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The Accidental Degeneracy Problem for Two Anyons Experiencing Coulomb or Oscillator Potential

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Abstract:

We show that the two-anyon system in an oscillator or Coulomb potential has the SU(2) or 0(3) symmetry respectively but with the restriction that only even values of the angular momentum m are allowed. Using these symmetries we algebraically obtain the bound state spectrum in both cases. Further, we also show that both of these problems are shape invariant and thereby offer a second method for calculating their spectrum algebraically.

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I. Introduction

Two of the most celebrated problems in quantum mechanics are the Coulomb and the oscillator potentials. It is well known that the bound state energy eigenvalue spectrum, in both the cases, has "accidental" degeneracies in any dimension $n(\geq 2)$. The origin of the degeneracies has also been well understood in terms of the symmetry group [1]. For example in the case of the oscillator potential in 3(2) dimensions, the symmetry group is SU(3) (SU(2)) while in the Coulomb case it is 0(4) (0(3)) respectively [1,2].

In last few years, anyons (which are objects in two space dimensions obeying statistics interpolating between bosons and fermions) have attracted lot of attention [3]. So far only the two-anyon quantum spectrum can be computed analytically (and that too in few cases), and as a result only second virial coefficient for an anyon gas has been calculated. Not surprisingly the problems of 2-anyons experiencing an oscillator [3] or Coulomb [4] potential can be done analytically, and one finds lots of degeneracies in the two spectra. It is then natural to inquire if one can find the corresponding symmetry of the 2-anyon problem in these two cases and further if one can obtain the energy eigenvalue spectrum using the symmetry group alone. This is the task that we have undertaken in this paper. In particular we show in Sec.II that for the problem of 2-anyons experiencing an oscillator potential, one can define three generators J_i (i = 1,2,3) which commute with the Hamiltonian and which satisfy the SU(2) algebra. Similarly, for the problem of two anyons experiencing a Coulomb potential, we show in Sec.III that we can define the Runge-Lenz two-vector (R_x, R_y) which along with the angular momentum satisfy the O(3) algebra. Further more, all the three generators commute with the Hamiltonian. The nontrivial point in both the cases is that the definition of the angular momentum operator in the presence of anyons is changed to $L = (xp_y - yp_x) + \hbar\alpha$. It may be noted that the eigenvalues of this operator are in general not interger valued. Nevertheless the eigenvalues of the canonical angular momentum operator $xp_{\bar{y}}yp_x$ are still integer valued. In Sec.IV we show that both problems can be cast in the supersymmetric form and the potentials are shape-invariant [5]. Using this shape invariance property, we immediately deduce the bound state spectrum in the two cases. Finally Sec.V is reserved for discussion.

II. Symmetry of the Two-Anyon Plus Oscillator Problem

The Hamiltonian for the harmonic oscillator problem in two-dimensions is given by [2]

$$2\mu H = p_x^2 + p_y^2 + \mu^2\omega^2(x^2 + y^2) \tag{2.1}$$

Its energy eigenvalue spectrum is

$$E_n = (n+1)\hbar\omega \; ; \quad n = 0, 1, 2, \cdots \tag{2.2}$$

with the corresponding degeneracy for every n being given by d=(n+1). It is well known that one can define three generators J_1, J_2, J_3 which commute with the above H and satisfy the SU(2) algebra [2]

$$[J_i, J_j] = i\varepsilon_{ijk}J_K; i, j, k = 1, 2, 3 \tag{2.3}$$

In particular, the explicit form of J_1, J_2, J_3 is

$$J_1 = \frac{1}{2}(a^+a - b^+b) \tag{2.4a}$$

$$J_2 = \frac{1}{2}(a^+b + b^+a) \tag{2.4b}$$

$$J_3 = \frac{i}{2}(b^+a - a^+b) \tag{2.4c}$$

where

$$a = \frac{\mu\omega x + ip_x}{\sqrt{2\hbar\mu\omega}}, \quad b = \frac{\mu\omega y + ip_y}{\sqrt{2\hbar\mu\omega}} \tag{2.5}$$

On using the fact that

$$[x, p_x] = i\hbar = [y, p_y] \tag{2.6}$$

it is easily seen that

$$\frac{1}{4}H^2 = (\vec{J}^2 + \frac{1}{4}\hbar^2)\omega^2 \tag{2.7}$$

From this equation, the spectrum (2.2) and the degeneracy follow by recognizing that the eigenvalue of $\vec{J}^2$ is j(j+1)$\hbar$, and it is (2j+1)-fold degenerate (j is a half-integer or interger).

Using the above discussion, we now show as to how the symmetry of the two-anyons experiencing oscillator potential can be uncovered. The key point of the whole discussion is the recognition of the fact that the interprolating statistics is a purely quantum mechanical effect, and that velocities are unchanged even in the presence of anyons. In other words even for 2-anyons with oscillator potential, the Hamiltonian (2.1), the operators a,b and hence the generators (2.4) remain unchanged when expressed in terms of coordinates x,y and velocities $\dot{x},\dot{y}$. On using the fact that for anyons $\dot{x},\dot{y}$ are given in terms of the momenta p_x and p_y by ($\alpha \equiv \theta/\pi, 0 \le \alpha \le 1$ with $\alpha = 0(1)$ corresponding to bosons (fermions))

$$\dot{x} = p_x - \frac{\hbar\alpha y}{r^2} \quad \text{and} \quad \dot{y} = p_y + \frac{\hbar\alpha x}{r^2} \tag{2.8}$$

we find that the relative Hamiltonian for 2-anyons expressing oscillator potential is:

$$2\mu H = \mu^2\omega^2(x^2+y^2) + \left(p_x - \frac{\hbar\alpha y}{r^2}\right)^2 + \left(p_y + \frac{\hbar\alpha x}{r^2}\right)^2 \tag{2.9}$$

One can again define the three generators J_1, J_2, J_3 as in eqs.(2.4) but now a,b are given by

$$a = \frac{\mu\omega x + i\left(p_x - \frac{\hbar\alpha y}{r^2}\right)}{\sqrt{2\hbar\mu\omega}}, \quad b = \frac{\mu\omega y + i\left(p_y + \frac{\hbar\alpha x}{r^2}\right)}{\sqrt{2\hbar\mu\omega}} \tag{2.10}$$

Remarkably enough the three generators so obtained commute with the Hamiltonian (2.9) and also satisfy the SU(2) algebra (2.3). In fact, after some lengthy but straightforward algebra, one can show that relation (2.7), relating H and $\vec{J}^2$, is also valid in the presence of anyons. One may then naively think that the spectrum is the same as that given by eq.(2.2). This however is not the case. Whereas in the absence of anyons, J_3 equals $\frac{1}{2\hbar}L$ (where L is the canonical angular momentum, i.e. $L \equiv xp_y - yp_x$), in the presence of anyons it follows from eqs.(2.4c) and (2.10) that J_3 is not the canonical

angular momentum but

$$J_3 = \frac{1}{2\hbar}(L + \hbar\alpha) \tag{2.11}$$

Put differently, anyons can be looked upon as bosons with long ranged interaction ($0 \le \alpha \le 1$). Hence the canonical angular momentum is still $xp_y - yp_x = i\hbar\frac{\partial}{\partial\phi}$. Its eigenvalues $\hbar m$ must not only be integer (so that the corresponding eigenfunction of $i\hbar\frac{\partial}{\partial\phi}$ is single valued) but must also be even ($m = 0, \pm 2, \pm 4, ...$) since one is working in a bosonic basis. As a result, following the discussion of Kretzschmar [6] and using eq.(2.7), the spectrum of 2-anyons experiencing oscillator potential is given by ($0 \le \alpha \le 1$)

$$E_n = (|2n+\alpha|+1)\hbar\omega; \quad n = 0, \pm 1, \pm 2, \dots \tag{2.12}$$

This can be shown to be identical to the known exact spectrum [4]. This argument also produces the correct degeneracies for any value of α. In particular again following the discussion of ref.[6], one finds that for noninteger α the Hilbert-space H can be split into two subspaces H_1 and H_2. The Hilbert -space H_1 contains eigenfunctions with eigenvalues of the form $(2n+1+\alpha)\hbar\omega$ (n = 0,1,2,....). For each of these eigenvalues, there are n+1 eigenfunctions distinguished from each other by the quantum number m ranging through the interval [0,2n] in twice the integer steps. Hilbert-space H_2, on the other hand, contains eigenfunctions with eigenvalues of the form $(2n+1-\alpha)\hbar\omega$ with n = 1,2..... For each of these eigenvalues, there are n eigenfunctions which are distinguished from each other by the quantum number m. This quantum number ranges in twice the integer steps through the interval $[-2n, -2]$.

III. Symmetry of the Two-Anyon Plus Coulomb Problem

The Hamiltonian for the Coulomb problem in two dimensions is given by [2]

$$H=\frac{1}{2\mu}(p_x^2+p_y^2)-\frac{e^2}{r} \qquad (3.1)$$

Its bound state energy eigenvalue spectrum is given by

$$E_n=-\frac{\mu e^4}{2\hbar^2(n+\frac{1}{2})^2} \quad ; \quad n=0,1,2,\dots \qquad (3.2)$$

and the corresponding degeneracy is (2n+1). Furthermore, it is known that one can define a Runge- Lenz two-vector (R_x, R_y) and orbital angular momentum L which together form the 0(3) algebra, and all these symmetry generators commute with the Hamiltonian of (3.1). In particular the generators of the 0(3) group are

$$R_x=\frac{1}{2\sqrt{-2\mu E}}\left[-Lp_y-p_yL+\frac{2\mu e^2 x}{r}\right] \qquad (3.3a)$$

$$R_y=\frac{1}{2\sqrt{-2\mu E}}\left[Lp_x+p_xL+\frac{2\mu e^2 y}{r}\right] \qquad (3.3b)$$

$$L=xp_y-yp_x \qquad (3.3c)$$

Here,-E represents the bound state energy eigenvalue corresponding to H as given by eq.(3.1). Using the commutation relations (2.6) it is easily checked that $(\vec{J}^2=R_x^2+R_y^2+L^2)$

$$-\frac{e^4\mu}{2E}=\vec{J}^2+\frac{\hbar^2}{4} \qquad (3.4)$$

From this relaton, we again obtain the eigenvalue spectrum (3.2) with the right degener-

acy of (2n+1), by recognizing that the group is 0(3) and that the eigenvalues of $\vec{J}^2$ are $j(j+1)\hbar^2$ with j = 0,1,2,......

The stratagy for obtaining the symmetry of the 2-anyon problem with the Coulomb potential is now identical to that in the last section. On recognizing the fact that the velocities $\dot{x},\dot{y}$ are unchanged, even in the presence of anyons, and further that in terms of momenta they are given by eq.(2.8), we see that the relative Hamiltonian for 2-anyons experiencing Coulomb potential is given by

$$H=\frac{1}{2\mu}\left[\left(p_x-\frac{\hbar\alpha y}{r^2}\right)^2+\left(p_y+\frac{\hbar\alpha x}{r^2}\right)^2\right]-\frac{e^2}{r} \qquad (3.5)$$

One finds that the modified generators given by

$$\hat{R}_x=\frac{1}{2\sqrt{-2\mu E}}\left[-\hat{L}\left(p_y+\frac{\hbar\alpha x}{r^2}\right)-\left(p_y+\frac{\hbar\alpha x}{r^2}\right)\hat{L}+\frac{2\mu e^2 x}{r}\right] \qquad (3.6a)$$

$$\hat{R}_y=\frac{1}{2\sqrt{-2\mu E}}\left[\hat{L}\left(p_x-\frac{\hbar\alpha y}{r^2}\right)+\left(p_x-\frac{\hbar\alpha y}{r^2}\right)\hat{L}+\frac{2\mu e^2 y}{r}\right] \qquad (3.6b)$$

$$\hat{L}=x\left(p_y+\frac{\hbar\alpha x}{r^2}\right)-y\left(p_x-\frac{\hbar\alpha y}{r^2}\right)=L+\hbar\alpha \qquad (3.6c)$$

commute with the new H of eq.(3.5) and that $\hat{R}_x$, $\hat{R}_y$ and $\hat{L}$ also satisfy the 0(3) algebra. As before one finds that $(\vec{J}^2=\hat{R}_x^2+\hat{R}_y^2+\hat{L}^2)$

$$-\frac{e^4\mu}{2E}=\vec{J}^2+\frac{\hbar^2}{4} \qquad (3.7)$$

As in the oscillator case, one has to remember that L is the canonical angular momentum whose eigenvalue m must be twice integer since one is working in the bosonic basis. Following ref.[6], it then follows that the energy eigenvalues of the Coulomb potential are given by

$$E = -\frac{\mu e^4}{2\hbar^2[|n+\alpha|+\frac{1}{2}]^2}, \quad n = 0, 1, \pm 2, \pm 3, \cdots \qquad (3.8)$$

For fractional value of $\alpha(\neq 1/2)$ the total Hilbert space H can again be split into two subspaces H_1 and H_2 where by definition H_1 is spanned by all eigenfunctions with $n \geq 0$ while H_2 is spanned by all eigenfunctions with $n \geq 0$. For a given positive eigenvalue n, we have $\frac{n}{2}+1$ or $\frac{(n+1)}{2}$ degenerate eigenfunctions depending on whether n is even or odd. These different eigenfunctions are distinquished from each other by the quantum number m which ranges in twice the integer steps through the interval [0,n] or [0,n-1] depending on if n is even or odd. Similarly for a given negative value of n, we have $\frac{n}{2}$ or $\frac{n-1}{2}$ degenerate levels depending on if $|n|$ is even or odd. These are distinguished from each other by the value of m i.e. $-|n| \leq m \leq -2$ or $-(|n|-1) \leq m \leq -2$ depending on whether n is even or odd. In the special case of semions ($\alpha = 1/2$), we see that the energy eigenvalues are the same in both subspaces H_1 and H_2. The corresponding eigenfunctions however differ by their quantum number n. As a result the degeneracy is increased to n+1 for any n(= 0,1,2.....), and there are level crossings at $\alpha = 1/2$ for every value of n (except n = 0).

IV. Shape Invariance of the Two-Anyon Potentials

We shall first show that the problem of two-anyons plus a Coulomb or oscillator potential can be cast in a supersymmetric form and that the partner potentials are infact shape invariant. We then deduce the spectrum of two-anyons with a Coulomb or

oscillator potential by making use of the fact that both problems have shape invaraint potentials.

Consider for example the 2-anyon harmonic oscillator Hamiltonian of eq.(2.1). On substituting the ansatz

$$\psi(r,\phi) = \frac{1}{\sqrt{r}} R(r) e^{im\phi} \qquad (4.1)$$

in the Schrödinger equation for this Hamiltonian, it is easily shown that R satisfies the equation

$$\hat{H} R(r) = E R(r) \qquad (4.2a)$$

where

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{1}{2}\mu\omega^2 r^2 + \frac{[(m+\alpha)^2 - \frac{1}{4}]\hbar^2}{2\mu r^2} \qquad (4.2b)$$

Let us now consider the operator

$$A = -\frac{\hbar}{\sqrt{2\mu}}\frac{d}{dr} - \frac{(|m+\alpha|+\frac{1}{2})\hbar}{\sqrt{2\mu}\, r} + \sqrt{\frac{\mu}{2}}\,\omega r \qquad (4.3)$$

Using (4.3), we find that the two supersymmetric partner potentials are [7]

$$H_B \equiv A^+ A = \hat{H} - \hbar\omega(|m+\alpha|+1) \qquad (4.4)$$

$$H_F \equiv A A^+ = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{1}{2}\mu\omega^2 r^2 - \hbar\omega(|m+\alpha|) + \frac{(|m+\alpha|+\frac{1}{2})(|m+\alpha|+\frac{3}{2})\hbar^2}{2\mu r^2}, \qquad (4.5)$$

Now, notice that the two potentials H_B and H_F are shape invariant i.e. [5]

$$H_F(|m+\alpha|) = H_B(|m+\alpha|+1)2\hbar\omega \tag{4.6}$$

In the notation of ref.[5], we find for this problem, that

$$a_0 = |m+\alpha|, \quad a_1 = |m+\alpha|+1$$
$$R(a_1) = (a_1 - a_0)2\hbar\omega \tag{4.7}$$

Thus, the energy eigenvalues of H_B are given by:

$$E_n^B = \sum_{k=1}^{n} R(a_k) = 2n\hbar\omega; \quad n = 0,1,2,\cdots \tag{4.8}$$

Hence, the energy eigenvalues of $\hat{H}$ (i.e. 2-anyons plus oscillator Hamiltonian) are given by

$$E_n = (2n + |m+\alpha| + 1)\hbar\omega \tag{4.9}$$

This is the exact answer [3].

Let us now show that the 2-anyon with a Coulomb interaction Hamiltonian is also shape invariant and deduce its spectrum. On substituting the ansatz (4.1) in the Schrödinger equation for the Coulomb Hamiltonian (3.1), it is easily shown that R satisfies equation (4.2a) where $\hat{H}$ is now given by

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} - \frac{e^2}{r} + \frac{[(m+\alpha)^2 - \frac{1}{4}]\hbar^2}{2\mu r^2} \tag{4.10}$$

Let us Consider the operator

$$A = -\frac{\hbar}{\sqrt{2\mu}}\frac{d}{dr} - \frac{(|m+\alpha|+\frac{1}{2})\hbar}{\sqrt{2\mu}\, r} + \frac{\sqrt{2\mu}\, e^2}{2\hbar(|m+\alpha|+\frac{1}{2})} \tag{4.11}$$

One immediately find, that the two supersymmetric partner potentials are

$$H_B \equiv A^+A = \hat{H} + \frac{\mu e^4}{2\hbar^2(|m+\alpha|+\frac{1}{2})^2} \tag{4.12}$$

$$H_F \equiv AA^+ = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{(|m+\alpha|+\frac{1}{2})(|m+\alpha|+\frac{3}{2})\hbar^2}{2\mu r^2} - \frac{e^2}{r} + \frac{\mu e^4}{2\hbar^2(|m+\alpha|+\frac{1}{2})^2} \tag{4.13}$$

Clearly H_B and H_F are shape invariant potentials i.e.

$$H_F(|m+\alpha|+\tfrac{1}{2}) = H_B(|m+\alpha|+\tfrac{3}{2}) + \frac{\mu e^4}{2\hbar^2(|m+\alpha|+\frac{1}{2})^2} - \frac{\mu e^4}{2\hbar^2(|m+\alpha|+\frac{3}{2})^2} \tag{4.14}$$

In the notation of ref[5], we have for this problem that:

$$a_0 = |m+\alpha| + \tfrac{1}{2}, \quad a_1 = |m+\alpha| + \tfrac{3}{2} \tag{4.15a}$$

$$R(a_1) = \frac{\mu e^4}{2\hbar^2}\left(\frac{1}{a_0^2} - \frac{1}{a_1^2}\right) \tag{4.15b}$$

Thus the energy eigenvalues of H_B are given by (n=0,1,2,....)

$$E_n^B = \sum_{k=1}^{n} R(a_k) = \frac{\mu e^4}{2\hbar^2}\left[\frac{1}{(|m+\alpha|+\frac{1}{2})^2} - \frac{1}{(|m+\alpha|+n+\frac{1}{2})^2}\right] \tag{4.16}$$

and the energy eigenvalues of $\hat{H}$, (i.e. the 2-anyon Hamiltonian plus a Coulomb interaction are given by [4]

$$E_n = -\frac{\mu e^4}{2\hbar^2(|m+\alpha|+n+\frac{1}{2})^2} \tag{4.17}$$

which is again the exact answer.

V.Discussion

In this paper we have shown that the 2-anyon system in a Coulomb or an Oscillator potential has the O(3) or SU(2) symmetries respectively. Using these symmetries, we have been able to obtain algebraically the bound state spectrum in the two cases. The real nontrivial question is if this discussion can be genealized to the case of n-anyons and if one can algebraically obtain the complete bound state spectrum of n-anyons in an the oscillator potential. Such a step would be very useful as the complete spectrum is still unknown. Further it would enable us to calculate the n'th virial coefficient of an anyon gas. We hope to attack this problem in the near future.

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