IOP Conf. Series: Journal of Physics: Conf. Series **1275** (2019) 012062 doi:10.1088/1742-6596/1275/1/012062

Some algebraically solvable two-dimensional dynamical systems with polynomial interactions

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Abstract. We tersely review a recently introduced technique to identify systems of two nonlinearly-coupled Ordinary Differential Equations (ODEs) *solvable by algebraic operations*; and we report some specific examples of this kind, namely systems of 2 first-order ODEs with polynomial right-hand sides,

$$\dot{x}_n = P^{(n)}(x_1, x_2) \ , \quad n = 1, 2 \ ,$$

satisfied by the 2 (possibly *complex*) dependent variables $x_n \equiv x_n(t)$. Here $P^{(n)}(x_1, x_2)$ indicates some specific *polynomial*. These examples are analogous, but different, from those previously reported.

1. Introduction

A technique to identify dynamical systems characterized by systems of *nonlinearly-coupled* Ordinary Differential Equations (ODEs) *solvable by algebraic operations* is based on the relations among the N *coefficients* $y_m(t)$ and the N *zeros* $x_n(t)$ of a time-dependent (monic) polynomial of degree N in the (complex) variable z:

$$p_N(z;t) = z^N + \sum_{m=1}^{N} \left[y_m(t) \ z^{N-m} \right] = \prod_{n=1}^{N} \left[z - x_n(t) \right] \ . \tag{1}$$

The basic idea is to consider "simple" time evolutions of the N *coefficients* $y_m(t)$—possibly *explicitly solvable* evolutions, maybe featuring remarkable properties such as *isochrony* (implying that *all* the N coefficients $y_m(t)$ are periodic in t with the same fixed period independent of the initial data); and to then consider the corresponding time evolutions of the N zeros $x_n(t)$, which are generally characterized by "more nonlinear" equations of motions, the solutions of which are then obtainable via the *algebraic* operation of computing the N zeros $x_n(t)$ of the polynomial $p_N(z;t)$ defined in terms of its N *coefficients* $y_m(t)$. This generally entails that the time evolution of the *zeros* $x_n(t)$ *inherits* properties (such as *isochrony*) of the evolution of the coefficients $y_m(t)$. This idea has a long history [1], and it has produced many developments, see for instance [2] [3] [4] [5] [6] and references therein.

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An additional recent development has explored the modifications of the approach outlined above which emerge if, rather than considering *generic* time-dependent polynomials, one focusses on *special* time-dependent polynomials featuring (for all time) *multiple zeros*: in particular, as a first step in this direction, results have been reported for the case of a polynomial featuring *one double zero* [7], then some significant progress has been made on the case of a polynomial featuring *one zero of arbitrary multiplicity* [8], and finally the most general case has been treated of a polynomial $p_M(z;t)$ of degree M featuring an *arbitrary* number N of *zeros* each of them of *arbitrary multiplicity* μ_n [9]:

$$p_M(z;t) = z^N + \sum_{m=1}^{M} \left[y_m(t)\ z^{M-m} \right] = \prod_{n=1}^{N} \left\{ [z - x_n(t)]^{\mu_n} \right\} , \tag{2}$$

of course with the N positive integers μ_n related to the positive integer parameters M and N by the relation

$$M = \sum_{n=1}^{N} (\mu_n) . \tag{3}$$

A second twist of this development restricted attention to the $N = 2$ case—i. e., polynomials (see (2)) featuring only 2 *zeros* [10] [9]. This stringent limitation has opened the way to the identification of *algebraically solvable two-dimensional dynamical systems with polynomial interactions*, namely systems characterized by the following systems of 2 nonlinearly coupled ODEs:

$$\dot{x}_n = P^{(n)}(x_1, x_2) , \quad n = 1, 2 . \tag{4}$$

Notation 1.1. Here and hereafter superimposed dots denote differentiations with respect to the dependent variable t ("time"), $x_n \equiv x_n(t)$ are the 2 (generally *complex*) dependent variables, and $P^{(n)}(x_1, x_2)$ are generally 2 polynomials (or "almost polynomials": see below) in these 2 dependent variables x_1, x_2. Parameters such a, b, c, α, β, γ (possibly equipped with subscripts) are time-independent; they can be arbitrarily assigned (up to explicitly indicated relations among them). Note that often, above and hereafter, the t-dependence of variables is *not* explicitly indicated (when this is unlikely to generate any misunderstanding). □

A few such systems have been identified and tersely discussed in [10] by taking as point of departure the case of a polynomial such as (2) with $M = 3$, and several such cases have been treated in [9] by focussing on the case with $M = 4$. In the present paper—after a terse reminder of the methodology to obtain these results—we focus again on the $M = 3$ case, reporting several solvable models of type (4) not previously identified. Analogous treatments of cases with $M > 4$ shall be performed by ourselves or by others in the future.

In the following **Section 2** we tersely review our notation and some results which are basic for the following treatment. In **Section 3** and its subsections several specific examples are reported (with some related results confined to **Appendix A**). The last **Section 4** outlines possible future developments.

2. Preliminaries

A basic tool of our treatment are the following definitions of the 3 variables $y_m \equiv y_m(t)$, $m = 1, 2, 3$ in terms of the 2 variables $x_n \equiv x_n(t)$, $n = 1, 2$:

$$y_1 = -(2x_1 + x_2) , \quad y_2 = x_1(x_1 + 2x_2) , \quad y_3 = -(x_1)^2 x_2 ; \tag{5}$$

they of course imply (see (2)) that x_1 and x_2 are the 2 *zeros*—with respective multiplicities 2 and 1—of the (monic) third-degree polynomial $p_3(z;t)$ with coefficients y_1, y_2, y_3:

$$p_3(z;t) = z^3 + \sum_{m=1}^{3} \left[y_m(t)\ z^{3-m} \right] = [z - x_1(t)]^2\ [z - x_2(t)] . \tag{6}$$

Note that this entails that x_1 and x_2 can be obtained—in terms of y_1 and y_2, or y_1 and y_3, or y_2 and y_3—by solving the following algebraic equations:

$$3\left(x_1\right)^2+2y_1x_1+y_2=0\ ,\quad x_2=-\left(2x_1+y_1\right)\ , \tag{7}$$

or

$$2\left(x_1\right)^3+y_1\left(x_1\right)^2+y_3=0\ ,\quad x_2=-\left(2x_1+y_1\right)\ , \tag{8}$$

or

$$\left(x_1\right)^3-y_2x_1-2y_3=0\ ,\quad x_2=\frac{-\left(x_1\right)^2+y_2}{2x_1}\ . \tag{9}$$

It can moreover be easily shown—or see [7] or [9] or [10]—that these formulas imply the following differential relations:

$$\dot{x}_1=-\frac{2x_1\dot{y}_1+\dot{y}_2}{2\left(x_1-x_2\right)}\ ,\quad \dot{x}_2=\frac{\left(x_1+x_2\right)\dot{y}_1+\dot{y}_2}{x_1-x_2}\ , \tag{10}$$

$$\dot{x}_1=\frac{-\left(x_1\right)^2\dot{y}_1+\dot{y}_3}{2x_1\left(x_1-x_2\right)}\ ,\quad \dot{x}_2=\frac{x_1x_2\dot{y}_1-\dot{y}_3}{x_1\left(x_1-x_2\right)}\ , \tag{11}$$

$$\dot{x}_1=\frac{x_1\dot{y}_2+2\dot{y}_3}{2x_1\left(x_1-x_2\right)}\ ,\quad \dot{x}_2=-\frac{x_1x_2\dot{y}_2+\left(x_1+x_2\right)\dot{y}_3}{\left(x_1\right)^2\left(x_1-x_2\right)}\ . \tag{12}$$

It is plain from these formulas that if the two variables $y_1\equiv y_1\left(t\right)$ and $y_2\equiv y_2\left(t\right)$, or $y_1\equiv y_1\left(t\right)$ and $y_3\equiv y_3\left(t\right)$, or $y_2\equiv y_2\left(t\right)$ and $y_3\equiv y_3\left(t\right)$ satisfy an *algebraically solvable* system of 2 first-order ODEs, say

$$\dot{y}_{m_1}=f_{m_1}\left(y_{m_1},y_{m_2}\right)\ ,\quad \dot{y}_{m_2}=f_{m_2}\left(y_{m_1},y_{m_2}\right) \tag{13}$$

with $m_1=1$, $m_2=2$ or $m_1=1$, $m_2=3$ or $m_1=2$, $m_2=3$, then the corresponding system of 2—generally nonlinearly-coupled, first-order—ODEs satisfied by $x_1\equiv x_1\left(t\right)$ and $x_2\equiv x_2\left(t\right)$ is as well *algebraically solvable*. Moreover, if the functions $f_{m_1}\left(y_{m_1},y_{m_2}\right)$ and $f_{m_2}\left(y_{m_1},y_{m_2}\right)$ are *appropriately chosen*, then the right-hand sides of the ODEs (10), (11), (12) become *polynomial* (or "*almost polynomial*": see below). In the following **Section 3** we report several *algebraically solvable* systems satisfied by the 2 dependent variables $x_1\equiv x_1\left(t\right)$ and $x_2\equiv x_2\left(t\right)$ obtained in this manner, i. e. by first replacing in the relevant equations (10), (11) and (12), the expressions of the time-derivatives of the relevant 2 variables y_m via (13) and subsequently replacing the expressions of these 2 variables y_m in terms of the 2 variables x_n via the relevant equations (5); of course with the functions $f_{m_1}\left(y_{m_1},y_{m_2}\right)$ and $f_{m_2}\left(y_{m_1},y_{m_2}\right)$ *appropriately chosen polynomials*(see below).

3. Results

Our first step is to identify 3 *solvable* systems of 2 evolution equations satisfied by the dependent variables $y_{m_1}\equiv y_{m_1}\left(t\right)$ and $y_{m_2}\equiv y_{m_2}\left(t\right)$. Their solvability is discussed in **Appendix A** of Ref. [10] and tersely reviewed below in **Appendix A**.

The first of these 3 systems—hereafter identified as **A1** (see **Appendix A**)—is characterized by the following 2 *uncoupled* ODEs:

$$\dot{y}_{m_1}=\sum_{\ell=0}^{L}\left[\alpha_\ell\left(y_{m_1}\right)^{\ell m_2+1}\right]\ ,\quad \dot{y}_{m_2}=\sum_{\ell=0}^{L}\left[\beta_\ell\left(y_{m_2}\right)^{\ell m_1+1}\right]\ . \tag{14}$$

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The second of these 3 systems—hereafter identified as **A2** (see **Appendix A**)—is characterized by the following 2 *coupled* ODEs:

$$\dot{y}_{m_1} = \sum_{\ell=0}^{L} \left[\alpha_\ell \left(y_{m_1}\right)^{\ell}\right] \ , \quad \dot{y}_{m_2} = y_{m_2} \sum_{\ell=1}^{L} \left[\beta_\ell \left(y_{m_1}\right)^{\ell-1}\right] + \sum_{\ell=0}^{L} \left[\gamma_\ell \left(y_{m_1}\right)^{\ell-1+(m_2/m_1)}\right] \ . \quad (15)$$

The third of these 3 systems—hereafter identified as **A3** (see **Appendix A**)—is characterized by the following 2 coupled ODEs:

$$\dot{y}_{m_1} = \alpha_0 + \alpha_1 y_{m_2} \ , \quad \dot{y}_{m_2} = \beta_0 \left(y_{m_1}\right)^{-1+m} + \beta_1 \left(y_{m_1}\right)^{-1+2m} \ ,$$
$$m = m_2/m_1 \ , \quad m_1 = 1 \ , \quad m_2 = 2, 3 \ . \quad (16)$$

Our next step is to list in the following 11 subsections 11 *algebraically solvable* systems of 2 nonlinearly-coupled ODEs (4) with *polynomial* (or "*almost polynomial*": see below) right-hand sides; in each case we identify the corresponding *appropriately chosen algebraically solvable* system of ODEs—see above and **Appendix A**—satisfied by the corresponding functions $y_{m_1}(t)$ and $y_{m_2}(t)$. But note that the algebraically solvable systems (4) thus identified are only 9, because 2 pairs of the systems identified below—although arrived at differently—are in fact *identical* (a phenomenon already noted in [9]).

3.1. Models of type ***A1***

In the 3 cases listed in this **Subsection** 3.1 the variables $y_{m_1}(t)$ and $y_{m_2}(t)$ are supposed to satisfy the system **A1** of 2 uncoupled ODEs (see (14) and **Appendix A**), with the indicated assignments of the various parameters.

Model A1.1: $m_1 = 1, \ m_2 = 2; \ L = 1; \ \alpha_0 = a, \ \alpha_1 = b; \ \beta_0 = 2a, \ \beta_1 = 6b;$

$$\dot{x}_1 = a x_1 + b x_1 \left[5 \left(x_1\right)^2 + 5 x_1 x_2 - \left(x_2\right)^2\right] \ ,$$
$$\dot{x}_2 = a x_2 - b \left[2 \left(x_1\right)^3 - 2 \left(x_1\right)^2 x_2 - 8 x_1 \left(x_2\right)^2 - \left(x_2\right)^3\right] \ . \quad (17)$$

Model A1.2: $m_1 = 1, \ m_2 = 3; \ L = 1; \ \alpha_0 = a, \ \alpha_1 = -2b; \ \beta_0 = 3a, \ \beta_1 = -162b;$

$$\dot{x}_1 = x_1 \left\{a + b \left[16 \left(x_1\right)^3 + 48 \left(x_1\right)^2 x_2 - 9 x_1 \left(x_2\right)^2 - \left(x_2\right)^3\right]\right\} \ ,$$
$$\dot{x}_2 = x_2 \left\{a - 2b \left[16 \left(x_1\right)^3 - 33 \left(x_1\right)^2 x_2 - 9 x_1 \left(x_2\right)^2 - \left(x_2\right)^3\right]\right\} \ . \quad (18)$$

Model A1.3: $m_1 = 2, \ m_2 = 3; \ L = 1; \ \alpha_0 = 2a, \ \alpha_1 = 2b; \ \beta_0 = 3a, \ \beta_1 = 81b;$

$$\dot{x}_1 = x_1 \left\{a + b \left(x_1\right)^3 \left[\left(x_1\right)^3 + 9 \left(x_1\right)^2 x_2 + 33 x_1 \left(x_2\right)^2 - 16 \left(x_2\right)^3\right]\right\} \ ,$$
$$\dot{x}_2 = x_2 \left\{a - b \left(x_1\right)^3 \left[2 \left(x_1\right)^3 + 18 \left(x_1\right)^2 x_2 - 15 x_1 \left(x_2\right)^2 - 32 \left(x_2\right)^3\right]\right\} \ . \quad (19)$$

3.2. Models of type ***A2***

In the 6 cases listed in this **Subsection** 3.2 the variables $y_{m_1}(t)$ and $y_{m_2}(t)$ are supposed to satisfy the system **A2** of 2 coupled ODEs (see (15) and **Appendix A**), with the indicated assignments of the various parameters.

Model A2.1: $m_1 = 1, \ m_2 = 2; \ L = 3; \ \alpha_0 = 3b_0, \ \alpha_1 = a_0 - 3b_1, \ \alpha_2 = -a_1 + 3b_2, \ \alpha_3 = a_2 - 3b_3;$ $\beta_\ell = (-1)^{\ell-1} 2a_{\ell-1}, \ \ell = 1, 2, 3; \ \gamma_\ell = (-1)^{\ell} 2b_\ell, \ \ell = 0, 1, 2, 3;$

$$\dot{x}_n = x_n \left(a_0 + a_1 X + a_2 X^2\right) - \left[b_0 + b_1 X + b_2 X^2 + b_3 X^3\right] \ ,$$
$$n = 1, 2; \ X = 2 x_1 + x_2 \ . \quad (20)$$

Model A2.2: $m_1 = 1,\ m_2 = 2;\ L = 3;\ \alpha_0 = 0,\ \alpha_\ell = c_\ell;\ \beta_\ell = 2c_\ell;\ \ell = 1, 2, 3$;

$$\dot{x}_n = x_n \left(c_1 + c_2 X + c_3 X^2 \right) \ , \quad n = 1, 2, \quad X = x_1 \left(x_1 + 2x_2 \right) \ . \tag{21}$$

Model A2.3: $m_1 = 1,\ m_2 = 3;\ L = 3;\ \alpha_0 = 9b_0,\ \alpha_\ell = (-1)^{\ell-1} (a_0 - 9b_\ell),\ \beta_\ell = (-1)^{\ell-1} 3a_{\ell-1},\ \ell = 1, 2, 3;\ \gamma_\ell = (-1)^\ell b_\ell,\ \ell = 0, 1, 2, 3.$

$$\begin{aligned}
\dot{x}_1 &= x_1 \left(a_0 + a_1 X + a_2 X^2 \right) - \left(\frac{5x_1 + x_2}{2x_1} \right) \left(b_0 + b_1 X + b_2 X^2 + +b_3 X^3 \right) \ , \\
\dot{x}_2 &= x_2 \left(a_0 + a_1 X + a_2 X^2 \right) - \left(\frac{4x_1 - x_2}{x_1} \right) \left(b_0 + b_1 X + b_2 X^2 + +b_3 X^3 \right) \ , \\
X &= 2x_1 + x_2 \ .
\end{aligned} \tag{22}$$

Note that the right hand sides of these ODEs (22) are *polynomial* only if *all* the coefficients b_ℓ vanish.

Model A2.4: $m_1 = 3,\ m_2 = 1;\ L = 3;\ \alpha_0 = 0,\ \alpha_\ell = (-1)^{\ell-1} 3c_\ell,\ \beta_\ell = (-1)^{\ell-1} c_{\ell-1},\ \ell = 1, 2, 3;\ \gamma_\ell = 0,\ \ell = 0, 1, 2, 3.$

$$\dot{x}_n = x_n \left(c_0 + c_1 X + c_2 X^2 \right) \ , \quad n = 1, 2, \quad X = (x_1)^2 x_2 \ . \tag{23}$$

Model A2.5: $m_1 = 2,\ m_2 = 3;\ L = 3;\ \alpha_0 = 0,\ \alpha_\ell = 2c_{\ell-1},\ \beta_\ell = 3c_{\ell-1},\ \ell = 1, 2, 3;\ \gamma_\ell = 0,\ \ell = 0, 1, 2, 3.$

$$\dot{x}_n = x_n \left(c_0 + c_1 X + c_2 X^2 \right) \ , \quad n = 1, 2, \quad X = x_1 \left(x_1 + 2x_2 \right) \ . \tag{24}$$

Model A2.6: $m_1 = 3,\ m_2 = 2;\ L = 3;\ \alpha_0 = 0,\ \alpha_\ell = (-1)^{\ell-1} 3c_{\ell-1},\ \beta_\ell = (-1)^{\ell-1} 2c_{\ell-1},\ \ell = 1, 2, 3;\ \gamma_\ell = 0,\ \ell = 0, 1, 2, 3.$

$$\dot{x}_n = x_n \left(c_0 + c_1 X + c_2 X^2 \right) \ , \quad n = 1, 2, \quad X = (x_1)^2 x_2 \ . \tag{25}$$

Remark 3.2-1. Note that the systems of ODEs (21) and (24) are identical, and likewise the systems of ODEs (23) and (25) are identical. □

3.3. Models of type ***A3***

In the 2 cases listed in this **Subsection** the variables $y_{m_1}(t)$ and $y_{m_2}(t)$ are supposed to satisfy the system **A3** of 2 coupled ODEs (see (16) and **Appendix A**), with the indicated assignments of the various parameters.

Model A3.1: $m_1 = 1,\ m_2 = 2;\ \alpha_0 = -3a,\ \alpha_1 = -9b;\ \beta_0 = -2a,\ \beta_1 = -2b;$

$$\begin{aligned}
\dot{x}_1 &= a + b \left[(x_1)^2 + 7x_1 x_2 + (x_2)^2 \right] \ , \\
\dot{x}_2 &= a + b \left[7 (x_1)^2 + 4x_1 x_2 - 2 (x_2)^2 \right] \ .
\end{aligned} \tag{26}$$

Model A3.2: $m_1 = 1,\ m_2 = 3;\ \alpha_0 = -18a,\ \alpha_1 = -486b;\ \beta_0 = -2a,\ \beta_1 = -2b;$

$$\begin{aligned}
\dot{x}_1 = {} & (x_1)^{-1} \{ a (5x_1 + x_2) \\
& + b \left[32 (x_1)^4 - 131 (x_1)^3 x_2 - 51 (x_1 x_2)^2 - 11 x_1 (x_2)^3 - (x_2)^4 \right] \} , \\
\dot{x}_2 = {} & (x_1)^{-1} \{ 2a (4x_1 - x_2) \\
& - 2b \left[32 (x_1)^4 + 112 (x_1)^3 x_2 - 51 (x_1 x_2)^2 - 11 x_1 (x_2)^3 - (x_2)^4 \right] \} .
\end{aligned} \tag{27}$$

4. Outlook

Space limitations do not allow to report here some natural developments of the results reported above, such as the identification of the *more general solvable* systems which obtain from those reported in the preceding **Section 3** via *linear* transformations of the dependent variables $x_n(t)$, say

$$x_1(t) = A_1 + A_{11} z_1(t) + A_{12} z_2(t) \ , \quad x_2(t) = A_2 + A_{21} z_1(t) + A_{22} z_2(t) \ , \tag{28}$$

with the 6 parameters $A_1, \ A_2, \ A_{11}, \ A_{12} \ , \ A_{21}, \ A_{22}$ *arbitrarily assigned*: the interested reader is referred to the analogous developments discussed in **Section 5** of Ref. [9].

And of course the investigation in various fields of applied mathematics of the *algebraically solvable* systems of 2 nonlinearly-coupled ODEs reported above is an interesting open task.

Acknowledgments

FP likes to thank the Physics Department of the University of Rome "La Sapienza" for the hospitality from April 2018 to January 2019 (during her sabbatical, when the results reported in this paper were obtained), and the organizers of DICE2018 (where these results were reported by her), especially Prof. Hans Thomas Elze.

Appendix A.

The 2 ODEs of the system **A1** (see (14)) are uncoupled, and each of them is clearly solvable by quadratures. Note in particular that the initial-value problem of the ODE

$$\dot{y} = ay + by^{m+1} \tag{A.1}$$

is given by the *explicit* formula

$$y(t) = y(0) \exp(at) \{1 + (b/a) [y(0)]^m [1 - \exp(mat)]\}^{-1} \ . \tag{A.2}$$

To solve the system **A2** one treats firstly the first of the 2 ODEs (15)—essentially as just above. Then one notes—see **Appendix A** of Ref. [10]—that the solution of the second ODE of the system (15) reads

$$y_{m_2}(t) = F(t) \left[y_{m_2}(0) + \int_0^t dt' \ \ [F(t')]^{-1} \sum_{\ell=0}^{L} \left\{ \gamma_\ell \, [y_{m_1}(t')]^{\ell-1+(m_2/m_1)} \right\} \right] \tag{A.3}$$

with

$$F(t) = \exp \left\{ \int_0^t \left[dt' \sum_{\ell=1}^{L} \left\{ \beta_\ell \, [y_{m_1}(t')]^{\ell-1} \right\} \right] \right\} \ . \tag{A.4}$$

Explicit solutions can be easily obtained in the following cases:

$$m_1 = 1 \ , \quad m_2 = 2, 3 \ , \quad m = m_2/m_1 \ , \tag{A.5}$$

$$\dot{y}_{m_1} = \alpha_0 + \alpha_1 y_{m_1} + \alpha_2 (y_{m_1})^2 \ , \tag{A.6}$$

$$\dot{y}_{m_2} = y_{m_2} (\beta_0 + \beta_1 y_{m_1}) + \gamma_0 (y_{m_1})^{-1+m} + \gamma_1 (y_{m_1})^m + \gamma_2 (y_{m_1})^{1+m} \ ; \tag{A.7}$$

$$\begin{aligned} y_{m_1}(t) &= \frac{y_{m_1}(0) [1 + (\Delta/\alpha_1) \tanh(\Delta t)] - 2 (\alpha_0/\alpha_1) \tanh(\Delta t)}{1 - \left\{ \left[2\alpha_2 y_{m_1}(0) + \Delta \right] / \alpha_1 \right\} \tanh(\Delta t)} \ , \\ \Delta^2 &= (\alpha_1)^2 - 4\alpha_0\alpha_2 \ , \end{aligned} \tag{A.8}$$

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$$y_{m_2}(t) = f(t)\left[y_{m_2}(0) + \int_0^t dt' \ \left[f(t')\right]^{-1} \sum_{\ell=0}^{L} \left\{\gamma_\ell \left[y_{m_1}(t')\right]^{\ell-1+m}\right\}\right] , \tag{A.9}$$

$$f(t) = \exp\left\{\int_0^t \left[dt' \sum_{\ell=1}^{L} \left\{\beta_\ell \left[y_{\tilde{m}_1}(t')\right]^{\ell-1}\right\}\right]\right\} . \tag{A.10}$$

Finally, to identify the solution of the system **A3** (see (16)) we note that time-differentiation of the first of its 2 ODEs entails that $y_{m_1}(t)$ satisfies the decoupled second-order ODE

$$\ddot{y}_{m_1} = \alpha_1 \left[\beta_0 \left(y_{m_1}\right)^{-1+m} + \beta_1 \left(y_{m_1}\right)^{-1+2m}\right] , \quad m = m_2/m_1 ; \tag{A.11}$$

hence (see (16)) $y_1(t)$ is an elliptic (for $m = 2$) or hyperelliptic (for $m = 3$) function; and likewise for $y_2(t)$ (see the first of the 2 ODEs (16)).

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