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Charged Dyson shells supported in curved spacetimes of spherically symmetric central compact objects

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Abstract It is proved that charged thin shells (charged Dyson shells) can be supported in unstable *static* equilibrium states around spherically symmetric central compact objects. The regime of existence of the composed central-compact-object-static-charged-shell configurations is characterized by the inequalities $\sqrt{m(m+2M)} < |q| < M + \sqrt{M^2+m^2}$, where $\{m, q\}$ are respectively the proper mass and the electric charge of the supported shell and M is the mass of the central compact object (a black hole or an horizonless compact star). We reveal the physically interesting fact that the supported charged shells become marginally-stable in the $|q|/\sqrt{m(m+2M)} \to 1^+$ limit, in which case the lifetime (instability timescale) of the composed system can be made arbitrarily large. Our analysis goes beyond the test shell approximation by properly taking into account the exact gravitational and electromagnetic self-interaction energies of the spherically symmetric shell in the curved spacetime. In particular, the existence of the composed compact-object-charged-shell static configurations in the Einstein–Maxwell theory is attributed to the non-linear electromagnetic self-energy of the supported shell.

1 Introduction

A Dyson shell is a hypothetical structure that in principle may be built around a compact star in order to collect the energy that it emits [1]. It is therefore of physical interest to ask whether massive thin shells that are made of test particles can remain in a static equilibrium state around central compact objects (like black holes and horizonless compact stars)?

It is well known that, within the framework of the standard Einstein–Maxwell theory, static neutral shells that are made of test particles cannot be supported at a finite radial distance above spherically symmetric charged compact objects. Intriguingly, however, it has recently been revealed in the physically important paper [2] that, in a generalized quasitopological nonlinear electrodynamic field theory, neutral shells that are made of test particles (with negligible self-gravity) can be supported in curved spacetimes of spherically symmetric charged compact objects.

Motivated by the intriguing results presented in [2], in the present compact paper we shall address the following physically important question: Going beyond the test particle approximation, is it possible to place a *static* thin Dyson shell around central compact objects in the standard Einstein–Maxwell field theory?

Below we shall explicitly prove that, for electrically *charged* shells, the answer to this interesting question is 'yes'. In particular, we shall reveal the fact that, due to the electromagnetic self-interaction that characterizes a charged compact shell, composed central-compact-object-static-charged-shell configurations in unstable equilibrium states do exist in the standard Einstein–Maxwell theory. Moreover, we shall explicitly prove that, by fine tuning the physical parameters of the composed system, the supported self-interacting charged shells can be made marginally-stable (with arbitrarily large lifetimes).

2 Description of the system

We consider a physical system which is composed of a central neutral compact object (a Schwarzschild black hole or an horizonless compact star) and a concentric spherically symmetric thin charged shell of radius R, proper mass m, and electric charge q.

The compactness parameter that characterizes the supporting central object is given by the dimensionless ratio [3]

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$$\mathcal{C} \equiv \frac{M}{R_{\text{central}}} \in (0, 1/2], \tag{1}$$

where M and R_{central} are respectively its mass and radius. A central supporting Schwarzschild black hole is characterized by the relation $\mathcal{C} = 1/2$, whereas the compactness parameters of spatially regular horizonless compact stars are restricted by the Buchdahl bound $\mathcal{C} < 4/9$ [4].

According to the Birkhoff theorem, the curved spacetime inside the spherically symmetric shell is described by the Schwarzschild line element [5,6]

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

In addition, according to the Birkhoff theorem the curved spacetime outside the static charged shell is described by the line element

$$ds^2 = -\Big\{1 - \frac{2[M + E(R)]}{r} + \frac{q^2}{r^2}\Big\}dt^2 + \Big\{1 - \frac{2[M + E(R)]}{r} + \frac{q^2}{r^2}\Big\}^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where $E(R)$ is the energy (energy as measured by asymptotic observers) of the charged shell.

As explicitly shown in [7,8], the radius-dependent energy $E(R)$ of the charged massive shell in the curved spacetime of the spherically symmetric central compact object can be deduced by solving its characteristic equation of motion

$$\sqrt{g_{\text{in}}(r = R) + \dot{R}^2} - \sqrt{g_{\text{out}}(r = R) + \dot{R}^2} = \frac{m}{R}, \tag{4}$$

where $\dot{R} \equiv dR/d\tau$ [9]. The radially dependent metric functions in (4) are given by the functional expressions [see Eqs. (2) and (3)]

$$g_{\text{in}}(r) = 1 - \frac{2M}{r} \tag{5}$$

and

$$g_{\text{out}}(r) = 1 - \frac{2[M + E(R)]}{r} + \frac{q^2}{r^2}. \tag{6}$$

3 Composed central-compact-object-charged-static-shell configurations

In the present section we shall explicitly prove that static self-interacting *charged* shells can be supported in spherically symmetric curved spacetimes of central compact objects. To this end, we shall analyze the functional behavior of the energy $E(R; M, m, q)$ that characterizes a static charged shell which is non-linearly coupled to a spherically symmetric central compact object.

The energy of the static shell can be obtained by substituting

$$\dot{R} \to 0 \tag{7}$$

into Eq. (4). In particular, taking cognizance of Eqs. (4)–(7), one obtains the functional expression

$$E(R; M, m, q) = m \cdot \sqrt{1 - \frac{2M}{R}} + \frac{q^2}{2R} - \frac{m^2}{2R} \tag{8}$$

for the radius-dependent total energy (as measured by asymptotic observers) of the charged massive shell in the curved spacetime of the central compact object.

Note that the various terms in the energy expression (8) have clear physical meanings. In particular, the first term on the r.h.s of (8), which is linear in the mass m of the shell, is its red-shifted mass–energy in the curved spacetime of the central compact object. The second term on the r.h.s of the energy expression (8), which is quadratic in the electric charge of the shell, represents its electromagnetic self-energy. The third term on the r.h.s of (8) represents the gravitational self-energy of the massive shell.

A static shell of radius $R = R_{\text{eq}}$ in a (stable or unstable) equilibrium state, if it exists, is characterized by the gradient relation

$$\frac{dE(R; M, m, q)}{dR} = 0 \quad \text{for} \quad R = R_{\text{eq}}. \tag{9}$$

We consider static shells that are located outside the surface of the central compact object, which implies the inequality

$$R > R_{\text{central}}. \tag{10}$$

In addition, we assume that there is no horizon (with $r_{\text{horizon}} \geq R > R_{\text{central}}$) that engulfs the composed central-compact-object-charged-shell static configuration. Taking cognizance of the curved line element (3), one deduces that this property of the composed compact-object-charged-shell system can be expressed in the form

$$1 - \frac{2[M + E(R)]}{R} + \frac{q^2}{R^2} > 0, \tag{11}$$

which yields the characteristic inequality [see Eq. (8)]

$$R > M + \sqrt{M^2 + m^2} \tag{12}$$

for the radius of the shell.

Substituting Eqs. (1), (8), (10), and (12) into Eq. (9), one finds that composed central-compact-object-charged-shell static configurations do exist in the standard Einstein–Maxwell theory in the dimensionless mass regime

$$\frac{m}{M} > \frac{\sqrt{1-2\mathcal{C}}}{\mathcal{C}} \tag{13}$$

with the charge-mass relation

$$\sqrt{m(m+2M)} < |q| < M + \sqrt{M^2+m^2}. \tag{14}$$

In particular, from Eqs. (8) and (9) one finds that static self-interacting shells are characterized by the $\{M, m, q\}$-dependent equilibrium radius

$$R = R_{\text{eq}}(M, m, q) = 2M \cdot \frac{(q^2-m^2)^2}{(q^2-m^2)^2 - 4M^2m^2}. \tag{15}$$

Substituting the analytically derived relation (15) for the radius of the charged static shell into Eq. (8), one obtains the functional expression

$$E(R = R_{\text{eq}}; M, m, q) = \frac{(q^2-m^2)^2 + 4M^2m^2}{4M(q^2-m^2)} \tag{16}$$

for the energy of the shell. Taking cognizance of Eqs. (8), (14), and (16), one deduces that the supported static charged shell is characterized by the relation

$$E(R = R_{\text{eq}}; M, m, q) > E(R \to \infty; M, m, q) = m. \tag{17}$$

The characteristic inequality (17) together with the fact that the radially-dependent energy expression (8) has one extremum point [see Eqs. (9) and (15)] imply that the static charged shell is in an unstable equilibrium state [10] (It is worth noting that in the next section we shall explicitly prove that, by fine tuning the physical parameters of the composed system, the instability timescale of the charged shells can be made arbitrarily large, thus making them marginally-stable).

Interestingly, inspection of the analytically derived expression (16) reveals the fact that the energy of the static charged shell is a monotonically increasing function of its electric charge parameter $|q|$. Thus, for given values of the physical parameters M and m, the energy of the shell is minimized for $|q|/\sqrt{m(m+2M)} \to 1^+$ [see (14)] to yield the simple asymptotic relation

$$E[R = R_{\text{eq}}; M, m, |q|/\sqrt{m(m+2M)} \to 1^+] \to m^+. \tag{18}$$

4 The instability timescale of the charged Dyson shells

In the previous section we have proved that spherically symmetric self-interacting charged Dyson shells can be supported in unstable static equilibrium states around central compact objects. In the present section we shall determine the characteristic lifetime (the instability timescale) of the composed central-compact-object-charged-shell static configurations. In particular, we shall reveal the interesting fact that, by fine tuning the physical parameters of the composed system, the lifetime of the charged shells can be made arbitrarily large.

The parameter-dependent instability timescale τ that characterizes radial perturbations of the unstable shells is given by the inverse Lyapunov exponent of the system [11]

$$\tau \equiv \lambda^{-1} = \left[\frac{E(R)}{g_{\text{in}}(R)} \cdot \sqrt{-\frac{2}{d^2E/dR^2}}\right]_{R=R_{\text{eq}}}. \tag{19}$$

Substituting Eqs. (5), (15), and (16) into Eq. (19), one finds the $\{M, m, q\}$-dependent functional expression

$$\tau(M, m, q) = \frac{(q^2-m^2)^{7/2}[(q^2-m^2)^2 + 4M^2m^2]}{Mm[(q^2-m^2)^2 - 4M^2m^2]^2} \tag{20}$$

for the characteristic instability timescale of the system.

Interestingly, inspection of Eq. (20) reveals the physically important fact that the lifetime (the characteristic instability timescale) of the charged shells can be made arbitrarily large by fine tuning the parameters of the composed physical system. In particular, one finds the asymptotic functional behavior

$$\tau(M, m, q) \to \frac{64\sqrt{2}(Mm)^{9/2}}{[(q^2-m^2)^2 - 4M^2m^2]^2} \to \infty \quad \text{for} \quad \frac{|q|}{\sqrt{m(m+2M)}} \to 1^+ \tag{21}$$

for the characteristic lifetime of a perturbed charged shell in the $|q|/\sqrt{m(m+2M)} \to 1^+$ limit. The asymptotic functional behavior (21) implies that self-interacting charged shells with the property $|q|/\sqrt{m(m+2M)} \to 1^+$, which are characterized by the relation $\tau \to \infty$, are marginally-stable.

5 Summary and discussion

It has recently been proved in the physically interesting work [2] that, in a nonlinear electrodynamic field theory, neutral test shells (with negligible self-gravity) can be supported in static equilibrium states in curved spacetimes of spherically symmetric charged compact objects.

Motivated by the intriguing observation made in [2], in the present compact paper we have raised the following physically interesting question: Can charged shells be supported in static equilibrium states around central compact objects in the standard Einstein–Maxwell field theory?

In order to address this question, we have analyzed the physical and mathematical properties of composed central-compact-object-static-charged-shell configurations. The main *analytical* results derived in this paper and their physical implications are as follows:

(1) We have revealed the fact that, within the framework of the standard Einstein–Maxwell theory, neutral compact objects (black holes and horizonless compact stars) of mass M can support spherically symmetric static *charged* shells of proper mass m and electric charge $q \neq 0$ whose radii are given by the remarkably compact functional expression [see Eq. (15)]

$$\frac{R}{2M} = \frac{(1-\alpha)^2}{(1-\alpha)^2 - 4\alpha\beta}, \tag{22}$$

where we have used here the dimensionless physical parameters [see Eq. (14)]

$$\alpha \equiv \frac{m^2}{q^2} < 1; \quad \beta \equiv \frac{M^2}{q^2}. \tag{23}$$

(2) It has been explicitly proved that the existence of the composed compact-object-charged-shell static configurations in the standard Einstein–Maxwell theory is a direct consequence of the electromagnetic self-interaction term [a non-linear energy term which is quadratic in the electric charge of the shell, see Eq. (8)] that characterizes the supported charged shells.

(3) It has been shown that the energies (as measured by asymptotic observers) of the supported charged shells are given by the dimensionless functional relation [see Eqs. (16) and (23)]

$$\frac{E}{M} = \frac{(1-\alpha)^2 + 4\alpha\beta}{4\beta(1-\alpha)}. \tag{24}$$

(4) We have shown that the composed central-compact-object-static-charged-shell configurations are in unstable equilibrium states in the sense that they are characterized by the maximally allowed energies,

$$\left[\frac{dE}{dR}\right]_{R=R_{\text{eq}}} = 0 \quad \text{and} \quad \left[\frac{d^2E}{dR^2}\right]_{R=R_{\text{eq}}} < 0, \tag{25}$$

for given values of the physical parameters $\{M, m, q\}$ that characterize the system.

(5) Finally, we have revealed the physically important fact that the characteristic lifetime (the instability timescale) of the supported charged shells becomes arbitrarily large in the dimensionless limit $|q|/\sqrt{m(m+2M)} \to 1^+$ [see (21)]:

$$\tau \to \infty \quad \text{for} \quad \frac{|q|}{\sqrt{m(m+2M)}} \to 1^+. \tag{26}$$

Thus, self-interacting static charged shells with the property $|q|/\sqrt{m(m+2M)} \to 1^+$ are marginally-stable.

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