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On Superization of Nonlinear Integrable Dynamical Systems

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Abstract: We study an interesting superization problem of integrable nonlinear dynamical systems on functional manifolds. As an example, we considered a quantum many-particle Schrödinger–Davydov model on the axis, whose quasi-classical reduction proved to be a completely integrable Hamiltonian system on a smooth functional manifold. We checked that the so-called "naive" approach, based on the superization of the related phase space variables via extending the corresponding Poisson brackets upon the related functional supermanifold, fails to retain the dynamical system super-integrability. Moreover, we demonstrated that there exists a wide class of classical Lax-type integrable nonlinear dynamical systems on axes in relation to which a superization scheme consists in a reasonable superization of the related Lax-type representation by means of passing from the basic algebra of pseudo-differential operators on the axis to the corresponding superalgebra of super-pseudodifferential operators on the superaxis.

Keywords: supersymmetry; super-differentiation; Lie superalgebra; algebra of pseudo-differential operators; coadjoint action; lax integrability; Lie-algebraic approach; gradient-holonomic scheme; casimir invariants; super-Poisson structure

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1. Introduction

Main modern field theoretic string theories of fundamental interactions are essentially based [1–7] on supersymmetric generalizations both of the space–time variables and canonical field variables, making possible to construct governing evolution systems free of singularities and nonphysical peculiarities. As often from the very beginning field equations are considered on the usual classical phase spaces, an important problem of constructing their corresponding supersymmetric extensions [3,8–10] arises and which, during past decades, has been treated by means of various [11–21] mathematical tools and approaches. In particular, within the two-dimensional completely integrable field theories, like Sin–Gordon, Thirring, Nonlinear Schödinger, Born–Infeld, and others, their supersymmetric integrable extensions were constructed by means of natural supersymmetric generalizations either of physically motivated reasonings [1,12,13,22–33] about the system evolution regarding the energy interaction's Hamiltonian structure or the related hidden supersymmetry's Lie-algebraic structure [11,20,29,34–43], which are responsible for their complete integrability. Being interested in more detailed analysis of these superization schemes, we considered a physically motivated [44–48] spatially one-dimensional quantum interacting many-particle model, described by the Hamiltonian operator

$$H_N = -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + 2\sum_{j=\overline{1,N}} n(x_j), \tag{1}$$

of $N \in \mathbb{N}$ charged Bose-particles, specified by the position dependent intensities $n(x_j) \in \mathbb{R}$ at points $x_j \in \mathbb{R}, j = \overline{1,N}$, and acting on the Hilbert space $L_2(\mathbb{R}^N;\mathbb{C})$ of the corresponding quantum states. In case of a medium with an infinite number of particles, the Hamiltonian operator should be naturally considered [44,45,49] within the secondary quantized Fock space representation

$$\hat{H} = \int_{\mathbb{R}} dx[\psi_x^+(x)\psi_x(x) + 2n(x)\psi^+(x)\psi(x)], \tag{2}$$

acting already on the tensor product Fock space $\Phi \otimes \Theta$, generated by a vacuum state $|0) \in \Phi \otimes \Theta$ and, in part, creation–annihilation operators $\psi^+(x), \psi(x) : \Phi \to \Phi$, respectively, satisfying the following canonical operator commutation brackets:

$$\begin{aligned} [\psi(x), \psi^+(y)] &= \delta(x-y), \\ [\psi(x), \psi(y)] &= 0 = [\psi^+(x), \psi^+(y)] \end{aligned} \tag{3}$$

being supplemented with the operator commutation brackets

$$\begin{aligned} [\psi(x), n(y)] &= 0 = [n(x), \psi^+(y)], \\ [n(x), n(y)] &= \partial\delta(x-y)/\partial x \end{aligned} \tag{4}$$

at arbitrary points $x, y \in \mathbb{R}$ for the intensity operator $n(x) : \Theta \to \Theta$, describing the simplest self-interacting quantum medium, whose quantum states are modeled by the related Fock space Θ. The corresponding Heisenberg evolution in time $t \in \mathbb{R}$ equations [44,45,49] for the dynamical operator variables $\psi(x), \psi^+(s)$, and $n(x) : \Phi \otimes \Theta \to \Phi \otimes \Theta$ read as

$$\begin{aligned} \partial\psi/\partial t &= \frac{1}{i}[\hat{H}, \psi] = i\psi_{xx} - 2n\psi, \\ \partial n/\partial t &= \frac{1}{i}[\hat{H}, n] = -2n(\psi\,\psi^+)_x, \\ \partial\psi^+/\partial t &= \frac{1}{i}[\hat{H}, \psi^+] = -i\psi^+_{xx} + 2n\psi^+, \end{aligned} \tag{5}$$

and were before intensively studied in [46,48] as a dynamical model for describing the mechanism of muscle contraction in living tissue. The obtained system of operator Schrödinger–Davydov-type Equation (5) allows the following quasi-classical Hamiltonian form

$$\begin{aligned} \partial\psi/\partial t &= \{H, \psi\}_P = i\psi_{xx} - 2n\psi, \\ \partial n/\partial t &= \{H, n\}_P = -2n(\psi\,\psi^*)_x, \\ \partial\psi^+/\partial t &= \{H, \psi^+\}_P = -i\psi^*_{xx} + 2n\psi^* \end{aligned} \tag{6}$$

endowed with the following quasi-classical Poisson brackets:

$$\begin{aligned} &\{\psi(x), \psi^*(y)\}_P = \delta(x-y), \{\psi(x), n(y)\}_P = 0 = \{n(x), \psi^*(y)\}_P, \\ &\{\psi(x), \psi(y)\}_P = 0 = \{\psi^*(x), \psi^*(y)\}_P\,, \{n(x), n(y)\}_P = \partial\delta(x-y)/\partial x \end{aligned} \tag{7}$$

at any points $x, y \in \mathbb{R}$ on a smooth functional manifold $M \subset \{(\psi, n, \psi^*) \in C^2(\mathbb{R}; \mathbb{C} \times \mathbb{R} \times \mathbb{C})\}$, easily following from (3) and (4) within the classical Dirac's correspondence [49] principle.

As was stated in [47,50,51], the derived hydrodynamic and Boltzmann–Vlasov-type kinetic equations there are related to System (7) and proved to be completely integrable Hamiltonian systems. Moreover, as will be demonstrated below, the above-derived non-linear quasi-classical Schrödinger–Davydov system (6) proves to also be a completely integrable bi-Hamiltonian flow [52] on the functional manifold M and whose possible superization schemes are analyzed in detail in our work below.

2. Quasi-Classical Integrability and a Simple Superization Scheme

Let us begin with analyzing the integrability of the above-derived quasi-classical Schrödinger–Davydov-type nonlinear dynamical system

$$\left.\begin{aligned} \partial\psi/\partial t &= i\psi_{xx} - 2n\psi, \\ \partial n/\partial t &= -2n(\psi\,\psi^*)_x \\ \partial\psi^*/\partial t &= -i\psi^*_{xx} + 2n\psi^* \end{aligned}\right\} := K[\psi, n, \psi^*], \tag{8}$$

with respect to the evolution parameter $t \in \mathbb{R}$, considered as a smooth vector field $K : M \to T(M)$ on the functional manifold M, via making use of the gradient-holonomic scheme, devised in [51,53,54]. As a first step, we need to demonstrate the existence of an infinite hierarchy of conservation laws and to state their commuting to each other with respect the Poisson bracket (7), presented above. Namely, for any smooth functionals $\gamma, \mu \in \mathcal{D}(M)$, their Poisson bracket is calculated via the expression

$$\{\gamma, \mu\}_P = (\mathrm{grad}\gamma | P\mathrm{grad}\mu), \tag{9}$$

where $\mathrm{grad} : \mathcal{D}(M) \to T^*(M)$ denotes the Gateau derivative with respect to the usual bilinear form $(\cdot|\cdot) : T^*(M) \times T(M) \to \mathbb{C}$ and the Poisson operator $P : T^*(M) \to T(M)$ is skew-symmetric, satisfying the following weak functional relationship:

$$\{(\psi(x), n(x), \psi^*(x))^\intercal, (\psi(y), n(y), \psi^*(y))\}_P = P\delta(x - y) \tag{10}$$

for any $x, y \in \mathbb{R}$, $\delta(x - y)$—the classical generalized Dirac delta-function, acting on an arbitrary continuous function $f \in C(\mathbb{R}; \mathbb{C})$ via the symbolic integral operation $f(x) := \int_{\mathbb{R}} \delta(x - y) f(y) dy$, is satisfied for all $x \in \mathbb{R}$. To calculate the infinite hierarchy of conservation laws for the vector field (8), it is enough to study special solutions to the governing linear Noether–Lax equation:

$$\varphi_t + K'^{,*}\varphi = 0, \tag{11}$$

where $K'^{,*} : T^*(M) \to T^*(M)$ denotes the adjoint to the Frechet derivative operator $K' : T(M) \to T(M)$ of the vector field (8) and a covector $\varphi \in T^*(M)$ can be chosen as

$$\varphi = (1, a, b)^\intercal \exp(-i\lambda^2 t + \partial^{-1}\sigma(x; \lambda)), \;\; \partial/\partial x \cdot \partial^{-1} = 1, \tag{12}$$

and the expressions

$$\sigma(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+ \cup \{-2,-1\}} \sigma_j[\psi, n, \psi^*]\lambda^{-j}, a(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} a_j[\psi, n, \psi^*]\lambda^{-j}, \tag{13}$$
$$b(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} b_j[\psi, n, \psi^*]\lambda^{-j},$$

are considered asymptotic with respect to an arbitrary complex parameter $\mathbb{C} \ni \lambda \to \infty$. Taking into account that

$$K'^{,*} = \begin{pmatrix} i\partial^2 - 2in & 2\psi^*\partial & 2in \\ -2i\psi & 0 & 2i\psi^* \\ 0 & 2\psi & 2in - i\partial^2 \end{pmatrix}, \quad (14)$$

one easily obtains a system of recurrent differential–algebraic relationships, giving rise to the following functional expressions:

$$\sigma_{-2} = -i, \sigma_{-1} = 1, \sigma_0 = \frac{1}{2}n, \sigma_1 = \psi^*\psi, \sigma_2 = \frac{1}{2}[n^2 - i(\psi^*\psi_x - \psi\psi_x^*)], \sigma_3 = \psi_x^*\psi_x + 2n\psi^*\psi, \quad (15)$$
$$\sigma_4 = \frac{1}{2}n^2 - 6(\psi^*\psi)^2 - \frac{1}{2}n_x^2 + 6in(\psi^*\psi_x - \psi\psi_x^*) - 2i(\psi^*\psi_{3x} - \psi\psi_{3x}^*), \ldots,$$

and so on. Since, owing to Representation (12), the quantity $\gamma(\lambda) := \int_{\mathbb{R}} dx\sigma(x;\lambda)$ is conservative with respect to the evolution parameter $t \in \mathbb{R}$ for all $\lambda \in \mathbb{C}$, we find that all functionals

$$H_0 = \frac{1}{2}\int_{\mathbb{R}} ndx, H_1 = \int_{\mathbb{R}} \psi^*\psi dx, H_2 = 1/2\int_{\mathbb{R}}[n^2 - i(\psi^*\psi_x - \psi\psi_x^*)]dx, H_3 = \int_{\mathbb{R}}(\psi_x^*\psi_x + 2n\psi^*\psi)dx, \quad (16)$$
$$H_4 = \frac{1}{2}\int_{\mathbb{R}}[n^2 - 12(\psi^*\psi)^2 - n_x^2 + 12in(\psi^*\psi_x - \psi\psi_x^*) - 4i(\psi^*\psi_{3x} - \psi\psi_{3x}^*)]dx, \ldots$$

are also conservative. To confirm now that the vector field (8) on the functional manifold M is Hamiltonian, it is enough within the gradient-holonomic scheme [53] to show that the respectively constructed conservation law

$$H_p = (\xi_p|(\psi_x, n_x, \psi_x^*)^\intercal) \quad (17)$$

for some suitably chosen $p \in \mathbb{N}$ generates the Poisson operator $P : T^*(M) \to T(M)$ for the flow (8) as

$$P = \left(\xi_p'^{,*} - \xi_p'\right)^{-1}. \quad (18)$$

For the case $p = 2$ one obtains that

$$\begin{aligned} H_2 = 1/2\int_{\mathbb{R}}[n^2 - i(\psi^*\psi_x - \psi\psi_x^*)]dx = & ((-i\psi^*, -\partial^{-1}n, i\psi)^\intercal|(\psi_x, n_x, \psi_x^*)^\intercal) = \\ = (\xi_2|(\psi_x, n_x, \psi_x^*)^\intercal), \quad & \xi_2 := (-i\psi^*, -\partial^{-1}n, i\psi)^\intercal, \end{aligned} \quad (19)$$

giving rise to the following Poisson operator:

$$P = \left(\xi_2'^{,*} - \xi_2'\right)^{-1} = \begin{pmatrix} 0 & 0 & i \\ 0 & \partial & 0 \\ -i & 0 & 0 \end{pmatrix}. \quad (20)$$

In a similar way, as above, for the case $p = 4$, one derives the second Poisson operator:

$$Q = \left(\xi_4'^{,*} - \xi_4'\right)^{-1} = \begin{pmatrix} -12\psi\partial^{-1}\psi & 4\partial\psi + 2\psi\partial & 12\psi\partial^{-1}\psi^* - 4i\partial^2 + 8in \\ 4\psi\partial + 2\partial\psi & -\partial^3 + 4n\partial + 4\partial n & 4\psi^*\partial + 2\partial\psi^* \\ -i & 4\partial\psi^* + 2\psi^*\partial & -12\psi^*\partial\psi \end{pmatrix}, \quad (21)$$

where, by definition, $H_4 = (\xi_4|(\psi_x, n_x, \psi_x^*)^\intercal)$. Moreover, one can check that the recurrent relationships

$$Q\mathrm{grad}H_j = 2P\mathrm{grad}H_{j+2} \quad (22)$$

hold for all $j \in \mathbb{Z}_+$, meaning that the Poisson operators (20) and (21) are compatible—that is, the affine sum $\lambda P + Q : T^*(M) \to T(M)$ is also a Poisson operator for all $\lambda \in \mathbb{C}$. The latter makes it possible to state that the infinite hierarchy of conservation laws (16) is such that they are commuting to each other with respect to both Poisson brackets

$$\{H_j, H_k\}_P = 0 = \{H_j, H_k\}_Q \tag{23}$$

for all $j,k \in \mathbb{Z}_+$. Since our dynamical system (8) allows the Hamiltonian representation

$$(\psi_t, n_t, \psi_t^*)^\intercal = \{H_3, (\psi, n, \psi^*)^\intercal)\}_P = -P\mathrm{grad}\, H_1[\psi, n, \psi^*], \tag{24}$$

coinciding with (6), we can formulate our first proposition.

Proposition 1. *The nonlinear Schrödinger–Davydov dynamical system (8) possesses an infinite hierarchy of conservation laws (16) commuting to each other and is an integrable bi-Hamiltonian flow on the functional manifold M.*

Remark 1. *Since the representation $(\psi_t, n_t, \psi_t^*)^\intercal = -Q\mathrm{grad}\, H_1[\psi, n, \psi^*]$ holds, one states that the dynamical system (8) is bi-Hamiltonian with respect to both Poisson structures (20) and (21) on the functional manifold M.*

Recall now the Poisson brackets (7) on the functional manifold M

$$\begin{aligned}\{\psi(x), \psi^*(y)\}_P = i\delta(x-y), \{\psi(x), n(y)\}_P = 0 = \{n(x), \psi^*(y)\}_P, \\ \{\psi(x), \psi(y)\}_P = 0 = \{\psi^*(x), \psi^*(y)\}_P, \{n(x), n(y)\}_P = \partial\delta(x-y)/\partial x\end{aligned} \tag{25}$$

at all points $x, y \in \mathbb{R}$ and observe that they are canonically ultra-local [45,55] except the field variable $n \in M$, depending on the delta-function derivative.

The latter, in particular, means that this field variable cannot be secondly quantized on some suitably chosen Fock space Θ. Nonetheless, this quantization can be performed, if carried out to superize the functional manifold M by means of the following scheme: $(\psi, n, \psi^*) \ni M \to (\tilde{\psi}, \tilde{n}, \tilde{\psi}^*) \simeq (\tilde{\psi}, \tilde{u}, \tilde{\psi}^*) \in \tilde{M} \in C^2(\mathbb{R}^{1|1}; \Lambda_0 \times \Lambda_1 \times \Lambda_0^*)$, where $\mathbb{R}^{1|1} := (x, \theta) \in \mathbb{R} \times \Lambda_1, \Lambda_0 \oplus \Lambda_1 := \Lambda^{(1|1)}$ is the classical one-dimensional Grassmann algebra over the complex field $\mathbb{C}$. To specify in more detail the superanalysis concepts used further, as well as the related superization scheme, we proceed below with some brief superanalysis preliminaries on functional supermanifolds.

3. Superanalysis Preliminaries on Superaxis $\mathbb{R}^{1|1}$

Consider the usual one-dimensional axis $\mathbb{R}^1$ and its supermanifold [8,9] extension $\mathbb{R}^{1|1}$ by means of coordinate variables $(x, \theta) \in \mathbb{R}^{1|1} \simeq \mathbb{R}^1 \times \Lambda_1$, specified by the $\mathbb{Z}_2$-graded Grassmann algebra $\Lambda^{(1|1)} = \Lambda_0 \oplus \Lambda_1$ over the field $\mathbb{C}$ with parities $p|_{\Lambda_s^{(1)}} = s, s \in \{0,1\}$, where $x \in \mathbb{R}^1$, $\theta \in \Lambda_1^{(1)}$ and $\theta^2 = 0$, respectively. An arbitrary smooth uniform function $f \in C^\infty(\mathbb{R}^{1|1}; \Lambda^{(1|1)}) \simeq C^\infty(\mathbb{R}^1; \Lambda^{(1|1)}) \times \Lambda^{(1)}, p(f) \in \{0,1\}$ is at point $(x, \theta) \in \mathbb{R}^1 \times \Lambda_1$, representable as

$$f(x, \theta) = f_0(x) + \theta f_1(x), \tag{26}$$

where the coefficients $f_0, f_1 \in C^\infty(\mathbb{R}^1; \Lambda^{(1|1)}_{k \bmod 2})$ and their parities $p(f_k) = p(f) + k\mathrm{mod}2$. The linear space of functions (26) over the $\mathbb{Z}_2$-graded Grassmann algebra $\Lambda^{(1)}$ generates the $\mathbb{Z}_2$-graded algebra $C^\infty(\mathbb{R}^{1|1}; \Lambda^{(1|1)})$, and the linear subspace of functions (26) with component $f_0(x) = 0, x \in \mathbb{R}^1$, generates its nilpotent ideal $J(\mathbb{R}^{1|1}; \Lambda^{(1)}) \subset C^\infty(\mathbb{R}^{1|1}; \Lambda^{(1|1)})$.

We also remark that the factor space $C^\infty(\mathbb{R}^{1|1};\Lambda^{(1|1)})/J(\mathbb{R}^{1|1};\Lambda^{(1)}) \simeq C^\infty(\mathbb{R}^1;\Lambda^{(1|1)})$ is equivalent to the space of coefficients of the algebra $C^\infty(\mathbb{R}^{1|1};\Lambda^{(1)})$.

Consider now a diffeomorphism of the superaxis $\mathbb{R}^{1|1}$, which is, by definition, a parity preserving the algebra automorphism of $C^\infty(\mathbb{R}^{1|1};\Lambda^{(1|1)})$, inducing a related homomorphism of $C^\infty(\mathbb{R}^1;\Lambda^{(1)})$.

Definition 1. *The corresponding linear space $\mathcal{G}(1,1) := Vect(\mathbb{R}^{1|1};\Lambda^{(1)})$ of vector fields on the superaxis $\mathbb{R}^{1|1}$ is, by definition, the Lie superalgebra of all derivations of the superalgebra $C^\infty(\mathbb{R}^{1|1};\Lambda^{1,1})$: that is, for any uniform vector field $a \in \mathcal{G}(1|1)$ with parity $p(a) \in \{0,1\}$, the condition $a(fg) = a(f)g + (-1)^{p(a)p(f)}fa(g)$ holds for any uniform function $f \in C^\infty(\mathbb{R}^{1|1};\Lambda^{(1)})$, $p(f) \in \{0,1\}$, and $g \in C^\infty(\mathbb{R}^{1|1};\Lambda^{(1)})$.*

The Lie superalgebra $\mathcal{G}(1|1)$-graded commutators of any uniform elements $a,b \in \mathcal{G}(1|1)$ can be recalculated as

$$[a,b](f) = a(b(f)) - (-1)^{p(a)p(b)}b(a(f)), \tag{27}$$

where $f \in C^\infty(\mathbb{R}^{1|1};\Lambda^{(1)})$. The above-constructed Lie superalgebra $\mathcal{G}(1|1)$ satisfies the Leibnitz super-commutator relationships

$$\begin{gathered}[a,b] = -(-1)^{p(a)p(b)}[b,a],\\ [a,[b,c]] = [[a,b],c] + (-1)^{p(a)p(b)}[b,[a,c]]\end{gathered}$$

for arbitrary a,b, and $c \in \mathcal{G}(1|1)$, and is generated by sections $\Gamma(\mathbb{R}^{1|1})$ of the tangent bundle $\left(T(\mathbb{R}^{1|1}),\pi,\mathbb{R}^{1|1}\right)$ over the superaxis $\mathbb{R}^{1|1}$, being equivalent to the free left $C^\infty(\mathbb{R}^{1|1};\Lambda^{(1)})$-module with basis $(\partial/\partial x;\partial/\partial\theta)$ and parities $p(\partial/\partial x) = 0, p(\partial/\partial\theta) = 1$, respectively.

Consider now an infinite-dimensional functional supermanifold$M^{n|1} \subset C^\infty(\mathbb{R}^{1|1};\Lambda^{(1)n})$, $n \in \mathbb{N}$, and a smooth vector field $K : M^{n|1} \to T(M^{n|1})$, where $T(M^{n|1})$ denotes the corresponding tangent space to $M^{n|1}$, and let $du/d\tau \in T(M^{n|1}), \tau \in \Lambda^{(1|1)}$, be the related one-parametric superflow on $M^{n|1}$, satisfying at each point $u \in M^{n|1}$ the relationship

$$du/d\tau = K[u], \tag{28}$$

called a dynamical supersystem on the supermanifold $M^{n|1}$.

Definition 2. *The above-introduced vector superfield $d/d\tau : M^{n|1} \to T(M^{n|1})$ with respect to the evolution superparameter $\tau \in \Lambda^{(1)}$ is called supersymmetric if it allows the following representation: $d/d\tau = \partial/\partial\theta + \theta\partial/\partial t$, where $t \in \mathbb{R}$ is a real evolution parameter.*

As we are interested in describing infinite-dimensional dynamical superflows (28), which are super-integrable Hamiltonian flows on the supermanifold $M^{n|1}, n \in \mathbb{N}$, that is possessing infinite hierarchies of commuting to each other conservation laws with respect to the corresponding Poisson structures on $M^{n|1}$, we need to provide at least sufficient conditions imposed on a superflow under regard allowing to state that it is superintegrable. The latter, in particular, means that this superflow is invariant with respect to some hidden supergroup symmetry, whose coadjoint orbits [53,55,56] prove to be equivalent to the superflow under regard. Within the gradient-holonomic approach, devised in [53,57], the existence of conservation laws to a given dynamical superflow (28) is ensured by solutions to the following Noether-Lax differential-functional equation

$$\partial\varphi/\partial\tau + K'^{,*}[u]\varphi = 0, \tag{29}$$

on the element $\varphi := \operatorname{grad} \gamma \in T^*(M^{n|1})$, where $K'^{,*}[u] : T^*(M^{n|1}) \to T^*(M^{n|1})$ denotes the usual adjoint Frechet derivative of the mapping $K : M^{n|1} \to T(M^{n|1})$ at a point $u \in M^{n|1}$ and $\gamma \in \mathcal{D}(M^{n|1})$ is a smooth functional on the supermanifold $M^{n|1}$ *a priori* invariant with respect to the superflow (28). To describe in more detail the supersymmetry properties of the superflow (28), we will consider the supersymmetric vector field $D_\theta := \partial/\partial\theta + \theta\partial/\partial x$ on the supermanifold $M^{n|1}$, which satisfies the following important for further properties:

$$D_\theta^2 = \partial/\partial x, \quad i_{D_\theta}\alpha^{(1)} = 0 \tag{30}$$

for all $(x,\theta) \in \mathbb{R}^{1|1}$, where $\alpha^{(1)} := dx + \theta d\theta \in T^*(\mathbb{R}^{1|1})$ denotes the canonical contact form [18,29,58,59] on the superaxis $\mathbb{R}^{1|1}$. In particular, the linear space $\mathcal{K}(1|1)$ over $\Lambda^{(1)}$ of all conformal vector fields $K_f : \mathbb{R}^{1|1} \to T(\mathbb{R}^{1|1})$ for $f \in C^\infty(\mathbb{R}^{1|1}; \Lambda^{(1)}) \simeq M_c^{1|1}$, leaving the contact form invariant, compile a conformal Lie superalgebra, whose coadjoint action on the naturally related adjoint space $\mathcal{K}(1|1)^*$ regarding the canonical bilinear form

$$(l|K_f) := \int_{\mathbb{R}^1} dx \int d\theta f(x,\theta) l(x,\theta) \tag{31}$$

for $(l, K_f) \in \mathcal{K}(1|1)^* \times \mathcal{K}(1|1)$, generates [29,53,55,59] the corresponding dynamical super-systems on the supermanifold $M_c^{1|1}$. Here, the super-integration $\int dx \int d\theta(\cdot)$ "measure" is defined [8] on the superaxis $\mathbb{R}^{1|1}$ via the following rules:

$$\int \theta d\theta = 1, \int d\theta = 0. \tag{32}$$

Moreover, taking into account that the adjoint space $\mathcal{K}(1|1)^*$ is Poissonian, carrying the canonical Lie-Poisson structure [55], this algebraic scheme makes it possible to construct on the supermanifold $M_c^{1|1}$ a special subset of so called super-Hamiltonian vector fields, which prove to be completely integrable within the classical Liouville-Arnold integrability type definition.

Returning now to our Hamiltonian system (24) on the functional supermanifold $\tilde{M} \subset C^\infty(\mathbb{R}^{1|1}; \times\Lambda_1^{(1)} \times \Lambda_0^{(1)*})$, let us analyze the two most natural ways of its superization:

The first one consists in constructing on the supermanifold $\tilde{M}$ a superflow

$$\tilde{K} : \tilde{M} \to T(\tilde{M}), \tag{33}$$

related with a suitably chosen super-connection [11,14,15,41] on the two-dimensional space-time $\mathbb{R}^2$ with the vanishing curvature, and whose structure supergroup leaves invariant the corresponding Casimir invariants and generates, respectively, an infinite hierarchy of conservation laws, reducing to *a priori* given quasi-classical conservation laws of the dynamical system (8) under regard. In particular, the supermanifold $\tilde{M}$ is equipped with the corresponding super-variables $(\tilde{\psi}, \tilde{u}, \tilde{\psi}^*) \in \tilde{M}$, which admit the following superalgebraic expansions:

$$\begin{aligned} &\tilde{\psi}(x,\theta) = \psi_0(x) + \theta\psi_1(x) \in \Lambda_0, \tilde{\psi}^*(x,\theta) = \psi_0^*(x) + \theta\psi_1^*(x) \in \Lambda_0^*, \\ &\tilde{u}(x,\theta) = u_1(x) + \theta u_0(x) \in \Lambda_0, \end{aligned} \tag{34}$$

entering in the corresponding Hamiltonian function

$$\tilde{H} = \int_{\mathbb{R}} dx \int d\theta [\tilde{\psi}^*_{\theta\theta}(x,\theta)\tilde{\psi}_{\theta\theta}(x,\theta) + 2\tilde{n}(x,\theta)\tilde{\psi}^*(x,\theta)\tilde{\psi}(x,\theta)], \tag{35}$$

This superization scheme is mainly based on the super-analysis techniques, first described in [7–9] and developed later, regarding integrable super-symmetric dynamical systems, in [4,11,14–18,20,28,59]. In particular, in [11], as well as in [14], there were proposed superization schemes, generalizing the classical associated zero curvature condition for the components of the corresponding super connection on the two-dimensional space-time $\mathbb{R}^2$, rigged with the Lie supergroup $SL(2|1)$, that allowed them to construct integrable supersymmetric Korteweg-deVries and Nonlinear Schrödinger type dynamical systems on supermanifolds.

The second one consists in the natural superization of a given quasi-classical Poisson structure to that on the corresponding functional supermanifold $\tilde{M}$ and was originally proposed in [1,13,60], which regarding to our quasi-classical Poisson brackets (25), leads to introducing the superfield $\tilde{n} := \tilde{u}_\theta = D_\theta \tilde{u}$ and next transforming the quasi-classical Poisson brackets (25) into the following ultra-canonical super-Poisson brackets

$$\{\tilde{\psi}(x,\theta),\tilde{\psi}^*(y,\eta)\}_{\tilde{P}} = i\delta(x,\theta|y,\eta), \{\tilde{\psi}(x,\theta),\tilde{u}(y,\eta)\}_{\tilde{P}} = 0 = \{\tilde{u}(x,\theta),\tilde{\psi}^*(y,\eta)\}_{\tilde{P}}, \\ \{\tilde{\psi}(x,\theta),\tilde{\psi}(y,\eta)\}_{\tilde{P}} = 0 = \{\tilde{\psi}^*(x,\theta),\tilde{\psi}^*(y,\eta)\}_{\tilde{P}}, \{\tilde{u}(x,\theta),\tilde{u}(y,\eta)\}_{\tilde{P}} = \delta(x,\theta|y,\eta) \tag{36}$$

at all super-points $(x,\theta),(y,\eta) \in \mathbb{R}^{1|1}$, where

$$\delta(x,\theta|y,\eta) = \delta(x-y-\theta\eta)(\theta-\eta) \tag{37}$$

denotes the supersymmetric Dirac delta-function, satisfying for any continuous super-function $\tilde{f} \in C^0(\mathbb{R}^{1|1};\Lambda)$ the determining relationship

$$\tilde{f}(x,\theta) := \int_{\mathbb{R}} dy \int d\eta \delta(x,\theta|y,\eta)\tilde{f}(y,\eta) \tag{38}$$

for all $(x,\theta) \in \mathbb{R}^{1|1}$ jointly with the mentioned above (32) Berezin integrals [8–10], assumed to be fulfilled, and which is appropriately applied to the super-generalized quasi-classical Hamiltonian function (35). Since the super-Poisson operator $\tilde{P} : T^*(\tilde{M}) \to T^*(M)$, corresponding to the Poisson superbrackets (36), acts on the cotangent superspace $T^*(\tilde{M})$, adjoint to the supersymmetric tangent space $T(\tilde{M})$, the latter can be endowed with the following super-bilinear form $(\cdot|\cdot) : T^*(\tilde{M}) \times T(\tilde{M}) \to \Lambda_0$, where for any $\tilde{f} \in T^*(\tilde{M}), \tilde{g} \in T(\tilde{M})$:

$$(\tilde{f}|\tilde{g}) := \int_{\mathbb{R}} dx \int d\theta \langle \tilde{f}(x,\theta)|\tilde{g}(x,\theta)\rangle_{\mathbb{E}^3}. \tag{39}$$

Having now applied the super-Poisson operator $\tilde{P} : T^*(\tilde{M}) \to T(\tilde{M})$ brackets (36) to the superized Hamiltonian function $H \in \mathcal{D}(M)$ in the form

$$\tilde{H} = \int_{\mathbb{R}} dx \int d\theta [\tilde{\psi}^*_{\theta\theta}(x,\theta)\tilde{\psi}_{\theta\theta}(x,\theta) + 2\tilde{u}_\theta(x,\theta)\tilde{\psi}^*(x,\theta)\tilde{\psi}(x,\theta)], \tag{40}$$

one derives the super-Hamiltonian system

$$\tilde{\psi}_t = \{\tilde{H},\tilde{\psi}\}_{\tilde{P}} = i\tilde{\psi}_{4\theta} - 2\tilde{u}_\theta\tilde{\psi}, \tilde{u}_t = -2(\tilde{\psi}^*\tilde{\psi})_\theta, \\ \tilde{\psi}^*_t = \{\tilde{H},\tilde{\psi}^*\}_{\tilde{P}} = -i\tilde{\psi}^*_{4\theta} + 2\tilde{u}_\theta\tilde{\psi}^*, \tag{41}$$

with respect to the real temporal parameter $t \in \mathbb{R}$, in relation to which one poses the following natural question:

Problem 1. *Does it inherit the classical integrability property of the Schrödinger–Davydov dynamical system (8) as considered on the functional supermanifold $\tilde{M}$? This will be analyzed in the section to follow.*

The question posed above is deeply motivated by the fact that the obtained superized Schrödinger–Davydov dynamical system (41) both reduces to its classical version (8) on the functional supermanifold M and simultaneously describes the related evolution of two anticommuting fermionic fields, nontrivially interacting with basic bosonic fields, defined by means of the expansions (34). The latter can be in part interpreted as the existence of an additional yet hidden inter-particle vacuum-based interaction, not taken into account a priori within the quantum Hamiltonian operator (2).

Remark 2. *The superization scheme that we applied to the quasi-classical Schrödinger–Davydov dynamical system (8) is strictly based on the superization of the classical field supermanifold M via the superization of the related non-ultralocal Poisson structure (25). The corresponding Hamiltonian equations (41), determining the evolution on the constructed functional supermanifold $\tilde{M}$, are those which presumably can describe additional hidden interparticle vacuum-based interaction of the quasi-classical fields under consideration, which are interesting for applications both in solid-state physics and modeling the muscle contraction mechanism in living tissue, as mentioned before.*

4. Superintegrability Analysis

Since we are interested in studying the superintegrability of the infinite-dimensional dynamical superflow (41) as a Hamiltonian flow on the supermanifold $\tilde{M}$, which possesses an infinite hierarchy of conservation laws commuting to each other with respect to the corresponding Poisson structures on $\tilde{M}$, we need to provide at least sufficient conditions imposed on a given superflow allowing us to state that it really passes them. The latter, in particular, means that this superflow is invariant with respect to some hidden supergroup symmetry, whose structure can be revealed within the effective gradient-holonomic approach, devised before in [51,53,54,57] and applied to diverse nonlinear dynamical systems. Concerning the super-integrability problem regarding the super-Hamiltonian system (41), we will rewrite it as the vector superfield

$$\left.\begin{aligned} \partial\tilde{\psi}/\partial t &= i\tilde{\psi}_{4\theta} - 2\tilde{u}_\theta\tilde{\psi}, \\ \partial\tilde{u}/\partial t &= -2(\tilde{\psi}^*\tilde{\psi})_\theta \\ \partial\tilde{\psi}^*/\partial t &= -i\tilde{\psi}^*_{4\theta} + 2\tilde{u}_\theta\tilde{\psi}^* \end{aligned}\right\} := \tilde{K}[\tilde{\psi},\tilde{u},\tilde{\psi}^*], \tag{42}$$

on the superized functional supermanifold $\tilde{M}$, and look within the gradient-holonomic approach [53,57] for special solutions to the corresponding Noether–Lax equation

$$\tilde{\varphi}_t + \tilde{K}'^{,*}\tilde{\varphi} = 0 \tag{43}$$

in the following asymptotic ax $\mathbb{C} \ni \lambda \to \infty$ form:

$$\tilde{\varphi} = (1,\tilde{a},\tilde{b})^\intercal \exp[-i\lambda^2 t + D_\theta^{-1}\tilde{\sigma}(x,\theta)], \tag{44}$$

where

$$\tilde{\sigma}(x,\theta;\lambda) \sim \sum_{j\in\mathbb{Z}_+\cup\{-2,-1\}} \tilde{\sigma}_j[\tilde{\psi},\tilde{u},\tilde{\psi}^*]\lambda^{-j}, \tilde{a}(x,\theta;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{a}_j[\tilde{\psi},\tilde{u},\tilde{\psi}^*]\lambda^{-j}, \tag{45}$$
$$\tilde{b}(x,\theta;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{b}_j[\tilde{\psi},\tilde{u},\tilde{\psi}^*]\lambda^{-j},$$

at arbitrary point $(x,\theta) \in \mathbb{R}^{1|1}$. Taking into account that the adjoint operator $\tilde{K}^{',*} : T^*(\tilde{M}) \to T^*(\tilde{M})$ is given by the expression

$$\tilde{K}^{',*} = \begin{pmatrix} iD_\theta^4 - 2i\tilde{u}_\theta & -2\tilde{\psi}^* D_\theta & 0 \\ 2iD_\theta\tilde{\psi} & 0 & -2iD_\theta\tilde{\psi}^* \\ 0 & -2\tilde{\psi}D_\theta - & 2i\tilde{u}_\theta - iD_\theta^4 \end{pmatrix}, \tag{46}$$

one obtains easily the following infinite recurrent system:

$$\begin{gathered} -i\delta_{j,-2} + \ D_\theta^{-1}\tilde{\sigma}_{j-k,\theta}\tilde{\sigma}_{k,\theta} + i\tilde{\sigma}_{j,x\theta} - 2i\tilde{u}_\theta\delta_{j,0} - 2\tilde{\psi}^*\tilde{a}_{j,\theta} + 2\tilde{\psi}\tilde{a}_{j-k}\tilde{a}_k = 0, \\ \tilde{a}_{j,\theta} - i\tilde{a}_{j+2} + \tilde{a}_{j-k,\theta}D_\theta^{-1}\tilde{\sigma}_{k,t} + 2i\tilde{\psi}_\theta\delta_{j,0} + 2i\tilde{\psi}\tilde{\sigma}_j - 2i\tilde{\psi}_\theta^*\tilde{b}_j - 2i\tilde{\psi}^*\tilde{b}_{j,\theta} - 2i\tilde{\psi}^*\tilde{b}_{j-k}\ \tilde{\sigma}_k = 0, \\ \tilde{b}_{j,\theta} - i\tilde{b}_{j+2} + \tilde{b}_{j-k,\theta}D_\theta^{-1}\tilde{\sigma}_{k,t} - 2\tilde{\psi}\tilde{a}_{j,\theta} + 2\tilde{\psi}\tilde{a}_{j-k,}\tilde{\sigma}_k + 2i\tilde{u}_\theta\tilde{b}_j - \\ -i(\tilde{b}_{j,xx} + 2\tilde{b}_{j-k}\tilde{\sigma}_{k,\theta} + \tilde{b}_{j-k}\tilde{\sigma}_{k,\theta x} + \tilde{b}_{j-k}\tilde{\sigma}_{k-s,\theta}\tilde{\sigma}_{s,\theta}) = 0 \end{gathered} \tag{47}$$

for all $j \in \mathbb{Z}_+ \cup \{-2,-1\}$. Trying to dissolve recurrently the above system (47), we obtain that first its coefficients are equal to

$$\begin{gathered} \tilde{\sigma}_{-1} = \theta, \tilde{\sigma}_0 = 0, \tilde{\sigma}_1 = \tilde{u}, \tilde{a}_0 = 0, \tilde{a}_1 = 2\tilde{\psi}\theta, \\ \tilde{a}_2 = 2\tilde{\psi}_\theta, \tilde{b}_0 = 0, \tilde{b}_1 = 0, b_2 = 0, \end{gathered} \tag{48}$$

but the second coefficient $\tilde{\sigma}_2$ satisfies the locally unsolvable differential–algebraic relationship

$$\tilde{\sigma}_{2,\theta} = -\frac{1}{2}\tilde{u}_{x\theta} + 3\tilde{\psi}^*\tilde{\psi}, \tag{49}$$

saying us that the recurrent system (47) fails to be infinitely continued. As an inference from this failure, we need to state that our naively constructed super-Hamiltonian system (41) does not possess an infinite hierarchy of conservation laws and is suitably not super-integrable on the superized functional supermanifold $\tilde{M}$. This negative result is also instructive, per se, informing us that a simple naive a priori superization of a classical integrable nonlinear dynamical system generally loses its integrability, or, in other words, "*Der Irrtum ist eine ebenso wichtige Lebensbedingung wie die Wahrheit*", i.e., "*Error is as important a condition for the progress of life as truth*" (by C.G. Jung: [61]).

In order to construct a more feasible and in some sense natural superization of the nonlinear dynamical Schrödinger-Davydov system (8), we first proceed to present its classical Lax-type operator representation, and then its suitably superized generalization, which will generate a priori integrable super-Hamiltonian flows, which we are interested in finding.

5. The Lax-Type Representation Scheme

We will start from the infinite hierarchy of gradient relationships (22) and observe that it can be rewritten as

$$Q\mathrm{grad}\,\gamma(\lambda) = 2\lambda^2 P\mathrm{grad}\,\gamma(\lambda), \tag{50}$$

where, by definition,

$$\gamma(\lambda) := \int_{\mathbb{R}} dx\sigma(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \lambda^{-j} \int_{\mathbb{R}} H_j[\psi, n, \psi^*]dx \tag{51}$$

is a generating as $\mathbb{C} \ni \lambda \to \infty$ function of conservation laws for the dynamical system (8), which can be identified [51,53,55] with the trace-functional of the monodromy matrix $S(x;\lambda) \in \text{End}\, \mathbb{E}^m, x \in \mathbb{R}$, naturally assigned to a matrix Lax-type "spectral" problem:

$$\partial f/\partial x = l[\psi, n, \psi^*; \lambda] f, \tag{52}$$

where $l[\psi, n, \psi^*; \lambda] \in \text{End}\, \mathbb{E}^m$ for some finite $m \in \mathbb{N}$ is considered—for brevity, 2π-periodic in $x \in \mathbb{R}$ and $f \in L_\infty(\mathbb{R}; \mathbb{E}^m)$. Namely, given

$$\gamma(\lambda) := \text{tr}\, S(x;\lambda), \tag{53}$$

where by definition, $S(x;\lambda) := F(x + 2\pi\pi, x; \lambda)$ and $F(x,y;\lambda) \in \text{End}\, \mathbb{E}^m$, $F(x,x;\lambda) = I$, $x \in \mathbb{R}$, denotes the fundamental matrix for the linear problem (52), depending on a point $(\psi, n, \psi^*) \in M$. Taking into account that the gradient element $\varphi(x;\lambda) := \text{grad}\gamma(\lambda) \in T^*(M)$ for all $(x;\lambda) \in \mathbb{R} \times \mathbb{C}$ satisfies the gradient relationship (50) and can be simultaneously represented as

$$\varphi(x;\lambda) = \text{tr}(l[\psi, n, \psi^*; \lambda]^{\prime,*} S(x;\lambda))), \tag{54}$$

where the monodromy matrix $S(x;\lambda) \in \text{End}\, \mathbb{E}^m$ solves [62] on the axis $\mathbb{R}$ the linear Novikov equation

$$\partial S/\partial x = [l, S], \tag{55}$$

one can suitably construct within the gradient-holonomic scheme [51,53] a finite set of differential algebraic matrix relationships on a searched mapping $l[\psi, n, \psi^*; \lambda] \in \text{End}\, \mathbb{E}^m$, tr $l[\psi, n, \psi^*; \lambda] = 0$, whose solution gives rise via simple enough but cumbersome calculations to the following result: $m = \dim l[\psi, n, \psi^*; \lambda] = 3$ and

$$l[\psi, n, \psi^*; \lambda] = \begin{pmatrix} 2i\lambda - \frac{in}{2\lambda} & \psi^* & -\frac{in}{2\lambda} \\ \frac{\psi}{2\lambda} & 0 & \frac{\psi}{2\lambda} \\ \frac{in}{2\lambda} & \psi^* & \frac{in}{2\lambda} - 2i\lambda \end{pmatrix}. \tag{56}$$

It is now easy to observe that the linear Lax-type spectral problem (52) reduces to the following pseudo-differential form:

$$-\partial^2 f/\partial x^2 + 2nf - 2i\psi^* \partial^{-1} \psi f = 4\lambda^2 f, \tag{57}$$

where $f \in W \subset L_\infty(\mathbb{R}; \mathbb{C})$ is a scalar function and $\lambda \in \mathbb{C}$ serves as a true spectral parameter.

Remark 3. *If one denotes the pseudo-differential expression from (57) as*

$$L := -\partial^2/\partial x^2 + 2n - 2i\psi^* \partial^{-1} \psi, \tag{58}$$

then one can construct [53,54,56,63] the same infinite hierarchy of conservation laws as (16) by means of the operator traces

$$H_j = \text{Tr}\left(L^{1/2} L^j\right), \tag{59}$$

where $L, L^{j/2} \in \Psi OP, j \in \mathbb{Z}_+$, *and* Tr $:\Psi OP \to \mathbb{C}$ *is the trace operation on the algebra* ΨOP *of pseudo-differential operators on the axis, coinciding with the integral over* $\mathbb{R}$ *of the functional coefficient at the inverse differentiation* ∂^{-1}.

The spectral problem (57) looks very interesting and represents [4,64,65] the Backlund-type operator transformation

$$DOP \ni L_0 \to L_0 + \alpha\psi^*\partial^{-1}\psi \in \Psi OP \tag{60}$$

from the algebra DOP of differential operators to that of pseudo-differential operators, ΨOP, where, by definition, ψ and $\psi^* \in W$ serve, respectively, as the eigenfunctions of the spectral problem

$$L_0\psi = \mu\psi \tag{61}$$

for some $\mu \in \mathbb{C}$ and its adjoint:

$$L_0^*\psi^* = \nu^*\ \psi^* \tag{62}$$

for some $\nu^* \in \mathbb{C}$.

Remark 4. *More details of this Backlund-type operator transformation (60) can be found in [64]. We should also mention here that the compatible pair of Poisson operators (20) (21) we found follows from the canonical Poisson bracket on the space $\Psi OP \times W \times W^*$ via the operator mapping (60).*

Since the above-obtained pseudo-differential operator (60) is a shifted classical Sturm–Liouville operator on the axis $\mathbb{R}$ of the second order, whose natural superization was first studied in [59], we can logically proceed to generalizing this result on the subject of the corresponding superization of the completely integrable Schrödinger–Davydov dynamical system under consideration.

6. Spectral Operator Problem and Related Superization Scheme

Let us consider the classical Sturm–Liouville operator expression

$$L_0 := -\partial^2/\partial x^2 + 2n(x) \tag{63}$$

on the real axis with a real potential $n(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ on the functional space W and its super-differential analog

$$\tilde{L}_0 := -D_\theta^3 + 2\tilde{n}(x,\theta) \tag{64}$$

on the super-axis, constructed in [59], where $\tilde{n}(x,\theta) \in \Lambda_1^{(1)}$ for all $(x,\theta) \in \mathbb{R}^{1|1}$. The super-differential spectral problem related to (64)

$$\tilde{L}_0\tilde{\psi} = (-D_\theta^3 + 2\tilde{n})\tilde{\psi} = \mu\tilde{\psi}, \tag{65}$$

where $\mu \in \Lambda_1$ and $\tilde{\psi} \in \tilde{W} \subset L_\infty(\mathbb{R}^{1|1};\Lambda_0)$, and its adjoint problem

$$\tilde{L}_0^*\tilde{\psi}^* = (D_\theta^3 + 2\tilde{n})\tilde{\psi} = \nu^*\tilde{\psi}, \tag{66}$$

where $\nu^* \in \Lambda_1^*$ and $\tilde{\psi}^* \in \tilde{W}^* \subset L_\infty(\mathbb{R}^{1|1};\Lambda_0^*)$ make it possible to superize the super-differential operator (64) as

$$\tilde{L}_0 \to \tilde{L} := -D_\theta^3 + 2\tilde{n} -\ 2i\tilde{\psi}^* D_\theta^{-1}\tilde{\psi} \tag{67}$$

by shifting on the Backlund transformed term $-2i\tilde{\psi}^* D_\theta^{-1}\tilde{\psi} \in s\Psi OP$ from the algebra $s\Psi OP$ of super-pseudo-differential operators. Based on the super pseudo-differential expression (67), one can calculate [20,26,27,31,32,36–39] the corresponding conserved super-laws as the following Casimir invariant functionals:

$$\tilde{H}_j = \mathrm{sTr}\left(\tilde{L}^{2j/3}\right), \tag{68}$$

for $j \in \mathbb{Z}_+$ regarding the related Lie superalgebra Lie$(s\Psi OP)$, where the super-trace operation sTr : $s\Psi OP \rightarrow \Lambda$ is defined as the super-integral over the super-axis $\mathbb{R}^{1|1}$ of the coefficient at the inverse super-differentiation D_θ^{-1}. In particular, taking into account that

$$\tilde{L}^{1/3} \sim -D_\theta + \tilde{w}_0^1 + (\tilde{w}_1^1 + \tilde{w}_1^0 D_\theta)\partial^{-1} + (\tilde{w}_2^1 + \tilde{w}_2^0 D_\theta)\partial^{-2} + \\ + (\tilde{w}_3^1 + \tilde{w}_3^0 D_\theta)\partial^{-3} + (\tilde{w}_4^1 + \tilde{w}_4^0 D_\theta)\partial^{-4}) + \ldots, \tag{69}$$

with the coefficients

$$\tilde{w}_1^1 = 2u, \tilde{w}_2^1 = 2/3(i\overline{\Psi}_\theta\Psi - 2i\overline{\Psi}\Psi_\theta - 2u_x), \tilde{w}_3^0 = 2/3(2i\overline{\Psi}\Psi_x + u_{x,\theta} - i\overline{\Psi}_\theta\Psi_\theta), \\ \tilde{w}_2^0 = -2/3(i\overline{\Psi}\Psi + u_\theta), \\ \tilde{w}_3^1 = 2/3(-i\overline{\Psi}_{x,\theta}\Psi - 2i\overline{\Psi}_\theta\Psi_x + 3i\overline{\Psi}\Psi_{x,\theta} + i\overline{\Psi}_x\Psi_\theta + u_{xx} - 4iu\overline{\Psi}\Psi + 2uu_\theta), \ldots \tag{70}$$

and so on, we can easily calculate the super-conservation laws:

$$\begin{aligned} &\tilde{H}_{1/2} = 0, \quad \tilde{H}_1 = 0, \quad \tilde{H}_{3/2} = -2i\int dx \int d\theta\overline{\Psi}\Psi, \\ &\tilde{H}_2 = 4/3\int dx \int d\theta[i(\overline{\Psi}\Psi)_\theta + u_x], \\ &\tilde{H}_{5/2} = 2/3i\int dx \int d\theta(2\overline{\Psi}\Psi_x - 2\overline{\Psi}_x\Psi - \overline{\Psi}_\theta\Psi_\theta), \\ &\tilde{H}_3 = 2i\int dx \int d\theta(\overline{\Psi}_{x,\theta}\Psi - \overline{\Psi}\Psi_{x,\theta}), \ldots, \\ &\tilde{H}_{9/2} = \int dx \int d\theta(6i\Psi_{xxx}\overline{\Psi} + 8i\Psi_{xx}\overline{\Psi}_x - 8\Psi_x\Psi\overline{\Psi}^2 + 8\Psi^2\overline{\Psi}_x\overline{\Psi} - \\ &\quad - 2i\Psi\overline{\Psi}_{xxx} - 4iu_\theta^{(1)}\Psi_x\overline{\Psi} + 4iu_\theta^{(1)}\Psi\overline{\Psi}_x - 2i\Psi_{\theta,xx}\overline{\Psi}_\theta - \\ &\quad - 2i\Psi_{\theta,x}\overline{\Psi}_{\theta,x} - 4\Psi_\theta\overline{\Psi}_\theta\Psi\overline{\Psi} - 4iu_x^{(1)}\overline{\Psi}_\theta\Psi + 4iu_x^{(1)}\Psi_\theta\overline{\Psi} + \\ &\quad + +4iu^{(1)}\overline{\Psi}_{\theta,x}\Psi - 4iu^{(1)}\Psi_{\theta,x}\overline{\Psi}) \\ &\tilde{H}_5 = \int dx \int d\theta(\tilde{\psi}_\theta^*\tilde{\psi}_{\theta\theta} - \tilde{\psi}_{\theta\theta}^*\tilde{\psi}_\theta + 2\tilde{\psi}^*\tilde{\psi}\tilde{n} - \tilde{\psi}_\theta^*\tilde{\psi}^*\ \tilde{\psi}_\theta\tilde{\psi}), \ldots \end{aligned} \tag{71}$$

and so on, which are invariant with respect to the super-evolution flow on $\tilde{M}$, which is equivalently represented [53,55,56] via the Lax-type dynamical super-operator flow

$$\partial\tilde{L}/\partial\tau = [\tilde{L}, \left(\tilde{L}^{10/3}\right)_+] \tag{72}$$

with respect to a super-temporal odd evolution parameter $\tau \in \Lambda_1$, where the sign "+" denotes the strictly nonnegative super-differential part of an expression in the brackets.

Remark 5. *Here, one must mention that the flow (72) is naturally interpreted [53–56,63] from the Lie-algebraic point of view as the coadjoint action of the operator Lie susperalgebra element* $\left(\tilde{L}^{10/3}\right)_+ \in$ Lie$(s\Psi OP_+)$ *on the element* $\tilde{L} \in$ Lie$(s\Psi OP)^*$*, where* Lie$(s\Psi OP_+)$ *denotes the nonnegative part of the natural direct sum splitting* Lie$(s\Psi OP) =$ Lie$(s\Psi OP_+) \oplus$ Lie$(s\Psi OP_-)$.

Having recalculated the flow (72) regarding the superized variables $(\tilde{\psi}, \tilde{n}, \tilde{\psi}^*) \in \tilde{M}$, one obtains the following Schrödinger–Davydov evolution flow:

$$\left.\begin{aligned} \partial\tilde{\psi}/\partial\tau &= i\tilde{\psi}_{\theta\theta\theta} - 2i\tilde{n}\tilde{\psi}^*\tilde{\psi} + i\tilde{\psi}^*\tilde{\psi}\tilde{\psi}_\theta \\ \partial\tilde{n}/\partial\tau &= -2\tilde{\psi}^*\tilde{\psi}\tilde{w}_0^1 - 2(\tilde{\psi}_\theta^*\tilde{\psi} - \tilde{\psi}^*\tilde{\psi}_\theta)\tilde{n}_\theta \\ \partial\tilde{\psi}^*/\partial\tau &= -i\tilde{\psi}_{\theta\theta\theta}^* + 2i\tilde{n}\tilde{\psi}^*\tilde{\psi} - i\tilde{\psi}^*\tilde{\psi}\tilde{\psi}_\theta \end{aligned}\right\} := \tilde{K}[\tilde{\psi}, \tilde{n}, \tilde{\psi}^*], \tag{73}$$

which is a super-Hamiltonian system with respect to the super-Poisson structure

$$\begin{aligned} &\{\tilde{\psi}(x,\theta),\tilde{\psi}^*(y,\eta)\}_{\tilde{P}} = i\delta(x,\theta|y,\eta), \{\tilde{\psi}(x,\theta),\tilde{n}(y,\eta)\}_{\tilde{P}} = \tilde{\psi}^*(x,\theta)\delta(x,\theta|y,\eta), \\ &\{\tilde{n}(x,\theta),\tilde{\psi}^*(y,\eta)\}_{\tilde{P}} = \tilde{\psi}(x,\theta)\delta(x,\theta|y,\eta), \{\tilde{n}(x,\theta),\tilde{n}(y,\eta)\}_{\tilde{P}} = D_\theta\delta(x,\theta|y,\eta), \\ &\{\tilde{\psi}(x,\theta),\tilde{\psi}(y,\eta)\}_{\tilde{P}} = 0 = \{\tilde{\psi}^*(x,\theta),\tilde{\psi}^*(y,\eta)\}_{\tilde{P}}, \{\tilde{u}(x,\theta),\tilde{u}(y,\eta)\}_{\tilde{P}} = \delta(x,\theta|y,\eta) \end{aligned} \tag{74}$$

on the functional supermanifold $\tilde{M}$—that is, $(\tilde{\psi}_\tau, \tilde{n}_\tau, \tilde{\psi}^*_\tau)^\intercal = \{\tilde{H}_{10}, (\tilde{\psi}, \tilde{n}, \tilde{\psi}^*)^\intercal\}_{\tilde{P}}$—coinciding with that of (73), where the evolution parameter $\tau \in \Lambda_1$, as mentioned above, is considered odd. The derived superflow (73) presents a searched superization regarding the quasi-classical integrable Schrödinger–Davydov dynamical system defined on the functional field manifold M. It is worth noting here that in some cases, one can anticipate that the corresponding super-evolution vector field $\tilde{K} : \tilde{M} \to T(\tilde{M})$ on the functional superfield manifold $\tilde{M}$ is represented as the supersymmetric super-differentiation $d/d\tau = \partial/\partial\theta + \theta\partial/\partial t$ with respect to the super-variable $\theta \in \Lambda_1$ and the real evolution parameter $t \in \mathbb{R}$. The latter suitably makes it possible to construct real-time evolution equations with respect to this temporal parameter $t \in \mathbb{R}$, thus representing the corresponding quasi-classical Schrödinger–Davydov dynamical system on the functional superfield manifold $\tilde{M}$, possessing interesting physical properties from application point of view. This dynamical supersystem provides an extraordinary example of a nonlinear integrable dynamical superflow with respect to a real evolution parameter, subject to which the Hamiltonian representation of the reduced vector field $d/dt : \tilde{M} \to T(\tilde{M})$ on the functional superfield manifold $\tilde{M}$ is still not clear and can be investigated in the future.

7. Conclusions

We have studied two interesting examples of the superization scheme regarding the classical Schrödinger–Davydov integrable nonlinear dynamical system on a functional manifold. In particular, we checked that the so-called "naive" approach, based on the superization of the phase space variables and extending the corresponding Poisson brackets upon the related functional supermanifold, fails to retain the dynamical system's super-integrability. Nonetheless, for a wide class of classical Lax-type integrable nonlinear dynamical systems on functional manifolds, a possible superization scheme consists in a reasonable superization of the related Lax-type representation by means of a transition from the basic algebra of pseudo-differential operators on the real axis to the corresponding superalgebra of super-pseudo-differential operators on the superaxis.

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