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The smearing function from the functional renormalization group

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Abstract. The smearing function has been proposed as a tool for writing down the one-loop correction to the Wilsonian effective action in the form of the Schwinger proper-time representation of the functional determinant. Due to the fact that the existence of the smearing function is only a postulate, it is interesting to explore if it is possible to obtain such a function from the functional renormalization group, which is a closely related formulation. In this work, we show that it is possible to obtain the smearing function from the functional renormalization group only in a special case, but not in general cases.

1. Introduction

Renormalization group is a tool for determining how physical quantities change with energy scales. There are many versions of the renormalization group, but the first version, that was motivated by the block-spin picture in statistical physics, is the Wilsonian renormalization group, in which the high-momentum modes are integrated out in the functional integral leaving the result that describes physics at low-energy scales [1]. In practical calculations, the Wilsonian renormalization group is hard to deal with, since it gives the recurrence relations of the coupling constants which are very tedious to solve. There is, however, another version of the renormalization group, which resembles the idea of block spin and gives a differential equation, instead of the recurrence relations, for determining how the physics changes with energy scales. Such a theory is known as the functional renormalization group [2], and the differential equation which lies at the very heart of it is known as the Wetterich equation.

Soon after the proposal of the functional renormalization group, another version of the renormalization group was proposed in [3], which seemed to be a hybrid version of the Wilsonian and the functional renormalization groups. Unlike the Wilsonian renormalization group, this version deals directly with the low-energy effective action obtained by integrating out the high-momentum modes. In doing so, the author of [3] proposed the existence of a smearing function, whose purpose is to extract only the low-momentum modes in the proper-time representation of the low-energy effective action at the one-loop level. The existence of such a smearing function, however, is questionable in that it filters the momentum modes indirectly via the proper-time variable, which does not seem to rely on any plausible mathematical concept. As the ideas of [3] and the functional renormalization group share something similar at the beginning of the formulations, it is interesting to see if it is possible that the smearing function comes out directly from the functional renormalization group. In this article, we will investigate this possibility, and will show explicitly that this can be done only in a special case, but not in general.

2. Functional renormalization group

In 1991, Christof Wetterich introduced the idea of an average field, defined as an integral of the fundamental field over a small region in space [4]. This average field thus looks like an extended object of finite size, and therefore the theory of such an object should describe the long-distance (or low-energy) physics, in analogous to the idea of block-spin renormalization group in statistical physics. This implies that such a theory may be treated, in some senses, as the low-energy effective theory of the original theory of the fundamental field. Unlike the Wilsonian effective action obtained by completely integrating out the high-momentum modes in the generating functional integral, it was found that the theory of the average field is obtained by adding a non-local term of the form

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, R_k(p)\phi(-p)\phi(p) \tag{1}$$

to the classical action, $S[\phi]$, where ϕ is the fundamental field of the theory and $R_k(p)$ is a regulator whose value is negligibly small when the momentum p is larger than the momentum scale k and is relatively large when $p < k$. Define the generating functional in Euclidean space,

$$Z_k[J] = \int D\phi\, \exp\left(-S[\phi] - \Delta S_k[\phi] + \int d^4x J(x)\phi(x)\right), \tag{2}$$

and the classical field,

$$\Phi(x) \equiv \langle\phi(x)\rangle = \frac{\delta \ln Z_k[J]}{\delta J(x)}, \tag{3}$$

one can calculate the corresponding effective action,

$$\Gamma_k[\Phi] = -\ln Z_k[J] + \int d^4x\, J(x)\Phi(x) - \Delta S_k[\Phi], \tag{4}$$

in the form of the loop expansion, with Φ being treated as the background field. As the form of $\Delta S_k[\phi]$ mimics the mass term, the regulator $R_k(p)$ plays the role of the additional mass term in the propagator, thereby damping out the low-momentum part ($p < k$) of the propagator while keeping the high-momentum part ($p > k$) almost intact. As a result, the low-momentum part of the propagator that appears in the loop expansion plays a negligible role, when comparing with the contributions from the high-momentum modes. By comparing this with the loop-expansion form of the Wilsonian effective action (obtained by first splitting the fundamental field into its low-momentum and high-momentum parts, and then integrating over the high-momentum modes while treating the low-momentum part as a background field), one may interpret $\Gamma_k[\Phi]$ as the low-energy effective action, and interpret $\Phi(x)$ as the low-energy field whose high-momentum modes ($p > k$) are almost negligible. We thus see that $\Phi(x)$ is exactly the average field introduced above, and, therefore, $\Gamma_k[\Phi]$ is called an average effective action, describing the physics at the momentum scale k.

Despite the fact that the direct calculation of $\Gamma_k[\Phi]$ is complicated, it turns out that $\Gamma_k[\Phi]$ satisfies the so-called Wetterich equation [5, 6],

$$\partial_k\Gamma_k[\Phi] = \frac{\hbar}{2}\mathrm{Tr}\left\{\frac{\partial_k R_k}{\Gamma_k^{(2)}[\Phi] + R_k}\right\}, \tag{5}$$

describing the change of $\Gamma_k[\Phi]$ with the momentum scale k. Here, $\Gamma_k^{(2)}[\Phi]$ is the second derivative of $\Gamma_k[\Phi]$ with respect to k. By making the ansatz of the form of $\Gamma_k[\Phi]$ based on some properties of the physical system under investigation, such as symmetries, people have obtained lots of interesting low-energy physics from the Wetterich equation, which tells us how physical quantities change with the momentum scale k [2, 7]. The theory based on Wetterich's idea is known as the theory of the functional renormalization group (FRG).

Instead of making the ansatz for $\Gamma_k[\Phi]$, one may evaluate it directly using the standard technique in quantum field theory. At one-loop level, it is well known that

$$\Gamma_k^{(1-\text{loop})}[\Phi] = S[\Phi] + \frac{1}{2}\text{Tr}\,\ln\left[\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right], \tag{6}$$

where $S_k[\phi] \equiv S[\phi] + \Delta S_k[\phi]$. The second term on the right-hand side can be conveniently expressed, using the Schwinger proper-time representation [8], as

$$\text{Tr}\,\ln\left[\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right] = -\int_0^\infty \frac{ds}{s}\,\text{Tr}\,\exp\left[-s\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right]. \tag{7}$$

Let us consider the case of ϕ^4 theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{8}$$

In the mean field approximation in which Φ is a constant,

$$\begin{aligned}
&\text{Tr}\,\exp\left[-s\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right] \\
&= \text{Tr}\,\exp\left[-s\left(\hat{P}^2 + m^2 + \frac{\lambda}{2}\Phi^2 + R_k(\hat{P})\right)\right] \\
&= \int d\Omega_3 \int_0^\infty \frac{p^3dp}{(2\pi)^4}\,\exp\left[-s\left(p^2 + m^2 + \frac{\lambda}{2}\Phi^2 + R_k(p)\right)\right],
\end{aligned} \tag{9}$$

where $\hat{P}$ is the momentum operator. We see that the integration over p may not be performed unless the form of $R_k(p)$ is simple enough.

3. Smearing function and its relation to the FRG

In [3], an idea similar to that of the Wilsonian renormalization group was proposed, in which the fundamental field is split into its high-momentum and low-momentum parts. After a delta function, which forces the low-momentum part to be equal to the average field $\Phi(x)$, has been inserted into the generating functional and the integration over the high-momentum modes has been performed, one obtains a low-energy effective action,

$$\Gamma_k^{(1-\text{loop})}[\Phi] = S[\Phi] + \frac{1}{2}\text{Tr}'\,\ln\left[\left.\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right], \tag{10}$$

to one-loop order, where $S[\phi]$ is the classical action and Tr' denotes the sum (or integral) over the high-momentum modes only. Due to the constraint on Tr', one cannot naively express the last term in equation (10) in the form of the Schwinger proper-time representation. Instead, it is tempting to write

$$\mathrm{Tr}'\,\ln\left[\left.\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right] = -\int_0^\infty \frac{ds}{s}\,\rho_k(s,\Lambda)\,\mathrm{Tr}\,\exp\left[-s\left.\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right], \tag{11}$$

where $\rho_k(s,\Lambda)$ is called the smearing function, which depends on the momentum scale k (inversely proportional to the size of the average field) and the momentum cutoff Λ of the theory. Note that there is no constraint on the trace, Tr, appearing in the integral anymore. Thus, the main reason for inserting the smearing function is to remove the constraint on the trace in such a way that all constraints on Tr' are taken care of by the smearing function. In general, it is required that $\rho_k(s,\Lambda)$ has the properties:

(i) $\rho_k(s,\Lambda)\to 0$ as $s\to 0$, to ensure the ultraviolet finiteness.

(ii) $\rho_k(s,\Lambda)\to 0$ as $k\to\Lambda$, to ensure that the effective action reduces to the original action in the absence of quantum fluctuations.

(iii) $\rho_k(s,\Lambda)\to 1$ as $k\to 0$, so that one obtains the full effective action when all quantum fluctuations are taken into account.

In practice, the explicit form of $\rho_k(s,\Lambda)$ suitable for the problem at hand is hard to find. Even worse, there is no definite proof that the speculation in equation (11) will work in all situations. Due to the fact that equations (7) and (11) play the same role in the one-loop effective action, it is interesting to explore the possibility of obtaining the smearing function from the FRG. Let us consider the ϕ^4 theory with a constant background field Φ. As suggested in [9], one may choose

$$R_k(p) = (k^2-p^2)\Theta(k^2-p^2), \tag{12}$$

with $\Theta(k^2-p^2)$ being the Heaviside step function, to ensure that the high-momentum modes, $p>k$, are integrated out completely. With the momentum cutoff Λ, equation (9) becomes

$$\begin{aligned}
&\mathrm{Tr}\,\exp\left[-s\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right]\\
&= \int d\Omega_3\int_0^\Lambda \frac{p^3dp}{(2\pi)^4}\,\exp\left[-s\left(p^2+m^2+\frac{\lambda}{2}\Phi^2+(k^2-p^2)\Theta(k^2-p^2)\right)\right]\\
&= \int \frac{d\Omega_3}{(2\pi)^4}\,\frac{1}{2s^2}\left[\left(1+sk^2+\frac{1}{2}(sk^2)^2\right)e^{-sk^2}-\left(1+s\Lambda^2\right)e^{-s\Lambda^2}\right]e^{-s(m^2+\lambda\Phi^2/2)}.
\end{aligned} \tag{13}$$

Here comes an important step. Observe that we may write

$$\begin{aligned}
\int \frac{d\Omega_3}{(2\pi)^4}\,\frac{1}{2s^2}e^{-s(m^2+\lambda\Phi^2/2)} &= \int d\Omega_3\int \frac{p^3dp}{(2\pi)^4}\,e^{-s(p^2+m^2+\lambda\Phi^2/2)}\\
&= \mathrm{Tr}\exp\left[-s\left(\hat{P}^2+m^2+\frac{\lambda}{2}\Phi^2\right)\right],
\end{aligned} \tag{14}$$

so that

$$\mathrm{Tr}\,\exp\left[-s\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right] = \rho_k(s,\Lambda)\,\mathrm{Tr}\exp\left[-s\left(\hat{P}^2+m^2+\frac{\lambda}{2}\Phi^2\right)\right], \tag{15}$$

where

$$\rho_k(s,\Lambda) = \left(1+sk^2+\frac{1}{2}(sk^2)^2\right)e^{-sk^2}-\left(1+s\Lambda^2\right)e^{-s\Lambda^2}. \tag{16}$$

Thus

$$\begin{aligned} \mathrm{Tr}\,\ln\left[\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right] &= -\int_0^\infty \frac{ds}{s}\,\rho_k(s,\Lambda)\,\mathrm{Tr}\exp\left[-s\left(\hat{P}^2+m^2+\frac{\lambda}{2}\Phi^2\right)\right] \\ &= -\int_0^\infty \frac{ds}{s}\,\rho_k(s,\Lambda)\,\mathrm{Tr}\,\exp\left[-s\left.\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\Phi}\right], \end{aligned} \tag{17}$$

which is precisely equation (11) upon identifying $\rho_k(s,\Lambda)$ as the smearing function. Note that $\rho_k(s,\Lambda)$ obtained in equation (16) does not satisfy the required properties of the smearing function unless we set $\Lambda\to\infty$, which is the natural momentum cutoff of the FRG. We thus obtain an example of obtaining the smearing function from the FRG in a very special case. Due to the fact that $R_k(p)$ appears inside the trace while the smearing function appears outside the trace in the one-loop correction to the effective action, we believe that the formalism in [3] and the FRG are not equivalent in general, and there is no general way of obtaining the smearing function from the FRG.

4. Conclusions

In this work, we have discussed and compared the low-energy effective action in the FRG and in the formalism of [3]. We were able to obtain the smearing function of [3] from the direct calculation in FRG only in a very special case, in which the background field is constant and the regulator takes a specific form. As it is not convincing that equation (11) is a correct representation of Tr', we suspect that the smearing function may not exist at all, and the impossibility of obtaining the smearing function from the FRG should be expected.

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