The Total Pair Production Cross Section in Hydrogen and Helium

Part II - Correction to the JLS value for σ_T

T. M. Knasel[+]

Laboratory for Nuclear Science
Massachusetts Institute of Technology
Cambridge, Massachusetts / USA

Abstract

Comparison of the Jost, Luttinger and Slotnick formula for the total pair production cross section to more recent theoretical work by Maximon, (for total coherent pair production), and by Mork (for total incoherent pair production) is presented. If one neglects the effect of atomic screening then the coherent and incoherent cross sections will be identical except for small corrections important only at low photon energies. The result of the comparison shows that the Jost, Luttinger and Slotnick formula (for coherent production) agrees with the formula of Maximon to better than 1 part per 1000 above 5 MeV. A very simple approximate formula for the total coherent (unscreened) pair production cross section is given. Finally a comparison of the difference between the Mork calculation for the unscreened incoherent cross section with the JLS formula gives the correction term that must be applied to the JLS formula in the incoherent case (called the retardation correction). The addition of screening correction and the retardation correction to the JLS formula then gives the most precise value for the total pair production . Comparison with recent experimental data shows good agreement.

[+]Present Address: Enrico Fermi Institute, University of Chicago
5630 Ellis Ave., Chicago, Illinois 60637 USA

Introduction

In part I of this paper the formula of Jost, Luttinger, and Slotnick[1] for the total cross-section for pair production was evaluated for the first time. It was shown that the values obtained were in good agreement with previous calculations of Bethe and of Bethe and Heitler in restricted energy regions. It is also possible to make a comparison with a recently derived formula of Maximon[2] (which does not include screening) and this will be done in the first section of this paper. Our second task will be to describe the various corrections that are necessary to be added to the JLS cross section to make this cross section directly comparable with experimental data. In particular at low energies a formula for the correction due to the retardation of the electron's field in incoherent pair production will be given.

1. Comparison of the JLS Formula to the Maximon Formula. (Without Screening)

Recently Maximon has computed the total pair production cross section in the Born approximation in a way involving no approximation of high energy behaviour.[2] His work however does not include screening (an effect due to the atomic electrons) a correction that will be fairly important as the photon energy increases above about 100 MeV. In principle his work and that of JLS should agree for all energies - if screening is neglected in the JLS case. As we have mentioned in Part I, up to now no evaluations of the JLS formula were available for such a check. In order to ascertain the correctness of these formulae, the JLS prescription was integrated with no screening correction over a large range of photon energies. The integration was carried out in a similar way to that mentioned in Part I. Table I presents the results of the JLS evaluation with no screening. The values up to 100 MeV are precise to .1%, and values up to 1000 MeV are precise to 1%. Also shown in Table I are the values of Maximon. The following comments can be made about the comparison.

a) For 3 MeV and above the JLS and Maximon values are in excellent agreement, consistent with the error in the numerical evaluation.

b) Below 3 MeV the JLS and Maximon formulae differ - this may be entirely attributable to the use of an approximate formula for R(t) in the JLS formula. As was discussed in Part I this introduces an error at very low photon energies.

Thus we can conclude that the evaluations of the two prescriptions for pair production give identical results. This confirmation of the JLS formula is important as it involves a comparison of two independently derived formulae, both without energy dependent approximations. In Part I it had been possible only to check the JLS formula at the two ends of the range of photon energies.

2. Simple Formula for the Total Cross Section in the High Energy Limit (Without Screening)

In the high energy limit and in the case of no screening, the formula for pair production can be evaluated by analytical integration, and thus an exact expression for the total cross sections can be obtained. Two such formulae have been calculated by Sorenssen,[2] and by Mork[3] - who also includes the effect of retardation (i.e. pair production in the field of an electron). In the Section 3 we shall study the retardation effect. For our present purposes we need only note that as $k \to \infty$ the retardation effect vanishes. Then in this limit the Mork formula as well as the Sorrensen formula (derived with no retardation terms, but with approximations good only at high energy) agree exactly, and this result should also agree with the JLS cross section with no screening. These formulas can be further simplified in a straightforward way when k is large[4] and we present here only the final result - namely both the Mork and Sorenssen formulas reduce to:

$$\begin{aligned} \sigma_T(\text{mbarn}) &= 1.80\ \ell n\ k - 3.43 \quad k \text{ in } m_e c^2 \text{ units} \\ &= 1.80\ \ell n\ k - 2.22 \quad k \text{ in MeV units.} \end{aligned} \tag{1}$$

Such a simple formula would not be expected to be too precise, however Table II presents the result of an evaluation of formula (1). Comparing with Table I, Column 2, the precise JLS evaluation, we find agreement to much better than 1% above 50 MeV. Formula (1) is expected to be increasingly more precise as k increases and is thus quite useful as a prediction of the total unscreened pair production cross section (the total cross section for a proton without atomic electron).

3. Calculation of an Approximate Formula for the Retardation Correction

Since a simple formula for σ_T can be quite close to the best theoretical estimates (when no screening is involved) a way is suggested to compute an approximation to the JLS formula to correct for retardation effects.

First let us briefly discuss the problem. The JLS formula is a correct prescription for pair production in the nuclear field. When the atomic electrons are considered there are two results

a) the nuclear pair production is reduced by atomic electron screening
b) the field of the atomic electrons themselves cause the photons to produce electron positron pairs.

Wheeler and Lamb and later Suh and Bethe showed that a form factor for electron field production replaces the nuclear form factor (describing nuclear shielding by electrons) in the formula for pair production.[4,5] This prescription when applied to the JLS formula is incomplete since the JLS formula does not contain any description of three types of effects that occur when a pair is produced in the field of an electron. These are exchange effects, compton (γ-e) effects and retardation effects. Mork has made an extensive study of these processes and concluded that exchange, and γ-e interactions decrease very rapidly in importance above threshold and are only 2% at ~7 MeV.[3] Retardation however has a very much larger effect. To detemine its magnitude we subtract the Mork formula for pair production with retardation, from the prediction of pair production alone (as given by Sorenssen). (Both containing no screening, a multiplicative effect that does not affect our present calculation). Our notation is such that

σ_T = pair production total cross section
σ_R = σ_T + Retardation correction
ω = photon energy in $m_e c^2$ units.
σ_0 = .579 mbarn.

$$\sigma_T = \frac{2\sigma_0}{\omega^3} \{[- (\tfrac{7}{9})\omega^3 + 2\omega^2 + \tfrac{8}{9}] (1 + 2\ell n \tfrac{\omega}{2}) +$$

$$[+ (\tfrac{28}{9}) \omega^3 - 4\omega^2 + \tfrac{8\omega}{3} - \tfrac{16}{9}] \ell n(\omega - 1) +$$

$$[- (\tfrac{88}{27}) \omega^3 + \tfrac{64}{9} \omega^2 - \tfrac{16}{9} \omega + \tfrac{32}{27}]\}$$

σ_T becomes keeping terms of order ω^{-1} and higher,

$$= \frac{2\sigma_0}{\omega^3}\{[-(\tfrac{7}{9})\omega^3 + 2\omega^2][1 - 2\ln 2 + 2\ln\omega] +$$

$$[(\tfrac{28}{9})\omega^3 - 4\omega^2]\ln\omega + [-(\tfrac{88}{27})\omega^3 + \frac{64}{9}\omega^2]\} .$$

Mork gives, for the total cross section with retardation effects included to the same approximation, i.e., terms in ω^{-1} included,

$$\sigma_R = \sigma_0\{3.111 \ln 2\omega - 8.074 - [1.333 (\ln 2\omega)^3$$

$$- 3(\ln 2\omega)^2 + 6.84 \ln 2\omega - 21.51]\,\omega^{-1}\} .$$

We then form the normalized difference

$$\frac{\sigma_T - \sigma_R}{\sigma_T} = \{(\frac{82}{9} - 4\ln 2) + 1.333\, G^3 - 3G^2$$

$$+ 6.84\, G - 21.51\} \times \frac{1}{\omega(3.111\, G - 8.014)} = \Delta \qquad (2)$$

with $G = \ln 2\omega$ and ω is in units of $m_e C^2 = 1$.

Using the above formula we have constructed Table III. The retardation effect above 1 GeV is less than 2%. This is interpreted of course, as 2%/(Z + 1) error in the total cross section (or 1/Z in relation to the coherent case). By the use of this formula, we can make this correction quite accurately.

By noting that lim $k \to \infty$, $\Delta \to 0$ we see that these results confirm the arguments of Suh and Bethe[5] that at sufficiently high energies the expression for triplet and for pair production should be identical (the screening terms are of course still different). Alternately, the conclusion of Joseph and Rohrlich[6] that about a -8% correction existed even at the highest energies is shown to be invalid. In Ref.4, the error in the Joseph and Rohrlich argument is shown. When their arguments are corrected, one arrives at the identical formulation presented here.

In order to check Equation (2) which contains approximations good only at large energies the explicit evaluation of the full Mork formula (taken from Ref.3) was subtracted from the results of the JLS formula (both with no screening) presented here. Table III shows this exact calculation of the difference, as well as the results of Equation (2). The results indicate that Equation (2) is capable of giving results precise to 2% in σ_T (unscreened) above 50 MeV (note 2% difference in σ_T(unscreened) yields 1% difference in σ_T(Hydrogen)).Table IV presents an evaluation of Eq.(2) in MeV units.

4. Radiative and Other Corrections

The radiative correction to the total pair production cross section has been calculated by Mork and Olsen.[7] They give the formula

$$\sigma_{Rad} = (.93 \pm .05)\%\ \sigma_T$$

$$\sigma = \sigma_T + \sigma_{RAD}$$

This is good in the limit of complete screening i.e. high energies. In the case of no screening, i.e. low energies the correction is a little larger being 1.12% at 15 MeV photon energy. For the purpose of providing accurate cross sections above 50 MeV one may add simply 0.93% to the total pair cross section as a very close approximation.

In addition to the corrections mentioned there is an uncertainty in the case of Hydrogen as to the effect in the screening calculation of the molecular form. The value of this correction term depends sensitively upon the particular wave function of Hydrogen molecule used. In a test of several molecular forms the author found that the correction was consistent with 0 ± .5% for a selection of several models. Further details of this correction will be published later.

5. Conclusion

In conclusion, it is possible now to present a table of total cross section values for Hydrogen and Helium that will be precise to about ± 0.5% for Hydrogen, and ±0.3% for Helium at all energies. These cross sections are computed by the following prescription.

$Z^2\sigma_T$(JLS, with coherent screening) +
$Z\{\sigma_T$(JLS, with incoherent screening) - Retardation correction$\}$ +
Radiative Correction = σ

The values σ_T(JLS) were presented in Part I of this work, and the retardation and radiative corrections are discussed in Part II. Using this prescription Table V has been constructed for selected photon energies. More significant figures than are justified are carried to the final answer in order to show how each correction is computed. Numbers from this table then represent the correct cross section for pair production in Hydrogen (± .5% accuarcy) and in Helium (± .3%). These numbers are useful in obtaining photon flux values in a number of high energy reactions. It should be noted that recent experimental work at DESY has obtained values for the total cross section for pair production in Hydrogen and Deuterium[8] (which should have the identical cross section to Hydrogen except for possible differences in the molecular wave function of Deuterium). Table VI and Figure 1 shows a comparison of these numbers, with the theoretical calculations presented here. In general the Hydrogen and Deuterium data are the same within statistical errors, showing to the level of about 1.0% in the total cross section that the molecular correction differences between the two different states are not observed. Comparison with the theoretical calculations presented here shows that the data tends to give a slightly larger cross section, by about .2 mb, or 1% on average, than theory would predict. Since this is only about one standard deviation, one must regard the comparison as showing agreement between experiment and theory. Refinement of the DESY measurements could in principle allow a .1% absolute measurement[4].

<u>Summary</u>

The theory of Quantum Electrodynamics can be used to make a very precise estimate of the total cross section for photon interactions which produce electron positron pairs. At high energies this is the predominant reaction contributing to the attenuation of photons. The most complete calculation of the pair production cross section had never been integrated nor numerically evaluated to compare with experiment. In Part I of this paper this formula was integrated including screening corrections necessary when applying the result to measurement in Hydrogen and in Helium. In this form the values so obtained were found to agree with two approximate cross section expressions, at the limits of very low and very high energy for the photon. In Part II the JLS values for no screening were compared to recent calculations of Maximon and found to agree exactly. In addition to screening two other small corrections are considered. For pairs produced in the field of an atomic electron an additional correction to the JLS formula, (which assumed a heavy

nucleus for recoil) is necessary (retardation correction). By using an expression for the unscreened total cross section with this correction (due to Mork) and subtracting a formula σ_T without this correction (both good at high energies) we were able to obtain a formula for the retardation correction, that agreed well with the difference of values given by the unscreened atomic electron formula for pair production (also due to Mork) and the JLS formula (both unscreened but good at all energies) The radiative correction has been already calculated and was applied here. The cross section for Hydrogen and Helium are given in a corrected form. Recent experimental data on Hydrogen confirms the cross sections presented here but only to ±1.0%. For other elements another experiment found agreement to 0.3%.[4] The JLS total pair production cross section has been verified by both theoretical and experimental cross-checks, and represents the most precise way to compute the total pair production cross section presently available.

Table VII shows the corrected values of σ_T for Hydrogen and Helium at various energies. The errors an ± 0.5%, and ±0.3% respectively, arising mainly from uncertainties in the screening corrections. These numbers are the best estimates currently available for the total pair production cross sections in Hydrogen and Helium.

TABLE I

Total Cross Section for Pair Production (no screening)
For a Z = 1 nucleus, with no atomic electrons. ϕ = 0.57938 mbarn

MeV Photon Energy	JLS mb	JLS ϕ units	Maximon ϕ units
2	0.197	0.340	0.3030
3	0.506	0.873	0.8716
4	0.815	1.407	1.416
5	1.094	1.889	1.900
6	1.343	2.318	2.328
8	1.764	3.044	3.052
10	2.110	3.641	3.647
20	3.255	5.617	5.618
50	4.855	8.380	8.377
100	6.092	10.514	10.511
200	7.336	12.662	12.659
500	8.986	15.510	15.507
1000	10.247	17.685	17.662

TABLE II

Evaluation of $\sigma_T(\text{mb}) = 1.80 \ln k - 2.22$ (k in MeV)

k (MeV)	σ_T (mb)
10	1.93
50	4.83
100	6.084
200	7.333
500	8.985
1000	10.253
5000	13.14
10000	14.39

TABLE III

$k(m_eC^2$ units)	Mork σ_T mb	JLS σ mb	δ mb	$(\delta/JLS)\%$	Equation (2) Δ %
50	2.77	3.68	0.91	24.7	26.3
60	3.12	3.99	.87	21.8	23.0
70	3.43	4.26	.83	19.5	20.6
80	3.69	4.50	.81	18.0	18.7
90	3.93	4.71	.78	16.6	17.2
100	4.14	4.89	.75	15.3	16.0
200	5.56	6.13	.57	9.3	9.6
300	6.38	6.86	.48	7.0	7.1
500	7.40	7.77	.37	4.8	4.9
1000	8.76	9.03	.27	3.0	2.9
5000	11.84	12.03	.19	1.6	0.8

TABLE IV

Evaluation of Formula (2)

k (MeV)	Δ %
20	31
30	23
50	16
70	13
100	9.8
120	8.6
140	7.6
160	6.9
200	5.9
300	4.3
400	3.5
500	2.9
750	2.1
1000	1.7
2000	.98
4000	.56
10000	.27

TABLE V

Cross Sections in mb/atom

k MeV	σ_T(JLS,H) (coherent)	σ_T(JLS,H) (incoherent)	Retardation correction	Radiation Correction	σ mb/atom
50.0	4.850	4.854	− 0.777	+ .089	9.0
100.0	6.052	6.088	− 0.596	+ .115	11.7
200.0	7.142	7.305	− 0.431	+ .140	14.2
400.0	8.001	8.414	− 0.295	+ .161	16.3
1000.0	8.741	9.543	− 0.162	+ .182	18.3
α	9.529	11.031	0	+ .206	20.7

[Correlated wave function used for He]

k MeV	$Z^2\sigma_T$(JLS, He) (coherent)	$Z\sigma_T$(JLS, He) (incoherent)	Retardation correction	Radiation Correction	σ mb/atom
50.0	19.312	9.706	− 1.562	+ .275	27.70
100.0	23.788	12.141	− 1.218	+ .347	35.10
200.0	27.468	14.426	− 0.852	+ .410	41.40
400.0	30.114	16.344	− 0.572	+ .459	46.30
1000.0	32.247	18.137	− 0.308	+ .508	50.50
α	34.386	20.275	0	+ .547	55.20

TABLE VI

Photon Energy GeV	Theory σ_T mb	Hydrogen σ_T mb	Deuterium σ_T mb	Average of H_2 and D_2
0.55	17.1	15.54 ± 0.5		15.54 ± 0.5
0.87	18.0	17.52 ± 0.8		17.52 ± 0.8
1.18	18.6	18.63 ± 0.9		18.63 ± 0.9
1.46	18.9	18.91 ± 0.20	18.89 ± 0.18	18.90 ± 0.14
1.98	19.2	19.06 ± 0.33	19.70 ± 0.23	19.38 ± 0.20
2.55	19.5	19.61 ± 0.28	19.62 ± 0.26	19.62 ± 0.20
2.99	19.7	19.57 ± 0.30	20.60 ± 0.19	20.08 ± 0.20
3.46	19.8	19.70 ± 0.24	19.97 ± 0.23	19.84 ± 0.17
3.98	19.9	20.02 ± 0.30	20.49 ± 0.21	20.26 ± 0.20
4.55	19.9	20.19 ± 0.33	20.34 ± 0.25	20.27 ± 0.20
4.99	20.0	19.58 ± 0.18	20.28 ± 0.15	19.93 ± 0.10
5.46	20.1	19.90 ± 0.25	20.25 ± 0.15	20.07 ± 0.15
5.98	20.1	20.17 ± 0.21	20.34 ± 0.20	20.25 ± 0.15
6.55	20.2	20.50 ± 0.24	20.76 ± 0.18	20.63 ± 0.15

Table VII

Corrected Total Cross Sections

Cross sections in mb/atom

Photon Energy k (MeV)	σ_T Hydrogen	σ_T Helium (Correlated w.f.)
100.0	11.66	35.1
150.0	13.15	39.0
175.0	13.69	40.3
200.0	14.15	41.4
300.0	15.45	44.5
400.0	16.28	46.3
500.0	16.85	47.6
600.0	17.28	48.5
700.0	17.62	49.2
800.0	17.88	49.7
900.0	18.10	50.2
1000.0	18.29	50.5
1250.0	18.65	51.3
1500.0	18.91	51.8
1750.0	19.11	52.1
2000.0	19.26	52.4
3000.0	19.65	53.2
4000.0	19.87	53.6
5000.0	20.02	53.9
8000.0	20.25	54.3
10000.0	20.33	54.4

References

1) L.C. Maximon; Journal of Research of the National Bureau of Standards 72B, 79 (1968)
(The total-cross-sections are taken from this paper - unfortunately no indication of the error of the values due to the calculations uncertainties is given -)

2) A. Sorenssen; Nuovo Cimento 38, 745 (1965)

3) K. J. Mork; Phy.Rev. 160, 1065 (1967)

4) See T.M. Knasel, Thesis Harvard University 1967, T.M. Knasel and J.K. Walker; Phy.Let. 27B, 1632 (1968).

5) K. S. Suh and H. A. Bethe; Phy.Rev. 115, 672 (1959)
J.A. Wheeler and W.E. Lamb; Phys.Rev.55, 858 (1939) and Phys.Rev. 101, 1836 (1956).

6) J. Joseph and F. Rohrlich; Rev.Mod.Phy. 30, 354 (1958)

7) K. J. Mork and H. Olsen; Phy.Rev. 140, B 1661 (1965) and 166, 1862 (1968)

8) D. Notz; Diplomarbeit Universität Hamburg.
"Measurement of the Pair Production Cross Section for Hydrogen and Deuterium between 1 and 7 GeV. I wish to thank Mr. Notz for showing me the results of his measurement prior to publication.

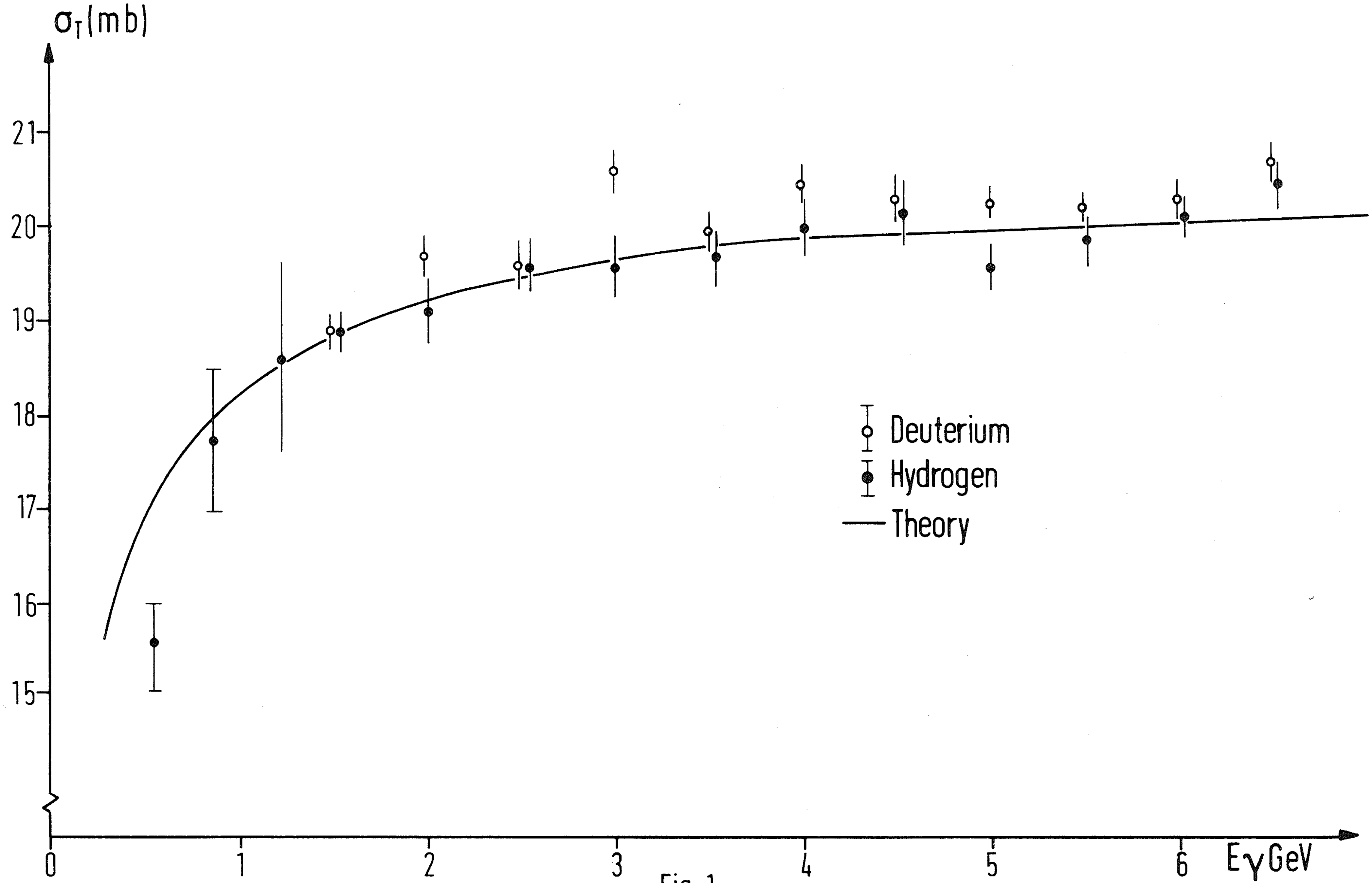
σT (mb)
21
20
19
18
17
16
15
0
1
2
3
4
5
6
Eγ GeV
Deuterium
Hydrogen
Theory

Fig. 1