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# Mixed Parity Variants of Apéry-Type Binomial Series and Level Four Colored Multiple Zeta Values

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## Abstract

In this paper, we study an Apéry-type series involving the central binomial coefficients  $\sum_{n_1 > \dots > n_d > 0} \frac{1}{4^{n_1}} \binom{2n_1}{n_1} n_1^{-s_1} \dots n_d^{-s_d}$  and its variations where the summation indices may have mixed parities and some or all “>” are replaced by “≥”, as long as the series are defined. These sums have naturally appeared in the calculation of massive Feynman integrals by the work of Jegerlehner, Kalmykov, and Veretin. We show that all these sums can be expressed as  $\mathbb{Q}$ -linear combinations of the real and/or imaginary parts of the colored multiple zeta values at level four, i.e., special values of multiple polylogarithms at fourth roots of unity. For example, our main theorem shows that when  $n_1^{-s_1}$  is replaced by  $(2n_1)^{-s_1}$  and other  $n_j^{-s_j}$ 's are replaced by either  $(2n_j)^{-s_j}$  or  $(2n_j + 1)^{-s_j}$ , then all the colored multiple zeta values can be chosen to have the same weight  $s_1 + \dots + s_d$ , but the weights of these values are only bounded by  $s_1 + \dots + s_d$  for general variant Apéry-type series of mixed parities. We also show that the corresponding series where  $\binom{2n_1}{n_1}/4^{n_1}$  is replaced by  $\binom{2n_1}{n_1}^2/16^{n_1}$  can be expressed in a similar way except for a possible extra factor of  $1/\pi$ , with the weight of the colored multiple zeta values similarly bounded.

**Keywords:** Apéry-type series; central binomial coefficient; colored multiple zeta value; multiple polylogarithm; iterated integral.

**MSC:** 11M32; 11B65; 11B37; 44A05; 33B30.

## 1. Introduction

In [1], we have considered the following Apéry-type inverse binomial series

$$\sum_{n_1 > \dots > n_d > 0} \frac{4^{n_1}}{\binom{2n_1}{n_1} n_1^{s_1} \dots n_d^{s_d}}$$

for all positive integers  $s_1 \geq 2, s_2, \dots, s_d \geq 1$ , and their variants with some or all of  $n_j$ 's replaced by  $2n_j \pm 1$  and some or all of “>” replaced by “≥”, as long as the series are defined. By generalizing Akhilesh's ideas in [2,3] we showed that the above sum can be expressed as  $\mathbb{Q}$ -linear combinations of the real and/or the imaginary parts of some colored multiple zeta values at level 4, i.e., multiple polylogarithms evaluated at 4th roots of unity. We also proved similar results after replacing  $4^{n_1}/\binom{2n_1}{n_1}$  by  $16^{n_1}/\binom{2n_1}{n_1}^2$ .



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In this paper, we will turn to another class of Apéry-type series defined by

$$\sum_{n_1 > \dots > n_d > 0} \frac{\binom{2n_1}{n_1}}{4^{n_1} n_1^{s_1} \dots n_d^{s_d}} \quad \text{and} \quad \sum_{n_1 > \dots > n_d > 0} \frac{\binom{2n_1}{n_1}^2}{16^{n_1} n_1^{s_1} \dots n_d^{s_d}} \tag{1}$$

and similar variants with the summation indices having mixed parities. We call these *Apéry-type (central) binomial series* (not inverse type). Instances of this type already appeared in Leshchiner’s work [4], extending Apéry’s formula. These sums have also appeared in the calculation of massive Feynman integrals [5]. Some of the ideas contained in this paper have already been applied to studying other types of infinite series in [6]. In [7,8], some similar finite sums have also been studied by different methods.

We remark that a particular type of odd variations already appeared implicitly in ([5] A.25). Indeed, it is easy to verify that  $(2n)a_n = (2n - 1)a_{n-1}$ , so that for any function  $S$

$$\frac{1}{2} \sum_{n \geq 0} \frac{a_n}{(n + 1)^c} S(n) = \sum_{n \geq 1} \frac{a_{n-1}}{(2n)n^{c-1}} S(n - 1) = \sum_{n > 0} \frac{a_n}{(2n - 1)n^{c-1}} S(n - 1).$$

### 1.1. Colored Multiple Zeta Values

Recall that for any  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  and  $N$ th roots of unity  $z_1, \dots, z_d$  the *colored multiple zeta value (CMZV)* at level  $N$  is defined by

$$\text{Li}_{\mathbf{s}}(z_1, \dots, z_d) := \sum_{n_1 > \dots > n_d > 0} \frac{z_1^{n_1} \dots z_d^{n_d}}{n_1^{s_1} \dots n_d^{s_d}}. \tag{2}$$

Conventionally, we call  $d$  the *depth* and  $|\mathbf{s}| = s_1 + \dots + s_d$  the *weight* of the CMZV defined by (2).

It is well known that (2) converges if and only if  $(s_1, z_1) \neq (1, 1)$  (see [9] and [10] Ch. 15), in which case, we say the CMZV is *admissible*. In fact, a quick calculation shows that if  $(s_1, z_1) = (1, 1)$ , then for any fixed  $n_2$ , the sum  $\sum_{n_1 > n_2} 1/n_1$  diverges as the tail of the harmonic series  $\text{Li}_1(1)$  which is itself non-admissible, and therefore  $\text{Li}_{\mathbf{s}}(z_1, \dots, z_d)$  diverges. On the other hand, if  $s_1 \geq 2$ , then (2) converges absolutely. Further, if  $s_1 = 1$  and  $z_1 \neq 1$  is some  $N$ th root of unity, then one can show the convergence of (2) by applying Abel’s method on summation (see, e.g., [11]).

As an admissible example of (2), the famous Catalan’s constant

$$G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2} = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{i^n (1 - (-1)^n)}{n^2} = \frac{1}{2i} (\text{Li}_2(i) - \text{Li}_2(-i)), \tag{3}$$

is the imaginary part of the CMZV  $\text{Li}_2(i)$  of weight 2 at level 4, where we have set  $i = \sqrt{-1}$ . We further note that

$$\frac{1}{2i} (\text{Li}_1(i) - \text{Li}_1(-i)) = \frac{1}{2i} \log \frac{1 - i}{1 + i} = \frac{1}{2i} \log i = \frac{\pi}{4},$$

which means that  $\pi$  is the imaginary part of  $4 \text{Li}_1(i)$ . In particular, we see that

$$G \in i\text{CMZV}_2^4 \quad \text{and} \quad \pi \in i\text{CMZV}_1^4,$$

where  $\text{CMZV}_w^N$  represent the  $\mathbb{Q}$ -span of all the CMZVs of weight  $w$  at level  $N$ . Conventionally, we set  $\text{CMZV}_0 = \mathbb{Q}$ .

Starting from the 1990s, mathematicians and theoretical physicists have been attracted to the study of colored multiple zeta values due to their frequent and sometimes unexpected appearance in quite a few different branches of mathematics and physics, in particular,

the computation of many Feynman diagrams (see, e.g., [12–24]). We point out also that in the literature, multiple zeta values (and their alternating version) are sometimes called Euler sums (see [25,26]).

### 1.2. Iterated Integrals

The CMZVs can be expressed by K.T. Chen’s iterated integrals (see [10] Sec. 2.1):

$$\text{Li}_s(z_1, \dots, z_d) = \int_0^1 a^{s_1-1} x_{\xi_1} \dots a^{s_d-1} x_{\xi_d},$$

where  $a = dt/t$ ,  $x_\xi = dt/(\xi - t)$ , and  $\xi_j = 1/(z_1 \dots z_j)$  for all  $j = 1, \dots, d$ . The theory of iterated integrals was developed firstly by K.T. Chen in the 1960s. It has played important roles in the study of algebraic topology and algebraic geometry in the past half century. Its simplest form is

$$\begin{aligned} \int_0^1 f_1(t)dt f_2(t)dt \dots f_p(t)dt &= \int_0^1 f_1(t)dt \circ f_2(t)dt \circ \dots \circ f_p(t)dt \\ &:= \int_{1 > t_1 > \dots > t_p > 0} f_1(t_1) f_2(t_2) \dots f_p(t_p) dt_1 dt_2 \dots dt_p. \end{aligned}$$

Hence, our conventional order of the iterated integrals is to integrate the last integral over  $[0, t]$  first, multiply this function on the penultimate 1-form, integrate over  $[0, t]$  again, and then iterate this integrating–multiplying step all the way to the front, with the integration over  $[0, 1]$  in the last step.

For example, the dilogarithm value  $\text{Li}_2(i)$  related to Catalan’s constant defined by (3) can be expressed as

$$\int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{-i-y} = \sum_{k=0}^\infty \int_0^1 \frac{i^{k+1} x^k dx}{k+1} = \sum_{k=0}^\infty \frac{i^{k+1}}{(k+1)^2} = \sum_{n=1}^\infty \frac{i^n}{n^2} = \text{Li}_2(i).$$

One can further extend the iterated integrals defined on  $[0, 1]$  to those over any piecewise smooth paths on the complex plane via pull-backs. We refer the interested readers to Chen’s original work [27,28] for more details.

### 1.3. Recursive Formulas

The key idea to compute the infinite sums in (1) and their even/odd-indexed variations is to derive some recursive formulas for the “ $n$ -tails” of a variable version of these series, which are given by the following (see also (14) and (18)–(21)). Here, an  $n$ -tail of a series means the complement of its  $n$ th partial sum so that for a convergent series, its  $n$ -tail vanishes as  $n \rightarrow \infty$ .

Set  $a_n(x) = \frac{1}{4^n} \binom{2n}{n} x^{2n}$  and  $a_n = a_n(1)$  for all  $n \geq 0$  where  $\binom{0}{0} = 1$ . For all  $s \geq 1$  and  $y \in (-\pi/2, \pi/2)$ , we have the following iterated integral expressions of  $n$ -tails:

$$\sum_{m > n} \frac{a_m(\sin y)}{(2m)^s} = \int_0^y (\cot t dt)^{s-1} (1 - \csc t dt \circ \sec t) a_n(\sin t) \tan t dt, \tag{4}$$

$$\sum_{m > n} \frac{a_m(\sin y)}{(2m+1)^s} = \csc y \int_0^y (\cot t dt)^{s-1} (1 - dt \circ \csc t \sec t) a_n(\sin t) \sin t \tan t dt, \tag{5}$$

$$\sum_{m \geq n} \frac{a_m(\sin y)}{(2m+1)^s} = \csc y \int_0^y (\cot t dt)^{s-1} (\csc t - dt \circ \sec t) a_n(\sin t) \tan t dt, \tag{6}$$

$$\sum_{m > n} \frac{a_m(\sin y)}{2m-1} = \cos y \int_0^y a_n(\sin t) \tan t \sec t dt, \tag{7}$$

$$\sum_{m>n} \frac{a_m(\sin y)}{(2m-1)^s} = \sin y \int_0^y (\cot t dt)^{s-2} (\cot^2 t dt) a_n(\sin t) \tan t \sec t dt \quad (s \geq 2). \tag{8}$$

In fact, in the above, we have used a notation generalizing that for the iterated integrals. See the beginning of the next section for details. To compute each of the even/odd/mixed variations of Apéry-type binomial series similar to (1), one can use (4)–(8) repeatedly to express it as an iterated integral, and then apply the change of variable  $t \rightarrow \sin^{-1}[(1-t^2)/(1+t^2)]$  to convert it to a  $\mathbb{Q}$ -linear combination of iterated integrals that only involve 1-forms of the shape  $dt/t$  and  $dt/(\zeta-t)$ , where  $\zeta^4 = 1$ , so that all the integrals are clearly in  $\text{CMZV}_{\leq |s|}^4 \otimes \mathbb{Q}[i]$ . See the related conversion rules in (46)–(48).

1.4. List of Notations

For the convenience of the readers, we provide the following list of notations we use throughout this paper.

(1) Series Let  $d \in \mathbb{N}$ ,  $s = (s_1, \dots, s_d) \in \mathbb{N}^d$ , and  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ .

$$\begin{aligned} a_n(x) &= \frac{1}{4^n} \binom{2n}{n} x^{2n}, \\ \alpha(s; t)_n &:= \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(t)}{(2n_1)^{s_1} \dots (2n_d)^{s_d}}, \\ \beta(s; t)_n &:= \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(t)}{(2n_1 + 1)^{s_1} \dots (2n_d + 1)^{s_d}}, \\ \beta^*(s; t)_n &:= \sum_{n_1 \geq \dots \geq n_d \geq n} \frac{a_{n_1}(t)}{(2n_1 + 1)^{s_1} \dots (2n_d + 1)^{s_d}}, \\ \gamma(s; t)_n &:= \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(t)}{(2n_1 - 1)^{s_1} \dots (2n_d - 1)^{s_d}}, \\ \Xi(l; s) &:= \sum_{n_1 \underset{1}{\succ} \dots \underset{d-1}{\succ} n_d \underset{d}{\succ} 0} \frac{a_{n_1}}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}}, \\ \Psi(l; s) &:= \sum_{n_1 \underset{1}{\succ} \dots \underset{d-1}{\succ} n_d \underset{d}{\succ} 0} \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} \end{aligned}$$

where  $l_j(n) = 2n, 2n \pm 1$ ,  $\underset{j}{\succ}$  is “ $\geq$ ” if  $l_j(n) = 2n + 1$  and is “ $>$ ” otherwise.

(2) Differential forms. Let  $\zeta$  and  $\zeta'$  be any fourth roots of unity, i.e.,  $\zeta^4 = \zeta'^4 = 1$ .

$$\begin{aligned} \mathbf{a} &= \frac{dt}{t}, \\ \mathbf{x}_\zeta &= \frac{dt}{\zeta - t}, \quad \mathbf{d}_{\zeta, \zeta'} = \mathbf{x}_\zeta - \mathbf{x}_{\zeta'}, \\ \mathbf{y} &= \mathbf{x}_{-i} + \mathbf{x}_i - \mathbf{x}_{-1} - \mathbf{x}_1, \\ \mathbf{z} &= -\mathbf{a} - \mathbf{x}_{-i} - \mathbf{x}_i, \\ F_s &= (\cot t dt)^{s-1} := \underbrace{(\cot t dt) \circ \dots \circ (\cot t dt)}_{s-1 \text{ times}}. \end{aligned}$$

(3) Terminology and acronym.

- CMZV: colored multiple zeta value, defined by (2);
- $\text{CMZV}_w^N$ : the  $\mathbb{Q}$ -vector space spanned by all the CMZVs of weight  $w$  at level  $N$ ;
- $\mathbf{bt}(l)$ : block type of  $l$ , defined as  $\alpha, \beta$ , or  $\gamma$  if  $l(n) = 2n, 2n + 1$ , or  $2n - 1$ , respectively;

- $\mathbf{bt}(l_1, \dots, l_d) := (\mathbf{bt}(l_1), \dots, \mathbf{bt}(l_d))$ : we may use this to write  $\Xi(\mathbf{bt}(\mathbf{l}); \mathbf{s}) = \Xi(\mathbf{l}; \mathbf{s})$  and  $\Psi(\mathbf{bt}(\mathbf{l}); \mathbf{s}) = \Psi(\mathbf{l}; \mathbf{s})$ ;
- $\alpha$ -block: a sum of the form  $\sum_{m>n} \frac{a_m(\sin t)}{(2m)^s}$ , or its integral expression (18);
- $\beta$ -block: a sum of the form  $\sum_{m>n} \frac{a_m(\sin t)}{(2m+1)^s}$ , or its integral expression (19);
- $\beta^*$ -block: a sum of the form  $\sum_{m \geq n} \frac{a_m(\sin t)}{(2m+1)^s}$ , or its integral expression (29);
- $\gamma$ -block: a sum of the form  $\sum_{m>n} \frac{a_m(\sin t)}{(2m-1)^s}$ , or its integral expressions (20) and (21).

1.5. Main Results of the Paper

In the next three sections, we will repeatedly apply the recursive Formulas (4)–(8) in the above to obtain the iterated integral expressions of three general Apéry-type binomial series. In Sections 4 and 5, we will analyze these iterated integrals carefully to prove one of the main results of this paper relating these sums to the CMZVs at level 4.

**Theorem 1** (cf. Theorems 6 and 8). *Let  $d \in \mathbb{N}$ ,  $\mathbf{s} \in \mathbb{N}^d$ , and  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ .*

- (a) *If  $\mathbf{bt}(l_1) = \alpha$  and  $\mathbf{bt}(l_j) = \alpha$  or  $\beta$  for all  $j \geq 2$ , then  $\Xi(\mathbf{l}; \mathbf{s}) \in \text{CMZV}_{|\mathbf{s}|}^4$ .*
- (b) *If  $\mathbf{bt}(l_1) = \beta$  and  $\mathbf{bt}(l_j) = \alpha$  or  $\beta$  for all  $j \geq 2$ , then  $\Xi(\mathbf{l}; \mathbf{s}) \in i\text{CMZV}_{|\mathbf{s}|}^4$ .*
- (c) *If  $\mathbf{bt}(l_1) = \gamma$  and  $s_1 = 1$ , then  $\Xi(\mathbf{l}; \mathbf{s}) = \Xi(l_2, \dots, l_d; s_2, \dots, s_d)$ .*
- (d) *If  $\mathbf{bt}(l_1) = \gamma$  and  $s_1 \geq 2$ , then  $\Xi(\mathbf{l}; \mathbf{s}) \in \text{CMZV}_{\leq |\mathbf{s}|-1}^4 \otimes \mathbb{Q}[i]$ .*
- (e) *Moreover, the claim in (d) still holds if one changes any of the strict inequalities  $n_j > n_{j+1}$  to  $n_j \geq n_{j+1}$  ( $n_{d+1} = 0$ ) and vice versa in the definition of  $\Xi(\mathbf{l}; \mathbf{s})$ , provided the series is defined.*

We then compute many examples (most of which are put in Appendix A) to show how the general procedure works. For the reader’s convenience, we have summarized all the main results of this section and examples of Theorem 8 in Tables 1 and 2.

In Section 6, we apply the beta integral to derive the iterated integral expressions for the corresponding Apéry-type binomial series in which  $a_n^2$  appears. The main result is the following.

**Theorem 2** (cf. Theorem 9). *Let  $d \in \mathbb{N}$ ,  $\mathbf{s} \in \mathbb{N}^d$ , and  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ .*

- (a) *If  $\mathbf{bt}(l_j) = \alpha$  or  $\beta$  for all  $j \geq 1$ , then we have  $\Psi(\mathbf{l}; \mathbf{s}) \in \frac{i}{\pi} \text{CMZV}_{|\mathbf{s}+1}^4$ .*
- (b) *More generally, if we relax the restriction on the index ordering of  $\Psi(\mathbf{l}; \mathbf{s})$  to allow “ $\succ_j$ ” to be either “ $\geq$ ” or “ $>$ ” for all  $j \geq 2$ , provided the series is defined, then the series lies in  $\frac{1}{\pi} \text{CMZV}_{\leq |\mathbf{s}|+\eta(\mathbf{s}, \mathbf{l})}^4 \otimes \mathbb{Q}[i]$ , where  $\eta(\mathbf{s}, \mathbf{l})$  is 0 or  $\pm 1$  as defined in (51).*

To demonstrate the ideas of Theorem 9, we then compute many examples, most of which are in Appendix B for ease of reading. For the reader’s convenience, we have summarized the results of Theorem 9 and its examples in Tables 3 and 4.

In the last two sections, we answer some questions in our paper [29], relate our work to some previously known results, and provide a few more enlightening examples. We end the paper by pointing out some possible future research directions.

2. Apéry-Type Central Binomial Series

In this section, we will consider the Apéry-type series defined in (1). We will need to extend Chen’s iterated integrals by combining 1-forms and functions as follows. For

any  $r \in \mathbb{N}$ , 1-forms  $f_1(t) dt, \dots, f_{r+1}(t) dt$ , and differentiable functions  $F_1(t), \dots, F_r(t), G_1(t), \dots, G_r(t)$ , we extend the definition of iterated integrals by setting recursively

$$\begin{aligned} & \int_a^b \left( \prod_{j=1}^r [F_j(t) + f_j(t) dt \circ G_j(t)] \right) \circ f_{r+1}(t) dt \\ & := \int_a^b \left( \prod_{j=1}^{r-1} [F_j(t) + f_j(t) dt \circ G_j(t)] \right) \circ (F_r(t) f_{r+1}(t)) dt \\ & + \int_a^b \left( \prod_{j=1}^{r-1} [F_j(t) + f_j(t) dt \circ G_j(t)] \right) \circ f_r(t) dt \circ G_r(t) f_{r+1}(t) dt, \end{aligned}$$

where

$$\prod_{j=1}^r \alpha_j = \alpha_1 \circ \dots \circ \alpha_r.$$

Our first result concerns the  $n$ -tails of the Apéry-type binomial series (1). To be consistent with all the major theorems of this paper, we formulate it as an even-indexed variation. For  $s \in \mathbb{N}$ , we define

$$p_s(t) := \tan t dt \left( \frac{dt}{\tan t} \right)^{s-1} (1 - \csc t dt \circ \sec t).$$

**Theorem 3.** For all  $n \in \mathbb{N}_0, s = (s_1, \dots, s_d) \in \mathbb{N}^d$ , and  $y \in [-\pi/2, \pi/2]$  we have

$$\begin{aligned} \alpha(s; \sin y)_n & := \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(\sin y)}{(2n_1)^{s_1} \dots (2n_d)^{s_d}} \\ & = \cot y \frac{d}{dy} \int_0^y p_{s_1} \circ \dots \circ p_{s_d} \circ a_n(\sin t) \tan t dt. \end{aligned}$$

Here, if  $y = \pm\pi/2$ , then the right-hand side of the above is defined to be the limit as  $y \rightarrow \pm\pi/2$ .

We will again call the sum  $|s| := s_1 + \dots + s_d$  the weight and  $d$  the depth of the series  $\alpha(s; \sin y)$ , respectively.

**Remark 1.** When  $y = \pm\pi/2$ , we must take limit  $y \rightarrow \pm\pi/2$  because  $\cot(\pm\pi/2) = 0$  while the first factor of  $\frac{d}{dy}(\dots)$  is  $\tan y$  which is undefined at  $\pm\pi/2$ . However,  $\cot y \tan y = 1$  is well defined at the limit as  $y \rightarrow \pm\pi/2$ .

**Proof.** The proof is in the same spirit as that of ([3] Thm. 4): first express the  $n$ -tail  $u_n(y)$  by an integral  $v_n(y)$  (in which the factor  $a_n(\sin t)$  magically appears again) and then iterate the procedure to arbitrary depth, where

$$\begin{aligned} u_n(y) & := \sum_{m>n} \frac{a_m(\sin y) \sec y}{2m-1} = \tan y \sum_{m>n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m-1} y}{2m-1}, \\ v_n(y) & := \int_0^y a_n(\sin t) \sec t \tan t dt = \int_0^y \binom{2n}{n} \frac{\sin^{2n-1} t}{4^n} \tan^2 t dt. \end{aligned}$$

We claim that  $u_n(y) = v_n(y)$ . Indeed, first, we have

$$\begin{aligned} \frac{d}{dy} (u_{n-1}(y) - u_n(y)) & = \frac{1}{4^n} \binom{2n}{n} \frac{d}{dy} \frac{\sin^{2n-1} y \tan y}{(2n-1)} \\ & = \binom{2n}{n} \frac{\sin^{2n-1} y}{4^n} \cdot \frac{(2n-1) + \sec^2 y}{2n-1} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dy} (v_{n-1}(y) - v_n(y)) &= \frac{\sin^{2n-3} y}{4^n} \tan^2 t \left( 4 \binom{2n-2}{n-1} - \binom{2n}{n} \sin^2 y \right) \\ &= \frac{\sin^{2n-1} y}{4^n} \binom{2n}{n} \sec^2 y \left( \frac{2n}{2n-1} - \sin^2 y \right) \\ &= \frac{\sin^{2n-1} y}{4^n} \binom{2n}{n} \sec^2 y \left( \frac{1}{2n-1} + \cos^2 y \right). \end{aligned}$$

Hence, for all  $n \geq 1$

$$u_{n-1}(y) - u_n(y) = v_{n-1}(y) - v_n(y)$$

since clearly the two sides both vanish when  $y = 0$ . Further, for all  $y \in (-\pi/2, \pi/2)$ ,

$$\lim_{n \rightarrow \infty} u_n(y) = \lim_{n \rightarrow \infty} v_n(y) = 0$$

since by Stirling’s formula

$$\frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}} \text{ as } n \rightarrow \infty.$$

By telescoping, we see immediately that  $u_n(y) = v_n(y)$ , or equivalently,

$$\sum_{m>n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m-1} y}{2m-1} = \cot y \int_0^y \binom{2n}{n} \frac{\sin^{2n-1} t}{4^n} \tan^2 t dt. \tag{9}$$

Differentiating (9), we obtain

$$\sum_{m>n} a_m \sin^{2m-2} y \cos y = \binom{2n}{n} \frac{\sin^{2n} y}{4^n} \sec y - \csc^2 y \int_0^y \binom{2n}{n} \frac{\sin^{2n} t}{4^n} \tan t \sec t dt. \tag{10}$$

Multiplying this by  $\sin y$  and integrating, we have

$$\sum_{m>n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m} y}{2m} = \int_0^y (1 - \csc t dt \circ \sec t) \binom{2n}{n} \frac{\sin^{2n} t}{4^n} \tan t dt.$$

Repeatedly multiplying these by  $\cot y$  and integrating, we have for all  $s \geq 1$

$$\sum_{m>n} \frac{a_m(\sin y)}{(2m)^s} = \int_0^y \left( \frac{dt}{\tan t} \right)^{s-1} (1 - \csc t dt \circ \sec t) a_n(\sin t) \tan t dt. \tag{11}$$

This shows the  $d = 1$  case of the theorem. To prove the general depth case, we now apply induction on  $d$ . Thus, assuming

$$\alpha(s; \sin y)_{n_d} = \cot y \frac{d}{dy} \int_0^y p_{s_1} \circ \dots \circ p_{s_{d-1}} \circ a_{n_d}(\sin t) \tan t dt,$$

we then obtain

$$\begin{aligned} \alpha(s; \sin y)_n &= \cot y \frac{d}{dy} \int_0^y p_{s_1} \circ \dots \circ p_{s_{d-1}} \circ \sum_{n_d>n} \frac{a_{n_d}(\sin t)}{(2n_d)^{s_d}} \tan t dt \\ &= \cot y \frac{d}{dy} \int_0^y p_{s_1} \circ \dots \circ p_{s_d} \circ a_n(\sin t) \tan t dt \end{aligned}$$

by taking  $m = n_d$  and  $s = s_d$  in (11). This completes the proof of the theorem.  $\square$

**Example 1.** If  $s_1 = \dots = s_d = 1$ , then we see that

$$\begin{aligned} \alpha(1; \sin y) &:= \sum_{n>0} \frac{1}{4^n} \binom{2n}{n} \frac{\sin^{2n} y}{2n} = \int_0^y (\csc t - \cot t) dt \\ &= \log \frac{\csc t}{\csc t + \cot t} \Big|_0^y = \log 2 - \log(1 + \cos y). \end{aligned}$$

Thus,  $\alpha(1; 1) = \log 2$ . When  $d = 2$ , we obtain

$$\begin{aligned} \alpha(1, 1; \sin y) &:= \int_0^y (\csc t dt \circ \sec t - \cot t dt) (\csc t - \cot t) \\ &= \int_0^y (\csc t - \cot t) dt (\csc t - \cot t) dt + \csc t dt \circ (\sec t - 1) (\csc t - \cot t) dt. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^u (\sec t - 1) (\csc t - \cot t) dt &= \int_0^u \frac{(1 - \cos t)^2 dt}{\sin t \cos t} \\ &= \int_0^u \frac{(\cos t - 1) d \cos t}{(1 + \cos t) \cos t} = \int_0^u \left( \frac{2}{1 + \cos t} - \frac{1}{\cos t} \right) d \cos t = \log \frac{(1 + \cos u)^2}{4 \cos u}. \end{aligned}$$

Setting  $a = \cos y$ , we have

$$\begin{aligned} \alpha(1, 1; \sin y) &:= \frac{1}{2} \left( \int_0^y (\csc t - \cot t) dt \right)^2 + \int_0^y \log \frac{(1 + \cos t)^2}{4 \cos t} \csc t dt \\ &= \frac{1}{2} \log^2 \frac{2}{1+a} + \int_0^y -\frac{1}{1-\cos^2 t} \log \frac{(1 + \cos t)^2}{4 \cos t} d \cos t \\ &= \frac{1}{2} \log^2 \frac{2}{1+a} + \int_a^1 \frac{1}{1-u^2} \log \frac{(1+u)^2}{4u} du \\ &= \frac{1}{2} \log^2 \frac{2}{1+a} + \frac{1}{2} \text{Li}_2(1-a) - \text{Li}_2\left(\frac{1-a}{2}\right) + \frac{1}{2} \text{Li}_2(-a) \\ &\quad + \log 2 \log(1+a) + \frac{1}{2} \log(a) \log(1+a) - \frac{1}{2} \log^2 2 - \frac{1}{2} \log^2(1+a) + \frac{1}{4} \zeta(2) \\ &= \frac{1}{2} \text{Li}_2(1-a) - \text{Li}_2\left(\frac{1-a}{2}\right) + \frac{1}{2} \text{Li}_2(-a) + \frac{1}{2} \log(a) \log(1+a) + \frac{1}{4} \zeta(2). \end{aligned}$$

Applying the identity  $\text{Li}_2(t) + \text{Li}_2(1-t) + \log t \log(1-t) = \zeta(2)$  with  $t = 1/2$ , we see that

$$\alpha(1, 1; 1) := \frac{1}{2} \log^2 2 + \frac{1}{4} \zeta(2) \approx 0.6514.$$

### 3. An Odd Variant

In this section, we turn to the Apéry-type binomial series similar to (1) but with summation indices restricted to odd numbers only. Set

$$\begin{aligned} \theta_s(t) &= \tan t dt \left( \frac{dt}{\tan t} \right)^{s-1} (1 - dt \circ \csc t \sec t), \\ q_s(t) &= \sec t dt \left( \frac{dt}{\tan t} \right)^{s-1} (\csc t - dt \circ \sec t). \end{aligned}$$

**Theorem 4.** For all  $n \in \mathbb{N}_0$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  and  $y \in [-\pi/2, \pi/2]$ , we have

$$\beta(\mathbf{s}; \sin y)_n := \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(\sin y)}{(2n_1 + 1)^{s_1} \dots (2n_d + 1)^{s_d}}$$

$$\begin{aligned}
 &= \csc y \cot y \frac{d}{dy} \int_0^y \theta_{s_1} \circ \dots \circ \theta_{s_d} \circ a_n(\sin t) \sin t \tan t \, dt, \\
 \beta^*(s; \sin y)_n &:= \sum_{n_1 \geq \dots \geq n_d \geq n} \frac{a_{n_1}(\sin y)}{(2n_1 + 1)^{s_1} \dots (2n_d + 1)^{s_d}} \\
 &= \cot y \frac{d}{dy} \int_0^y q_{s_1} \circ \dots \circ q_{s_d} \circ a_n(\sin t) \tan t \, dt.
 \end{aligned}$$

Here, when  $y = \pm\pi/2$ , we understand the right-hand side of the above as the limit when  $y \rightarrow \pm\pi/2$ .

**Remark 2.** When  $y = \pm\pi/2$ , we must take limit  $y \rightarrow \pm\pi/2$  because  $\cot(\pm\pi/2) = 0$  while the first factor of  $\frac{d}{dy}(\dots)$  is  $\sec y$  which is undefined at  $\pm\pi/2$ . However,  $\cot y \sec y = \csc y$  is well defined at the limit  $y \rightarrow \pm\pi/2$ .

**Remark 3.** The introduction of both  $\beta$ -sums and  $\beta^*$  is one of the key ideas of this paper. Their relation is similar to that between the classical multiple zeta values and the multiple zeta star values in which the difference lies in whether we allow summation indices to be equal to each other. Some results look neater and more concise with one type of sums while others are prettier and more manageable with the other.

**Proof.** From (10), we obtain

$$\sum_{m>n} a_m \sin^{2m-2} y \cos y = \binom{2n}{n} \frac{\sin^{2n} y}{4^n} \sec y - \csc^2 y \int_0^y \binom{2n}{n} \frac{\sin^{2n} t}{4^n} \tan t \sec t \, dt. \tag{12}$$

Multiplying this by  $\sin^2 y$  on both sides and integrating, we see that

$$\sum_{m>n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m+1} y}{2m+1} = \int_0^y (1 - dt \circ \csc t \sec t) \binom{2n}{n} \frac{\sin^{2n+1} t}{4^n} \tan t \, dt.$$

Repeatedly multiplying this by  $\cot y$  and integrating, we have for all  $s \geq 1$

$$\sum_{m>n} \frac{a_m(\sin y) \sin y}{(2m+1)^s} = \int_0^y \left(\frac{dt}{\tan t}\right)^{s-1} (1 - dt \circ \csc t \sec t) a_n(\sin t) \sin t \tan t \, dt. \tag{13}$$

This proves the  $d = 1$  case of the  $\beta$ -sum in the theorem

On the other hand, multiplying (12) by  $\sin^2 y$  and adding  $\frac{1}{4^n} \binom{2n}{n} \sin^{2n} y \cos y$  on both sides, we obtain

$$\sum_{m \geq n} \frac{1}{4^m} \binom{2m}{m} \sin^{2m} y \cos y = \binom{2n}{n} \frac{\sin^{2n} y}{4^n} \sec y - \int_0^y \binom{2n}{n} \frac{\sin^{2n} t}{4^n} \tan t \sec t \, dt.$$

Integrating we obtain

$$\sum_{m \geq n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m+1} y}{2m+1} = \int_0^y (\csc t - dt \circ \sec t) \binom{2n}{n} \frac{\sin^{2n+1} t}{4^n} \sec t \, dt.$$

Repeatedly multiplying these by  $\cot y$  and integrating, we have for all  $s \geq 1$

$$\sum_{m \geq n} \frac{a_m(\sin y) \sin y}{(2m+1)^s} = \int_0^y \left(\frac{dt}{\tan t}\right)^{s-1} (\csc t - dt \circ \sec t) a_n(\sin t) \sin t \sec t \, dt. \tag{14}$$

This shows that the theorem holds for  $\beta^*$ -sum when  $d = 1$ .

To prove the general case of the theorem for the  $\beta$ - and the  $\beta^*$ -sums, we can apply induction on  $d$  and using the special case with  $m = n_d$  and  $s = s_d$  in (13) and (14), respectively. This concludes the proof of the theorem.  $\square$

### 4. Another Odd Variant

In this section, we study another variant of the Apéry-type binomial series similar to (1). Even though these series look very similar to those in the previous section, they are fundamentally different and therefore require different approaches to handle them. Set

$$\varphi_s(t) = \begin{cases} \tan t dt, & \text{if } s = 1; \\ \tan^2 t dt \circ (\cot t dt)^{s-2} \circ \cot^2 t dt, & \text{if } s \geq 2. \end{cases}$$

**Theorem 5.** For all  $n \in \mathbb{N}_0$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  and  $y \in [-\pi/2, \pi/2]$ , we have

$$\begin{aligned} \gamma(\mathbf{s}; \sin y)_n &:= \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(\sin y)}{(2n_1 - 1)^{s_1} \dots (2n_d - 1)^{s_d}} \\ &= \cos y \cot y \frac{d}{dy} \int_0^y \varphi_{s_1} \circ \dots \circ \varphi_{s_d} \circ a_n(\sin t) \tan t \sec t dt. \end{aligned}$$

Here, when  $y = \pm\pi/2$ , we understand the right-hand side of the above as the limit when  $y \rightarrow \pm\pi/2$ .

**Proof.** By (9), we have

$$\sum_{m > n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m-1} y}{2m-1} = \cot y \int_0^y \binom{2n}{n} \frac{\sin^{2n-1} t}{4^n} \tan^2 t dt.$$

For any  $s \geq 2$ , we can repeatedly multiply this equation by  $\cot y$  and integrate to obtain

$$\sum_{m > n} \frac{1}{4^m} \binom{2m}{m} \frac{\sin^{2m-1} y}{(2m-1)^s} = \int_0^y \left( \frac{dt}{\tan t} \right)^{s-2} \frac{dt}{\tan^2 t} \binom{2n}{n} \frac{\sin^{2n-1} t}{4^n} \tan^2 t dt. \tag{15}$$

This proves the  $d = 1$  case of the theorem. To prove the general case, we can apply induction on  $d$  and use the special case with  $m = n_d$  and  $s = s_d$  in (15). This concludes the proof of the theorem.  $\square$

**Example 2.** If  $s_1 = \dots = s_d = 1$ , then we see that

$$\begin{aligned} \gamma(\mathbf{s}; \sin y) &:= \sum_{n_1 > \dots > n_d > 0} \frac{a_{n_1}(\sin y)}{(2n_1 - 1) \dots (2n_d - 1)} \\ &= \cos y \int_0^y (\tan t dt)^{d-1} \circ \tan t \sec t dt = 1 - \cos y \sum_{j=1}^{d-1} \frac{\log^j |\sec y|}{j!}. \end{aligned}$$

In particular, we see that  $\gamma(\{1\}^d; 1) = 1$  for all  $d \geq 1$ , where  $\{s\}^d$  means repeating string  $s$  exactly  $d$  times.

It turns out that the evaluation  $\gamma(\{1\}^d; 1) = 1$  is just a very special case of the following result on the  $n$ -tails  $\gamma(\{1\}^d; 1)_n$ , which in turn easily leads to Theorem 6.

**Corollary 1.** For all  $n \in \mathbb{N}_0$  and  $d \in \mathbb{N}$ , we have

$$\gamma(\{1\}^d; 1)_n = \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}}{(2n_1 - 1) \dots (2n_d - 1)} = a_n.$$

**Proof.** For any  $y \in (-\pi/2, \pi/2)$  and  $d \in \mathbb{N}$ , we set

$$f_d(y) = \int_0^y (\tan t dt)^{d-1} \circ a_n(\sin t) \tan t \sec t dt.$$

Since  $f'_{d+1}(y) = (\tan y)f_d(y)$  if  $d \geq 2$ , by Theorem 5, we have

$$\begin{aligned} \gamma(\mathbf{s}; 1)_n &= \lim_{y \rightarrow (\pi/2)^-} (\cos y)f_d(y) \\ &= \lim_{y \rightarrow (\pi/2)^-} \frac{f_d(y)}{\sec y} = \lim_{y \rightarrow (\pi/2)^-} \frac{f'_d(y)}{\sec y \tan y} = \lim_{y \rightarrow (\pi/2)^-} \frac{f_{d-1}(y)}{\sec y} \end{aligned}$$

by L'Hôpital's rule. Thus, repeatedly applying L'Hôpital's rule finally yields

$$\gamma(\{1\}^d; 1)_n = \lim_{y \rightarrow (\pi/2)^-} \frac{f_1(y)}{\sec y} = \lim_{y \rightarrow (\pi/2)^-} \frac{f'_1(y)}{\sec y \tan y} = \lim_{y \rightarrow (\pi/2)^-} a_n(\sin y) = a_n,$$

as desired.  $\square$

**Theorem 6.** Suppose  $d \in \mathbb{N}$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ . Let  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ . Then

$$\sum_{n_0 > n_1 \succ \dots \succ n_d > 0} \frac{a_{n_0}}{(2n_0 - 1)l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \sum_{n_1 \succ \dots \succ n_d > 0} \frac{a_{n_1}}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}},$$

where " $\succ$ " can be either " $\geq$ " or " $>$ ", provided the series is defined.

**Proof.** This follows from Corollary 1 immediately.  $\square$

Theorem 6 enables us to consider fewer leading parity cases in Theorem 8.

### 5. Mixed Parity Variations of Apéry-Type Binomial Series

We now turn to the variants of Apéry-type binomial series in which each summation index is restricted to either even or odd numbers.

#### 5.1. Evaluate Even/Odd Variants of Apéry-Type Series by Iterated Trigonometric Integrals

For any  $r \in \mathbb{N}$ , 1-forms  $f_1(t) dt, \dots, f_r(t) dt$ , and any differentiable function  $F(t)$ , we extend the iterated integral notation to define

$$\begin{aligned} f_1(t) dt \circ \overleftarrow{F}(t) \circ f_2(t) dt &= F(t) f_1(t) dt \circ f_2(t) dt, \\ \int_0^y \overleftarrow{F}(t) \circ f_1(t) dt \circ \dots \circ f_r(t) dt &= F(y) \int_0^y f_1(t) dt \circ \dots \circ f_r(t) dt. \end{aligned}$$

Define the *block type* of  $l$  by  $\mathbf{bt}(l) = \alpha, \beta$  and  $\gamma$  if  $l(n) = 2n, 2n + 1$  and  $2n - 1$ , respectively. For  $s \in \mathbb{N}$ , set  $F_s = (\cot t dt)^{s-1}$  and

$$\begin{aligned} \lambda_{\alpha,s}(t) &= F_s(\tan t dt - \csc t dt d \sec t), \\ \lambda_{\beta,s}(t) &= \overleftarrow{\csc t} \circ F_s(\sin t \tan t dt - dt d \sec t), \\ \lambda_{\gamma,1}(t) &= \overleftarrow{\cos t} \circ \tan t \sec t dt, \\ \lambda_{\gamma,s}(t) &= \overleftarrow{\sin t} \circ F_{s-1} \frac{dt}{\tan^2 t} \tan t \sec t dt \quad (s \geq 2). \end{aligned}$$

**Theorem 7.** Suppose  $d \in \mathbb{N}$  and  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ . Let  $y \in (-\pi/2, \pi/2)$ . Then, for any  $l_1(n) = 2n, 2n \pm 1$  and  $l_2(n), \dots, l_d(n) = 2n, 2n + 1$ , we have the  $n$ -tails

$$\sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(\sin y)}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \int_0^y \lambda_{\mathbf{bt}(l_1), s_1} \circ \dots \circ \lambda_{\mathbf{bt}(l_d), s_d} \circ \overleftarrow{a_n(\sin t)}. \tag{16}$$

Using non-trigonometric 1-forms, we obtain, for all  $x \in (-1, 1)$ ,

$$\sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(x)}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \int_0^x \Lambda_{\mathbf{bt}(l_1), s_1} \circ \dots \circ \Lambda_{\mathbf{bt}(l_d), s_d} \circ \overleftarrow{a_n(t)}, \tag{17}$$

where  $\Lambda_{\mathbf{bt}(l), s}$  are obtained from  $\lambda_{\mathbf{bt}(l), s}$  by applying the substitution  $t \rightarrow \sin^{-1} t$ .

**Proof.** By (11), (13), and (15), we readily obtain

$$\sum_{m > n} \frac{a_m(\sin y)}{(2m)^s} = \int_0^y F_s(1 - \csc t \, dt \circ \sec t) a_n(\sin t) \tan t \, dt, \tag{18}$$

$$\sum_{m > n} \frac{a_m(\sin y)}{(2m + 1)^s} = \csc y \int_0^y F_s(1 - dt \circ \csc t \sec t) a_n(\sin t) \sin t \tan t \, dt. \tag{19}$$

$$\sum_{m > n} \frac{a_m(\sin y)}{2m - 1} = \cos y \int_0^y a_n(\sin t) \tan t \sec t \, dt, \tag{20}$$

$$\sum_{m > n} \frac{a_m(\sin y)}{(2m - 1)^s} = \sin y \int_0^y F_{s-1} \frac{dt}{\tan^2 t} a_n(\sin t) \tan t \sec t \, dt \quad (s \geq 2). \tag{21}$$

Thus,

$$\lambda_{\alpha, s}(t) = F_s(\tan t \, dt - \csc t \, dt \circ \sec t) = F_s(\csc t \, dt \circ \sec t - \cot t \, dt), \tag{22}$$

$$\lambda_{\beta, s}(t) = \overleftarrow{\csc t} \circ F_s(\sin t \tan t \, dt - dt \circ \sec t) = \overleftarrow{\csc t} \circ F_s(dt \circ \sec t - \cos t \, dt), \tag{23}$$

$$\lambda_{\gamma, 1}(t) = \overleftarrow{\cos t} \circ d \sec t, \tag{24}$$

$$\lambda_{\gamma, s}(t) = \overleftarrow{\sin t} \circ F_{s-1} \frac{dt}{\tan^2 t} d \sec t = \overleftarrow{\sin t} \circ F_{s-1} \left( \frac{dt}{\sin t \tan t} - \frac{dt}{\tan^2 t} \circ \sec t \right) \quad (s \geq 2). \tag{25}$$

Indeed, all the above identities follow directly from the definition. We will demonstrate (22) in detail to illustrate the ideas. We only need to show that for any fixed number  $c$  and variable  $x$ ,

$$\int_c^x \tan t \, dt - \int_c^x (\sec t - \sec c) \csc t \, dt = \int_c^x \sec c \csc t \, dt - \int_c^x \cot t \, dt.$$

This is clear when  $x = c$ . Thus, it suffices to show the two sides have the same derivative, which is straight-forward to verify using the trigonometric identity  $\tan t + \cot t = \sec t \csc t$ . Similarly, (23) follows from the trigonometric identity  $\sin t \tan t + \cos t = \sec t$  while (25) follows from  $\sec t / \tan^2 t = 1 / (\sin t \tan t)$ .

For convenience, we call the expression on both sides of (18) (resp. (19), resp. (20), and (21)) an  $\alpha$ -block (resp.  $\beta$ -block, resp.  $\gamma$ -block). In (16), each  $s_j$  corresponds to (a variation of) such a block. We find that after starting with a block in (18)–(21), we can repeatedly apply (18)–(21) to insert all the middle blocks until the end. This concludes the constructive proof of the theorem.  $\square$

5.2. Evaluate Even/Odd Variations of Apéry-Type Binomial Series by CMZVs at Level 4

Let  $d \in \mathbb{N}$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ . Let  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ . Set

$$\Xi(\mathbf{l}; \mathbf{s}) := \sum_{n_1 \succ_j \dots \succ_{d-1} n_d \succ_d 0} \frac{a_{n_1}}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}}, \tag{26}$$

where  $\succ_j$  is “ $\geq$ ” if  $l_j(n) = 2n + 1$  and is “ $>$ ” otherwise. Due to Theorem 6, we can always assume that if the leading block is a  $\gamma$ -block, then  $s_1 \geq 2$ .

**Theorem 8.** Let  $\text{CMZV}_w^4$  be the  $\mathbb{Q}$ -vector space spanned by all the CMZVs of weight  $w$  at level 4. Set

$$\delta(l) = \begin{cases} 0, & \text{if } l(n) = 2n; \\ 1, & \text{otherwise,} \end{cases} \quad \nu(l) = \begin{cases} 0, & \text{if } l_1(n) = 2n - 1; \\ 1, & \text{otherwise.} \end{cases}$$

- (a) If  $l_1(n) \neq 2n - 1$ , then we have  $\Xi(\mathbf{l}; \mathbf{s}) \in i^{\delta(l_1)} \text{CMZV}_{|\mathbf{s}|}^4$ .
- (b) Suppose  $(\mathbf{bt}(l_1), s_1) \neq (\gamma, 1)$ . Then, we have  $\Xi(\mathbf{l}; \mathbf{s}) \in \text{CMZV}_{|\mathbf{s}|-\nu(l_1)}^4 \otimes \mathbb{Q}[i]$ .
- (c) Moreover, the claim in (b) still holds if one changes any of the strict inequalities  $n_j > n_{j+1}$  in (26) to  $n_j \geq n_{j+1}$  ( $n_{d+1} = 0$ ) and vice versa, provided the series is defined. In particular,

$$\sum_{n_1 \succ \dots \succ_{n_d} \succ 0} \frac{a_{n_1}}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} \in \text{CMZV}_{\leq |\mathbf{s}|-\nu(l_1)}^4 \otimes \mathbb{Q}[i], \tag{27}$$

where “ $\succ$ ” can be either “ $\geq$ ” or “ $>$ ”, provided the series is defined.

**Proof.** We first claim that we may reduce the sums in the first two cases to those sums where  $l_j(n) = 2n - 1$  appears only when  $j = 1$ , if it ever appears. We can prove this by induction on the depth in exactly the same way as was used in the proof of ([1] Thm. 4.2(b)) and thus we omit it here.

(a) Recall that (11) and (14) provide us the following iterative structure:

$$\sum_{m > n} \frac{a_m(\sin y)}{(2m)^s} = \int_0^y F_s(1 - \csc t \, dt \circ \sec t) a_n(\sin t) \tan t \, dt, \tag{28}$$

$$\sum_{m \geq n} \frac{a_m(\sin y)}{(2m + 1)^s} = \csc y \int_0^y F_s(\csc t - dt \circ \sec t) a_n(\sin t) \tan t \, dt, \tag{29}$$

where  $F_s = (\cot t \, dt)^{s-1}$ . The key idea is to use (28) and (29) repeatedly to express (26) as an iterated integral and then use the change of variable  $t \rightarrow \sin^{-1}[(1 - t^2)/(1 + t^2)]$  to convert this iterated integral to a  $\mathbb{Q}$ -linear combination of iterated integrals that are clearly in  $\text{CMZV}_{|\mathbf{s}|}^4$ .

To begin, similarly to the proof of Theorem 7, we call the iterated integral of form (28) in an iteration an  $\alpha$ -block, and the iterated integral of form (29) in an iteration a  $\beta^*$ -block. The extra function  $\csc y$  before the  $\beta^*$ -block brings the main complication into this process since it changes the shape of the block in front of it.

The first reduction step is that we may assume  $\beta$ -block does not appear. Indeed, the equality

$$\sum_{m > n} f(m, n) = \sum_{m \geq n} f(m, n) - f(n, n)$$

clearly decomposes all  $\beta$ -blocks into  $\beta^*$ -blocks.

We now assume only  $\alpha$ - and  $\beta^*$ -blocks appear in the sum, since  $l_1(n) \neq 2n - 1$  by assumption. Then, there are only four different types of the first block:

**1st-block**  $\alpha \circ \alpha$ : first block is an  $\alpha$ -block, followed by another  $\alpha$ -block or nothing:

$$F_s(1 - \csc t dt \circ \sec t) \tan t dt = F_s(\tan t dt - \csc t \sec t dt + \csc t dt \circ \sec t) = F_s(\csc t dt \circ \sec t - \cot t dt). \tag{30}$$

**1st-block**  $\alpha \circ \beta^*$ : first block is an  $\alpha$ -block, followed by a  $\beta^*$ -block:

$$F_s(1 - \csc t dt \circ \sec t) \sec t dt = F_s(\sec t dt - \csc t dt d \tan t) = F_s(\csc t dt \circ \tan t). \tag{31}$$

**1st-block**  $\beta^* \circ \alpha$ : first block is a  $\beta^*$ -block, followed by an  $\alpha$ -block:

$$F_s(\csc t - dt \circ \sec t) \tan t dt = F_s(\sec t dt - dt d \sec t) = F_s(dt \circ \sec t). \tag{32}$$

**1st-block**  $\beta^* \circ \beta^*$ : first block is  $\beta^*$ -block, followed by another  $\beta^*$ -block:

$$F_s(\csc t - dt \circ \sec t) \sec t dt = F_s(\csc t \sec t dt - dt d \tan t) = F_s(\cot t dt + dt \circ \tan t).$$

The 1-forms appearing in the above are listed as follows:

$$\csc t dt, \cot t dt, dt. \tag{33}$$

Moreover, the observation below will be crucial later, as follows:

(\*) The 1-form  $dt$  appears only when a  $\beta^*$ -block appears first and it always has a trailing  $\circ \sec t$  (resp.  $\circ \tan t$ ) if it is followed by an  $\alpha$ -block (resp.  $\beta^*$ -block).

Hence, all the blocks after the first may (or may not) be multiplied by either  $\tan t$  or  $\sec t$ .

Similarly, for the middle blocks (i.e., neither initial nor end), we have the following four cases:

**Mid-block**  $\alpha \circ \alpha$  :  $(1 \text{ or } \sec t)F_s(\csc t dt \circ \sec t - \cot t dt), \tag{34}$

**Mid-block**  $\alpha \circ \beta^*$  :  $(1 \text{ or } \sec t)F_s(\csc t dt \circ \tan t), \tag{35}$

**Mid-block**  $\beta^* \circ \alpha$  :  $(1 \text{ or } \tan t)F_s(dt \circ \sec t), \tag{36}$

**Mid-block**  $\beta^* \circ \beta^*$  :  $(1 \text{ or } \tan t)F_s(\cot t dt + dt \circ \tan t). \tag{37}$

Therefore, the following additional 1-forms may appear:

$$\tan t dt, \sec t \csc t dt. \tag{38}$$

We now turn to the ending blocks. Since

$$(\tan t dt - \csc t dt \int_0^t \sec x \tan x dx) = (\tan t dt - (\sec t - 1) \csc t dt) = (\csc t - \cot t) dt,$$

$$(\tan t \csc t dt - dt \int_0^t \sec x \tan x dx) = (\sec t dt - (\sec t - 1) dt) = dt,$$

we may have the following forms for the end block:

$s_d \geq 2$ , **End-block**  $\alpha$  :  $(1 \text{ or } \sec t)F_s(\csc t - \cot t) dt, \tag{39}$

$s_d \geq 2$ , **End-block**  $\beta^*$  :  $(1 \text{ or } \tan t)F_s dt. \tag{40}$

If  $s = 1$ , then since

$$\tan t(\csc t - \cot t) = \sec t - 1, \quad \sec t(\csc t - \cot t) = \csc t(\sec t - 1)$$

we see that the end block has the form

$$s_d = 1, \text{ End-block } \alpha : \quad (\csc t - \cot t) dt \text{ or } \csc t(\sec t - 1) dt, \tag{41}$$

$$s_d = 1, \text{ End-block } \beta^* : \quad dt \text{ or } \tan t dt. \tag{42}$$

Under the change of variable  $t \rightarrow \sin^{-1}[(1 - t^2)/(1 + t^2)]$ , we have

$$\begin{aligned} dt &\rightarrow id_{-i,i}, & \cot t dt &\rightarrow y, & \tan t dt &\rightarrow z, \\ \csc t dt &\rightarrow d_{-1,1}, & \sec t dt &\rightarrow -a, & \sec t \csc t dt &\rightarrow y + z, \end{aligned} \tag{43}$$

where  $x_\zeta = dt/(\zeta - t)$ ,  $d_{\zeta,\zeta'} = x_\zeta - x_{\zeta'}$ ,  $y = x_{-i} + x_i - x_{-1} - x_1$  and  $z = -a - x_{-i} - x_i$ . We see that under the above change of variable, which reverses the order of the 1-forms, the 1-form  $a$  does not appear at the end (see (33)). On the other hand, the only 1-forms that can appear at the beginning are

$$\begin{aligned} dt &\rightarrow id_{-i,i}, & \tan t dt &\rightarrow z, & (\csc t - \cot t) dt &\rightarrow d_{-1,1} - y = 2x_{-1} - x_{-i} - x_i, \\ \sec t dt &\rightarrow -a, & \csc t(\sec t - 1) dt &\rightarrow y + z - d_{-1,1} = -a - 2x_{-1}. \end{aligned}$$

The key observation is that  $x_1$  does not appear at the beginning. Consequently, all the iterated integrals are convergent and provide the real or the imaginary part of some admissible CMZVs at level 4.

To determine exactly whether it is the real or the imaginary part, we need to count the number of  $id_{-i,i}$ 's, the only 1-form that has the imaginary coefficient  $i$ , which is produced only by the original 1-form  $dt$  in the trigonometric iterated integral expression. To do this, we break into two cases, guided by the crucial observation ( $\star$ ) above. To save space, we denote by  $N(dt)$  the number of 1-form  $dt$  in the trigonometric iterated integral expression of the sum in (26).

(A) The starting block is  $\beta^*$ . When followed by another  $\beta^*$ -block, the trailing  $\tan t$  can only be combined with two other forms, namely,  $\cot t dt$  or  $dt \circ \tan t$  (see (37)), which produces  $dt^2$  or  $dt(\tan t dt) \circ \tan t$ . Repeating this until the end block if no  $\alpha$ -block appears, or until a transition  $\beta^*$ - $\alpha$  block chain emerges (36), we see that either (i) there are even number of  $dt$ 's without trailing  $\circ \tan t$  or (ii) there are odd number of  $dt$ 's with a trailing  $\circ \tan t$ . If  $\alpha$ -block does not appear at all, then the end block is given by (40) ( $s = s_d \geq 2$ ) and (42) ( $s = s_d = 1$ ) so that the parity changes in case (i) or the parity does not change in case (ii). If there is a transition  $\beta^*$ - $\alpha$  block chain, then there is a trailing  $\circ \sec t$  produced while there are odd number of  $dt$ 's in front of  $\circ \sec t$ . By case (B) below, we see that the number of  $dt$ 's after this transition must be even. To summarize, we see that  $N(dt)$  is always odd if the starting block is  $\beta^*$ .

(B) The starting block is  $\alpha$ . If there is no  $\beta^*$ -block, then clearly  $N(dt) = 0$  by (30), (34), (39), and (41). Suppose  $\beta^*$ -block does appear. By (36) and (37), we see that this block produces either

- (i) an even number of  $dt$ 's with a trailing  $\circ \sec t$  when followed by an  $\alpha$ -block, or
- (ii) an even number of  $dt$ 's with a trailing  $\circ \tan t$  when followed by a  $\beta^*$ -block, or
- (iii) an odd number of  $dt$ 's without a trailing function.

Case (i) leads to no  $dt$ 's until the next  $\beta^*$ -block appears. For the other two cases, we can argue exactly as in case (A) above and show that these repeat until the end or until a transition  $\alpha$ - $\beta^*$  block chain (back to case (i)). Repeating the above argument in the

three cases (i)–(iii), we see that until the end, we still have the same three cases. If the end block is  $\alpha$ , then we must be back in case (i) and by (39) and (41), this block does not produce the 1-form  $dt$  so that  $N(dt)$  is even. If the end the block is  $\beta^*$ , then it has the form  $(\tan t)F_s dt$  in case (ii) and  $F_s dt$  in case (iii). In case (ii), either two  $dt$ 's or no  $dt$  is produced so it does not change the parity of  $N(dt)$ . In case (iii), one  $dt$  is produced which changes  $N(dt)$  to even. To summarize, we see that  $N(dt)$  is always even if the starting block is  $\alpha$ .

Furthermore, (b) and (c) can be proved by the same proof as that for ([1] Thm. 4.2(b–d)). We may first reduce the general case to the case where if  $\gamma$ -block appears then it only appears as the first block, in which case we can assume the weight of this block  $s \geq 2$  by Theorem 6. We now further consider two subcases: (i) the second block is an  $\alpha$ -block of weight  $b$  and (ii) the second block is a  $\beta^*$ -block of weight  $b$ . Thus, we have two kinds of iterated integrals to consider, as follows:

$$\begin{aligned} (i) : & \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} \tan t \sec t dt F_{b-1} (\csc t - \cot t) dt \dots, \\ (ii) : & \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} \sec^2 t dt F_{b-1} dt \dots \end{aligned}$$

The claim in the theorem then follows immediately from the computation in Examples A5 and A6. Indeed, from the examples, we see that the claims in the theorem hold when the first two blocks are given by either  $\gamma$ - $\alpha$ -block or  $\gamma$ - $\beta^*$ -block chains. But if there are more blocks after these two, the argument in case (a) applies since these additional ones are either  $\alpha$ -blocks or  $\beta^*$ -blocks.

This completes the proof of the theorem.  $\square$

**Remark 4.** The theorem generalizes the first inclusion relation in ([29] Thm. 9.6).

**Corollary 2.** Suppose  $d \in \mathbb{N}$ ,  $s = (s_1, \dots, s_d) \in \mathbb{N}^d$ . Let  $l_1(n), \dots, l_d(n) = 2n, 2n + 1$ . Then, we have

$$\begin{aligned} \sum_{n_1 > n_2 \succ \dots \succ n_d \succ_d 0} \frac{a_{n_1}}{n_1^{s_1} l_2(n_2)^{s_2} \dots l_d(n_d)^{s_d}} &\in \text{CMZV}_{|s|}^4, \\ \sum_{n_1 \geq n_2 \succ \dots \succ n_d \succ_d 0} \frac{a_{n_1}}{(2n_1 + 1)^{s_1} l_2(n_2)^{s_2} \dots l_d(n_d)^{s_d}} &\in i\text{CMZV}_{|s|}^4, \\ \sum_{n_1 \succ \dots \succ n_d \succ_d 0} \frac{a_{n_1}}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} &\in \text{CMZV}_{|s|}^4 \otimes \mathbb{Q}[i], \end{aligned} \tag{44}$$

where  $\succ_d$  is “ $\geq$ ” or “ $>$ ” depending on whether  $l_d(n) = 2n + 1$  or  $2n$ , and all other  $\succ$ 's are either “ $\geq$ ” or “ $>$ ”, as long as the series are defined.

**Proof.** The proof easily follows from the fact that using the Principle of Inclusion and Exclusion, we may convert  $\geq$  to  $>$  (and vice versa) using the partial fractions.  $\square$

**Remark 5.** Since the proofs of Theorem 8 and Corollary 2 are both constructive, we see that every sum of the form (26), or more generally of the form (44), can be computed exactly in terms of the CMZVs at level 4.

We summarize the results of Theorems 6 and 8 and Corollary 2 in Table 1. Set

$$\Xi(l; s; \text{Rel}) := \sum_{n_1 \succ \dots \succ n_d \succ 0} \frac{\binom{2n_1}{n_1}}{4^{n_1} l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}},$$

where  $\text{Rel} = \{>, \dots, >\}$  represents any fixed choice of all the relations “ $>$ ” which can be either “ $\geq$ ” or “ $>$ ”, provided the series is defined. Also, put  $\mathbf{l}' = (l_2, \dots, l_d)$  and  $\mathbf{s}' = (s_2, \dots, s_d)$ .

**Table 1.** Summary of Theorem 8, where  $\alpha/\beta$  means either  $\alpha$  or  $\beta$ .

| $\mathbf{bt}(\mathbf{l}_1, \dots, \mathbf{l}_d)$              | Rel                          | $\Xi(\mathbf{l}; \mathbf{s}; \text{Rel}) \in$      |
|---|------------------------------|--|
| $(\alpha, \{\alpha/\beta\}^{d-1})$                            | $\{>, >, \dots, >, >_d\}$    | $\text{CMZV}_{ s }^4$                              |
| $(\beta, \{\alpha/\beta\}^{d-1})$                             | $\{\geq, >, \dots, >, >_d\}$ | $i\text{CMZV}_{ s }^4$                             |
| $(\{\alpha/\beta\}^d)$  | $\{>, \dots, >\}$            | $\text{CMZV}_{ s }^4 \otimes \mathbb{Q}[i]$        |
| $(\gamma, \{\alpha/\beta/\gamma\}^{d-1}), (s_1 = 1)$          | $\{>, \text{Rel}'\}$         | $= \Xi(\mathbf{l}'; \mathbf{s}'; \text{Rel}')$     |
| $(\gamma, \{\alpha/\beta/\gamma\}^{d-1}), (s_1 \geq 2)$       | $\{>_1, \dots, >_d\}$        | $i\text{CMZV}_{ s -1}^4$                           |
| $(\alpha/\beta, \{\alpha/\beta/\gamma\}^{d-1})$               | $\{>, \dots, >\}$            | $\text{CMZV}_{\leq  s }^4 \otimes \mathbb{Q}[i]$   |
| $(\gamma/\beta, \{\alpha/\beta/\gamma\}^{d-1}), (s_1 \geq 2)$ | $\{>, \dots, >\}$            | $\text{CMZV}_{\leq  s -1}^4 \otimes \mathbb{Q}[i]$ |

5.3. Typical Examples of Even/Odd Variations of Apéry-Type Binomial Series

In this subsection, we will compute a few examples of the form

$$\Xi(\mathbf{bt}(\mathbf{l}); \mathbf{s}) = \Xi(\mathbf{l}; \mathbf{s}) := \sum_{\substack{n_1 > \dots > \\ 1 \quad d-1 \quad d}} \sum_{n_d > 0} \frac{a_{n_1}}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}}, \quad a_{n_1} = \frac{\binom{2n_1}{n_1}}{4^{n_1}}.$$

We first convert these even/odd variations of Apéry-type binomial series to the CMZVs at level 4 by Theorem 8, and then we use Au’s package [30] to simplify the expressions. We also note that under the change of variable

$$t \rightarrow \sin^{-1} t \quad \text{followed by} \quad t \rightarrow \frac{1-t^2}{1+t^2} \tag{45}$$

we have

$$\cot t \, dt \rightarrow \omega_0 := \frac{dt}{t} \rightarrow y, \quad \csc t \, dt \rightarrow \omega_3 := \frac{dt}{t\sqrt{1-t^2}} \rightarrow d_{-1,1}, \tag{46}$$

$$dt \rightarrow \omega_1 := \frac{dt}{\sqrt{1-t^2}} \rightarrow id_{-i,i}, \quad \sec t \, \csc t \, dt \rightarrow \omega_{20} := \frac{dt}{t(1-t^2)} \rightarrow y + z, \tag{47}$$

$$\tan t \, dt \rightarrow \omega_2 := \frac{t \, dt}{1-t^2} \rightarrow z, \quad \sec t \, dt \rightarrow \omega_8 := \frac{dt}{1-t^2} \rightarrow -a. \tag{48}$$

**Example 3.** For a single  $\alpha$ -block or  $\beta^*$ -block, and any  $s \in \mathbb{N}$ , we have (39) and (40)

$$\Xi(\alpha; s) = \sum_{n \geq 0} \frac{a_n}{(2n)^s} = \int_0^{\pi/2} F_s(\csc t - \cot t) \, dt = (-1)^s \int_0^1 (2x_{-1} - x_{-i} - x_i) y^{s-1},$$

$$\Xi(\beta; s) = \sum_{n \geq 0} \frac{a_n}{(2n+1)^s} = \int_0^{\pi/2} F_s \, dt = (-1)^s i \int_0^1 d_{-i,i} y^{s-1}$$

by (46)–(48). Here, we point out that after change of variable (45), we need to multiply  $(-1)^w$  when reversing the interval back to  $[0, 1]$ , where  $w$  is the weight of the value, i.e., the number of 1-forms in the iterated integral. In particular, when  $s \leq 3$ , we have

$$\Xi(\alpha; 1) = \log 2, \quad \Xi(\alpha; 2) = \frac{\pi^2 - 12 \log^2 2}{24}, \quad \Xi(\alpha; 3) = \frac{4 \log^3 2 - \pi^2 \log 2 + 6\zeta(3)}{24}, \tag{49}$$

$$\Xi(\beta; 1) = \frac{\pi}{2}, \quad \Xi(\beta; 2) = \frac{\pi \log 2}{2}, \quad \Xi(\beta; 3) = \frac{\pi}{48} (\pi^2 + 12 \log^2 2). \tag{50}$$

**Example 4.** For a double  $\beta^*$ -block, for any  $s \in \mathbb{N}$  we have, by (37) and (40)

$$\begin{aligned} \Xi(\beta, \beta; s, 1) &= \sum_{n_1 \geq n_2 \geq 0} \frac{a_{n_1}}{(2n_1 + 1)^s (2n_2 + 1)} \\ &= \int_0^{\pi/2} F_s(\cot t \, dt \, dt + dt \, \tan t \, dt) = (-1)^{s-1} i \int_0^1 (\mathbf{d}_{-i,i} \mathbf{y} + \mathbf{z} \mathbf{d}_{-i,i}) \mathbf{y}^{s-1}. \end{aligned}$$

When  $s \leq 3$ , we obtain

$$\begin{aligned} \Xi(\beta, \beta; 1, 1) &= \pi \log 2, \\ \Xi(\beta, \beta; 2, 1) &= \frac{3\pi \log^2 2}{4}, \\ \Xi(\beta, \beta; 3, 1) &= \frac{\pi}{48} (\pi^2 \log 2 + 16 \log^3 2 + 3\zeta(3)). \end{aligned}$$

**Example 5.** We now consider the series given by a  $\beta^*$ - $\alpha$ -block chain. For any  $s \in \mathbb{N}$ , we obtain

$$\begin{aligned} \Xi(\beta, \alpha; s, 1) &= \sum_{n_1 \geq n_2 > 0} \frac{a_{n_1}}{(2n_1 + 1)^s (2n_2)} = \int_0^{\pi/2} F_s \, dt \circ \sec t \sum_{n_2 > 0} \frac{a_{n_2}(\sin t)}{2n_2} \quad (\text{by (32)}) \\ &= \int_0^{\pi/2} F_s \, dt (\sec t \, \csc t \, dt - \csc t \, dt) = (-1)^s i \int_0^1 (\mathbf{a} + 2\mathbf{x}_{-1}) \mathbf{d}_{-i,i} \mathbf{y}^{s-1}. \end{aligned}$$

by (41). When  $s = 1$ , we obtain

$$\begin{aligned} \Xi(\beta, \alpha; 1, 1) &= 2\text{Im}(2 \text{Li}_{1,1}(-1, -i) - \text{Li}_2(i)) = 2G - \frac{1}{2} \pi \log 2 \approx 0.7431, \\ \Xi(\beta, \alpha; 2, 1) &= \frac{3\pi^3}{32} - 4\text{Im} \text{Li}_3\left(\frac{1+i}{2}\right) - \frac{3\pi \log^2 2}{8} \approx 0.0605, \end{aligned}$$

where  $G$  is Catalan’s constant defined by (3). By (49) and (50)

$$\begin{aligned} \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 + 1)(2n_2)} &= \Xi(\beta, \alpha; 1, 1) - \sum_{n > 0} \frac{a_n}{(2n + 1)(2n)} \\ &= \Xi(\beta, \alpha; 1, 1) - \log 2 + \frac{\pi}{2} - 1 \\ &= 2G - \frac{1}{2} \pi \log 2 - \log 2 + \frac{1}{2} \pi - 1 \approx 0.6207. \end{aligned}$$

Let  $H_n = 1 + 1/2 + \dots + 1/n$  be the  $n$ th harmonic number. By combining the above examples, we have

$$\sum_{n \geq 0} \frac{a_n H_{2n}}{2n + 1} = \Xi(\beta, \alpha; 1, 1) + \Xi(\beta, \beta; 1, 1) - \Xi(\beta; 2) = 2G$$

which is consistent with a formula on ([31] p. 10). Similarly, all the four formulas on the top of ([31], p. 10) can be verified.

We have computed more examples with different types of blocks and put them in Appendix A. For the convenience of the readers, we have summarized all these examples in Table 2. Recall that the block type  $\mathbf{bt}(l) = \alpha, \beta, \gamma$  if  $l(n) = 2n, 2n + 1, 2n - 1$ , respectively. Further, set  $\mathbf{bt}(l_1, \dots, l_d) = (\mathbf{bt}(l_1), \dots, \mathbf{bt}(l_d))$ .

**Table 2.** Summary of Examples 3–5, A1–A8, where  $V = \text{CMZV}^4$  and  $\delta_{c,1}$  is the Kronecker symbol.

| $\mathbf{bt}(l)$             | $s$                          | $\Xi(l; s) \in$                          | Example # |
|------------------------------|------------------------------|--|-----------|
| $(\alpha/\beta)$             | $s$                          | $V_s$                                    | 3         |
| $(\beta^*, \beta^*)$         | $(s, 1)$                     | $iV_{s+1}$                               | 4         |
| $(\beta^*, \alpha)$          | $(s, 1)$                     | $iV_{s+1}$                               | 5         |
| $(\alpha, \beta^*)$          | $(s, 1)$                     | $V_{s+1}$                                | A1        |
| $(\beta^*, \alpha, \alpha)$  | $(s, 1, 1)$                  | $iV_{s+2}$                               | A2        |
| $(\beta^*, \alpha, \beta^*)$ | $(s, 1, 1)$                  | $iV_{s+2}$                               | A3        |
| $(\alpha, \gamma)$           | $(s, 1)$                     | $V_{\leq s+1}$                           | A4        |
| $(\gamma, \alpha)$           | $(s, 1)$                     | $iV_{\leq s} + V_1$                      | A5        |
| $(\gamma, \alpha)$           | $(s, b), s \geq 2, b \geq 2$ | $iV_{\leq s+b-1} + V_b$                  | A5        |
| $(\gamma, \beta^*)$          | $(s, 1), b \geq 2$           | $iV_{\leq s} + V_1$                      | A6        |
| $(\gamma, \beta^*)$          | $(s, b), s \geq 2, b \geq 2$ | $iV_{\leq s+b-1}$                        | A6        |
| $(\gamma, \gamma, \beta^*)$  | $(s, b, c)$                  | $iV_{\leq s+b+c-1} + \delta_{c,1}V_1$    | A7        |
| $(\gamma, \alpha, \gamma)$   | $(s, b, c)$                  | $V_{\leq s+b+c-1} \otimes \mathbb{Q}[i]$ | A8        |

In general, we can apply Theorem 8 to compute even every possible variation of  $\Xi(l; s)$  in which the index ordering can be change to either strict or non-strict inequalities, in terms of the CMZVs at level 4. However, the general case needs the complicated process of using partial fractions, which is demonstrated in Example A8. Also, from Table 2, we can see clearly that the weight drops only when the leading block is a  $\gamma$ -block.

### 6. Apéry-Type Binomial Series Involving Squares of Central Binomial Coefficients

In this section, we will consider another class of Apéry-type binomial series by replacing  $a_n$  by  $a_n^2$  in all the series appearing in the previous sections.

**Theorem 9.** Let  $d \in \mathbb{N}, s = (s_1, \dots, s_d) \in \mathbb{N}^d$ . Let  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ .

(a) Let  $l_1(n), \dots, l_d(n) = 2n, 2n + 1$ . Then, we have

$$\Psi(l; s) := \sum_{\substack{n_1 \succ_1 \dots \succ_{d-1} n_d \succ_d 0}} \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} \in \frac{i}{\pi} \text{CMZV}_{|s|+1}^4,$$

where  $\succ_j$  is “ $\geq$ ” if  $l_j(n) = 2n + 1$ , and is “ $>$ ” otherwise.

(b) More generally, if  $l_1(n), \dots, l_d(n) = 2n, 2n \pm 1$ , then we have

$$\sum_{n_1 \succ_1 n_2 \succ_2 \dots \succ_{d-1} n_d \succ_d 0} \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} \in \frac{1}{\pi} \text{CMZV}_{\leq |s| + \eta(s_1, l)}^4 \otimes \mathbb{Q}[i],$$

where “ $\succ$ ” is either “ $\geq$ ” or “ $>$ ” provided the series is defined, and

$$\eta(s_1, l) = \begin{cases} 0, & \text{if } \mathbf{bt}(l_1) = \gamma, s_1 = 1 \text{ and } d = 1; \\ 0, & \text{if } \mathbf{bt}(l_1) = \gamma, s_1 = 1 \text{ and } \mathbf{bt}(l_2) = \alpha; \\ -1, & \text{if } \mathbf{bt}(l_1) = \gamma, s_1 = 1 \text{ and } \mathbf{bt}(l_2) \neq \alpha; \\ -1, & \text{if } \mathbf{bt}(l_1) = \gamma, s_1 \geq 2; \\ 1, & \text{if } \mathbf{bt}(l_1) \neq \gamma. \end{cases} \tag{51}$$

**Proof.** The key observation is to apply the Wallis integral

$$\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} \sin^{2n} t dt = \frac{1}{2} B\left(n + \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{2} a_n.$$

In the following proof, we drop the restriction on the summation indices to save space.

(i) When  $l_1(n) = 2n + 1$  by (16), we see that the sum

$$\sum \frac{a_{n_1}(\sin y)}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \csc y \int_0^y F_{s_1}(\sin t \tan t dt - dt d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}.$$

Thus, integrating over  $(0, \pi/2)$  and dividing by  $\pi/2$ , we obtain

$$\sum \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \frac{2}{\pi} \int_0^{\pi/2} \csc t dt F_{s_1}(\sin t \tan t dt - dt d \sec t) \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}. \tag{52}$$

The claim follows immediately since there are odd number of  $dt$ 's in this case, as shown in the proof of Theorem 8(a).

(ii) If  $l_1(n) = 2n$ , then we see that

$$\sum \frac{a_{n_1}(\sin y)}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \int_0^y F_{s_1}(\tan t dt - \csc t dt d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}.$$

Thus, integrating over  $(0, \pi/2)$  and dividing by  $\pi/2$ , we obtain

$$\sum \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \frac{2}{\pi} \int_0^{\pi/2} dt F_{s_1}(\tan t dt - \csc t dt d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}. \tag{53}$$

The corollary holds as well in this case as the number of  $dt$ 's is changed to odd because of the leading  $dt$ , since originally, as shown in the proof of Theorem 8(a), the number of  $dt$ 's was even. The appearance of  $\eta(s_1, l) = 0$  is due to the special behavior of  $\alpha$ -block as manifested by Example A12.

(iii) If  $l_1(n) = 2n - 1$ , then there are two cases. By (16), if  $s_1 = 1$ , then we have

$$\sum \frac{a_{n_1}(\sin y)}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \cos y \int_0^y (d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}.$$

Hence, integrating over  $(0, \pi/2)$  and dividing by  $\pi/2$ , we obtain

$$\sum \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \frac{2}{\pi} \int_0^{\pi/2} \cos t dt (d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}.$$

If  $s_1 \geq 2$ , then

$$\sum \frac{a_{n_1}(\sin y)}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \sin y \int_0^y F_{s-1} \frac{dt}{\tan^2 t} (d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}. \tag{54}$$

Thus, integrating over  $(0, \pi/2)$  and dividing by  $\pi/2$ , we obtain

$$\sum \frac{a_{n_1}^2}{l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}} = \frac{2}{\pi} \int_0^{\pi/2} \sin t dt F_{s-1} \frac{dt}{\tan^2 t} (d \sec t) \circ \prod_{j=2}^d \lambda_{l_j, s_j} \circ \overleftarrow{a_n(\sin t)}. \tag{55}$$

To complete the proof of the theorem, we only need to pay attention to the CMZV weight increasing in (a) and weight drop phenomenon in (b) and (c) associated with a leading  $\gamma$ -block (i.e.,  $\eta(l_1) = -1$  when  $\mathbf{bt}(l_1) = \gamma$ ). The first phenomenon in (a) is obvious from (i) and (ii) above. By (iii), it is also easy to see the weight cannot increase. We can show

that, in fact, the weight may drop by 1, and we can carry out a case-by-case study using the examples given in the next section and the Appendix. The two cases with  $\eta(s_1, \mathbf{l}) = 0$  are given by Example 8 ( $\mathbf{bt}(l_1) = \gamma, s_1 = 1$  and  $d = 1$ ), Example A13 ( $\mathbf{bt}(l_1, l_2) = (\gamma, \alpha), (s_1, s_2) = (1, s), s \geq 2$ ), and Example A14 ( $\mathbf{bt}(l_1, l_2) = (\gamma, \alpha), (s_1, s_2) = (1, 1)$ ). One can quickly find these in Table 4. This concludes the proof of the theorem.  $\square$

We summarize the results of Theorem 9 in Table 3. Set

$$\Psi(\mathbf{l}; \mathbf{s}; \text{Rel}) := \sum_{n_1 > \dots > n_d > 0} \frac{\binom{2n_1}{n_1}^2}{16^{n_1} l_1(n_1)^{s_1} \dots l_d(n_d)^{s_d}}.$$

**Table 3.** Summary of Theorem 9, where  $V = \text{CMZV}^4$  and  $\alpha/\beta$  means either  $\alpha$  or  $\beta$ .

| $\mathbf{bt}(\mathbf{l})$                         | Rel   | $\Psi(\mathbf{l}; \mathbf{s}; \text{Rel}) \in$                    |
|---|---|---|
| $(\{\alpha/\beta\}^d)$                            | $\{\underset{1}{\gamma}, \dots, \underset{d}{\gamma}\}$                       | $\frac{i}{\pi} V_{ s +1} V$                                       |
| $(\alpha/\beta, \{\alpha/\beta/\gamma\}^{d-1})$   | $\{\underset{1}{\gamma}, \underset{2}{\gamma}, \dots, \underset{d}{\gamma}\}$ | $\frac{1}{\pi} V_{\leq  s +1} V \otimes \mathbb{Q}[i]$            |
| $(\gamma, \beta, \{\alpha/\beta/\gamma\}^{d-2})$  | $\{\underset{1}{\gamma}, \underset{2}{\gamma}, \dots, \underset{d}{\gamma}\}$ | $\frac{1}{\pi} V_{\leq  s -1} V \otimes \mathbb{Q}[i]$            |
| $(\gamma, \emptyset, \dots, \emptyset)$           | $\{>\}$   | $\frac{1}{\pi} (V_{\leq  s -1} + iV_1)$                           |
| $(\gamma, \alpha, \{\alpha/\beta/\gamma\}^{d-2})$ | $\{\underset{1}{\gamma}, \underset{2}{\gamma}, \dots, \underset{d}{\gamma}\}$ | $\frac{1}{\pi} V_{\leq \max\{ s -1, 2\}} V \otimes \mathbb{Q}[i]$ |

In the following, we present a few examples illustrating the ideas in Theorem 9. We will adopt the convention  $\Psi(\mathbf{bt}(\mathbf{l}); \mathbf{s}) = \Psi(\mathbf{l}; \mathbf{s})$ .

**Example 6.** We consider the series given by an  $\alpha$ -block. From (39), (41), and (53), we have

$$\begin{aligned} \Psi(\alpha; \mathbf{s}) &= \sum_{n>0} \frac{a_n^2}{(2n)^s} = \frac{2}{\pi} \int_0^{\pi/2} dt F_s(\csc t - \cot t) dt \\ &= (-1)^s \frac{2i}{\pi} \int_0^1 (2x_{-1} - x_{-i} - x_i) y^{s-1} d_{-i,i}. \end{aligned}$$

Therefore,

$$\sum_{n>0} \frac{a_n^2}{2n} = -\frac{2i}{\pi} \int_0^1 (2x_{-1} - x_{-i} - x_i) d_{-i,i} = \frac{2}{\pi} (\pi \log 2 - 2G) \approx 0.22, \tag{56}$$

$$\begin{aligned} \sum_{n>0} \frac{a_n^2}{(2n)^2} &= \frac{2i}{\pi} \int_0^1 (2x_{-1} - x_{-i} - x_i) y d_{-i,i} \\ &= \frac{2}{\pi} \left( \frac{3}{16} \pi^3 - 8 \text{Im Li}_3 \left( \frac{1+i}{2} \right) - \frac{3}{4} \pi \log^2 2 \right) \approx 0.077, \end{aligned} \tag{57}$$

$$\begin{aligned} \sum_{n>0} \frac{a_n^2}{(2n)^3} &= -\frac{2i}{\pi} \int_0^1 (2x_{-1} - x_{-i} - x_i) y^2 d_{-i,i} \\ &= \frac{2}{\pi} \left( 24\beta(4) - 32 \text{Im Li}_4 \left( \frac{1+i}{2} \right) - \frac{3}{8} \pi^3 \log 2 + \frac{\pi}{4} (2 \log^3 2 + \zeta(3)) \right) \approx 0.034. \end{aligned}$$

**Example 7.** We consider the series given by a  $\beta^*$ -block. From (40), (42), and (52), we have

$$\Psi(\beta; \mathbf{s}) = \sum_{n \geq 0} \frac{a_n^2}{(2n+1)^s} = \frac{2}{\pi} \int_0^{\pi/2} \csc t dt F_s dt = (-1)^{s+1} \frac{2i}{\pi} \int_0^1 d_{-i,i} y^{s-1} d_{-1,1}.$$

Thus,

$$\begin{aligned} \sum_{n \geq 0} \frac{a_n^2}{2n+1} &= \frac{2i}{\pi} \int_0^1 d_{-i,i} d_{-1,1} = \frac{4G}{\pi} \approx 1.1662, \\ \sum_{n \geq 0} \frac{a_n^2}{(2n+1)^2} &= -\frac{2i}{\pi} \int_0^1 d_{-i,i} y d_{-1,1} = \frac{2}{\pi} \left( \frac{3}{16} \pi^3 - 8 \operatorname{Im} \operatorname{Li}_3 \left( \frac{1+i}{2} \right) + \frac{\pi}{4} \log^2 2 \right) \approx 1.0379, \\ \sum_{n \geq 0} \frac{a_n^2}{(2n+1)^3} &= \frac{2i}{\pi} \int_0^1 d_{-i,i} y^2 d_{-1,1} \\ &= \frac{2}{\pi} \left( \operatorname{Im} \operatorname{Li}_4 \left( \frac{1+i}{2} \right) - 24\beta(4) + \frac{3}{8} \pi^3 \log 2 + \frac{\pi}{6} \log^3 2 \right) \approx 1.01. \end{aligned}$$

**Example 8.** We consider the series given by a single  $\gamma$ -block. From (9), we have

$$\Psi(\gamma; 1) = \sum_{n > 0} \frac{a_n^2}{2n-1} = \frac{2}{\pi} \int_0^{\pi/2} \cos t dt d \sec t \tag{58}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} (1 - \cos t) dt = \frac{2}{\pi} \left( \frac{\pi}{2} - 1 \right) \approx 0.3633. \tag{59}$$

Note that there is no weight drop here so it is one of the two reasons we need to introduce  $\eta(s_1, \mathbf{l}) = 0$  in Theorem 9. The other reason is given in Examples A13 and A14. For all  $s \geq 2$

$$\begin{aligned} \Gamma_s := \Psi(\gamma; s) &= \sum_{n > 0} \frac{a_n^2}{(2n-1)^s} = \frac{2}{\pi} \int_0^{\pi/2} \sin t dt F_{s-1} \frac{dt}{\tan^2 t} d \sec t \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin t dt F_{s-1} (\csc^2 t - 1) dt d \sec t \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin t dt F_{s-1} (d(-\cot t) (d \sec t) - dt d \sec t) \\ &= -\Gamma_{s-1} + \frac{2}{\pi} \int_0^{\pi/2} \sin t dt F_{s-1} (\sec t dt - dt d \sec t) \\ &= -\Gamma_{s-1} + \frac{2}{\pi} \int_0^{\pi/2} \sin t dt F_{s-1} dt = -\Gamma_{s-1} + I_{s-1} \\ &= \frac{2}{\pi} \left( (-1)^s \int_0^{\pi/2} \sin t dt \frac{dt}{\tan^2 t} d \sec t + \sum_{j=2}^{s-1} (-1)^{s-1-j} I_j \right), \end{aligned}$$

where

$$\begin{aligned} I_j &= \int_0^{\pi/2} \sin t dt F_j dt = \int_0^{\pi/2} \cos t \cot t dt F_{j-1} dt = \int_0^{\pi/2} \csc t dt F_{j-1} dt - I_{j-1} \\ &= (-1)^{j-1} \int_0^{\pi/2} \sin t dt dt + \sum_{k=1}^{j-1} (-1)^{j-1-k} \int_0^{\pi/2} \csc t dt F_k dt, \\ &= (-1)^{j-1} + \sum_{k=1}^{j-1} (-1)^j i \int_0^1 d_{-i,i} y^{k-1} d_{-1,1}. \end{aligned}$$

Hence, for all  $s \geq 2$ , we obtain

$$\Gamma_s = \sum_{n > 0} \frac{a_n^2}{(2n-1)^s} = (-1)^s \frac{2}{\pi} \left( s - \frac{\pi}{2} - i \sum_{k=2}^{s-1} (s-k) \int_0^1 d_{-i,i} y^{k-2} d_{-1,1} \right).$$

Consequently,

$$\sum_{n > 0} \frac{a_n^2}{(2n-1)^2} = \frac{2}{\pi} \left( 2 - \frac{\pi}{2} \right) \approx 0.2732,$$

$$\sum_{n>0} \frac{a_n^2}{(2n-1)^3} = \frac{2}{\pi} \left( \frac{\pi}{2} + 2G - 3 \right) \approx 0.2563.$$

We have computed more examples with different types of blocks and put them in Appendix B. For the convenience of the readers, we have summarized all these examples in Table 4.

**Table 4.** Summary of Examples 6–8, A9–A17, where  $V = CMZV^4$ .

| $bt(l)$                    | $s$                      | $\Psi(l; s) \in$                           | Example #         |
|----------------------------|--------------------------|--|-------------------|
| $(\alpha)$                 | $(s)$                    | $\frac{i}{\pi} V_{s+1}$                    | 6                 |
| $(\beta^*)$                | $(s)$                    | $\frac{i}{\pi} V_{s+1}$                    | 7                 |
| $(\gamma)$                 | $(s), s \geq 2$          | $\frac{1}{\pi} (\mathbb{Q} + iV_{<s})$     | 8                 |
| $(\{\gamma\}^d)$           | $(\{1\}^d)$              | $\frac{1}{\pi} (V_{<d} + iV_1)$            | 8 ( $d = 1$ ), A9 |
| $(\gamma, \gamma)$         | $(s, 1)$                 | $\frac{1}{\pi} (V_{\leq 1} + iV_{\leq s})$ | A10               |
| $(\{\gamma\}^d)$           | $(2, \{1\}^{d-1})$       | $\frac{1}{\pi} (V_{<d} + iV_1)$            | A11               |
| $(\gamma, \alpha)$         | $(2, 1)$                 | $\frac{1}{\pi} (V_1 + iV_{\leq 2})$        | A12               |
| $(\{\gamma\}^d, \alpha)$   | $(\{1\}^d, s), s \geq 2$ | $\frac{1}{\pi} (V_{<s+d} + iV_{s,s+1})$    | A13               |
| $(\{\gamma\}^d, \alpha)$   | $(\{1\}^{d+1})$          | $\frac{1}{\pi} (V_{\leq d} + iV_{\leq 2})$ | A14               |
| $(\gamma, \alpha)$         | $(s, 2)$                 | $\frac{1}{\pi} (V_2 + iV_{\leq s+1})$      | A15               |
| $(\gamma, \alpha, \alpha)$ | $(s, 1, 1), s \geq 2$    | $\frac{1}{\pi} (V_2 + iV_{\leq s+1})$      | A16               |
| $(\{\gamma\}^d, \beta)$    | $(\{1\}^{d+1})$          | $\frac{1}{\pi} V_{\leq d}$                 | A17               |

**Corollary 3.** We have

$$\sum_{n=1}^{\infty} \frac{a_n^2 H_n^{(3)}}{2n-1} = \frac{2}{\pi} (16G + 4\pi - 8\pi \log 2 - \zeta(3) - 8).$$

**Proof.** By partial fractions, we have

$$\frac{1}{(2n-1)(2n)^3} = \frac{1}{2n-1} - \frac{1}{(2n)^3} - \frac{1}{(2n)^2} - \frac{1}{2n}.$$

Thus, by (58) and (A3)

$$\begin{aligned} \frac{1}{8} \sum_{n=1}^{\infty} \frac{a_n^2 H_n^{(3)}}{2n-1} &= \sum_{n=1}^{\infty} \frac{a_n^2}{(2n-1)(2n)^3} + \sum_{n>m>0} \frac{a_n^2}{(2n-1)(2m)^3} \\ &= \frac{2}{\pi} \left( \frac{\pi}{2} - 1 \right) + V_1(3) - \sum_{n=1}^{\infty} \frac{a_n^2}{(2n)^3} - \sum_{n=1}^{\infty} \frac{a_n^2}{(2n)^2} - \sum_{n=1}^{\infty} \frac{a_n^2}{2n}. \end{aligned}$$

Now, the last three sums are given in Example 6. The corollary follows immediately.  $\square$

### 7. A Corollary and Some Related Sums from Other Works

In this last section, we first answer affirmatively a few questions we posted at the end of [29]. For  $k \in \mathbb{N}^d$  and  $l \in \mathbb{N}^e$ , we define

$$\begin{aligned} \zeta_n(k) &:= \sum_{n \geq m_1 > \dots > m_d > 0} \frac{1}{m_1^{k_1} \dots m_d^{k_d}}, \\ t_n(l) &:= \sum_{n \geq r_1 > \dots > r_e > 0} \frac{1}{(2r_1-1)^{l_1} \dots (2r_e-1)^{l_e}}. \end{aligned}$$

**Corollary 4.** For all  $m \in \mathbb{N}$ ,  $p \in \mathbb{N}_{\geq 2}$ ,  $q \in \mathbb{N}_{\geq 3}$ , and all compositions of positive integers  $\mathbf{k}$  and  $\mathbf{l}$  (including the cases  $\mathbf{k} = \emptyset$  or  $\mathbf{l} = \emptyset$ ), we have

$$\begin{aligned} \text{(a)} \quad & \sum_{n=1}^{\infty} a_n \frac{\zeta_n(\mathbf{k})t_n(\mathbf{l})}{n^p} \in \text{CMZV}_{|k|+|\mathbf{l}|+p}^4 & \text{(b)} \quad & \sum_{n=1}^{\infty} a_n^2 \frac{\zeta_n(\mathbf{k})t_n(\mathbf{l})}{n^q} \in \frac{i}{\pi} \text{CMZV}_{|k|+|\mathbf{l}|+q+1}^4 \\ \text{(c)} \quad & \sum_{n=0}^{\infty} a_n \frac{\zeta_n(\mathbf{k})t_n(\mathbf{l})}{(2n+1)^p} \in i \text{CMZV}_{|k|+|\mathbf{l}|+p}^4 & \text{(d)} \quad & \sum_{n=0}^{\infty} a_n^2 \frac{\zeta_n(\mathbf{k})t_n(\mathbf{l})}{(2n+1)^q} \in \frac{i}{\pi} \text{CMZV}_{|k|+|\mathbf{l}|+q+1}^4. \end{aligned}$$

**Proof.** Write

$$\zeta_n(\mathbf{k}) = \sum_{n \geq m_1 > \dots > m_d > 0} \frac{1}{m_1^{k_1} \dots m_d^{k_d}}, \quad t_n(\mathbf{l}) = \sum_{n > r_1 > \dots > r_e \geq 0} \frac{1}{(2r_1+1)^{l_1} \dots (2r_e+1)^{l_e}}.$$

We only need to note the following facts: (i) for any summation index  $m$  for  $\zeta_n(\mathbf{k})$  and summation index  $r$  for  $t_n(\mathbf{l})$ , there are only two possibilities:  $m > r$  or  $r \geq m$ ; (ii) we can re-write

$$\sum_{n > r_1} \frac{1}{(2n+1)^q(2r_1+1)^{l_1}} = \sum_{n \geq r_1} \frac{1}{(2n+1)^q(2r_1+1)^{l_1}} - \frac{1}{(2n+1)^{q+l_1}}$$

and obtain similar identities when  $n$  and  $r_1$  are replaced by  $r_j$  and  $r_{j+1}$ . Therefore, we see that (a) and (c) (resp. (b) and (d)) are special cases of Theorem 8(a) (resp. Theorem 9(a)). □

We now turn to some identities we found in the literature. For  $n, k \in \mathbb{N}$ , it is conventional to define the generalized harmonic numbers by

$$H_n^{(k)} := \zeta_n(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots + \frac{1}{n^k}.$$

From the examples above and those contained in Appendix B, we can derive immediately the following identity which also appeared in ([32] Thm. 2.7) and [33]:

$$\begin{aligned} \sum_{n>0} \frac{a_n^2 H_{2n}}{(2n-1)^2} &= \sum_{n>0} \frac{a_n^2}{(2n-1)^3} + \sum_{n>m>0} \frac{a_n^2}{(2n-1)^2(2m-1)} + \sum_{n>m>0} \frac{a_n^2}{(2n-1)^2(2m)} \\ &+ \sum_{n>0} \frac{a_n^2}{(2n-1)^2} - \sum_{n>0} \frac{a_n^2}{(2n-1)} + \sum_{n>0} \frac{a_n^2}{(2n)} \\ &= \frac{2}{\pi} \left( \left( \frac{\pi}{2} + 2G - 3 \right) + \left( 3 - \frac{\pi}{2} - 2 \log 2 \right) + \left( 2G - \pi \log 2 + \pi - 4 \log 2 \right) \right. \\ &\quad \left. + \left( 2 - \frac{\pi}{2} \right) - \left( \frac{\pi}{2} - 1 \right) + \left( \pi \log 2 - 2G \right) \right) \\ &= \frac{4G - 12 \log 2 + 6}{\pi}. \end{aligned}$$

Similarly, we can also verify

$$\sum_{n>0} \frac{a_n^2 H_{2n}}{2n-1} = \frac{2}{\pi} (3 \log 2 - 1)$$

([32] Thm. 2.5) and ([34] Thm. 5.15),

$$\sum_{n>0} \frac{a_n^2 H_n}{2n-1} = \frac{8 \log 2 - 4}{\pi}$$

([35] Thm. 1),

$$\sum_{n>0} \frac{a_n^2 H_n}{(2n-1)^2} = \frac{12 - 16 \log 2}{\pi}$$

([35] Thm. 2) and ([31], p. 10),

$$\sum_{n>0} \frac{a_n^2 (H_n^2 + H_n^{(2)})}{2n-1} = \frac{4\pi}{3} - \frac{32 \log^2 2 - 32 \log 2 + 16}{\pi}$$

([35] Thm. 4).

For the last equation, we may use the stuffle relation

$$H_n^2 = 2\zeta_n(1, 1) + H_n^{(2)}. \tag{60}$$

As a further application, we can derive ([34] Thm. 5.12) as follows. Noting that  $(2n-1)a_{n-1} = 2na_n$  for all  $n \in \mathbb{N}$ , we obtain by shifting index  $n \rightarrow n-1$

$$\begin{aligned} \sum_{n>0} \frac{a_n^2}{(n+1)} H_n^{(2)} &= \sum_{n>k>0} \frac{4na_n^2}{(2n-1)^2 k^2} = \sum_{n>k>0} \frac{2a_n^2}{(2n-1)k^2} + \sum_{n>k>0} \frac{2a_n^2}{(2n-1)^2 k^2} \\ &= 8V_1(2) + 8W_2 = \frac{2}{\pi} \left( 16G + \frac{\pi^2}{3} - 8\pi \log 2 \right), \end{aligned}$$

where  $V_1(2)$  and  $W_2$  are given by Examples A13 and A15, respectively. Similarly, using the stuffle relation (60) and the identity

$$\begin{aligned} \sum_{n>0} \frac{a_n^2}{(n+1)} \zeta_n(1, 1) &= \sum_{n>k>m>0} \frac{4na_n^2}{(2n-1)^2 km} \\ &= \sum_{n>k>m>0} \frac{2a_n^2}{(2n-1)km} + \sum_{n>k>m>0} \frac{2a_n^2}{(2n-1)^2 km} \\ &= 8Y_1 + 8Y_2 = \frac{2}{\pi} \left( 16\pi \log^2 2 - 16G - \pi^2 + 8\pi \log 2 \right), \end{aligned}$$

where  $Y_1$  and  $Y_2$  are given by Example A16, we can confirm ([34] Thm. 5.12) immediately.

### 8. Conclusions and Future Plan

In this paper, by using iterated integrals, we have demonstrated that the Apéry-type binomial series

$$\sum_{n_1>\dots>n_d>0} \frac{\binom{2n_1}{n_1}}{4^{n_1} (2n_1)^{s_1} \dots (2n_d)^{s_d}} \quad \text{and} \quad \sum_{n_1>\dots>n_d>0} \frac{\binom{2n_1}{n_1}^2}{4^{2n_1} (2n_1)^{s_1} \dots (2n_d)^{s_d}}$$

can be expressed as  $\mathbb{Q}$ -linear combinations of the real and/or imaginary parts of the colored multiple zeta values at level 4, with an extra factor of  $1/\pi$  for the squared version. The same claim still holds if some or all the indices  $2n_j$  are replaced by  $2n_j \pm 1$  and “>” replaced by “ $\geq$ ” as long as the series converge.

From numerical evidence, it seems that similar results can be obtained if  $4^{n_1}$  is replaced by  $8^{n_1}$ ,  $12^{n_1}$ , or  $16^{n_1}$ ; however, the level must be increased significantly. Currently, our method only shows that these series can be expressed in terms of multiple polylogarithms at suitable algebraic points. In the related work [36], we have studied the alternating versions of the series treated in this paper and those series in which the binomial coefficients appear on the denominators.

On the other hand, one can introduce more binomial factors besides the central ones as was conducted in [37–42]. It would be very interesting to find a unified way to deal with all the different types of binomial series including the alternating ones. We plan to return to this in the future.

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### Appendix A. Additional Examples of Theorem 8

In this appendix, we will compute some more intricate examples by applying the main ideas of Theorem 8. We are particularly interested in expressing these sums by the CMZVs with the appropriate weights. We have summarized these in Table 2. In the following, it is handy to use the notation  $\Xi(\mathbf{bt}(\mathbf{l}); s) = \Xi(\mathbf{l}; s)$  to keep track of all the different types of sums we compute.

**Example A1.** We now consider the series given by an  $\alpha$ - $\beta^*$ -block chain. By (34) and (31), we have

$$\begin{aligned} \Xi(\alpha, \beta; s, 1) &= \sum_{n_1 > n_2 \geq 0} \frac{a_{n_1}}{(2n_1)^s (2n_2 + 1)} \\ &= \int_0^{\pi/2} F_s \csc t \, dt \tan t \, dt = (-1)^{s+1} \int_0^1 \mathbf{zd}_{-1,1} \mathbf{y}^{s-1} \end{aligned}$$

by (46)–(48). Hence,

$$\begin{aligned} \Xi(\alpha, \beta; 1, 1) &= \int_0^1 \mathbf{zd}_{-1,1} = \frac{\pi^2}{8} \approx 1.2337, \\ \Xi(\alpha, \beta; 2, 1) &= - \int_0^1 \mathbf{zd}_{-1,1} \mathbf{y} = \frac{1}{8} (7\zeta(3) - \pi^2 \log 2) \approx 0.1966. \end{aligned}$$

The next two examples show that, as predicted by Theorem 8, if  $\gamma$ -block does not appear, then all the CMZVs should have the same weight as that of the Apéry-type binomial series.

**Example A2.** For a  $\beta^*$ - $\alpha$ - $\alpha$ -block chain, by (32), (34), and (41), we have

$$\begin{aligned} \Xi(\beta, \alpha, \alpha; s, 1, 1) &= \sum_{n_1 \geq n_2 > n_3 > 0} \frac{a_{n_1}}{(2n_1 + 1)^s (2n_2) (2n_3)} \\ &= \int_0^{\pi/2} F_s \, dt (\sec t \csc t \, dt) ((\sec t \csc t - \csc t) \, dt) - \int_0^{\pi/2} F_s (\csc t \, dt) ((\csc t - \cot t) \, dt) \\ &= i(-1)^s \int_0^1 (a + 2x_{-1})(a + x_{-1} + x_1) \mathbf{d}_{-i,i} \mathbf{y}^{s-1} - i(-1)^s \int_0^1 (2x_{-1} - x_{-i} - x_i) \mathbf{d}_{-1,1} \mathbf{d}_{-i,i} \mathbf{y}^{s-1} \end{aligned}$$

by (46)–(48). Thus,

$$\begin{aligned} \Xi(\beta, \alpha, \alpha; 1, 1, 1) &= -i \int_0^1 \left[ (a + 2x_{-1})(a + x_{-1} + x_1) - (2x_{-1} - x_{-i} - x_i) d_{-1,1} \right] d_{-i,i} \\ &= 2G \log 2 - \frac{\pi^3}{6} + 8\text{Im Li}_3 \left( \frac{1+i}{2} \right) \approx 0.6627, \\ \Xi(\beta, \alpha, \alpha; 2, 1, 1) &= i \int_0^1 \left[ (a + 2x_{-1})(a + x_{-1} + x_1) - (2x_{-1} - x_{-i} - x_i) d_{-1,1} \right] d_{-i,i} y \\ &= 26\beta(4) - \frac{25}{96} \pi^3 \log 2 - 32\text{Im Li}_4 \left( \frac{1+i}{2} \right) - 4\text{Im Li}_3 \left( \frac{1+i}{2} \right) \log 2 \\ &\quad + \pi \left( \frac{5}{24} \log^3 2 - \frac{7}{16} \zeta(3) \right) \approx 0.0122, \end{aligned}$$

where  $\beta$  is the Dirichlet beta function

$$\beta(s) = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^s}.$$

**Example A3.** For a  $\beta^*-\alpha-\beta^*$ -block chain, by (32), (35), and (42), we have

$$\begin{aligned} \Xi(\beta, \alpha, \beta; s, 1, 1) &= \sum_{n_1 \geq n_2 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1+1)^s (2n_2)(2n_3+1)} \\ &= \int_0^{\pi/2} F_s dt (\sec t \csc t dt) (\tan t dt) = i(-1)^s \int_0^1 z(y+z) d_{-i,i} y^{s-1} \end{aligned}$$

by (46)–(48). Thus, we have

$$\begin{aligned} \Xi(\beta, \alpha, \beta; 1, 1, 1) &= -i \int_0^1 z(y+z) d_{-i,i} = \frac{\pi^3}{24} \approx 1.2919, \\ \Xi(\beta, \alpha, \beta; 2, 1, 1) &= i \int_0^1 z(y+z) d_{-i,i} y = \frac{1}{24} \pi^3 \log 2 - \frac{7}{32} \pi \zeta(3) \approx 0.0694, \\ \Xi(\beta, \alpha, \beta; 3, 1, 1) &= -i \int_0^1 z(y+z) d_{-i,i} y^2 \\ &= \frac{\pi}{11520} \left( 73\pi^4 + 480\pi^2 \log^2 2 - 120 \left( 2 \log^4 2 + 48 \text{Li}_4 \left( \frac{1}{2} \right) + 63\zeta(3) \log 2 \right) \right) \approx 0.0141. \end{aligned}$$

**Example A4.** We now consider series given by an  $\alpha-\gamma$ -block chain. In general, for all  $s \in \mathbb{N}$

$$\begin{aligned} \Xi(\alpha, \gamma; s, 1) &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1)^s (2n_2 - 1)} \\ &= \sum_{n_1 > n_2 \geq 0} \frac{a_{n_1}}{(2n_1)^s (2n_2 + 1)} - \sum_{n > 0} \frac{a_n}{2n - 1} + \sum_{j=1}^s \sum_{n > 0} \frac{a_n}{(2n)^j}. \end{aligned}$$

From Example 2, Example 3, and Example A1

$$\Xi(\alpha, \gamma; s, 1) = (-1)^{s-1} \int_0^1 z d_{-1,1} y^{s-1} - 1 - \sum_{j=0}^{s-1} (-1)^j \int_0^1 (2x_{-1} - x_i - x_{-i}) y^j.$$

We can also compute  $\Xi(\alpha, \gamma; s, 1)$  directly as follows. By (9) and (11), we have

$$\begin{aligned} &\Xi(\alpha, \gamma; s, 1) \\ &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1)^s (2n_2 - 1)} \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\pi/2} F_s(\sin t dt - \csc t dt \tan t dt) \tan t \sec t dt \\
 &= \int_0^{\pi/2} F_s(\tan t dt - \csc t dt \tan t \sec t dt) - \int_0^{\pi/2} F_s(\sin t dt - \csc t dt \tan t dt) \\
 &= \int_0^{\pi/2} F_s(\tan t dt - \csc t \sec t dt + \csc t dt) - \int_0^{\pi/2} F_s(\sin t dt - \csc t dt \tan t dt) \\
 &= (-1)^{s-1} \int_0^1 (x_i + x_{-i} - 2x_{-1} + zd_{-1,1})y^{s-1} - I_s,
 \end{aligned}$$

where  $I_1 = \int_0^{\pi/2} \sin t dt = 1$  and for all  $s \geq 2$

$$\begin{aligned}
 I_s &= \int_0^{\pi/2} F_s \sin t dt = \int_0^{\pi/2} F_s(1 - \cos t) dt = I_{s-1} + \int_0^{\pi/2} F_{s-1}(\cot t - \csc t dt \\
 &= I_{s-1} + (-1)^s \int_0^1 (2x_{-1} - x_i - x_{-i})y^{s-2} = 1 + \sum_{j=0}^{s-2} (-1)^j \int_0^1 (2x_{-1} - x_i - x_{-i})y^j.
 \end{aligned}$$

Thus,

$$\Xi(\alpha, \gamma; s, 1) = (-1)^{s-1} \int_0^1 zd_{-1,1}y^{s-1} - 1 - \sum_{j=0}^{s-1} (-1)^j \int_0^1 (2x_{-1} - x_i - x_{-i})y^j.$$

In particular,

$$\begin{aligned}
 \Xi(\alpha, \gamma; 1, 1) &= \int_0^1 (zd_{-1,1} + x_i + x_{-i} - 2x_{-1}) - 1 = \frac{\pi^2}{8} + \log 2 - 1 \approx 0.9268, \\
 \Xi(\alpha, \gamma; 2, 1) &= \frac{1}{24} (\pi^2(1 - 3 \log 2) + 21\zeta(3)) - \frac{1}{2} \log^2 2 + \log 2 - 1 \approx 0.0608, \\
 \Xi(\alpha, \gamma; 3, 1) &= \frac{\pi^2}{192} (\pi^2 + 12 \log^2 2 - 8 \log 2 + 8) + \frac{\log^2 2}{6} (\log 2 - 3) \\
 &\quad - \frac{\log 2}{8} (7\zeta(3) - 8) + \frac{\zeta(3) - 4}{4} \approx 0.0097.
 \end{aligned}$$

**Example A5.** We now consider series given by a  $\gamma$ - $\alpha$ -block chain. By Theorem 6, we have

$$\Xi(\gamma, \alpha; 1, 1) = \Xi(\alpha; 1) = \log 2 \quad \text{and} \quad \Xi(\gamma, \alpha; 1, b) = \Xi(\alpha; b) \quad \forall b \in \mathbb{N}.$$

Next, for any  $s \geq 2$  by (15) and (41), we obtain

$$\begin{aligned}
 &\Xi(\gamma, \alpha; s, 1) \\
 &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)} = \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} \tan t \sec t dt (\csc t - \cot t) dt \\
 &= -\Xi(\gamma, \alpha; s - 1, 1) + \int_0^{\pi/2} F_{s-1} (\sec t dt - dt d \sec t) (\csc t - \cot t) dt \\
 &= -\Xi(\gamma, \alpha; s - 1, 1) + \int_0^{\pi/2} F_{s-1} dt (\sec t \csc t - \csc t) dt \\
 &= -\Xi(\gamma, \alpha; s - 1, 1) - (-1)^s i \int_0^1 (a + 2x_{-1}) d_{-i,i} y^{s-2} \\
 &= (-1)^s \Xi(\gamma, \alpha; 2, 1) - (-1)^s \sum_{j=1}^{s-2} i \int_0^1 (a + 2x_{-1}) d_{-i,i} y^j \\
 &= (-1)^s \int_0^1 (2x_{-1} - x_i - x_{-i}) - (-1)^s \sum_{j=0}^{s-2} i \int_0^1 (a + 2x_{-1}) d_{-i,i} y^j.
 \end{aligned}$$

Note the highest CMZV weight drops by 1 as predicted by Theorem 8(b). For example,

$$\begin{aligned} \Xi(\gamma, \alpha; 2, 1) &= 2G - \frac{1}{2}\pi \log 2 - \log 2 \approx 0.0499, \\ \Xi(\gamma, \alpha; 3, 1) &= \frac{3\pi^3}{32} - 4\text{Im Li}_3\left(\frac{1+i}{2}\right) - \frac{\pi \log 2}{8}(3 \log 2 - 4) + \log 2 - 2G \approx 0.01. \end{aligned}$$

For all  $b \geq 2$ , we have

$$\begin{aligned} &\Xi(\gamma, \alpha; 2, b) \\ &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^2(2n_2)^b} = \int_0^{\pi/2} \frac{dt}{\tan^2 t} (d \sec t) (\cot t dt) F_{b-1}(\csc t - \cot t) dt \\ &= \int_0^{\pi/2} (\csc t \cot t dt \cot t dt - \cot^2 t dt \csc t dt) F_{b-1}(\csc t - \cot t) dt \\ &= \int_0^{\pi/2} ((\csc t - 1) \cot t dt - (\csc^2 t - 1) dt \csc t dt) F_{b-1}(\csc t - \cot t) dt \\ &= \int_0^{\pi/2} (dt \csc t dt - \cot t dt) F_{b-1}(\csc t - \cot t) dt \\ &= -i(-1)^b \int_0^1 (2x_{-1} - x_i - x_{-i}) y^{b-2} d_{-1,1} d_{-i,i} - (-1)^b \int_0^1 (2x_{-1} - x_i - x_{-i}) y^{b-1}. \end{aligned}$$

More generally, setting  $\Xi'(s, b) = \Xi(\gamma, \alpha; s, b)$ , we see that for all  $s \geq 3, b \geq 2$ ,

$$\begin{aligned} &\Xi(\gamma, \alpha; s, b) \\ &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s(2n_2)^b} = \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} \tan t \sec t dt F_b(\csc t - \cot t) dt \\ &= -\Xi'(s - 1, 1) + \int_0^{\pi/2} F_{s-1} (\sec t dt - dt d \sec t) F_b(\csc t - \cot t) dt \\ &= -\Xi'(s - 1, b) + \int_0^{\pi/2} F_{s-1} dt \sec t F_b(\csc t - \cot t) dt \\ &= -\Xi'(s - 1, b) + \int_0^{\pi/2} F_{s-1} dt \csc t dt F_{b-1}(\csc t - \cot t) dt \\ &= -\Xi'(s - 1, b) - (-1)^{s+b} i \int_0^1 (2x_{-1} - x_i - x_{-i}) y^{b-2} d_{-1,1} d_{-i,i} y^{s-2} \\ &= (-1)^s \Xi'(2, b) - (-1)^{s+b} i \sum_{j=1}^{s-2} \int_0^1 (2x_{-1} - x_i - x_{-i}) y^{b-2} d_{-1,1} d_{-i,i} y^j \\ &= -(-1)^{s+b} \int_0^1 (2x_{-1} - x_i - x_{-i}) \left( y^{b-1} + i \sum_{j=0}^{s-2} y^{b-2} d_{-1,1} d_{-i,i} y^j \right) \\ &\in \text{CMZV}_{\leq s+b-1}^4 \otimes \mathbb{Q}[i] \end{aligned}$$

by induction.

**Example A6.** We now consider series given by a  $\gamma$ - $\beta^*$ -block chain. By Theorem 6, we have

$$\Xi(\gamma, \beta; 1, 1) = \sum_{n_1 > n_2 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)(2n_2 + 1)} = \Xi(\beta; 1) = \frac{\pi}{2}.$$

For any  $s \geq 2$  by (15) and (42), we obtain

$$\Xi(\gamma, \beta; s, 1) = \sum_{n_1 > n_2 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s(2n_2 + 1)}$$

$$= \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} \sec^2 t dt = \int_0^{\pi/2} F_s dt - I_{s-1},$$

where for all  $s \geq 2$

$$\begin{aligned} I_s &= \int_0^{\pi/2} F_s \frac{dt}{\tan^2 t} \tan t dt = \int_0^{\pi/2} F_s \csc^2 t dt \tan t dt - F_s dt \tan t dt \\ &= \int_0^{\pi/2} F_s dt - F_s dt \tan t dt - I_{s-1}. \end{aligned}$$

Note that

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \frac{dt}{\tan^2 t} \tan t dt = \int_0^{\pi/2} \csc^2 t dt \tan t dt - \int_0^{\pi/2} dt \tan t dt \\ &= \int_0^{\pi/2} [-\cot u]_t^{\pi/2} \tan t dt - \int_0^{\pi/2} dt \tan t dt \\ &= \int_0^{\pi/2} dt - \int_0^{\pi/2} dt \tan t dt = \frac{\pi}{2} - i \int_0^1 z d_{-i,i}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Xi(\gamma, \beta; s, 1) &= i \int_0^1 \sum_{j=2}^s (-1)^{s-j} (F_j dt + F_{j-1} dt \tan t dt) - (-1)^s \frac{\pi}{2} \\ &= (-1)^s \left( i \int_0^1 \sum_{j=2}^s (d_{-i,i} y^{j-1} + z d_{-i,i} y^{j-2}) - \frac{\pi}{2} \right). \end{aligned}$$

Thus, the highest CMZV weight again drops by 1 as predicted by Theorem 8(b). In particular,

$$\begin{aligned} \Xi(\gamma, \beta; 2, 1) &= i \int_0^1 d_{-i,i} y + z d_{-i,i} - \frac{\pi}{2} = \pi \left( \log 2 - \frac{1}{2} \right) \approx 0.6067, \\ \Xi(\gamma, \beta; 3, 1) &= -i \int_0^1 (d_{-i,i} (y^2 + y + 1) + z d_{-i,i} (y + 1)), \\ &= \frac{\pi(2 - 4 \log 2 + 3 \log^2 2)}{4} \approx 0.5252, \\ \Xi(\gamma, \beta; 4, 1) &= i \int_0^1 (d_{-i,i} (y^3 + y^2 + y + 1) + z d_{-i,i} (y^2 + y + 1)), \\ &= \pi \left( \log 2 + \frac{\pi^2}{48} \log 2 - \frac{3}{4} \log^2 2 + \frac{1}{3} \log^3 2 - \frac{1}{2} + \frac{1}{16} \zeta(3) \right) \approx 0.5072. \end{aligned}$$

Next, for all  $b \geq 2$ , we have

$$\begin{aligned} \Xi(\gamma, \beta; 2, b) &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^2 (2n_2 + 1)^b} \\ &= \int_0^{\pi/2} \frac{dt}{\tan^2 t} d(\tan t) \cot t dt F_{b-1} dt \\ &= \int_0^{\pi/2} (\cot t dt \cot t dt - \cot^2 t dt dt) F_{b-1} dt \\ &= \int_0^{\pi/2} (\cot t dt \cot t dt - (\csc^2 t - 1) dt dt) F_{b-1} dt \\ &= \int_0^{\pi/2} (\cot t dt \cot t dt + dt dt - \cot t dt) F_{b-1} dt \\ &= i(-1)^b \int_0^1 d_{-i,i} y^{b-2} (d_{-i,i}^2 - y^2 - y). \end{aligned}$$

In particular,

$$\Xi(\gamma, \beta; 2, 2) = \frac{\pi}{24} (\pi^2 + 6 \log 2 (\log 2 - 2)) \approx 0.5804.$$

More generally, setting  $\Xi''(s, b) = \Xi(\gamma, \beta; s, b)$  for all  $s \geq 2$ , we see that for all  $s, b \geq 2$ ,

$$\begin{aligned} & \Xi(\gamma, \beta; s, b) \\ &= \sum_{n_1 > n_2 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b} = \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} d(\tan t) F_b dt \\ &= -\Xi''(s - 1, b) + \int_0^{\pi/2} F_{s-1} (\csc t \sec t dt - dt d(\tan t)) F_b dt \\ &= -\Xi''(s - 1, b) + \int_0^{\pi/2} F_{s-1} (\csc t \sec t dt - \tan t dt + dt \circ \tan t) F_b dt \\ &= -\Xi''(s - 1, b) + \int_0^{\pi/2} F_{s+b-1} dt - F_{s-1} dt dt F_{b-1} dt \\ &= (-1)^s \Xi''(2, b) + i(-1)^{s+b} \sum_{j=1}^{s-2} \int_0^1 d_{-i,i} (y^{b-2} d_{-i,i}^2 y^j - y^{j+b}) \\ &= -i(-1)^{s+b} \int_0^1 d_{-i,i} y^{b-2} y + i(-1)^{s+b} \sum_{j=0}^{s-2} \int_0^1 d_{-i,i} (y^{b-2} d_{-i,i}^2 y^j - y^{j+b}) \\ &\in i\text{CMZV}_{\leq s+b-1}^4. \end{aligned}$$

In particular,

$$\Xi(\gamma, \beta; 3, 2) = \frac{\pi}{4} \left( \frac{1}{6} \pi^2 (\log 2 - 1) + \frac{1}{2} \log 2 - \log^2 2 + \frac{1}{3} \log^3 2 + \frac{1}{8} \zeta(3) \right) \approx 0.5202.$$

**Example A7.** We now consider the series given by a  $\gamma$ - $\gamma$ - $\beta^*$ -block chain. In general, for any  $s, b, c \in \mathbb{N}$ , we have

$$\begin{aligned} & \Xi(\gamma, \gamma, \beta; s, b, c) \\ &= \sum_{n_1 > n_2 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 - 1)^b (2n_3 + 1)^c} \\ &= \sum_{n_1 > n_2 + 1 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b (2n_3 + 1)^c} \\ &= \sum_{n_1 > n_2 + 1 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b (2n_3 + 1)^c} \\ &= \sum_{n_1 > n_2 + 1, n_2 \geq n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b (2n_3 + 1)^c} \\ &= \left( \sum_{n_1 > n_2, n_2 \geq n_3 \geq 0} - \sum_{n_1 = n_2 + 1, n_2 \geq n_3 \geq 0} \right) \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b (2n_3 + 1)^c} \\ &= \sum_{n_1 > n_2 \geq n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b (2n_3 + 1)^c} - \sum_{n_1 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^{s+b} (2n_3 + 1)^c}. \end{aligned}$$

We are thus reduced to the case with the only  $\gamma$ -block appearing at the beginning, as demonstrated in the proof of Theorem 8.

In fact, we may also compute these sums directly. By Theorem 6, we have

$$\Xi(\gamma, \gamma, \beta; 1, 1, 1) = \Xi(\beta; 1) = \frac{\pi}{2}.$$

For any  $s \geq 2$  by (15) and (42), we obtain

$$\begin{aligned} \Xi(\gamma, \gamma, \beta; s, 1, 1) &= \sum_{n_1 > n_2 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 - 1)(2n_3 + 1)} \\ &= \int_0^{\pi/2} F_{s-1} \frac{dt}{\tan^2 t} \tan t dt \sec^2 t dt dt. \end{aligned}$$

If  $s = 2$ , we obtain

$$\begin{aligned} \Xi(\gamma, \gamma, \beta; 2, 1, 1) &= \sum_{n_1 > n_2 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^2 (2n_2 - 1)(2n_3 + 1)} \\ &= \int_0^{\pi/2} \frac{dt}{\tan^2 t} \tan t dt \sec^2 t dt dt = \int_0^{\pi/2} \frac{dt}{\tan^2 t} (\tan^2 t dt dt - \tan t dt \tan t dt) \\ &= \int_0^{\pi/2} \frac{dt}{\tan^2 t} (\sec^2 t dt dt - dt dt - \tan t dt \tan t dt) \\ &= \int_0^{\pi/2} \cot t dt dt - (\csc^2 t - 1) dt (\tan t dt + dt dt + \tan t dt \tan t dt) \\ &= \int_0^{\pi/2} \cot t dt dt + \int_0^{\pi/2} [\cot u]_t^{\pi/2} (\tan t dt + dt dt + \tan t dt \tan t dt) \\ &\quad + \int_0^{\pi/2} dt (\tan t dt + dt dt + \tan t dt \tan t dt) \\ &= \int_0^{\pi/2} \cot t dt dt - \int_0^{\pi/2} (dt + \cot t dt dt + dt \tan t dt) \\ &\quad + \int_0^{\pi/2} dt (\tan t dt + dt dt + \tan t dt \tan t dt) \\ &= \int_0^{\pi/2} dt (dt dt + \tan t dt \tan t dt - 1) \\ &= i \int_0^1 \tilde{d}^3_{-i,i} + \tilde{d}_{-i,i} - z z \tilde{d}_{-i,i} = \frac{\pi(\pi^2 + 6 \log^2 2 - 12)}{24} \approx 0.0984. \end{aligned}$$

If  $s \geq 3$ , then

$$\begin{aligned} &\Xi(\gamma, \gamma, \beta; s, 1, 1) \\ &= \int_0^{\pi/2} F_{s-1} d(-\cot t) \tan t dt d(\tan t) dt - F_{s-1} dt \tan t dt d(\tan t) dt \\ &= -\Xi(\gamma, \gamma, \beta; s - 1, 1, 1) + \int_0^{\pi/2} F_{s-1} (dt - dt \tan t dt) d(\tan t) dt \\ &= -\Xi(\gamma, \gamma, \beta; s - 1, 1, 1) + \int_0^{\pi/2} F_{s-1} [(\tan t dt - dt \tan^2 t dt) dt \\ &\quad - (dt - dt \tan t dt) \tan t dt] \\ &= -\Xi(\gamma, \gamma, \beta; s - 1, 1, 1) + \int_0^{\pi/2} F_{s-1} [dt dt dt + dt \tan t dt \tan t dt] \\ &= -\Xi(\gamma, \gamma, \beta; s - 1, 1, 1) - (-1)^s i \int_0^1 [z z \tilde{d}_{-i,i} - \tilde{d}^3_{-i,i}] y^{s-2} \\ &= (-1)^s \Xi(\gamma, \gamma, \beta; 2, 1, 1) - (-1)^s \sum_{j=1}^{s-2} i \int_0^1 [z z \tilde{d}_{-i,i} - \tilde{d}^3_{-i,i}] y^j \\ &= (-1)^s i \tilde{d}_{-i,i} - (-1)^s \sum_{j=0}^{s-2} i \int_0^1 [z z \tilde{d}_{-i,i} - \tilde{d}^3_{-i,i}] y^j. \end{aligned}$$

Hence, the highest CMZV weight again drops by 1 as predicted by Theorem 8(b). In particular,

$$\begin{aligned} & \Xi(\gamma, \gamma, \beta; 3, 1, 1) \\ &= \frac{\pi(2\pi^2 \log 2 - 4\pi^2 + 24 \log^3 2 - 24 \log^2 2 - 15\zeta(3) + 48)}{96} \approx 0.0207. \end{aligned}$$

**Example A8.** In this example, we explicitly reduce a  $\gamma$ - $\alpha$ - $\gamma$ -block chain to the case with the only  $\gamma$ -block appearing at the beginning. For all  $s, b, c \in \mathbb{N}$ ,

$$\begin{aligned} & \Xi(\gamma, \alpha, \gamma; s, b, c) \\ &= \sum_{n_1 > n_2 > n_3 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^b (2n_3 - 1)^c} \\ &= \sum_{n_1 > n_2 > n_3 + 1 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^b (2n_3 + 1)^c} \\ &= \sum_{n_1 > n_2 > n_3 + 1, n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^b (2n_3 + 1)^c} \\ &= \left( \sum_{n_1 > n_2 > n_3 \geq 0} - \sum_{n_1 > n_2 = n_3 + 1, n_3 \geq 0} \right) \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^b (2n_3 + 1)^c} \\ &= \sum_{n_1 > n_2 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^b (2n_3 + 1)^c} - \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^b (2n_2 - 1)^c}. \end{aligned}$$

By partial fractions, we obtain

$$\frac{1}{x^b(x-1)^c} = \sum_{j=1}^c \binom{-b}{c-j} \frac{1}{(x-1)^j} + \sum_{j=1}^b (-1)^{b+c-j} \binom{-c}{b-j} \frac{1}{x^j}.$$

For example, we have

$$\begin{aligned} & \Xi(\gamma, \alpha, \gamma; s, 2, 2) \\ &= \sum_{n_1 > n_2 > n_3 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2)^2 (2n_3 + 1)^2} \\ & \quad - \sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s} \left( \frac{1}{(2n_2)^2} - \frac{2}{2n_2} + \frac{1}{(2n_2 - 1)^2} - \frac{2}{2n_2 - 1} \right). \end{aligned}$$

Yet, for any  $a, b \in \mathbb{N}$

$$\sum_{n_1 > n_2 > 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 - 1)^b} = \sum_{n_1 > n_2 \geq 0} \frac{a_{n_1}}{(2n_1 - 1)^s (2n_2 + 1)^b} - \sum_{n > 0} \frac{a_n}{(2n_1 - 1)^{s+b}}.$$

Therefore, we have reduced  $\Xi(\gamma, \alpha, \gamma; s, b, c)$  to the case where  $\gamma$ -block only appears at the very beginning.

### Appendix B. Additional Examples of Theorem 9

In this appendix, we will present a few more complicated examples to illustrate the main ideas of Theorem 9. We are mainly interested in describing these sums using the CMZVs with the appropriate weights. We have summarized these findings in Table 4. In the following, we will use the convenient notation  $\Psi(\mathbf{bt}(I); s) = \Psi(I; s)$ .

**Example A9.** We consider the series given by a  $\{\gamma\}^d$ -block chain, with weight equal to 1 for every block. From (9) and (54), we have

$$\begin{aligned} \Gamma^{\circ d} &:= \Psi(\{\gamma\}^d; \{1\}^d) = \sum_{n_1 > \dots > n_d > 0} \frac{a_{n_1}^2}{(2n_1 - 1) \dots (2n_d - 1)} \\ &= \frac{2}{\pi} \int_0^{\pi/2} \cos t \, dt (\tan t \, dt)^{d-1} d \sec t \\ &= \frac{2}{\pi} \int_0^{\pi/2} \cos t \, dt (\tan t \, dt)^{d-2} (\tan t \sec t - \tan t) dt \\ &= \Gamma^{\circ(d-1)} - \frac{2}{\pi} A_{d-1} = \Gamma^{\circ 1} - \frac{2}{\pi} \sum_{k=1}^{d-1} A_k, \end{aligned}$$

where  $A_0 = 1$  and for all  $d \geq 1$

$$\begin{aligned} A_d &= \int_0^{\pi/2} \cos t \, dt (\tan t \, dt)^d = A_{d-1} - (-1)^d \int_0^1 z^{d-1} (x_{-i} + x_i) \\ &= A_0 + \sum_{j=0}^{d-1} (-1)^j \int_0^1 z^j (x_{-i} + x_i). \end{aligned} \tag{A1}$$

Observe that by (58) in Example 8, we obtain  $\Gamma^{\circ 1} = \Gamma_1 = 1 - \frac{2}{\pi}$ . Therefore, for all  $d \geq 1$ , we have

$$\Gamma^{\circ d} := \frac{2}{\pi} \left( \frac{\pi}{2} - \sum_{k=0}^{d-1} A_k \right) = \frac{2}{\pi} \left( \frac{\pi}{2} - d - \sum_{j=0}^{d-2} (-1)^j (d-1-j) \int_0^1 z^j (x_{-i} + x_i) \right).$$

We see that the weight drops by 1 as predicted by Theorem 9(b). In particular, when  $d = 2$

$$\sum_{n_1 > n_2 > 0} \frac{a_{n_1}^2}{(2n_1 - 1)(2n_2 - 1)} = \frac{2}{\pi} \left( \frac{\pi}{2} + \log 2 - 2 \right) \approx 0.168.$$

**Example A10.** For a  $\gamma$ - $\gamma$ -block chain with first  $\gamma$ -block having weight  $> 1$ , we have

$$T_2 := \Psi(\{\gamma\}^2; 2, 1) = \sum_{n_1 > n_2 > 0} \frac{a_{n_1}^2}{(2n_1 - 1)^2(2n_2 - 1)} = S_2 = \frac{2}{\pi} \left( 3 - \frac{\pi}{2} - 2 \log 2 \right)$$

by (A2). For all  $s \geq 3$ , by (9) and (55)

$$\begin{aligned} T_s &:= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}^2}{(2n_1 - 1)^s(2n_2 - 1)} = \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt (\cot t \, dt)^{s-2} \cot^2 t \, dt \tan t \, dt d \sec t \\ &= -T_{s-1} + \frac{2}{\pi} (B_{s-2} - C_{s-2}) = (-1)^s T_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} (B_k - C_k), \end{aligned}$$

where

$$\begin{aligned} B_0 &= \int_0^{\pi/2} \sin t \, dt dt d \sec t = \int_0^{\pi/2} \cos t (\sec t - 1) \, dt = \frac{\pi}{2} - 1 \\ B_s &= \int_0^{\pi/2} \left( \sin t \, dt (\cot t \, dt)^s dt d \sec t \right) = \int_0^{\pi/2} \left( \cos t \cot t \, dt (\cot t \, dt)^{s-1} dt d \sec t \right) \\ &= -B_{s-1} + \int_0^{\pi/2} \left( \csc t \, dt (\cot t \, dt)^{s-1} dt d \sec t \right) \end{aligned}$$

and similarly

$$\begin{aligned}
 C_0 &= \int_0^{\pi/2} \left( \sin t dt dt \tan t dt d \sec t \right) = \int_0^{\pi/2} \cos t dt (d \sec t - \tan t dt) \\
 &= \int_0^{\pi/2} (1 - \cos t - \tan t + \sin t \tan t) dt \\
 &= \int_0^{\pi/2} (1 - 2 \cos t + \sec t - \tan t) dt = \frac{\pi}{2} - 2 - \int_0^1 (x_{-i} + x_i) = \frac{\pi}{2} - 2 + \log 2, \\
 C_s &= \int_0^{\pi/2} \left( \sin t dt (\cot t dt)^s dt \tan t dt d \sec t \right) \\
 &= \int_0^{\pi/2} \left( (\cos t \cot t dt) (\cot t dt)^{s-1} dt \tan t dt d \sec t \right) \\
 &= \int_0^{\pi/2} \left( (\csc t - \sin t) dt (\cot t dt)^{s-1} dt (\tan t \sec t - \tan t) dt \right) \\
 &= -C_{s-1} + \int_0^{\pi/2} \left( \csc t dt (\cot t dt)^{s-1} dt (d \sec t - \tan t dt) \right) \\
 &= -C_{s-1} + \int_0^{\pi/2} \left( \csc t dt (\cot t dt)^{s-1} dt d \sec t \right) \\
 &\quad - \int_0^{\pi/2} \left( \csc t dt (\cot t dt)^{s-1} dt \tan t dt \right).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 B_s - C_s &= -(B_{s-1} - C_{s-1}) + \int_0^{\pi/2} \left( \csc t dt (\cot t dt)^{s-1} dt \tan t dt \right) \\
 &= -(B_{s-1} - C_{s-1}) + (-1)^s i \int_0^1 z d_{-i,i} y^{s-1} d_{-1,1} \\
 &= (-1)^s (B_0 - C_0) + (-1)^s \sum_{j=0}^{s-1} i \int_0^1 z d_{-i,i} y^j d_{-1,1}.
 \end{aligned}$$

Since  $B_0 - C_0 = 1 - \log 2$ , we obtain

$$\begin{aligned}
 T_s &= (-1)^s T_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} \left( (-1)^k (1 - \log 2) + (-1)^k \sum_{j=0}^{k-1} i \int_0^1 z d_{-i,i} y^j d_{-1,1} \right) \\
 &= (-1)^s T_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} \left( (-1)^k (1 - \log 2) + (-1)^k \sum_{j=0}^{k-1} i \int_0^1 z d_{-i,i} y^j d_{-1,1} \right) \\
 &= (-1)^s \frac{2}{\pi} \left( 3 - \frac{\pi}{2} - 2 \log 2 + (s-2)(1 - \log 2) + \sum_{j=0}^{s-3} i (s-2-j) \int_0^1 z d_{-i,i} y^j d_{-1,1} \right) \\
 &= (-1)^s \frac{2}{\pi} \left( 1 - \frac{\pi}{2} + s(1 - \log 2) + i \sum_{j=0}^{s-3} (s-2-j) \int_0^1 z d_{-i,i} y^j d_{-1,1} \right).
 \end{aligned}$$

In particular,

$$\begin{aligned}
 T_3 &= -\frac{2}{\pi} \left( 4 - \frac{\pi}{2} - 3 \log 2 + i \int_0^1 z d_{-i,i} d_{-1,1} \right) \\
 &= \frac{2}{\pi} \left( \frac{3}{16} \pi^3 - 8 \operatorname{Im} \operatorname{Li}_3 \left( \frac{1+i}{2} \right) + \frac{1}{4} \pi (2 + \log^2 2) - G \log^2 2 + \log^3 2 - 4 \right) \approx 0.0646, \\
 T_4 &= \frac{2}{\pi} \left( 5 - \frac{\pi}{2} - 4 \log 2 + 2i \int_0^1 z d_{-i,i} d_{-1,1} + i \int_0^1 z d_{-i,i} y d_{-1,1} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\pi} \left( 64 \operatorname{Im} \operatorname{Li}_4 \left( \frac{1+i}{2} \right) + 8 \operatorname{Im} \operatorname{Li}_3 \left( \frac{1+i}{2} \right) \log 2 + \frac{1}{12} \pi (\log^3 2 - 6 - 6 \log^2 2) - 50 \beta(4) \right. \\
 &\quad \left. + 16 \operatorname{Im} \operatorname{Li}_3 \left( \frac{1+i}{2} \right) + \frac{3}{16} \pi^3 (\log^3 2 - 2) - \log^4 2 + G \log^4 2 + 5 \right) \approx 0.00202.
 \end{aligned}$$

**Example A11.** We consider the series given by a  $\{\gamma\}^d$ -block chain with leading weight two and trailing weight one blocks. By (9) and (55)

$$\begin{aligned}
 S_d &:= \Psi(\{\gamma\}^d; 2, \{1\}^{d-1}) = \sum_{n_1 > n_2 > \dots > n_d > 0} \frac{a_{n_1}^2}{(2n_1 - 1)^2 (2n_2 - 1) \dots (2n_d - 1)} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt \cot^2 t \, dt (\tan t \, dt)^{d-1} \, d \sec t \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -\sin t \, dt \, dt (\tan t \, dt)^{d-1} \, d \sec t - \cos t \, dt (\tan t \, dt)^{d-1} \, d \sec t \right. \\
 &\quad \left. + \sin t \, dt \, dt (\tan t \, dt)^{d-2} \, d \sec t \right) \\
 &= -2X_d + X_{d-1} \\
 &= \frac{2}{\pi} \left( d + 1 - \frac{\pi}{2} + 2 \sum_{j=0}^{d-2} (-1)^j (d-1-j) \int_0^1 z^j (\mathbf{x}_{-i} + \mathbf{x}_i) \right. \\
 &\quad \left. - \sum_{j=0}^{d-3} (-1)^j (d-2-j) \int_0^1 z^j (\mathbf{x}_{-i} + \mathbf{x}_i) \right) \\
 &= \frac{2}{\pi} \left( d + 1 - \frac{\pi}{2} + \sum_{j=0}^{d-2} (-1)^j (d-j) \int_0^1 z^j (\mathbf{x}_{-i} + \mathbf{x}_i) \right).
 \end{aligned}$$

This shows the weight may drop by two in this special case. In particular,

$$\begin{aligned}
 S_2 &= \frac{2}{\pi} \left( 3 - \frac{\pi}{2} - 2 \log 2 \right) \approx 0.0273, \\
 S_3 &= \frac{2}{\pi} \left( \log^2 2 - \frac{\pi^2}{12} - 2 \log 2 - \frac{\pi}{2} + 4 \right) \approx 0.0049.
 \end{aligned} \tag{A2}$$

**Example A12.** We consider the series given by a  $\gamma$ - $\alpha$ -block chain with  $\alpha$ -block having weight one. If the  $\gamma$ -block has weight one which is followed by a  $\alpha$ -block then there is no weight drop from the Example A13. Yet, with the higher weight  $\gamma$ -block, the weight drop pattern resumes (i.e.,  $\eta(s_1, l) = -1$  in Theorem 9):

$$\begin{aligned}
 \Psi(\gamma, \alpha; 2, 1) &= \sum_{n_1 > n_2 > 0} \frac{a_{n_1}^2}{(2n_1 - 1)^2 (2n_2)} = \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt \cot^2 t \, dt \, d \sec t (\csc t - \cot t) \, dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -\sin t \, dt \, dt \, d \sec t (\csc t - \cot t) \, dt \right. \\
 &\quad \left. - \cos t \, dt \, d \sec t (\csc t - \cot t) \, dt + \sin t \, dt \, \sec t \, dt (\csc t - \cot t) \, dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -2 \cos t \, dt \, d \sec t (\csc t - \cot t) \, dt + dt (\csc t - \cot t) \, dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -dt (\csc t - \cot t) \, dt + 2 \cos t \, dt (\csc t \sec t - \csc t) \, dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -dt (\csc t - \cot t) \, dt + 2(1 - \sin t) (\csc t \sec t - \csc t) \, dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -dt (\csc t - \cot t) \, dt + 2(\csc t \sec t - \csc t - \sec t + 1) \, dt \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\pi} \left( \pi - \int_0^1 i(2x_{-1} - x_{-i} - x_i) d_{-i,i} - 4x_{-1} \right) \\
 &= \frac{2}{\pi} (2G - \pi \log 2 + \pi - 4 \log 2) \approx 0.0148.
 \end{aligned}$$

**Example A13.** We consider the series given by a  $(\{\gamma\}^d, \alpha)$ -block chain with all  $\gamma$ -blocks having weight 1 and the trailing  $\alpha$ -block having weight  $s \geq 2$ . Put

$$V_d(s) := \Psi(\{\gamma\}^d, \alpha; \{1\}^d, s) = \sum_{n_1 > \dots > n_d > m > 0} \frac{a_{n_1}^2}{(2n_1 - 1) \dots (2n_d - 1)(2m)^s}.$$

We have by (39) and (54)

$$\begin{aligned}
 V_1(s) &:= \sum_{n > m > 0} \frac{a_n^2}{(2n - 1)(2m)^s} = \frac{2}{\pi} \int_0^{\pi/2} \cos t dt (d \sec t)(\cot t dt)^{s-1} (\csc t - \cot t) dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left[ dt (\cot t dt)^{s-1} (\csc t - \cot t) dt \right. \\
 &\quad \left. - (1 - \sin t) \csc t dt (\cot t dt)^{s-2} (\csc t - \cot t) dt \right] \\
 &= \frac{2(-1)^s}{\pi} \int_0^1 (2x_{-1} - x_{-i} - x_i) y^{s-2} (-iy d_{-i,i} - d_{-1,1} + i d_{-i,i}) \\
 &\in \frac{i}{\pi} \text{CMZV}_{s+1}^4 + \frac{1}{\pi} \text{CMZV}_s^4 \otimes \mathbb{Q}[i]
 \end{aligned}$$

Notice that there is no weight drop as predicted by Theorem 9, in which case we have  $\eta(s_1, \mathbf{l}) = 0$ . As some special cases,

$$\begin{aligned}
 V_1(2) &= \frac{2}{\pi} \left( \frac{3\pi^3}{16} - \frac{\pi \log 2}{4} (3 \log 2 - 4) - 8 \text{Im Li}_3 \left( \frac{1+i}{2} \right) - 2G - \frac{\pi^2}{24} \right) \approx 0.0352, \\
 V_1(3) &= 24\beta(4) + \frac{1}{16} (\pi^3 (3 - 6 \log 2) + 2\zeta(3)) + \frac{\pi}{4} (2 \log^3 2 - 3 \log^2 2 + \zeta(3)), \\
 &\quad - 8 \text{Im Li}_3 \left( \frac{1+i}{2} \right) - 16 \text{Im Li}_4 \left( \frac{1+i}{2} \right) \approx 0.0244. \tag{A3}
 \end{aligned}$$

For any  $d, s \geq 2$ , we have, by (9), (39), and (54)

$$\begin{aligned}
 V_d(s) &= \sum_{n_1 > \dots > n_d > m > 0} \frac{a_{n_1}^2}{(2n_1 - 1) \dots (2n_d - 1)(2m)^s} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \cos t dt (\tan t dt)^{d-1} (d \sec t)(\cot t dt)^{s-1} (\csc t - \cot t) dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \cos t dt (\tan t dt)^{d-2} [d(\sec t) - \tan t dt \circ \sec t](\cot t dt)^{s-1} (\csc t - \cot t) dt \\
 &= V_{d-1}(s) - \frac{2}{\pi} L_{d,s} = V_1(s) - \frac{2}{\pi} \sum_{k=2}^d L_{k,s},
 \end{aligned}$$

where

$$\begin{aligned}
 L_{1,s} &:= \int_0^{\pi/2} \cos t dt \csc t dt (\cot t dt)^{s-2} (\csc t - \cot t) dt \\
 &= \int_0^{\pi/2} (1 - \sin t) \csc t dt (\cot t dt)^{s-2} (\csc t - \cot t) dt \\
 &= \int_0^1 (2x_{-1} - x_{-i} - x_i) y^{s-2} (d_{-1,1} - i d_{-i,i}),
 \end{aligned}$$

and for all  $d, s \geq 2$

$$\begin{aligned}
 L_{d,s} &:= \int_0^{\pi/2} \cos t dt (\tan t dt)^{d-1} \csc t dt (\cot t dt)^{s-2} (\csc t - \cot t) dt \\
 &= \int_0^{\pi/2} (1 - \sin t) \tan t dt (\tan t dt)^{d-2} \csc t dt (\cot t dt)^{s-2} (\csc t - \cot t) dt \\
 &= L_{d-1,s} + \int_0^{\pi/2} (\tan t - \sec t) dt (\tan t dt)^{d-2} \csc t dt (\cot t dt)^{s-2} (\csc t - \cot t) dt \\
 &= L_{d-1,s} + (-1)^s \int_0^1 (2x_{-1} - x_{-i} - x_i) y^{s-2} d_{-1,1} z^{d-2} (x_{-i} + x_i) \\
 &= L_{1,s} + \sum_{j=0}^{d-2} (-1)^j \int_0^1 (2x_{-1} - x_{-i} - x_i) y^{s-2} d_{-1,1} z^j (x_{-i} + x_i) \\
 &\in \frac{1}{\pi} \text{CMZV}_{\leq s+d-1}^4.
 \end{aligned}$$

Hence,

$$V_d(2) = V_1(2) - \frac{2}{\pi} \left( (d-1)L_1 + \sum_{j=0}^{d-2} (s-1-j)(-1)^j \int_0^1 (2x_{-1} - x_{-i} - x_i) d_{-1,1} z^j (x_{-i} + x_i) \right).$$

Thus, there is no weight drop in this case. In particular,

$$\begin{aligned}
 V_2(2) &= \frac{2}{\pi} \left( \frac{3\pi^3}{16} - G(\pi+4) - 8\text{Im Li}_3\left(\frac{1+i}{2}\right) + \frac{\pi^2}{24}(\log 2 - 2) \right. \\
 &\quad \left. - \frac{\pi \log 2}{4}(3 \log 2 - 8) + \frac{35}{16}\zeta(3) \right) \approx 0.017.
 \end{aligned}$$

**Example A14.** We consider the series given by a  $\{\gamma\}^d$ - $\alpha$ -block chain with  $\alpha$ -block having weight one. For all  $d \geq 1$ , we have by (9), (41), and (54)

$$\begin{aligned}
 U_d &:= \Psi(\{\gamma\}^d, \alpha; \{1\}^{d+1}) = \sum_{n_1 > \dots > n_d > m > 0} \frac{a_{n_1}^2}{(2n_1 - 1) \dots (2n_d - 1)(2m)} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \cos t dt (\tan t dt)^{d-1} d \sec t (\csc t - \cot t) dt = U_{d-1} - D_{d-1} = U_0 - \sum_{k=0}^{d-1} D_k,
 \end{aligned}$$

where by (56)

$$\begin{aligned}
 U_0 &:= \Psi(\alpha; 1) = \frac{2}{\pi} (\pi \log 2 - 2G), \\
 D_0 &= \frac{2}{\pi} \int_0^{\pi/2} \cos t dt \csc t (\sec t - 1) dt = \frac{2}{\pi} \int_0^{\pi/2} (1 - \sin t) \csc t (\sec t - 1) dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} (\csc t \sec t - \csc t - \sec t + 1) dt = \frac{2}{\pi} \left( \frac{\pi}{2} + \int_0^1 2x_{-1} \right) = \frac{2}{\pi} \left( \frac{\pi}{2} - 2 \log 2 \right)
 \end{aligned}$$

and for all  $d \geq 1$

$$\begin{aligned}
 D_d &= \frac{2}{\pi} \int_0^{\pi/2} \left( \cos t dt (\tan t dt)^d \csc t (\sec t - 1) dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( (1 - \sin t) \tan t dt (\tan t dt)^{d-1} \csc t (\sec t - 1) dt \right) \\
 &= D_{d-1} + \frac{2}{\pi} \int_0^{\pi/2} \left( (\tan t - \sec t) dt (\tan t dt)^{d-1} \csc t (\sec t - 1) dt \right)
 \end{aligned}$$

$$\begin{aligned}
 &= D_{d-1} - (-1)^d \frac{2}{\pi} \int_0^1 (a + 2x_{-1}) z^{d-1} (x_{-i} + x_i) \\
 &= D_0 + \frac{2}{\pi} \sum_{j=0}^{d-1} (-1)^j \int_0^1 (a + 2x_{-1}) z^j (x_{-i} + x_i).
 \end{aligned}$$

Hence, for all  $d \geq 1$ ,

$$U_d := U_0 - dD_0 - \frac{2}{\pi} \sum_{j=0}^{d-2} (-1)^j (d - 1 - j) \int_0^1 (a + 2x_{-1}) z^j (x_{-i} + x_i).$$

Thus, the weight drops by 1, as predicted by Theorem 9(b), except for the special case when  $d = 1$  when there is no weight drop. In particular,

$$\begin{aligned}
 U_1 &= \frac{2}{\pi} \left( \pi \log 2 - 2G - \frac{\pi}{2} + 2 \log 2 \right) \\
 U_2 &= \frac{2}{\pi} \left( \pi \log 2 - 2G - \pi + 4 \log 2 - \int_0^1 (a + 2x_{-1})(x_{-i} + x_i) \right) \\
 &= \frac{2}{\pi} \left( \pi \log 2 - 2G - \pi + 4 \log 2 + \frac{\pi^2}{12} - \frac{3}{2} \log^2 2 \right) \approx 0.0499.
 \end{aligned}$$

Note that  $U_1$  is one of the reasons we need to introduce  $\eta(s_1, \mathbf{l}) = 0$  in Theorem 9 since there is no weight drop in this special case.

**Example A15.** We consider series given by a  $\gamma$ - $\alpha$ -block chain with  $\alpha$ -block having weight two. We have  $W_1 := V_1(2)$ , and by (39) and (55)

$$\begin{aligned}
 W_2 &:= \Psi(\gamma, \alpha; 2, 2) = \sum_{n>m>0} \frac{a_n^2}{(2n - 1)^2 (2m)^2} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt \cot^2 t \, dt (d \sec t) (\cot t \, dt) (\csc t - \cot t) \, dt \\
 &= -W_1 + \int_0^{\pi/2} \sin t \, dt (\sec t \, dt - dt \, d \sec t) \cot t \, dt (\csc t - \cot t) \, dt \\
 &= -W_1 + \int_0^{\pi/2} \cos t \, dt \csc t \, dt (\csc t - \cot t) \, dt \\
 &= -W_1 + \int_0^{\pi/2} (1 - \sin t) \csc t \, dt (\csc t - \cot t) \, dt \\
 &= \frac{2}{\pi} \int_0^1 i(2x_{-1} - x_{-i} - x_i) y d_{-i,i} + 2(2x_{-1} - x_{-i} - x_i) d_{-1,1} - 2i(2x_{-1} - x_{-i} - x_i) d_{-i,i} \\
 &= \frac{2}{\pi} \left( 4G + \frac{\pi^2}{12} - \frac{3\pi^3}{16} + 8\text{Im Li}_3 \left( \frac{1+i}{2} \right) + \frac{\pi \log 2}{4} (3 \log 2 - 8) \right) \approx 0.0064.
 \end{aligned}$$

For all  $s \geq 3$ ,

$$\begin{aligned}
 W_s &:= \Psi(\gamma, \alpha; s, 2) = \sum_{n>m>0} \frac{a_n^2}{(2n - 1)^s (2m)^2} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt (\cot t \, dt)^{s-2} \cot^2 t \, dt (d \sec t) \cot t \, dt (\csc t - \cot t) \, dt \\
 &= -W_{s-1} + \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt (\cot t \, dt)^{s-2} (\sec t \, dt - dt \, d \sec t) \cot t \, dt (\csc t - \cot t) \, dt \\
 &= -W_{s-1} + \frac{2}{\pi} N_{s-2} = (-1)^s W_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} N_k,
 \end{aligned}$$

where

$$\begin{aligned}
 N_0 &= \int_0^{\pi/2} \sin t dt dt \csc t dt (\csc t - \cot t) dt = \int_0^{\pi/2} \cos t dt \csc t dt (\csc t - \cot t) dt \\
 &= \int_0^{\pi/2} (1 - \sin t) \csc t dt (\csc t - \cot t) dt = \int_0^1 (2x_{-1} - x_{-i} - x_i)(d_{-1,1} - i d_{-i,i})
 \end{aligned}$$

and for all  $s \geq 1$

$$\begin{aligned}
 N_s &= \int_0^{\pi/2} \sin t dt dt (\cot t dt)^s dt \csc t dt (\csc t - \cot t) dt \\
 &= \int_0^{\pi/2} \cos t \cot t dt (\cot t dt)^{s-1} dt \csc t dt (\csc t - \cot t) dt \\
 &= -N_{s-1} + \int_0^{\pi/2} \csc t dt (\cot t dt)^{s-1} dt \csc t dt (\csc t - \cot t) dt \\
 &= -N_{s-1} - (-1)^s i \int_0^1 (2x_{-1} - x_{-i} - x_i) d_{-1,1} d_{-i,i} y^{s-1} d_{-1,1} \\
 &= (-1)^s N_0 - (-1)^s \sum_{j=0}^{s-1} i \int_0^1 (2x_{-1} - x_{-i} - x_i) d_{-1,1} d_{-i,i} y^j d_{-1,1}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 W_s &= (-1)^s W_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} N_k \\
 &= (-1)^s W_2 + (-1)^s \frac{2}{\pi} \left( (s-2)N_0 - \sum_{j=0}^{s-3} (s-2-j)i \int_0^1 (2x_{-1} - x_{-i} - x_i) d_{-1,1} d_{-i,i} y^j d_{-1,1} \right).
 \end{aligned}$$

In particular,

$$\begin{aligned}
 W_3 &= -W_2 - \frac{2}{\pi} \left( N_0 - i \int_0^1 (2x_{-1} - x_{-i} - x_i) d_{-1,1} d_{-i,i} d_{-1,1} \right) \\
 &= \frac{2}{\pi} \left( 6\beta(4) + \frac{3\pi^3}{16} - 8\text{Im Li}_3 \left( \frac{1+i}{2} \right) - 6G + \frac{\pi^2}{24} (2G-3) \right. \\
 &\quad \left. - \frac{3\pi \log^2 2}{4} + \log^3 2 - \frac{7}{4} \zeta(3) \right) \approx 0.0016.
 \end{aligned}$$

**Example A16.** We consider the series given by a  $\gamma$ - $\alpha$ - $\alpha$ -block chain with both  $\alpha$ -blocks having weight one. We have, by (34), (39), and (54)

$$\begin{aligned}
 Y_1 &:= \Psi(\gamma, \alpha, \alpha; 1, 1, 1) = \sum_{n>k>m>0} \frac{a_n^2}{(2n-1)(2k)(2m)} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \cos t dt d \sec t (\csc t dt \circ \sec t - \cot t dt) (\csc t - \cot t) dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( dt \csc t dt (\sec t \csc t - \csc t) dt - dt \cot t dt (\csc t - \cot t) dt \right. \\
 &\quad \left. - \cos t dt (\sec t \csc t dt \circ \sec t - \csc t dt) (\csc t - \cot t) dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( dt \csc t dt (\sec t \csc t - \csc t) dt - dt \cot t dt (\csc t - \cot t) dt \right. \\
 &\quad \left. - (1 - \sin t) (\sec t \csc t dt \circ \sec t - \csc t dt) (\csc t - \cot t) dt \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( dt \csc t dt \left( \sec t \csc t - \csc t \right) dt - dt \cot t dt \left( \csc t - \cot t \right) dt \right. \\
 &\quad \left. - \left( \sec t \csc t - \sec t \right) dt \left( \sec t \csc t - \csc t \right) dt + \left( \csc t - 1 \right) dt \left( \csc t - \cot t \right) dt \right) \\
 &= \frac{2}{\pi} \int_0^1 \left( i(a + 2x_{-1})d_{-1,1}d_{-i,i} + i(2x_{-1} - x_{-i} - x_i)y d_{-i,i} - (a + 2x_{-1})(x_{-1} + x_1) \right. \\
 &\quad \left. + (2x_{-1} - x_{-i} - x_i)(d_{-1,1} - i d_{-i,i}) \right) \\
 &= \frac{2}{\pi} \left( 16\text{Im Li}_3 \left( \frac{1+i}{2} \right) - \frac{19\pi^3}{48} + \frac{\pi^2}{8} + \frac{\pi}{2} \log 2(\log 2 - 2) - 2\log^2 2 + G(4\log 2 + 2) \right) \\
 &\approx 0.044.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 Y_2 &:= \Psi(\gamma, \alpha, \alpha; 2, 1, 1) = \sum_{n>k>m>0} \frac{a_n^2}{(2n-1)^2(2k)(2m)} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \sin t dt \cot^2 t dt d \sec t \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( -\sin t dt dt d \sec t \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \right. \\
 &\quad \left. - \cos t dt d \sec t \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \right. \\
 &\quad \left. + \sin t dt \sec t dt \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \right) \\
 &= -Y_1 + \frac{2}{\pi} \int_0^{\pi/2} \left( -dt \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \right. \\
 &\quad \left. + \cos t dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \right. \\
 &\quad \left. + dt \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \right) \\
 &= -Y_1 + \frac{2}{\pi} \int_0^{\pi/2} \left( (1 - \sin t) \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \right) \\
 &= -Y_1 + \frac{2}{\pi} \int_0^{\pi/2} \left( \left( \sec t(\csc t - 1) dt \circ \sec t + (1 - \csc t) dt \right) \left( \csc t - \cot t \right) dt \right) \\
 &= -Y_1 + \frac{2}{\pi} \int_0^1 \left( (a + 2x_{-1})(x_{-1} + x_1) + (2x_{-1} - x_{-i} - x_i)(i d_{-i,i} - d_{-1,1}) \right) \\
 &= \frac{2}{\pi} \left( -16\text{Im Li}_3 \left( \frac{1+i}{2} \right) + \frac{19\pi^3}{48} - \frac{\pi^2}{4} + \frac{\pi}{2} (4\log 2 - \log^2 2) + 4\log^2 2 - 4G(\log 2 + 2) \right) \\
 &\approx 0.0022.
 \end{aligned}$$

For all  $s \geq 3$ , by (40) and (55)

$$\begin{aligned}
 Y_s &:= \Psi(\gamma, \alpha, \alpha; s, 1, 1) = \sum_{n>k>m>0} \frac{a_n^2}{(2n-1)^s(2k)(2m)} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^{s-2} \cot^2 t dt d \sec t \left( \csc t dt \circ \sec t - \cot t dt \right) \left( \csc t - \cot t \right) dt \\
 &= -Y_{s-1} + \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^{s-2} \left( \sec t dt - dt d \sec t \right) \left( \csc t dt \circ \sec t - \cot t dt \right) \\
 &\quad \circ \left( \csc t - \cot t \right) dt \\
 &= -Y_{s-1} + \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^{s-2} dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt
 \end{aligned}$$

$$= -Y_{s-1} + \frac{2}{\pi} M_{s-2} = (-1)^s Y_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} M_k,$$

where

$$\begin{aligned} M_0 &= \int_0^{\pi/2} \sin t dt dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \\ &= \int_0^{\pi/2} \cos t dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \\ &= \int_0^{\pi/2} (1 - \sin t) \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \\ &= \int_0^{\pi/2} \left( (\sec t \csc t - \sec t) dt \circ \sec t + (1 - \csc t) dt \right) \left( \csc t - \cot t \right) dt \\ &= \int_0^1 (a + 2x_{-1})(x_{-1} + x_1) + (2x_{-1} - x_{-i} - x_i)(i d_{-i,i} - d_{-1,1}) \end{aligned}$$

and for all  $s \geq 1$

$$\begin{aligned} M_s &= \int_0^{\pi/2} \sin t dt (\cot t dt)^s dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \\ &= \int_0^{\pi/2} \cos t \cot t dt (\cot t dt)^{s-1} dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \\ &= -M_{s-1} + \int_0^{\pi/2} \csc t dt (\cot t dt)^{s-1} dt \left( \sec t \csc t dt \circ \sec t - \csc t dt \right) \left( \csc t - \cot t \right) dt \\ &= -M_{s-1} + (-1)^s i \int_0^1 \left( (a + 2x_{-1})(a + x_{-1} + x_1) - (2x_{-1} - x_{-i} - x_i) d_{-1,1} \right) d_{-i,i} y^{s-1} d_{-1,1} \\ &= (-1)^s M_0 - (-1)^s i \sum_{j=0}^{s-1} \int_0^1 \left( (a + 2x_{-1})(a + x_{-1} + x_1) - (2x_{-1} - x_{-i} - x_i) d_{-1,1} \right) d_{-i,i} y^j d_{-1,1}. \end{aligned}$$

Hence,

$$\begin{aligned} Y_s &= (-1)^s Y_2 + \frac{2}{\pi} \sum_{k=1}^{s-2} (-1)^{s-k} M_k = (-1)^s Y_2 + (-1)^s \frac{2}{\pi} \left( (s-2)M_0 \right. \\ &\quad \left. - \sum_{j=0}^{s-3} (s-2-j)i \int_0^1 \left( (a + 2x_{-1})(a + x_{-1} + x_1) - (2x_{-1} - x_{-i} - x_i) d_{-1,1} \right) d_{-i,i} y^j d_{-1,1} \right). \end{aligned}$$

In particular,

$$\begin{aligned} Y_3 &= -Y_2 - \frac{2}{\pi} \left( M_0 - \int_0^1 \left( (a + 2x_{-1})(a + x_{-1} + x_1) - (2x_{-1} - x_{-i} - x_i) d_{-1,1} \right) d_{-i,i} d_{-1,1} \right) \\ &= \frac{2}{\pi} \left( \frac{3\pi^2}{8} - 10\beta(4) + 16\text{Im Li}_3 \left( \frac{1+i}{2} \right) + 8\text{Im Li}_4 \left( \frac{1+i}{2} \right) + 8\text{Im Li}_3 \left( \frac{1+i}{2} \right) \log 2 \right. \\ &\quad \left. - \frac{3\pi^3}{96} (38 + 15 \log 2) - 6 \log^2 2 + G \left( 6 - \frac{\pi^2}{4} + 4 \log^2 2 + 4 \log 2 \right) \right. \\ &\quad \left. + \pi \left( \frac{1}{2} \log^2 2 - \frac{5 \log^3 2}{24} - 3 \log 2 + \frac{7}{4} \zeta(3) \right) \right) \approx 0.00024. \end{aligned}$$

**Example A17.** We consider the series given by a  $(\{\gamma\}^d, \beta^*)$ -block chain, with all  $\gamma$ -blocks having weight one. When  $d = 1$ , by (42) and (54)

$$X_1 := \Psi(\gamma, \beta; 1, 1) = \sum_{n>m \geq 0} \frac{a_n^2}{(2n-1)(2m+1)} = \frac{2}{\pi} \int_0^{\pi/2} \cos t dt d(\tan t) dt$$

$$\begin{aligned}
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( \sin t \, dt \, dt - \cos t \, dt \, \tan t \, dt \right) = \frac{2}{\pi} \int_0^{\pi/2} \left( \cos t \, dt - (1 - \sin t) \, \tan t \, dt \right) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} (\sec t - \tan t) \, dt = \frac{-2}{\pi} \left( \int_0^1 (x_{-i} + x_i) \right) = \frac{2 \log 2}{\pi}.
 \end{aligned}$$

Suppose  $d \geq 2$ . Then, by (9), (42), and (54)

$$\begin{aligned}
 X_d &:= \Psi(\{\gamma\}^d, \beta; \{1\}^{d+1}) = \sum_{n_1 > \dots > n_d > m \geq 0} \frac{a_{n_1}^2}{(2n_1 - 1) \dots (2n_d - 1)(2m + 1)} \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \cos t \, dt \, (\tan t \, dt)^{d-1} \, d(\tan t) \, dt \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left( \cos t \, dt \, (\tan t \, dt)^{d-2} (\sec^2 t - 1) \, dt \, dt - \cos t \, dt \, (\tan t \, dt)^d \right) \\
 &= X_{d-1} - \frac{2}{\pi} (E_{d-2} + A_d),
 \end{aligned}$$

where  $A_d$  is defined by (A1),  $E_0 = \pi^2/8 - 1$  and for all  $d \geq 1$

$$\begin{aligned}
 E_d &= \int_0^{\pi/2} \cos t \, dt \, (\tan t \, dt)^d \, dt \, dt = \int_0^{\pi/2} (1 - \sin t) \, \tan t \, dt \, (\tan t \, dt)^{d-1} \, dt \, dt \\
 &= E_{d-1} + \int_0^{\pi/2} (\tan t - \sec t) \, dt \, (\tan t \, dt)^{d-1} \, dt \, dt = E_{d-1} + (-1)^d \int_0^1 x_{-i,i}^2 z^{d-1} (x_{-i} + x_i) \\
 &= E_0 - \sum_{j=0}^{d-1} (-1)^j \int_0^1 x_{-i,i}^2 z^j (x_{-i} + x_i),
 \end{aligned}$$

Thus, for all  $d \geq 2$ , by (A1)

$$\begin{aligned}
 X_d &= X_1 - \frac{2}{\pi} \left( \sum_{k=0}^{d-2} E_k + \sum_{k=2}^d A_k \right) = \frac{2}{\pi} \left( d \log(2) - (d-1) \frac{\pi^2}{8} \right. \\
 &\quad \left. + \sum_{j=0}^{d-3} (-1)^j (d-2-j) \int_0^1 x_{-i,i}^2 z^j (x_{-i} + x_i) - \sum_{j=1}^{d-1} (-1)^j (d-j) \int_0^1 z^j (x_{-i} + x_i) \right).
 \end{aligned}$$

We see the weight drops by 1, as predicted by Theorem 9(b). In particular,

$$X_2 = \frac{2}{\pi} \left( 2 \log 2 - \frac{\pi^2}{12} - \frac{1}{2} \log^2 2 \right) \approx 0.206.$$

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