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Topological transition on a conformal manifold for the quantum Ising model with a longer range interaction

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An attempt is made to show the topological transition on conformal manifold for quantum Ising model with longer range interaction. This model Hamiltonian system has different gapped phases with different topological indices and also different quantum critical lines depending on the presence and absence of the transverse field. We also present the central charge for the different regimes of the parameter space. We show explicitly on the interplay of criticality, topology and central charge (C) in the presence and absence of transverse field and also non-universal character of C. We show how the Lifshitz transition occurs in the presence of transverse field and changes the universality class of the central charge in presence of transverse field. We show explicitly the existence of conformal field theory (CFT) criticality and non-CFT criticality. We have presented an explicit calculations to find the relation between the polynomial function and the Anderson-pseudo spin model Hamiltonian. Our results are far more enrich than the existence results of noninteracting fermionic many body system. This work not only provides a new perspective in topological state of conformal field theory but also for low dimensional quantum many body system.

The physics of a statistical system is scale invariant at criticality. It is then natural to generalize the hypothesis of global scale invariance to an invariance under a local coordinate dependent scaling[1–4]. This conception is enormously successful in lower dimensional quantum many particle system with many new and important results. From the mathematical stand point conformal symmetry applies to continuum theories and therefore one can apply to the critical phenomena is formulated in the language of quantum field theory. Universality in critical phenomena is one of the most important concepts in statistical physics. At the critical point, the long-distance properties are governed by a fixed point in terms of renormalization Group (RG), which is nothing but a scale-invariant (conformal) field theory[5–15].

Two dimensional CFT has been developed with the seminal paper of Belavian, Polyakov and Zamolochikov[12,13]. It has been proven to be an extremely rich in mathematical physics with three main application in string theory, two-dimensional critical system and application to mathematics in general and group theory. In the present study, we do the CFT for 2 (= 1 + 1) dimensional critical phenomena. The physics of quantum criticality in low dimensional (1 + 1) quantum many body system can be studied by using conformal field theory[1–4]. At the quantum critical point the most important parameter is the central charge, which is related with the quantum fluctuations at zero temperature. The central charge can be calculated in many ways. At zero temperature it can be calculated from the zeros of the polynomial of the model Hamiltonian system.

There are plethora of quantum many body systems, topological quantum many body systems are a subclass which do not possess any local order parameter, its topological phases are characterization in terms of a topological invariant number (W)[1,3,4] which do not change under small perturbations until the energy gap closes. Change in the topological invariant number indicates the topological phase transition. Our main intention is to study the topological transition in conformal manifold.

Motivation of the study is the following:

The study and the results of gapped phase are in great success for topological state of quantum matter. But a comprehensive understanding of the gapless phases and their transition is not upto that level of gapped phase. We study in the space of two (= 1 + 1) dimensional quantum critical phase with spinless fermionic degrees of freedom [Eqs. (2) and (4)] described by a continuous family of conformal field theories (CFTs), also known as the conformal manifold[16]. We would like to study topological transition in this conformal manifold.

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The minimal model of CFT is not sufficient to describe the central charge behaviour for the quantum Ising model with longer range interaction[1–3,15]. The quantum Ising model has two gapped phases, one is topologically phase and the other is non-topological and only a single quantum critical point which is topologically active. For this case there is no evidence of quantum critical lines and multicritical points. But for our model Hamiltonian system is quite different it has several gapped phases with topological and non-topological character. This model Hamiltonian has also several gapless quantum critical lines along with multicritical points, depending on the presence or absence of transverse field. It is well known that in the gapped phase there are no massless degrees of freedom but it is gapless for the quantum critical lines. As we understand C measure the gapless degrees of freedom[3]. It is common believe that one has to introduce another degree of freedom, i.e, the central charge of the bulk conformal field theory, in addition to the winding number, in order to fully characterize the topological phase of a gapless BDI system[16–19] (In BDI symmetry class, time reversal symmetry, charge conjugation symmetry and chiral symmetry has preserved, which may possibly be extended to the remaining chiral classes and the BdG classes[4,9,10]; in Cartan nomenclature, it is also called chiral orthogonal symmetry). Now our question is that how it reflects in our study. In the present study, we have three quantum critical lines with specific values of C, which corresponds to the conformal manifold of the present problem.

We also show explicitly that there is a difference between the quantum criticality and topological criticality. In quantum critical lines, values of C are the same along that critical lines. But we also observe the topological invariant number may not be same along that quantum critical line, it may show a transition from a region of parameter space to the other. The main motivation of this study is to find the interplay of criticality and topology in presence and absence of transverse field. We also motivate to search, whether there is any quantum critical line which has finite central charge but topologically partially or completely trivial. In this study we raise the question whether there is any possibility to find the emergence of quantum Lifshitz transition. We also raise the question whether there is any possibility non-CFT criticality? What is the difference between the CFT criticality and non-CFT criticality? We also raise the question, whether there is any relation between polynomial function and Anderson pseudospin Hamiltonian.

Model Hamiltonian and Its Variant

We consider transverse field Ising model with three spin interaction[20–27].

$$H = -\sum_i \left(\lambda_1 \sigma_i^z \sigma_{i-1}^z + \lambda_2 \sigma_i^x \sigma_{i-1}^z \sigma_{i+1}^z + \mu \sigma_i^x\right), \tag{1}$$

where $\sigma^{x,z}$ are Pauli matrices.

We do the Jordan-Wigner transformation[1–3] $\sigma_i^x = 1 - 2c_i^\dagger c_i$ and $\sigma_i^z = -\prod_{j<i}(1-2c_j^\dagger c_j)(c_i + c_i^\dagger)$, the model Hamiltonian can be written in spinless fermionic form as

$$H = -\mu \sum_{i=1}^{N} (1 - 2c_i^\dagger c_i) - \lambda_1 \sum_{i=1}^{N-1} (c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + h.c) - \lambda_2 \sum_{i=2}^{N-1} (c_{i-1}^\dagger c_{i+1} + c_{i+1} c_{i-1} + h.c), \tag{2}$$

where nearest neighbor superconducting gap is equal to nearest neighbor hopping amplitude (λ_1) and next nearest neighbor superconducting gap is equal to next nearest neighbor hopping amplitude (λ_2). In this equation, $c_i^\dagger (c_i)$ is creation (annihilation) fermionic operator and $h.c$ represents the Hermitian conjugate. It is a one-dimensional mean-field model for a triplet superconductor. The three spin interaction added to the transverse field Ising model can be physically realized in realistic Hamiltonians since the term is generated through real-space renormalization group treatments[21] .

This model has been studied previously in different contexts[25–27] The model was first introduced by the authors of Ref.[21] to study the persistence of quantum criticality at high temperature in correlated systems. The authors of Ref.[22] has studied the physics of Majorana zero modes in the gapped phases of this model with both broken and unbroken time-reversal symmetry.

This model Hamiltonian has studied extensively not only in the context of quantum spin system but also from the perspective of topological state of quantum matter. But in the present study, we solve this model Hamiltonian from the different perspective of CFT[5,24,26,27].

The model Hamiltonian can be expressed in terms of pseudo spin-vector. The transition can be verified by investigating behavior of pseudo spin-vector in the parameter space[26,27]. Thus the model Hamiltonian is,

$$\mathscr{H}(k) = \boldsymbol{\chi}(k).\boldsymbol{\sigma} = \chi_z(k)\sigma_z - \chi_y(k)\sigma_y, \tag{3}$$

where $\chi_z(k) = -2\lambda_1 \cos k - 2\lambda_2 \cos 2k + 2\mu$, and $\chi_y(k) = 2\lambda_1 \sin k + 2\lambda_2 \sin 2k$. The excitation spectra can be obtained as

$$E_k = \pm\sqrt{\chi_z^2(k) + \chi_y^2(k)}. \tag{4}$$

This Hamiltonian represent a set of pseudospins ($\boldsymbol{\sigma}$) in a two dimensional magnetic field (χ). The analytical relations of the pseudospin are the following.

$\sigma_k{}^- = (\sigma_k{}^+)^\dagger = c_k c_{-k}\ \sigma_k{}^z = (1/2)(c_k{}^\dagger c_k + c_{-k}{}^\dagger c_{-k} - 1)$

These analytical relations of , σ_k, satisfy the $SU(2)$ algebra and also the commutation relations are the following:

$[\sigma_k{}^z, \sigma_{k1}{}^\pm] = \pm\delta_{k,k1}\sigma_{k1}{}^\pm$. $[\sigma_k{}^+, \sigma_{k1}{}^-] = 2\delta_{k,k1}\sigma_{k1}{}^z$.

We find three quantum critical lines for this model Hamiltonian, $\lambda_2 = \mu + \lambda_1$, $\lambda_2 = \mu - \lambda_1$ and $\lambda_2 = -\mu$, obtained for momentum $k_0 = \pm\pi$, $k_0 = 0$ and $k_0 = \cos^{-1}(-\lambda_1/2\lambda_2)$ respectively. The topological angle can be written as $\phi_k = \tan^{-1}(\chi_y(k)/\chi_z(k))$.

The above model Hamiltonian can also be expressed in terms of Majorana operators.

$$H = -\mu\sum_{i=1}^{N} b_i a_i - \lambda_1 \sum_{i=1}^{N-1} b_i a_{i+1} - \lambda_2 \sum_{i=2}^{N-1} b_{i-1} a_{i+1} \tag{5}$$

We use the Majorana fermion operators in terms of Dirac fermion operators as

$a_i = c_i{}^\dagger + c_i$, $\tilde{a_i} = -i(c_i{}^\dagger - c_i)$, and also use the following relation $\{a_i, a_j\} = 2\delta_{i,j}$ and $\{a_i, \tilde{a_j}\} = 0$ to obtain the Hamiltonian in Majorana basis.

The general form of the above Hamiltonian can be written as,

$$H = -i\sum_{\alpha=0,1,2}\left(\sum_{j=1}^{N-\alpha}\gamma_\alpha b_j a_{j+\alpha}\right), \tag{6}$$

where γ_0, γ_1 and γ_2 are respectively μ, λ_1 and λ_2 for our model Hamiltonian system.

For this model Hamiltonian, the authors of Ref.[23] have already investigated the underlying structure of the integral definition and necessary modifications of topological invariant number (*W*) with the constraint of bulk boundary correspondence to be valid at criticality. In general, the definition of the *W* can be written as the integral evaluation of Berry connection over the Brillouin zone[23–27]. *W* is valid only in gapped phases where it takes integer values. The author have modified to understand the underlying structure of the integral form of the topological invariant number at criticality. They have proposed a method where one has to separate out the integer and fractional parts from the integral form to get a one to one correspondence between topological invariant number and number of Majorana zero mode at criticality.

Topological invariant number at a criticality can be obtained by omitting an infinitesimal neighborhood of a gap closing point. Thus in this study, we give more emphasize on the CFT study for this model Hamiltonian.

A few basic aspect of central charge and essential calculations for this model Hamiltonian

The central charge counts the number of massless degrees of freedom. For the gapped system, there are no low energy degrees of freedom and thus the central charge is zero. There are different physical significance between the integer and fractional central charge. Theories with integer central charge have free field representation. Theories with half-integer central charge, example of quantum Ising model (2D Ising model) have a free Majorana representation at the transition point. is roughly a measure of the number of degrees of freedom of the model considered. The central charge labels each universality class of the critical theory[8–15].

One can write the polynomial function corresponding to the Hamiltonian. Every BDI Hamiltonian is equivalent to a particular polynomial. In the present study, we will extract many interesting physics from the zeros of the polynomial. Equation (6) can be brought to the complex form using Fourier transformation, $f(k) = \sum_{\alpha=0,1,2}\gamma_\alpha e^{ik\alpha}$[19,23], with $\zeta = e^{ik}$. $f(\zeta)$ is homomorphic function away from the possible pole at the origin. The modulus of this function ($|f(e^{ik})|$) gives the single particle energy of momentum *k*, more precisely, one can write $f_k = \epsilon_k e^{i\phi_k}$ (ϵ_k and ϕ_k are real values). The phase angle (arg (fe^{ik})) is the angle required in Bogoliubov rotation to define the quasiparticle. We will see in the due course of this study that this simple behaviour of this function will generate interplay between the topological phase, quantum criticality and central charge of the model Hamiltonian.

One can write the model Hamiltonian as,

$$f(\zeta) = \sum_{\alpha=0,1,2}\gamma_\alpha\zeta^\alpha = -\mu + \lambda_1\zeta + \lambda_2\zeta^2, \tag{7}$$

where γ_0, γ_1 and γ_2 are respectively $-\mu$, λ_1 and λ_2. The zero solution of the above equation is

$$\zeta_{1,2} = \frac{-\lambda_1 \pm \sqrt{\lambda_1{}^2 + 4\mu\lambda_2}}{2\lambda_2}. \tag{8}$$

The topological invariant (*W*) and the *C* can be calculated by using the number of zeros of the solution inside and on circumference of the unit circle. The locations of zeros indicate different state of the system. Zeros inside, on and outside the unit circle are respectively for the topological, topological transition and non-topological phases of the system.

Now we explain explicitly why the non-degenerate zero of the $f(\zeta)$ function is treating the CFT physics but the degenerate zero of $f(\zeta)$ fails to do that. We have already discussed that the $f(\zeta)$ is proportional to the energy dispersion. If $f(k)$ is nondegenerate, the energy dispersion ($\epsilon(k) \propto (k - k_0)$) is a relativistic dispersion. In order to treat the model Hamiltonian system using CFT, Lorentz symmetry has to be manifested, i.e., it must have a relativistic dispersion and a dynamical critical exponent $z = 1$. For the case of m degenerate zeros, it implies that for a non-linear dispersion, $\epsilon(k)$ behave as $\sim (k - k_0)^z$, here is the breakdown of Lorentz invariance, for this situation CFT can not be used.

The topological property of the system can be captured by the winding number (W) which counts the number of edge modes in the corresponding gapped or critical phases. The critical properties of the system captured from the central charge C which measures the CFT of the corresponding criticality. Due to the one-to-one correspondence between the Hamiltonian and the associated complex function $f(\zeta)$, the central charge can be obtained using the zeros ($\zeta = 1, 2, 3$) in the complex plane if they are non-degenerate. Among the zeros of the complex function $f(\zeta)$, that lie on the unit circle carries the information that the system is at criticality. Therefore, the number of zeros on the unit circle determines the value of C as

$$C = \frac{1}{2}(\text{number of zeros on the unit circle}). \quad (9)$$

Non-degenerate zeros $e^{\pm ik_0}$ on the unit circle $|z| = 1$ contributes a massless Majorana fermion field theory. This suggests that there must be an intimate connection between the number of zeros and the degrees of freedom in a CFT, measured by the C. The fact that each Majorana fermion contributes with central charge $C = 1/2$. In particular this means that systems with 2C zeros on the unit circle result in a total central charge C. In our calculation with zeros $z_0 = e^{\pm ik_0}$, we expect the central charge $C = 1$ (which is indeed true since the central charge is additive under the presence of two Majorana fermion contributes with $C = 1/2$)

We will use this formula to generate and explain Figs. 1, 2, 3, 4, 5, 6, 7, and 8 of the next section. We also raise the question, whether there is any relation between polynomial function and Anderson pseudospin Hamiltonian. The main motivation of this study is to find the interplay of criticality and topology in presence and absence of transverse field.

Results and discussions

Results for the central charge study

In Fig. 1, we present the results of quantum Ising model for benchmarking our results. This figure consists of two panels. The upper one presents quantum critical phase diagram for the quantum Ising model. The point 'b' ($\lambda_1 = \mu$) is the transition point, where system transit from non-topological phase to topological phase. At the transition point $C = 1/2$ but for the gapped phase $C = 0$. The lower panel present the polynomial solutions [Eq. (8)] for quantum Ising model. We find zeros of the polynomial solution on the circumference of the circle which indicates the transition.

It is well known to us that the Ising model has no phase transition but the quantum Ising model has the transition when the magnitude of the coupling and the transverse field is the same. The transition point is topologically active with massless Majorana fermion mode ($C = 1/2$)[1–3] otherwise system has the massive Majorana fermion with $C = 0$.

Now we present the results to show one can further get new and important results for the longer range interaction over the quantum Ising model.

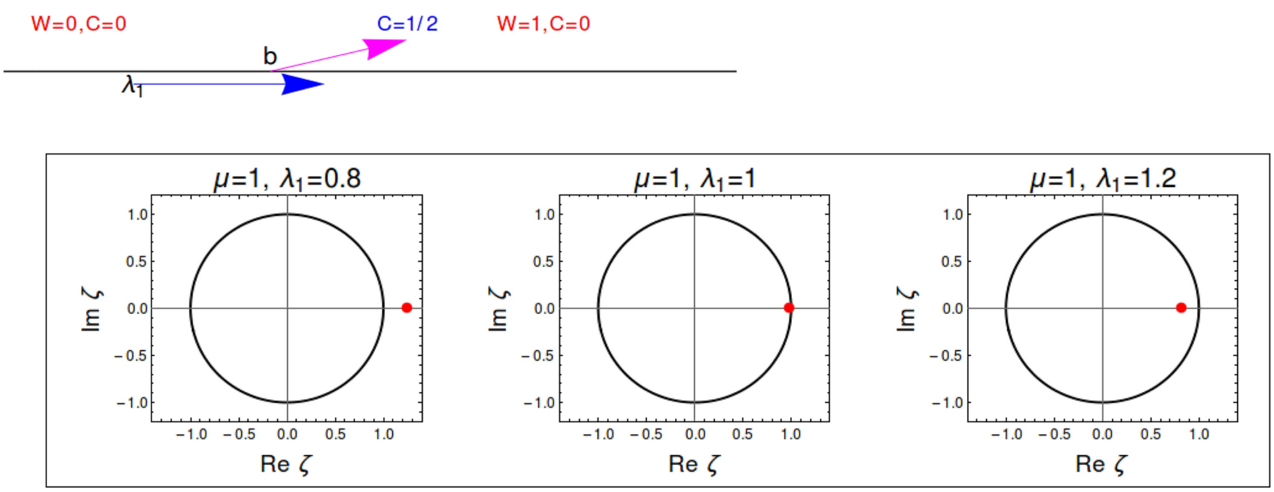

Fig. 1. The upper panel represents quantum critical phase diagram for quantum Ising model. The lower panel presents zeros of polynomial function solution [Eq. (8)].

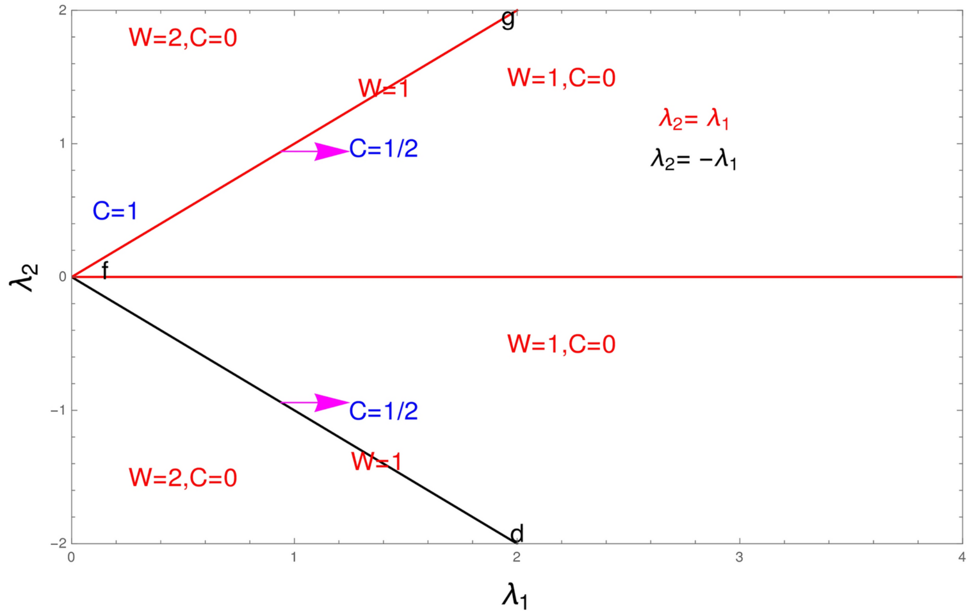

Fig. 2. Topological phase diagram of model Hamiltonian for $\mu = 0$. Line 'fg' represents the critical line $\lambda_2 = \mu + \lambda_1$ line 'fd' represents the critical line $\lambda_2 = \mu - \lambda_1$. Point 'f' is the multicritical which differentiate between three distinct gapped phases with $W = 1, 2$ and $C = 1$ is the central charge for this multicritical point.

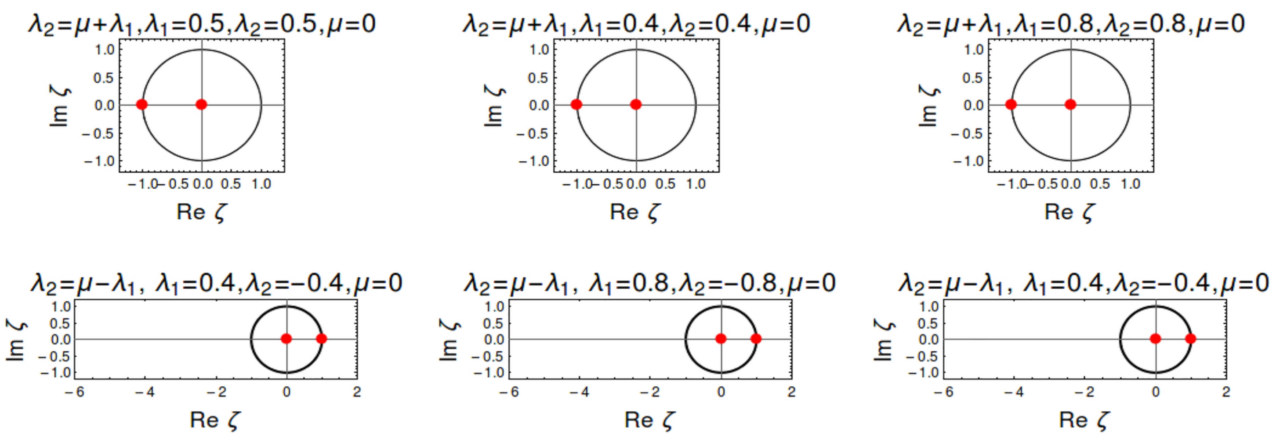

Fig. 3. This figure presents the zeros of the polynomial function [Eq. (8)]. This figure consists of two panels. The upper and the lower panels are respectively for the quantum critical line $\lambda_2 = \mu + \lambda_1$ and $\lambda_2 = \mu - \lambda_1$. For all panels we consider $\mu = 0$.

In Fig. 2, we present the complete phase diagram of topological state and CFT for our model Hamiltonian system without transverse field ($\mu = 0$). This phase diagram consists of three gapped phase ($C = 0$) and two quantum critical lines 'fg' and 'fd' ($\lambda_2 = \lambda_1$, $\lambda_2 = -\lambda_1$). Here only one multicritical point ('f') with $z = 1$. Value of C is 1/2 for the all quantum critical lines and the value of $C = 1$ is for the quantum critical point 'f'. The value of W (= 1) is the same for the all quantum critical line. The most interesting feature is that the physics of CFT is applicable for the whole phase diagram, because the Lorentz symmetry preserve for the whole range of phase diagram. We observe that all quantum critical lines are in the topological phase and there is no topological quantum phase transition along any quantum critical line. There is no topological transition for $\lambda_2 = 0$ line. This phase diagram is symmetric with respect to $\lambda_2 = 0$. It is well known that the central charge labels each universality class of critical theory. For this case two quantum critical lines have the same winding number and same central charge, thus the universality class of the critical lines of the theory is the same. Thus for this case quantum criticality and topological quantum criticality is the same.

Figure 3, We present the results for our model Hamiltonian system without transverse field. We present zeros of the polynomial solution in the unit circle. This figure consists of two panels, the upper, and lower panels

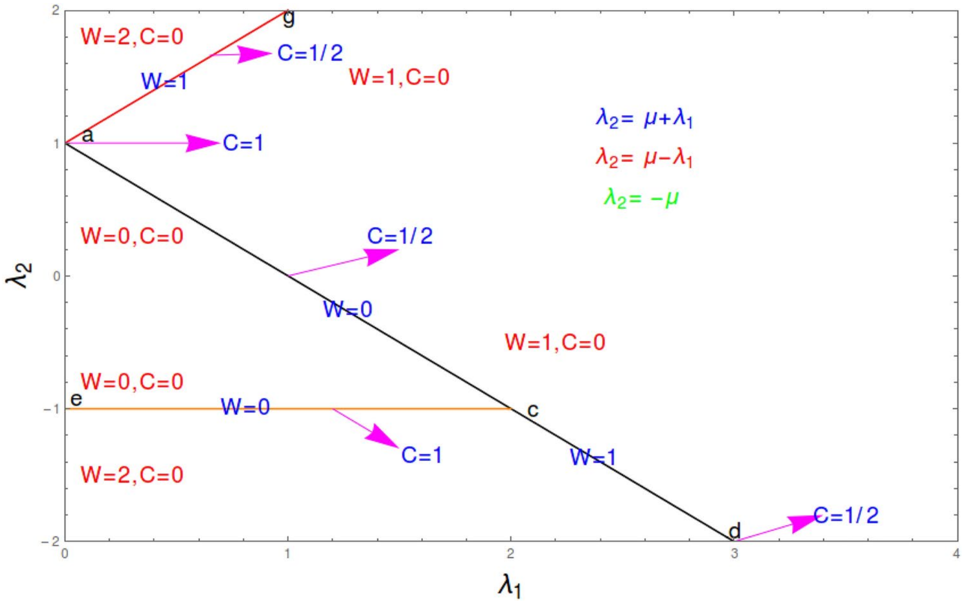

Fig. 4. Topological phase diagram of model Hamiltonian for $\mu = 1$. Line 'ad' represents the critical line $\lambda_2 = \mu - \lambda_1$ (blue line), line 'ec' represents the critical line $\lambda_2 = -\mu$ (yellow line) and line 'ag' represents the critical line $\lambda_2 = \mu + \lambda_1$ (red line). Points 'a' and 'c' are multi-critical points which differentiate between three distinct gapped phases with $W = 0, 1, 2$ and C is the central charge.

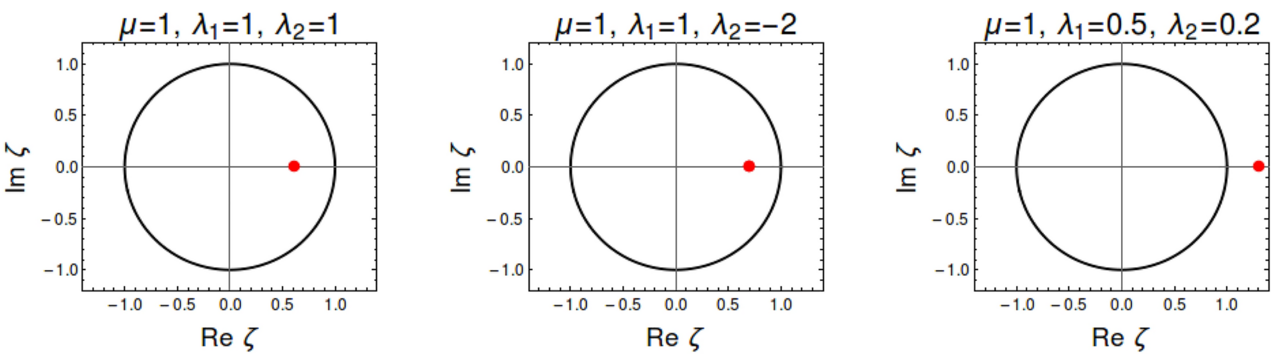

Fig. 5. This figure panel presents the zeros of the polynomial function [Eq. (8)]. This figure consists of three figures for different values λ_1 and λ_2 as depicted in figures. Here we set $\mu = 1$. This figure present different gapped phases with zero central charge.

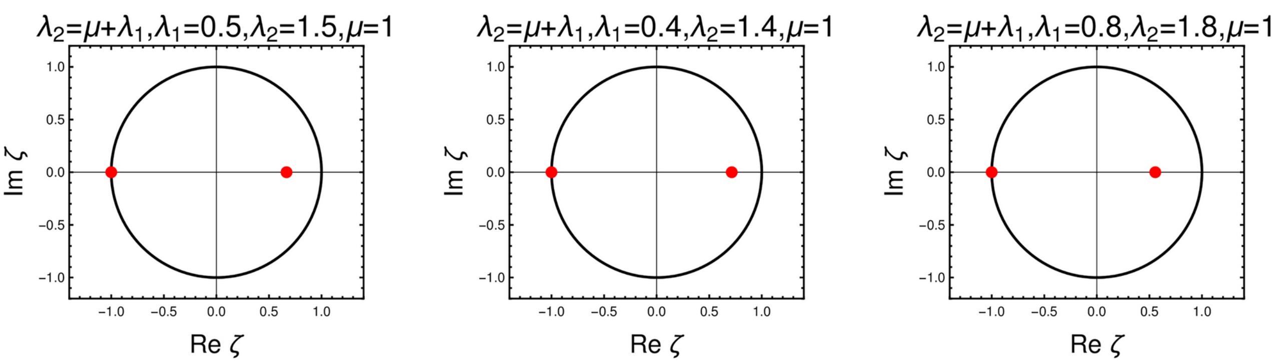

Fig. 6. This figure panel presents the zeros of the polynomial function [Eq. (8)]. This figure presents the quantum critical line $\lambda_2 = \mu + \lambda_1$. Here we set $\mu = 1$ for different values of λ_1.

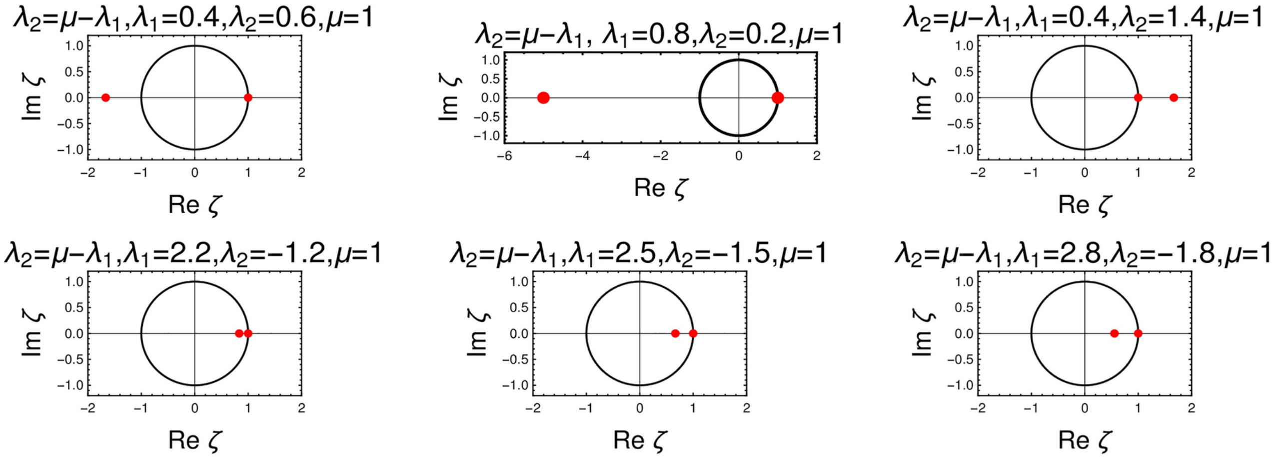

Fig. 7. This figure panel presents the zeros of the polynomial function [Eq. (8)]. This figure panel is for the quantum critical line $\lambda_2 = \mu - \lambda_1$. For all panels we consider $\mu = 1$. This figure consists of two panels the upper and lower panel are respectively for $\lambda_1 < 2\mu$ and $\lambda_1 > 2\mu$.

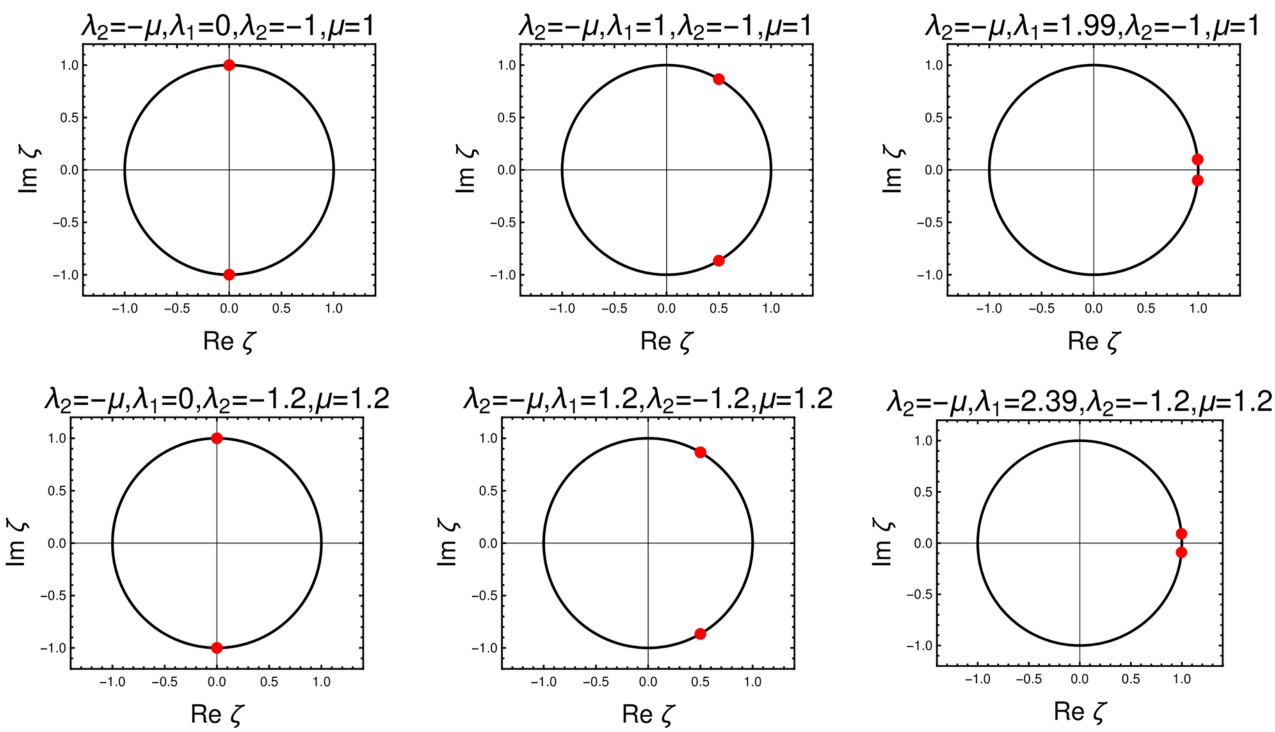

Fig. 8. This figure panel presents the zeros of the polynomial function [Eq. (8)] for the quantum critical line $\lambda_2 = -\mu$. The upper and the lower row are respectively for $\mu = 1$ and 1.2.

are respectively for $\lambda_2 = \lambda_1$, $\lambda_2 = -\lambda_1$. We observe that system is in the topological phase ($W = 1$) and the $C = 1/2$ for both quantum critical lines.

In Fig. 4, we present the complete phase diagram of topological state and also CFT for our model Hamiltonian system with transverse field and also quantum Ising model. It reveals from this figure that the presence of transverse field gives many new important results over the absence of transverse field.

This phase diagram consists of four gapped topological phases separated by the three quantum critical lines, which we depict in this figure. This model Hamiltonian has two quantum multicritical points ('a' and 'c'), point 'a' is topologically trivial but non-zero central charge ($C = 1$), the point 'c' is another multicritical point where the CFT is not define which we will discuss in detail as we proceed in the manuscript. We have already mentioned in the previous section that the central charge for the gapped system is zero. This system consists of three quantum critical lines as depicted, values of *C* for the quantum critical lines 'ag' and 'ad' is 1/2. But the topological character is different , according to the least topological invariant number W is 1 for the quantum critical line 'ag' and 'cd'

(a part of the 'ad' quantum critical line). But for the part of the quantum critical line, 'ac', W is zero. The other quantum critical line is 'ec', where W is zero but $C = 1$. It reveals from our study that the value of C is zero for the all gapped phase, which we will also discuss in the due course of this manuscript.

Two quantum critical lines 'ag' and 'ad' have the same C, i.e., 1/2. The winding number of the quantum critical line, 'ag', is 1. But a part of the quantum critical, 'ac', $W = 0$ in $\lambda_1 < 2\mu$, whereas $W = 1$ for the quantum critical line 'cd', where $\lambda_1 > 2\mu$. Thus the central charge, which labels each universality class of critical line, is violating here as per our considerations the topological quantum criticality. The two regions, topologically active and another is topologically trivial, are separated by a topologically active multicritical point ('c') of this quantum critical line. Thus the topological quantum criticality and conventional quantum criticality are not the same. There is also some interesting physics for this quantum critical line, 'ec', C of this critical line is 1 but $W = 0$ for the whole quantum critical line.

In Fig. 5, we present the different gapped phase for our model Hamiltonian system for $\mu = 1$. We set different values λ_1 and λ_2 as depicted in figure. In this figure, we present the polynomial mapping solution of our model Hamiltonian [Eq. (8)]. This figure panel consists of three figures. It reveals from our study that the left and middle figure (in both figures $\lambda_1 = 1$), system is in the topological state with $C = 0$. In the right figure, system is in the non-topological phase and also $C = 0$. Thus the results of this figure is consistent with eq.9 and also with the basic criteria of finite C.

Figure 6, presents the results for the quantum critical line, $\lambda_2 = \mu + \lambda_1$. In this figure, we present the polynomial mapping solution of our model Hamiltonian [Eq. (8)]. These three figures are for different values of λ_1 and λ_2 but satisfy the condition of quantum critical line, $\lambda_2 = \mu + \lambda_1$. It reveals from this figure panel that for this quantum critical line system is always in the topological phase ($W = 1$).

Figure 7, presents the results for the quantum critical line, $\lambda_2 = \mu - \lambda_1$. In this figure, we present the polynomial mapping solution of our model Hamiltonian [Eq. (8)]. These figures consists of two figures panel for different values of λ_1 and λ_2 but satisfy the condition of quantum critical line, $\lambda_2 = \mu - \lambda_1$. It reveals from the upper panel for this quantum critical line system is always in the non-topological phase ($W = 0$). A single red dot always on the circumference on the circle, i.e, $C = 1/2$. Here we consider the parameter space $\lambda_1 < 2\mu$. The second panel of this figure is for $\lambda_1 > 2\mu$, it reveals from study that for this region of parameter space, system is always in the topological phase ($W = 1$) and also $C = 1/2$.

Figure 8, presents the results for the quantum critical line, $\lambda_2 = -\mu$. In this figure, we present the polynomial mapping solution of our model Hamiltonian [Eq. (8)]. This figure panel consists of two rows, the upper and the lower rows are respectively $\mu = 1$ and 1.2. We present the results for different values of λ_1, for this quantum critical line, there is no topological phase ($W = 0$). We also observe that $C = 1$ for this quantum critical line. The most important feature, we observe that the two red dots are closing to each other as we increase the value of λ_1, when they merge (at $\lambda_1 = 2$), i.e., the degenerate at that point $z = 2$, CFT is not applicable at that point. In the lower panel, we observe the same behaviour of red dots for different values of μs.

Results for energy dispersion and scaling relation

Figure 9 presents the energy dispersion for our model Hamiltonian. This figure panel consists of two rows for two different values of μ as depicted in the figure and each panel consists of three figures. The left, right and middle figures are respectively for the three quantum critical points. These dispersion are based on Eq. (4). In the upper panel, the left figure is for the quantum Ising model at the topological quantum phase transition with $W = 1$ and $C = 1/2$ (point 'b' of fig. 1), where both the dispersion branches meet at the origin.

The middle figure is for the dispersion (point 'a' of Fig. 4, $W = 0$ and $C = 1$), it is one of the multicritical point, where the dispersion branches meet phase at the origin and also at the Brillouin zone boundary. Thus it has no topological phase but with integer value of C. The right figure is for the another multicritical point ('c'), which is non-conformal quantum critical point, where the physics of CFT is not applicable. One of the most interesting features that we obtain a flat dispersion close to the origin. In the lower panel, for the different value of μ, 'we observe the same behaviour of energy dispersion.

Figure 10 presents the scaling relation of energy dispersion for our model Hamiltonian. This figure panel consists of two panels for two different values of μ as depicted in figure and each panel consists of three figures. The left, right and middle figures are respectively for the three quantum critical points. We present the scaling relation of energy dispersion near to the quantum critical points for both panels. We extract the dynamical critical exponent (z2) from that energy dispersion. It reveals from our study that $z = 1$ is for the left and middle but for the right figure it is 2.

This entails the fact that there is a topological quantum critical point where the Lifshitz universality class with $z = 2$, between two distinct gapless phases through muticritical point. The Lifshitz transition that emerges in our study only for the presence of transverse field for both rows of this figure panel. At this multicritical point system is in non-CFT criticality. To the best of our knowledge this is the first study of the existence conformal criticality and non-conformal criticality for this model Hamiltonian system. We observe the same behaviour of dispersion scaling for different values of μ.

A general comparison of the results with physical explanation: richness of the present study

Benchmarking the existence results: For the sake of completeness, at first we discuss, previously existing results on the CFT study.

Our model Hamiltonian belongs to the $(1 + 1)$ dimensional universality class. We want to give an impact about the richness of this work. It is well known that in two dimensional universality class of CFT is specified by the C and the spectrum of the primary field (h^+, h^-). The unitarily condition leads to the following expression for C $(0 < C < 1)$[1,2,11,15].

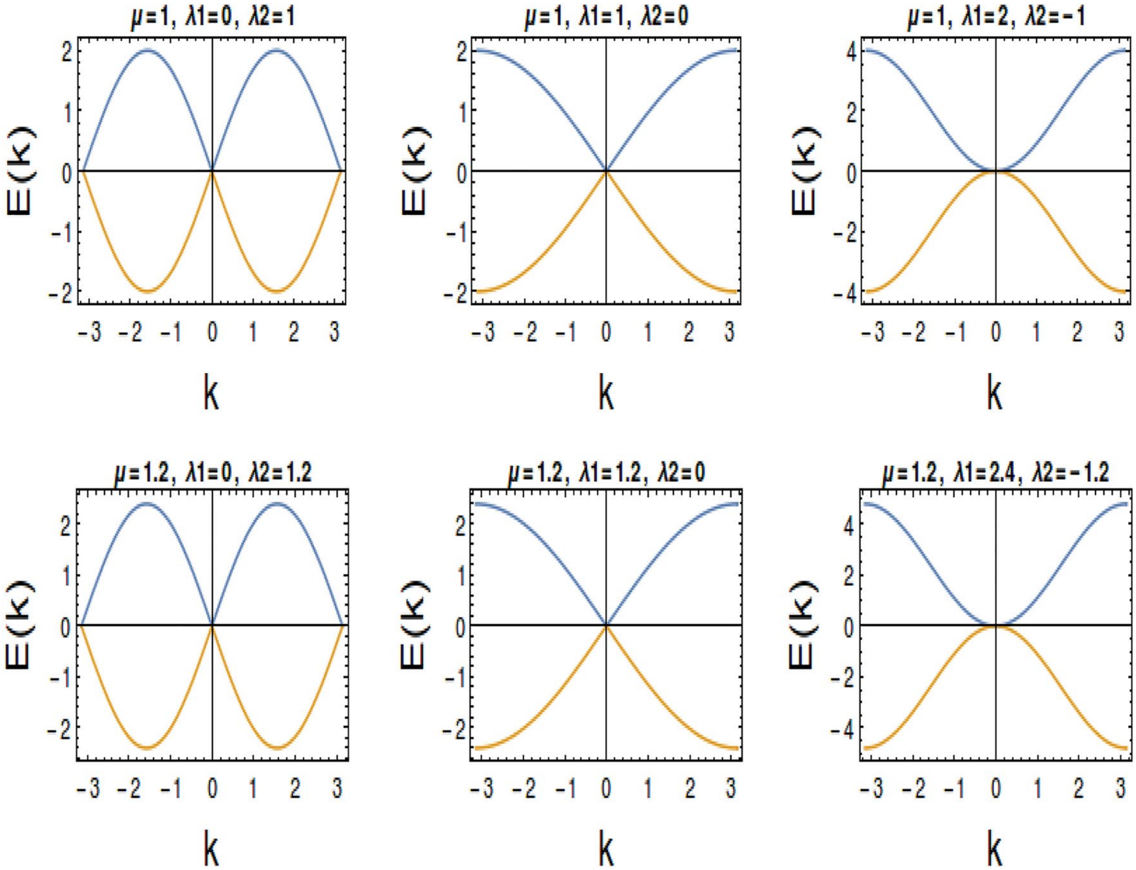

Fig. 9. This figure panel consists of two rows and each panel consists three figures for energy dispersion [Eq. (4)]. The upper and the lower rows are respectively for the $\mu = 1$ and $\mu = 1.2$. The left figure is for the quantum Ising model, middle and right figures are for quantum Ising model with longer range interaction.

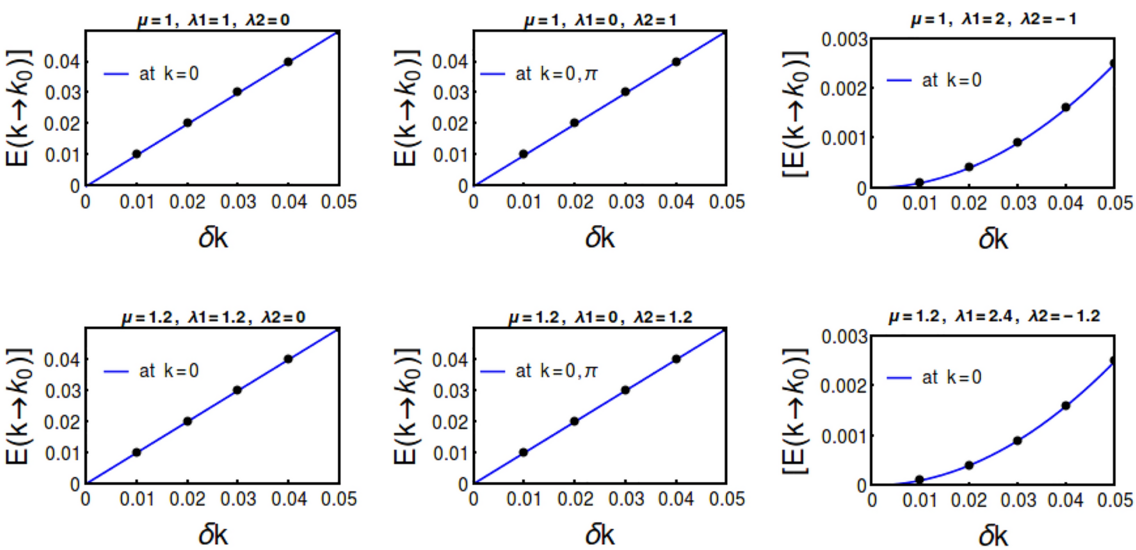

Fig. 10. This figure panel consists of two rows and each panel consists three figures. The upper and the lower rows are respectively for the $\mu = 1$ and $\mu = 1.2$. These figures are corresponding scaling analysis for the different quantum critical points. The left figure is for the quantum Ising model, middle and right figures are for quantum Ising model with longer range interaction.

$$C = 1 - \frac{6}{m(m+1)}, \quad (10)$$

where $m = 3, 4, 5$. The number of primary field is finite for this range of *C*. The conformal weights are rational numbers which are determined by the Kac formula. different values of *m* corresponds to the different physical systems, for $m = 3$, it is the results for quantum Ising model. The *C* is 1/2 for the quantum Ising model, we also achive these results, thus we benchmark the existing result, which we present in Fig. 1 .

Hallmarking of the present results:

The most important and new results for the model Hamiltonian [Eq. (2)] with three parameters ($\lambda_1, \lambda_2, \mu$). The number of quantum critical lines is three. The *C* of two quantum critical lines is 1/2 but for the another is 1 ($\lambda_2 = -\mu$). Thus the all quantum critical lines are not in the same universality class. A part of the quantum critical line ($\lambda_2 = 0$) is topologically trivial phase ($W = 0$) and the other part of critical line is topologically non-trivial ($W = 1$). Thus the topological quantum criticality is not the same with conventional quantum criticality.

Our model Hamiltonian [Eq. (2)] is the spinless fermionic system. It is well-known in the literature[2,3] that the value of *C* is 1 for the spinless fermionic system. It reveals from our study that situations occur at the multicritical point 'a' and the other is for the entire quantum critical ($\lambda_2 = -\mu$). Thus it is clear that at the multicritical point, 'a', and also for the quantum critical line ($\lambda_2 = -\mu$) system shows the behaviour of spinless Dirac fermions. We already find $C = 0$ is for the gapped phase for our model Hamiltonian system and also $C = 1/2$ for two quantum critical lines. Apart from that there is a topological transition along a quantum critical line. Thus the all quantum critical lines and multicritical points are not providing the physics of Majorana fermion. These new and important hallmarking results for this model Hamiltonian, make this study not only new perspective in topological state of quantum matter but also from low dimensional quantum many body system with Dirac fermion physics.

It is not possible to study quantum Ising model in presence of three sites spin interaction with this minimal model. Thus the present study based on the zeros of the polynomial solution gives the further advancement of the results for quantum Ising model with longer range interaction.

We observe that at the meeting point of two quantum critical lines ('ag' and 'ad' lines of Fig. 4), *C* is 1 (= 1/2 + 1/2). This value of *C* occurs for both presence and absence of transverse field. It is well known in the literature that each Majorana field contribute 1/2. At the junction point two Majorana fields are added up. Thus we reproduce the existing results for quantum Ising chain through our present study of zeros of the polynomial solution. We also observe the presence of Lifshitz transition for finite values μ.

Now we present important feature of our model Hamiltonian without any transverse field ($\mu = 0$). It is well known that central charge labels each universality class of the critical theory. For this case, system has two quantum critical lines, the central charge is the same for both quantum critical lines, thus it satisfies the condition of the universality class as one conjecture from CFT. For this case topological criticality and the quantum criticality is the same for this model Hamiltonian system. '

A mathematical analysis of polynomial function (*f*(*z*)) from the perspective of Anderson-pseudo spin Hamiltonian

We have already expressed our model Hamiltonian in different presentations. One of the presentation is the Anderson pseudo spin Hamiltonian [Eq. (3)], where one can find the topological properties from the parametric plot study of the coefficient of $\chi_y(k)$ and $\chi_z(k)$, i.e., how the pseudo spin rotate in the unit circle in the parametric space[26,27]. This is nothing but geometric presentation of the topological state of the model Hamiltonian system. As we have understood that the position of the origin of the parametric plot determine the topological phase. We have also presented our model Hamiltonian in polynomial presentation, where the zeros of the polynomial function has presented in the unit circle in the complex plane [Eq. (8)]. Now our main intention is to find whether there is any relation between the polynomial function (*f*(*z*)) and the Anderson-pseudo spin Hamiltonian. We also raise the question whether there is any relation between the zeros of the polynomial solution on the circumference of the unit circle and the origin of the parametric plot. We raise this question because both of them are the geometric representation of the model Hamiltonian. The relation between the polynomial function and the parametric plot is the following (detail derivation is relegated to the "Method" section)

$$f(z) = \pm(\chi_z + \chi_y). \quad (11)$$

In Fig. 11 presents the parametric plot for our model Hamiltonian system without transverse field ($\mu = 0$). This figure consists of two panels. The upper and the lower panel are respectively for the quantum critical line $\lambda_2 = \lambda_1$ and $\lambda_2 = -\lambda_1$. It reveals from the results of upper panel that the parametric plot touches the origin and also encloses the origin for a single time. Thus for this quantum critical line $W = 1$ and $C = 1/2$. It reveals from the results of lower panel that the parametric plot touches the origin and also enclose the origin for a single time but the morphology of the parametric plot is opposite. Thus the for this quantum critical line $W = 1$ and $C = 1/2$. These results are consistent with with the zeros of the polynomial solution. Thus it appears from this study that *C* can calculated how many times the parametric plot touches the origin. At the origin both $\chi_y(k)$ and $\chi_z(k)$ are zero, system has the gapless excitations at this point, the central charge measure the gapless degrees of freedom. Thus our results are physically consistent.

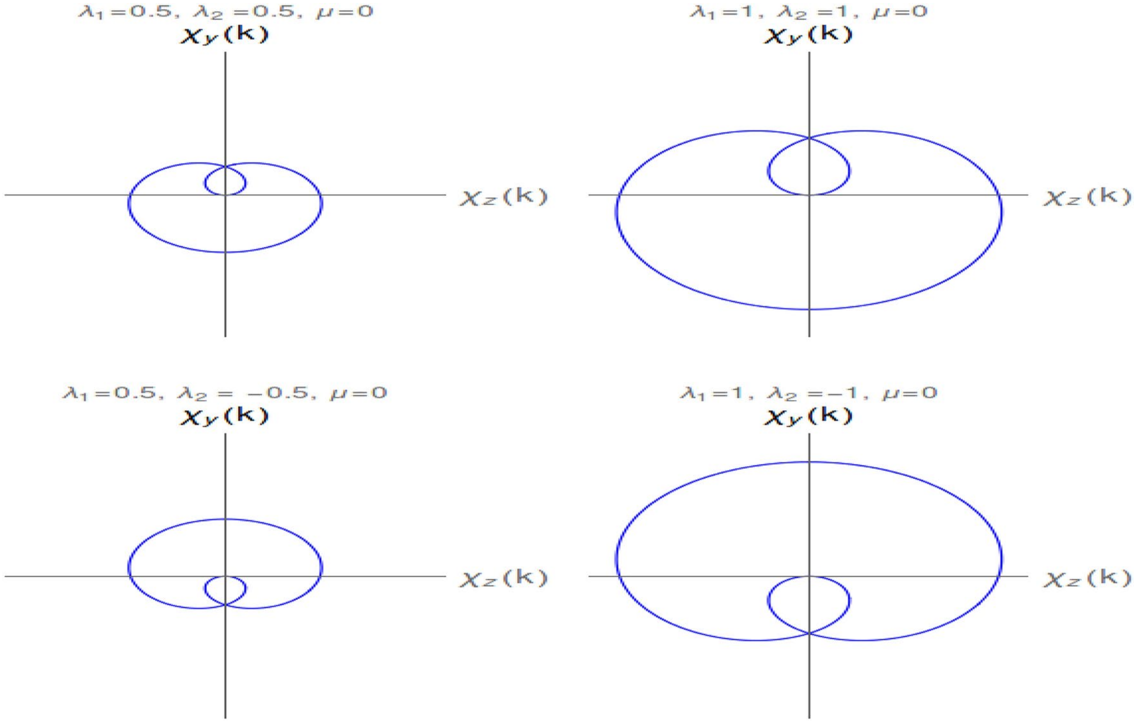

Fig. 11. We present the parametric plot based on the analysis of Anderson pseudospin Hamiltonian [Eq. (3)]. This figure panel consists of two rows in absence of transverse field for different regions of parameter space. The upper and lower panels are respectively for $\lambda_2 = \lambda_1$ and $\lambda_2 = -\lambda_1$ respectively.

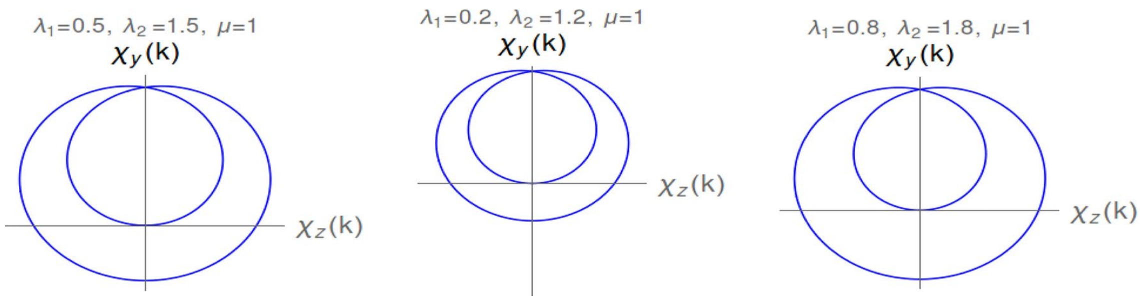

Fig. 12. We present the parametric plot based on the analysis of Anderson pseudospin Hamiltonian [Eq. (33)]. This figure panel consists of three figures for different values of λ_1 and λ_2 as depicted in figure but $\mu = 1$ is fixed. This figure panel is for the multicritical line $\lambda_2 = \mu + \lambda_1$.

In Fig. 12 , we present the parametric plot study for the quantum critical line $\lambda_2 = \mu + \lambda_1$. It reveals from our study that one of the curves encircle the origin and also touches the origin. It implies that $W = 1$ and system is in gapless quantum critical phase and and also $C = 1/2$.

In Fig. 13 presents the parametric plot study for the quantum critical line $\lambda_2 = -\mu$. This figure consists of two panels, here we set $\mu = 1$. The upper and the lower panel of this figure are respectively for $\lambda \leq 1$ and $\lambda > 1$. In the lower panel due to the higher values of λ_1, system approaches the topologically active multicritical point. It reveals from the figures of the upper row that the parametric curve touches the origin twice thus the C is 1. In the lower panel as the value of λ_1 increases the part of the parametric curve which touches the origin reduces the size and very close to the multicritical it just touch as a single point. Here the parametric curve touches origin twice.

In Fig. 14 presents the parametric plot study for the quantum critical line $\lambda_2 = \mu - \lambda_1$. This figure consists of two panels, here we set $\mu = 1$. The upper and the lower panel are respectively $\lambda_1 < 2\mu$ and $\lambda_1 > 2\mu$. It reveals from the study of upper panel that the parametric plot touches the origin only one time but not encloses the origin, thus for this part of the quantum critical line is topologically inactive but $C = 1/2$. In the lower panel, parametric plot encircles the origin and also touches the origin only once, thus for this part of the quantum critical line, it is topologically active and also $C = 1/2$. Thus it is clear from this analysis one to one correspondence between the parametric plot study and the polynomial solution in the complex plane. One can

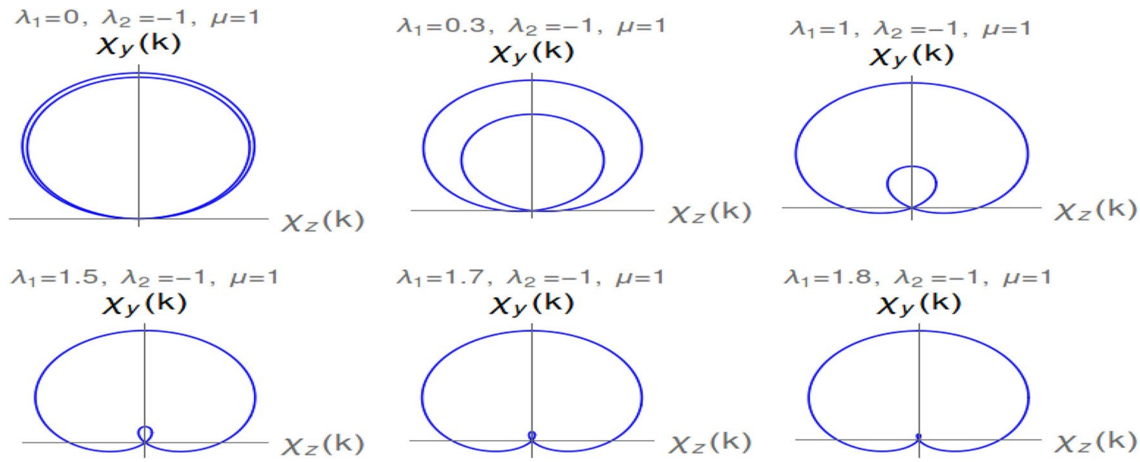

Fig. 13. We present the parametric plot based on the analysis of Anderson pseudospin Hamiltonian [Eq. (3)]. This figure panel consists of two rows in presence of transverse field for the quantum critical line $\lambda_2 = -\mu$. In this figure we set, $\mu = 1$ for different values of λ_1 and λ_2 as depicted in frame labels.

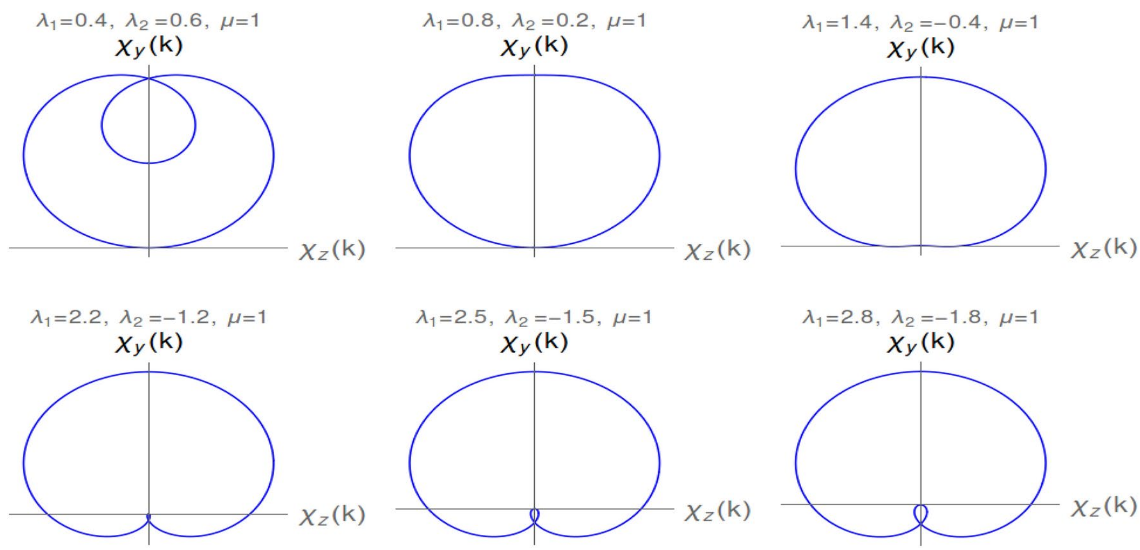

Fig. 14. We present the parametric plot based on the analysis of Anderson pseudospin Hamiltonian [Eq. (3)]. This figure panel consists of two rows, the upper and lower rows are respectively for $\lambda_1 < 2\mu$ and $\lambda_1 > 2\mu$. In this figure we set, $\mu = 1$ for different values of λ_1 and λ_2 as depicted in frame labels.

conclude that the central charge of the parametric plot is the one-half of the how many times, parametric curve touch the origin.

Discussion

We have presented the results of conformal field theory study for quantum Ising model with longer range interaction in presence and absence of transverse field. Different quantum critical lines have shown different topological and central charge behaviour. We have shown evidence of topological transition in conformal manifold. We have also shown for certain region of parameter space topological criticalty and quantum criticality are the same but for certain region of parameter it is not the same. We have shown explicitly that the central charge is zero for the gapped phase of the system but for the gapless quantum critical lines is finite. We have found quantum Lifshitz transition for quantum Ising model with longer range interaction in presence of transverse field. This quantum Lifshitz transition occurs at the non-conformal quantum critical point. We have observed the universality of central charge for certain quantum critical lines. We have also shown one to one correspondence between Anderson pseudo spin Hamiltonian and the polynomial function. This work not only provides a new perspective in topological state of conformal field theory but also for low dimensional quantum many body system.

Methods

Our starting point is the Anderson pseudo spin Hamiltonian [Eq. (3)]

$$\mathscr{H}(k) = \chi_z(k)\sigma_z - \chi_y(k)\sigma_y, \tag{12}$$

where $\chi_z(k) = -2\lambda_1 \cos k - 2\lambda_2 \cos 2k + 2\mu$, and $\chi_y(k) = 2\lambda_1 \sin k + 2\lambda_2 \sin 2k$.

$$\mathscr{H}(k) = \begin{bmatrix} \chi_z(k) & i\chi_y(k) \\ i\chi_y(k) & -\chi_z(k) \\ . & \end{bmatrix}, \tag{13}$$

One can diagonalize the above model Hamiltonian by using a unitary transformation.

$$U = e^{-i\alpha_k \sigma_x/2}. \tag{14}$$

One can also write the unitary operator in terms of matrix presentation .

$$U = \begin{bmatrix} \cos\alpha_k/2 & -i\sin\alpha_k/2 \\ i\sin\alpha_k/2 & -\cos\alpha_k/2 \end{bmatrix}, \tag{15}$$

One can finally diagonalize the Hamiltonian as

$$U\mathscr{H}U^\dagger = \begin{bmatrix} \chi_z \cos\alpha_k + \chi_y \sin\alpha_k & -i\chi_y cos\alpha_k + i\chi_z \sin\alpha_k \\ i\chi_y \sin\alpha_k - i\chi_z sin\alpha_k & -\chi_z \cos\alpha_k - \chi_y \sin\alpha_k \end{bmatrix} \tag{16}$$

$$\begin{aligned} U\mathscr{H}U^\dagger &= \begin{bmatrix} \chi_z \cos\alpha_k + \chi_y \sin\alpha_k & -i\chi_y cos\alpha_k + i\chi_z \sin\alpha_k \\ i\chi_y \sin\alpha_k - i\chi_z sin\alpha_k & -\chi_z \cos\alpha_k - \chi_y \sin\alpha_k \end{bmatrix} \\ &= \begin{bmatrix} E(k) & 0 \\ 0 & E(k) \\ . & \end{bmatrix} \end{aligned} \tag{17}$$

We get the following mathematical relations by using the above two matrices

$\chi_z \cos\alpha_k + \chi_y \sin\alpha_k = E(k)$

and $\chi_y cos\alpha_k = \chi_z \sin\alpha_k$.

Now we get the following relations by using the above two equations

$\cos\alpha_k = \pm\chi_z/E(k)$

$\sin\alpha_k = \pm\chi_y/E(k)$

Finally we write the above two equations as, $\cos\alpha_k + i\sin\alpha_k = e^{i\alpha_k} = \pm(\chi_z/|E(k)| + i\chi_y/|E(k)|)$

$|E|e^{i\alpha_k} = \pm(\chi_z + i\chi_y)$

Left hand side of the above equation is nothing but the *f*(*k*) thus one can write

$f(z) = \pm(\chi_z + i\chi_y)$

We derive this relation by using the Ref.[9,26].

Data availability

All data generated or analysed during this study are included in this published article.

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Author contributions

S.S. identified and solved the problem, also write the manuscript and generate all the figures. Finally S.S submitted the manuscript.

Declarations

Competing interests

The authors declare no competing interests.

Additional information

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