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Classical and quantum superfield invariants in $\mathcal{N} = (1, 1), 6D$ SYM theory

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Abstract. We give an overview of recent results on the classical and quantum superfield invariants of $\mathcal{N} = (1, 1), 6D$ supersymmetric Yang-Mills theory in the off-shell $\mathcal{N} = (1, 0)$ and on-shell $\mathcal{N} = (1, 1), 6D$ harmonic superspaces.

1. Introduction

For the last few years, maximally extended supersymmetric gauge theories (with 16 supersymmetries) are under intensive study. These can be represented by the following chain

$$\mathcal{N} = 4, 4D \implies \mathcal{N} = (1, 1), 6D \implies \mathcal{N} = (1, 0), 10D.$$

Among them, the $\mathcal{N} = 4, 4D$ SYM (Super Yang-Mills) theory is most known. It is UV finite and perhaps completely integrable at the quantum level [1]. On the other hand, $\mathcal{N} = (1, 1), 6D$ SYM is not renormalizable by formal counting (the coupling constant is dimensionful) but is also expected to possess various unique properties. In particular, it respects the so-called “dual conformal symmetry” like its $4D$ counterpart [2]. It provides the effective theory descriptions of some particular low energy sectors of string theory, such as D5-brane dynamics. The full effective action of D5-brane is expected to be of non-abelian Born-Infeld type, generalizing the microscopic $\mathcal{N} = (1, 1)$ SYM action [3]. $\mathcal{N} = (1, 1)$ SYM is anomaly free [4], as distinct from $\mathcal{N} = (1, 0)$ SYM. The $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (1, 0)$ SYM theories are analogs of $\mathcal{N} = 8$ supergravity which is also formally non-renormalizable, so their study in the quantum domain can shed more light on the quantum properties of the latter theory.

Recent perturbative explicit calculations in $\mathcal{N} = (1, 1)$ SYM (treated as a low-energy limit of type II superstrings) show a lot of cancelations of the UV divergencies which cannot be expected in advance. The theory is UV-finite up to 2 loops, while at 3 loops only a single-trace counterterm of canonical dim 10 is required. The allowed double-trace counterterms do not appear [5] - [7]. To explain this, one needs new non-renormalization theorems. As usual, the maximally supersymmetric off-shell formulations are needed to clarify these issues.

However, maximum that one can achieve in $6D$ is off-shell $\mathcal{N} = (1, 0)$ supersymmetry¹. The most natural off-shell formulation of $\mathcal{N} = (1, 0)$ SYM is achieved in harmonic $\mathcal{N} = (1, 0), 6D$ superspace (HSS) [9, 10] as a generalization of $\mathcal{N} = 2, 4D$ HSS [11, 12]. In HSS, the $\mathcal{N} = (1, 1)$

¹ The maximal off-shell supersymmetry with 16 supercharges is claimed to be attainable in the “pure spinor” superfield formalism [8], but here we limit our attention to the standard superspaces.



SYM theory action can be schematically presented as a sum $[\mathcal{N} = (1, 1) \text{ SYM}] = [\mathcal{N} = (1, 0) \text{ SYM} + 6D \text{ hypermultiplet}]$, with the second hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetry.

In order to know the putative structure of the effective action and candidate counterterms for $\mathcal{N} = (1, 1)$ SYM theory, one needs to learn how to construct higher-dimension $\mathcal{N} = (1, 1)$ invariants in terms of $\mathcal{N} = (1, 0)$ superfields. In the “brute-force” method one starts with the appropriate $\mathcal{N} = (1, 0)$ SYM invariant and then completes it to $\mathcal{N} = (1, 1)$ invariant by adding the proper hypermultiplet terms. In practice, it is very cumbersome technically, though the situation is somewhat simplified by the fact that for finding all admissible superfield counterterms it is enough to stay *on shell*².

In [13] there was developed a new approach to constructing higher-dimension $\mathcal{N} = (1, 1)$ invariants, based on the concept of on-shell $\mathcal{N} = (1, 1)$ harmonic superspace [14].

The hidden supersymmetry tells us nothing about the precise coefficients before the $\mathcal{N} = (1, 1)$ invariants constructed in one or another way. To determine them, one should reproduce them from the superfield perturbation theory. The first steps towards this goal were recently undertaken in [15] - [17].

In the talk I will briefly address all these issues.

2. 6D superspaces and superfields

2.1. Basic superspaces

- Standard $\mathcal{N} = (1, 0)$, 6D superspace [18] is defined as the following set of coordinates:

$$z = (x^M, \theta_i^a), \quad M = 0, \dots, 5, \quad a = 1, \dots, 4, \quad i = 1, 2, \quad (1)$$

with Grassmann pseudoreal θ_i^a .

- Harmonic $\mathcal{N} = (1, 0)$, 6D superspace [9, 10] is obtained by adding $SU(2)$ harmonics to (1):

$$Z := (z, u) = (x^M, \theta_i^a, u^{\pm i}), \quad u_i^- = (u_i^+)^*, \quad u^{+i} u_i^- = 1, \quad u^{\pm i} \in SU(2)_R / U(1). \quad (2)$$

- *Analytic* $\mathcal{N} = (1, 0)$, 6D superspace has half the number of Grassmann coordinates as compared to (2):

$$\zeta := (x_{(\text{an})}^M, \theta^{+a}, u^{\pm i}) \subset Z, \quad x_{(\text{an})}^M = x^M + \frac{i}{2} \theta_k^a \gamma_{ab}^M \theta_l^b u^{+k} u^{-l}, \quad \theta^{\pm a} = \theta_i^a u^{\pm i}. \quad (3)$$

Basic differential operators in the analytic basis of 6D HSS are defined as :

$$\begin{aligned} D_a^+ &= \partial_{-a}, \quad D_a^- = -\partial_{+a} - 2i\theta^{-b} \partial_{ab}, \\ D^0 &= u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^{+a} \partial_{+a} - \theta^{-a} \partial_{-a} \\ D^{++} &= \partial^{++} + i\theta^{+a} \theta^{+b} \partial_{ab} + \theta^{+a} \partial_{-a}, \quad D^{--} = \partial^{--} + i\theta^{-a} \theta^{-b} \partial_{ab} + \theta^{-a} \partial_{+a}, \end{aligned} \quad (4)$$

where $\partial_{\pm a} \theta^{\pm b} = \delta_a^b$ and $\partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}$, $\partial^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}$.

2.2. Basic superfields

- The basic geometric object of $\mathcal{N} = (1, 0)$ SYM theory is the analytic gauge connection V^{++} :

$$\nabla^{++} = D^{++} + V^{++}, \quad \delta V^{++} = -\nabla^{++} \Lambda, \quad \Lambda = \Lambda(\zeta).$$

² To avoid a possible confusion, by this we merely mean that the on-shell vanishing counterterms can be absorbed into the microscopic action by a field redefinition. No equations of motion are actually assumed for the involved (super)fields.

- The second harmonic (non-analytic) connection V^{--} covariantizes the second harmonic derivative:

$$\nabla^{--} = D^{--} + V^{--}, \quad \delta V^{--} = -\nabla^{--} \Lambda.$$

It is related to V^{++} by the harmonic flatness condition

$$\begin{aligned} [\nabla^{++}, \nabla^{--}] &= D^0 \Rightarrow D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0 \\ \Rightarrow V^{--} &= V^{--}(V^{++}, u^\pm). \end{aligned}$$

One can make use of the analytic gauge freedom to choose the Wess-Zumino gauge:

$$V^{++} = \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^+)_a^3\lambda^{-a} - 3(\theta^+)^4\mathcal{D}^{--}.$$

Here A_{ab} is the gauge field, $\lambda^{-a} = \lambda^{ai}u_i^-$ is the gaugino and $\mathcal{D}^{--} = \mathcal{D}^{ik}u_i^-u_k^-$, where $\mathcal{D}^{ik} = \mathcal{D}^{ki}$, are the auxiliary fields. This is just the irreducible contents of the $\mathcal{N} = (1, 0)$ vector (gauge) multiplet.

- Having V^{--} , it is straightforward to define the covariant spinor and vector derivatives

$$\begin{aligned} \nabla_a^- &= [\nabla^{--}, D_a^+] = D_a^- + \mathcal{A}_a^-, \quad \nabla_{ab} = \frac{1}{2i}[D_a^+, \nabla_b^-] = \partial_{ab} + \mathcal{A}_{ab}, \\ \mathcal{A}_a^-(V) &= -D_a^+V^{--}, \quad \mathcal{A}_{ab}(V) = \frac{i}{2}D_a^+D_b^+V^{--}, \\ [\nabla^{++}, \nabla_a^-] &= D_a^+, \quad [\nabla^{++}, D_a^+] = [\nabla^{--}, \nabla_a^-] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0, \end{aligned}$$

and, next, the covariant superfield strengths

$$\begin{aligned} [D_a^+, \nabla_{bc}] &= \frac{i}{2}\varepsilon_{abcd}W^{+d}, \quad [\nabla_a^-, \nabla_{bc}] = \frac{i}{2}\varepsilon_{abcd}W^{-d}, \\ W^{+a} &= -\frac{1}{6}\varepsilon^{abcd}D_b^+D_c^+D_d^+V^{--}, \quad W^{-a} := \nabla^{--}W^{+a}, \\ \nabla^{++}W^{+a} &= \nabla^{--}W^{-a} = 0, \quad \nabla^{++}W^{-a} = W^{+a}, \\ D_b^+W^{+a} &= \delta_b^aF^{++}, \quad F^{++} = \frac{1}{4}D_a^+W^{+a} = (D^+)^4V^{--}, \\ \nabla^{++}F^{++} &= 0, \quad D_a^+F^{++} = 0. \end{aligned}$$

- The hypermultiplet superfield has off shell an infinite number of auxiliary fields coming from its expansion over harmonic variables

$$q^{+A}(\zeta) = q^{iA}(x)u_i^+ - \theta^{+a}\psi_a^A(x) + \text{An infinite tail of auxiliary fields}, \quad A = 1, 2.$$

2.3. $\mathcal{N} = (1, 0)$ superfield actions

The $\mathcal{N} = (1, 0)$ SYM action in 6D HSS was invented by Zupnik [10]:

$$\begin{aligned} S^{SYM} &= \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr} \int d^6x d^8\theta du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)}, \\ \delta S^{SYM} &= 0 \Rightarrow F^{++} = 0, \end{aligned}$$

where $1/(u_1^+ u_2^+), \dots$ are the harmonic distributions defined in [11, 12].

The hypermultiplet action (with q^{+A} in the adjoint representation of gauge group) reads

$$\begin{aligned} S^q &= -\frac{1}{2f^2} \text{Tr} \int d\zeta^{(-4)} q^{+A} \nabla^{++} q_A^+, \quad \nabla^{++} q_A^+ = D^{++} q_A^+ + [V^{++}, q_A^+], \\ \delta S^q &= 0 \Rightarrow \nabla^{++} q^{+A} = 0. \end{aligned}$$

The $\mathcal{N} = (1, 0)$ superfield form of the $\mathcal{N} = (1, 1)$ SYM action is just a sum of the two actions defined above:

$$S^{(V+q)} = S^{\text{SYM}} + S^q = \frac{1}{f^2} \left(\int dZ \mathcal{L}^{\text{SYM}} - \frac{1}{2} \text{Tr} \int d\zeta^{(-4)} q^{+A} \nabla^{++} q_A^+ \right),$$

$$\delta S^{(V+q)} = 0 \Rightarrow F^{++} + \frac{1}{2} [q^{+A}, q_A^+] = 0, \quad \nabla^{++} q^{+A} = 0.$$

It respects invariance under the second hidden $\mathcal{N} = (0, 1)$ supersymmetry:

$$\delta V^{++} = \epsilon^{+A} q_A^+, \quad \delta q^{+A} = -(D^+)^4 (\epsilon_A^- V^{--}), \quad \epsilon_A^\pm = \epsilon_{aA} \theta^{\pm a}.$$

3. $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ invariants of higher dimension

The problem was how to construct higher-dimension $\mathcal{N} = (1, 1)$ invariants from the $\mathcal{N} = (1, 0)$ gauge superfield strength W^{+a} and the hypermultiplet superfield q^{+A} . Firstly, it was solved by direct calculations.

- $d = 6$: In the pure SYM case the invariant of this dimension is uniquely constructed as [19]

$$S_{\text{SYM}}^{(6)} = \frac{1}{2g^2} \text{Tr} \int d\zeta^{(-4)} du \left(F^{++} \right)^2 \sim \text{Tr} \int d^6 x [(\nabla^M F_{ML})^2 + \dots]$$

Does its off-shell completion to an off-shell $\mathcal{N} = (1, 1)$ invariant exist? The answer is NO, only an expression whose $\mathcal{N} = (0, 1)$ variation vanishes *on-shell* can be found. It is unique up to two real parameters

$$\mathcal{L}^{d=6} = \frac{c_0}{2g^2} \text{Tr} \int du d\zeta^{(-4)} \left(F^{++} + \frac{1}{2} [q^{+A}, q_A^+] \right) \left(F^{++} + 2\beta [q^{+A}, q_A^+] \right)$$

But it vanishes on-shell by itself! Thus the non-vanishing on-shell counterterms of canonical dimension 6 are absent, and this proves the *one-loop finiteness* of $\mathcal{N} = (1, 1)$ SYM.

Recently, $d = 6$ counterterms were found by the explicit quantum calculations in $\mathcal{N} = (1, 0)$ superspace [15] - [17]. It was shown that they vanish *off-shell*, without any use of the equations of motion, just due to vanishing of the corresponding numerical coefficients!

- $d = 8$: All $\mathcal{N} = (1, 0)$ superfield terms of such dimension in the pure $\mathcal{N} = (1, 0)$ SYM theory prove to vanish on the gauge fields mass shell, in accord with the old statement of ref. [21]. Could adding the hypermultiplet terms somehow change this?

Our analysis showed that there exist NO $\mathcal{N} = (1, 0)$ supersymmetric off-shell invariants of the dimension 8 which would respect the on-shell $\mathcal{N} = (1, 1)$ invariance.

This means that $\mathcal{N} = (1, 1)$ SYM theory is at least *on-shell* finite at two loops. It is still an open question whether it is *off-shell* finite, i.e. whether the coefficients of the candidate counterterms are vanishing, like at one loop (now under examination).

Surprisingly, the $d = 8$ superfield expression which is non-vanishing on shell and respects an on-shell $\mathcal{N} = (1, 1)$ supersymmetry can be constructed by *giving up* the requirement of *off-shell* $\mathcal{N} = (1, 0)$ supersymmetry.

An example of such an invariant in $\mathcal{N} = (1, 0)$ SYM is very simple

$$\tilde{S}_1^{(8)} \sim \text{Tr} \int d\zeta^{(-4)} \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d}.$$

Since $D_a^+ W^{+b} = \delta_a^b F^{++}$, this derivative vanishes on shell, *i.e.* when $F^{++} = 0$. Thus, W^{+a} is an analytic superfield, when disregarding the terms proportional to the equations of motion, and the above action respects $\mathcal{N} = (1, 0)$ supersymmetry on shell. Also, a double-trace on-shell invariant exists. Both such on-shell invariants admit $\mathcal{N} = (1, 1)$ completions.

Though the nontrivial on-shell $\underline{d} = 8$ invariants exist, they cannot appear as counterterms for the $\mathcal{N} = (1, 1)$ SYM theory. The reason is that they do *not* possess the off-shell $\mathcal{N} = (1, 0)$ supersymmetry which the physically relevant counterterms should obey within the manifestly $\mathcal{N} = (1, 0)$ invariant supergraph technique. The non-existence of such counterterms agrees with the old component consideration of ref. [22].

Apart from the fact that such $\underline{d} = 8$ terms cannot appear as *counterterms* in $\mathcal{N} = (1, 1)$ SYM theory, they can appear, e.g., as *finite quantum corrections* to the effective Wilsonian action. For the pure $\mathcal{N} = (1, 0)$ SYM theory this was observed in [23, 24].

It was desirable to work out some simple and systematic way of constructing higher-order on-shell $\mathcal{N} = (1, 1)$ supersymmetric invariants. This becomes possible within the on-shell harmonic $\mathcal{N} = (1, 1)$ superspace.

4. $\mathcal{N} = (1, 1)$ on-shell harmonic superspace

The first step in constructing such a superspace is to extend $\mathcal{N} = (1, 0)$ superspace to $\mathcal{N} = (1, 1)$ one,

$$z = (x^{ab}, \theta_i^a) \Rightarrow \hat{z} = (x^{ab}, \theta_i^a, \hat{\theta}_a^A).$$

The double set of covariant spinor derivatives appears,

$$\nabla_a^i = \frac{\partial}{\partial \theta_i^a} - i\theta^{bi} \partial_{ab} + \mathcal{A}_a^i, \quad \hat{\nabla}^{aA} = \frac{\partial}{\partial \hat{\theta}_{Aa}} - i\hat{\theta}_b^A \partial^{ab} + \hat{\mathcal{A}}^{aA}.$$

The defining constraints of $\mathcal{N} = (1, 1)$ SYM in this extended superspace read [18, 21]:

$$\begin{aligned} \{\nabla_a^{(i}, \nabla_b^{j)}\} &= \{\hat{\nabla}^{a(A}, \hat{\nabla}^{bB)}\} = 0, \quad \{\nabla_a^i, \hat{\nabla}^{bA}\} = \delta_a^b \phi^{iA} \\ \Rightarrow \nabla_a^{(i} \phi^{j)A} &= \hat{\nabla}^{a(A} \phi^{B)i} = 0 \quad (\text{By Bianchis}). \end{aligned}$$

Next, we are led to define $\mathcal{N} = (1, 1)$ HSS with the double set of $SU(2)$ harmonics [14]:

$$Z = (x^{ab}, \theta_i^a, u_k^\pm) \Rightarrow \hat{Z} = (x^{ab}, \theta_i^a, \hat{\theta}_b^A, u_k^\pm, u_A^\pm)$$

Then we pass to the analytic basis and choose the “hatted” spinor derivatives short, $\nabla^{\hat{+}a} = D^{\hat{+}a} = \frac{\partial}{\partial \theta_a^+}$. The $\mathcal{N} = (1, 1)$ SYM constraints are rewritten in $\mathcal{N} = (1, 1)$ HSS as

$$\begin{aligned} \{\nabla_a^+, \nabla_b^+\} &= 0, \quad \{D^{\hat{+}a}, D^{\hat{+}b}\} = 0, \quad \{\nabla_a^+, D^{\hat{+}b}\} = \delta_a^b \phi^{++}, \\ [\nabla^{\hat{+}\hat{+}}, \nabla_a^+] &= 0, \quad [\tilde{\nabla}^{++}, \nabla_a^+] = 0, \quad [\nabla^{\hat{+}\hat{+}}, D^{\hat{+}a}] = 0, \quad [\tilde{\nabla}^{++}, D^{\hat{+}a}] = 0, \\ [\tilde{\nabla}^{++}, \nabla^{\hat{+}\hat{+}}] &= 0. \end{aligned}$$

Here

$$\begin{aligned} \nabla_a^+ &= D_a^+ + \mathcal{A}_a^+(\hat{Z}), \quad \tilde{\nabla}^{++} = D^{++} + \tilde{V}^{++}(\hat{\zeta}), \quad \nabla^{\hat{+}\hat{+}} = D^{\hat{+}\hat{+}} + V^{\hat{+}\hat{+}}(\hat{\zeta}), \\ \hat{\zeta} &= (x_{\text{an}}^{ab}, \theta^{\pm a}, \theta_c^\pm, u_i^\pm, u_A^\pm). \end{aligned}$$

The starting point of solving these constraints is to fix, using the $\Lambda(\hat{\zeta})$ gauge freedom, the WZ gauge for the second harmonic connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$

$$V^{\hat{+}\hat{+}} = i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\hat{\mathcal{A}}^{ab} + \varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\varphi_d^A u_A^{\hat{-}} + \varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}\mathcal{D}^{AB}u_A^{\hat{-}}u_B^{\hat{-}},$$

with $\hat{\mathcal{A}}^{ab}$, φ_d^A and $\mathcal{D}^{(AB)}$ being some $\mathcal{N} = (1, 0)$ harmonic superfields.

Then the above constraints are reduced to some harmonic equations which can be explicitly solved. The crucial point is the requirement that the vector $6D$ connections in the sectors of hatted and unhatted variables **are identical** to each other.

As the final result, we have obtained that the first harmonic connection V^{++} coincides precisely with the standard $\mathcal{N} = (1, 0)$ one, $V^{++} = V^{++}(\zeta)$, while the dependence of all other geometric $\mathcal{N} = (1, 1)$ objects on the variables with “hat” proves to be strictly fixed

$$\begin{aligned} V^{\hat{+}\hat{+}} &= i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\mathcal{A}^{ab} - \frac{1}{3}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}D_d^+ q^{-\hat{-}} + \frac{1}{8}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, q^{-\hat{-}}] \\ \phi^{+\hat{+}} &= q^{+\hat{+}} - \theta_a^{\hat{+}}W^{+a} - i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\nabla^{ab}q^{+\hat{-}} + \frac{1}{6}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}[D_d^+ q^{-\hat{-}}, q^{+\hat{-}}] \\ &\quad + \frac{1}{24}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, [q^{+\hat{-}}, q^{-\hat{-}}]]. \end{aligned}$$

Here, $q^{+\hat{\pm}} = q^{+A}(\zeta)u_A^{\hat{\pm}}$, $q^{-\hat{\pm}} = q^{-A}(\zeta)u_A^{\hat{\pm}}$ and W^{+a} , $q^{\pm A}$ are just the $\mathcal{N} = (1, 0)$ superfields used previously. In the process of solving the constraints, there appeared the analyticity conditions for q^{+A} , as well as the full set of the superfield equations of motion

$$\nabla^{++}q^{+A} = 0, \quad F^{++} = \frac{1}{4}D_a^+ W^{+a} = -\frac{1}{2}[q^{+A}, q_A^+].$$

The basic advantage of using the constrained $\mathcal{N} = (1, 1)$ strengths $\phi^{\pm\hat{+}}$ for constructing various invariants is their extremely simple transformation rules under the hidden $\mathcal{N} = (0, 1)$ supersymmetry

$$\delta\phi^{\pm\hat{+}} = -\epsilon_a^{\hat{+}}\frac{\partial}{\partial\theta_a^{\hat{+}}}\phi^{\pm\hat{+}} - 2i\epsilon_a^{\hat{-}}\theta_b^{\hat{+}}\partial^{ab}\phi^{\pm\hat{+}} - [\Lambda^{(comp)}, \phi^{\pm\hat{+}}],$$

where $\Lambda^{(comp)}$ is some common composite gauge parameter which does not contribute under Tr .

5. Invariants in $\mathcal{N} = (1, 1)$ superspace

The previous single-trace on-shell $\underline{d} = 8$ invariant admits a simple rewriting in $\mathcal{N} = (1, 1)$ superspace

$$S_{(1,1)} = \int dud\zeta^{(-4)}\mathcal{L}_{(1,1)}^{+4}, \quad \mathcal{L}_{(1,1)}^{+4} = -\text{Tr} \frac{1}{4} \int d\hat{\zeta}^{(-4)}d\hat{u}(\phi^{+\hat{+}})^4, \quad d\hat{\zeta}^{(-4)} \sim (D^{\hat{-}})^4$$

$$\delta\mathcal{L}_{(1,1)}^{+4} = -2i\partial^{ab}\text{Tr} \int d\hat{\zeta}^{(-4)}d\hat{u} \left[\epsilon_a^{\hat{-}}\theta_b^{\hat{+}}\frac{1}{4}(\phi^{+\hat{+}})^4 \right].$$

The double-trace $\underline{d} = 8$ invariant can also be easily constructed.

Now it is easy to construct the single- and double-trace $\underline{d} = 10$ invariants possibly responsible for the 3-loop counterterms

$$\begin{aligned} S_1^{(10)} &= \text{Tr} \int dZ d\hat{\zeta}^{(-4)}d\hat{u}(\phi^{+\hat{+}})^2(\phi^{-\hat{+}})^2, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}, \\ S_2^{(10)} &= - \int dZ d\hat{\zeta}^{(-4)}d\hat{u} \text{Tr}(\phi^{+\hat{+}}\phi^{-\hat{+}}) \text{Tr}(\phi^{+\hat{+}}\phi^{-\hat{+}}). \end{aligned}$$

These are $\mathcal{N} = (1, 1)$ extensions of the $\mathcal{N} = (1, 0)$ SYM invariants $\sim \varepsilon_{abcd} \text{Tr} (W^{+a} W^{-b} W^{+c} W^{-d})$, $\sim \varepsilon_{abcd} \text{Tr} (W^{+a} W^{-b}) \text{Tr} (W^{+c} W^{-d})$.

It is notable that the single-trace $\underline{d} = 10$ invariant admits a representation as an integral over the *full* $\mathcal{N} = (1, 1)$ superspace

$$S_1^{(10)} \sim \text{Tr} \int dZ d\hat{Z} \phi^{+\hat{+}} \phi^{-\hat{-}}, \quad \phi^{-\hat{-}} = \nabla^{\hat{-}\hat{-}} \phi^{-\hat{+}}.$$

On the other hand, the double-trace $\underline{d} = 10$ invariant *cannot* be written as the full integral and so it looks as being UV protected.

This could partly explain why in the perturbative calculations of the amplitudes in the $\mathcal{N} = (1, 1)$ SYM single-trace 3-loop divergence is seen, while no double-trace structures at the same order were observed [6, 7]

However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1, 0)$ supergraphs should generate invariants in the off-shell $\mathcal{N} = (1, 0)$ superspace, not in the on-shell $\mathcal{N} = (1, 1)$ superspace. So some additional piece of reasoning is needed to explain the absence of the double-trace divergences.

6. Quantum $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM

For calculating various $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ invariants, including counterterms, there was an urgent need to formulate self-consistent $\mathcal{N} = (1, 0)$ superfield perturbation techniques: superpropagators, background field method, etc. All that was recently given in a few papers by Buchbinder, Ivanov, Merzlikin and Stepanyantz, [15] - [17]. These methods were used to prove the one-loop **off-shell** finiteness of $\mathcal{N} = (1, 1)$ SYM theory formulated in terms of $\mathcal{N} = (1, 0)$ superfields.

The basic idea of the background field method is to split the relevant superfields into the sum of the “background” superfields V^{++}, Q^+ and the “quantum” ones v^{++}, q^+ ,

$$V^{++} \rightarrow V^{++} + f v^{++}, \quad q^+ \rightarrow Q^+ + q^+, \quad (5)$$

and then to expand the action in a power series in quantum fields.

By skipping details, the $\mathcal{N} = (1, 0)$, $6D$ SYM theory in the background field approach is described by the three quantum superfield ghosts: two fermionic Faddeev-Popov ghosts **b** and **c** together with the single bosonic Nielsen-Kallosh ghost φ , in addition to the quantum v^{++} and q^+ superfields. We started with the model in which hypermultiplet belongs to an arbitrary representation R of gauge group, not just to adjoint.

After integrating, in the functional integral, over quantum superfields, the following representation for the one-loop quantum correction to the classical action is obtained

$$\begin{aligned} \Gamma^{(1)}[V^{++}, Q] = & \frac{i}{2} \text{Tr} \ln \left\{ \widehat{\square}^{AB} - 2f^2 \tilde{Q}^{+m} (T^A G_{(1,1)} T^B)_m{}^n Q_n^+ \right\} - \frac{i}{2} \text{Tr} \ln \widehat{\square} \\ & - i \text{Tr} \ln (\nabla^{++})_{\text{Adj}}^2 + \frac{i}{2} \text{Tr} \ln (\nabla^{++})_{\text{Adj}}^2 + i \text{Tr} \ln \nabla_{\text{R}}^{++}, \end{aligned}$$

where subscripts Adj and R mean that the corresponding operators are taken in the adjoint representation and that of the hypermultiplet and

$$\widehat{\square} = \frac{1}{2} (D^+)^4 (\nabla^{--})^2$$

is the covariant Box operator.

The complete one-loop divergent part of the effective action reads

$$\Gamma_{div}^{(1)}[V^{++}, Q^+] = \frac{C_2 - \frac{d_R}{d_G} C_2(R)}{3(4\pi)^3 \varepsilon} \text{tr} \int d\zeta^{(-4)} du (F^{++})^2 - \frac{2if^2}{(4\pi)^3 \varepsilon} \int d\zeta^{(-4)} du \tilde{Q}^+ (C_2 - C_2(R)) F^{++} Q^+.$$

The coefficients of the $(F^{++})^2$ and $\tilde{Q}^+ F^{++} Q^+$ terms in the divergent part of one-loop effective action are proportional to the differences between the second order Casimir operator C_2 for adjoint representation of gauge group and the operators $T(R) = \frac{d_R}{d_G} C_2(R)$ and $C_2(R)$ for the hypermultiplet representation R , respectively.

Since $\mathcal{N} = (1, 1)$, $6D$ supersymmetric Yang-Mills theory involves the hypermultiplet in **adjoint** representation of gauge group, with $d_R = d_G$, $C_2(R) = C_2$, the divergent part vanishes for this case. Hence, the $\mathcal{N} = (1, 1)$ SYM theory is one-loop finite, and there is no need to use the equations of motion to prove this property.

For any other choice of the hypermultiplet irrep, it does not vanish even on shell, so in general the theory is divergent already at one loop. The pure $\mathcal{N} = (1, 0)$ SYM corresponds to $C_2(R) = 0$ and the one-loop divergency is vanishing on the equation of motion $F^{++} = 0$, in accord with the old result by Howe and Stelle [21].

7. Summary and outlook

- Off-shell $\mathcal{N} = (1, 0)$ and on-shell harmonic $\mathcal{N} = (1, 1)$, $6D$ superspaces can be efficiently used to construct higher-dimensional invariants in the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM theories.
- $\mathcal{N} = (1, 1)$ SYM constraints were solved in terms of harmonic $\mathcal{N} = (1, 0)$ superfields. This allowed to explicitly construct the full set of the superfield dimension $d = 8$ and $d = 10$ invariants with $\mathcal{N} = (1, 1)$ on-shell supersymmetry.
- All $\underline{d=6}$ $\mathcal{N} = (1, 1)$ invariants are at least on-shell vanishing, proving the UV finiteness of $\mathcal{N} = (1, 1)$ SYM at one loop.
- The off-shell $\underline{d=8}$ $\mathcal{N} = (1, 1)$ invariants are absent. Assuming that the $\mathcal{N} = (1, 0)$ supergraphs yield integrals over the full $\mathcal{N} = (1, 0)$ harmonic superspace, this means the absence of two-loop counterterms.
- Two $\underline{d=10}$ invariants were explicitly constructed as integrals over the whole $\mathcal{N} = (1, 0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over $\mathcal{N} = (1, 1)$ superspace, while the double-trace one cannot. This property combined with an additional reasoning could explain why the double-trace invariant is UV protected.
- The quantum techniques for $\mathcal{N} = (1, 0)$ SYM theory was worked out and used to show that $\mathcal{N} = (1, 1)$ SYM theory is one-loop finite off shell, without need in eqs. of motion.

7.1. Further lines of study

In conclusion, we outline some further possible lines of study:

(a) To construct the next $d \geq 12$ invariants in the $\mathcal{N} = (1, 1)$ SYM theory with the help of the on-shell $\mathcal{N} = (1, 1)$ harmonic superspace techniques (Buyukli & Ivanov, in preparation);

(b) To reproduce higher-dimensional invariants from the quantum superfield perturbation theory, to examine whether $\mathcal{N} = (1, 1)$ SYM theory is two-loop finite off shell (Buchbinder *et al*, in preparation);

(c) To work out the quantum superfield perturbation theory directly in $\mathcal{N} = (1, 1)$ double-harmonized superspace;

(d) To apply the same methods for constructing the Born-Infeld action with manifest off-shell $\mathcal{N} = (1, 0)$ and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetries. To check the hypothesis that such an action could be identified with the full quantum effective action of the $\mathcal{N} = (1, 1)$ SYM;

(e) Applications in supergravity?

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