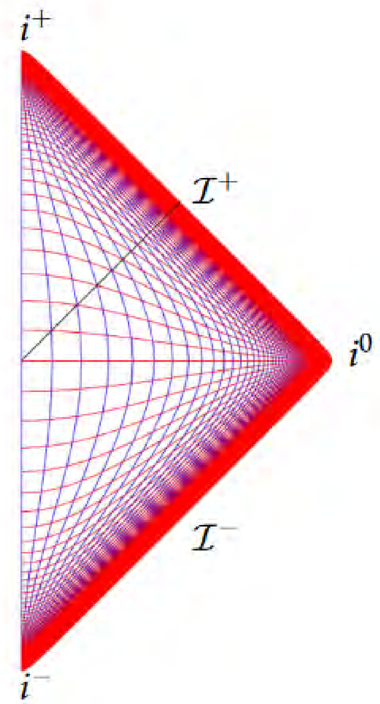

Figure 2.4: Penrose diagram of Minkowski spacetime.[53]

The Penrose diagram for Kruskal spacetime is,

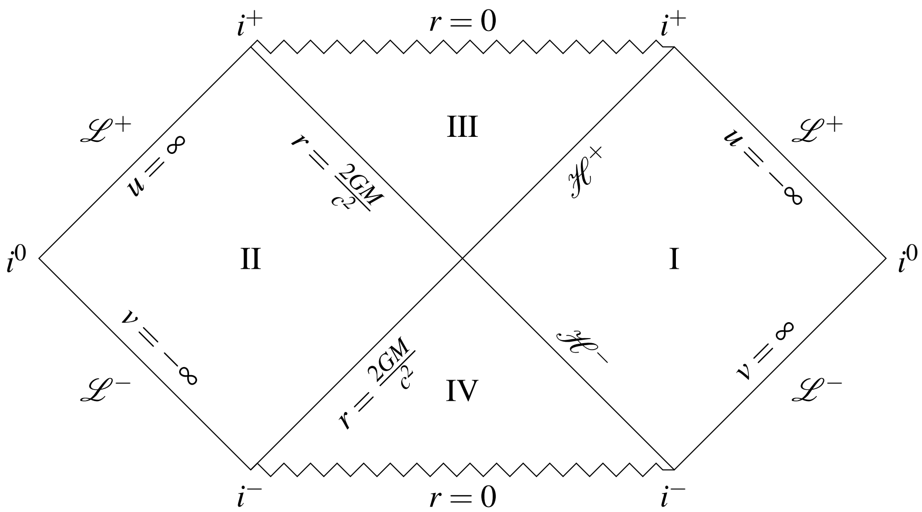

Figure 2.5: Penrose diagram for Kruskal geometry

The Penrose diagram allows us to give a more rigorous definition of a black hole. Although, we will be a bit neglectful here about the rigor part but simply give a flavor of the definition. Let us now turn to the diagram. The Penrose diagram representing Kruskal Geometry incorporates both the Black hole and White hole structures as we know Kruskal geometry is the maximally extended Schwarzschild solution[53]. If we look at the diagram, we see that the diagonal lines represent Black hole and White hole horizon ($r = \frac{2GM}{c^2}$) whereas the jagged lines represents the singularity of Black hole and White hole($r = 0$). The spacetime is divided by the horizon into four parts. The right quadrant 'I' represents the asymptotically flat exterior region of the Black Hole, the upper quadrant 'III' represents the Black Hole interior, the left quadrant 'II' represents another copy of the asymptotically flat region of quadrant 'I'. However, notice that it is causally disconnected from the region on the right. The bottom quadrant 'IV' represents a White hole. From the diagram we can perceive that the black hole region is defined to be the set of points that can not send a signal to $\mathscr{L}^+$. Which is not the case for Minkowski space, where every photon ends at $\mathscr{L}^+$. If there were no singularities at $r = 0$, U and V would have a diamond-shaped Pen-rose diagram, shown in figure 2.4 which is a 2D Minkowski space. Furthermore, In figure 2.5, The future event horizon, $\mathscr{H}^+$ is the boundary of the Black Hole region. Equivalently, The boundary of the causal past of $\mathscr{L}^+$ is the future event horizon, $\mathscr{H}^+$. To summarize, if we want to define a black hole we need to know the entirety of the spacetime. To illustrate, if we run light rays backwards from $\mathscr{L}^+$ then the event horizon is defined by the boundaries of these light rays. Furthermore, if we take a space like slice Λ at some moment in time then we won't be able to define a black hole region that refers only to that spacelike slice[53]. This means that unless an observer knows the entire future evolution of the spacetime, they can't really know if they're inside a black hole. Equivalently, we can also define the white hole region to be that part of spacetime that cannot receive signals from $\mathscr{L}^-$.

2.5 The No Hair Theorem

The picture of black holes does not stop here. It extends to stationary,non-stationary,charged black holes. A stationary black hole is one in which the metric is not invariant under the parity, meaning that the black hole has some angular momentum that keeps it rotating. A suitable metric for such black holes is the Kerr metric. Furthermore, black holes may carry electric charge and in principle a magnetic mono-pole moment. A charged, rotating black hole in a vacuum can be described by the Kerr-Newman metric. The Kerr-Newman metric[3] in Boyer-Lindquist coordinates is,

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2 sin^2\theta}{\Sigma} dt^2 - 2a sin^2\theta \frac{r^2 + a^2 - \Delta}{\Sigma} dt d\phi \\ &+ \frac{(r^2+a^2)^2 - \Delta}{\Sigma} sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned} \tag{2.43}$$

where,

$$\Sigma = r^2 + a^2 cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2 + e^2$$

If we inspect the above metric we can see that the three parameters a, M and e describe the individual black hole. Here, a is the total angular momentum J per unit mass, $a = \frac{J}{M}$ and e is expressed in terms of electric and magnetic charges Q and P as, $e = \sqrt{Q^2 + P^2}$.

Many uniqueness theorems have emerged due to the above established metric. In 1967, a conjecture by Israel states that for asymptotically flat, static, vacuum black holes spacetime that are "suitably regular" on and outside the event horizon, the spacetime must be isometric to the Schwarzschild solution [4]. This idea was further extended by theorems by Carter in 1971 [6], Robinson in 1975 [10], Bunting and Masood in 1987 [14], and Sudarsky and Wald in 1992 [33].

The no-hair theorem was conjectured by Carter and Robinson, stating that the three parameters that occur in the Kerr-Newman metric—mass M, total angular momentum J, and electromagnetic charge e—can be used to characterize all stationary electro-vacuum black hole solutions. The Kerr-Newman metric is the most general stationary, asymptotically flat isolated black hole solution to Einstein's field equations, as demonstrated by the combination of the different uniqueness theorems.

2.6 The Laws of Black Hole Mechanics

In 1973, J.M.Bardeen, B.Carter and S.W.Hawking published their ground breaking paper titled 'The Four Laws of Black Hole Mechanics' [7]. In their paper, they formulated that black holes are analogous to thermodynamic black bodies. In the sense that they have a defined temperature and entropy just like a black body.

Zeroth-Law

The zeroth law of black hole mechanics states that the surface gravity, κ of a stationary black hole is constant on the future event horizon $\mathscr{H}^+$. This is analogous to the zeroth law of thermodynamics which states that thermodynamic systems in equilibrium have constant temperature.

First-Law

The first law of black hole mechanics is an expression of the conservation of energy in black holes. The first law gives us the following expression,

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ \tag{2.44}$$

Here, $M, A, J,$ and Q represent the mass, the area, the total angular momentum and charge of the black hole respectively. Furthermore, κ is the surface gravity on $\mathscr{H}^+$, Ω_H is the angular velocity, and Φ_H is the electric surface potential. The terms $\frac{\kappa}{8\pi}$ and A is analogous to temperature and entropy in thermodynamics.

Second-Law

The second law of black hole mechanics known as 'Hawking's area theorem' , states that the area A of the event horizon of each black hole does not decrease with time,

$$dA \geq 0 \tag{2.45}$$

which is clearly analogous to the second law of thermodynamics which states that the total entropy of a closed system does not decrease with time.

Third-Law

The third law of black hole mechanics states that it is impossible to reduce the surface gravity κ to zero in any finite number of operations. This is analogous to the third law of thermodynamics, which states that absolute zero temperatures cannot be obtained by a finite number of operations.

Although some of the laws have been challenged by many theorists throughout recent developments [54] they were the first step towards Hawking's radiation that gave birth to the information paradox.

The Bekenstein-Hawking entropy, a concept of thermodynamic entropy for black holes, was developed from the laws mentioned above, Which has the following form,

$$S_{BH} = \frac{c^3 A}{4\hbar G} \tag{2.46}$$

Here, $c, \hbar, A$ and G are speed of light, reduced Planck's constant, area of the future event horizon and the universal gravitational constant respectively. This result will be of great importance in deriving one of the parameters of the proposed wave-function in chapter 7.

Chapter 3

Conditions Regarding Niceness of the Black Hole Geometry

One local theory is quantum field theory. When local quantum field theory is taken into account outside of a black hole, modes fall inside the black hole and are radiated out. This is known as the Hawking process. We will discuss the Hawking process a bit in the next section. If we look for a model for the universe, the standard model is somewhat being complete except it does not include gravity. The problem of non-renormalizability arises when we attempt to quantify gravity. Consequently, there isn't a comprehensive theory of gravity that is consistent with experiments. Hence why, we have to be careful about the quantum effects of gravity before doing physics. As a result, we confine our attention to the solar system limit and the niceness condition proposed by Mathur in [30] in this chapter. It is merely a threshold beyond which the effects of quantum gravity can be disregarded. As, we do not have a quantum theory of gravity we must work within a limit where we can neglect the quantum effects of gravity. In addition, we do work with quantum gravity effects in the solar system limit but they are negligibly small. Mathur goes on to apply 'N' niceness conditions of the solar system limit while disregarding the implications of quantum gravity. Along with introducing the idea of pair production as a result of the distortion of such slices, he also shows how these slices evolve. The concept of Hawking radiation results from pair creation at the horizon. He refers to the well-known information paradox as the "Hawking Theorem" with the assumptions N (Niceness Condition). Given that this dissertation is closely related to Mathur's conclusion, we offer the Niceness Conditions and beautiful slices of the Black hole geometry that were described in [30].

3.1 The Solar System Limit

We do not need to worry about quantum gravity in routine experiments, as stated above and in Mathur's work[30]. We believe there is a suitable limit where the effects of quantum gravity become negligible and a local, well-defined approximate evolution equation becomes feasible. This limit was termed the "solar system physics" limit by Mathur, or just the "solar system limit." He defined the solar system limit as follows:

Definition of Solar System Limit:

There must be a set of "niceness conditions" N with a small parameter ϵ so that a known, local evolution equation can describe physics with arbitrarily high accuracy when ϵ is made arbitrarily small. In other words, we can specify the quantum state on an initial spacelike slice under conditions N, and the state on subsequent slices is given by a Hamiltonian evolution operator. Additionally, as the distance between two regions increases to infinity , the influence of the state in

one region on the evolution in another must decrease to zero.(locality)
We have specified these N conditions in the next section.

3.2 The Nice Slices found by Mathur

The nice slices should have qualities that would allow us to effectively avoid Planck scale physics, since quantum field theory fails at the Planck scale where we expect the quantum gravity effects to be particularly prominent. We remain in the semi-classical realm given the following characteristics (minimal to no impacts of quantum gravity):

($N1$) We define our quantum state on a spacelike slice. On our slice, we expect the intrinsic curvature 'R' to be much smaller than Planck scale. (i.e $R << \frac{1}{l_p^2}$). Here, l_p is the Planck length which is the smallest unit of length[34].

($N2$) We also require that the slice is sitting on a 4-dimensional spacetime and the extrinsic curvature 'K' of the slice is small compared to the Planck scale.(i.e $K << \frac{1}{l_p^2}$)

($N3$) In addition, we require that the 4-curvature of spacetime near the spacelike slice to be small everywhere.

($N4$) Furthermore, we require that any matter on the slices should not have ultra-Planckian property. In contrast, the wavelength of any quanta on the slices should have wavelength longer than Planck length(ie.$\lambda >> l_p$). In addition, the energy density and momentum density should be much smaller than the planck density, l_p^{-4}.

($N5$) In order to maintain our semi-classical domain, evolution to subsequent slices should likewise meet the aforementioned requirements.

Now if we evolve the slice we encounter something interesting. The subsequent slices will experience pair creation since the vacuum state on one slice won't always be a vacuum state on the subsequent slice. The generated pairs are in the following state:

$$|\Psi\rangle_{pair} = Ce^{\gamma\hat{c}^{\dagger}\hat{b}^{\dagger}} |0\rangle_b |0\rangle_c \tag{3.1}$$

Where, γ is a number of order unity. We can capture the essence of the above state in the following state as well,

$$|\Psi\rangle_{pair} = \frac{1}{\sqrt{2}} (|0\rangle_b |0\rangle_c + |1\rangle_b |1\rangle_c) \tag{3.2}$$

Here, the b quanta is the Hawking radiated partner and the c quanta is the black hole infalling partner. We shall be working with the latter wave function throughout the rest of the paper. Now, let us turn to slices on the black hole in the next section.

3.3 Nice Slices of the Black Hole Geometry

We consider an initial spacelike slice where the black hole is represented by a matter state $|\Psi\rangle_M$. Then the state on the complete slice is,

$$|\Psi\rangle_{pair} \equiv |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}} (|0\rangle_b |0\rangle_c + |1\rangle_b |1\rangle_c) \tag{3.3}$$

As we have seen in the previous sections, black holes have spacelike singularity inside the horizon. If our spacelike slice somehow intersects with the singularity then the N niceness conditions will be violated. Therefore, we need to specify a spacetime region that can satisfy all the niceness conditions stated above. Furthermore, the specified region will let us capture the essence of the Hawking radiation. Now, we make one spacelike slice with 3 parts,

1. For $r > \frac{4GM}{c^2}$ we let the slice be t=constant.

2. Inside the black hole where $r < \frac{2GM}{c^2}$ we require the spacelike slices to be r=constant rather than t=constant as we have seen the axes flip inside the hole. Furthermore, we restrict the slice to $\frac{MG}{2c^2} < r < \frac{3MG}{2c^2}$. Then the slice is not near the horizon $r = \frac{2GM}{c^2}$ or the singularity $r = 0$.

3. Now we join these parts with smooth connector segments. Then the connected region must satisfy the niceness conditions as well.

This connected region gives us one complete spacelike slice that Mathur denotes as S_1. The evolution of this slice is discussed in the next section.

3.4 Evolution to Later Slices

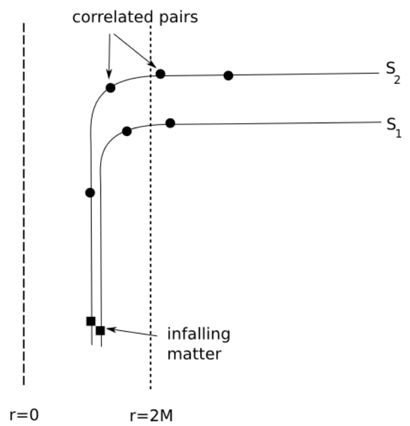

Figure 3.1: Slice evolution in black hole geometry.[30]

Let us now make a later spacelike slice as shown in figure 3.1 which Mathur calls S_2 with the following characteristics,

1. For the slice $r > \frac{4GM}{c^2}$ which was the slice with t being constant therefore, we evolve it in time and get, $t_1 = t + \Delta$.

2. For the r= constant slice we take $r_1 = r + \delta$, where $\delta << \frac{M}{c^2}$.

3. We again join the slices of $r_1 = constant$ and $t_1 = constant$ with smooth connector segments. Notice that the r_1 slice of S_2 is longer than the r slice of S_1. This is required as, the connector segment has to join the r=constant part to the t=constant part. Since, t= constant part has been evolved in time in the later slice we require that r_1 is longer than r to join r_1 with t_1.

Moreover, figure 3.1 gives us an intuitive picture of the Hawking radiated partner that we shall call 'b' quanta and the infalling partner that we shall denote as 'c' quanta. We will be working with these quanta throughout the rest of this paper. The Hawking radiation found in Hawking's original calculation was for the 'b' quanta we see in the figure. Now that we have defined our niceness conditions on the complete slices of black hole geometry, we are ready to jump into the famous information paradox.

Chapter 4

The Information Paradox

There has been various conflicts among the theory of relativity and quantum mechanics throughout the century. While many believe that one of them must be wrong as they are both quite inconsistent with each other, there has been many efforts on having a more unified theory such as string theory. Nevertheless, no one has established any ground breaking research to nullify any of the theories. On the other hand, Einstein himself was not very fond of the quantum theory as his famous saying on the matter goes, 'God does not play dice with the universe'. In any case, both relativity and quantum mechanics work in their own regime quite well. The problem arises when we try to combine them together. Thus, scientists think maybe there is another piece to the puzzle; 'a quantum theory of gravity'. In addition, It is difficult to resolve the conundrum that using conventional gravitational properties in a quantum environment violates quantum mechanics.
Now, let's attempt to comprehend the true nature of the paradox. First of all, unitarity is a basic postulate in quantum mechanics. It states that the time evolution of a quantum state is represented by a unitary operator. Furthermore, As we shall show later in this chapter, a unitary transformation on a pure state must result to a pure state, but this is not the case for black holes. The information loss is closely related to unitarity. Naively speaking, the information that gets inside the black hole is trapped there forever from the known universe. That is the paradox as no information can be created or be lost by the fundamental principle of physics. Now one might argue that information loss is found all over nature starting from burning papers to human life itself. When a paper is burnt the information on it is completely lost right? The answer is, 'No'. The information that the paper contained is encoded in the ash of the paper and the radiated photons. On the other hand, the burning atoms of the paper are entangled with the radiated photons. Then one might argue that the entanglement entropy is increasing, does that not violate unitarity as it won't preserve its pure state anymore? Again it's a 'No' . Initially, the entanglement entropy goes up then slowly after a page time the entanglement entropy goes to zero as shown in the figure 4.1 . As, the radiated product forms a pure state and saves unitarity. We will see that for a black hole that is not the case. In a black hole, the entanglement entropy of the radiated quanta and infalling quanta keeps increasing after each successive step, leading to a final mixed state after the evaporation of black hole which indeed violates unitarity. This was the establishment of Hawking's paper 'Breakdown of Predictability in Gravitational Collapse' published in 1976[11]. We will shed light on this matter next.

4.1 The Hawking Process

In 1973, Stephen Hawking developed a theoretical argument on the radiation from the vicinity of a black hole[9]. He applied quantum field theory and considered a scaler field near a black hole which resulted naturally to the continuous energy radiation. In quantum field theory, ordinary space is filled with vacuum fluctuations in electromagnetic fields which consists of pairs of photons being created at one event and recombining at another event. From the look of it, they seem to violate conservation of energy. nonetheless, if they last less than $\Delta t = \frac{\hbar}{2\Delta E}$ where $\hbar$ is the reduced Planck's constant, Δt is the time of their existence, ΔE is the amount of violation then they do not violate any physical law at large scale. Let's consider a pair of photons, one with positive energy and another with negative energy. In flat spacetime, the negative energy photon would not be able to propagate freely, so it will necessarily combine with the positive energy photon and annihilate within time Δt. However, if they are produced just outside the event horizon of a black hole, it has a chance of crossing the horizon before the time Δt elapses; once inside the black hole it can propagate freely. This negative energy photon decreases the mass of the black hole due to the mass-energy equivalence principle. Hawking showed that the photons that come out as the radiated photons with positive energy have the spectrum characteristics of a black body with temperature $T_H = \frac{\hbar}{8\pi k_B M}$[12]. This result will turn out to violate the unitarity of quantum mechanics as we shall see next.

4.2 The Fundamentals of the Paradox and Entanglement Entropy

The degree to which states or systems within a system are entangled or associated with one another is indicated by the entanglement entropy. Furthermore, it lets us asses the states of a system better in a sense that it will let us see which systems evolved into a pure or mixed state and saved unitarity. To illustrate, let us think of a pure system M consisting two systems A and B. We can use the Von Neumann entanglement entropy[22], $S(A) = -Tr[\rho_A log(\rho_A)]$, where ρ_A is the reduced density matrix, $S(A)$ is the entanglement entropy of system A with system B. if $S(A) = 0$ then there is no entanglement between the two systems. Thus the system M evolved to a pure state and unitarity is saved. On the other hand, if $S(A) \neq 0$ then the system is entangled and we might have a mixed evolved state, which in this case violates unitarity. Observe that we must compute the reduced density matrix before we can determine the entanglement entropy. We calculate the reduced density matrix by taking partial trace over all the other systems. We will now explore these concepts of a bell pair state representing a vacuum state outside the event horizon of a black hole. The quantum state outside the black hole as shown in Mathur's paper [30] is then given by,

$$|\Psi\rangle_L \approx |\psi\rangle_M \otimes \frac{1}{\sqrt{2}} \left(|0\rangle_{b_1} |0\rangle_{c_1} + |1\rangle_{b_1} |1\rangle_{c_1} \right) \tag{4.1}$$

Here, $|\psi\rangle_M$ is the initial spacelike slice that represent the matter state of the black hole. The density matrix for the entire state is given by,

$$\begin{aligned} \rho &= |\Psi\rangle_L \langle\Psi| \\ \rho &= \frac{1}{2} \left(|0\rangle_{b_1} \langle 0| \otimes |0\rangle_{c_1} \langle 0| + |0\rangle_{b_1} \langle 1| \otimes |0\rangle_{c_1} \langle 1| + |1\rangle_{b_1} \langle 0| \otimes |1\rangle_{c_1} \langle 0| + |1\rangle_{b_1} \langle 1| \otimes |1\rangle_{c_1} \langle 1| \right) \otimes |\psi\rangle_M \langle\psi| \end{aligned} \tag{4.2}$$

If we now calculate the reduced density matrix for b_1 quanta of (4.1) we get the following,

$$\begin{aligned}
\rho_{b_1} =& tr_{c_1} tr_M(\rho) \\
=& tr_{c_1} \langle \psi_M|\Psi\rangle \langle \Psi|\psi_M\rangle \\
=& tr_{c_1} \left[\frac{1}{2}\left(|0\rangle_{b_1}\langle 0|\otimes|0\rangle_{c_1}\langle 0| + |0\rangle_{b_1}\langle 1|\otimes|0\rangle_{c_1}\langle 1| + |1\rangle_{b_1}\langle 0|\otimes|1\rangle_{c_1}\langle 0| + |1\rangle_{b_1}\langle 1|\otimes|1\rangle_{c_1}\langle 1|\right)\right] \\
=& \langle 0| \left[\frac{1}{2}\left(|0\rangle_{b_1}\langle 0|\otimes|0\rangle_{c_1}\langle 0| + |0\rangle_{b_1}\langle 1|\otimes|0\rangle_{c_1}\langle 1| + |1\rangle_{b_1}\langle 0|\otimes|1\rangle_{c_1}\langle 0| + |1\rangle_{b_1}\langle 1|\otimes|1\rangle_{c_1}\langle 1|\right)\right]|0\rangle + \\
& \langle 1| \left[\frac{1}{2}\left(|0\rangle_{b_1}\langle 0|\otimes|0\rangle_{c_1}\langle 0| + |0\rangle_{b_1}\langle 1|\otimes|0\rangle_{c_1}\langle 1| + |1\rangle_{b_1}\langle 0|\otimes|1\rangle_{c_1}\langle 0| + |1\rangle_{b_1}\langle 1|\otimes|1\rangle_{c_1}\langle 1|\right)\right]|1\rangle \\
=& \frac{1}{2}\left(|0\rangle_{b_1}\langle 0| + |1\rangle_{b_1}\langle 1|\right)
\end{aligned}$$

Let's denote the states $|0\rangle$ and $|1\rangle$ as following,

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

and

$$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

therefore we have,

$$\begin{aligned}
|1\rangle\langle 1| =& \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\
=& \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}
\end{aligned}$$

Similarly,

$$\begin{aligned}
|0\rangle\langle 0| =& \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} \\
=& \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}
\end{aligned}$$

Finally, we get the reduced density matrix for b_1 quanta to be,

$$\begin{aligned}
\rho_{b_1} =& \frac{1}{2}\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\right] \\
\rho_{b_1} =& \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}
\end{aligned} \tag{4.3}$$

Now if we compute the entanglement of b_1 with (M, c_1) we obtain,

$$\begin{aligned}
S_{entanglement}(b_1) =& -tr(\rho_{b_1}\ln\rho_{b_1}) \\
S_{entanglement}(b_1) =& -tr\left[\begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\ln\frac{1}{2} & 0\\0 & \ln\frac{1}{2}\end{pmatrix}\right] \\
S_{entanglement}(b_1) =& -tr\left[\begin{pmatrix}\frac{1}{2}\ln\frac{1}{2} & 0\\0 & \frac{1}{2}\ln\frac{1}{2}\end{pmatrix}\right] \\
S_{entanglement}(b_1) =& -\ln\frac{1}{2} \\
S_{entanglement}(b_1) =& \ln 2
\end{aligned} \tag{4.4}$$

If we now again evolve our slice to the next spacelike slice then due to the stretching in r=constant part we will again have new pair creations (b_2, c_2). Our new state will look something like the following,

$$|\Psi\rangle_L \approx |\psi\rangle_M \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{b_1}|0\rangle_{c_1} + |1\rangle_{b_1}|1\rangle_{c_1}\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{b_2}|0\rangle_{c_2} + |1\rangle_{b_2}|1\rangle_{c_2}\right) \tag{4.5}$$

The reduced density matrix for the set $\{b_1, b_2\}$ is,

$$\begin{aligned}
\rho_{(b_1,b_2)} =& tr_{c_2} tr_{c_1} tr_M |\Psi_L\rangle\langle\Psi_L| \\
=& tr_{c_2}\left[\frac{1}{2}\left(|0\rangle_{b_1}\langle 0| + |1\rangle_{b_1}\langle 1|\right) \otimes \frac{1}{2}\left(|0\rangle_{b_2}|0\rangle_{c_2} + |1\rangle_{b_2}|1\rangle_{c_2}\right)\left({}_{b_2}\langle 0|_{c_2}\langle 0| + {}_{b_2}\langle 1|_{c_2}\langle 1|\right)\right] \\
=& \frac{1}{2}\left(|0\rangle_{b_1}\langle 0| + |1\rangle_{b_1}\langle 1|\right) \otimes \frac{1}{2}\left(|0\rangle_{b_2}\langle 0| + |1\rangle_{b_2}\langle 1|\right) \\
=& \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\
=& \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}
\end{aligned}$$

if we compute the entanglement of the set $\{b_1, b_2\}$ with the set $\{M, c_1, c_2\}$ we obtain,

$$\begin{aligned}
S_{entanglement}(b_1, b_2) =& -tr[\rho_{(b_1,b_2)} \ln \rho_{(b_1,b_2)}] \\
=& -tr\left[\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \ln\frac{1}{4} & 0 & 0 & 0 \\ 0 & \ln\frac{1}{4} & 0 & 0 \\ 0 & 0 & \ln\frac{1}{4} & 0 \\ 0 & 0 & 0 & \ln\frac{1}{4} \end{pmatrix}\right] \\
=& -tr\left[\begin{pmatrix} \frac{1}{4}\ln\frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4}\ln\frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4}\ln\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4}\ln\frac{1}{4} \end{pmatrix}\right] \\
=& -\ln\frac{1}{4} \\
=& 2\ln 2
\end{aligned}$$

Similar as before after N such steps, we will have the sate of the evolved slice as,

$$|\Psi\rangle_L \approx |\psi\rangle_M \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{b_1}|0\rangle_{c_1} + |1\rangle_{b_1}|1\rangle_{c_1}\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{b_2}|0\rangle_{c_2} + |1\rangle_{b_2}|1\rangle_{c_2}\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{b_N}|0\rangle_{c_N} + |1\rangle_{b_N}|1\rangle_{c_N}\right)$$

Then the space $\{b_1, b_2, \ldots, b_N\}$ is entangled with the rest of the system $(M, c_1, \ldots, c_N)$ with,

$$S_{entanglement} = N \ln 2 \tag{4.6}$$

We see that the 'b' quanta are maximally entangled with the rest of the system. The b quanta are heavily entangled with essentially nothing if the black hole disappears after N steps. This scenario tells us that the wave function disappears completely after N steps. Then we are left with nothing but the density matrix to describe the state. Another issue is that, as the particles fall into the black hole it's mass decreases and after enough steps it might decrease to Planck size before evaporation. Then our niceness conditions will not hold and we can not evolve the states any further after such steps. If on the other hand, the black hole is completely evaporated then we will see a violation of quantum mechanics and if instead it is stopped at Planck size then that would lead to 'Remnant Scenario'. We will discuss these phenomena next.

4.3 The Violation of Quantum Mechanics

The time evolution operator in quantum mechanics evolves a state to its later state. the time evolution operator is generally known as $\hat{U}(t) = e^{-\frac{it\hat{H}}{\hbar}}$, which is unitary as anyone can see[38]. Therefore, if we evolve a pure state in time, we will end up with a pure state. Let us see this mathematically,
Suppose, we have a state $|\psi\rangle$, then the density matrix of the state is given as, $\rho = |\psi\rangle\langle\psi|$. Now, If we apply the unitary operator on this state then it evolves to a state, $|\psi'\rangle = \hat{U}(t)|\psi\rangle$, the adjoint of this state is $\langle\psi'| = \langle\psi|\hat{U}^\dagger(t)$. Now the density operator is, $\rho' = |\psi'\rangle\langle\psi'| = \hat{U}(t)|\psi\rangle\langle\psi|\hat{U}^\dagger(t)$. Let us look at the trace of this density matrix square,

$$\begin{aligned}
Tr(\rho'^2) =& Tr(\rho'\rho') \\
=& Tr\left[\hat{U}(t)|\psi\rangle\langle\psi|\hat{U}^\dagger(t)\hat{U}(t)|\psi\rangle\langle\psi|\hat{U}^\dagger(t)\right] \\
=& Tr\left[\hat{U}(t)\rho\hat{U}^\dagger(t)\hat{U}(t)\rho\hat{U}^\dagger(t)\right] \\
=& Tr\left[\hat{U}(t)\rho\rho\hat{U}^\dagger(t)\right] \\
=& Tr\left[\hat{U}(t)\rho^2\hat{U}^\dagger(t)\right] \\
=& Tr\left[\rho^2\hat{U}(t)\hat{U}^\dagger(t)\right] \\
=& Tr(\rho^2)
\end{aligned}$$

therefore, we see that the trace of the density matrix square remains unchanged under unitary transformation. In other words, the time evolution of a state do not change the nature of the state. A pure that has, $Tr(\rho^2) = 1$ which will still remain the same if we evolve the state into a later state. Same is true for a mixed state for which $Tr(\rho^2) < 1$.However, for black holes we do not get this result as we started with a pure state and we are getting a mixed state after the Black Hole has evaporated as the radiated quanta are entangled with basically nothing if the Black Hole has evaporated, meaning that we do not have a quantum state function anymore to represent the final state instead we have a density matrix. In fact, the radiated quanta are maximally entangled. This is the essence behind the violation of unitarity in quantum mechanics.
Another issue is the information loss. We initially started with the mass state $|\Psi\rangle_M$ and the pair state $|\psi\rangle_{(b_1,c_1)}$. After N steps we are left with only b quanta states, which does not give us the choice to evolve back to its past state as we do not know for sure where the rest of the states went. This is another violation in quantum mechanics, as any state that can be evolved into its later state must contain the complete information about its past state so that it can be evolved backwards in time.

4.4 Remnants

The spacelike slices that we have considered with the N niceness conditions stops evolving when the black hole becomes Planck sized. A Planck sized black hole is called a remnant. We give the following two definition of remnants; The first one is given by Mathur in [30].Another definition is given by Yen Chin Ong in [42].

Definition A:

If a collection of objects with mass and size less than the specified constraints $m < m_{remnants}$ and $l < l_{remnants}$ exist yet permit strong entanglement with systems far from the object, we will declare that our gravity theory contains remnants.

Definition B:

Under Hawking evaporation, a remnant is a localized late stage of a black hole that is either (i) completely stable or (ii) long lived. Remnants do not violate quantum mechanics. The information that gets into the remnant stays there forever and is not lost. Since we seek a finite number of states in a finitely bounded energy and mass state, remnants are actually "unwanted stuff." The number of distinct remnant species must be limitless for remnants to be able to track almost anything that may create an infinitely big Black Hole. Due to the limitless number of potential objects that could circulate in the loop, remnants can potentially result in loop divergences. Some theories try to solve this by considering the exterior of the remnant to be small but the interior to be extremely large. However, theorists dealing with the information paradox are not very fond of remnants. As you can see from the second definition, remnants can also be long lived. However, long lived remnants are also pretty stable as they take almost forever to evaporate compared to the evaporation of a black hole, and they emit information as they evaporate slowly. Furthermore, A remnant may not have a horizon because the concept of spacetime is not entirely obvious at that scale. Susskind in [20] argues from an entropy perspective that, it is necessary to assume that Black Holes possess an infinitely higher entropy beyond and above the typical Bekenstein-Hawking entropy. Even at asymptotic distances, this extra entropy results in an abundant remnant scenario in Rindler space. As a result, he comes to the conclusion that he had no choice but to abandon the notion of remnants. Several remnants proposals are provided in [42]. Some other popular ones can also be found in [13] [2] [5] for interested readers. A diagram of long-lived remnant can be found in [42] which is something as the following,

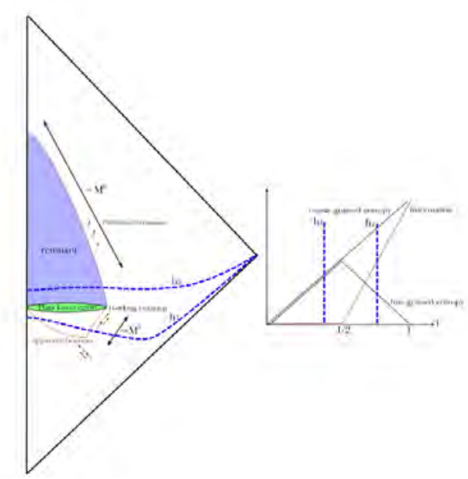

Figure 4.1: Page curve and Penrose diagram for long-lived remnant.[42]

4.5 Possible Resolutions So Far

String theory is currently the most widely recognized explanation of quantum gravity. The 'Fuzzball solution' is the solution that string theory provides. Regular horizons are not attainable with fuzzball solutions. Instead they suggest that the topology of the horizon is non-trivial. Furthermore, They consist of energy microstates and counting these microstates lead us to the Bekenstein entropy. Considering the microstates at strong coupling we find that the microstates never develops a horizon. Instead we get a fuzzball like object that radiates from its surface. Thus we do not get the rising entanglement and there is no information paradox as it suggests. However, one of the main criticism of this is not having an event horizon. Furthermore, even if you can count microstates you still run into the infalling problem since curvatures are low at the horizon, it appears that a collapsing shell will keep shrinking towards $r = 0$, leaving a vacuum region outside the horizon. Furthermore, The probability for the shell to tunnel into one of the fuzzball microstates is very low [52].

Juan Maldacena laid the foundation for the AdS/CFT correspondence principle, an illustration of the holographic principle, in his notorious paper [21], published in 1997. According to the holographic principle, the boundary of a surface encodes all information about the volume it encloses[52][29]. Susskind postulated that any genuine theory of quantum gravity for a given volume of space should allow a complete description based just on the space's lower dimensional boundary, based on the principle[19]. Quantum field theories that remain invariant under conformal transformations are known as conformal field theories. The maximally symmetric solution to Einstein's field equations in n-dimensions with a negative cosmological constant Λ is the n-dimensional de-sitter space AdS_n. Any asymptotically AdS_n spacetime that can be characterized by a quantum theory of gravity is dual to a $(n-1)$ dimensional conformal field theory that resides on the conformal boundary of the AdS_n bulk and has topology $\mathbb{R} \times S^{n-2}$, according to the AdS/CFT correspondence. CFT states are dual to any AdS black hole. Since AdS incorporates quantum gravity that is dual to a unitary quantum field theory , we may determine the temperature and entropy of AdS black holes by comparing them to the CFT's temperature and the number of CFT excited states at that temperature. Since AdS is dual to a CFT, black holes in it should be unitary. However, this isn't true. In [30], Mathur provided a persuasive argument for why the AdS/CFT correspondence cannot escape the contradiction. He basically argues that the Ads/CFT correspondence is an abstract argument in resolving the information paradox as Black holes reside in the Ads and unitarity is preserved in CFT. However, we need a quantum description of gravity in Ads to actually resolve the paradox as there is no formation of Black Holes in CFT so there is no direct evidence of information recovery either. If we apply Hawking theorem in CFT we will still run into the information paradox problem without studying the formation/evaporation process of the Black Hole in detail.

Another resolution to the black hole information paradox was proposed by AMPS [39] which is known as the firewall solution. The firewall resolution recommends breaking the entanglement between outgoing and ingoing Hawking partners right away in order to create a firewall and release a significant quantity of energy. However, this violates Einstein's equivalence principle[17] as there is something special happening at the event horizon and you will definitely experience the immense energy of the horizon if you are inside a box crossing the horizon in free fall. In contrast, many of the theorists today do no regard the firewall resolution as relevant as the previous two.

The final wave function that vanishes is the primary issue with the information paradox, not entanglement or the horizon. Disappearance of the wave function leads to the violation of unitarity. Instead, one expect entropy to behave like the burning paper, initially going up then after a certain page-time it starts to go down to eventually zero as shown in the page curve of figure 4.1. In [18] [15][16], Page offered several intriguing theories on this Page time and Page Curve. The majority of modern physicists mostly concur that unitarity must be preserved, and that non-local information must be released to do this. We will explore non-locality in chapter 6.

Chapter 5

The Leading Order Hawking State

Why can't we regard black holes as hot bodies? The reason for this is that radiation from hot bodies is released by the components that compose the hot body, but radiation from black holes is caused by the stretching of a portion of spacelike slice. [30][9]. We examined the status of the generated pairs "to leading order" in the preceding section. We saw that we couldn't avoid Hawking's conclusion if this were the pair's true state. Nevertheless, any leading order calculation will always have small corrections. Nonetheless, Hawking's conclusion cannot be avoided by these small corrections. We investigate the correction in the this chapter.

The leading order Hawking state presented in Mathur's paper[30] is,

$$S^{(1)} = \frac{1}{\sqrt{2}}\left(|0\rangle_{b_{n+1}}|0\rangle_{c_{n+1}} + |1\rangle_{b_{n+1}}|1\rangle_{c_{n+1}}\right) \tag{5.1}$$

With the above state we saw that the spacelike slice after N steps had an entanglement entropy of $N\ln 2$. Thus, it could not escape Hawking's conclusion. Mathur denotes the state of the modes at time step t_n as $|\Psi_{M,c},\psi_b(t_n)\rangle$. Here,

- $\Psi_{M,c}$ denotes the mass of the black hole and also all the infalling c quanta that have been created at all earlier steps.
- ψ_b denotes all the radiated b quanta that are created at all earlier steps.

The state is not a product state as they are entangled. We can also write the state at timestep t_n as,

$$|\Psi_{M,c},\psi_b(t_n)\rangle = \sum_{m,n} C_{mn}\psi_m\chi_n \tag{5.2}$$

Where,

- ψ_m is an orthonormal basis state for the quanta (M,c) inside the black hole.
- χ_n is an orthonormal basis state for the radiated quanta b outside the black hole.

We can perform a unitary transformation on ψ_i,χ_j to get,

$$|\Psi_{M,c},\psi_b(t_n)\rangle = \sum_{i} C_i\psi_i\chi_i \tag{5.3}$$

The reduced density matrix for b quanta is then,

$$
\begin{aligned}
\rho_b =& |\Psi\rangle \langle\Psi| \\
=& \sum_i \sum_j C_i C_j^* |\psi_i\rangle |\chi_j\rangle \langle\psi_i| \langle\chi_j| \\
=& \sum_i \sum_j C_i C_j^* |\chi_j\rangle |\psi_i\rangle \langle\psi_i| \langle\chi_j| \\
=& \sum_j C_i C_j^* |\chi_j\rangle \delta_{ij} \langle\chi_j| \\
=& \sum_i C_i C_i^* |\chi_i\rangle \langle\chi_i| \\
=& |C_i|^2 \delta_{ij}
\end{aligned}
$$

Therefore, the entanglement at time t_n is,

$$
\begin{aligned}
& S_{entanglement}(t_n) = -\,tr(\rho_{b_n} \ln \rho_{b_n}) \\
\implies & S_{entanglement}(t_n) = -\sum_i |C_i|^2 \ln |C_i|^2
\end{aligned}
\tag{5.4}
$$

Furthermore, in the leading order evolution we would have at time step t_{n+1},

$$
|\Psi_{M,c}, \psi_b(t_n)\rangle \to |\Psi_{M,c}, \psi_b(t_n)\rangle \left[\frac{1}{\sqrt{2}} \left(|0\rangle_{b_{n+1}} |0\rangle_{c_{n+1}} + |1\rangle_{b_{n+1}} |1\rangle_{c_{n+1}} \right) \right] \tag{5.5}
$$

where the state of the freshly formed pair is indicated by the term enclosed in box brackets. The most general deformation in this leading order state will be shown in the following section, along with the resulting entanglement entropy.

5.1 Small Correction to the Hawking Leading Order State

The current effort is to determine whether taking into account a large number of small corrections can result in the entanglement entropy decreasing and ultimately reaching zero, so resolving the paradox. Let us consider another state vector,

$$
S^{(2)} = \frac{1}{\sqrt{2}} \left(|0\rangle_{b_{n+1}} |0\rangle_{c_{n+1}} - |1\rangle_{b_{n+1}} |1\rangle_{c_{n+1}} \right) \tag{5.6}
$$

We consider the evolution at timestep t_{n+1}. Mathur in addition, considered that the b quanta that have been created at all earlier steps is not affected by the evolution at this step. This is an important remark. The reason behind this is that the b quanta have left the vicinity of the black hole and unless there is some non-local interaction between the hole and the radiated quanta, new pair creation is not affected by the previously radiated quanta. Therefore, the evolution to time step t_{n+1} occurs as follows,

$$
|\chi_i\rangle \to |\chi_i\rangle \tag{5.7}
$$

and

$$
|\psi_i\rangle \to |\psi_i^{(1)}\rangle \otimes |S^{(1)}\rangle + |\psi_i^{(2)}\rangle \otimes |S^{(2)}\rangle \tag{5.8}
$$

We have in the leading order evolution,

$$
|\psi_i^{(1)}\rangle = |\psi_i\rangle \, and \, |\psi_i^{(2)}\rangle = 0 \tag{5.9}
$$

Furthermore, note that the unitarity of evolution requires,

$$
||\psi_i^{(1)}||^2 + ||\psi_i^{(2)}||^2 = 1 \tag{5.10}
$$

Now, at time step t_{n+1} we have the complete state as,

$$
\begin{aligned}
&|\Psi_{M,c},\psi_b(t_{n+1})\rangle = \sum_i C_i \left[|\psi_i^{(1)}\rangle \otimes |S^{(1)}\rangle + |\psi_i^{(2)}\rangle \otimes |S^{(2)}\rangle\right] |\chi_i\rangle \\
\implies &|\Psi_{M,c},\psi_b(t_{n+1})\rangle = \left[\sum_i C_i |\psi_i^{(1)}\rangle |\chi_i\rangle\right] \otimes |S^{(1)}\rangle + \left[\sum_i C_i |\psi_i^{(2)}\rangle |\chi_i\rangle\right] \otimes |S^{(2)}\rangle \\
\implies &|\Psi_{M,c},\psi_b(t_{n+1})\rangle = |\Lambda^{(1)}\rangle \otimes |S^{(1)}\rangle + |\Lambda^{(2)}\rangle \otimes |S^{(2)}\rangle
\end{aligned}
\tag{5.11}
$$

Here, we have defined the states,

$$
|\Lambda^{(1)}\rangle = \sum_i C_i |\psi_i^{(1)}\rangle |\chi_i\rangle \tag{5.12}
$$

and

$$
|\Lambda^{(2)}\rangle = \sum_i C_i |\psi_i^{(2)}\rangle |\chi_i\rangle \tag{5.13}
$$

Normalization of the wave-function requires that,

$$
\begin{aligned}
&\langle\Psi_{M,c},\psi_b(t_{n+1})|\Psi_{M,c},\psi_b(t_{n+1})\rangle = 1 \\
\implies &\langle\Lambda^{(1)}|\Lambda^{(1)}\rangle + \langle\Lambda^{(2)}|\Lambda^{(2)}\rangle = 1 \\
\implies &||\Lambda^{(1)}||^2 + ||\Lambda^{(2)}||^2 = 1
\end{aligned}
\tag{5.14}
$$

Mathur called the correction small if,

$$
||\Lambda^{(2)}||^2 < \varepsilon, \varepsilon << 1 \tag{5.15}
$$

He considered this as, the new pairs are mostly created in the $S^{(1)}$ state. If there is no such bound then the correction is of order unity. However, the correction must be small as we considered the effect of the quanta is already small on the spacelike slice due to the N niceness conditions. Therefore, even though the amplitude of $S^{(2)}$ can be large the probability of finding the orthogonal state $S^{(2)}$ when the pair is created has to be much less than unity.

5.2 Mathur's Entropy Bounds

The entanglement entropy at timestep t_n is given by (5.16). Mathur denotes it as S_0, which is,

$$
S_0 = -\sum_i |C_i|^2 \ln |C_i|^2 \tag{5.16}
$$

Now, the aim is to show that at the next timestep the entanglement entropy of the b quanta with the hole (M,c) increases if we apply the condition of small correction states above. Mathur in his paper, took the help of three lemmas to prove one crucial theorem that states that the entanglement entropy increases at each stage of evolution by at least an amount $\ln 2 - 2\varepsilon$.

5.2.1 Entropy of multiple subsystems

A recap on some preliminary entropy concept is not a bad idea. Suppose, we have a system consisting three subsystems A,B and C. Then $S(A) \equiv -tr\rho_A \ln \rho_A$. Where, ρ_A is the reduced density matrix for subsystem A which we can get by tracing out B and C from the density matrix of the whole system. Moreover, S(A) gives the entanglement entropy of subsystem A with the rest of the system. Similarly, S(A+B) is the entanglement entropy of subsystem A+B with C which we get by taking the union of subsystems A and B's reduced density matrix.

5.2.2 Subsystems of the black hole system

Prior to discussing the lemmas and theorem, let's examine the black hole system's subsystem as described by Mathur in [30].

1. the radiated quanta $\{b_1, b_2, \dots, b_n\}$ in time step t_n is denoted as {b}. Furthermore, We always assume that the dynamics of pair production at the subsequent timestep are not affected by quanta released at previous steps.

2. the black hole interior consists of the state (M,{c}) where M is the mass state and {c} is the infalling quanta set $\{c_1, c_2, \dots, c_n\}$. We are now attempting to examine the implications of the pair formed at time step t_{n+1}, which can weakly interact with (M,{c}) to induce entanglement that was absent in the leading order Hawking state.

3. At timestep t_{n+1}, we write the pair that will be formed as $p \equiv (c_{n+1}, b_{n+1})$.

4. At time step t_n we have the entanglement entropy, $S_{\{b\}} = S_0$. Since, we assumed that the earlier emitted quanta can not be influenced by the newly created pair, we will still have at time step t_{n+1}, $S_{\{b\}} = S_0$, the same entanglement entropy of the {b} quanta at time step t_{n+1} as found at t_n.

5.2.3 Lemma 1

If the small correction (5.15) is true then the entanglement of the pair 'p' specified above with the remainder of the system is bounded by,

$$S(p) \equiv -tr\rho_p \ln \rho_p < \varepsilon. \tag{5.17}$$

proof:

The density matrix for 'p' pair calculated by Mathur is,

$$\rho_p = \begin{pmatrix} \langle \Lambda^{(1)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(1)} | \Lambda^{(2)} \rangle \\ \langle \Lambda^{(2)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(2)} | \Lambda^{(2)} \rangle \end{pmatrix} \tag{5.18}$$

Then the entanglement of $p \equiv (b_{n+1}, c_{n+1})$ with the rest of the system is given by,

$$S_{entanglement}(p) = -tr[\rho_p \ln \rho_p] \tag{5.19}$$

$$S_{entanglement}(p) = -tr \left[\begin{pmatrix} \langle \Lambda^{(1)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(1)} | \Lambda^{(2)} \rangle \\ \langle \Lambda^{(2)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(2)} | \Lambda^{(2)} \rangle \end{pmatrix} \begin{pmatrix} \ln \langle \Lambda^{(1)} | \Lambda^{(1)} \rangle & \ln \langle \Lambda^{(1)} | \Lambda^{(2)} \rangle \\ \ln \langle \Lambda^{(2)} | \Lambda^{(1)} \rangle & \ln \langle \Lambda^{(2)} | \Lambda^{(2)} \rangle \end{pmatrix} \right] \tag{5.20}$$

Now, let us consider Mathur's correction that,

$$||\Lambda^{(2)}|| < \varepsilon, \varepsilon << 1$$

From which we can say that,

$$\langle \Lambda^{(2)} | \Lambda^{(2)} \rangle = ||\Lambda^{(2)}||^2 \equiv \varepsilon_1^2 < \varepsilon^2 \tag{5.21}$$

Due to normalization we know that,

$$||\Lambda^{(1)}||^2 + ||\Lambda^{(2)}||^2 = 1 \tag{5.22}$$

Which implies that,

$$\langle \Lambda^{(1)} | \Lambda^{(1)} \rangle = ||\Lambda^{(1)}||^2 \equiv 1 - \varepsilon_1^2 < 1 - \varepsilon^2 \tag{5.23}$$

By Schwartz inequality[58] we also get,

$$\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle \equiv \varepsilon_2 < \varepsilon \tag{5.24}$$

Then the entanglement of $p \equiv (b_{n+1}, c_{n+1})$ with the rest of the system is given by,

$$\begin{aligned}
S_{entanglement}(p) &= -tr\left[\begin{pmatrix}\langle\Lambda^{(1)}|\Lambda^{(1)}\rangle & \langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\\ \langle\Lambda^{(2)}|\Lambda^{(1)}\rangle & \langle\Lambda^{(2)}|\Lambda^{(2)}\rangle\end{pmatrix}\begin{pmatrix}\ln\langle\Lambda^{(1)}|\Lambda^{(1)}\rangle & \ln\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\\ \ln\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle & \ln\langle\Lambda^{(2)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\\
&= -tr\left[\begin{pmatrix}1-\varepsilon_1^2 & \varepsilon_2\\ \varepsilon_2 & \varepsilon_1^2\end{pmatrix}\begin{pmatrix}\ln(1-\varepsilon_1^2) & \ln\varepsilon_2\\ \ln\varepsilon_2 & \ln\varepsilon_1^2\end{pmatrix}\right]\\
&= -tr\left[\begin{pmatrix}(1-\varepsilon_1^2)\ln(1-\varepsilon_1^2)+\varepsilon_2\ln\varepsilon_2 & (1-\varepsilon_1^2)\ln\varepsilon_2+\varepsilon_2\ln\varepsilon_1^2\\ \varepsilon_2\ln(1-\varepsilon_1^2)+\varepsilon_1^2\ln\varepsilon_2 & \varepsilon_2\ln\varepsilon_2+\varepsilon_1^2\ln\varepsilon_1^2\end{pmatrix}\right]\\
&= -\left[(1-\varepsilon_1^2)\ln(1-\varepsilon_1^2)+2\varepsilon_2\ln\varepsilon_2+\varepsilon_1^2\ln\varepsilon_1^2\right]\\
&= -\left[-\varepsilon_1^2\ln(1-\varepsilon_1^2)+\ln(1-\varepsilon_1^2)+2\varepsilon_2\ln\varepsilon_2+\varepsilon_1^2\ln\varepsilon_1^2\right]\\
&= -\left[\varepsilon_1^4+\frac{1}{2}\varepsilon_1^6+\cdots-\varepsilon_1^2-\frac{1}{2}\varepsilon_1^4-\frac{1}{3}\varepsilon_1^6-\cdots+2\varepsilon_2\ln\varepsilon_2+\varepsilon_1^2\ln\varepsilon_1^2\right]\\
&= -\left[-\varepsilon_1^2+2\varepsilon_2\ln\varepsilon_2+2\varepsilon_1^2\ln\varepsilon_1+O(\varepsilon_1^4)\right] < \varepsilon
\end{aligned}$$

Here we have used the expansion

$$\ln(1-x^2) \approx -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \ldots \tag{5.25}$$

Which is smaller than ε if $x = \varepsilon_1$. We can thus write this result as,

$$S(p) < \varepsilon \tag{5.26}$$

where the pair formed at timestep t_{n+1} is $p \equiv (b_{n+1}, c_{n+1})$. This suggests that there is weak entanglement between the entire pair p and the remainder of the system.

5.2.4 Lemma 2

The entanglement entropy at timestep t_n minus a small value ε is less than the entanglement entropy of the system $(\{b\}+p)$.

$$S(\{b\}+p) > S_0 - \varepsilon \tag{5.27}$$

Proof:

We can use the sub-additivity theorem of entropy of two systems A and B,

$$S(A+B) \geq |S(A)-S(B)| \tag{5.28}$$

let $A = b$ and $B = p$ we arrive at,

$$S(\{b\}+p) \geq S(\{b\}) - S(p) \tag{5.29}$$

$$\implies S(\{b\}+p) \geq S_0 - \varepsilon \tag{5.30}$$

Here, we have used the previous lemma's result $-S(p) > -\varepsilon$ and the entanglement entropy at timestep t_n from (5.16).

5.2.5 Lemma 3

With a small correction to "the leading order Hawking state," the entanglement entropy is reduced by ε.

$$S(c_{n+1}) > \ln 2 - \varepsilon \tag{5.31}$$

Proof:

The complete state of (5.11) can be written as,

$$|\Psi_{M,c},\psi_b(t_{n+1})\rangle = \left[|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle\right] + \left[|1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right] \tag{5.32}$$

the reduce density matrix of c_{n+1} is,

$$\begin{aligned}
\rho_{c_{n+1}} &= tr_{b_{n+1}}(\rho_p)\\
\rho_{c_{n+1}} &= \langle 0|_{b_{n+1}}\rho_p|0\rangle_{b_{n+1}} + \langle 1|_{b_{n+1}}\rho_p|1\rangle_{b_{n+1}}\\
\rho_{c_{n+1}} &= |0\rangle_{c_{n+1}}\langle 0|\frac{1}{2}\left[|\Lambda^{(1)}+\Lambda^{(2)}\rangle\langle\Lambda^{(1)}+\Lambda^{(2)}|\right] + |1\rangle_{c_{n+1}}\langle 1|\frac{1}{2}\left[|\Lambda^{(1)}-\Lambda^{(2)}\rangle\langle\Lambda^{(1)}-\Lambda^{(2)}|\right]\\
\rho_{c_{n+1}} &= \begin{pmatrix}\frac{1}{2}|\Lambda^{(1)}+\Lambda^{(2)}\rangle\langle\Lambda^{(1)}+\Lambda^{(2)}| & 0\\ 0 & \frac{1}{2}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\langle\Lambda^{(1)}-\Lambda^{(2)}|\end{pmatrix}
\end{aligned} \tag{5.33}$$

Now, let

$$|\lambda^{(1)}\rangle = |\Lambda^{(1)}+\Lambda^{(2)}\rangle \tag{5.34}$$

and

$$|\lambda^{(2)}\rangle = |\Lambda^{(1)}-\Lambda^{(2)}\rangle \tag{5.35}$$

Then the density matrix of (5.39) turns out,

$$\rho_{c_{n+1}} = \begin{pmatrix}\frac{1}{2}|\lambda^{(1)}\rangle\langle\lambda^{(1)}| & 0\\ 0 & \frac{1}{2}|\lambda^{(2)}\rangle\langle\lambda^{(2)}|\end{pmatrix} \tag{5.36}$$

Taking trace over λ,

$$\rho_{c_{n+1}} = \begin{pmatrix}\frac{1}{2}\langle\lambda^{(1)}|\lambda^{(1)}\rangle & 0\\ 0 & \frac{1}{2}\langle\lambda^{(2)}|\lambda^{(2)}\rangle\end{pmatrix} \tag{5.37}$$

As by the spectral decomposition theorem,

$$\begin{aligned}
&\langle\lambda^{(1)}|\lambda^{(1)}\rangle\langle\lambda^{(1)}|\lambda^{(1)}\rangle + \langle\lambda^{(2)}|\lambda^{(1)}\rangle\langle\lambda^{(1)}|\lambda^{(2)}\rangle\\
&= \langle\lambda^{(1)}|\lambda^{(1)}\rangle\langle\lambda^{(1)}|\lambda^{(1)}\rangle + \langle\lambda^{(1)}|\lambda^{(2)}\rangle\langle\lambda^{(2)}|\lambda^{(1)}\rangle\\
&= \langle\lambda^{(1)}|\left[|\lambda^{(1)}\rangle\langle\lambda^{(1)}| + |\lambda^{(2)}\rangle\langle\lambda^{(2)}|\right]|\lambda^{(1)}\rangle\\
&= \langle\lambda^{(1)}|\lambda^{(1)}\rangle
\end{aligned}$$

Hence, replacing back (5.34) and (5.35) into (5.39), we have the density matrix of c_{n+1} ,

$$\rho_{c_{n+1}} = \begin{pmatrix}\frac{1}{2}\langle(\Lambda^{(1)}+\Lambda^{(2)})|(\Lambda^{(1)}+\Lambda^{(2)})\rangle & 0\\ 0 & \frac{1}{2}\langle(\Lambda^{(1)}-\Lambda^{(2)})|(\Lambda^{(1)}-\Lambda^{(2)})\rangle\end{pmatrix} \tag{5.38}$$

Using $\langle\Lambda^{(1)}|\Lambda^{(1)}\rangle = 1-\varepsilon_1^2$ and$\langle\Lambda^{(2)}|\Lambda^{(2)}\rangle = \varepsilon_1^2$,we get the reduced density matrix for the c_{n+1} quanta found by Mathur's calculation,

$$\begin{aligned}
\rho_{c_{n+1}} =&\frac{1}{2}\begin{pmatrix}\langle\Lambda^{(1)}|\Lambda^{(1)}\rangle+\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle+\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle+\langle\Lambda^{(2)}|\Lambda^{(2)}\rangle & 0\\ 0 & \langle\Lambda^{(1)}|\Lambda^{(1)}\rangle-\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle-\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle+\langle\Lambda^{(2)}|\Lambda^{(2)}\rangle\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix}1-\varepsilon_1^2+\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle+\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle+\varepsilon_1^2 & 0\\ 0 & 1-\varepsilon_1^2-\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle-\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle+\varepsilon_1^2\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix}1+\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle+\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle & 0\\ 0 & 1-\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle-\langle\Lambda^{(2)}|\Lambda^{(1)}\rangle\end{pmatrix}\\
=&\frac{1}{2}I+\begin{pmatrix}\mathrm{Re}\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -\mathrm{Re}\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}+O(\varepsilon^2)
\end{aligned} \tag{5.39}$$

Then the entanglement of (c_{n+1}) with the remainder of the system is given by,

$$\begin{aligned}
&S_{entanglement}(c_{n+1}) = -tr[\rho_{c_{n+1}}\ln\rho_{c_{n+1}}]\\
=&-tr\left[\frac{1}{2}I+\begin{pmatrix}Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\left[\ln\left[\frac{1}{2}I+\begin{pmatrix}Re(\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle) & 0\\ 0 & Re(-\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle)\end{pmatrix}\right]\right]\\
=&-tr\left[\frac{1}{2}I+\begin{pmatrix}Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\left[\ln\left[\begin{pmatrix}\frac{1}{2}+Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & \frac{1}{2}-Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\right]\\
=&-tr\left[\frac{1}{2}I+\begin{pmatrix}Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\left[\begin{pmatrix}\ln(\frac{1}{2}+Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle) & 0\\ 0 & \ln(\frac{1}{2}-Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle)\end{pmatrix}\right]\\
=&-tr\left[\frac{1}{2}I+\begin{pmatrix}Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\left[\begin{pmatrix}\ln\frac{1}{2}+\ln(1+2Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle) & 0\\ 0 & \ln\frac{1}{2}+\ln(1-2Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle)\end{pmatrix}\right]\\
=&-tr\left[\frac{1}{2}I+\begin{pmatrix}Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\left[\begin{pmatrix}\ln\frac{1}{2} & 0\\ 0 & \ln\frac{1}{2}\end{pmatrix}+\begin{pmatrix}\ln(1+2Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle) & 0\\ 0 & \ln(1-2Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle)\end{pmatrix}\right]\\
=&-tr\left[\frac{1}{2}I+\begin{pmatrix}Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0\\ 0 & -Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix}\right]\left[\begin{pmatrix}\ln\frac{1}{2} & 0\\ 0 & \ln\frac{1}{2}\end{pmatrix}+\begin{pmatrix}2[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]-2[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2+\dots & 0\\ 0 & -2[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]-2[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2+\dots\end{pmatrix}\right]\\
=&-tr\left[\begin{pmatrix}\frac{1}{2}\ln\frac{1}{2} & 0\\ 0 & \frac{1}{2}\ln\frac{1}{2}\end{pmatrix}+\begin{pmatrix}[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 & 0\\ 0 & [Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2\end{pmatrix}\right]+\text{higher order terms}\\
=&\ln 2-2[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2+\text{higher order terms}\\
=&\ln 2-2\varepsilon_2^2+O(\varepsilon_2^3)+\cdots>\ln 2-\varepsilon
\end{aligned}$$

Therefore, we have proved the lemma, $S(c_{n+1})>\ln 2-\varepsilon$.

5.2.6 Theorem

Using the above three lemmas we can now prove an important theorem proposed by Mathur [30], *At time step t_n, the set $\{b\}$'s total entanglement entropy with the black hole is S_0. Suppose that the next emitted pair deviates from the leading order hawking state by a very tiny amount less than ε at time step t_{n+1}. If this small correction is valid, the bound provides the entanglement entropy of the released quanta $\{b\}$ with b_{n+1} at timestep t_{n+1},*

$$S(\{b\}+b_{n+1})>S_0+\ln 2-2\varepsilon \tag{5.40}$$

which asserts that if the deviations from the leading order Hawking state are minimal, the entanglement entropy of the released quanta must inevitably rise with each emission.

Proof:

Suppose, we have three systems A,B and C. The strong sub-additivity theorem[24] of entropy states that,

$$S(A+B)+S(B+C) \geq S(A)+S(C) \tag{5.41}$$

If we set $A=\{b\}, B=b_{n+1}$ and $C=c_{n+1}$,

$$\begin{aligned}
&S(\{b\}+b_{n+1})+S(b_{n+1}+c_{n+1}) \geq S(\{b\})+S(c_{n+1}) \\
&S(\{b\}+b_{n+1})+S(p) \geq S(\{b\})+S(c_{n+1}) \\
&S(\{b\}+b_{n+1}) \geq S(\{b\})+S(c_{n+1})-S(p) \\
&S(\{b\}+b_{n+1}) \geq S_0+\ln 2-\varepsilon-\varepsilon \\
&S(\{b\}+b_{n+1}) \geq S_0+\ln 2-2\varepsilon
\end{aligned} \tag{5.42}$$

Here, we have used the result from lemma 1 and lemma 3 and equation (5.16) to prove the theorem. From this theorem we can clearly see that the entanglement entropy does not decrease with time. Therefore, the small correction do not fix the paradox.

5.3 Generalized Mathur's Bound

Taking a look at the following state,

$$|\Psi\rangle_{n+1}=\sum_{i=0}^{2^n-1} a_i\,|i\rangle_b\,|i\rangle_c \otimes \frac{1}{\sqrt{2}}\left(exp(S_{i,n,0})\,|0\rangle_b\,|0\rangle_c+exp(S_{i,n,1})\,|1\rangle_b\,|1\rangle_c\right) \tag{5.43}$$

Mishkat, Mahbub, Abdul Matin, Moinul and Avik Roy generalized the Mathur's bound[46]. The states of the n outgoing and n ingoing quanta are shown here by $|i\rangle_b$ and $|i\rangle_c$, respectively. If the previous pairs are given as $|i\rangle_b\,|i\rangle_c$, the amplitude that observes the new pair in the state $|j\rangle_b\,|j\rangle_c$ is called $exp(S_{i,n,j})$. $|S_{i,n,j}|$ is a tiny value if the correction is small. They demonstrated that entropy cannot be reduced by even a strong link to the state under consideration. Furthermore, they found an upper bound in addition to Mathur's bound that,

$$0 \leq \Delta \leq \log 2 \tag{5.44}$$

Also, note that normalization requires,

$$\sum_{i=0}^{2^n-1} |a_i|^2=1 \tag{5.45}$$

and

$$\sum_{j=0}^{1} exp(2S_{i,n,j})=2 \tag{5.46}$$

Moreover, they also generalized the three lemmas and the theorem in [46]. This is important as it shows that no matter what correction we choose to the leading order hawking state we do not get a decrease in entropy.

Chapter 6

Locality and Non-locality

Before going into the final chapter, we should look at locality and non-locality in physics. Both of the terms are basically for interactions between two objects that can be particles, waves, matter, anything. This is important because we do not want to violate causality. According to Einstein's special theory of relativity, nothing can travel faster than the speed of light; not even information. If such a situation happens where anything exceeds the speed of light then causality is violated. Therefore, for a particle to have an influence on another particle at a distance, it must obey the laws of causality. In other words, they can not have instantaneous effect on each other. There are two ways particles can influence each other without violating causality; either by preserving locality or by non-local interaction. We shall explore them in the next two sections.

6.1 Locality in Physics

Locality in physics is a classical concept. It asserts that only an object's immediate environment has a direct impact on it. The idea behind this is that for a cause at one point to have an effect at another point must mediate the action through something in space between those points. This 'something' can be a wave or particle or any mediating matter. Furthermore, an event at one point can not cause a simultaneous result at another point. This is due to the limitation on maximum speed of causal influence can travel(the speed of light c) imposed by special relativity. Let us look at a light-cone where two points are causally disconnected.

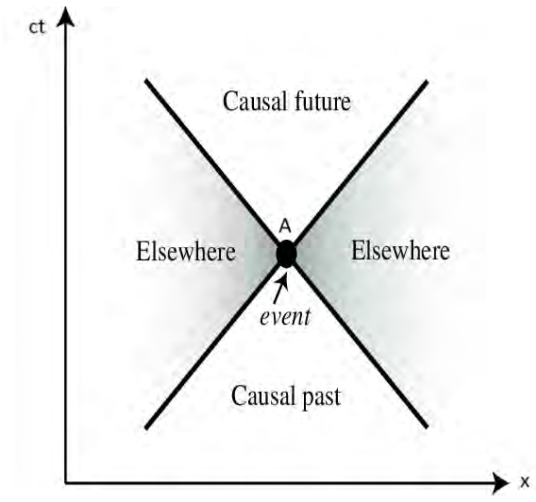

Figure 6.1: Causal relationship of events in a light cone

In the figure 6.1, all the events that are outside the light-cone namely, elsewhere are causally disconnected from the event 'A' at the center as for the events in elsewhere region to actually have any influence on event A, they must transmit information faster than the speed of light, violating causality. On the other hand, causal future and causal past can send information to event A without violating causality. If the information is conveyed through a medium such as a wave or particle then that conserve locality. A theory that conserves locality is called a local theory[48][59]. Almost all of the theories in classical physics are local theories, starting from Newtonian mechanics to electro-dynamics to special and general theory of relativity.

6.2 Non-Locality in Physics

Non-locality is basically action at a distance. Newton himself was not very fond of the idea that matter can affect other matter without any direct contact. In addition, he allowed physical interaction to be instantaneous. Later on, Einstein proposed that gravity is not actually non-contact. In general relativity, gravity is a phenomenon of the spacetime curvature. The existence of mass is what causes this curvature. In general, space-time's curvature at the volume's boundary will increase with the amount of mass contained within a particular volume of space.[44] The curvature varies to represent the shifting positions of mass-containing objects as they travel through spacetime. In some situations, changes in this curvature caused by accelerating particles spread like gravitational waves. Due to this curvature in spacetime, smaller objects orbit around the larger mass. Thus, gravity is a local theory in classical physics that does not allow instantaneous effect.
The locality breaks down at quantum level. When we look at black holes, the spacetime characteristics switch inside the hole and thus for any information to actually come out of the hole it will necessarily violate causality. To avoid such thing to happen there is something called a hidden variable that connects the two causally disconnected part. Look at the diagram below,

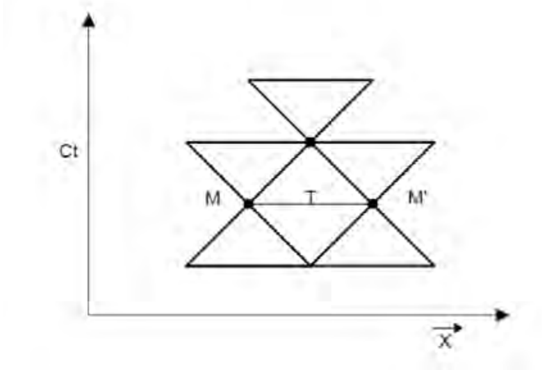

Figure 6.2: Light-cone of M, M' and T

The light-cones of M and M' are causally disconnected. However, T is connected causally to both M and M'. Therefore, one can send information from M to M' through T without the violation of causality. T here acts as a hidden variable. We expect something similar in case of the black hole as well except we do not know what this hidden variable is. One might need to have new assumptions for a consistent description of black hole as Giddings puts it in [26]. The Hawking radiated quanta and infalling quanta are entangled. Therefore, the b quanta(radiated quanta) contains information from inside the hole. This transfer of information is carried out non locally without violating causality as explained above. This non local information transfer can happen without direct contact. We assume our information is carried out non-locally.
Giddings in [26],[31],[36],[35],[41],[40] argued that non-locality is just a natural property of strong gravity. According to Giddings, Locality in QFT can be understood by the statement,

The observables commute at spacelike separations when a spacelike slice is divided into two non-overlapping sections. However he adds, particles are made or destroyed by QFT measurements; if two particles with a high enough total energy are attempted to be measured in a small enough location, the backreaction of the generated particles severely distorts the region's metric and causal structure. He therefore comes to the conclusion that there is no explicit declaration of locality in such situations. Additionally, he suggests that these factors point to a general principle of nonlocality. The uncertainty principle, which states that in some situations one cannot describe physics in terms of classical phase space degrees of freedom and must instead use the quantum-mechanical wavefunction, is strongly analogous to a nonlocality principle that states that in some situations one cannot describe physics in terms of local degrees of freedom. In particular, a basic standard for determining if D-dimensional locality is valid while attempting to investigate a pair of quanta that are characterized by wavepackets with roughly definite coordinates x and y and momenta p and q is,

$$|x-y|^{D-3} \geq G_D|p+q| \tag{6.1}$$

However, consideration of such non-locality does not resolve the paradox as we have seen in the previous chapter. We shall resort to a specific wave function along with non-locality that might resolve the paradox in the final chapter.

Chapter 7

A Resolution to the Paradox

We have seen so far that considering bell pair states as our initial wave function on the spacelike slice do not resolve the information paradox in anyway even if we consider a larger span of the vector space. In addition, we have seen that there exists an entropy bound, $0 \leq S_{entanglement} \leq \ln 2$, which clearly states that the entanglement entropy can not decrease at any successive step. Furthermore, so far we have worked with local spacelike slices without considering the effects of the radiated quanta onto the new pair creation. We shall workout the entanglement of the radiated quanta at each successive step and observe the entropy bound next. Then we will propose a different wave function that can have non-local interaction with the previously created pairs and observe its entanglement entropy. We will later see that the newly found entanglement entropy can indeed decrease over time resolving the paradox. However, to achieve such goal we will need to further restrain our wave function to some new conditions as we will see later in this chapter.

7.1 The Leading Order Hawking State and its Page Curve Solution

In chapter 4, we have considered the state on the spacelike slice at timestep t_{n+1} as,

$$|\Psi_{M,c}, \psi_b(t_{n+1})\rangle = |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} \frac{1}{\sqrt{2}} |\Lambda^{(1)} + \Lambda^{(2)}\rangle + |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \frac{1}{\sqrt{2}} |\Lambda^{(1)} - \Lambda^{(2)}\rangle \tag{7.1}$$

Here, we have considered the small correction to the leading order Hawking state by Mathur. Our goal is to show that whether the entanglement entropy between the radiated pairs with the rest of the system follow a page like curve or not. Mathur in [30] showed that the entanglement entropy of the radiated pairs with the rest of the system at each successive step increases, we will further show what happens if we consider a page like curve to the state he considered. Furthermore, we will consider all the niceness conditions hold and the subsystems of the black hole system remains unchanged. However, we won't fix the entropy of the radiated quanta at timestep t_n as Mathur did. Now, if the spacelike slices followed a page like curve then after reaching a certain point the entanglement entropy must start to decrease to eventually zero. Let us consider that after 'N' steps the entanglement entropy reaches a certain point then after another 'M' steps it decreases to eventually zero following,

$$S_N + S_M = 0 \tag{7.2}$$

In chapter 4, we found that the entanglement entropy of b_{n+1} quanta with the rest of the system at timestep t_{n+1} to be,

$$S_{entanglement}(t_{n+1}) = \ln 2 - 2[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 + \text{higher order terms} \tag{7.3}$$

Note that we have used the same density matrix of (5.39) as the density matrix of b_{n+1} as they are essentially the same. We shall now derive a bound that shows that considering small corrections to the leading order Hawking state can not lead to a page like curve. First, let us consider the state on the spacelike slice at timestep t_{n+2} as,

$$\begin{aligned} |\Psi_{M,c},\psi_b(t_{n+2})\rangle = & \left[|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right]\otimes \\ & \left[|0\rangle_{c_{n+2}}|0\rangle_{b_{n+2}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{n+2}}|1\rangle_{b_{n+2}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right] \end{aligned} \tag{7.4}$$

Here, we have retained the small correction in the next timestep. Similar calculations as before can show that the reduced density matrix for (b_{n+1},b_{n+2}) is,

$$\begin{aligned} \rho_{(b_{n+1},b_{n+2})} = & \frac{1}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix} Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle & 0 & 0 & 0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle\end{pmatrix} \\ & + \begin{pmatrix} [Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 & 0 & 0 & 0\\ 0 & -[Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 & 0 & 0\\ 0 & 0 & -[Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 & 0\\ 0&0&0&[Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2\end{pmatrix} \\ & + O(\varepsilon^2) + O(\varepsilon^4) \end{aligned} \tag{7.5}$$

Thus, the entanglement entropy of (b_{n+1},b_{n+2}) with the rest of the system is given by,

$$S_{entanglement}(t_{n+2}) = 2\ln 2 - 4[Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 + \text{higher order terms} \tag{7.6}$$

Furthermore, after N such steps the wave function becomes,

$$\begin{aligned} |\Psi_{M,c},\psi_b(t_N)\rangle = & \left[|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right]\otimes \\ & \left[|0\rangle_{c_{n+2}}|0\rangle_{b_{n+2}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{n+2}}|1\rangle_{b_{n+2}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right]\otimes \\ & \dots \\ & \otimes\left[|0\rangle_{c_N}|0\rangle_{b_N}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_N}|1\rangle_{b_N}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right] \end{aligned} \tag{7.7}$$

The entanglement entropy of $(b_{n+1},b_{n+2},\dots b_N)$ with the rest of the system is then given by,

$$S_N = N\ln 2 - 2N[Re\,\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 + \text{higher order terms} \tag{7.8}$$

After this step we are assuming that the entanglement entropy starts to decrease. Let us look at the state in the time step t_{N+1},

$$\begin{aligned} |\Psi_{M,c},\psi_b(t_{N+1})\rangle = & \left[|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right]\otimes \\ & \left[|0\rangle_{c_{n+2}}|0\rangle_{b_{n+2}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{n+2}}|1\rangle_{b_{n+2}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right]\otimes \\ & \dots \\ & \otimes\left[|0\rangle_{c_N}|0\rangle_{b_N}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_N}|1\rangle_{b_N}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right] \\ & \otimes\left[|0\rangle_{c_{N+1}}|0\rangle_{b_{N+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}+\Lambda^{(2)}\rangle + |1\rangle_{c_{N+1}}|1\rangle_{b_{N+1}}\frac{1}{\sqrt{2}}|\Lambda^{(1)}-\Lambda^{(2)}\rangle\right] \end{aligned} \tag{7.9}$$

The entanglement entropy of $(b_{n+1}, b_{n+2}, \ldots b_N, b_{N+1})$ with the rest of the system becomes,

$$S_{N+1} = (N+1)\ln 2 - 2(N+1)[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 + \text{higher order terms} \tag{7.10}$$

Hence, after 'M' such steps we have the entanglement entropy of $(b_{n+1}, b_{n+2}, \ldots b_N, b_{N+1}, \ldots b_M)$ with the rest of the system as,

$$S_M = (N+M)\ln 2 - 2(M+N)[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 + \text{higher order terms} \tag{7.11}$$

For physical significance, we further assume that after N steps the small correction state dominates and the leading order Hawking state becomes small, i.e $\Lambda^{(1)} \sim \varepsilon$ and $\Lambda^{(2)} \sim O(1)$.Then by the condition (5.24) we get,

$$S_N = N\ln 2 - 2N\varepsilon_2^2 + \text{higher order terms} \tag{7.12}$$

Since, we are assuming that our entanglement entropy decreases from $(N+1)$ step to M step, we can use the equation (7.2) and deduce that $S_M = 0$ and get,

$$\begin{aligned}
& S_M = 0 \\
& (N+M)\ln 2 + 2(M+N)[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 + \text{higher order terms} = 0 \\
& S_N = -M\ln 2 + 2M[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 \\
& N\ln 2 - 2N\varepsilon_2^2 = -M\ln 2 + 2M[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 \\
& N\ln 2 + M\ln 2 - 2M[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 = 2N\varepsilon_2^2 \\
& \ln 2 + \frac{M}{N}\ln 2 - \frac{2M}{N}[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 = 2\varepsilon_2^2 \\
& \frac{\ln 2}{2} + \frac{M}{2N}\ln 2 - \frac{M}{N}[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 = \varepsilon_2^2 < \varepsilon \\
& \frac{\ln 2}{2} + \frac{M}{2N}\ln 2 - \varepsilon < \frac{M}{N}[Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2 \\
& \frac{N\ln 2}{2M} + \frac{1}{2}\ln 2 - \frac{N}{M}\varepsilon < [Re\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle]^2
\end{aligned} \tag{7.13}$$

From the above result we see that the leading order Hawing state dominates even if we consider infinite M steps. Therefore, no such scenario can happen where we can follow a page-like curve where the leading order Hawking state dominates in N steps and then it becomes small decreasing the entropy. Hence, we get to the same conclusion as [30] that the entanglement entropy of the radiated quanta at any successive step can not decrease to zero leaving the paradox at bay.

7.2 The Proposed Wave Function

One of the key assumptions made by Mathur in [30] is that interactions among the created pairs happen locally. Such locality is violated when we consider that the entanglement after N steps goes down in order to follow a page-like curve. This is why we saw in (7.13) that the leading order Hawking state can not be of order ε. Nevertheless, we saw that considering QFT like locality and unitarity will still lead to the information paradox. In addition, considering small correction does not resolve the paradox either. Therefore, we need to consider a different wave function. In addition, we assume non-local interaction among the created pairs and see where that lead us to. The usual entangled bell pair wave function is like the following,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_b|0\rangle_c + |1\rangle_b|1\rangle_c) \tag{7.14}$$

and in n time steps our proposed toy-model of a wave function would take the following form,

$$|\Psi\rangle_n = \bigotimes_n \frac{\exp\left(\sum_{i=0}^{n} \mu_i G - \alpha_i r_i\right)}{\sqrt{2}} \left(|0\rangle_{b_n} |0\rangle_{c_n} + |1\rangle_{b_n} |1\rangle_{c_n}\right) \tag{7.15}$$

Note that when the terms in the exponential become zero we are left with the initial leading order Hawking state. Furthermore, we have non local interaction terms embedded inside the exponential that will essentially resolve the paradox as we shall see. Let us look at the terms in the exponential,

- The term μ is the rate at which the internal mass states of the black hole decreases.
- G is the usual gravitational constant in 4-dimension.
- The term α can be thought of as the rate of particle production.
- r is the distance of each emitted quanta ('b' quanta considered previously) from the horizon.

The non-local gravitational interaction between the freshly formed particle pairs and the mass state inside the black hole is provided by the total term μG. On the other hand, the non-local gravitational effects between succeeding hawking radiated particles generated at each time step are provided by the quantity αr. Furthermore, we expect newly formed particles to have some effect on the already emitted particles. Whereas, Mathur assumed that the newly created particles have no effect on the previously radiated particles as they have already left the vicinity of the black hole. This is because he assumed local interaction among the pairs so that when the radiated pairs left the vicinity of the black hole, they can be collected outside and no effect from the newly created pairs can occur. However, since we are considering non-local gravitational interaction between the radiated pairs, we can expect action at a distance from the newly created pairs. One point to mention is that this is an ansatz. The motivation behind the terms in the exponential comes from the fact that we need a correlation between the mass state of the black hole and the radiated quanta, which is the first term μG and the correlation between the radiated quanta and the infalling quanta is the second term αr. The minus sign in front of the second term indicates that the non-local gravitational effect between the correlated pairs decreases over distance. Furthermore, we will see that choosing such wave-function can lead to a decrease in the entanglement entropy which is the purpose of this dissertation.

7.3 The Entanglement Entropy

We shall now see how this toy model follows a page-like curve. First of all, notice that for the very first particle production the wave function is given by the leading order Hawking state of (7.14). This is because prior to this state there are no other pairs to have any local effect on the produced pairs. The wave function in (7.24) comes into play after the first pair production.
Let us denote,

$$A_n = \sum_{i=1}^{n} (\mu_i G - \alpha_i r_i) \tag{7.16}$$

The wave function in (7.24) then becomes,

$$|\Psi\rangle_n = \bigotimes_n \frac{\exp(A_n)}{\sqrt{2}} \left(|0\rangle_{b_n} |0\rangle_{c_n} + |1\rangle_{b_n} |1\rangle_{c_n}\right) \tag{7.17}$$

The reduced density matrix for the emitted particles b_n is as follows,

$$\begin{aligned}
\rho_{b_n} &= tr_{c_n}(|\Psi\rangle_n \langle\Psi|) \\
\rho_{b_n} &= \bigotimes_n \frac{1}{2} exp(2A_n) \left[|0\rangle_{b_n} \langle 0| + |1\rangle_{b_n} \langle 1|\right] \\
\rho_{b_n} &= \frac{1}{2^n} exp(2A_n) \delta_{ij} \delta_{kl} \ldots \delta_{mn}
\end{aligned} \tag{7.18}$$

Therefore, at timestep t_n, the entanglement entropy of all the b_n emitted particles with the rest of the system is as follows,

$$
\begin{aligned}
S_n =& -tr(\rho_{b_n} \ln \rho_{b_n}) \\
S_n =& -tr\left(\frac{1}{2^n} exp(2A_n)\delta_{ij}\delta_{kl}\ldots\delta_{mn} \ln\left[\frac{1}{2^n} exp(2A_n)\delta_{ij}\delta_{kl}\ldots\delta_{mn}\right]\right) \\
S_n =& -tr\left(\frac{1}{2^n} exp(2A_n) \ln\left[\frac{1}{2^n} exp(2A_n)\right]\delta_{ij}\delta_{kl}\ldots\delta_{mn}\right) \\
S_n =& -tr\left(\frac{1}{2^n} exp(2A_n) \ln\left(\frac{1}{2^n}\right)\delta_{ij}\delta_{kl}\ldots\delta_{mn} + \frac{1}{2^n} exp(2A_n) \ln\left[exp(2A_n)\right]\delta_{ij}\delta_{kl}\ldots\delta_{mn}\right) \\
S_n =& -tr\left(\frac{1}{2^n} exp(2A_n) \ln\left(\frac{1}{2^n}\right)\delta_{ij}\delta_{kl}\ldots\delta_{mn} + 2^{1-n} A_n exp(2A_n)\delta_{ij}\delta_{kl}\ldots\delta_{mn}\right) \\
S_n =& -tr\left(2^{1-n} A_n exp(2A_n)\delta_{ij}\delta_{kl}\ldots\delta_{mn} - \frac{n}{2^n} exp(2A_n)\ln(2)\delta_{ij}\delta_{kl}\ldots\delta_{mn}\right) \\
S_n =& \frac{n2^n}{2^n} exp(2A_n)\ln 2 - 2^n 2^{1-n} A_n exp(2A_n) \\
S_n =& n exp(2A_n)\ln 2 - 2A_n exp(2A_n)
\end{aligned} \tag{7.19}
$$

from the above result we see that the entropy is not maximally entangled as it was for the leading order hawking state calculation in (4.6). However, the emitted particles are still highly entangled with the rest of the system. We expect this to change over time because when the second term in A_n dominates, the term in the exponential becomes negative leading to a decrease in the entanglement entropy.
At the next time step, we expect the wave function to be,

$$
|\Psi\rangle_{n+1} = \bigotimes_{n+1} \frac{exp\,(A_{n+1})}{2}\left(|0\rangle_{b_{n+1}}|0\rangle_{c_{n+1}} + |1\rangle_{b_{n+1}}|1\rangle_{c_{n+1}}\right) \tag{7.20}
$$

the reduced density matrix describing the set (b_{n+1}) is given as,

$$
\begin{aligned}
\rho_{(b_{n+1})} =& tr_{c_{n+1}}(|\Psi_{n+1}\rangle\langle\Psi_{n+1}|) \\
=& \bigotimes_{n+1} \frac{1}{2^{n+1}} exp\,(2A_{n+1})\left(|0\rangle_{b_{n+1}}\langle 0| + |1\rangle_{b_{n+1}}\langle 1|\right) \\
=& \frac{1}{2^{n+1}} exp\,(2A_{n+1})\,\delta_{ij}\delta_{kl}\ldots\delta_{mn}\delta_{ab}
\end{aligned}
$$

Similar calculations as before shows that the entanglement entropy of the set (b_{n+1}) with the rest of the system after t_{n+1} steps is as follows,

$$
S_n + S_1 = (n+1)exp(2A_{n+1})\ln 2 - 2A_{n+1}exp(2A_{n+1}) \tag{7.21}
$$

We are assuming that after 'n' time steps the entanglement entropy starts to decrease and in the next 'm' time steps the entanglement entropy decreases tensed to zero leading to a page-like curve till the black hole evaporates completely. This is shown in the figure 7.1, where we see that the entanglement entropy increases in the first n time steps then slowly it decreases downward in the next 'm' time step following a page like curve.[1]
Hence, after 'm' such steps we get the entanglement entropy as,

$$
S_n + S_m = (n+m)exp(2A_m)\ln 2 - 2A_m exp(2A_m) \tag{7.22}
$$

[1]The values of m are taken from the interval $[1,m]$, where $m \in \mathbb{Z}^+$.

Since, we require that the entropy goes to zero as the hole evaporates. Then by (7.2) we have,

$$\begin{aligned} &S_n + S_m = 0 \\ &(n+m)exp(2A_m)\ln 2 + 2A_m exp(2A_m) = 0 \\ &nexp(2A_m)\ln 2 = -mexp(2A_m)\ln 2 - 2A_m exp(2A_m) \end{aligned} \tag{7.23}$$

The equation in (7.23) gives us a relation between the time steps n and m. This relation will help us generate the numerical model for our toy model in future section.

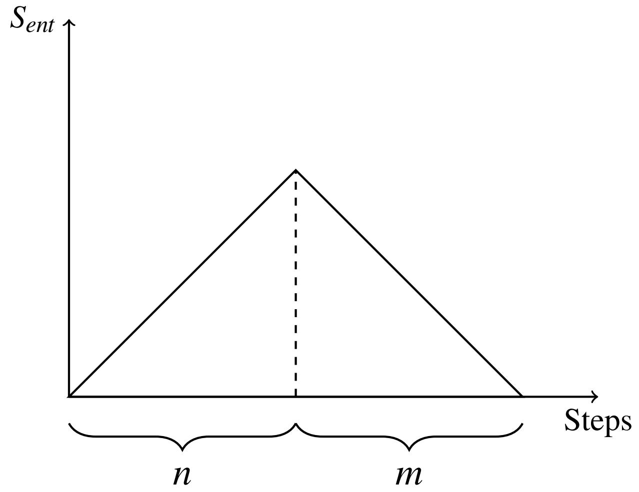

Figure 7.1: Page Curve

7.4 Dimensions of μ and α

In natural units we set $c = \hbar = 1$ which leads to $[M] = \frac{1}{[T]}$ and $[L] = [T]$.
The wave function that we have proposed looks like,

$$|\Psi\rangle_n = \bigotimes_n \frac{\exp\left(\sum_{i=0}^{n} \mu_i G - \alpha_i r_i\right)}{\sqrt{2}} \left(|0\rangle_{b_n} |0\rangle_{c_n} + |1\rangle_{b_n} |1\rangle_{c_n}\right) \tag{7.24}$$

Where we have denoted the term in the exponential as,

$$A_n = \sum_{i=0}^{n} \mu_i G - \alpha_i r_i \tag{7.25}$$

Since exponential should be dimensionless we want A_n to be dimensionless.
In contrast, to make A_n dimensionless we need both of the terms in (7.25) to be dimensionless. However, by definition, μ is the mass decrease rate of the black hole. Therefore, it should have, $\mu = \frac{dm}{dt} = \frac{[M]}{[T]} = \frac{1}{[T]^2}$. Let us then check for μ,

$$\begin{aligned} [\mu] &= \frac{1}{[G]} \\ &= \frac{[M][T]^2}{[L]^3} \\ &= \frac{[T]^2}{[T]^3[T]} \\ &= \frac{1}{[T]^2} \end{aligned} \tag{7.26}$$

Which is what we expect. Similarly for α,

$$\begin{aligned}[\alpha] =& \frac{1}{[r]}\\ =& \frac{1}{[L]}\\ =& \frac{1}{[T]}\end{aligned} \tag{7.27}$$

Notice that the dimensional analysis above prescribes both α and μ reflects rate terms which is what we have proposed in the first place.

7.5 Relationship Between μ and α

The non-local interaction terms in the proposed wave-function of (7.24) is vaguely defined in the sense that we do not know how μ and α change over time. In addition, we do not have an expression relating μ and α either. In order to capture the complete Toy-model, we need to first find a relationship between μ and α, then an expression suggesting their change over time. We shall be focusing on the relation in this section.
For the $i = 1$ case, we know that our wave function takes the form of (7.14) which is the leading order hawking state. For which we do not have any correlation between μ and α as they do not play any role in this state. However, as stated before, after the first pair emission μ and α will be non-zero and we will have their impact on the wave function. In contrast, we shall try to find a relationship between μ and α from the normalization condition,

$$\langle \Psi_i | \Psi_j \rangle_n = \delta_{ij} \Lambda_n^m (UV) \tag{7.28}$$

Notice that the proposed wave function is not normalizable to unity as we do not have a well defined expression of α. In contrast, the proposal given is a conjecture since we need a full description of quantum gravity in order to incorporate the effects of quantum gravity into α. This is one the limitations of the wave function. Therefore, we take the normalization to be some number Λ_{UV}(has dimension of length) that is relevant in our energy scale and we expect the wave function to be well defined within that domain. Furthermore, m represents the dimension of the space in which we are normalizing our wave function. For instance, if we are in a spherical symmetric system, m =3 as we are integrating over volume dv. Taking the inner product of (7.24) we get,

$$\begin{aligned}\langle \Psi | \Psi \rangle_n =& \int_0^\infty (4\pi)^{n-1} \exp\left(\sum_{i=1}^{n-1} 2\mu_i G\right) \prod_{i=1}^{n-1} r_i^2 \exp\left(-2\alpha_i r_i\right) dr_i\\ \langle \Psi | \Psi \rangle_n =& (4\pi)^{n-1} \exp\left(\sum_{i=1}^{n-1} 2\mu_i G\right) \int_0^\infty \prod_{i=1}^{n-1} r_i^2 \exp\left(-2\alpha_i r_i\right) dr_i\\ \langle \Psi | \Psi \rangle_n =& (4\pi)^{n-1} \exp\left(\sum_{i=1}^{n-1} 2\mu_i G\right) \prod_{i=1}^{n-1} \frac{2!}{(2\alpha_i)^3}\end{aligned} \tag{7.29}$$

Applying the normalization condition of (7.28) to (7.29) we get,

$$\begin{aligned}
\prod_{i=1}^{n-1} \Lambda_n^3(UV)\frac{(2\alpha_i)^3}{2!} =&(4\pi)^{n-1}\exp\left(\sum_{i=1}^{n-1} 2\mu_i G\right) \\
\prod_{i=1}^{n-1} \ln\left[\Lambda_n^3(UV)\frac{(2\alpha_i)^3}{2!}\right] =&\ln\left[(4\pi)^{n-1}\exp\left(\sum_{i=1}^{n-1} 2\mu_i G\right)\right] \\
\prod_{i=1}^{n-1} \ln\left[\frac{(2\alpha_i)^3}{2!}\right] + \ln\left[\Lambda_n^3(UV)\right] =&(n-1)\ln 4\pi + \sum_{i=1}^{n-1} 2\mu_i G \\
\implies \sum_{i=1}^{n-1} \mu_i =&\prod_{i=1}^{n-1} \frac{\ln\left[\frac{(2\alpha_i)^3}{2!}\right]}{2G} - \frac{(n-1)\ln 4\pi}{2G} + \frac{\ln\left[\Lambda_n^3(UV)\right]}{2G}
\end{aligned} \tag{7.30}$$

Now that we have established a relation between μ and α, let us look at what we yield at for different values of n.
For the case $n = 1$ i.e the first pair emission, we have $\mu_0 = \alpha_0 = 0$.
For $n = 2$ i.e second pair emission,

$$\begin{aligned}
\mu_1 =&\frac{\ln\left[\frac{(2\alpha_1)^3}{2!}\right]}{2G} - \frac{\ln 4\pi}{2G} + \frac{\ln[\Lambda_1^3(UV)]}{2G} \\
\mu_1 =&\frac{\ln\left[\frac{(2\alpha_1)^3\Lambda_1^3(UV)}{8\pi}\right]}{2G} \\
\mu_1 =&\frac{\ln\left[\frac{(\alpha_1)^3\Lambda_1^3(UV)}{\pi}\right]}{2G}
\end{aligned} \tag{7.31}$$

For $n = 3$ i.e third pair emission, we have,

$$\begin{aligned}
\mu_1 + \mu_2 =&\frac{\ln\left[\frac{(2\alpha_1)^3(2\alpha_2)^3}{2!2!}\right]}{2G} - 2\frac{\ln 4\pi}{2G} + \frac{\ln[\Lambda_1^3(UV)\Lambda_2^3(UV)]}{2G} \\
\mu_1 + \mu_2 =&\frac{\ln\left[\frac{(2\alpha_1)^3(2\alpha_2)^3}{2!2!}\right]}{2G} - \frac{\ln 16\pi^2}{2G} + \frac{\ln[\Lambda_1^3(UV)\Lambda_2^3(UV)]}{2G} \\
\mu_1 + \mu_2 =&\frac{\ln\left[\frac{(2\alpha_1)^3(2\alpha_2)^3\Lambda_1^3(UV)\Lambda_2^3(UV)}{64\pi^2}\right]}{2G} \\
\mu_2 =&\frac{\ln\left[\frac{(\alpha_1)^3(\alpha_2)^3\Lambda_1^3(UV)\Lambda_2^3(UV)}{\pi^2}\right]}{2G} - \mu_1 \\
\mu_2 =&\frac{\ln\left[\frac{(\alpha_1)^3(\alpha_2)^3\Lambda_1^3(UV)\Lambda_2^3(UV)}{\pi^2}\right]}{2G} - \frac{\ln\left[\frac{(\alpha_1)^3\Lambda_1^3(UV)}{\pi}\right]}{2G} \\
\mu_2 =&\frac{\ln\left[\frac{(\alpha_2)^3\Lambda_2^3(UV)}{\pi}\right]}{2G}
\end{aligned} \tag{7.32}$$

Similarly, for $n = 4$ case i.e the fourth pair emission,

$$\mu_3 =\frac{\ln\left[\frac{(\alpha_3)^3\Lambda_3^3(UV)}{\pi}\right]}{2G} \tag{7.33}$$

Therefore, we can establish that,

$$\mu_i = \frac{\ln\left[\frac{(\alpha_i)^3\Lambda_i^3(UV)}{\pi}\right]}{2G} \tag{7.34}$$

The expression above gives us an idea of how μ is dependent on α. Furthermore, both μ and α terms are dynamical as they are dependent on the size of the black hole. We shall explore this feature in the next section. In addition, we can see that the dimensions of μ and α checks out. Furthermore, we assume that both α and μ goes to zero as the black hole evaporates as physically there is no mass to decrease and there will be no Hawking pairs either.

7.6 Expressions for μ and α

If we treat our black hole as a classical black hole then we can establish an expression for μ from the hawking radiation. Since classical black holes are treated as thermodynamic black bodies, we can estimate a rate of change of the mass by simply applying hawking radiation and thermodynamics to the system. From black body radiation, the Stefan-Boltzmann radiation law is given as,

$$\frac{dE}{dt} = A\sigma T_H^4 \tag{7.35}$$

Here, $\frac{dE}{dt}$ is the rate of radiated energy of the black hole, A is the area of the black hole, σ is the Stefan- Boltzmann constant and T_H is the hawking temperature. The area of the black hole, $A = 4\pi r^2$ if we treat it as a spherical mass, where r is the radius of black hole extended to the horizon. Since the horizon is found at $\frac{2GM}{c^2}$, $r = \frac{2GM}{c^2}$. Furthermore, the hawking temperature is $T_H = \frac{\hbar c^3}{8\pi k_B GM}$. For a black hole, the decrease in mass is being radiated away as photons in forms of energy. Therefore, From the Einstein's mass-energy equivalence principle, $E = Mc^2$ we can get the following relation,

$$-\frac{dM}{dt} = \frac{1}{c^2}\frac{dE}{dt} \tag{7.36}$$

Where, the negative sign incorporates mass loss due to radiated energy. Putting all these together we get,

$$\begin{aligned}
-\frac{dM}{dt} =& \frac{A\sigma T_H^4}{c^2} \\
\frac{dM}{dt} =& -\frac{4\pi(2GM)^2\hbar^4 c^{12}(\pi^2 k_B^4)}{(8\pi k_B GM)^4(60\hbar^3 c^8)} \\
M^2 dM =& -\kappa dt \\
\int_{M_0}^{M} M^2 dM =& -\int_0^t \kappa dt \\
M^3 - M_0^3 =& -\kappa t \\
M^3 =& M_0^3 - \kappa t \\
M(t) =& \sqrt[3]{M_0^3 - \kappa t}
\end{aligned} \tag{7.37}$$

Here, $\kappa = \frac{\hbar c^4}{15360\pi G^2}$ is a constant, M_0 is the initial mass of the black hole and $M(t)$ is the mass of the black hole at any particular time t. The above result was derived by Hawking in [12] using thermodynamic principles that was used to estimate the lifetime of a black hole.
Let us now define μ as $\mu = -\frac{dM}{dt}$.Similar calculation in (7.37) reveals,

$$\begin{aligned}
\mu =& \frac{A\sigma T_H^4}{c^2} \\
\implies \mu(t) =& \frac{\kappa}{M(t)^2}
\end{aligned} \tag{7.38}$$

we see that $\mu \sim \frac{1}{M^2}$. The above result makes sense as we expect the rate of mass decrease to be slower for massive black holes and dimensionally it checks out as well. In addition, note that μ is a

dynamical term in our wave function. The relation in (7.38) will help us in generating a numerical model in the next section. From (7.34) we can now get an expression for α as well.
From (7.34) we can find that,

$$\alpha_i = \sqrt[3]{\frac{\pi \exp(2G\mu_i)}{\Lambda_i^3(UV)}} \tag{7.39}$$

The above result in (7.39) tells us how particle emission rate is dependent on mass decrease rate. Furthermore, due to the exponential in the expression, we expect higher particle emission rate as the black hole loses its mass.
Now that we have well defined terms for the wave function in (7.24) we can generate a numerical model that will show us how entropy decreases over time leading to a page-curve solution.

7.7 A Numerical Model of the Wave Function

The toy-model that we have proposed will look something like figure 7.2. Here $r = 0$ is the singularity at the center and $r = \frac{2GM}{c^2}$ is the horizon. The particles that are outside the horizon are the radiated quanta or emitted particles. the emitted particles are at different distance 'r_n' as we can see. One point to note here is that the distance of the emitted particle, 'r' is an independent parameter. To determine the values of r precisely we need new physics i.e a quantum theory of black holes perhaps as we do not yet have a theory for non-local effects of gravity or quantum gravity for black holes. Furthermore, so far from QFT we know that quantum fluctuations happen randomly in space-time. Therefore, in our model we skip the calculation of 'r' and estimate random values of r_n numerically.

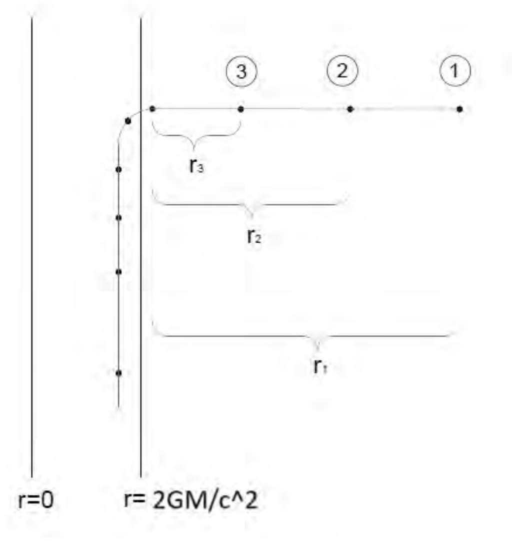

Figure 7.2: Toy-Model

7.7.1 Parameters of the Model

To develop a numerical model we need to properly define all the parameters involved. For starters, let us consider the equation of (7.38) and generalize it for i'th particle emission as,

$$\mu_i(t) = \frac{\kappa}{M_i(t)^2} \tag{7.40}$$

where, $M_i(t)$ is the mass of the black hole at time t_i,

$$M_i(t) = \sqrt[3]{M_0^3 - \kappa t_i} \tag{7.41}$$

Furthermore, from (7.39) we have the relation,

$$\alpha_i = \sqrt[3]{\frac{\pi \exp(2G\mu_i)}{\Lambda_i^3(UV)}} \tag{7.42}$$

Consecutively, The relation in (7.16) forms to,

$$\begin{aligned}
A_n &= \sum_{i=0}^{n} (\mu_i G - \alpha_i r_i) \\
A_n &= \sum_{i=0}^{n} \left(\mu_i G - \sqrt[3]{\frac{\pi \exp(2G\mu_i)}{\Lambda_i^3(UV)}} r_i \right) \\
A_n &= \sum_{i=0}^{n} \left(x_i - \frac{r_i}{\Lambda_i(UV)} \sqrt[3]{\pi exp(2x_i)} \right)
\end{aligned} \tag{7.43}$$

Here, we have set $x_i = \mu_i G$. The entanglement entropy found previously is as follows,

$$S_n = n exp(2A_n) \ln 2 - 2A_n exp(2A_n) \tag{7.44}$$

With the defined parameters above we will constructively generate a numerical model that will show the decrease in entanglement entropy over time. It might seem as both x_i and r_i are independent parameters but, in truth, we have only two independent parameters, namely, time (t) embedded in μ_i and the distance of the emitted particles from the horizon (r). We will see next how we should tune such parameters to get the desired result.

7.7.2 Restrictions on the Parameters

We need our model to be physically realistic in accordance with astrophysics. To achieve such a goal we need to implicate the following restrictions on the parameters we have found so far,

- The dynamic mass term of the black hole, $M_i(t)$, should always be greater than the Planck mass i.e. $(M_i(t) >> m_p)$. Otherwise, we will encounter remnant scenario or extremal black hole case, which we are trying to avoid as we do not want quantum effects to dominate in our model.

- The initial black hole mass M_0 should be higher than $1.8\mathrm{M}_\odot$ as the Chandrasekhar limit suggests [49][51]. However, later developments suggest that black holes can have any mass [54]. In addition, we set the initial values, μ_0=α_0=0 as we want the wave function to be the leading order hawking state of (7.14) for the creation of the first pair.

- Black holes have extremely long life time. In fact, some black holes live as long as the age of the universe! Therefore, appropriately setting the time interval in which we will observe the evolution of the entanglement entropy is crucial. We need longer time intervals to observe the change in entropy. If the time interval is too small or too large, we might miss the evolution of entropy completely.

- The entanglement entropy can only decrease if A_n decreases over time. We will see this in the next section. Therefore, we need to properly adjust the values of r_i so that we can indeed get a decrease in entropy. We can do this because QFT permits random quantum fluctuations in space-time resulting in random particle pair creation at random distances from the black hole.

7.7.3 Qualitative Discussion of the Model

The toy-model of a wave function that we have proposed in (7.24) is a decreasing wave function as the terms in it suggest. However, the entanglement entropy found in (7.19) has some complications. If we reorganize the equation a bit we end up with,

$$S_n = (n \ln 2 - 2A_n) \exp(2A_n) \tag{7.45}$$

Here, we have two product terms. the one with the exponential however will vary much faster than the other term. Therefore, when the term in the exponential gets much smaller we expect a rapid decrease in the entanglement entropy. To illustrate, let us look at some numerical examples.

Early Stage:

Let us look at the entanglement entropy at the start of the black hole's lifetime. Suppose that the black hole has initial mass $M_0 = 5\mathrm{M}_{\odot}$.Then from equation (7.43) we have,

$$x = 2.64 \times 10^{-57} \tag{7.46}$$

then,

$$exp(2x) \approx 1 \tag{7.47}$$

Thus,

$$A = -1.4645918875615234 \tag{7.48}$$

here and later in the next examples, we will use $r_i = \Lambda_i(UV) = 1m$ for simplicity. Therefore we get,

$$S_1 = (\ln 2 + 2.929182)\, exp(-2.929182) \approx 0.19357993306 \tag{7.49}$$

Mid Stage:

Let us now see what happens to the entanglement entropy at the mid life of the black hole. For simplicity we exclude the sum in the terms. The life time of the black hole can be equated from equation (7.37),

$$t = 2.5227 \times 10^{77} s \tag{7.50}$$

The midlife will be at, $t_{1/2} = 1.26135 \times 10^{77} s$. Again from equation (7.37) we can get the mass of the black hole at its midlife to be,

$$M_{1/2} \approx 3.965\mathrm{M}_{\odot} \tag{7.51}$$

Then,

$$x_{1/2} \simeq 4.2029530141576115 \times 10^{-57} \tag{7.52}$$

Thus,

$$A_n = -1.4645918875615234 \tag{7.53}$$

Therefore we get,

$$S_n = (n \ln 2 + 2.929182)\, exp(-2.929182) \approx (n \ln 2 + 2.929182)\, 0.05344073479 \tag{7.54}$$

Which suggests that it is a steep line of increasing entanglement entropy. suppose we reach this stage at n=50th particle emission. then,

$$S_{50} \approx 2.00865237075 \tag{7.55}$$

Which is indeed increasing.

Later Stage:

Suppose the black hole is at a stage where its mass becomes $M \approx 10^{\frac{5}{2}} kg$. Then ,

$$x \approx 2.643022799 \quad (7.56)$$

and

$$A_n \approx -5.886998244499 \quad (7.57)$$

Therefore,

$$S_n = 0.00000763343\,(n\ln 2 + 11.7829721448) \quad (7.58)$$

suppose we reach this stage at 100th particle emission, then,

$$S_{100} \approx 0.00061905354 \quad (7.59)$$

Which shows a significant decrease in the entanglement entropy! Therefore, we see that the proposed toy-model behaves the way we want it to. Furthermore, Let us suppose that we are almost at the end of the black holes lifetime. Keeping the restrictions on the parameters stated in 6.6.2, we are assuming that the mass of the black hole is, $M_t = 10^4 kg$. then,

$$x \approx 26.4398 \quad (7.60)$$

then,

$$exp(2x) \approx 9.2324823 \times 10^{22} \quad (7.61)$$

Which is a huge number! Therefore,

$$A_t = -66194607.9254 \quad (7.62)$$

Therefore we get,

$$S_t = 0 \quad (7.63)$$

The exponential term decreases abruptly and we are left with no entanglement at all. Therefore, we see that after a certain time the entanglement entropy definitely starts to decrease as the exponential term dictates over the other product term and eventually, abruptly the entanglement entropy goes to zero even way before the black hole has evaporated. The wave function therefore, caries out information exponentially leaving the entanglement eventually out of the equation.

Chapter 8

conclusion

We have demonstrated that taking into account non-local gravitational interaction is a first step in resolving the information paradox connected to black holes. There are several opportunities for more research because the formulation of quantum extremal surfaces and the search for a quantum gravity using quantum field theory are still in their infancy. Because of its uncertain nature, the gravitational interaction term (α) has not been calculated in this dissertation. This model provides a continuation of Mathur's work in [30] where, we have shown that considering a drastic change of the quantum system does indeed lead to a decrease in the entanglement entropy. The black hole information paradox has been one of the longest unresolved problems in physics. It is compared by some theorists to the problem of ultra-violate catastrophe in classical physics that gave birth to quantum physics. It is therefore, believed that new physics is required to solve the information paradox. The importance of formulating a quantum theory of gravity thus can not be stressed any further.

We started this dissertation by introducing some of the necessary background on the information paradox. Then we re-derived some of the important foundings of Mathur in [30]. Furthermore, we have shown that considering any small correction to the leading order Hawking state can not lead to a page-curve resolution. Finally, we showed that the proposed wave function indeed decreases the entanglement entropy after a specific page time; at a specific mass of the black hole ($10^{\frac{5}{2}}$kg).The very specific mass arises due to a numerical modeling. In general, we expect the entanglement to decrease at smaller black hole mass as we expect the quantum effects to dominate over that energy scale, making quantum gravitational effects much stronger. Therefore, the decrease in entropy at some very late stage of the black holes lifetime is not unexpected.

Finally, let us address some of the limitations of the proposed wave-function. While our toy-model presented above does show a decrease in entropy there are certain subtleties that needs to be addressed.

One of the limitations of the presented wave function is that at each successive time step we get only one particle pair creation. We have considered this to handle numerical calculations in a more manageable manner. If further extension to the wave function is required it is suggested to span the vector space in a suitable manner.

Another issue is the manifestation of non-local gravitational interaction terms in our model. Since we still do not have a full description of quantum gravity up to the date this manifestation is only a conjecture.

Furthermore, we run into the normalization problem where we have specified the term $\Lambda(UV)$ in (7.28) in order to incorporate the effects of quantum gravity in our model. We are ambiguous about how $\Lambda(UV)$ behaves as it is only a conjecture. Therefore. for simple calculation purpose we have omitted the specific value of $\Lambda(UV)$ and set it to unity. Notice that the unitarity of the the wave-function depends on the value of $\Lambda(UV)$. Therefore, a quantum theory of gravity is required

to manifest unitarity of our model.
One of the other issues is the calculation of α. We have only established how α is related to μ. However, due to its unknown nature , we were not able to derive a full expression of α based on non-local gravitational interaction.
Finally, we need to address the change in entanglement entropy. Notice that the entanglement entropy keeps rising until the black hole reaches a certain mass; for our model it is at $10^{\frac{5}{2}} kg$, which is almost at the end of black holes lifetime. Although it is expected that the decrease in the entanglement entropy happens much later in black hole lifetime, the page curve for this model does not behave nicely as we reach at an abrupt decrease of the entropy right at the end of the black holes lifetime. In comparison with the existing solutions such as the Ads/CFT correspondence this result might incorporate that the missing puzzle in semi-classical theory(Quantum Gravity Theory) is playing a vital role in the unitarity of the quantum system.
Furthermore, our model is not aligned with string theory as we are working in a semi-classical regime. In hindsight, our model is still a conjecture that shows that the entanglement entropy can be decreased considering the specific conditions such as the drastic deviation of the wave function from the leading order Hawking state. Hence, we agree with Mathur[30] that no small correction to the leading order Hawking state can lead to a decrease in entropy.
A graph can be generated with the numerical model prescribed above to specify page time and the page curve itself which can be an extension to this work.

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