Tolman-Oppenheimer-Volkov conditions beyond spherical symmetry

José P. Mimoso

Departamento de Física and Instituto de Astrofísica e Ciências do Espaço,
Faculdade de Ciências da Universidade de Lisboa,
Campo Grande, Ed. C8 1749-016 Lisboa, Portugal
E-mail: jpmimoso@fc.ul.pt

Alan Maciel

Centro de Matemática, Computação e Cognição, Universidade Federal do ABC,
Avenida dos Estados 5001, CEP 09210-580, Santo André, São Paulo, Brazil
E-mail: alan.silva@ufabc.edu.br

Morgan Le Delliou

Institute of Theoretical Physics, School of Physical Science and Technology, Lanzhou University,
No. 222, South Tianshui Road, Lanzhou, Gansu 730000, P R China
E-mail: delliou@lzu.edu.cn, delliou@ift.unesp.br

The TOV equation is usually interpreted as the relativistic counterpart of the classical condition for hydrostatic equilibrium, and characterises the static equilibrium of bound, spherical distributions of matter such as stars. In the present work we aim at showing that a generalised TOV equation also determines the equilibrium of models endowed with other symmetries besides spherical. We resort to the dual null formalism applied to spacetimes with two dimensional spherical, planar and hyperbolic symmetries, and consider a perfect fluid as the source. Static configurations assume the existence of a time-like Killing vector field orthogonal to the surfaces of symmetry, and homogeneous dynamical solutions arise when the Killing field is space-like. In order to treat equally all the aforementioned cases, we discuss the definition of a quasi-local energy for the spacetimes with planar and hyperbolic foliations, since the Hawking-Hayward definition only applies to compact foliations. This procedure enables us to translate our geometrical formalism to the fluid dynamics language in a unified way, to find the generalised TOV equation, for the three cases when the solution is static, and to obtain the evolution equation, for the homogeneous spacetime cases. Remarkably, we show that the static solutions which are not spherically symmetric violate the weak energy condition (WEC). We also show that the counterpart of the TOV equation $\rho+P=0$, defines a cosmological constant-type behaviour, both in the hyperbolic and spherical cases. This implies a violation of the strong energy condition in both cases, added to the above mentioned violation of the weak energy condition in the hyperbolic case. We illustrate our unified treatment obtaining analogs of Schwarzschild interior solution, for an incompressible fluid $\rho = \rho_0$ constant.

Keywords: Gravitational collapse; hydrostatic equilibrium.

1. Introduction

The Tolman-Oppenheimer-Volkoff (TOV) equation[1,2] characterises spherical objects in equilibrium in the framework of General Relativity. It appears as the relativistic counterpart of the classical condition for hydrostatic equilibrium. Given the relevance of the spherical configurations in astrophysics, many solutions were investigated,[3–15] and more specifically, different formalisms[16–20] and solutions generating techniques have been developed.[21, 22] Extensions of the TOV equation have also been investigated in the framework of modified gravity theories.[23–29]

Yet a unified characterization of the underlying features of the TOV equation has attracted little attention, and this is what concerns us in the present work. There is a widespread perception of the TOV equation as being restricted to the spherically symmetric case, but this is a misled assumption. So, in the present work we derive a generalised TOV equation that also characterises the equilibrium of models endowed with other symmetries besides the spherical.[a]

We adopt the dual null formalism[b] that offers a description of the spacetime based on the properties of the optical flow, and has the significant advantage of being a coordinate free formalism. In the literature it has been considered in connection with the behaviour of dynamical black holes,[35, 36] the definition of energy in more general geometries,[37] the gravitational collapse of fluids,[38] and even with the definition of generalized horizons in modified gravity.[39] The dual null formalism has also been useful to explicit the “linear” behavior of gravity for sources that satisfy the hypotheses of the Birkhoff theorem.[31]

Here, we apply the dual null formalism to analyse, in a unified way, the spacetimes which admit a codimension-two foliation with constant curvature leaves.[40] This comprises the spherical, planar and hyperbolic symmetries, sourced by a perfect fluid. Aside from the Killing vector fields that are tangent to those surfaces of symmetry, we assume the existence of an additional symmetry generated by a Killing vector field orthogonal to those surfaces at each event. A particular case of this setup, where the symmetry is spherical and the Killing vector is timelike, corresponds to the spherically symmetric perfect fluid in hydrostatic equilibrium, which leads us to the well-known TOV equation. As we will show this celebrated equation arises most naturally in the dual-null framework which, moreover, allows us to generalize it for the planar and hyperbolic cases. This generalisation of the geometry underlying the TOV equation, stepping beyond spherical symmetry, is novel and leads to consequences which are far from trivial.[40]

Since in planar and hyperbolic geometries the spatial hypersurfaces are open, this extension requires the novel introduction of a mass-energy “parameter”. In fact one needs to promote a generalization of the Misner-Sharp/Hawking-Hayward definitions of the mass-energy distribution, which overcomes the problem of the divergence of the latter quantities due to the natural threading with infinite surfaces.

---

[a] A similar prejudice applying to the Birkhoff theorem, has been addressed with its generalization to more general geometries by C. Bona,[30] and more recently by us in.[31]

[b] We follow here the nomenclature coined by Sean A. Hayward[32–34] although in some references this formalism is referred to as double-null.

Finally, our investigation of the planar and hyperbolic geometries reveals the violation of the weak energy condition in order to maintain hydrostatic equilibrium. This takes the form of negative effective mass, physically translating repulsive curvature effects, which suggest a link to repulsive source models, as those proposed to mimic dark energy, generate bouncing universes, or support classical wormholes.[41, 42]

For completness, when the metric is characterised by a spacelike, rather than a timelike Killing vector, we have spatially homogeneous spacetimes that correspond to some of the Bianchi spacetimes, or Kantowski-Sachs, as expected. The hydrostatic equilibrium on those spacetimes is only possible when their source is a cosmological constant in the non flat cases, also implying the violation of energy conditions.

2. The dual null formalism

We consider metrics that have a codimension-two maximally symmetric foliation, and can be written as

$$\mathrm{d}s^2 = N_{ab}\mathrm{d}x^a\mathrm{d}x^b + Y^2(x^c)\left(\mathrm{d}\theta^2 + S_\epsilon^2\mathrm{d}\phi^2\right) , \tag{1}$$

where

$$S_\epsilon = \begin{cases} \sin\theta, \text{ for } \epsilon = 1 \\ 1, \text{ for } \epsilon = 0 \\ \sinh\theta, \text{ for } \epsilon = -1 \end{cases} ,$$

and where we divide the tangent space $\mathcal{T}$ at each event in two orthogonal subspaces $\mathcal{T} = \mathcal{N} \oplus \mathcal{S}$. Here $\mathcal{S}$ is the subspace generated by the orbits of (θ, ϕ) and $\mathcal{N}$, the subspace of $\mathcal{T}$ orthogonal to $\mathcal{S}$. The x^a coordinates are chosen orthogonal to $\mathcal{S}$, which gives the metric in the warped sum form of Eq. (1).

We denote $s_{ab} = Y^2\gamma_{ab}$ the induced metric in each leaf of the foliation where $Y(x^c)$ is the warp factor. Evidently, $\gamma_{ab} := \delta_a^\theta\delta_b^\theta + S_\epsilon^2\delta_a^\phi\delta_b^\phi$ has constant curvature and does not depend on the coordinates x^a which identify each leaf Σ_{x^c}, defined as the locus spanned by the orbits of θ and ϕ for fixed x^c. We define an orthonormal two dimensional basis $(n^a\,, e^a)$ for $\mathcal{N}$, whose induced metric is N_{ab}, according to Eq. (1). This basis satisfies

$$-n^a n_a = e^a e_a = 1\,, \quad n^a e_a = n^a s_{ab} = e^a s_{ab} = 0\,. \tag{2}$$

We may also define a dual null basis for the same subspace from n^a and e^a by

$$\begin{aligned} k^a &= \frac{1}{2}\left(n^a + e^a\right)\,, \quad l^a = \frac{1}{2}\left(n^a - e^a\right)\,, \\ n^a &= k^a + l^a\,, \quad e^a = k^a - l^a\,, \end{aligned} \tag{3}$$

which satisfies

$$k^a k_a = l^a l_a = 0\,, \quad k^a l_a = -\frac{1}{2}\,. \tag{4}$$

The metric g_{ab} can be written as

$$g_{ab} = \frac{2}{k^c l_c} k_{(a} l_{b)} + s_{ab} \,. \tag{5}$$

We associate the null expansion for each null vector as follows

$$\Theta_k = \frac{1}{2} s^{ab} \mathcal{L}_k s_{ab} = \frac{1}{2} Y^{-2} \gamma^{ab} \mathcal{L}_k Y^2 \gamma_{ab} = \frac{2}{Y} k^a \partial_a Y \,. \tag{6}$$

We may extend the definition of null expansion to timelike and spacelike vectors in $\mathcal{N}$, calling it the two-expansion, since it measures the rate of variation of area, as in the null case. We may define the mean curvature form $\mathcal{K}_a = \partial_a \ln Y^2$, such that, we obtain for the two-expansion $\Theta_{(u)}$ of any vector u^a in $\mathcal{N}$

$$\Theta_{(u)} = u^a \mathcal{K}_a \,. \tag{7}$$

We describe our spacetimes by means of the behaviour of the null expansion, casting the Einstein equations, $G_{ab} = 8\pi T_{ab}$ in terms of expansions, i.e., by writing the Raychaudhuri equations and constraint equations[35, 43–45]

$$\mathcal{L}_k \Theta_{(k)} = \nu_k \Theta_{(k)} - \frac{\Theta_{(k)}^2}{2} - 8\pi T_{ab} k^a k^b \,, \tag{8a}$$

$$\mathcal{L}_l \Theta_{(l)} = \nu_l \Theta_{(l)} - \frac{\Theta_{(l)}^2}{2} - 8\pi T_{ab} l^a l^b \,, \tag{8b}$$

$$\begin{aligned} \mathcal{L}_k \Theta_{(l)} + \mathcal{L}_l \Theta_{(k)} &= -\Theta_{(l)} \nu_k - \Theta_{(k)} \nu_l - \\ 2\,\Theta_{(k)}\,\Theta_{(l)} &+ \epsilon \frac{2\,k^a l_a}{Y^2} + 16\pi\, T_{ab} k^a l^b \,, \end{aligned} \tag{8c}$$

where we included the inaffinities ν_k and ν_l, defined as

$$\nu_k = \frac{1}{k^c l_c} l^b k^a \nabla_a k_b \qquad \nu_l = \frac{1}{k^c l_c} k^b l^a \nabla_a l_b \,. \tag{9}$$

In the latter equations, for T_{ab}, if we take the source to be a perfect fluid[c], then the energy momentum tensor reduces to

$$T_{ab} = \rho n_a n_b + P(e_a e_b + s_{ab}) \,. \tag{10}$$

where we adapt our vector basis to the fluid source, such that n^a gives its flow. In the latter decomposition ρ is the energy density, and P is the isotropic pressure, both measured by an observer moving with 4-velocity n^a. By construction, the flow n^a is always orthogonal to the surfaces of symmetry and will be characterized by two quantities

$$\mathcal{A} = e^a \dot{n}_a = e^a n^b \nabla_b n_a \,, \qquad \mathcal{B} = e^a n'_a = e^a e^b \nabla_b n_a \,. \tag{11}$$

The scalar $\mathcal{A}$ gives us the acceleration of the flow, a positive sign meaning that the acceleration is outwards in the spherical, compact case. The scalar $\mathcal{B}$ gives the

[c]The equations in the case of a general fluid are given by us in Ref.[46].

change of direction of n^a as we travel along e^a. It is the $e - e$ component of the extrinsic curvature K_{ab} of the 3-space orthogonal to this flow, since

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}\,, \tag{12}$$

where $h_{ab} = g_{ab} + n_a n_b$. We may also write

$$h_{ab} = e_a e_b + Y^2 \gamma_{ab}\,, \tag{13}$$

which gives

$$K_{ab} = \mathcal{B}\, e_a e_b + \frac{\Theta_{(n)}}{2}\, Y^2 \gamma_{ab}\,. \tag{14}$$

The trace of Eq. (14) gives us the flow volumetric expansion $\Theta_3 = \nabla_a n^a = K^a_a$ as

$$\Theta_3 = \mathcal{B} + \Theta_{(n)}\,. \tag{15}$$

In order to relate our quantities with the flow scalars, we compute the shear scalar σ, by taking the symmetric traceless part of K_{ab}. We obtain

$$\sigma = \frac{\Theta_{(n)}}{6} - \frac{\mathcal{B}}{3}\,, \tag{16}$$

which implies

$$\frac{\Theta_3}{3} + \sigma = \frac{\Theta_{(n)}}{2}\,, \tag{17}$$

in agreement with the result obtained in Ref.[38].

Using the inaffinities of the null basis vectors, $\mathcal{A}$ and $\mathcal{B}$ can be expressed as

$$\mathcal{A} = \nu_k - \nu_l\,, \qquad \mathcal{B} = \nu_k + \nu_l\,. \tag{18}$$

Contracting the conservation of the energy-momentum tensor with e_b (Euler equation in[47]) we get

$$e_b \nabla_a T^{ab} = (\rho + P)\dot{n}^b e_b + e^a \nabla_a P = 0 \Rightarrow$$
$$\mathcal{A} = -\frac{e^a \partial_a P}{\rho + P}\,. \tag{19}$$

3. Orthogonal Killing vector

We now assume that our metric has a Killing vector orthogonal to maximally symmetric surfaces, as we are interested in static configurations. Our symmetry requirements imply that it commutes with the symmetry generators on the foliation. We denote this hypersurface orthogonal Killing vector field χ^a. It satisfies the Killing equation,

$$\mathcal{L}_\chi g_{ab} = 0\,. \tag{20}$$

Proposition 1. If a spacetime is described by a metric of the form (1) and admits an orthogonal Killing vector $\chi^a \in \mathcal{N}$, then $\Theta_\chi = 0$.

Proof. We may write, from Eq. (6), $\Theta_\chi = \frac{1}{2} s^{ab} \mathcal{L}_\chi s_{ab} = \frac{1}{2} Y^{-2} \gamma^{ab} \mathcal{L}_\chi Y^2 \gamma_{ab}$, and

$$g_{ab} = N_{ab} + Y^2 \gamma_{ab} \,. \tag{21}$$

Then

$$\begin{aligned} 0 = Y^{-2}\gamma^{ab}\mathcal{L}_\chi g_{ab} &= Y^{-2}\gamma^{ab}\mathcal{L}_\chi N_{ab} + 2\Theta_\chi \\ &= -Y^{-2} N^{ab} \mathcal{L}_\chi \gamma_{ab} + 2\Theta_\chi \,. \end{aligned} \tag{22}$$

However

$$\mathcal{L}_\chi \gamma_{ab} = 0 \,, \tag{23}$$

since χ^a does not admit components in $\mathcal{S}$ and γ_{ab} doesn't depend on coordinates along $\mathcal{N}$. Therefore, Eq. (22) implies that $\Theta_\chi = 0$. □

Consequently, if there is an extra symmetry with orbits orthogonal to those of the maximally symmetric leaves of the foliation, the two-expansion of its generator vanishes. This also implies that if $\mathrm{d}Y$ is spacelike, then χ_a is timelike and vice-versa. If $\mathrm{d}Y$ is null, the Killing vector will also be null.

4. Extension of the mass-energy parameter to open geometries

There is a widely known mass-energy definition suitable to the spherically symmetric case, namely the Misner-Sharp mass-energy,[48, 49] that is defined regardless of asymptotic assumptions. However, as we also intend to analyze nonspherical spacetimes in this work, we are led to consider a more general mass-energy definition, namely the Hawking-Hayward's (hereafter HH).[37, 50] The HH mass-energy gives the mass-energy content inside a closed compact surface in terms of an integral over that surface, in a manner similar to the Gauss law in Newtonian gravity. This quasilocal mass-energy has been explored in different contexts, such as seen in Refs.[51–53].

Under our symmetry assumptions the Hawking-Hayward mass-energy enclosed by Σ is reduced to

$$M_\Sigma = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \int_\Sigma \left[\mathcal{R} - \frac{1}{k^a l_a} \Theta_{(k)} \Theta_{(l)} \right] \mathrm{d}\Sigma \tag{24}$$

where $\mathcal{R}$ is the two-dimensional Ricci scalar and A is the area of Σ. We have included a factor of $-\dfrac{1}{k^a l_a}$ in the optical scalars part of the mass-energy, compared with the formula present in Ref.[37], in order to take account of our different normalization of the null normals. Our symmetry assumptions imply that the only non-vanishing optical scalar on Σ is the null expansion.

Since we assume that Σ is maximally symmetric, we have $\mathcal{R} = \frac{2\epsilon}{Y^2}$. We also have

$$\Theta_{(k)}\Theta_{(l)} = k^{(a}l^{b)}\partial_a \ln Y^2 \partial_b \ln Y^2 = \frac{k^c l_c}{2} g^{ab}\partial_a \ln Y^2 \partial_b \ln Y^2 =$$

$$\frac{1}{2}k^c l_c ||\mathrm{d}\ln Y^2||^2 = k^c l_c \frac{2}{Y^2}||\mathrm{d}Y||^2 \Rightarrow \frac{\Theta_{(k)}\Theta_{(l)}}{k^c l_c} = \frac{2}{Y^2}||\mathrm{d}Y||^2 \,, \tag{25}$$

where we used Eq. (5) in the second step.

For the spherical case $\epsilon = 1$ and $A = 4\pi Y^2$, we obtain the known interpretation of $||\mathrm{d}Y||$ in terms of the Misner-Sharp mass-energy, which coincides with the Hawking-Hayward one

$$M_\Sigma = \frac{Y}{2}\left(1 - ||\mathrm{d}Y||^2\right) \Leftrightarrow ||\mathrm{d}Y||^2 = 1 - \frac{2M_\Sigma}{Y} \,. \tag{26}$$

In this work, we aim at treating all three symmetry types in the same manner. In the planar and hyperbolic cases ($\epsilon = 0$ and $\epsilon = -1$, respectively), the Hawking-Hayward mass is not conveniently defined for the integration domain set by our preferred foliation, as it requires a closed compact surface which is absent in the latter cases. Therefore, we need a mass-energy definition which might be equivalent to the HH mass-energy, but suitable to deal with (instead of adapted for) non compact domains in order to take advantage of the planar or hyperbolic symmetry.

We can make such an extension of the HH mass-energy, as long as their boundary correspond to a pair of symmetric two-surfaces of symmetry Σ corresponding to the same warp factor Y. Of course, those domains are infinite and have an infinite mass-energy content in general. However, as they are homogeneous along the surfaces of symmetry, we can successfully adapt the HH mass-energy definition in order to obtain a finite *mass-energy parameter* with those cases. They then describe an infinite mass-energy distribution, homogeneous along the surfaces of symmetry, with a finite density.

We proceed by first making the replacement

$$\frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \to \frac{Y}{4\pi\kappa}, \tag{27}$$

in order to keep its dimensionality, and eliminating the explicit dependence on the area of Σ. Evidently, by setting $\kappa = 4$ we recover the Hawking-Hayward mass-energy in the spherical case. This step is justified by the fact that originally this factor was introduced to correct the dimensionality of the mass-energy, and to make it match the Arnowitt-Deser-Misner (ADM) mass,[54] where both are well defined. Since our symmetric spacetimes allow an "areal scalar" as the warp factor Y, we can replace $\sqrt{A}$ by Y as the quantity with dimension of length associated to each surface of symmetry.

We then define the quasi-local mass-energy parameter $\mu(Y)$ by

$$M_\Sigma = \frac{\mu(Y)}{4\pi} \int S_\epsilon(\theta)\, \mathrm{d}\theta \mathrm{d}\phi\,, \tag{28}$$

and we write

$$\frac{Y}{4\pi\kappa}\left[\mathcal{R} - \frac{\Theta_{(k)}\Theta_{(l)}}{k^c l_c}\right]\int_\Sigma \mathrm{d}\Sigma = \frac{Y}{4\pi\kappa}\left[2\epsilon - 2||\mathrm{d}Y||^2\right]\int S_\epsilon(\Theta)\mathrm{d}\theta\mathrm{d}\phi\,. \tag{29}$$

Equating Eqs. (28) and (29) and eliminating the improper area integral on both sides we derive

$$||\mathrm{d}Y||^2 = \epsilon - \frac{\kappa\mu(Y)}{2Y}\,. \tag{30}$$

Equation (30) coincides with the known mass function [see Eq. (15.7a) in[55]] which appears as we integrate the Einstein equations of specific spacetimes with metrics of the form (1) for planar and hyperbolic symmetries. From now on, we will consider Eq. (30) with the choice $\kappa = 4$ as the mass-energy definition.

An alternative route to Eq. (30) can be obtained by computing the HH mass-energy in a finite domain, symmetric with respect to the central plane or wire, $Y = 0$, and taking the limit where the domain tends to be the whole surface. The finite integration domain consist of the union of

(1) a subset of the Σ_Y, that we denote Γ_r, bounded by a circle γ_r of radius r on the (θ, ϕ) coordinate plane and
(2) a compact surface given by the surfaces Δ_r defined by γ_r transported along Y orbits.

It forms a closed surface, corresponding to a part of a cylinder bounded by $Y =$ constant surfaces in the space of coordinates (Y, θ, ϕ). Therefore, the HH mass-energy enclosed by those surfaces will by finite, and given by

$$M_r = \frac{1}{8\pi}\sqrt{\frac{A_r}{16\pi}}\left(\int_{\Gamma_r}(\dots)S_\epsilon \mathrm{d}\theta\mathrm{d}\phi + \int_{\Delta_r}(\dots)\mathrm{d}\Delta\right) \tag{31}$$

where (...) replaces the integrand of Eq. (24). In the limit $r \to \infty$, the first integral in Eq. (31) scales as r^2 while the second one scales as r. This means that, in the limit $r \to \infty$, and repeating the replacement in Eq. (27) we obtain

$$\frac{M_r}{A_r} \to \frac{\mu(Y)}{4\pi Y^2}\,. \tag{32}$$

This relation is particularly adequate to obtain the extension of the TOV equation for the open geometries, as it will become apparent in the following section.

5. Evolution equations

5.1. *Timelike Killing vector*

We assume $\chi^a \chi_a < 0$. In this case, the spacetime is static, and $n^a \sim \chi^a$. Therefore, from Proposition 1, $\Theta_{(n)} = 0$ everywhere, and $\mathrm{d}Y$ is spacelike, since it is orthogonal to n^a. If $\Theta_{(n)}$ vanishes everywhere, this means that the fluid has no radial velocity, therefore we are dealing with a static fluid with a flow parallel to the Killing vector field.

In order to characterize its static equilibrium, we realise the derivative of the flow 2-expansion along the flow itself is vanishing

$$\mathcal{L}_n \Theta_{(n)} = 0\,, \tag{33}$$

since $\Theta_{(n)} = 0$ everywhere.

We may write $\mathcal{L}_n \Theta_{(n)}$ in terms of the null expansions as

$$\mathcal{L}_n \Theta_{(n)} = \mathcal{L}_k \Theta_{(k)} + \mathcal{L}_l \Theta_{(l)} + \mathcal{L}_l \Theta_{(k)} + \mathcal{L}_k \Theta_{(l)}\,. \tag{34}$$

Substituting the Eqs. (8a), (8b), and (8c), we obtain

$$\begin{aligned} \mathcal{L}_n \Theta_{(n)} = -\frac{(\Theta_{(k)} + \Theta_{(l)})^2}{2} - \\ \Theta_{(k)}\Theta_{(l)} - \frac{\epsilon}{Y^2} - 8\pi T_{ab} e^a e^b + \mathcal{A}(\Theta_{(k)} - \Theta_{(l)})\,. \end{aligned} \tag{35}$$

Recall that we are assuming $\Theta_{(n)} = \Theta_{(k)} + \Theta_{(l)} = 0$, and that $\Theta_{(k)} - \Theta_{(l)} = \Theta_{(e)}$, using Eqs. (7) and (3). We identify here $\Theta_{(k)}\Theta_{(l)}$ as the mass term, since it equals $\dfrac{2}{Y^2}||\mathrm{d}Y||^2 = \dfrac{2}{Y^2}\left(\epsilon - \dfrac{2\mu(Y)}{Y}\right)$.

Taking the source to be a perfect fluid, (10), and contracting the divergence of the energy-momentum tensor with e_b (Euler equation in[47]) we get

$$\begin{aligned} e_b \nabla_a T^{ab} = (\rho + P)\dot{n}^b e_b + e^a \nabla_a P = 0 \Rightarrow \\ \mathcal{A} = -\frac{e^a \partial_a P}{\rho + P}\,. \end{aligned} \tag{36}$$

Since $\Theta_{(n)} = 0$, e^a is proportional to ∂_Y, and as e^a is normalized, we have

$$e_a = \frac{1}{||\mathrm{d}Y||}\partial_a Y\ , \quad e^a = ||\mathrm{d}Y||(\partial_Y)^a\,, \tag{37}$$

which gives us

$$\mathcal{A}\Theta_e = -||\mathrm{d}Y||^2\,\frac{2}{Y}\frac{\partial_Y P}{\rho + P}\,. \tag{38}$$

Therefore, replacing $||\mathrm{d}Y||^2$ by its meaning in terms of mass, $\mathcal{L}_n \Theta_{(n)} = 0$ corresponds to[40]

$$\left(\frac{\epsilon}{Y^2} - \frac{2\mu(Y)}{Y^2}\right)\frac{\partial_Y P}{\rho + P} = -\frac{1}{Y}\frac{\mu(Y)}{Y^2} - 4\pi P\,, \tag{39}$$

that reveals that the generalization of the concept of mass-energy given by (32) is particularly suited to avoid the caveats of the divergence of that quantity if it were defined in accordance to the canon of the spherically symmetric models.

Of course the latter equation (39) can alternatively be cast under a more familiar form as

$$\frac{\partial_Y P}{\rho + P} = -\left(\frac{\mu(Y)}{Y^2} + 4\pi P Y\right)\left(\epsilon - \frac{2\mu(Y)}{Y}\right)^{-1}, \tag{40}$$

which is what we denote as *the unified TOV equation.* It reduces to the well-know TOV equation for spherically symmetric spacetimes when $\epsilon = 1$, and it corresponds to the equation of hydrostatic equilibrium for planar and hyperbolic geometries, in the cases where $\epsilon = 0$ and $\epsilon = -1$, respectively. This underlines the fact that the TOV equation is a hydrostatic equilibrium equation, and not an equation of state, as it is sometimes erroneously stated.

In order to determine $\mu(Y)$ we consider the $\mathcal{L}_e\Theta_{(e)}$ Raychaudhuri equation

$$\mathcal{L}_e\Theta_{(e)} = \mathcal{B}\Theta_{(n)} - \frac{\Theta_{(e)}^2}{2} + \frac{1}{4}\left(\Theta_{(n)}^2 - \Theta_{(e)}^2\right) + \frac{\epsilon}{Y^2} - 8\pi T_{ab}n^a n^b\,, \tag{41}$$

which, by using $\Theta_{(n)} = 0$, and Eq. (37) lead us to

$$||\mathrm{d}Y||\partial_Y\left(\frac{2}{Y}||\mathrm{d}Y||\right) = -\frac{3}{Y^2}||\mathrm{d}Y||^2 + \frac{\epsilon}{Y^2} - 8\pi\rho\,. \tag{42}$$

Substituting Eq. (30) into Eq. (42), we obtain

$$\partial_Y\mu = 4\pi\rho Y^2\,, \tag{43}$$

which looks like the mass-energy equation of spherical symmetry. Here, it should be interpreted as the mass-energy equation in the spherical case, and as a mass-energy parameter equation in the planar and hyperbolic cases. Furthermore, Eqs. (43) and (30) imply that, if the *weak energy condition (WEC)*[56] *holds*, only the spherically symmetric case admits static regular solutions. Indeed, as those solutions require $||\mathrm{d}Y||^2 > 0$, that implies $\mu < 0$ for $\epsilon \leq 0$ and, as in regular spacetimes,

$$\mu(Y) = 4\pi\int_0^Y \rho(y)y^2\mathrm{d}y\,, \tag{44}$$

this imposes $\rho < 0$.

With Eq. (43), the last requirement to solve Eq. (40) is the equation of state of the fluid, $f(\rho, P) = 0$ which should come from specific physical considerations.

5.2. *Spacelike Killing vector: The cosmological cases*

In the case of a spacelike Killing vector, $\mathrm{d}Y$ is timelike, the flow n_a is orthogonal to the Killing vector, and the unitary base vector e^a is parallel to it. This imposes no constraint on the sign of μ according to Eq. (30), and thus there is no need to violate energy conditions in order to consider these solutions, thorougly studied in cosmology.[57]

Combining the dynamical equation given by $\mathcal{L}_e\Theta_{(e)} = 0$, and Proposition 1, we have $\Theta_{(e)} = 0$. Replacing Eq. (16) in Eq. (41), we derive

$$\frac{3}{4}\Theta_{(n)}^2 - 3\sigma\Theta_{(n)} + \frac{\epsilon}{Y^2} = 8\pi\rho\,, \tag{45}$$

and, using Eq. (17), we can express this Eq. (45) in terms of the volume expansion Θ_3 obtaining

$$\frac{\Theta_3^2}{3} - 3\sigma^2 = 8\pi\rho - \frac{\epsilon}{Y^2}\,, \tag{46}$$

which corresponds to the generalised Friedmann constraint equation for the evolution of a homogeneous and anisotropic universe. In the case $\sigma = 0$, we may identify $\Theta_3 = 3H$, and we recover the usual Friedmann equation for the flat ($\epsilon = 0$) and open ($\epsilon = -1$) spatially isotropic universes. Notice though that $\sigma = 0$ also yields anisotropic, cosmological solutions when the matter content is not a perfect fluid.[58]

The $\mathcal{L}_n\Theta_{(n)}$ Raychaudhuri equation gives the evolution of $\Theta_{(n)}$. Using Eq. (35) we obtain

$$\mathcal{L}_n\Theta_{(n)} = -\frac{3}{4}\Theta_{(n)}^2 - \frac{\epsilon}{Y^2} - 8\pi T_{ab}e^a e^b\,, \tag{47}$$

so that subtracting Eq. (45) from Eq. (47), we further derive

$$\mathcal{L}_n\Theta_{(n)} = -3\sigma\Theta_{(n)} - 8\pi\left(\rho + P\right)\,, \tag{48}$$

which, together with an equation of state relating ρ and P closes our system. By adding half of the Eq. (48) with one third of Eq. (46), we obtain:

$$\mathcal{L}_n\left(\frac{\Theta_3}{3}\right) + \left(\frac{\Theta_3}{3}\right)^2 = \\ -2\sigma\left(\frac{\Theta_3}{3} + \sigma\right) - \frac{\epsilon}{3Y^2} - \frac{4\pi}{3}\left(\rho + 3P\right)\,. \tag{49}$$

Those homogeneous and anisotropic spacetimes belong to a subclass of Bianchi models,[59,60] with the case $\epsilon = 0$ corresponding to Bianchi type I universes, $\epsilon = -1$ corresponding to the Bianchi type III models, and $\epsilon = 1$ to the Kantowski-Sachs spacetimes.[61]

In this work we are mainly focused on the hydrostatic equilibrium situations. Thus, our interest will be directed to understanding whether it is possible to find a correspondence between the TOV equation of static equilibrium, and some condition applying to the spatially homogeneous models.

Imposing staticity amounts in the present case to have $\Theta_3 = 0$, $\mathcal{L}_n\Theta_3 = 0$, and $\sigma = 0$ in Eqs. (46) and (49). Reconciling the reduced equations simply requires

$$\rho + P = 0\,, \tag{50}$$

in the $\epsilon = \pm 1$ cases, and has no realisation when $\epsilon = 0$, as $\rho = 0$ from Eq.(46). Hence we conclude that the TOV condition interpreted as a cornerstone of stability yields the well-known equation of state characterising a cosmological constant in the non flat cases. In hindsight, one could have anticipated this result which emerges here in a self-consistent way. Moreover we see that strong energy condition (SEC) is violated for both cases $\epsilon = \pm 1$, whilst the WEC is additionally violated for the $\epsilon = -1$ case, as follows from Eq. (46).

6. An illustration: Incompressible fluid solutions

Using our unified TOV equation, Eq. (40), we may look for static perfect fluid solutions for all three symmetries considered here. By choosing a timelike coordinate T along the flow, making $n_a = -\alpha(Y)\mathrm{d}T$, and the warp factor Y, we obtain the following line element in the (T, Y) coordinates:

$$\mathrm{d}s^2 = -\alpha^2(Y)\mathrm{d}T + \frac{\mathrm{d}Y^2}{\epsilon - \frac{2\mu(Y)}{Y}} + Y^2\mathrm{d}\Omega_\epsilon \,, \tag{51}$$

where $\mathrm{d}\Omega_\epsilon = \left(\mathrm{d}\theta^2 + S_\epsilon^2\mathrm{d}\phi^2\right)$ and the functions α and μ will be given by solving Einstein equations, i.e., Eqs. (40) and (30).

Here, we will apply our unified treatment to find the analogs of Schwarzschild interior solution, that is, we will use the equation of state of an incompressible fluid $\rho = \rho_0$ constant. It is important to note that, as we have discussed in Sec. 5.1, the static solutions with $\epsilon \neq 1$ violate the WEC, therefore we should take $\rho_0 < 0$ in those cases.

Equation (36) implies

$$\frac{\alpha'}{\alpha} = -\frac{P'}{\rho + P} \Rightarrow \alpha = \frac{c_0}{\rho_0 + P} \,, \tag{52}$$

where c_0 is an integration constant that can be set by rescaling the time coordinate and the prime denotes Y differentiation.

Equation (43) gives us

$$\mu(Y) = \frac{4\pi\rho_0 Y^3}{3} \,, \tag{53}$$

which we replace in Eq. (40) to obtain

$$P(Y) = \rho_0 \left(\frac{2\sqrt{|\epsilon - \frac{Y_s}{Y_g}|}}{3\sqrt{|\epsilon - \frac{Y_s}{Y_g}|} - \sqrt{|\epsilon - \frac{Y_s Y^2}{Y_s^3}|}} - 1 \right) . \tag{54}$$

where Y_g is the analog of the radius of the object and is the least positive number that satisfy $P(Y_g) = 0$, $Y_s = \dfrac{8\pi\rho_0 Y_g^3}{3}$ is the analog of the Schwarzschild radius, although it can not be interpreted as a location since it will be a negative number. This gives

$$\alpha = \frac{1}{2}\left(3\sqrt{\left|\epsilon - \frac{Y_s}{Y_g}\right|} - \sqrt{\left|\epsilon - \frac{Y_s Y^2}{Y_g^3}\right|}\right) \tag{55}$$

which has a similar form to the interior Schwarzschild solution, where we only change the sign of the mass-energy parameter and change the value of ϵ in the formula. Of course the physical properties are very distinct, since the solutions violate the WEC.

In Fig. 1 we compare the pressure for the three cases. From the slope of the curves, we notice that only the hyperbolic case presents $P' > 0$, compensating the repulsive gravity force in this setup. This is the opposite of the more familiar situations presented in the spherical and planar cases, where gravity is attractive, with $P' < 0$ sustaining the weight of the configuration. We can also see that the planar case admits a positive pressure for $0 < Y < Y_g$. That means that, as long as mass-energy is negative, we may have static plane configurations over a finite Y interval. On the other hand, the hyperbolic solution only admits positive pressure for $Y > Y_g$, so there is no analog of the Schwarzschild interior solution for this foliation, although it can be interpreted as an exterior fluid solution to an internal void. It can thus be matched to a hyperbolic vacuum solution for $Y < Y_g$, found as a particular case in Ref.[31]:

$$\mathrm{d}s^2 = -\left(\frac{2m}{Y} - 1\right)\mathrm{d}t^2 + \frac{\mathrm{d}Y^2}{\frac{2m}{Y} - 1} + Y^2(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,, \tag{56}$$

where the parameter $m = |\mu|$. The peculiarities of the hyperbolic solutions with regard to the energy conditions are also found in the work.[42]

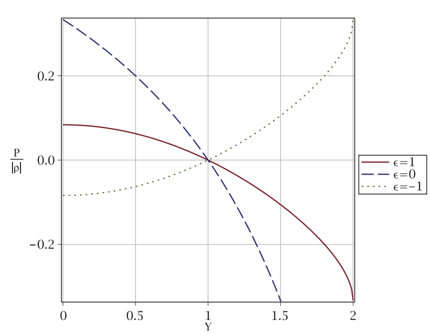

Fig. 1. Pressure as function of Y for $Y_g = 1$ and $|Y_s| = 0.25$ for $\epsilon = 1$, $\epsilon = 0$ and $\epsilon = -1$.

The equation of state consisting of a negative energy density with a positive pressure might be achieved by some kind of phantom field, but a Lagrangian description of the fluid is beyond the scope of this work. However, our simple incompressible model, with constant energy density, but varying pressure, is reminiscent of a constant time surface of a McVittie or Shah-Vaidya spacetime, which admits a Lagrangian scalar field as source.[62,63] This suggests the possibility that there exist field models in the literature which can source the solutions presented in this paper.

We notice that the planar solutions may also represent a subclass of cylindrical solutions (see Refs.[64,65]) if we select one coordinate along the plane to be periodic. Thus, our planar solution may be interpreted as a static cylinder of fluid with boundary given by $Y = Y_g$. At this surface, the solution must be matched with a vacuum solution.

Actually, all fluid configurations found can be matched with the corresponding static solutions presented in Ref.[31] which arise from applying Birkhoff theorem for external fluid sources that satisfy the hypotheses of the theorem. In those cases there is a matter content present outside, the most common examples being an electromagnetic field, and a cosmological constant. Therefore the matching surface will correspond to a surface where $P(Y)$ matches the pressure of the exterior solution, in a manner similar to the way in which an uncompressible charged sphere is matched to a Reissner-Nordström solution in Ref.[66].

7. Conclusion

In this paper we analysed spacetimes with a two-dimensional maximally symmetric foliation sourced by a perfect fluid. We proved that in those cases, if there is a Killing vector orthogonal to the leaves, its two-expansion vanishes, which allows us to simplify our dynamical equations in terms of the two-expansion of a unit vector orthogonal to the Killing field.

When the Killing vector χ^a is timelike, we find that the flow lines must be tangent to χ^a, and as this is true at all times, the equations describe a hydrostatic equilibrium, governed by a (generalised) TOV equation. When the Killing vector is spacelike, we have instead a spatially homogeneous dynamical spacetime. The result is a subclass of Bianchi universes, with only one shear degree of freedom. The corresponding equation gives the evolution of expansion and shear scalars.

Our approach relates the mass parameter to the geometrically defined quasi-local mass-energies of Misner-Sharp and Hawking-Hayward by slightly changing its definition in order to apply it to our infinite mass-energy distributions. This innovation is in itself a step towards addressing the open issue of defining mass/energy in gravitation and cosmology, c.f the recent works of,[67] and others[68,69] on this subject.

Using these concepts we could recover the physical interpretation of the geometrical quantities appearing in the equilibrium/evolution equations, translate the dual null formalism to the more usual relativistic fluid dynamics framework, and show that the TOV equation arises as a particular case of those equations. Henceforth the generalizations of the TOV equation appear automatically by just setting $\epsilon = 0$ or -1 accordingly.

From this treatment it emerges the fact that the only static fluid solutions that satisfy the WEC are the spherical ones, as the other two cases require a negative energy density.

In what regards the spatially homogeneous spacetimes, the hydrostatic equilibrium condition also implies a violation of the SEC[70, 71] for the non planar solutions, constraining the equation of state for the perfect fluid to be that of a cosmological constant[d].

In order to illustrate the analogy between the planar, hyperbolic and spherical cases we studied the static solutions for an incompressible fluid. We found that, besides the known case of spherically symmetric spacetimes, we can obtain a static interior fluid configuration only in the case of planar symmetric spacetimes. In the hyperbolic case, the static configuration is an exterior solution that can surround an inner vacuum region.

Our unified way to describe three classes of spacetimes foliated with codimension-two leaves of constant curvature leads the way to further generalizations that we will address elsewhere.

Acknowledgements

The work of M.Le D. has been supported by Lanzhou University starting fund and by the Fundamental Research Funds for the Central Universities (Grant No.lzujbky-2019-25). The work of JPM was supported by FCT/MCTES through national funds (PIDDAC) by the grants UID/FIS/04434/2019, UIDB/04434/2020 and UIDP/04434/2020. as well as by CERN/FIS-PAR/0037/2019 and PTDC/FIS-OUT/29048/2017. The authors wish to thank Xu Yumeng for helpful discussions.

References

1. R. C. Tolman, Effect of inhomogeneity on cosmological models, *Proc. Natl. Acad. Sci. U.S.A.* **20**, 169 (March 1934).
2. J. R. Oppenheimer and G. M. Volkoff, On massive neutron cores, *Phys. Rev.* **55**, 374 (February 1939).
3. W. H. Zurek and D. N. Page, Black-hole thermodynamics and singular solutions of the Tolman-Oppenheimer-Volkoff equation, *Phys. Rev.* **D29**, 628 (1984).
4. M. S. R. Delgaty and K. Lake, Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations, *Comput. Phys. Commun.* **115**, 395 (1998).
5. S. Berger, R. Hojman and J. Santamarina, General Exact Solutions of Einstein Equations for Static Perfect Fluids With Spherical Symmetry, *J. Math. Phys.* **28**, p. 2949 (1987).
6. G. Fodor, Generating spherically symmetric static perfect fluid solutions (2000).
7. K. Dev and M. Gleiser, Anisotropic stars: Exact solutions, *Gen. Rel. Grav.* **34**, 1793 (2002).

[d]There are several arguments in the literature suggesting the canonical energy conditions should be abandoned as a criterion of viability.[72]

8. S. Rahman and M. Visser, Space-time geometry of static fluid spheres, *Class. Quant. Grav.* **19**, 935 (2002).
9. V. Gorini, U. Moschella, A. Yu. Kamenshchik, V. Pasquier and A. A. Starobinsky, Tolman-Oppenheimer-Volkoff equations in presence of the Chaplygin gas: stars and wormhole-like solutions, *Phys. Rev.* **D78**, p. 064064 (2008).
10. L. Zou, F.-Y. Li and H. Wen, Exact solutions of the Einstein-Maxwell equations in charged perfect fluid spheres for the generalized Tolman-Oppenheimer-Volkoff equations, *Int. J. Mod. Phys.* **D22**, p. 1350009 (2013).
11. L. Herrera and W. Barreto, General relativistic polytropes for anisotropic matter: The general formalism and applications, *Phys. Rev.* **D88**, p. 084022 (2013).
12. F. Özel and P. Freire, Masses, Radii, and the Equation of State of Neutron Stars, *Ann. Rev. Astron. Astrophys.* **54**, 401 (2016).
13. L. Herrera and A. Di Prisco, Self-similarity in static axially symmetric relativistic fluids, *Int. J. Mod. Phys.* **D27**, p. 1750176 (2017).
14. V. Faraoni, Embedding black holes and other inhomogeneities in the universe in various theories of gravity: a short review, *Universe* **4**, p. 109 (2018).
15. Y. X. Martins, L. F. A. Campos, D. d. S. P. Teixeira and R. J. Biezuner, Existence and Classification of Pseudo-Asymptotic Solutions for Tolman-Oppenheimer-Volkoff Systems, *Annals Phys.* **409**, p. 167929 (2019).
16. J. D. V. Arbañil, J. P. S. Lemos and V. T. Zanchin, Polytropic spheres with electric charge: compact stars, the Oppenheimer-Volkoff and Buchdahl limits, and quasiblack holes, *Phys. Rev.* **D88**, p. 084023 (2013).
17. S. Carloni and D. Vernieri, Covariant Tolman-Oppenheimer-Volkoff equations. I. The isotropic case, *Phys. Rev.* **D97**, p. 124056 (2018).
18. S. Carloni and D. Vernieri, Covariant Tolman-Oppenheimer-Volkoff equations. II. The anisotropic case, *Phys. Rev.* **D97**, p. 124057 (2018).
19. A. A. Isayev, Comment on "Covariant Tolman-Oppenheimer-Volkoff equations. II. The anisotropic case", *Phys. Rev.* **D98**, p. 088503 (2018).
20. P. Papadopoulos and J. A. Font, Relativistic hydrodynamics on space - like and null surfaces: Formalism and computations of spherically symmetric space-times, *Phys. Rev.* **D61**, p. 024015 (2000).
21. K. Lake, All static spherically symmetric perfect-fluid solutions of Einstein's equations, *Phys. Rev. D* **67**, p. 104015 (May 2003).
22. P. Boonserm, M. Visser and S. Weinfurtner, Solution generating theorems for the TOV equation, *Phys. Rev.* **D76**, p. 044024 (2007).
23. G. J. Olmo, H. Sanchis-Alepuz and S. Tripathi, Stellar Structure Equations in Extended Palatini Gravity, *Phys. Rev.* **D86**, p. 104039 (2012).
24. K. Glampedakis, G. Pappas, H. O. Silva and E. Berti, Post-Tolman-Oppenheimer-Volkoff formalism for relativistic stars, *Phys. Rev.* **D92**, p. 024056 (2015).
25. R. Jain, B. G. Sidharth and C. Corda, On the Jeans Theorem and the 'Tolman-Oppenheimer-Volkoff Equation' in R^2 Gravity, *Adv. High Energy Phys.* **2016**, p. 2601741 (2016).
26. H. Velten, A. M. Oliveira and A. Wojnar, A free parametrized TOV: Modified Gravity from Newtonian to Relativistic Stars, *PoS* **MPCS2015**, p. 025 (2016).
27. A. Wojnar and H. Velten, Equilibrium and stability of relativistic stars in extended theories of gravity, *Eur. Phys. J.* **C76**, p. 697 (2016).
28. K. A. Bronnikov, J. C. Fabris, O. F. Piattella and E. C. Santos, Static, spherically symmetric solutions with a scalar field in Rastall gravity, *Gen. Rel. Grav.* **48**, p. 162 (2016).

29. A. V. Astashenok, S. D. Odintsov and A. de la Cruz-Dombriz, The realistic models of relativistic stars in $f(R) = R + \alpha R^2$ gravity, *Class. Quant. Grav.* **34**, p. 205008 (2017).
30. C. Bona, A new proof of the generalized birkhoff theorem, *Journal of Mathematical Physics* **29**, 1440 (1988).
31. A. Maciel, M. Le Delliou and J. P. Mimoso, Revisiting the Birkhoff theorem from a dual null point of view, *Phys. Rev.* **D98**, p. 024016 (2018).
32. S. A. Hayward, Dual - null dynamics, *Ann. Inst. H. Poincare Phys. Theor.* **59**, 399 (1993).
33. S. A. Hayward, Dual-null dynamics of the einstein field, *Classical and Quantum Gravity* **10**, 779 (apr 1993).
34. S. A. Hayward, The general solution to the einstein equations on a null surface, *Classical and Quantum Gravity* **10**, 773 (apr 1993).
35. S. A. Hayward, General laws of black-hole dynamics, *Phys. Rev. D* **49**, 6467 (June 1994).
36. J. M. Senovilla, Trapped surfaces, *Int. J. Mod. Phys. D* **20**, p. 2139 (2011).
37. S. A. Hayward, Quasilocal gravitational energy, *Phys. Rev. D* **49**, 831 (January 1994).
38. A. Maciel, M. Le Delliou and J. P. Mimoso, Dual null formalism for the collapse of fluids in a cosmological background, *Phys. Rev.* **D92**, p. 083525 (2015).
39. A. Maciel, Quasilocal approach to general universal horizons, *Phys. Rev. D* **93**, p. 104013 (2016).
40. A. Maciel, M. Le Delliou and J. P. Mimoso, New perspectives on the TOV equilibrium from a dual null approach, *Classical and Quantum Gravity* **37**, p. 125005 (June 2020).
41. C. Cattoen and M. Visser, Necessary and sufficient conditions for big bangs, bounces, crunches, rips, sudden singularities, and extremality events, *Class.Quant.Grav.* **22**, 4913 (2005).
42. F. S. N. Lobo and J. P. Mimoso, Possibility of hyperbolic tunneling, *Phys. Rev.* **D82**, p. 044034 (2010).
43. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
44. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).
45. S. Carroll, *Spacetime and Geometry* (Addison-Wesley Publishing Company, San Francisco, 2004).
46. A. Maciel, M. Le Delliou and J. P. Mimoso, New perspectives on the TOV equilibrium from a dual null approach, *Class. Quant. Grav.* **37**, p. 125005 (2020).
47. J. P. Mimoso, M. Le Delliou and F. C. Mena, Separating expansion from contraction in spherically symmetric models with a perfect fluid: Generalization of the tolman-oppenheimer-volkoff condition and application to models with a cosmological constant, *Phys. Rev. D* **81**, p. 123514 (June 2010).
48. C. W. Misner and D. H. Sharp, Relativistic equations for adiabatic, spherically symmetric gravitational collapse, *Phys. Rev.* **136**, B571 (October 1964).
49. S. A. Hayward, Gravitational energy in spherical symmetry, *Phys. Rev. D* **53**, 1938 (1996).
50. V. Faraoni, Is the Hawking quasilocal energy "Newtonian"?, *Symmetry* **7**, 2038 (2015).
51. S. A. Hayward, Unified first law of black-hole dynamics and relativistic thermodynamics, *Classical Quantum Gravity* **15**, 3147 (1998).
52. A. Prain, V. Vitagliano, V. Faraoni and M. Lapierre-Léonard, Hawking–Hayward quasi-local energy under conformal transformations, *Class. Quant. Grav.* **33**, p. 145008 (2016).

53. V. Faraoni and J. Coté, Quasilocal mass and multipole expansion in scalar-tensor gravity (2019).
54. R. L. Arnowitt, S. Deser and C. W. Misner, Dynamical structure and definition of energy in general relativity, *Phys. Rev.* **116**, 1322 (1959).
55. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge Monographs on Mathematical Physics, 2nd edn. (Cambridge University Press, Cambridge, England, 2009).
56. P. Martin-Moruno and M. Visser, Classical and semi-classical energy conditions, *Fundam. Theor. Phys.* **189**, 193 (2017).
57. G. F. R. Ellis and H. van Elst, Cosmological models: Cargese lectures 1998, *NATO Sci. Ser. C* **541**, 1 (1999).
58. J. P. Mimoso and P. Crawford, Shear - free anisotropic cosmological models, *Class. Quant. Grav.* **10**, 315 (1993).
59. L. Bianchi, On the three-dimensional spaces which admit a continuous group of motions, *General Relativity and Gravitation* **33**, 2171 (2001).
60. G. F. R. Ellis and M. A. H. MacCallum, A Class of homogeneous cosmological models, *Commun. Math. Phys.* **12**, 108 (1969).
61. R. Kantowski and R. K. Sachs, Some spatially homogeneous anisotropic relativistic cosmological models, *J. Math. Phys.* **7**, p. 443 (1966).
62. E. Abdalla, N. Afshordi, M. Fontanini, D. C. Guariento and E. Papantonopoulos, Cosmological black holes from self-gravitating fields, *Phys. Rev. D* **89**, p. 104018 (May 2014).
63. D. C. Guariento, A. Maciel, M. M. C. Mello and V. T. Zanchin, Charged cosmological black holes: a thorough study of a family of solutions, *Phys. Rev.* **D100**, p. 104050 (2019).
64. F. C. Mena, Cylindrically symmetric models of gravitational collapse to black holes: A short review, *Int. J. Mod. Phys.* **D24**, p. 1542021 (2015).
65. K. Bronnikov, N. O. Santos and A. Wang, Cylindrical Systems in General Relativity (2019).
66. J. D. V. Arbañil, J. P. S. Lemos and V. T. Zanchin, Incompressible relativistic spheres: Electrically charged stars, compactness bounds, and quasiblack hole configurations, *Phys. Rev.* **D89**, p. 104054 (2014).
67. V. Faraoni, Quasilocal energy in modified gravity, *Classical Quantum Gravity* **33**, p. 015007 (2016).
68. H. Barzegar, P. T. Chrusciel and M. Hörzinger, Energy in higher-dimensional spacetimes, *Phys. Rev.* **D96**, p. 124002 (2017).
69. M.-T. Wang and S.-T. Yau, Quasilocal mass in general relativity, *Phys. Rev. Lett.* **102**, p. 021101 (2009).
70. S. Capozziello, F. S. N. Lobo and J. P. Mimoso, Energy conditions in modified gravity, *Phys. Lett.* **B730**, 280 (2014).
71. S. Capozziello, F. S. N. Lobo and J. P. Mimoso, Generalized energy conditions in Extended Theories of Gravity, *Phys. Rev.* **D91**, p. 124019 (2015).
72. C. Barcelo and M. Visser, Twilight for the energy conditions?, *Int. J. Mod. Phys.* **D11**, 1553 (2002).