

The Plebański-Demiański class of black holes and quantum tunneling

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Abstract. The Plebański-Demiański solution of Einstein's field equations represents a general class of black hole solutions. The thermodynamic quantities and other results for this spacetime reduce to those for the Kerr-Newman, Reissner-Nordström, and Schwarzschild black holes. Hawking radiations of charged and uncharged fermions and scalar particles have been studied for this class of black holes using the semi-classical approach of quantum tunneling. Working out the tunneling probability of these particles across the event horizons using complex path integration yields Hawking temperature as well.

1. The Plebański-Demiański solution

The Plebański-Demiański class of solutions [1] of Einstein's equations includes the famous Kerr-Newman rotating black hole, the Kerr-NUT, the Reissner-Nordström, and the Schwarzschild black holes as special cases. This also contains the particular solution for accelerating black holes [2, 3, 4, 5]. The general form of the metric contains seven parameters which characterize the mass M , electric and magnetic charges e and g respectively, Kerr-like rotation parameter a which is equal to angular momentum per unit mass i.e. $a = J/M$, the NUT (Newman-Unti-Tamburino) parameter l , acceleration of the source α and the cosmological constant Λ . We write the general Plebański-Demiański metric in the following notation [2, 3]

$$ds^2 = \frac{1}{\bar{\Omega}^2} \left\{ -\frac{Q}{\rho^2} [dt - (a \sin^2 \theta + 4l \sin^2 \frac{\theta}{2}) d\phi]^2 + \frac{\rho^2}{Q} dr^2 + \frac{\rho^2}{P} d\theta^2 + \frac{P}{\rho^2} \sin^2 \theta [adt - (r^2 + (a+l)^2) d\phi]^2 \right\}, \quad (1)$$

where

$$\begin{aligned} \bar{\Omega} &= 1 - \frac{\alpha}{\omega} (l + a \cos \theta) r, \\ \rho^2 &= r^2 + (l + a \cos \theta)^2, \\ P &= 1 - a_3 \cos \theta - a_4 \cos^2 \theta, \\ Q &= (\omega^2 k + e^2 + g^2) - 2Mr + \epsilon r^2 - 2\alpha \frac{n}{\omega} r^3 - (\alpha^2 k + \frac{\Lambda}{3}) r^4, \\ a_3 &= 2\alpha \frac{a}{\omega} M - 4\alpha^2 \frac{al}{\omega^2} (\omega^2 k + e^2 + g^2) - 4\frac{\Lambda}{3} al, \end{aligned}$$

$$a_4 = -\alpha^2 \frac{a^2}{\omega^2} (\omega^2 k + e^2 + g^2) - \frac{\Lambda}{3} a^2,$$

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha \frac{l}{\omega} M - (a^2 + 3l^2) \left[\frac{\alpha^2}{\omega^2} (\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3} \right], \quad (2)$$

$$n = \frac{\omega^2 k l}{a^2 - l^2} - \alpha \frac{a^2 - l^2}{\omega} M + (a^2 - l^2) l \left[\frac{\alpha^2}{\omega^2} (\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3} \right], \quad (3)$$

$$k = \left[1 + 2\alpha \frac{l}{\omega} M - 3\alpha^2 \frac{l^2}{\omega^2} (e^2 + g^2) - l^2 \Lambda \right] \left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2 \right)^{-1}. \quad (4)$$

Here ϵ and k are arbitrary real parameters, n is the Plebański-Demiański parameter and ω is the twist. When α, l, g and Λ vanish the line element reduces to the Kerr-Newman solution. If the electric charge and rotation parameter also vanish, we get the Schwarzschild solution. Thus the metric represents a complete class of accelerating, rotating and charged black holes. We note that the metric is singularity free if $|a| < |l|$, and it has a Kerr-like ring singularity at $\rho = 0$ when $|a| \geq |l|$. We will see later that the following form of the above metric is very convenient for many purposes [6, 7, 8, 9]

$$ds^2 = -f(r, \theta) dt^2 + \frac{dr^2}{g(r, \theta)} + \Sigma(r, \theta) d\theta^2 + K(r, \theta) d\phi^2 - 2H(r, \theta) dt d\phi, \quad (5)$$

where the functions f, g, Σ, K and H have the following definitions

$$f(r, \theta) = \left(\frac{Q - a^2 P \sin^2 \theta}{\rho^2 \Omega^2} \right), \quad (6)$$

$$g(r, \theta) = \frac{Q \Omega^2}{\rho^2}, \quad (7)$$

$$\Sigma(r, \theta) = \left(\frac{\rho^2}{P \Omega^2} \right), \quad (8)$$

$$K(r, \theta) = \left(\frac{\sin^2 \theta \left[P(r^2 + a^2)^2 - a^2 Q \sin^2 \theta \right]}{\rho^2 \Omega^2} \right), \quad (9)$$

$$H(r, \theta) = \left(\frac{a \sin^2 \theta \left[P(r^2 + a^2) - Q \right]}{\rho^2 \Omega^2} \right). \quad (10)$$

The electromagnetic vector potential is given by [10]

$$A = \frac{-er \left[dt - a \sin^2 \theta d\phi \right] - g \cos \theta \left[a dt - (r^2 + a^2) d\phi \right]}{r^2 + a^2 \cos^2 \theta}. \quad (11)$$

The event horizons are obtained by putting $g(r, \theta) = 0$, yielding

$$r = \frac{1}{\alpha \cos \theta}, r = \pm \frac{1}{\alpha}, \text{ and } r_{\pm} = M \pm \sqrt{M^2 - a^2 - e^2 - g^2}, \quad (12)$$

where r_{\pm} represent the outer and inner horizons similar to the Kerr-Newman black hole. The other horizons are acceleration horizons. We restrict ourselves only to non-extremal black holes i.e. when the quantity under the radical sign in Eq. (12) is positive. The angular velocity at the black hole horizon [6, 7]

$$\Omega_h = \frac{H(r_+, \theta)}{K(r_+, \theta)}, \quad (13)$$

becomes

$$\Omega_h = \frac{a}{(r_+^2 + a^2)}. \quad (14)$$

We define another function which will be needed later

$$F(r, \theta) = f(r, \theta) + \frac{H^2(r, \theta)}{K(r, \theta)}.$$

Using the values of f , K and H this takes the form

$$F(r, \theta) = \frac{PQ\rho^2}{\Omega^2 [P(r^2 + a^2)^2 - a^2Q \sin^2 \theta]}. \quad (15)$$

In this article we review some of the work done on this interesting class of solutions, particularly with reference to their thermodynamical properties and Hawking radiations. Results on Hawking radiations for charged and uncharged fermions and scalar particles are given. The Hamilton-Jacobi method of quantum tunneling and WKB approximation have been used to solve the Dirac and the Klein-Gordon equations. The tunneling probabilities of particles for crossing the event horizons have been worked out by complex path integration. We note that all the results for the general case reduce to the particular black holes, mentioned above, in appropriate limits.

2. Thermodynamics

In order to discuss thermodynamics of the class of black holes we have described in the first section, we first consider the special case of non-accelerating black holes. For this we put $\alpha = 0 = \Lambda$ in metric (1), so that $\omega^2 k = a^2 - l^2$ and therefore $\epsilon = 1$, $n = l$, $P = 1$, and we get [3]

$$ds^2 = \frac{Q}{\rho^2} [dt - (a \sin^2 \theta + 4l \sin^2 \frac{\theta}{2}) d\phi]^2 - \frac{\rho^2}{Q} dr^2 - \rho^2 d\theta^2 - \frac{\sin^2 \theta}{\rho^2} [adt - (r^2 + (l + a)^2) d\phi]^2, \quad (16)$$

where

$$\rho^2 = r^2 + (l + a \cos \theta)^2, \quad Q = (a^2 - l^2 + e^2 + g^2) - 2Mr + r^2, \quad (17)$$

which is, in fact, the Kerr-Newman-NUT solution. Putting $Q = 0$ gives the location of inner and outer horizons of the black hole as [3]

$$r_{\pm} = M \pm \sqrt{M^2 + l^2 - a^2 - e^2 - g^2}, \quad (18)$$

where $M^2 \geq a^2 + e^2 + g^2 - l^2$.

The ergosphere of this black hole is obtained by

$$g_{tt} = 0. \quad (19)$$

$$r_+ \leq r(\theta) \leq r_a, \quad (20)$$

where r_a is the outer horizon of the corresponding Reissner-Nordström black hole with magnetic and NUT charges g and l respectively.

We find the surface gravity for this black hole by using the angular velocity [11]

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}}. \quad (21)$$

Thus we get

$$\Omega_h = \frac{a}{r_+^2 + (l+a)^2}. \quad (22)$$

or

$$\Omega_h = \frac{a}{2M^2 + 2l^2 + 2al - e^2 - g^2 + 2M\sqrt{M^2 + l^2 - a^2 - e^2 - g^2}}, \quad (23)$$

and, therefore, the surface gravity becomes [12]

$$\kappa_h = \frac{1}{2a}\Omega_h \left. \frac{dQ}{dr} \right|_{r=r_+}. \quad (24)$$

The Hawking temperature $T = \kappa_h/2\pi$ of the black hole becomes

$$T = \frac{1}{2\pi} \left[\frac{\sqrt{M^2 + l^2 - a^2 - e^2 - g^2}}{2M^2 + 2l^2 + 2al - e^2 - g^2 + 2M\sqrt{M^2 + l^2 - a^2 - e^2 - g^2}} \right], \quad (25)$$

When l and g vanish we get the temperature for the Kerr-Newman black hole.

Next, we consider accelerating black holes which are free of NUT parameter, l . Putting $l = 0$, $k = 1$, in Eq. (1) we note that $\omega = a$, $a_3 = 2\alpha M$, $a_4 = -\alpha^2(a^2 + e^2 + g^2) - \Lambda a^2/3$. This metric contains six arbitrary parameters M , e , g , α , a and Λ . In the following work we put the cosmological constant equal to zero. We see that these black holes have the inner and outer horizons identical to those of the Kerr-Newman black hole

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - e^2 - g^2}, \quad (26)$$

and acceleration horizons at $r = 1/\alpha$ and $r = 1/\alpha \cos \theta$.

The surface gravity of accelerating and rotating black holes is given by [13]

$$\kappa_h = \frac{\bar{\Omega}^2}{2\rho^2} \left[\frac{d\bar{Q}}{dr} \right]. \quad (27)$$

As $\bar{\Omega} \neq 0$ at $r = r_+$ we get

$$\kappa_h = \frac{\bar{\Omega}^2}{2\rho^2} 2[(r - M)(1 - \alpha^2 r^2) - \alpha^2 r(r^2 - 2Mr + a^2 + e^2 + g^2)]. \quad (28)$$

Since at the outer horizon, $r^2 - 2Mr + a^2 + e^2 + g^2 = 0$, therefore, putting the value of ρ^2 and $\bar{\Omega}$, we get

$$\kappa_h = \frac{(r_+ - M)}{(r_+^2 + a^2)} (1 - \alpha r_+)^3 (1 + \alpha r_+). \quad (29)$$

Using Eq. (26) this can be written as explicitly as

$$\kappa_h = \frac{[1 - \alpha(M + \sqrt{M^2 - a^2 - e^2 - g^2})]^3 [1 + \alpha(M + \sqrt{M^2 - a^2 - e^2 - g^2})]}{(\sqrt{M^2 - a^2 - e^2 - g^2})^{-1} [2M^2 - e^2 - g^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2}]}. \quad (30)$$

We see that from Eq. (29) the surface gravity will vanish at the acceleration horizon, $r = 1/\alpha$. If l and g vanish, this gives the surface gravity for the Kerr-Newman black hole. The relation

(30) can also be written in terms of the inner and outer horizons. For this we use the relation $r_+ - r_- = (r_+ - M) - (r_- - M) = 2(r_+ - M)$ in Eq. (29) to get [13]

$$\kappa_h = \frac{r_+ - r_-}{2(r_+^2 + a^2)}(1 - \alpha r_+)^3(1 + \alpha r_+). \quad (31)$$

In this case the angular velocity from Eq. (21) becomes

$$\Omega = \frac{a[\bar{Q} - P(r^2 + a^2)]}{\bar{Q}a^2 \sin^2 \theta - P(r^2 + a^2)^2}, \quad (32)$$

which at the horizon takes the form

$$\Omega_h = \frac{a}{2M^2 - e^2 - g^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2}}. \quad (33)$$

We find that the acceleration parameter, α , is only permitted in the following range [13]

$$\alpha < \frac{1}{M + \sqrt{M^2 - a^2 - e^2 - g^2}}. \quad (34)$$

3. Hawking radiation of Dirac particles

Soon after the discovery of thermal radiations from black holes by Hawking [14, 15], different techniques were adopted to study their mathematical and physical properties. This, in fact, gave rise to a new area of investigations which lies at the interface of general relativity, quantum mechanics and thermodynamics. The methods used to study these radiations include the null geodesic method, Hamilton-Jacobi tunneling method [16, 17, 18, 19], the method of dimensional reduction [20, 21] and the so-called anomaly technique [22, 23]. The main idea is that a pair of particles is created near the event horizon of the black hole and from this pair the negative energy particle tunnels inwards. This decreases the mass of the black hole. The positive energy particle tunnels out and constitutes Hawking radiation.

Hawking radiation from the Plebański-Demiański class has been studied [7, 8, 9] using the Hamilton-Jacobi method with particular reference to accelerating and rotating black holes. In this section we briefly review the tunneling method for Dirac particles from these black holes and present results. The main idea is to solve the Dirac equation in the background of the spacetime in Eq. (5). We know that in covariant form the Dirac equation with mass of fermions, m , can be written as

$$\iota\gamma^\mu(D_\mu)\Psi + \frac{m}{\hbar}\Psi = 0, \quad (35)$$

where

$$D_\mu = \partial_\mu + \Omega_\mu, \quad (36)$$

$$\Omega_\mu = \frac{-1}{8}\Gamma_\mu^{\alpha\beta}[\gamma^\alpha, \gamma^\beta], \quad (37)$$

and $[\gamma^\alpha, \gamma^\beta]$ satisfy the commutation relations

$$[\gamma^\alpha, \gamma^\beta] = -[\gamma^\beta, \gamma^\alpha] \quad \text{if } \alpha \neq \beta, \quad [\gamma^\alpha, \gamma^\beta] = 0 \quad \text{if } \alpha = \beta. \quad (38)$$

We choose the following form of γ^μ

$$\gamma^t = \sqrt{\frac{(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta)\Omega^2}{PQ\rho^2}}\gamma^0, \gamma^r = \sqrt{\frac{Q\Omega^2}{\rho^2}}\gamma^3, \gamma^\theta = \sqrt{\frac{P\Omega^2}{\rho^2}}\gamma^1, \quad (39)$$

$$\gamma^\phi = \frac{\rho\Omega\gamma^2}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta}} + \frac{a(P(r^2+a^2) - Q)\gamma^0}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta)}, \quad (40)$$

where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad (41)$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad (42)$$

and the Pauli sigma matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (43)$$

To solve the Dirac equation in this set up we take an *ansatz* of the following form for the spin-up and spin-down solutions

$$\Psi_\uparrow(t, r, \theta, \phi) = \begin{pmatrix} A(t, r, \theta, \phi)\xi_\uparrow \\ B(t, r, \theta, \phi)\xi_\uparrow \end{pmatrix} \exp\left[\frac{iI_\uparrow(t, r, \theta, \phi)}{\hbar}\right], \quad (44)$$

$$\Psi_\downarrow(t, r, \theta, \phi) = \begin{pmatrix} C(t, r, \theta, \phi)\xi_\downarrow \\ D(t, r, \theta, \phi)\xi_\downarrow \end{pmatrix} \exp\left[\frac{iI_\downarrow(t, r, \theta, \phi)}{\hbar}\right]. \quad (45)$$

Here A, B, C, D are arbitrary functions of the coordinates, ξ_\uparrow and ξ_\downarrow are the eigenvectors of σ^i , and I_\uparrow and I_\downarrow are the corresponding actions. On using Eq. (38) the Dirac equation takes the form

$$(i\gamma^t\partial_t + i\gamma^r\partial_r + i\gamma^\theta\partial_\theta + i\gamma^\phi\partial_\phi)\Psi + \frac{m}{\hbar}\Psi = 0. \quad (46)$$

We substitute the above *ansatz* into the Dirac equation and compute it term by term. Dividing by the exponential term and neglecting the terms with \hbar we obtain the following four equations

$$0 = -B\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow + \sqrt{\frac{\Omega^2 Q}{\rho^2}}\partial_r I_\uparrow + \frac{a(P(r^2+a^2) - Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta)}\partial_\phi I_\uparrow\right] + Am, \quad (47)$$

$$0 = -B\left[\sqrt{\frac{\Omega^2 P}{\rho^2}}\partial_\theta I_\uparrow + \frac{i\rho\Omega}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta}}\partial_\phi I_\uparrow\right], \quad (48)$$

$$0 = A\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow - \sqrt{\frac{\Omega^2 Q}{\rho^2}}\partial_r I_\uparrow + \frac{a(P(r^2+a^2) - Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta)}\partial_\phi I_\uparrow\right] + Bm, \quad (49)$$

$$0 = A\left[\sqrt{\frac{\Omega^2 P}{\rho^2}}\partial_\theta I_\uparrow + \frac{i\rho\Omega}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta}}\partial_\phi I_\uparrow\right]. \quad (50)$$

Considering the Killing vectors of the background spacetime we take the following separation of variables for the action [7]

$$I_\uparrow = -Et + J\phi + W(r, \theta). \quad (51)$$

Here E and J are the energy and angular momentum of the emitted particle. Substituting this into the above set of four equations yields

$$0 = -B \left[\frac{-E}{\sqrt{F(r, \theta)}} + \sqrt{\frac{\Omega^2 Q}{\rho^2}} W'(r, \theta) + \frac{a(P(r^2 + a^2) - Q)}{\sqrt{F(r, \theta)}(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta)} J \right] + Am, \quad (52)$$

$$0 = B \left[\sqrt{\frac{\Omega^2 P}{\rho^2}} W_\theta(r, \theta) + \frac{\iota \rho \Omega}{\sin \theta (\sqrt{P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta}} J \right], \quad (53)$$

$$0 = A \left[\frac{-E}{\sqrt{F(r, \theta)}} - \sqrt{\frac{\Omega^2 Q}{\rho^2}} W'(r, \theta) + \frac{a(P(r^2 + a^2) - Q)}{\sqrt{F(r, \theta)}(P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta)} J \right] + Bm, \quad (54)$$

$$0 = A \left[\sqrt{\frac{\Omega^2 P}{\rho^2}} W_\theta(r, \theta) + \frac{\iota \rho \Omega}{\sin \theta \sqrt{P(r^2 + a^2)^2 - Qa^2 \sin^2 \theta}} J \right]. \quad (55)$$

Solving these equations as explained in Ref. [7] we obtain

$$W_+(r) = \int \frac{(E - \Omega_h J)(r_+^2 + a^2)}{2(r - r_+)(r_+ - M)(1 - \alpha^2 r_+^2)}, \quad (56)$$

which on integration around the pole $r = r_+$ gives

$$W(r) = \pi \iota \frac{(E - \Omega_h J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}, \quad (57)$$

$$ImW = \pi \frac{(E - \Omega_h J)(r_+^2 + a^2)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}. \quad (58)$$

So, the emission and absorption probabilities of fermions become

$$P_{emission} = \exp[-2ImI] = \exp[-2(ImW_+ + Im\Theta)], \quad (59)$$

$$P_{absorption} = \exp[-2ImI] = \exp[-2(ImW_- + Im\Theta)]. \quad (60)$$

Since $ImW_+ = -ImW_-$, we see that the total probability that the particle tunnels from inside the event horizon to outside is

$$\Gamma \sim \frac{P_{emission}}{P_{absorption}} = \exp[-4ImW_+], \quad (61)$$

or

$$\Gamma = \exp\left[-\frac{2\pi(E - \Omega_h J)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}\right]. \quad (62)$$

Comparing this with the Boltzmann factor $\Gamma = \exp[-\beta E]$ where $\beta = 1/T_h$ we find that the Hawking temperature [24, 25] is given by

$$T_h = \frac{(r_+ - M)(1 - \alpha^2 r_+^2)}{2\pi(r_+^2 + a^2)}, \quad (63)$$

where r_+ is the event horizon. We note that formulae (62) and (63) reduce to the tunneling probability and temperature of the Kerr black hole [26] if acceleration vanishes. Further, we can recover relations for the Schwarzschild black hole if angular momentum also vanishes.

Proceeding as before we work out the tunneling probability at the acceleration horizon, $r_\alpha = 1/\alpha$, and find that the Hawking temperature at this horizon comes out to be

$$T_h = \frac{\alpha(\alpha^2 a^2 - 2M\alpha + 1)}{2\pi(1 + \alpha^2 a^2)}. \quad (64)$$

4. Hawking radiation of charged fermions

Now we will solve the Dirac equation for accelerating and rotating black holes with electric and magnetic charges. The covariant Dirac equation with charge q is given by

$$i\gamma^\mu \left(D_\mu - \frac{iq}{\hbar} A_\mu \right) \Psi + \frac{m}{\hbar} \Psi = 0, \quad (65)$$

where m is the mass of the fermion particles.

Taking an *ansatz* similar to the one in the previous section for uncharged fermions and following the same steps we get the following set of four equations to be solved.

$$0 = -B \left[\frac{1}{\sqrt{F(r, \theta)}} \partial_t I_\uparrow + \sqrt{g(r, \theta)} \partial_r I_\uparrow + \frac{H(r, \theta)}{K(r, \theta) \sqrt{F(r, \theta)}} \partial_\phi I_\uparrow - \frac{1}{\sqrt{F(r, \theta)}} q A_t - \frac{H(r, \theta)}{K(r, \theta) \sqrt{F(r, \theta)}} q A_\phi \right] + mA, \quad (66)$$

$$0 = -B \left[\frac{1}{\sqrt{\Sigma(r, \theta)}} \partial_\theta I_\uparrow + \frac{i}{\sqrt{K(r, \theta)}} \partial_\phi I_\uparrow - \frac{i}{\sqrt{K(r, \theta)}} q A_\phi \right], \quad (67)$$

$$0 = A \left[\frac{1}{\sqrt{F(r, \theta)}} \partial_t I_\uparrow - \sqrt{g(r, \theta)} \partial_r I_\uparrow + \frac{H(r, \theta)}{K(r, \theta) \sqrt{F(r, \theta)}} \partial_\phi I_\uparrow - \frac{1}{\sqrt{F(r, \theta)}} q A_t - \frac{H(r, \theta)}{K(r, \theta) \sqrt{F(r, \theta)}} q A_\phi \right] + mB, \quad (68)$$

$$0 = -A \left[\frac{1}{\sqrt{\Sigma(r, \theta)}} \partial_\theta I_\uparrow + \frac{i}{\sqrt{K(r, \theta)}} \partial_\phi I_\uparrow - \frac{i}{\sqrt{K(r, \theta)}} q A_\phi \right]. \quad (69)$$

Solving these as before we get the probabilities of crossing the horizon in each direction as [8]

$$P_{emission} \propto \exp[-2ImI] = \exp[-2(ImR_+ + Im\Theta)], \quad (70)$$

$$P_{absorption} \propto \exp[-2ImI] = \exp[-2(ImR_- + Im\Theta)]. \quad (71)$$

Thus the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma \propto \frac{P_{emission}}{P_{absorption}} = \frac{\exp[-2(ImR_+ + Im\Theta)]}{\exp[-2(ImR_- + Im\Theta)]},$$

or

$$\Gamma = \exp[-2(ImR_+ - ImR_-)],$$

which in terms of the event horizon becomes

$$\Gamma = \exp\left[\frac{-2\pi(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}\left(E - \Omega_h J - \frac{qer_+}{(r_+^2 + a^2)}\right)\right]. \quad (72)$$

Comparing this with the Boltzmann factor as before we find the Hawking temperature as

$$T_h = \frac{(r_+ - M)(1 - \alpha^2 r_+^2)}{2\pi(r_+^2 + a^2)}. \quad (73)$$

Here again if acceleration vanishes we recover the tunneling probability and temperature for the Kerr-Newman black hole [26]. Taking rotation equal to zero yields the expression for the Reissner-Nordström black hole.

5. Hawking radiation of scalar particles

Black holes emit scalar particles besides other particles [27]. Now we study contribution of scalar fields towards Hawking radiation for the class of black holes at hand. For this purpose the Klein-Gordon equation

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi - \frac{m^2}{\hbar^2}\Phi = 0, \quad (74)$$

for the scalar field Φ and mass of the scalar particle m will be solved [9]. In order to apply WKB approximation we assume the following solution

$$\Phi(t, r, \theta, \phi) = \exp\left[\frac{i}{\hbar}I(t, r, \theta, \phi) + I_1(t, r, \theta, \phi) + O(\hbar)\right]. \quad (75)$$

Using symmetries of the background spacetime we take the following separation of variables in the action

$$I = -Et + W(r) + J\phi, \quad (76)$$

where E and J are the energy and angular momentum of the emitted particle. Using this action the Klein-Gordon equation at the horizon, $r = r_+$, takes the form

$$0 = -\frac{1}{F(r_+, \theta)}\left(E - \frac{H(r_+, \theta)}{K(r_+, \theta)}J\right)^2 + \frac{J^2}{K(r_+, \theta)} + g(r_+, \theta)(W'(r))^2 + m^2. \quad (77)$$

Using Eqs. (6) to (10) and (15) in this, and solving by integration gives [9]

$$W_+(r) = \frac{i\pi(r_+^2 + a^2)(E - \Omega_h J)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}, \quad (78)$$

$$ImW_+(r) = \frac{\pi(r_+^2 + a^2)(E - \Omega_h J)}{2(r_+ - M)(1 - \alpha^2 r_+^2)}. \quad (79)$$

The resulting tunneling probability becomes

$$\Gamma = \exp\left[-2\pi\frac{(r_+^2 + a^2)(E - \Omega_h J)}{(r_+ - M)(1 - \alpha^2 r_+^2)}\right]. \quad (80)$$

As before we compare this with the Boltzmann factor to write the Hawking temperature which is the same as in Eq. (73). We again note that the results for *simpler* black holes can be recovered from this. If we put acceleration equal to zero the temperature for the Kerr-Newman black hole is obtained [26]. If α, a, e and g vanish we get Hawking temperature for the Schwarzschild black hole [28] as $T_h = 1/8\pi M$.

For charged scalar particles the outer horizon for accelerating and rotating charged black holes is $r_+ = M + \sqrt{M^2 - a^2 - e^2 - g^2}$. The Klein-Gordon equation for the scalar field Φ with charge q is given by

$$g^{\mu\nu} \left(\partial_\mu - \frac{iq}{\hbar} A_\mu \right) \left(\partial_\nu - \frac{iq}{\hbar} A_\nu \right) \Phi - \frac{m^2}{\hbar^2} \Phi = 0, \quad (81)$$

where A_μ is the vector potential. Using an *ansatz* similar to the one in the uncharged case, the above equation becomes

$$g^{\mu\nu} (\partial_\mu I - qA_\mu) (\partial_\nu I - qA_\nu) + m^2 = 0. \quad (82)$$

Following the same procedure as before we obtain the tunneling probability as

$$\Gamma = \exp \left[-\frac{2\pi (r_+^2 + a^2)}{(r_+ - M) (1 - \alpha^2 r_+^2)} \left(E - \Omega_h J - \frac{qer_+}{(r_+^2 + a^2)} \right) \right], \quad (83)$$

which is the same as in the case of Dirac particles [8]. As before, the Hawking temperature is given by

$$T_h = \frac{(r_+ - M) (1 - \alpha^2 r_+^2)}{2\pi (r_+^2 + a^2)}. \quad (84)$$

The value of r_+ in this case is given by Eq. (12).

6. Conclusion

The Plebański-Demiański family of solutions of Einstein's field equations contains many well-known black hole solutions. These, among others, include the Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman and Kerr-Newman-NUT solutions. Thus the general results obtained for this family reduce to these particular cases. The family also contains solutions for accelerating and rotating black holes. Here we have reviewed the thermodynamic properties of these solutions. We see that in the case of the Kerr-Newman-NUT black hole the behaviour of temperature is similar to that of the Kerr-Newman black hole if the NUT parameter has less magnitude than that of the sum of the electric and magnetic charges and the rotation parameter. For accelerating and rotating black holes there is a limit on the magnitude of acceleration as well.

Various techniques have been used to study Hawking radiations from black holes. This has been done, for example, by using the Newman-Penrose formalism, the null-geodesic and the so-called Hamilton-Jacobi methods. Here we have used the method of quantum tunneling, which is a variant of the Hamilton-Jacobi method, and complex path integration technique. Radiations from black holes are considered as a phenomenon of quantum tunneling across their event horizons. Using these semi-classical methods we have studied the radiation of charged and uncharged Dirac particles and scalar particles from the Plebański-Demiański class of black holes. The tunneling probabilities for the incoming and outgoing particles have been calculated by solving the Dirac and Klein-Gordon equations in the framework of WKB approximation.

Our procedure works out Hawking temperature for these black configurations as well. Apart from giving results for the case of accelerating and rotating black holes, our formulae reduce to those for the well-known black holes also. We get formulae for the Kerr-Newman black hole when

the NUT parameter vanishes, and the Kerr black hole, when the electric and magnetic charges vanish too. When angular momentum is also zero the thermodynamical quantities reduce to those of the simplest case of the Schwarzschild black hole.

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