The Application of Differential Geometry to Classical and Quantum Gravity

Clive Gene Wells
University of Cambridge

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Churchill College

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TO

MY PARENTS

PREFACE

All of the work described in this dissertation is believed to be original except where explicit reference is made to other authors. No part of this dissertation has been, or is being, submitted for any other qualification other than the degree of Doctor of Philosophy in the University of Cambridge.

Some of the work included in this dissertation is the result of collaborations with my supervisor, Prof. Gary Gibbons, with Dr Chris Fewster and with Dr Chris Fewster and Dr Atsushi Higuchi at the University of York. Specifically Chapters 6 and 7 arose from work done in collaboration with Prof. Gary Gibbons and has been published or submitted for publication, [1, 2]. Chapter 8 is result of collaboration with Dr Chris Fewster and has been published [3]. Chapter 9 arose from work done in collaboration with Chris Fewster and Dr Atsushi Higuchi, [4]. The sections detailing the treatment of the classical limit are an exception and are likely to be published separately. Any errors contained in the sections detailed above are entirely my own responsibility.

I would like to take the opportunity at this point to offer my thanks and appreciation to those people who have been influential in the writing of this dissertation. In particular it is a pleasure to thank my PhD supervisor, Prof. Gary Gibbons for all his help and a huge number of discussions we have had on this field of study.

Secondly I would like to thank Dr Chris Fewster who first introduced me to the study of chronology violating spacetimes and has been inspirational to work with, both as a fellow student and after he had graduated.

Signed.. Dated..

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1. INTRODUCTION

There are few concepts in mathematics and theoretical physics more important than that of symmetry. For hundreds of years Galilean, or more recently Lorentzian, symmetry has been a cornerstone in the development of our understanding of nature. Physicists have frequently found that it is by understanding the symmetries that leave a physical system invariant that allow them to make sensible hypotheses about its underlying structure. Within the last century, symmetries, as exemplified by group theory, have been an indispensable tool for the study of high energy particle physics as well as solid state physics, crystallography and quantum chemistry to name only a few areas of current scientific research.

Within the context of modern metric theories of gravitation, continuous symmetries of spacetime may be elucidated by searching for Killing vectors. When we postulate the existence of a spacetime Killing vector field the equations of motion themselves have added symmetry. These are hidden, or internal, symmetries that can be made manifest by the process of dimensional reduction; that is by regarding some of the metric components as describing new matter content on an effective spacetime of a lower dimension. From a purely mathematical viewpoint this procedure has great elegance giving rise as it does to a harmonic mapping system. The effective matter content may then be interpreted in terms of coordinates on another manifold; the target space of the harmonic map. For pure gravity, Einstein-Maxwell theory and the truncation of Superstring theory we will be discussing, these target manifolds have constant negative curvature. By examining more closely these target spaces we can discover their symmetries. Understanding and parameterizing these symmetries in turn leads us to a reinterpretation of the effective matter content in our theory. We are thereby led to the powerful method of exact solution generating techniques. Given such a highly non-linear set of differential equations it seems at first quite remarkable that a few simple transformations involving no more than simple algebraic manipulations can give us new and complicated solutions from old.

The idea of dimensional reduction is not a new one. Kaluza-Klein theory attempts to model the four-dimensional spacetime of everyday experience as the dimensional reduction of a spacetime in five, or sometimes more, dimensions. The theory unifies gravitation and electromagnetism, at least within a classical context, but requires the existence of a new field which we call the dilaton. The unification of electromagnetism and gravity isn't quite what we expect from our understanding of the physics in the laboratory insofar as there is a very special form of coupling between the dilaton and the electromagnetic field. One may view the coupling as giving rise to an effective polarization of the vacuum. That is to say we should distinguish between the electric field and the electric displacement and between the magnetic field and the magnetic induction. The effective permittivities and permeabilities are such that the speed of light that one would measure remains unaltered.

Kaluza-Klein theory is not alone in exhibiting this sort of behaviour. String theory too includes such a dilaton field. A problem immediately arises if we postulate the existence of extra dimensions, namely that we do not observe them. The accepted solution to this puzzle is to assume the extra dimensions are compactified, usually as a circle of very small radius. When dimensionally reducing from more than five dimensions more possibilities are available in the compactification procedure. Within the context of String theory there is a special radius, related to the string tension that will allow for enhanced symmetry; it is in this way that non-abelian gauge symmetries can be incorporated within a quantum theory coupled with dimensional reduction.

When looking for a quantum theory of gravity most of the usual methods for quantization are fraught with conceptual difficulties. One reason for this arises because of the special rôle time, or more generally the causal structure, is given in the quantization procedure. Fortunately the path integral formulation is free from most of these problems, though it is not without its own difficulties. Postulating a path integral, or sum over histories formulation to quantum gravity is useful in that some ideas can be tested without knowing the details of the full quantum theory. In particular certain semi-classical processes can be investigated and provide hints on what is likely and unlikely to be a feature of the full theory.

One area where there has been some debate is whether topology change is an essential ingredient in a quantum theory of gravitation. Within the context of a path integral formulation one might suppose that one sums over all metrics with a given fixed topology. Others argue that it is metrics plus topologies, what we might call geometries, that one should sum over. To try to resolve this question we can examine semi-classical processes and see if we can find any that represent topology change. A semi-classical process exhibiting such a phenomenon would surely imply that the full quantum theory must allow for it and that the supposition of a path integral summation over only metrics was doomed. Perhaps the most remarkable example (for most of us) of topology change is the idea that the universe as whole tunnelled from nothing.

On a slightly less grand scale, there seem to be solutions in the semi-classical theory that represent the pair production of black hole monopoles and anti-monopoles. To get an idea of the rate of production of these monopoles we need to use something like a saddle point approximation in the path integral. It is necessary therefore to know that there are no other saddle points nearby. More formally we want to know about the uniqueness of the instanton representing the pair creation process.

The black hole uniqueness theorem for the Kerr-Newman solution remained an unsolved problem for the best part of a decade until it was completed in the early 1980s when Bunting and Mazur exploited the special features present on the target space of the appropriate harmonic mapping problem. It was only through an understanding the symmetry that progress could be made. The analogous vacuum Einstein theory problem had been solved by a remarkable piece of trial and error manipulation of the field equations by Robinson. The establishing of such a suitable divergence identity from the field equations to prove a uniqueness result will be familiar to anyone who understands the proof of the uniqueness of solutions to Laplace's equation. We will be presenting a new proof of the black hole divergence identity which exploits a complex manifold construction of the effective electromagnetic Lagrangian. The advantage of the new proof is that it neatly fits in with the exact solution generating techniques that we will also be discussing.

The problem of proving the uniqueness of the instanton amounts to proving the uniqueness of a class of accelerating black holes confined inside a Faraday flux tube. This is by no means a simple task. Not only is the solution is rather complicated, but more importantly it has a different overall topology to that of the Kerr-Newman black hole. The presence of an acceleration horizon complicates matters considerably, effectively introducing a new boundary into the harmonic mapping problem. The domain on which the harmonic mapping problem is played out is rectangular. The boundary conditions on the horizons we need to impose are not very stringent, however the analysis is hampered by the fact that two rectangles are in general not conformally homeomorphic, and thus we cannot compare two candidate solutions that are defined on conformally inequivalent domains. We may contrast this situation with the analysis of the uniqueness of the Kerr-Newman black hole. The two dimensional domain we consider in that proof is a semi-infinite rectangle. By scaling and translation any two such domains may be made to coincide.

The trigonometric functions are invariant under a translation by 2π, they therefore exhibit a discrete translational symmetry. Considerably more such symmetry is possible. Amongst the meromorphic functions there are non-constant functions that are invariant under translation by elements of a lattice. These are the elliptic functions. The fundamental period parallelogram may be chosen in many different ways. One special case is when we have a rectangular lattice so that the functions are naturally defined on a rectangle in the complex plane. These maps are appropriate for the analysis of the accelerating black hole uniqueness problem. Elliptic function theory is perhaps not as well known as it might be to the physics community though from a mathematical point of view it has many remarkable and fascinating results.

Hitherto the black hole uniqueness theorems have not allowed for the presence of an acceleration horizon, in extending them in this direction we have made a significant extension. So too is the extension to the Superstring, or $N = 4$ Supergravity theories. By paying careful attention to the internal symmetries present when we require the solution to be static we can develop a formalism to prove black hole uniqueness results in these theories. String theories are one of the most favoured routes to a theory of quantum gravity. The low energy effective gravitational theory can be found and its black hole solutions shed new light on possible non-perturbative aspects of these theories.

There exist analogous instanton solutions representing the black hole monopole-anti-monopole pair creation in these theories. As in the Einstein-Maxwell theory the uniqueness of these instantons is an important question. We can use the remarkable fact that for a suitable truncation of these theories the appropriate harmonic mapping problem is identical to two copies of that which we find for vacuum General Relativity and the all-important divergence identity is readily at hand. We need to marry both new extensions to the uniqueness theorems to prove the uniqueness of these instantons which have surprisingly until now been overlooked.

It is by no means the case that the uniqueness theorem formalism developed for the Superstring theory is only applicable to accelerating black holes in electromagnetic flux tubes. The theory has many applications though as yet rotating solutions fall beyond its scope, as do the theories when incorporating an axionic field. Incorporating these would involve establishing a suitable divergence identity of which we have only found a special case. The positivity of the divergence result might be problematical to establish, as the symmetric space has a signature with more than one timelike direction. Both the Mazur method and the method given in the text rely on a single timelike direction. In contrast Bunting's result may well be the best method to establish such a result, relying as it does much more on the negative curvature of the target space, than on the exact details of the sigma-model. The uniqueness theorems we shall prove are important in their own right but may also be regarded as illustrative of how we may use the theory to prove uniqueness theorems subject to different asymptotic conditions.

Supersymmetry is another symmetry that is favoured by theoretical physicists. In such theories the extremal black holes solutions are typically supersymmetric. These satisfy the appropriate Bogomol'nyi bound and a state of anti-gravity can exist, that is the attractive forces due to the even spin fields are held in equipoise with the repulsion due to the odd spin ones. The basic result is that well-behaved solutions in such theories should be attractive at large distances. This is related to the idea that a congruence of null geodesics passing at large distance from a source should converge due to the gravitation influence. Using this focussing property, Penrose, Sorkin and Woolgar have proved a version of the positive mass theorem. It is fundamentally different from the existing proofs, and by mirroring the proof of their result in an auxiliary five dimensional spacetime we can establish new anti-gravity bounds on the four-dimensional solutions of dilaton gravity.

The special coupling the dilaton has to the electromagnetic field in such theories may be interpreted in terms of an effective polarization of the vacuum. A simplified model of this system is to neglect the gravitational effects and turn to the flat space version of this theory. In such a way we can isolate the salient points of this aspect of the gravitational theory, without needing to solve the full set of equations. In this limit the dilaton coupling constant may be rescaled to unity. The cosmological defects in the theory are more easily found than in the gravitational version and in particular cosmic string, domain wall and Dirac monopole solutions may be found. The cosmic string solutions are particular interesting in Dilation Electrodynamics. It turns out that magnetic flux may become confined inside regions where the effective permeability is large compared to infinity. When one repeats the calculation for a massive dilaton one finds that there are solutions in which the flux becomes unconfined. In this way we can speculate a natural solution to the monopole problem. Monopoles typically terminating cosmic string-like solutions would be rapidly accelerated towards each other and annihilate in a phase where the dilaton were massless. The breaking of supersymmetry is widely believed by string theorists to lead to a massive dilaton. In this phase the magnetic flux may become unconfined and leave no apparent defect to the future observer.

The stationary rotating black hole solutions in Einstein-Maxwell theory are described by the Kerr-Newman Solution. In this solution the singularity is a ring within the event horizon. Also within the horizon there is a region where Closed Timelike Curves (CTC's) are present. This need not cause much concern as the exterior region is causally disconnected from what happens inside the black hole. However the possibility of CTC's present within a region accessible to experiment has prompted some discussion recently. One might hope that the concept of causality comes from an analysis of the physics, rather than having it as an added hypothesis. It is therefore worth studying spacetimes where chronology violation may occur, and see to what extent physics throws up obstacles to their existence. It is conceivable that at a classical level that self-consistent histories be allowed. However there is a problem that the initial value problem in such spacetimes is ill-posed in general, and there may be many classical solutions for a given initial data set. It is therefore a good idea to try to understand the quantum mechanics in such spacetimes; this may resolve some of the classical ambiguities and might provide an arena for the play-off between the geometry (or rather the causal structure) and quantum theory.

One particular problem with CTC spacetimes is the question of the unitarity of the evolution operator. Work has been done which suggests that in such spacetimes an evolution operator will not in general be unitary. There are two approaches one might take. The first is to try to repair the theory by in some way extending the theory so that unitarity is restored. The second way is to question the calculation of the evolution operator and try to get a better understanding of the fundamental quantum mechanics.

Both approaches have been tried. There are a number of proposals that try to repair unitarity. One we shall be investigating is to look to the theory of unitary dilations. In this theory we have an auxiliary inner product space (which may be indefinite, such spaces are called *Krein Spaces*). The general idea is that the Hilbert space on which we have assumed the quantum theory in being played out is actually only a subspace of a larger Krein space. Ordinarily the degrees of freedom not represented by the Hilbert space are inaccessible to observations. To do this we restrict the form of observables on the total Krien spaces. However the evolution operator is a unitary operator between the relevant Krien spaces, though not considered as (the projection of) the operator on the Hilbert space. In effect the extra degrees of freedom allow somewhere for parts of the wavefunction to hide from view. The total evolution is unitary, at the expense of introducing indefinite inner product spaces into the discussion. The idea of a unitary dilation is motivated by the simple geometric observation that any linear transformation of the real line is the projection of an orthogonal transformation (called an *orthogonal dilation* of the original mapping) in a larger (possibly indefinite) inner product space. To see this, note that any linear contraction on the line may be regarded as the projection of a rotation in the plane: the contraction in length along the x-axis, say, being balanced by a growth in the y-component. Similarly, a linear dilation on the line may be regarded as the projection of a Lorentz boost in two dimensional Minkowski space. The cost of introducing indefinite inner product spaces may be too high; that is as yet unclear. However there is a nice analogy with the Klien paradox. The probabilities one calculates in this case may become negative, on account of the spacetime *signature*. The problem is not however with the treatment of spacetime, but rather a signal that a further level of quantization is required to understand the system of fields one is working with. It seems reasonable that the necessity of introducing a Krein space may well be signalling a similar situation. The problem is universal in that it is not due to the details of any particular field, so the negative probabilities will come from the indefinite signature of the inner product space rather than from the spacetime. It might be noted that a burst of particle production has been suggested by Hawking to prevent a CTC region from occurring. In a sense then the Unitary Dilation Proposal seems to be well suited to account for the possibility.

The other route one might try is to re-calculate the evolution using different quantum mechanical tools. The various quantization procedures we have at present all give the same answers under normal situations. It is interesting to find out whether one or other is unusually well suited to discuss CTC models. If it proves to be the case then perhaps we will have a better understanding of quantum theory in situations that are beyond experiment. In particular, if one method were favoured over the others that might be the one to think about when trying to construct quantum gravity.

We will be investigating the CTC models within the formalism of the Quantum Initial Value Problem (QIVP). This is not in itself a quantization procedure, but with operator ordering conventions we are able to press home the analysis of a number of CTC model spacetimes and compare our answers with those obtained using path integral techniques. The idea is to investigate the quantum theory by setting up an operator-valued initial data set and trying to find an evolution that obeys the operator version of the equations of motion. We adopt this approach as it does not presuppose the solutions will obey the Canonical Commutation (Anti-Commutation) Relations (CCR's/CAR's). This is in contrast with many quantization procedures which automatically guarantees that they are preserved (and often require the existence of quantities that are not well-defined in our models). We say that the QIVP is well-posed if an initial data set defined to the past of the nonchronal region and satisfying the CCR/CAR's is evolved in such a way that the solution to the future of the nonchronal region also forms a representation of the CCR/CAR's. As we shall see this more general setting allows us to consider systems where unitarity is not necessarily preserved.

We shall be considering a number of simple CTC model spacetimes, of a type introduced by Politzer, these comprise of a number of discrete spatial positions and a continuous timelike coordinate that has been subject to appropriate identification, to produce some CTC's, and investigate a simple linear field. In this case the evolution is unitary, and in agreement with the path integral. More interestingly, the treatment of the interacting version of the model using the QIVP formalism yields different results from that obtained from the path integral. We suggest that this is due to the inclusion of too large a set of paths in the path integral method that has been proposed.

In one particular model we study, consisting of two spatial points we discover that the QIVP technique leads to a unitary evolution whereas the path integral does not. It has been suggested that the non-unitary of the S-matrix is not the physically relevant quantity, so it's non-uniqueness is unimportant. Instead it is the superscattering operator, mapping initial to final density matrices that one should be computing. We shall be examining a system where the evolution rule cannot be transcribed into the language of a superscattering operator with the usual properties, in particular the positive definiteness of the evolved density matrix is violated; this leads to negative probabilities.

One of the motivating factors in studying the quantum theory for CTC models is to try to see if quantum mechanics resolves the classical ambiguities present in the initial value problem. Using a coherent states approach we will show that the quantum theory is well-defined in these models, and investigate the classical limit. For certain coupling strengths the classical non-uniqueness is resolved unambiguously within the quantum theory. However, matters are not always so straightforward. There seem to be other coupling strengths were no classical limit is physically relevant. Perhaps more interesting still are those situations where a classical limit does exist, but does not correspond to a solution to the classical equations of motion and boundary conditions. In a sense the classical solution is a superposition of modes that achieve consistency only after a finite number of traversals of the CTC.

One suggestion that has been discussed is that CTC models may induce, possibly unlimited, particle production. We will examine a lattice model spacetime with a unitary S-matrix where this is indeed the case. This model is loosely based about Thirring's Model, with suitable CTC identifications. It is interesting to compare lattice model calculations with the analogous arguments for the continuum. A suitable limit of the lattice model and the treatment of the continuum limit yield identical results, this gives us some confidence in supposing that the conclusions from the consideration of our rather simplified CTC models can be taken over to a more realistic setting.

This dissertation has ten chapters and one appendix. In Chap. 2 we discuss internal symmetry transformations, explaining how to use dimensional reduction to exhibit some underlying symmetry of Einstein's equations. In this chapter we also give a construction of the Poincaré and Bergmann metrics, which naturally arise from consideration of pure Einstein theory and Einstein-Maxwell theory respectively. This construction will be the starting point for our new proof of the Bunting/Mazur result. The Double Ehlers' transformation for the superstring and $N = 4$ supergravity theories is derived in Chap. 2, this will be important later in our discussions on black hole superstring uniqueness theorems.

In Chap. 3 we present a number of important exact solutions that we shall be studying. In particular we will be meeting the C-metric and Ernst solution. We shall also be looking at the Weyl coordinate system, investigating some exact solutions derived by considering an internal symmetry transformation. Later on we will be exploiting elliptic function theory to write the C-metric and Ernst solution in terms of coordinates that will be highly advantageous to us in Chap. 4. In addition there is an appendix to Chap. 3 that quickly reviews the properties of the Jacobi elliptic functions we shall need and sets out our conventions.

In Chap. 4 we prove the uniqueness of the Ernst solution and C-metric by exploiting the theory of Riemann surfaces and elliptic functions. We also present a new proof of the positivity of the divergence required in that proof using the complex hyperboloid construction of the Bergmann metric from Chap. 2. The absence of a proof of this result was the reason why the Kerr-Newman black hole uniqueness theorem remained unproved for many years. Finally we discuss the issue of black hole monopole pair creation and explain how our uniqueness theorem has a bearing on that problem.

Chap. 5 is devoted to extending the black hole uniqueness theorems to the static, axisymmetric, and axion-free, truncation of superstring theory. We introduce the Stringy C-metric and Stringy Ernst solution. These provide the basis for a set of new instantons that mediate black hole pair production in that theory. By using what we have learnt from Chap. 2 and Chap. 4 we then establish the uniqueness of these solutions.

Suitably truncated String theory and Kaluza-Klein theory are both examples of dilaton gravity. Starting from the effective Lagrangian for dilaton gravity we will find in Chap. 6 new bounds on the ADM mass of the spacetime in terms of the dilaton and electromagnetic charges. This is the anti-gravity bound and is derived by generalizing the proof of the positive mass theorem given by Penrose, Sorkin and Woolgar.

In Chap. 7 we investigate the cosmic string-like solutions to flat space dilaton electrodynamics. We also present solutions representing other topological defects, such as domain walls and Dirac monopoles.

Part II of this dissertation is concerned with the problems of CTC's, and in particular the properties of quantum fields in chronology violating spacetimes. In Chap. 8 we attempt to provide a possible mechanism whereby the loss of unitary that has been noted by various authors may be repaired. This is the Unitary Dilation Proposal. We also put another proposal made by Anderson on a more rigorous footing and comment on an operational problem that arises.

Chap. 9 is where we discuss the Quantum Initial Value Problem for a class of fields in specific chronology violating spacetimes. We mention the classical non-uniqueness of the problem and how quantum theory (at least with specific operator ordering) can remove this ambiguity. A large section is dedicated to the classical limit of our quantum theory and reveal some rather interesting results. Throughout we will be comparing our answers with those obtained using path integral methods and noting that the methods do not agree. We therefore deduce that the path integral approach needs some modification in these CTC spacetimes; as the current methods do not provide solutions to the quantum field equations.

In Chap. 10 we perform a calculation of the particle creation due to the presence of a system of CTC's. The calculation yields an ultraviolet divergence in the total particle number and therefore a divergence in the stress energy tensor. This is in support of the Chronology Protection Conjecture. We also compare the results of the QIVP and path integral methods and find that for this model the results do agree, in contrast with the models in Chap. 9.

Finally we include one appendix, where we derive a few formulae that are necessary for the dimensional reduction calculations in a number of the chapters in Part I.

Part I

BLACK HOLES AND DILATONS

2. INTERNAL SYMMETRY TRANSFORMATIONS

2.1 Introduction

One striking aspect of vacuum General Relativity is its underlying simplicity. Looked at from the point of view of trying to propose a generally covariant action functional it is hard to imagine how a more simple theory describing the gravitational field by the curvature of spacetime might be constructed. A very natural extension to pure gravity is obtained when we consider the gravitational field to have sources that are described by a harmonic mapping Lagrangian. If we postulate the existence of a Killing vector field then Einstein-Maxwell theory naturally falls into this form when we consider the process of dimensional reduction.

In this chapter we will be studying a number of harmonic mappings and exploiting their symmetries. We begin in Sect. 2.2 by investigating the dimensional reduction of pure gravity. This leads us to the $SL(2,\mathbb{R})/SO(2)$ σ-model found by Geroch [5]. The generalization to include an electromagnetic field is treated in Sect. 2.3; the construction presented there yields an $SU(1,2)/S\left(U(1)\times U(2)\right)$ σ-model. The target space metrics for these theories are the Poincaré and Bergmann metrics respectively. A geometrical construction of these is presented in Sect. 2.4. The construction presented there will be important in establishing a new proof of the Bunting and Mazur result [6, 7, 8].

Having established the relationship between linear transformations in an auxiliary complex space and internal symmetries of the fields arising from the dimensional reduction procedure, we derive the Ehlers' transformation appropriate to vacuum General Relativity [9] in Sect. 2.5, and in the section following the transformations we find for its electromagnetic extension. This includes a derivation of the Harrison transformation [10] which will be important in subsequent chapters. We will show how the Harrison and

Ehlers' transformations are 1-parameter subgroups of a larger eight dimensional group of transformations mapping new solutions from old [11], however various gauge and obvious scaling symmetries are included and these hold no interest for our investigations.

Superstring theory is a theory that has been very popular as a possible approach to unifying General Relativity and Quantum Theory. In Sect. 2.7 we applying similar techniques to those employed for pure gravity and Einstein-Maxwell theory to derive some internal symmetry transformations for a particular truncation of this theory. The $N = 4$ Supergravity theory has the same possible truncation, so our result is also valid in that theory. The particular form of the effective action after performing a dimensional reduction is highly reminiscent of what we find for pure gravity. In particular this allows us to use an Ehlers' transformation, or more precisely a *Double Ehlers' Transformation*, as the effective Lagrangian consists of two copies of what we find for pure gravity. Later in Chap. 5 we will use this fact to prove black hole uniqueness results in the Superstring theory.

2.2 Pure Gravity

In this section we investigate the effective Lagrangian arising from pure gravity after a dimensional reduction on a Killing vector field $K = \partial/\partial t$. Our starting point is to write the metric in the form

$$\boldsymbol{g} = -V\left(\mathbf{d}t + \boldsymbol{A}\right) \otimes \left(\mathbf{d}t + \boldsymbol{A}\right) + V^{-1}\gamma_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j \tag{2.2.1}$$

and introduce the twist 1-form $\boldsymbol{\omega}$ associated with the vector field K. We set

$$\boldsymbol{\omega} = *(\boldsymbol{k} \wedge \mathbf{d}\boldsymbol{k}) \ , \tag{2.2.2}$$

so that $\boldsymbol{k} = -V\left(\mathbf{d}t + \boldsymbol{A}\right)$. Let us define $\boldsymbol{H} = \mathbf{d}\boldsymbol{A}$. We write $\boldsymbol{H}$ rather than $\boldsymbol{F}$ here so as not to cause notational difficulties when we come to look at the analogous arguments in the presence of an electromagnetic field. We therefore have

$$\boldsymbol{H} = \mathbf{d}\left(\frac{\boldsymbol{k}}{i_K\boldsymbol{k}}\right) \tag{2.2.3}$$

and

$$\frac{*\boldsymbol{\omega}}{i_K\boldsymbol{k}} = \boldsymbol{k}\wedge\boldsymbol{H}\ . \tag{2.2.4}$$

Note also that $i_K\boldsymbol{H} = 0$ from Eq. (2.2.3) and the equation $\mathcal{L}_K\boldsymbol{k} = 0$. Thus,

$$\begin{aligned}
\frac{*\left(\boldsymbol{\omega}\wedge *\boldsymbol{\omega}\right)}{\left(i_K\boldsymbol{k}\right)^2} &= *\left(*\left(\boldsymbol{k}\wedge\boldsymbol{H}\right)\wedge\boldsymbol{k}\wedge\boldsymbol{H}\right) \\
&= *\left(i_K*\boldsymbol{H}\wedge\boldsymbol{k}\wedge\boldsymbol{H}\right) \\
&= -\left(i_K\boldsymbol{k}\right)*\left(\boldsymbol{H}\wedge *\boldsymbol{H}\right). \qquad (2.2.5)
\end{aligned}$$

In index notation, and in terms of the three metric γ_{ij} this result states:

$$\frac{2\omega_i\omega_j\gamma^{ij}}{V^2} = H_{ij}H_{kl}\gamma^{ik}\gamma^{jl}. \tag{2.2.6}$$

We may now calculate $\mathbf{d}\boldsymbol{\omega}$:

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega} &= -*\boldsymbol{\delta}\left(\boldsymbol{k}\wedge\mathbf{d}\boldsymbol{k}\right) \\
&= *\mathcal{L}_K\mathbf{d}\boldsymbol{k} + *\left(\boldsymbol{k}\wedge\boldsymbol{\delta}\mathbf{d}\boldsymbol{k}\right) \\
&= 2*\left(\boldsymbol{k}\wedge\boldsymbol{R}(\boldsymbol{k})\right), \qquad (2.2.7)
\end{aligned}$$

where $\boldsymbol{R}(\boldsymbol{k})$ is the Ricci 1-form associated with $\boldsymbol{k}$. For pure gravity $R_{ab} = 0$ and hence (at least locally) we may introduce a twist potential, ω defined by $\boldsymbol{\omega} = \mathbf{d}\omega$.

We are now in a position to dimensionally reduce the Lagrangian density on the Killing vector K. Eq. (A.31) of Appendix A gives the appropriate expression involving

$\boldsymbol{H}$ and V. We find after a suitable transcription of notation that

$$\mathcal{L} = \sqrt{|\gamma|}\left({}^3R - \frac{1}{2}\gamma^{ij}\left[\frac{\nabla_i V \nabla_j V + \nabla_i \omega \nabla_j \omega}{V^2}\right]\right) \ , \tag{2.2.8}$$

in which a total divergence has been discarded. We recognize this as a harmonic mapping (or sigma model) Lagrangian

$$\mathcal{L} = \sqrt{|\gamma|}\left({}^3R - 2\gamma^{ij}G_{AB}\frac{\partial \phi^A}{\partial x^i}\frac{\partial \phi^B}{\partial x^j}\right), \qquad \left(\phi^A\right) = \begin{pmatrix} V \\ \omega \end{pmatrix}, \tag{2.2.9}$$

with

$$\boldsymbol{G} = G_{AB}\mathbf{d}\phi^A \otimes \mathbf{d}\phi^B = \frac{\mathbf{d}V \otimes \mathbf{d}V + \mathbf{d}\omega \otimes \mathbf{d}\omega}{4V^2} \ . \tag{2.2.10}$$

The metric $\boldsymbol{G}$ is the metric on the target space of the harmonic map. This target space is the half-plane $V > 0$. It is advantageous to define a complex coordinate $\epsilon = V + i\omega$, and use a Möbius transformation to map the half-plane to the unit disc:

$$\xi = \frac{1+\epsilon}{1-\epsilon} \ . \tag{2.2.11}$$

The metric $\boldsymbol{G}$ now takes the form

$$\boldsymbol{G} = G_{AB}\mathbf{d}\phi^A \otimes \mathbf{d}\phi^B = \frac{\mathbf{d}\xi \otimes_S \mathbf{d}\overline{\xi}}{\left(1 - |\xi|^2\right)^2} \ . \tag{2.2.12}$$

Here we write $\otimes_S$ for the symmetrized tensor product. This metric can be recognized as the Poincaré metric on the unit disc. We shall give a geometrical construction of the Poincaré metric in Sect. 2.4, which will be a guide for how to treat more complicated harmonic maps that arise when we investigate other theories we will be interested in that couple to certain matter fields.

2.3 Einstein-Maxwell Theory

In this section we look at the dimensional reduction of the Einstein-Maxwell Lagrangian on a Killing vector $K = \partial/\partial t$. In units where Newton's constant is taken to be unity, the Lagrangian density is given by

$$\mathcal{L} = \sqrt{|g|}\left(R - F_{ab}F^{ab}\right) \ . \tag{2.3.1}$$

The field $\boldsymbol{F}$ being derived from a vector potential: $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$. We will be assuming that the Maxwell field obeys the appropriate symmetry condition: $\mathcal{L}_K \boldsymbol{F} = 0$. The exactness of $\boldsymbol{F}$ implies that

$$\mathbf{d}i_K\boldsymbol{F} = 0. \tag{2.3.2}$$

It is now convenient to introduce the electric and magnetic fields by

$$\boldsymbol{E} = -i_K\boldsymbol{F} \qquad \text{and} \qquad \boldsymbol{B} = i_K{*}\boldsymbol{F} \ . \tag{2.3.3}$$

Notice that $i_K\boldsymbol{E} = i_K\boldsymbol{B} = 0$ as a consequence of the general result $(i_K)^2 = 0$. The electromagnetic field tensor may be decomposed in terms of the electric and magnetic fields as follows:

$$\boldsymbol{F} = \frac{-\boldsymbol{k}\wedge\boldsymbol{E} - *\left(\boldsymbol{k}\wedge\boldsymbol{B}\right)}{i_K\boldsymbol{k}} \ . \tag{2.3.4}$$

The Lagrangian for the electromagnetic interaction, proportional to $F_{ab}F^{ab}$, may be written in terms of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields,

$$\begin{aligned}
\tfrac{1}{2}F_{ab}F^{ab} &= *\left(\boldsymbol{F}\wedge *\boldsymbol{F}\right) \\
&= \frac{*\left(\boldsymbol{E}\wedge *\boldsymbol{E} - \boldsymbol{B}\wedge *\boldsymbol{B}\right)}{i_K\boldsymbol{k}} \\
&= \frac{|\mathbf{B}|^{\mathbf{2}} - |\mathbf{E}|^{\mathbf{2}}}{V} \ .
\end{aligned} \tag{2.3.5}$$

Two of Maxwell's equations, namely the ones involving the divergence of $\mathbf{B}$ and the curl of $\mathbf{E}$ arise not from the Lagrangian, but rather from the exactness of $\boldsymbol{F}$. In order to find these equations we evaluate $i_K{*}\mathbf{d}\boldsymbol{F}=0$:

$$\begin{aligned} i_K{*}\mathbf{d}\boldsymbol{F} &= i_K\left[*\left(\mathbf{d}\left(\frac{\boldsymbol{k}}{i_K\boldsymbol{k}}\right)\wedge\boldsymbol{E}\right)-\boldsymbol{\delta}\left(\boldsymbol{k}\wedge\left(\frac{\boldsymbol{B}}{i_K\boldsymbol{k}}\right)\right)\right] \\ &= (i_K\boldsymbol{k})\,\boldsymbol{\delta}\left(\frac{\boldsymbol{B}}{i_K\boldsymbol{k}}\right)-\frac{i_E\boldsymbol{\omega}}{i_K\boldsymbol{k}} \end{aligned} \tag{2.3.6}$$

or in vector notation

$$\nabla.\left(\frac{\mathbf{B}}{V}\right)+\frac{\boldsymbol{\omega}.\mathbf{E}}{V^2}=\mathbf{0}\ . \tag{2.3.7}$$

This equation is a constraint on the fields. We also notice that $\mathbf{d}\boldsymbol{F}=0$ implies together with the symmetry condition, that $i_K\mathbf{d}\boldsymbol{F}=-\mathbf{d}i_K\boldsymbol{F}=-\mathbf{d}\boldsymbol{E}=0$ and hence that locally we may write $\boldsymbol{E}=\mathbf{d}\Phi$. In order to progress we will also need to know about the divergence of $\boldsymbol{\omega}/V^2$. Firstly observe that

$$\mathbf{d}\left(\frac{\boldsymbol{k}}{i_K\boldsymbol{k}}\right)=i_K\boldsymbol{C} \tag{2.3.8}$$

for some 3-form $\boldsymbol{C}$. This follows from the fact that when we apply i_K to the 2-form on the left hand side we get zero (as previously mentioned in the last section). Decomposing the left hand side into 'electric' and 'magnetic' parts we see that the 'electric' part is zero. This leads to

$$\begin{aligned} -\nabla.\left(\frac{\boldsymbol{\omega}}{V^2}\right)=\boldsymbol{\delta}\left(\frac{\boldsymbol{\omega}}{V^2}\right) &= *\left[\mathbf{d}\left(\frac{\boldsymbol{k}}{i_K\boldsymbol{k}}\right)\wedge\mathbf{d}\left(\frac{\boldsymbol{k}}{i_K\boldsymbol{k}}\right)\right] \\ &= *(i_K\boldsymbol{C}\wedge i_K\boldsymbol{C}) \\ &= *i_K(\boldsymbol{C}\wedge i_K\boldsymbol{C})=0, \end{aligned} \tag{2.3.9}$$

as the last equation involves the inner product of a 5-form, which automatically vanishes. The constraint equation may therefore be written as a divergence:

$$\nabla.\left(\frac{\mathbf{B}}{V}-\frac{\boldsymbol{\omega}\Phi}{V^2}\right)=\mathbf{0}\ . \tag{2.3.10}$$

In order to impose the constraint we need to make use of a Legendre transformation. To this end we introduce a Lagrangian multiplier, Ψ. After discarding total divergences the Lagrangian to vary is given by

$$\mathcal{L} = \sqrt{|g|}\left(R + 2\left[\frac{|\nabla\Phi|^2 - \mathbf{B}^2}{V} + \frac{2\,\mathbf{B}.\nabla\Psi}{V} - \frac{\boldsymbol{\omega}.(\Phi\nabla\Psi - \Psi\nabla\Phi)}{V^2}\right]\right). \tag{2.3.11}$$

Varying with respect to $\mathbf{B}$ we conclude that $\mathbf{B} = \nabla\Psi$ and performing the dimensional reduction to three dimensions we find

$$\mathcal{L} = \sqrt{\gamma}\left({}^3R - 2\left[\frac{|\nabla V|^2 + \omega_i\omega^i}{4\,V^2} - \frac{|\nabla\Phi|^2 + |\nabla\Psi|^2}{V} - \frac{\boldsymbol{\omega}.(\Phi\nabla\Psi - \Psi\nabla\Phi)}{V^2}\right]\right) \ . \tag{2.3.12}$$

All indices in the above equation are raised and lowered using γ_{ij} and its inverse.

As in the case of pure gravity we try to introduce a twist potential by examining $\mathbf{d}\boldsymbol{\omega}$. Eq. (2.2.7) and Einstein's equation in the presence of an electromagnetic field yield

$$\mathbf{d}\boldsymbol{\omega} = 16\pi{*}\,(\boldsymbol{k} \wedge \boldsymbol{T}(\boldsymbol{k})) \tag{2.3.13}$$

where the stress-energy 1-form associated with $\boldsymbol{k}$, denoted $\boldsymbol{T}(\boldsymbol{k})$, is given by

$$\boldsymbol{T}(\boldsymbol{k}) = \frac{1}{4\pi}\left({*}(i_K\boldsymbol{F} \wedge {*}\boldsymbol{F}) - \frac{1}{2}{*}(\boldsymbol{F} \wedge {*}\boldsymbol{F})\boldsymbol{k}\right). \tag{2.3.14}$$

Substituting Eq. (2.3.14) into (2.3.13), we find

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega} &= 4{*}\,(\boldsymbol{k} \wedge {*}\,(i_K\boldsymbol{F} \wedge {*}\boldsymbol{F})) \\
&= 4 i_K\boldsymbol{F} \wedge i_K{*}\boldsymbol{F} \\
&= -4\,\boldsymbol{E} \wedge \boldsymbol{B} \ .
\end{aligned} \tag{2.3.15}$$

It then follows that

$$\mathbf{d}(\boldsymbol{\omega} + 2(\Phi\mathbf{d}\Psi - \Psi\mathbf{d}\Phi)) = 0, \tag{2.3.16}$$

and we may locally introduce a potential, ω, with $\mathbf{d}\omega = \boldsymbol{\omega} + 2(\Phi\mathbf{d}\Psi - \Psi\mathbf{d}\Phi)$. It turns out to be highly useful to combine the two potentials, Φ and Ψ into a single complex potential, $\psi = \Phi + i\Psi$. Using this complex potential we have

$$i\boldsymbol{\omega} = i\mathbf{d}\omega - \psi\mathbf{d}\overline{\psi} + \overline{\psi}\mathbf{d}\psi \ . \tag{2.3.17}$$

We are now in a position to define the Ernst potential [12], ϵ by, $\epsilon = V - |\psi|^2 + i\omega$. Then clearly

$$\mathbf{d}\epsilon + 2\psi\mathbf{d}\overline{\psi} = \mathbf{d}V + i\boldsymbol{\omega} \ . \tag{2.3.18}$$

Substituting this into Eq. (2.3.12) we get

$$\begin{aligned}
\mathcal{L} &= \sqrt{|\gamma|}\left({}^3R - 2\left[\frac{|\nabla\epsilon + 2\psi\nabla\overline{\psi}|^2}{4V^2} - \frac{|\nabla\psi|^2}{V} + \frac{i\boldsymbol{\omega}.(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi})}{V^2}\right]\right) \\
&= \sqrt{|\gamma|}\left({}^3R - 2\left[\frac{|\nabla\epsilon + 2\overline{\psi}\nabla\psi|^2}{4V^2} - \frac{|\nabla\psi|^2}{V}\right]\right) \qquad (2.3.19) \\
&= \sqrt{|\gamma|}\left({}^3R - 2\gamma^{ij}G_{AB}\frac{\partial\phi^A}{\partial x^i}\frac{\partial\phi^B}{\partial x^j}\right) \qquad (\phi^A) = \begin{pmatrix}\epsilon \\ \psi\end{pmatrix} \ . \qquad (2.3.20)
\end{aligned}$$

with the harmonic mapping target space metric G_{AB} given by

$$\boldsymbol{G} = G_{AB}\mathbf{d}\phi^A \otimes \mathbf{d}\phi^B = \frac{\left(\mathbf{d}\epsilon + 2\overline{\psi}\mathbf{d}\psi\right) \otimes_S \left(\mathbf{d}\overline{\epsilon} + 2\psi\mathbf{d}\overline{\psi}\right)}{4V^2} - \frac{\mathbf{d}\psi \otimes_S \mathbf{d}\overline{\psi}}{V} \ . \tag{2.3.21}$$

This metric is conveniently written in terms of new variables with

$$\xi = \frac{1+\epsilon}{1-\epsilon}; \qquad\qquad \eta = \frac{2\psi}{1-\epsilon}. \tag{2.3.22}$$

The metric $\boldsymbol{G}$ then takes the form

$$\boldsymbol{G} = \frac{(1-|\eta|^2)\,\mathbf{d}\xi \otimes_S \mathbf{d}\overline{\xi} + (1-|\xi|^2)\,\mathbf{d}\eta \otimes_S \mathbf{d}\overline{\eta} + \xi\overline{\eta}\mathbf{d}\overline{\xi}\otimes_S\mathbf{d}\eta + \overline{\xi}\eta\mathbf{d}\xi \otimes_S \mathbf{d}\overline{\eta}}{\left(1-|\xi|^2-|\eta|^2\right)^2} \ . \tag{2.3.23}$$

This is the Bergmann metric, and is the natural generalization of the Poincaré metric in higher dimensions, as we shall see in the following sections there is an interesting $SU(1,2)$ action preserving this metric. With these transformations we will be able to generate new solutions from old, in both the pure gravity and Einstein-Maxwell theories.

2.4 The Poincaré and Bergmann Metrics

The Poincaré and Bergmann metrics have simple geometrical constructions. The Poincaré metric is the natural metric to put on the unit disc, as its isometries are precisely those Möbius maps that leave the unit disc invariant. Our starting point is the vector space $\mathbb{C}^{n+1}$. For the Poincaré metric, $n = 1$, whilst for the Bergmann metric $n = 2$. We will be using complex coordinates $z_0, z_1, ..., z_n$. Let us write

$$\eta = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{2.4.1}$$

and define an indefinite inner product using $\langle w, z \rangle = w^\dagger \eta z$. We therefore have $\mathbf{d}\boldsymbol{s} = \|\mathbf{d}z\|$. This metric induces a metric on the hyperboloid defined by

$$\|z\|^2 = -1. \tag{2.4.2}$$

Fig. 2.1 shows how we may project from any point on the hyperboloid to a point on the unit disc (or ball if $n \geq 2$). In the diagram each point on the hyperboloid corresponds to a circle, as two points differing by a phase are projected to the same point on the disc. The disc sits in the space at $z_0 = 1$, touching at the lowest point on the hyperboloid given by Eq. (2.4.2).

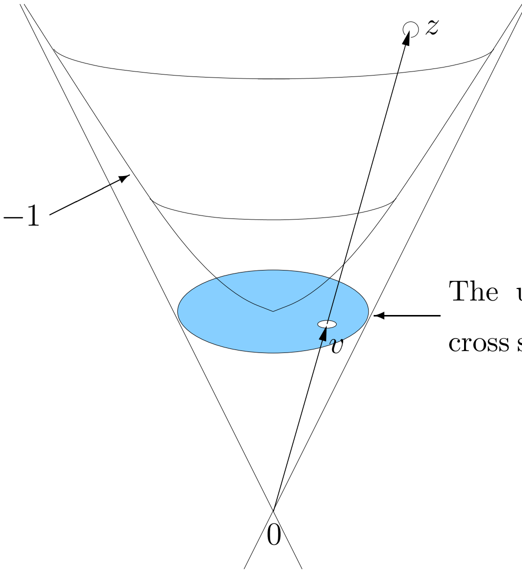

Fig. 2.1: *Construction of the projection mapping from the hyperboloid $\|z\|^2 = -1$ to the unit disc. The induced metric on the disc from this construction is the Poincaré metric, or in higher dimensions the Bergmann metric.*

We will find it convenient to write

$$\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} = re^{it} \begin{pmatrix} 1 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}, \qquad r^{-2} = 1 - v^\dagger v. \tag{2.4.3}$$

Using $2\mathbf{d}r = (\mathbf{d}v^\dagger v + v^\dagger \mathbf{d}v)r^3$, we quickly establish that

$$\mathbf{d}\boldsymbol{s}^2 = \mathbf{d}z^\dagger \eta \mathbf{d}z = -\left[\mathbf{d}t + \boldsymbol{A}\right]^2 + r^2 \mathbf{d}v^\dagger \mathbf{d}v + r^4 \mathbf{d}v^\dagger v v^\dagger \mathbf{d}v, \tag{2.4.4}$$

and

$$\boldsymbol{A} = \frac{i}{2} r^2 \left(v^\dagger \mathbf{d}v - \mathbf{d}v^\dagger v\right). \tag{2.4.5}$$

We also notice that $\mathbf{d}t + \boldsymbol{A}$ may be written as

$$\mathbf{d}t + \boldsymbol{A} = \frac{i}{2}\left(\mathbf{d}z^\dagger \eta z - z^\dagger \eta \mathbf{d}z\right). \tag{2.4.6}$$

Hence we find that the metric can then be expressed as

$$\mathbf{d}\boldsymbol{s}^2 = -\left[\mathbf{d}t + \boldsymbol{A}\right]^2 + g_{a\bar{b}}\mathbf{d}v^a\mathbf{d}\overline{v}^{\bar{b}}. \tag{2.4.7}$$

We call $g_{a\bar{b}}$ the Bergmann metric for $n = 2$, or the Poincaré metric for $n = 1$. We remark in passing that these metrics are Kähler, and one may define the symplectic 2-form $\boldsymbol{\Omega}$ by

$$\boldsymbol{\Omega} = -i\left(r^2\mathbf{d}v^\dagger \wedge \mathbf{d}v + r^4\mathbf{d}v^\dagger v \wedge v^\dagger \mathbf{d}v\right). \tag{2.4.8}$$

A simple calculation then shows,

$$\mathbf{d}\boldsymbol{\Omega} = 0. \tag{2.4.9}$$

As with all Kähler metrics we may derive the metric and symplectic 2-form from a Kähler potential, K, where in this case

$$K = \log r \tag{2.4.10}$$

with

$$\mathbf{d}\boldsymbol{s}^2 = 2\,\mathbf{d}v^\dagger\left(\frac{\partial^2 K}{\partial v \partial v^\dagger}\right)\mathbf{d}v \tag{2.4.11}$$

and

$$\boldsymbol{\Omega} = 2i\,\boldsymbol{d}\overline{\boldsymbol{d}}K, \tag{2.4.12}$$

where $\boldsymbol{d}$ and $\overline{\boldsymbol{d}}$ are the exterior derivatives with respect to holomorphic and anti-holomorphic coordinates respectively.

We will now proceed to investigate the isometries of the hyperboloid with this metric, acting with elements of $U(1,n)$. We see that if $g \in U(1,n)$ (so that $g^\dagger \eta g = \eta$) then we will have that

$$\mathbf{d}t + \boldsymbol{A} \mapsto \frac{i}{2}\left(\mathbf{d}z^\dagger g^\dagger \eta g z - z^\dagger g^\dagger \eta g \mathbf{d}z\right) = \frac{i}{2}\left(\mathbf{d}z^\dagger \eta z - z^\dagger \eta \mathbf{d}z\right) = \mathbf{d}t + \boldsymbol{A}. \tag{2.4.13}$$

So the elements of $U(1,n)$ will generate isometries of the Bergmann or Poincaré metric. When we project onto the domain $v^\dagger v < 1$ we get isometries from the elements of $SU(1,n)$, as a mere change of phase gives rise to the same isometry of the Bergmann metric, it just generates a translation of the t-coordinate. The group of isometries acts transitively on the domain, and hence we draw the conclusion that the curvature must be constant. It is useful to look at the stabilizer of some point, for simplicity (and without loss of generality) we take the origin. We observe that if

$$g \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = re^{it} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad g \in SU(1,n), \tag{2.4.14}$$

then $g \in S(U(1) \times U(n))$, so we may identify the domain with the (symmetric) space of left cosets $SU(1,n)/S\left(U(1) \times U(n)\right)$.

In this section we have seen how to construct the Bergmann metric in terms of a suitable projection and an auxiliary complex vector space. The electrovac system can be expressed in this language by defining

$$z = \begin{pmatrix} 1-\epsilon \\ 1+\epsilon \\ 2\psi \end{pmatrix} \tag{2.4.15}$$

with the Lagrangian written as

$$\mathcal{L} = \sqrt{\gamma}\left({}^3R - 2\frac{\| (\nabla z)^\perp \|^2}{\|z\|^2} \right). \tag{2.4.16}$$

We have defined the orthogonal component of ∇z as $(\nabla z)^\perp = \nabla z - \langle z, \nabla z\rangle z/\|z\|^2$. It is clear that there is much symmetry in this system and we will be exploiting this fact in the next few sections as well as in Sect. 4.5 when we present a new proof of the relevant divergence identity vital for the construction of black hole uniqueness results.

2.5 The Ehlers' Transformation (Pure Gravity)

In Sect. 2.2 we found that the metric on the target space of the harmonic mapping for pure gravity reduced on a timelike Killing vector K took the form

$$\boldsymbol{G} = G_{AB}\mathbf{d}\phi^A \otimes \mathbf{d}\phi^B = \frac{\mathbf{d}\xi \otimes_S \mathbf{d}\overline{\xi}}{(1-|\xi|^2)^2}. \tag{2.5.1}$$

We may apply the isometries that we found in the previous section and re-interpret in terms of the twist and 'Newtonian' potentials. As we have seen the Poincaré metric has a natural interpretation in terms of an $SU(1,1)/S(U(1)\times U(1))$ harmonic mapping system. This space is isomorphic to $SL(2,\mathbb{R})/SO(2)$ which was first noticed by Geroch [5]. We may act with an element $A \in SU(1,1)$ by means of the transformation

$$re^{it}\begin{pmatrix} 1-\epsilon \\ 1+\epsilon \end{pmatrix} \mapsto re^{it}A\begin{pmatrix} 1-\epsilon \\ 1+\epsilon \end{pmatrix}. \tag{2.5.2}$$

There are two obvious isometries, a scaling of the Killing vector,

$$\epsilon \mapsto e^{\theta}\epsilon \tag{2.5.3}$$

which corresponds to the boost A,

$$A = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \tag{2.5.4}$$

and a gauge transformation of the twist potential:

$$\epsilon \mapsto \epsilon + it \tag{2.5.5}$$

corresponding to the $U(1,1)$ matrix

$$B = \begin{pmatrix} 1-it/2 & -it/2 \\ it/2 & 1+it/2 \end{pmatrix}. \tag{2.5.6}$$

The remaining degree of freedom is usefully expressed by considering the action of

$$\begin{pmatrix} 1 - it/2 & it/2 \\ -it/2 & 1 + it/2 \end{pmatrix} , \tag{2.5.7}$$

this gives rise to the Ehlers' transformation [9]:

$$\epsilon \mapsto \frac{\epsilon}{1 + it\epsilon}. \tag{2.5.8}$$

These maps allow us to generate new solutions to Einstein's equations from previously known solutions when that solution possesses a Killing vector. In the case of pure gravity there is a one-parameter set of new solutions that can be derived in this way, $SU(1,1)$ is a three-dimensional Lie group but as we have seen, gauge transformations of the scalar potential and a rescaling of the Killing vector account for two of these degrees of freedom.

2.6 The Transformations for Einstein-Maxwell Theory

The Lie group $SU(1,2)$ is defined to be the set of all matrices

$$SU(1,2) = \{B \in GL(3,\mathbb{C}) \mid B^{\dagger}\eta B = \eta,\ \det B = 1\} \tag{2.6.1}$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.6.2}$$

with a group structure derived by using the standard matrix multiplication.

It is highly useful to define the involutive automorphism $\sigma : SU(1,2) \to SU(1,2)$ by $\sigma(A) = \eta A \eta$. We are already aware of some of the isometries of the Bergmann metric, for instance we may add a constant to the twist potential:

$$\begin{aligned} \epsilon &\mapsto \epsilon + it, \\ \psi &\mapsto \psi, \end{aligned} \tag{2.6.3}$$

where t is real. This transformation corresponds to the matrix

$$A = \begin{pmatrix} 1 - it/2 & -it/2 & 0 \\ it/2 & 1 + it/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.6.4}$$

with action defined by

$$r' e^{it'} \begin{pmatrix} 1 \\ \xi' \\ \eta' \end{pmatrix} = r e^{it} A \begin{pmatrix} 1 \\ \xi \\ \eta \end{pmatrix} . \tag{2.6.5}$$

The matrix $\sigma(A)$ generalizes the Ehlers' transformation:

$$\begin{aligned} \epsilon &\mapsto \frac{\epsilon}{1 + it\epsilon} , \\ \psi &\mapsto \frac{\psi}{1 + it\epsilon} . \end{aligned} \tag{2.6.6}$$

Another obvious isometry of the Bergmann metric results from making gauge transformations to the electric and magnetic potentials:

$$\epsilon \mapsto \epsilon - 2\overline{\beta}\psi - |\beta|^2, \tag{2.6.7}$$

$$\psi \mapsto \psi + \beta. \tag{2.6.8}$$

This arises from considering the $SU(1,2)$-matrix

$$B = \begin{pmatrix} 1+|\beta|^2/2 & |\beta|^2/2 & \overline{\beta} \\ -|\beta|^2/2 & 1-|\beta|^2/2 & -\overline{\beta} \\ \beta & \beta & 1 \end{pmatrix}, \qquad \beta \in \mathbb{C}. \tag{2.6.9}$$

The matrix $\sigma(B)$ gives rise to the Harrison transformation [10]:

$$\begin{aligned} \epsilon &\mapsto \Lambda^{-1}\epsilon, \\ \psi &\mapsto \Lambda^{-1}(\psi + \beta\epsilon), \\ \Lambda &= 1 - 2\overline{\beta}\psi - |\beta|^2\epsilon. \end{aligned} \tag{2.6.10}$$

Finally to complete a set of eight generators for the group consider the combined rescaling of the Killing vector and electromagnetic duality rotation:

$$\begin{aligned} \epsilon &\mapsto |\alpha|^2\epsilon, \\ \psi &\mapsto \alpha\psi \qquad \alpha \in \mathbb{C}. \end{aligned} \tag{2.6.11}$$

which corresponds to the matrix

$$C = \begin{pmatrix} \left(\alpha^{-1}+\overline{\alpha}\right)/2 & \left(\alpha^{-1}-\overline{\alpha}\right)/2 & 0 \\ \left(\alpha^{-1}-\overline{\alpha}\right)/2 & \left(\alpha^{-1}+\overline{\alpha}\right)/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.6.12}$$

The matrix $\sigma(C)$ corresponds to a redefinition of the parameter α and hence does not give rise to any new transformations.

2.7 The $N = 4$ Supergravity and Superstring Theories

Let us now turn to a truncated theory arising from the bosonic sector of the $N = 4$ Supergravity and Superstring Theories. These theories possesses a dilaton with coupling parameter equal to unity, as well as electric and magnetic potentials. For simplicity we will restrict attention to the static truncation of the harmonic map. The $N = 4$ theory possesses an axionic field, and six $U(1)$ gauge fields that combined have an $SO(6)$ invariance. Together with a suitable duality rotation it is possible to reduce the theory to one with just two $U(1)$ gauge fields, one purely electric, the other purely magnetic. At this point the axion decouples and can be consistently set equal to zero. What remains can be written in terms of an effective single electromagnetic field (with both electric and magnetic parts), see Gibbons [13] for further details. The Lagrangian density can then be written:

$$\mathcal{L} = \sqrt{|g|}\left(R - 2|\nabla\phi|^2 - e^{-2\phi}F_{ab}F^{ab}\right) \ . \tag{2.7.1}$$

After a dimensional reduction on a spacelike axial Killing vector field $m = \partial/\partial\varphi$ it takes the form:

$$\mathcal{L} = \sqrt{\gamma}\left({}^3R - 2\left(\frac{|\nabla X|^2}{4X^2} + |\nabla\phi|^2 + \frac{e^{-2\phi}|\nabla\psi_e|^2}{X} + \frac{e^{2\phi}|\nabla\psi_m|^2}{X}\right)\right) \tag{2.7.2}$$

where

$$\begin{aligned} \boldsymbol{g} &= X\mathbf{d}\varphi\otimes\mathbf{d}\varphi + X^{-1}\gamma_{ij}\mathbf{d}x^i\otimes\mathbf{d}x^j, && (2.7.3)\\ \mathbf{d}\psi_e &= -i_m\boldsymbol{F}, && (2.7.4)\\ \mathbf{d}\psi_m &= e^{-2\phi}i_m{*}\boldsymbol{F}, && (2.7.5)\end{aligned}$$

${}^3R$ is the Ricci scalar of the metric γ_{ij} and the metric γ_{ij} has been used to perform the contractions in Eq. (2.7.2). The Hodge dual in Eq. (2.7.5) is that from the four-dimensional metric (2.7.3). In order to derive Eq. (2.7.2) we have needed to perform a Legendre transform, which has the effect of changing the sign of the $|\nabla\psi_m|^2$ term from what one might have naïvely expected. The justification for this follows in a similar manner to the derivation of Eq. (2.3.12). We now define new coordinates

$$X_+ = X^{1/2} e^{\phi} \qquad \text{and} \qquad X_- = X^{1/2} e^{-\phi}. \tag{2.7.6}$$

Together with the electrostatic potentials $\psi_+ = \sqrt{2}\psi_e$ and $\psi_- = \sqrt{2}\psi_m$. The metric on the target space of the harmonic map is given by

$$G_{AB}\mathbf{d}\phi^A \otimes \mathbf{d}\phi^B = \frac{\mathbf{d}X_+ \otimes \mathbf{d}X_+ + \mathbf{d}\psi_+ \otimes \mathbf{d}\psi_+}{X_+^2} + \frac{\mathbf{d}X_- \otimes \mathbf{d}X_- + \mathbf{d}\psi_- \otimes \mathbf{d}\psi_-}{X_-^2}. \tag{2.7.7}$$

We remark that this precisely takes the form of two copies of the Lagrangian for pure gravity. We will be exploiting this fact in Chap. 5 in relation to the Black Hole Uniqueness Theorems. For the moment we merely note that we can perform independent Ehlers' transformations to both X_+ and X_- to derive new solutions.

2.7.1 The Double Ehlers' Transformation

Performing independent Ehlers' transformations to the system yield the following:

$$X \mapsto \frac{X}{\left[1+\beta^2\left(Xe^{2\phi}+\psi_+^2\right)\right]\left[1+\gamma^2\left(Xe^{-2\phi}+\psi_-^2\right)\right]}; \tag{2.7.8}$$

$$e^{2\phi} \mapsto e^{2\phi}\frac{1+\gamma^2\left(Xe^{-2\phi}+\psi_-^2\right)}{1+\beta^2\left(Xe^{2\phi}+\psi_+^2\right)}; \tag{2.7.9}$$

$$\psi_+ \mapsto \frac{\psi_+ + \beta\left(Xe^{2\phi}+\psi_+^2\right)}{1+\beta^2\left(Xe^{2\phi}+\psi_+^2\right)}; \tag{2.7.10}$$

$$\psi_- \mapsto \frac{\psi_- + \gamma\left(Xe^{-2\phi}+\psi_-^2\right)}{1+\gamma^2\left(Xe^{-2\phi}+\psi_-^2\right)}. \tag{2.7.11}$$

For the transformation from a vacuum solution we have the slightly simpler form:

$$X \mapsto \frac{X}{\left(1+\beta^2 X\right)\left(1+\gamma^2 X\right)}; \tag{2.7.12}$$

$$e^{2\phi} \mapsto \frac{1+\gamma^2 X}{1+\beta^2 X}; \tag{2.7.13}$$

$$\psi_+ \mapsto \frac{\beta X}{1+\beta^2 X}; \tag{2.7.14}$$

$$\psi_- \mapsto \frac{\gamma X}{1+\gamma^2 X}. \tag{2.7.15}$$

In particular if we apply this to Minkowski space we generate the Stringy Melvin Universe:

$$\boldsymbol{g} = \left(1+\beta^2 r^2 \sin^2\theta\right)\left(1+\gamma^2 r^2 \sin^2\theta\right)\left(-\mathbf{d}t\otimes\mathbf{d}t + \mathbf{d}r\otimes\mathbf{d}r + r^2\mathbf{d}\theta\otimes\mathbf{d}\theta\right) + \frac{r^2\sin^2\theta\mathbf{d}\varphi\otimes\mathbf{d}\varphi}{\left(1+\beta^2 r^2 \sin^2\theta\right)\left(1+\gamma^2 r^2 \sin^2\theta\right)}; \tag{2.7.16}$$

$$e^{2\phi} = \frac{1+\gamma^2 r^2 \sin^2\theta}{1+\beta^2 r^2 \sin^2\theta}; \tag{2.7.17}$$

$$\boldsymbol{A} = -\sqrt{2}\gamma r\cos\theta\mathbf{d}t + \frac{\beta r^2\sin^2\theta\mathbf{d}\varphi}{\sqrt{2}\left(1+\beta^2 r^2 \sin^2\theta\right)}. \tag{2.7.18}$$

This solution represents the stringy generalization of Melvin's Universe. Whereas in Melvin's universe the electric and magnetic fields can be transformed into one another by a simple duality rotation without affecting the metric (meaning often that we need only consider a purely magnetic or electric universe), the stringy universe of necessity involves both electric and magnetic fields. These fields are parallel and provide a repulsive force to counterbalance the attractive force of the spin zero dilaton and spin two graviton fields. The Stringy Melvin Universe will be important to us as it will model a strong electromagnetic field in string theory and we will be considering the mediation of the pair production of suitable black hole monopoles by such fields in Chap. 5.

3. WEYL COORDINATES, THE C-METRIC AND THE ERNST SOLUTION

The mathematics of the axisymmetric stationary Einstein theory has some remarkable and rather unexpected results. In particular it has an interesting relationship to Newtonian gravity. To understand this relationship it is advantageous to use a particular coordinate system, known as Weyl coordinates. The representation of familiar vacuum solutions can be re-interpreted in terms of the associated Newtonian systems. Doing so may shed new light on the solution, as the superposition principle that is inherent in Newtonian gravity finds a translation into the vacuum axisymmetric stationary Einstein theory. In this chapter we will be interested in a number of exact solutions. Starting from the interpretation of the Schwarzschild solution and Rindler space, we can construct a solution representing an accelerating black hole. This is the vacuum C-metric. Allied to this is a charged version in the Einstein-Maxwell system. We will be discussing its form in Sect. 3.2. The C-metric and more especially a derivative of it, the Ernst solution will be important for the consideration of the quantum gravity process of black hole monopole pair creation. It is therefore important to understand a little about these solutions. In Sect. 3.3 we discuss the derivation of the Ernst solution in terms of the Harrison transform introduced in Chap. 2. The other important exact solution for our discussion is Melvin's Magnetic Universe, this solution represents a uniform magnetic field in Einstein-Maxwell theory. It is the energy in the magnetic field that can give rise to the black hole monopole pair creation process.

In Sects. 3.4 and 3.5 we make use of elliptic function theory to write the Ernst solution and C-metric in terms of new coordinates that are highly advantageous to the problem of finding black hole uniqueness theorems for these solutions. Appendix 3.A provides some introductory material on the functions we shall be using, and concisely sets out our conventions. It turns out that the use of elliptic functions is extraordinarily useful in describing these solutions, the complexity of some of the expressions given in terms of these function can be deceptive, and often arises from taking the real or imaginary parts of simple analytic functions. In Sect. 3.6 we prove some awkward technical lemmas to help us understand the relationship between the parameters of the solutions when represented in terms of elliptic functions and those quantities that are easily determined by examining the asymptotic behaviour, or the behaviour close to the axis of symmetry.

3.1 Weyl Coordinates and the Vacuum C-metric

In this section we will review a number of features of axisymmetric static vacuum solutions to Einstein's equations. With these premises, Einstein's equations take on a particularly pleasing form. We start by writing the metric as follows:

$$\boldsymbol{g} = -e^{2U}\mathbf{d}t \otimes \mathbf{d}t + e^{-2U}\left(e^{2\gamma}\left(\mathbf{d}x^1 \otimes \mathbf{d}x^1 + \mathbf{d}x^2 \otimes \mathbf{d}x^2\right) + \rho^2 \mathbf{d}x^3 \otimes \mathbf{d}x^3\right). \tag{3.1.1}$$

The spacetime possesses Killing vectors $K = \partial/\partial t$ and $m = \partial/\partial x^3$. Proceeding to compute the Einstein equations, we find

$$\begin{aligned} {}^{(2)}\nabla^2 \rho &= 0, & (3.1.2) \\ {}^{(2)}\nabla.\left(\rho\, {}^{(2)}\nabla U\right) &= 0 & (3.1.3) \end{aligned}$$

and

$$-D^2\rho + 2D\rho\, D\gamma - 2\rho(DU)^2 = 0, \tag{3.1.4}$$

where

$${}^{(2)}\nabla = \left(\frac{\partial}{\partial x^1}\,, \frac{\partial}{\partial x^2}\right), \tag{3.1.5}$$

and

$$D = \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\ . \tag{3.1.6}$$

The quantity ρ is the norm of the Killing bivector:

$$\rho^2 = -\|\boldsymbol{k} \wedge \boldsymbol{m}\|^2, \tag{3.1.7}$$

where

$$\boldsymbol{k} = e^{2U}\mathbf{d}t \quad \text{and} \quad \boldsymbol{m} = e^{-2U}\rho^2\mathbf{d}x^3. \tag{3.1.8}$$

Anticipating the results of Sects. 4.2 and 4.3 there is a considerable simplification if we take $x^1 = \rho$ and $x^2 = z$, where z is the harmonic conjugate to ρ with respect to the metric

$$\mathbf{d}x^1 \otimes \mathbf{d}x^1 + \mathbf{d}x^2 \otimes \mathbf{d}x^2. \tag{3.1.9}$$

The Einstein equations now become

$$^{(3)}\nabla^2 U \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{\partial^2 U}{\partial z^2} = 0, \tag{3.1.10}$$

i.e., Laplace's equation in cylindrical polar coordinates in $\mathbb{R}^3$. The equations for γ reduce to

$$\frac{\partial\gamma}{\partial\rho} = \rho\left[\left(\frac{\partial U}{\partial\rho}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right], \tag{3.1.11}$$

$$\frac{\partial\gamma}{\partial z} = 2\rho\frac{\partial U}{\partial\rho}\frac{\partial U}{\partial z}. \tag{3.1.12}$$

The integrability condition for γ is automatically satisfied whenever $^{(3)}\nabla^2 U = 0$. We note that it is not entirely trivial to assume that ρ and z provide a good coordinate system for the solution, in Sect. 4.3 we will show the validity of this procedure under suitable conditions (and also in the presence of an electromagnetic field).

Einstein's equations are non-linear and we cannot simply add up solutions of the equations to get new ones. However as Laplace's equation is linear we may superpose solutions in exactly the same way as for Newtonian gravity. In the present case we need to solve a linear equation. The non-linearity of Einstein's equations manifests itself in

Eq. (3.1.11) and (3.1.12). However, we may solve in principle for γ by quadrature.

It is interesting to see what various known solutions to Einstein's equations look like in the Weyl formulation. The Schwarzschild solution turns out to be represented by the potential due to a uniform rod lying along the axis between two points z_1 and z_2, of mass per unit length of 1/2, i.e.,

$$U = \tfrac{1}{2}\log(R_2 - (z - z_2)) - \tfrac{1}{2}\log(R_1 - (z - z_1)), \qquad (3.1.13)$$

where

$$R_i^2 = \rho^2 + (z - z_i)^2 \qquad \text{for } i = 1, 2. \qquad (3.1.14)$$

We may look on the solution as the linear superposition of two semi-infinite line masses of linear density 1/2 and −1/2. The Rindler spacetime is just flat space written in terms of accelerating coordinates. In this formulation the spacetime is that derived from considering the potential from a single semi-infinite line mass of density 1/2.

Bonnor [14] shows that another vacuum solution with which we will be concerned for much of this present chapter has a simple interpretation in terms of such rods. A finite rod is to be interpreted as a particle/black hole, a semi-infinite line mass as a source at infinity responsible for causing an acceleration. We may superpose the two solutions. Let

$$e^{2U} = \frac{c^2(R_S - (z - z_S))(R_A - (z - z_A))}{R_N - (z - z_N)} \qquad (3.1.15)$$

with

$$z_S < z_N < z_A \ . \qquad (3.1.16)$$

The segment $[z_S, z_N]$ represents a spherical particle, z_A determines the acceleration. The solution has a nodal singularity on the axis, we may eliminate the singularity from one section of the axis by a suitable choice of c. We cannot eliminate the singularity from both sections of the axis simultaneously. If we leave the conical singularity between the finite rod and the semi-infinite line mass we speak of a *cosmic strut*, the semi-infinite rod pushing the particle along. The other option is to leave the conical singularity between the particle and infinity, in this case one says that we have a *cosmic string* pulling the particle along. These characterizations give a physical interpretation to the nodal singularity and suggest they are responsible for the particle's acceleration.

In the analytically extended version of the solution, we have two such particles either connected by a *cosmic spring* – the particles accelerate towards each other until a critical moment when they start to recede, or they have a pair of cosmic strings that bring the particles to a halt and then begin to accelerate them away from one another.

The solution we have been describing is called *the Vacuum C-metric*, we will have more to say about its electromagnetic generalization later. In particular we will be concerned with giving a new physical motivating force for the acceleration.

We may apply the methods of the previous chapter to generalize the Weyl system to include any number of positive or negative energy scalar fields. The Lagrangian density after a dimensional reduction on the timelike Killing vector yields

$$\mathcal{L} = \sqrt{\gamma}\left({}^3R - 2\gamma^{ij}G_{AB}\frac{\partial\psi^A}{\partial x^i}\frac{\partial\psi^B}{\partial x^j}\right). \tag{3.1.17}$$

where

$$\gamma_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j = e^{2\gamma}\left(\mathbf{d}x^1 \otimes \mathbf{d}x^1 + \mathbf{d}x^2 \otimes \mathbf{d}x^2\right) + \rho^2\mathbf{d}x^3 \otimes \mathbf{d}x^3, \tag{3.1.18}$$

$$\left(\psi^A\right) = \begin{pmatrix} U \\ \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}, \tag{3.1.19}$$

$$(G_{AB}) = \begin{pmatrix} 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}, \tag{3.1.20}$$

where the diagonal element is $+1$ if the appropriate scalar has positive energy and -1 if it has negative energy.

The Einstein equations for this system are:

$$^{(3)}\nabla^2 \psi^A = 0 \tag{3.1.21}$$

and

$$\frac{\partial \gamma}{\partial \rho} = \rho\, G_{AB} \left[\frac{\partial \psi^A}{\partial \rho} \frac{\partial \psi^B}{\partial \rho} - \frac{\partial \psi^A}{\partial z} \frac{\partial \psi^B}{\partial z} \right] \,, \tag{3.1.22}$$

$$\frac{\partial \gamma}{\partial z} = 2\rho\, G_{AB} \frac{\partial \psi^A}{\partial \rho} \frac{\partial \psi^B}{\partial z} \,. \tag{3.1.23}$$

Again the integrability condition is satisfied whenever Eq. (3.1.21) holds. In Sect. 4.4 we will demonstrate that the integrability condition is always satisfied for a general harmonic mapping system. Any isometry of $\mathbb{R}^{n+1}$ with the appropriate flat metric, G_{AB}, gives rise to new solutions of the system.

One can, for instance, use a boost to add a negative energy scalar field to any of the solutions that we have been discussing.

As we have seen there are many ways of generating new solutions of Einstein's equations from existing solutions, one can add together Weyl solutions and use isometries of the target spaces of harmonic mapping Lagrangians to provide an enormous number of new solutions.

3.2 The Charged C-metric

The Vacuum C-metric has a history going back as far as 1918 [15], its electromagnetic generalization was discovered in 1970 by Kinnersley and Walker [16]. It is to be noted however that this generalization is not simply a Harrison Transformation on the timelike Killing vector as is the case for charging up the Schwarzschild solution to get the Riessner-Nordstrøm black hole. Later, in Sect. 3.3 we will be applying the Harrison transform to the charged C-metric but using the *angular* Killing vector – This is Ernst's solution. To begin with we describe the solution determined by Kinnersley and Walker:

$$\boldsymbol{g} = r^2 \left(\frac{\mathbf{d}x \otimes \mathbf{d}x}{G(x)} - \frac{\mathbf{d}y \otimes \mathbf{d}y}{G(y)} + G(x)\mathbf{d}\alpha \otimes \mathbf{d}\alpha + G(y)\mathbf{d}t \otimes \mathbf{d}t \right) \tag{3.2.1}$$

with

$$Ar = (x-y)^{-1}, \tag{3.2.2}$$

$$G(x) = 1 - x^2 - 2\widetilde{m}x^3 - \widetilde{g}^2 x^4, \tag{3.2.3}$$

$$\widetilde{m} = mA \quad \text{and} \quad \widetilde{g} = gA. \tag{3.2.4}$$

The case $g = 0$ is the vacuum solution discussed in the previous section. If we take the limit $A \to 0$, we discover that the solution reduces to the Riessner-Nordstrøm solution where m and g are the mass and charge of the black hole. We remark that m is not the ADM mass (unless $A = 0$). The ADM mass is zero, as the ADM 4-momentum is invariant under boosts and rotations and therefore must be zero. The quantity A is the acceleration of the world-line $r = 0$ when m and g are zero. We conclude that the C-metric represents an accelerating black hole. The charged C-metric has a nodal singularity, we will eliminate the singularity representing the cosmic string, thus leaving the cosmic strut intact.

Let us label the roots of the quartic equation $G(x) = 0$ as x_i in descending order (we are considering the case when we have four real roots) $x_4 < x_3 < x_2 < 0 < x_1$. We shall restrict attention to the following ranges for the coordinates.

$$x \in [x_2, x_1] \tag{3.2.5}$$

$$y \in [x_3, x_2] \tag{3.2.6}$$

$$\phi \in [0, 2\pi) \tag{3.2.7}$$

$$t \in (-\infty, \infty) \tag{3.2.8}$$

where ϕ is defined by

$$\phi = \frac{G'(x_2)}{2}\alpha. \tag{3.2.9}$$

The range of ϕ has been chosen to eliminate the cosmic string.

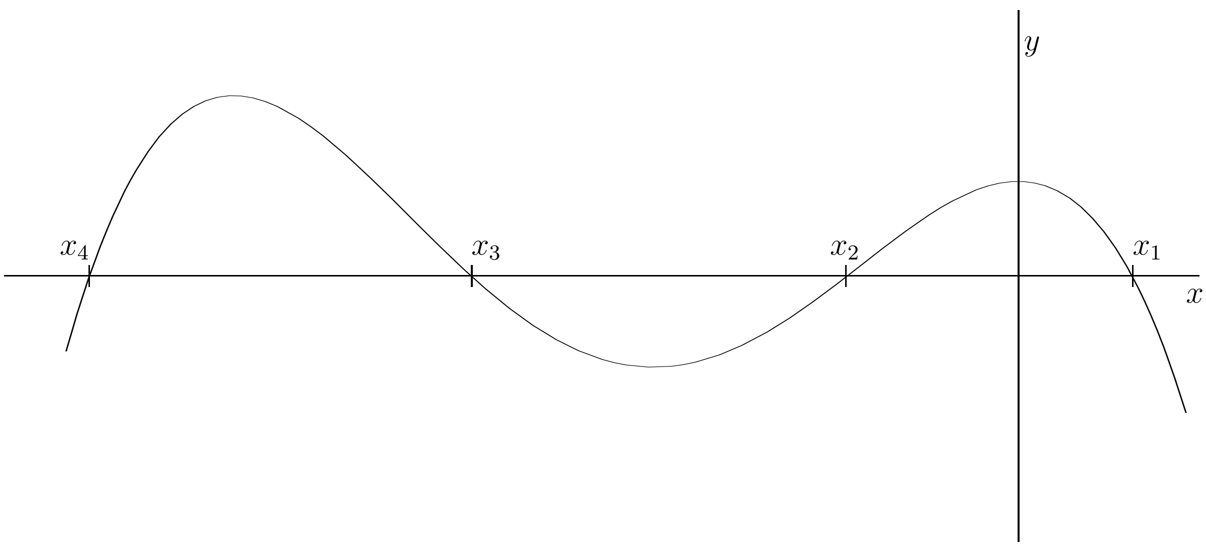

Fig. 3.1: *The graph of quartic function $y = G(x)$*

This means that $0 < r < \infty$, the singularity as $r \to 0$ corresponds to $y \to -\infty$ whilst $r \to \infty$ corresponds to the point $x = x_2,\ y = x_2$. There are two horizons that interest us: a black hole event horizon at $y = x_3$ and an acceleration horizon at $y = x_2$. In addition there is an inner horizon at $y = x_4$. With these choices the cosmic strut appears as the section of the axis $x = x_1$, $x_3 < y < x_2$.

From the metric we may read off the norm of the Killing bivector, ρ, for the C-metric:

$$\rho = r^2\sqrt{-G(x)G(y)}. \tag{3.2.10}$$

We have imposed the condition that $G(x) = 0$ have four real roots, this condition defines a region in the parameter space $(\widetilde{m}, \widetilde{g})$ shown in Fig. 3.3.

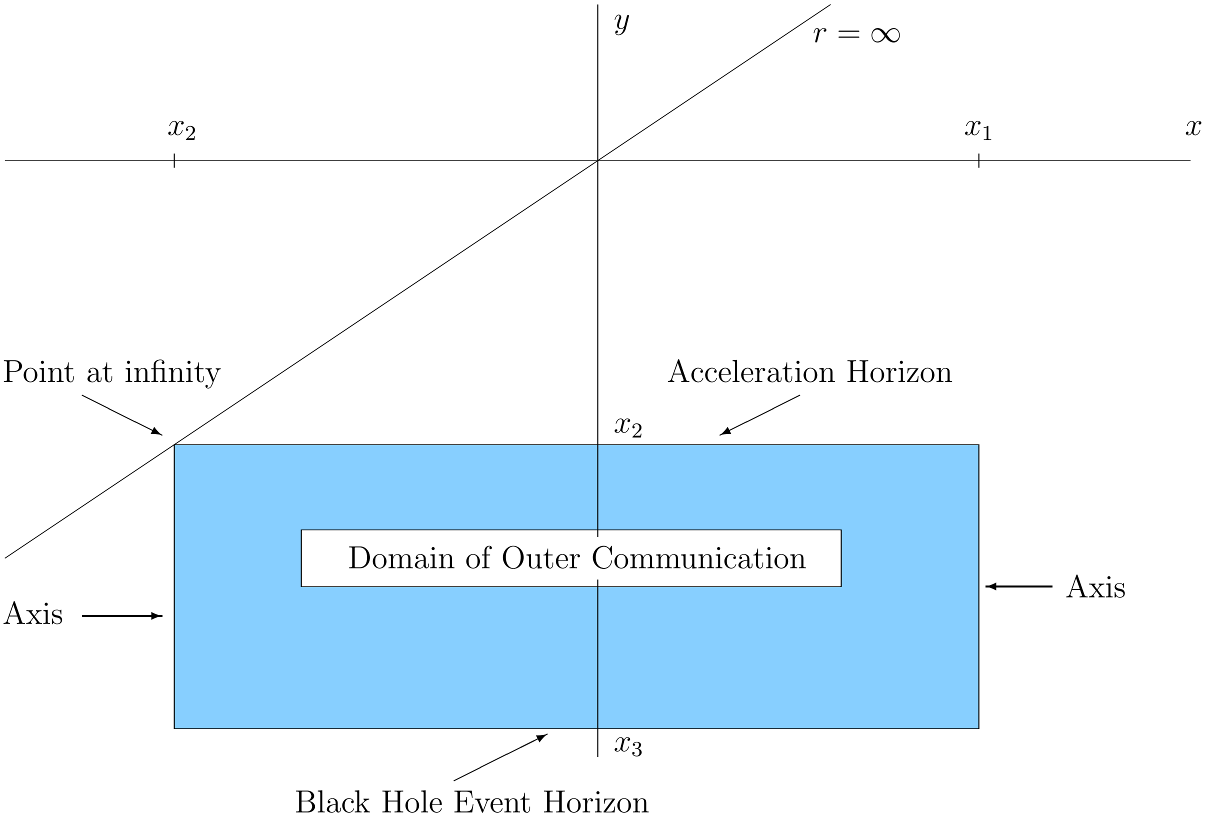

Fig. 3.2: *The Horizon Structure of the C-metric*

It is a simple matter now to determine the Ernst potentials derived from the angular Killing vector for the C-metric. They are presented below:

$$\epsilon = -r^2 G(x) - \frac{\widetilde{g}^2}{A^2}(x - x_2)^2, \quad (3.2.11)$$

$$\psi = -i\frac{\widetilde{g}}{A}(x - x_2). \quad (3.2.12)$$

$$(3.2.13)$$

3.3 Melvin's Magnetic Universe and The Ernst Solution

In this section we look at the result of performing a Harrison transformation Eq. (2.6.10) on Minkowski space and on the C-metric. We will be applying the transformation derived

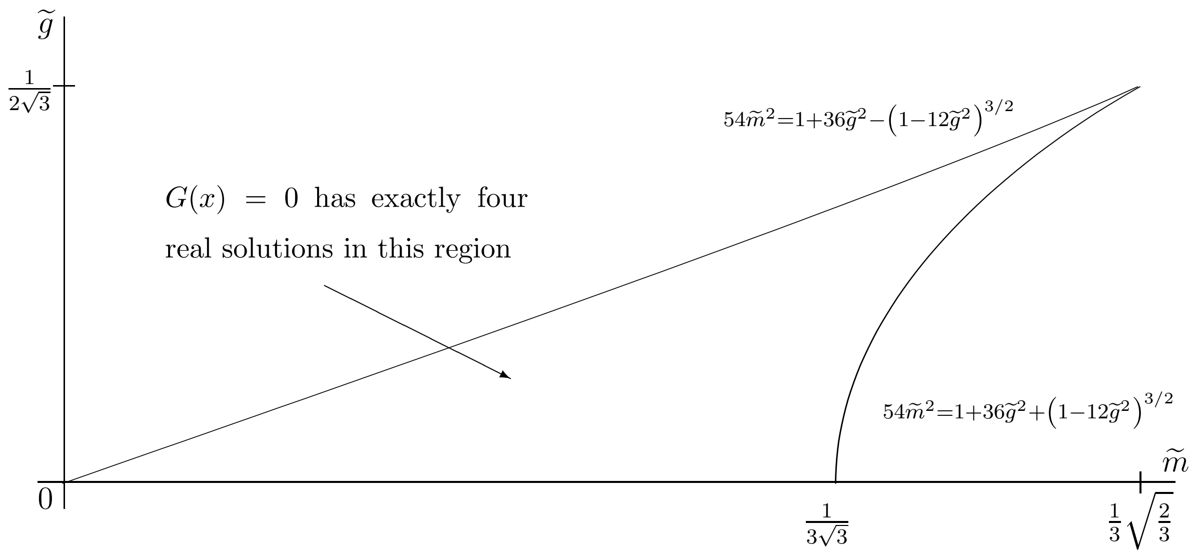

Fig. 3.3: *The Parameter space for* $\widetilde{m}, \widetilde{g}$

from consideration of the angular Killing vector $\partial/\partial\phi$.

Let us write Minkowski space in terms of cylindrical polar coordinates, thus

$$\boldsymbol{g} = -\mathbf{d}\widetilde{t} \otimes \mathbf{d}\widetilde{t} + \mathbf{d}r \otimes \mathbf{d}r + r^2\mathbf{d}\phi \otimes \mathbf{d}\phi + \mathbf{d}x \otimes \mathbf{d}x. \tag{3.3.1}$$

The Ernst potentials derived from the angular Killing vector are

$$\epsilon = -r^2, \tag{3.3.2}$$

$$\psi = 0. \tag{3.3.3}$$

Performing the Harrison transformation gives the new Ernst Potentials:

$$\begin{aligned}\epsilon &\mapsto -r^2\left(1+\tfrac{1}{4}B_0^2r^2\right)^{-1}, &(3.3.4)\\ \psi &\mapsto -\frac{i}{2}B_0r^2\left(1+\tfrac{1}{4}B_0^2r^2\right)^{-1}, &(3.3.5)\end{aligned}$$

and hence the new metric is

$$\begin{aligned}\boldsymbol{g} &= \Lambda^2(-\mathbf{d}\widetilde{t}\otimes\mathbf{d}\widetilde{t}+\mathbf{d}r\otimes\mathbf{d}r+\mathbf{d}x\otimes\mathbf{d}x)+r^2\Lambda^{-2}\mathbf{d}\phi\otimes\mathbf{d}\phi, &(3.3.6)\\ \Lambda &= 1+\tfrac{1}{4}B_0^2r^2. &(3.3.7)\end{aligned}$$

The electromagnetic field is given by

$$\boldsymbol{F}=\mathbf{d}\left(\frac{B_0r^2\mathbf{d}\phi}{2\left(1+\frac{1}{4}B_0^2r^2\right)}\right)=\frac{B_0r\mathbf{d}r\wedge\mathbf{d}\phi}{\left(1+\frac{1}{4}B_0^2r^2\right)^2}, \qquad (3.3.8)$$

This solution is Melvin's Magnetic Universe [17]. The Melvin solution represents a uniform tube of magnetic lines of flux in stable equilibrium with gravity. The transverse magnetic pressure balancing the attractive gravitational force.

We now proceed to apply the Harrison transformation to the C-metric, the new Ernst potentials are easily derived:

$$\begin{aligned}\epsilon &= -\Lambda^{-1}\left(r^2G(x)+\frac{\widetilde{g}^2}{A^2}(x-x_2)^2\right), &(3.3.9)\\ \psi &= -i\Lambda^{-1}\left(\frac{\widetilde{g}}{A}(x-x_2)+\frac{B}{2}\left(r^2G(x)+\frac{\widetilde{g}^2}{A^2}(x-x_2)^2\right)\right), &(3.3.10)\end{aligned}$$

with

$$\Lambda=\left(1+\frac{B\widetilde{g}}{2A}(x-x_2)\right)^2+\frac{1}{4}B^2r^2G(x). \qquad (3.3.11)$$

The metric and electromagnetic field tensor are transformed into (Ernst [18]):

$$\boldsymbol{g} = \Lambda^2 r^2 \left(\frac{\mathbf{d}x \otimes \mathbf{d}x}{G(x)} - \frac{\mathbf{d}y \otimes \mathbf{d}y}{G(y)} + G(y)\mathbf{d}t \otimes \mathbf{d}t \right) + r^2 G(x) \Lambda^{-2} \mathbf{d}\alpha \otimes \mathbf{d}\alpha, \tag{3.3.12}$$

$$\boldsymbol{F} = i\mathbf{d}\psi \wedge \mathbf{d}\alpha. \tag{3.3.13}$$

The great advantage of performing a Harrison Transformation to the C-metric is that it allows us to eliminate the nodal singularity from the entire axis. We do this by carefully choosing the parameter B, it turns out that the condition to be achieve this is given by:

$$\left(\frac{1}{\Lambda^2} \frac{dG(x)}{dx} \right) \Bigg|_{x=x_2} + \left(\frac{1}{\Lambda^2} \frac{dG(x)}{dx} \right) \Bigg|_{x=x_1} = 0. \tag{3.3.14}$$

In the limit $mA, gA, gB \ll 1$ this equation reduces to Newton's Second Law,

$$gB = mA \tag{3.3.15}$$

The Ernst Solution represents a black hole monopole undergoing a uniform acceleration due to the presence of a cosmological magnetic field. This solution has an electric counterpart, obtained by performing a duality transformation to the solution.

3.4 The Ernst Solution in terms of Elliptic Functions

In this section we will draw on the properties of elliptic functions. For a brief summary of all the results we will need and to establish our conventions, see Appendix 3.A.

As we remarked previously the norm of the Killing bivector, ρ for the C-metric (and Ernst solution) is simply given by:

$$\rho = r^2 \sqrt{-G(x)G(y)}\ , \tag{3.4.1}$$

where

$$\begin{aligned} Ar &= (x-y)^{-1}, && (3.4.2)\\ G(x) &= 1 - x^2 - 2\widetilde{m}x^3 - \widetilde{g}^2x^4, && (3.4.3)\\ \widetilde{m} &= mA, \qquad \widetilde{g} = gA. && (3.4.4)\end{aligned}$$

The induced metric on the two-dimensional space of orbits of the group action generated by the symmetries from the Killing vectors, $\mathcal{M}_{\mathrm{II}}$ has the form

$$\boldsymbol{g}_{\mathrm{II}} = \frac{\mathbf{d}x \otimes \mathbf{d}x}{G(x)} - \frac{\mathbf{d}y \otimes \mathbf{d}y}{G(y)} \ . \tag{3.4.5}$$

We may calculate z, the harmonic conjugate to ρ, from the Cauchy-Riemann equations:

$$\sqrt{G(x)}\frac{\partial\rho}{\partial x} = \sqrt{-G(y)}\frac{\partial z}{\partial y} \ , \qquad \sqrt{-G(y)}\frac{\partial\rho}{\partial y} = -\sqrt{G(x)}\frac{\partial z}{\partial x} \ . \tag{3.4.6}$$

This leads to

$$z = \frac{1}{2}r^2(G(x)+G(y)) + \frac{1}{2}g^2(x+y)^2 + \frac{m}{A}(x+y) + \text{constant.} \tag{3.4.7}$$

We shall denote by z_A, z_N and z_S the images of the acceleration horizon, and the north and south poles of event horizon. It is also useful to define

$$k^2 = \frac{z_N - z_S}{z_A - z_S} = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_2-x_4)} \ . \tag{3.4.8}$$

The quantity k turns out to be the modulus of many of the elliptic functions we shall be using. We will transform coordinates so that

$$\frac{\chi}{M} = \int_{x_2}^{x} \frac{dt}{\sqrt{G(t)}} \ , \qquad \text{and} \qquad \frac{\eta}{M} = \int_{y}^{x_2} \frac{dt}{\sqrt{-G(t)}} \ . \tag{3.4.9}$$

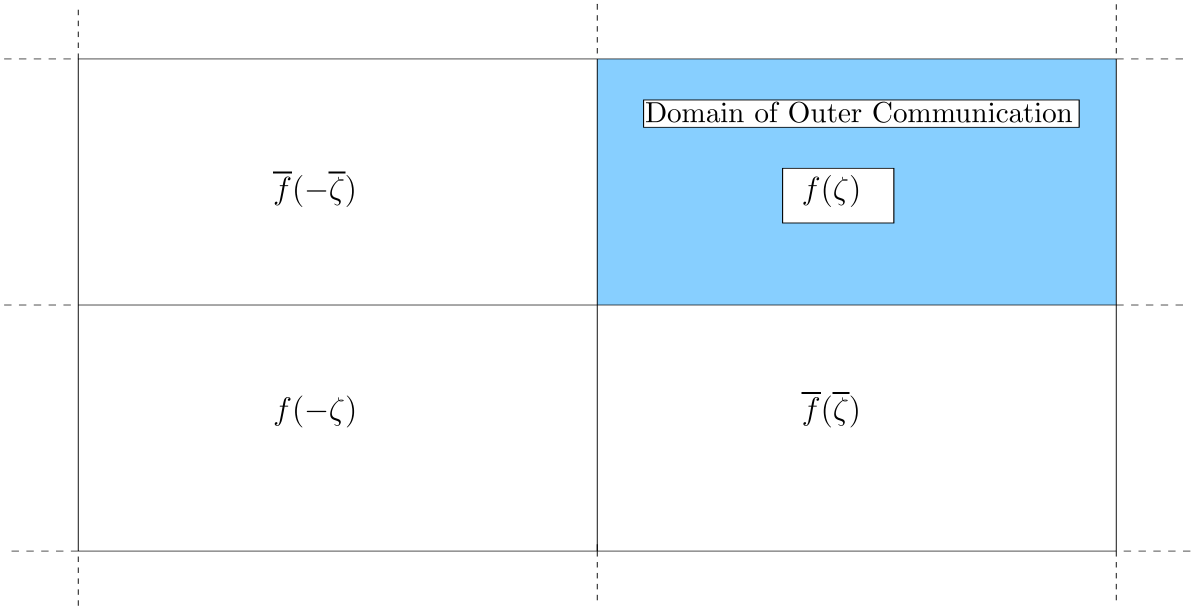

Fig. 3.4: *Using Schwarz reflection to extend the analytic function f.*

The value of M is given by

$$M^2 = e_1 - e_3 \tag{3.4.10}$$

where $e_i = \wp(\omega_i)$, the Weierstrass $\wp$-function being formed with the invariants g_2 and g_3 of G given by

$$g_2 = \frac{1 - 12\widetilde{g}^2}{12}\,, \tag{3.4.11}$$

$$g_3 = \frac{1 + 36\widetilde{g}^2 - 54\widetilde{m}^2}{216}\,. \tag{3.4.12}$$

Letting $\zeta = \chi + i\eta$ we have that $f(\zeta) = z(\zeta) - i\rho(\zeta)$ is an analytic function defined on $\mathcal{M}_{\mathrm{II}}$, the (two-dimensional section of) the domain of outer communication which in the present case is a rectangle in the complex ζ-plane. We now use Schwarz reflection in the boundaries (where $\rho = 0$). See Fig. 3.4.

On each rectangle $f(\zeta)$ takes the value indicated (and by the permanence of functional relations under analytic continuation they apply everywhere). We may proceed to reflect in the new boundaries, what we find is that $f(\zeta) = f(\zeta + 2K)$ and $f(\zeta) = f(\zeta + 2iK')$ where

$$\frac{K}{M} = \int_{x_2}^{x_1} \frac{dt}{\sqrt{G(t)}} \quad \text{and} \quad \frac{K'}{M} = \int_{x_3}^{x_2} \frac{dt}{\sqrt{-G(t)}} \tag{3.4.13}$$

and hence f is an even doubly periodic meromorphic function, i.e., a map between two compact Riemann surfaces, namely a torus, T and the Riemann sphere $\mathbb{C}^\infty$.

Applying the Valency theorem, we deduce that f is exactly n-1 for some n (and $n \geq 2$ as the sphere and torus are not homeomorphic). We can find n by examining the pre-image of infinity, $f^{-1}\{\infty\} = \{0\}$. We must work out the multiplicity, a simple calculation tells us:

$$f(\zeta) = \frac{2L^2}{\zeta^2} + O(1) \tag{3.4.14}$$

with $M = AL$. There is a second order pole at $\zeta = 0$. Therefore $f : T \to \mathbb{C}^\infty$ is exactly 2-1. Clearly f restricted to $\mathcal{M}_{\mathrm{II}}$, $f|_{\mathcal{M}_{\mathrm{II}}} : \mathcal{M}_{\mathrm{II}} \to \{z - i\rho | \rho > 0\}$ is 1-1.

As the map f is a doubly periodic even meromorphic function, another application of the Valency theorem shows that any analytic map : $T \to \mathbb{C}^\infty$ can be expressed in terms of the Weierstrass $\wp$-function and its derivative. Our map is especially simple

$$f(\zeta) = 2L^2(\wp_\Omega(\zeta) + \alpha), \qquad \alpha \text{ some real constant,} \tag{3.4.15}$$

$$\Omega = 2K\mathbb{Z} + 2iK'\mathbb{Z}. \tag{3.4.16}$$

Without loss of generality we set $\alpha = 0$. The critical points of $\wp_\Omega(\zeta)$ are the four corners of $\mathcal{M}_{\mathrm{II}}$ where $\wp'_\Omega(0) = \infty$ and $\wp'_\Omega(K) = \wp'_\Omega(iK') = \wp'_\Omega(K + iK') = 0$, this follows from the observation that the mapping fails to be conformal at these points, or alternatively by noticing it as a particular property of the $\wp$-function. We remark that $\wp'_\Omega(\zeta)$ is exactly 3-1 and we have three points where $\wp'_\Omega(\zeta) = 0$ and three (coincident) points where $\wp'_\Omega(\zeta) = \infty$. Hence we have found all the critical points of the map. Except at the critical points, the function $f|_{\mathcal{M}_{\mathrm{II}}}$ is invertible and (ρ, z) provide a coordinate system for the domain.

We now write the solution in terms of the coordinates (χ, η), after defining $\kappa = G'(x_2)$ and

$$q = \frac{\kappa \widetilde{g}}{AM^2} = \frac{\kappa \widetilde{g} L}{M^3} \tag{3.4.17}$$

we have

$$\sqrt{G(x)} = \frac{-G'(x_2)\wp'(\chi/M)}{4\left(\wp(\chi/M) - \frac{1}{24}G''(x_2)\right)^2} = \frac{\kappa \,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi}{2M\left(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi\right)^2} \tag{3.4.18}$$

$$x - x_2 = \frac{G'(x_2)}{4\left(\wp(\chi/M) - \frac{1}{24}G''(x_2)\right)} = \frac{\kappa\,\mathrm{sn}^2\chi}{4M^2\left(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi\right)} \tag{3.4.19}$$

and

$$\sqrt{-G(y)} = \frac{\kappa\,\mathrm{sn}\,\eta\,\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta}{2M\left(1 - D\,\mathrm{sn}^2\eta\right)^2} \tag{3.4.20}$$

$$y - x_2 = \frac{-\kappa\,\mathrm{sn}^2\eta}{4M^2\left(1 - D\,\mathrm{sn}^2\eta\right)} \tag{3.4.21}$$

for the constant $D = (1 + k'^2)/3 - G''(x_2)/24M^2$.

The metric takes the form

$$\boldsymbol{g} = -V\mathbf{d}t \otimes \mathbf{d}t + X\mathbf{d}\phi \otimes \mathbf{d}\phi + \Sigma\left(\mathbf{d}\chi \otimes \mathbf{d}\chi + \mathbf{d}\eta \otimes \mathbf{d}\eta\right) \tag{3.4.22}$$

where

$$X = \frac{4L^2(1 - D\,\mathrm{sn}^2\eta)^2\,\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi}{\Lambda^2\left(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi\right)^2\left(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi\right)^2} \tag{3.4.23}$$

$$V = \frac{4\Lambda^2 L^2\left(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi\right)^2\mathrm{sn}^2\eta\,\mathrm{cn}^2\eta\,\mathrm{dn}^2\eta}{\left(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi\right)^2\left(1 - D\,\mathrm{sn}^2\eta\right)^2} \tag{3.4.24}$$

$$\Sigma = \frac{16\Lambda^2 L^2\left(\mathrm{cn}^2\eta + D\,\mathrm{sn}^2\eta\right)^2(1 - D\,\mathrm{sn}^2\eta)^2}{\kappa^2\left(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi\right)^2} \tag{3.4.25}$$

$$\Lambda = \left(1 + \frac{B_0 q\,\mathrm{sn}^2\chi}{8(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)}\right)^2 + \frac{{B_0}^2 L^2\left(1 - D\,\mathrm{sn}^2\eta\right)^2\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi}{\left(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi\right)^2\left(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi\right)^2} \tag{3.4.26}$$

$$\rho = \frac{4L^2\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,\mathrm{sn}\,\eta\,\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta}{\left(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi\right)^2} \tag{3.4.27}$$

and

$$z - i\rho = 2L^2\wp(\chi + i\eta). \tag{3.4.28}$$

We have written B_0 for the Harrison transformation parameter above and reserve B for the magnetic potential. The $\wp$-function is with respect to the lattice $2K\mathbb{Z} + 2iK'\mathbb{Z}$, so that $2L^2 = z_A - z_S$. In addition the magnetic field is given by

$$\boldsymbol{F} = \mathbf{d}B \wedge \mathbf{d}\phi \tag{3.4.29}$$

with

$$B = \frac{1}{\Lambda}\left(\frac{q\,\mathrm{sn}^2\,\chi}{4(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)} + \frac{2B_0L^2\,(1 - D\,\mathrm{sn}^2\,\eta)^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi}{(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2} + \frac{B_0q^2\,\mathrm{sn}^4\,\chi}{32\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)^2}\right). \tag{3.4.30}$$

We will need to investigate the behaviour of X, B and ρ near the axis $\chi = 0$, we find

$$X = O(\chi^2) \tag{3.4.31}$$

$$B = O(\chi^2) \tag{3.4.32}$$

and

$$\rho = \frac{4L^2\,\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta}{\mathrm{sn}^3\,\eta}\,\chi + O\left(\chi^3\right). \tag{3.4.33}$$

Near the other axis $\chi = K$ we discover, setting $u = K - \chi$,

$$X = O(u^2), \tag{3.4.34}$$

$$B = \frac{2}{B_0 + 8D/q} + O(u^2) \tag{3.4.35}$$

and

$$\rho = 4L^2k'^2\,\mathrm{sn}\,\eta\,\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta\,u + O\left(u^3\right). \tag{3.4.36}$$

Near infinity, setting $\chi = R^{-1/2}\sin\theta$ and $\eta = R^{-1/2}\cos\theta$, we have

$$X = \frac{4}{{B_0}^4 L^2 \sin^2\theta}\frac{1}{R} + +O\left(\frac{1}{R^2}\right) \tag{3.4.37}$$

and

$$B = \frac{2}{B_0} - \frac{2}{{B_0}^3 L^2 \sin^2\theta}\frac{1}{R} + O\left(\frac{1}{R^2}\right). \tag{3.4.38}$$

Finally we find that ρ behaves as

$$\rho = 4L^2 \sin\theta\cos\theta R + O\left(\frac{1}{R}\right). \tag{3.4.39}$$

3.5 The C-metric in terms of Elliptic Functions

As a special case of the previous section we set the cosmological magnetic field B_0 to zero. Doing so leads to a different behaviour near infinity. We find that

$$X = O\left(\chi^2\right) \tag{3.5.1}$$

$$B = O\left(\chi^2\right) \tag{3.5.2}$$

close to the axis $\chi = 0$. Near the other axis $u = 0$ with $u = K - \chi$,

$$X = O\left(u^2\right) \tag{3.5.3}$$

$$B = \frac{q}{4D} + O\left(u^2\right) \tag{3.5.4}$$

whilst near infinity the behaviour is quite different from that of the Ernst Solution, and we have

$$X = 4L^2 \sin^2\theta\, R + O(1) \tag{3.5.5}$$

$$B = \frac{q}{4\sin^2\theta}\frac{1}{R} + O\left(\frac{1}{R^2}\right) \tag{3.5.6}$$

We will therefore need to impose different boundary conditions to prove the uniqueness of this solution. This will be done in Subsect. 4.6.2.

3.6 Determination of the Parameters of the Ernst Solution

We now present a couple of technical lemmas that will enable us to determine the parameters $\widetilde{m}$ and $\widetilde{g}$ from an Ernst solution by looking closely at its behaviour on the axis and as one goes off towards infinity. This is important for our discussion of the uniqueness theorems in the next chapter. Given a candidate spacetime we need to find an Ernst solution that coincides asymptotically (and to the right order) on the axis and off towards infinity. In addition to complete the uniqueness result we need to have both solutions defined on a common domain. This means that the quantity k defined by Eq. (3.4.8) must be the same for each solution. If we can find such an Ernst solution then we may use a divergence identity to prove uniqueness in a similar way as one does to show the uniqueness of solutions to Laplace's equation, only here the divergence identity is rather more complicated.

The boundary conditions we will need determine B_0 directly. The quantities L and q/D may be regarded as given. In addition, as we have just remarked we may assume knowledge of k the modulus of the elliptic functions.

We break the proof into two lemmas. Firstly we prove that the parameters D and k uniquely determine $\widetilde{m}$ and $\widetilde{g}$.

Lemma. *Given the modulus $k \in (0,1)$ and $D \in [0,k']$, where k' is the complementary modulus there exist values of $\widetilde{m}$ and $\widetilde{g}$ such that*

$$g(\chi) = \frac{\kappa \operatorname{sn}\chi \operatorname{cn}\chi \operatorname{dn}\chi}{2M(\operatorname{cn}^2\chi + D\operatorname{sn}^2\chi)^2} = \sqrt{G(x)} = \sqrt{1 - x^2 - 2\widetilde{m}x^3 - \widetilde{g}^2x^4} \tag{3.6.1}$$

where

$$\frac{dx}{\sqrt{G(x)}} = \frac{d\chi}{M}, \tag{3.6.2}$$

the values of M and κ being determined from D and k.

Proof: To begin with we investigate the turning points of $g(\chi)$ for $\chi \in [K/2, K]$. We therefore differentiate:

$$\begin{aligned} \frac{2M}{\kappa}\frac{dg(\chi)}{d\chi} &= \frac{\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi}{(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)^2} - \frac{\mathrm{sn}^2\chi\,\mathrm{dn}^2\chi}{(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)^2} \\ &\quad - \frac{k^2\,\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi}{(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)^2} - \frac{4(D-1)\,\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi}{(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)^3}\ . \end{aligned} \tag{3.6.3}$$

Setting this equal to zero we find that

$$D = \frac{3\,\mathrm{sn}^2\chi_0\,\mathrm{cn}^2\chi_0\,\mathrm{dn}^2\chi_0 + \mathrm{cn}^4\chi_0\,\mathrm{dn}^2\chi_0 - k^2\,\mathrm{sn}^2\chi_0\,\mathrm{cn}^4\chi_0}{3\,\mathrm{sn}^2\chi_0\,\mathrm{cn}^2\chi_0\,\mathrm{dn}^2\chi_0 + \mathrm{sn}^4\chi_0\,\mathrm{dn}^2\chi_0 + k^2\,\mathrm{sn}^4\chi_0\,\mathrm{cn}^2\chi_0}\ . \tag{3.6.4}$$

We shall now prove that $D \in [0, k']$ is in one to one correspondence with the values $\chi_0 \in [K/2, K]$. On this region $\mathrm{sn}^2\chi_0$ varies monotonically from $1/(1+k')$ to unity. We make the substitutions:

$$\mathrm{sn}^2\chi_0 = S \tag{3.6.5}$$

$$\mathrm{cn}^2\chi_0 = 1 - S \tag{3.6.6}$$

$$\mathrm{dn}^2\chi_0 = 1 - k^2 S \tag{3.6.7}$$

Note that $D(S = 1/(1+k')) = k'$ and $D(S = 1) = 0$. We now prove that $D(S)$ is monotonic decreasing:

$$\frac{dD}{dS} = -\frac{h(k,S)}{S^2\left[3 - 2(1+k^2)S + k^2S^2\right]^2} \tag{3.6.8}$$

where the function $h(k, S)$ is defined by

$$\begin{aligned} h(k,S) &= 3 - 4(1+k^2)S + 2(2k^4+k^2+2)S^2 - 4k^2(1+k^2)S^3 + 3k^4S^4 \\ &= (1-k^2S^2)^2 + 2[1-(1+k^2)S+k^2S^2]^2 + 2(1-k^2)^2S^2 \geq 0 \end{aligned} \tag{3.6.9}$$

with equality if and only if $S = 1$ and $k = 1$. This establishes the strict monotonicity. Hence we may write $\chi_0 = \chi_0(D)$. Next we calculate M^2 by using $G''(0) = -2$, i.e.,

$$2\,\frac{d^2 \log g(\chi)}{d\chi^2}\bigg|_{\chi=\chi_0} = -\frac{2}{M^2}\ . \tag{3.6.10}$$

This equation can be written (on eliminating D) as

$$\sqrt{\operatorname{sn}\chi_0 \operatorname{cn}\chi_0 \operatorname{dn}\chi_0}\,\frac{d^2}{d\chi_0^2}\left(\frac{1}{\sqrt{\operatorname{sn}\chi_0 \operatorname{cn}\chi_0 \operatorname{dn}\chi_0}}\right) = \frac{1}{2M^2} \tag{3.6.11}$$

in terms of the variable S introduced earlier we have,

$$\frac{1}{M^2} = \frac{(1-k^2S^2)^2 + 2[1-(1+k^2)S+k^2S^2]^2 + 2(1-k^2)^2S^2}{2S(1-S)(1-k^2S)} \tag{3.6.12}$$

The function M^2 is monotonically decreasing on $S \in [1/(1+k'), 1]$ i.e., on $\chi_0 \in [K/2, K]$ with $M^2(1) = 0$ and

$$M^2_{max} = \frac{1}{1+k'^2}. \tag{3.6.13}$$

The derivative with respect to S of M^2 is given by

$$\frac{d\,M^2}{dS} = \frac{6(1-k^2S^2)(k'^2+k^2(1-S)^2)(1-S(1+k'))(1-S(1-k'))}{\left((1-k^2S^2)^2 + 2[1-(1+k^2)S+k^2S^2]^2 + 2(1-k^2)^2S^2\right)^2} \geq 0. \tag{3.6.14}$$

We have equality only when $\chi_0 = K/2$.

Having found χ_0 and M^2 we may read off κ by noting that $G(0) = 1$, thus

$$\kappa = \frac{2M(\operatorname{cn}^2\chi_0 + D\operatorname{sn}^2\chi_0)^2}{\operatorname{sn}\chi_0 \operatorname{cn}\chi_0 \operatorname{dn}\chi_0}. \tag{3.6.15}$$

We may now go on to find the value of $\widetilde{g}$. We use the relation (3.A.7) that

$$\begin{aligned} 1-12\widetilde{g}^2 &= \tfrac{1}{2}M^4\left[(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2+(\epsilon_3-\epsilon_1)^2\right] \\ &= M^4(1-k^2+k^4) \\ &= M^4(1-k'^2+k'^4). \end{aligned} \tag{3.6.16}$$

We have already seen that $M^2 \leq 1/(1+k'^2)$. Hence the RHS of Eq. (3.6.16) is bounded above by

$$1-\frac{3k'^2}{(1+k'^2)^2} \leq 1. \tag{3.6.17}$$

That is to say the value $\widetilde{g}$ is uniquely determined. We now make use of the discriminant expression (3.A.8) to write

$$\frac{(\epsilon_1-\epsilon_2)^2(\epsilon_2-\epsilon_3)^2(\epsilon_3-\epsilon_1)^2}{\left((\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2+(\epsilon_3-\epsilon_1)^2\right)^3} = \frac{{g_2}^3-27{g_3}^2}{54{g_2}^3} \tag{3.6.18}$$

i.e.,

$$\frac{54k^4k'^4}{(1+k^4+k'^4)^3} = \frac{(1-12\widetilde{g}^2)^3-(1+36\widetilde{g}^2-54\widetilde{m}^2)^2}{(1-12\widetilde{g}^2)^3}. \tag{3.6.19}$$

This determines $\widetilde{m}$. Observe that the LHS takes values between $[0,1]$ attaining its upper bound only when $k^2=1/2$. We take

$$54\widetilde{m}^2 = 1+36\widetilde{g}^2-\left[1-\frac{54k^4k'^4}{(1+k^4+k'^4)^3}\right]^{1/2}(1-12\widetilde{g}^2)^{3/2} \tag{3.6.20}$$

for $k^2 \leq 1/2$ and

$$54\widetilde{m}^2 = 1+36\widetilde{g}^2+\left[1-\frac{54k^4k'^4}{(1+k^4+k'^4)^3}\right]^{1/2}(1-12\widetilde{g}^2)^{3/2} \tag{3.6.21}$$

when $k^2 \geq 1/2$. This is because when $k \to 0$ our solutions lie on the line given by Eq. (3.6.20) while when $k \to 1$ they satisfy Eq. (3.6.21). Continuity then determines which solution to take as we increase k from zero to one. □

Thus it suffices to find D from the quantities directly read off from the boundary conditions. These being L and q/D. For the next Lemma it is useful to define

$$\Delta = \frac{q}{DL} = \frac{\kappa \widetilde{g}}{M^3 D} \tag{3.6.22}$$

which we may assume is given.

Lemma. *Given the modulus k and the quantity Δ^2 defined by*

$$\Delta^2(D) = \frac{\kappa^2 \widetilde{g}^2}{M^6 D^2}, \tag{3.6.23}$$

we can invert to give $D = D(\Delta^2)$ provided

$$\Delta^2 \geq \frac{16(1 + 3k'^2 + k'^4)}{(1 + k')^2}\,. \tag{3.6.24}$$

Proof: Firstly we show that

$$\Delta^2(k') = \frac{16(1 + 3k'^2 + k'^4)}{(1 + k')^2} \tag{3.6.25}$$

rising monotonically to infinity. To see that Δ^2 is increasing on $S \in [1/(1 + k'), 1]$, we examine its derivative. We find

$$\frac{d\Delta^2}{dS} = \frac{64(1 - (1 - k')S)((1 + k')S - 1)f(k, S)h(k, S)}{S^2(1 - S)^2\left(1 - k^2S^2 + 2k'^2S\right)^3\left(1 - S^2 + k'^2 + \left(1 + k^2\right)(1 - S)^2\right)^3} \tag{3.6.26}$$

where we have defined

$$\begin{aligned} f(k, S) = \quad & -3 + \left(16 - 20k^2\right) S + \left(36 - 36k^2 + 28k^4\right) S^2 \\ & + \left(8 + 48k^2 + 44k^4 - 8k^6\right) S^3 + \left(-4 - 12k^2 - 50k^4 - 44k^6 - 4k^8\right) S^4 \\ & + \left(8k^2 + 36k^6 + 24k^8\right) S^5 + \left(-4k^4 + 4k^6 - 20k^8\right) S^6 + 4k^8S^7 - k^8S^8. \end{aligned} \tag{3.6.27}$$

We may write $f(k,S)$ in an explicitly non-negative form for $S,k\in[0,1]$,

$$
\begin{aligned}
f(k,S) = \quad & 1+2\left(1-S+5k'^2S\right)^2+2\left(2+11k^2+11k'^2k^2\right)S^2 \\
& +4\left(18k^2+5k^4+2k'^6\right)S^3(1-S)^5 \\
& +\left(4k^2k'^6+24k'^4+12k'^2+256k^2k'^2\right. \\
& \qquad \left.+96k^4k'^2+28k^2+318k^4\right)S^4(1-S)^4 \\
& +\left(440k^2k'^2+220k^4k'^2+8k^8+460k^4+64\right)S^5(1-S)^3 \\
& +\left(28+28k'^2+560k'^2k^2+232k'^2k^4+364k^4+28k^8\right)S^6(1-S)^2 \\
& +\left(24+216k'^2k^2+100k^4k'^2+128k^4+20k^8\right)S^7(1-S) \\
& +\left(1+3k'^8+28k^2k'^2+28k^2\right)S^8\geq 0.
\end{aligned} \tag{3.6.28}
$$

Thus we have proved that the derivative of Δ^2 is non-negative on the required domain and as it is clearly non-constant the derivative has isolated zeros (being analytic in S), therefore we may conclude that k and the value of Δ^2 in the range $[\Delta^2(k'),\infty)$ uniquely determine the mass and charge parameters, $\widetilde{m}$ and $\widetilde{g}$ for a suitable Ernst solution. Having done so we may then construct M which in turn determines the acceleration from the relation $A=M/L$. Thus we have one constraint on the range of the parameters representing the Ernst solution when we write it in terms of the elliptic functions introduced, namely

$$\Delta^2 \geq \frac{16\left(1+3k'^2+k'^4\right)}{\left(1+k'\right)^2}. \tag{3.6.29}$$

It remains an open question whether there exists other solutions to the Einstein-Maxwell system that behave asymptotically like the Ernst solutions that violate this condition which have no naked singularities or other serious defects.

3.A Appendix: Elliptic Integrals and Functions

In the next chapter we will be presenting a black hole uniqueness theorem for the Ernst Solution. It turns out that Elliptic integrals and the Weierstrass and Jacobi elliptic functions provide a valuable tool in establishing that result. We will be describing the solution in terms of new coordinates related to our previous ones by elliptic functions. In this appendix we will establish our conventions and collect together most of the general mathematical results concerning these functions which we will be using. These results are predominantly taken from Whittaker and Watson [19], where proofs may be found.

Just as the sine and cosine functions can be regarded as functions on a circle, when we have a doubly periodic function we may form the quotient of $\mathbb{C}$ by its period set. Let us call the period set Ω. When we quotient $\mathbb{C}$ by the lattice Ω we produce with a torus. In general two different lattices produce conformally inequivalent tori. For our purposes we will only need to consider lattices of the form $2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}$, where ω_1 is real and ω_3 is purely imaginary.

The Valency Theorem states that a non-constant analytic function between two compact Riemann Surfaces is exactly n-1 for some n, which we call its *valency*. We shall be applying this result when one of the compact Riemann surfaces is the Riemann Sphere and the other one is of these tori.

The first function we will need is the Weierstrass $\wp$-function, this is a doubly periodic meromorphic function with a second order pole at the origin, defined by

$$\wp_\Omega(\zeta) = \frac{1}{\zeta^2} + \sum_{\omega\in\Omega\setminus\{0\}} \left(\frac{1}{(\zeta-\omega)^2} - \frac{1}{\omega^2} \right). \tag{3.A.1}$$

We will define $\omega_2 = \omega_1 + \omega_3$ and $e_i = \wp(\omega_i)$. The $\wp$-function obeys the differential equation:

$$\wp'(\zeta)^2 = 4(\wp(\zeta) - e_1)(\wp(\zeta) - e_2)(\wp(\zeta) - e_3). \tag{3.A.2}$$

This is easily established by noting that the ratio of the LHS and the RHS has no poles and is therefore, by the Valency Theorem, constant, the constant is determined by examining what happens as $\zeta \to 0$. Looking at this limit we see

$$e_1 + e_2 + e_3 = 0. \tag{3.A.3}$$

The differential equation is then given by

$$\wp'(\zeta)^2 = 4\wp(\zeta)^3 - g_2\wp(\zeta) - g_3. \tag{3.A.4}$$

We will call g_2 and g_3 the *invariants* of the $\wp$-function. Note that

$$e_1e_2 + e_2e_3 + e_3e_1 = -\tfrac{1}{4}g_2 \qquad \text{and} \tag{3.A.5}$$

$$e_1e_2e_3 = \tfrac{1}{4}g_3. \tag{3.A.6}$$

Therefore

$$(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2 = \tfrac{3}{2}g_2 \tag{3.A.7}$$

and

$$(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = \frac{{g_2}^3 - 27{g_3}^2}{16}. \tag{3.A.8}$$

Integrating the differential equation we find the elliptic integral

$$\zeta = \int_z^\infty \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} \tag{3.A.9}$$

has the solution $z = \wp(\zeta)$. We also point out that the Weierstrass $\wp$-function has the scaling property:

$$\wp(M\zeta; M\Omega) = M^{-2}\wp(\zeta; \Omega). \tag{3.A.10}$$

The invariants of $\wp$ for $M\Omega$ being $g_2' = M^{-4}g_2$, $g_3' = M^{-6}g_3$.

We may evaluate elliptic integrals of the form

$$\int_{x_0}^{x} \frac{dt}{\sqrt{f(t)}} \tag{3.A.11}$$

for any quartic $f(t) = a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4$ with a root x_0, in terms of an appropriate $\wp$ function. The invariants of which are given by

$$g_2 = a_0a_4 - 4a_1a_3 + 3{a_2}^2 \tag{3.A.12}$$

$$g_3 = a_0a_2a_4 + 2a_1a_2a_3 - {a_2}^3 - a_0{a_3}^2 - {a_1}^2a_4. \tag{3.A.13}$$

The result being:

$$x - x_0 = \frac{f'(x_0)}{4\left(\wp(\zeta) - \frac{1}{24}f''(x_0)\right)} \tag{3.A.14}$$

and

$$\sqrt{f(x)} = \frac{-f'(x_0)\wp'(\zeta)}{4\left(\wp(\zeta) - \frac{1}{24}f''(x_0)\right)^2}. \tag{3.A.15}$$

The Weierstrass function will be extremely useful in what follows. It is also highly useful to define the Jacobi elliptic functions that in some sense generalize the trigonometric functions.

$$\operatorname{sn}\zeta = \sqrt{\frac{e_1 - e_3}{\wp(\zeta/M) - e_3}}, \tag{3.A.16}$$

$$\operatorname{cn}\zeta = \sqrt{\frac{\wp(\zeta/M) - e_1}{\wp(\zeta/M) - e_3}}, \tag{3.A.17}$$

$$\operatorname{dn}\zeta = \sqrt{\frac{\wp(\zeta/M) - e_2}{\wp(\zeta/M) - e_3}}, \tag{3.A.18}$$

(3.A.19)

with $M = \sqrt{e_1 - e_3}$. These functions obey certain algebraic and differential identities. We introduce the *modulus*, k defined by

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} \tag{3.A.20}$$

and the *complementary modulus*, k' defined by $k'^2 = 1 - k^2$. For the situation we will be considering the modulus will be real-valued and in the range $[0, 1]$. The algebraic identities we need are

$$\mathrm{cn}^2\,\zeta = 1 - \mathrm{sn}^2\,\zeta, \tag{3.A.21}$$

$$\mathrm{dn}^2\,\zeta = 1 - k^2\,\mathrm{sn}^2\,\zeta. \tag{3.A.22}$$

Whilst the derivatives are given by

$$\frac{d}{d\zeta}\,\mathrm{sn}\,\zeta = \mathrm{cn}\,\zeta\,\mathrm{dn}\,\zeta, \tag{3.A.23}$$

$$\frac{d}{d\zeta}\,\mathrm{cn}\,\zeta = -\,\mathrm{sn}\,\zeta\,\mathrm{dn}\,\zeta, \tag{3.A.24}$$

$$\frac{d}{d\zeta}\,\mathrm{dn}\,\zeta = -k^2\,\mathrm{sn}\,\zeta\,\mathrm{cn}\,\zeta. \tag{3.A.25}$$

The functions $\mathrm{sn}\,\zeta$ and $\mathrm{cn}\,\zeta$ are periodic with periods $4K$ whilst that of $\mathrm{dn}\,\zeta$ has period $2K$ where

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{3.A.26}$$

It is useful to define the analogous quantity associated with the complementary modulus, namely:

$$K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}. \tag{3.A.27}$$

After rescaling so that $\epsilon_i = e_i/M^2$ we find that

$$\epsilon_1 = \frac{1}{3}(1 + k'^2), \tag{3.A.28}$$

$$\epsilon_2 = \frac{1}{3}(k^2 - k'^2), \tag{3.A.29}$$

$$\epsilon_3 = -\frac{1}{3}(1 + k^2). \tag{3.A.30}$$

The Jacobi functions obey the addition rules:

$$\operatorname{sn}(\zeta + C) = \frac{\operatorname{sn}\zeta \operatorname{cn} C \operatorname{dn} C + \operatorname{sn} C \operatorname{cn}\zeta \operatorname{dn}\zeta}{1 - k^2 \operatorname{sn}^2\zeta \operatorname{sn}^2 C} , \tag{3.A.31}$$

$$\operatorname{cn}(\zeta + C) = \frac{\operatorname{cn}\zeta \operatorname{cn} C - \operatorname{sn}\zeta \operatorname{sn} C \operatorname{dn}\zeta \operatorname{dn} C}{1 - k^2 \operatorname{sn}^2\zeta \operatorname{sn}^2 C} , \tag{3.A.32}$$

$$\operatorname{dn}(\zeta + C) = \frac{\operatorname{dn}\zeta \operatorname{dn} C - k^2 \operatorname{sn}\zeta \operatorname{sn} C \operatorname{cn}\zeta \operatorname{cn} C}{1 - k^2 \operatorname{sn}^2\zeta \operatorname{sn}^2 C} . \tag{3.A.33}$$

In particular when $C = K$ we can use $\operatorname{sn} K = 1$, $\operatorname{cn} K = 0$, $\operatorname{dn} K = k'$ together with $\operatorname{sn} 0 = 0$, $\operatorname{cn} 0 = \operatorname{dn} 0 = 1$ and the fact that $\operatorname{sn}\zeta$ is an odd function of ζ whilst both $\operatorname{cn}\zeta$ and $\operatorname{dn}\zeta$ are even to deduce

$$\operatorname{sn}(K - \zeta) = \frac{\operatorname{cn}\zeta}{\operatorname{dn}\zeta} , \tag{3.A.34}$$

$$\operatorname{cn}(K - \zeta) = \frac{k' \operatorname{sn}\zeta}{\operatorname{dn}\zeta} , \tag{3.A.35}$$

$$\operatorname{dn}(K - \zeta) = \frac{k'}{\operatorname{dn}\zeta} . \tag{3.A.36}$$

Although these formulae are valid for all complex values of ζ it will be convenient to write $\zeta = \chi + i\eta$ and to be able to decompose the elliptic functions into real and imaginary parts in terms of functions of χ and η. The addition formulae allow us to do this provided we know the values of the functions evaluated on a purely imaginary argument. For this we use Jacobi's imaginary transform:

$$\operatorname{sn} i\eta = i \frac{\operatorname{sn}\eta}{\operatorname{cn}\eta}, \tag{3.A.37}$$

$$\operatorname{cn} i\eta = \frac{1}{\operatorname{cn}\eta}, \tag{3.A.38}$$

$$\operatorname{dn} i\eta = \frac{\operatorname{dn} \eta}{\operatorname{cn} \eta}, \tag{3.A.39}$$

where importantly the elliptic functions on the RHS of each of the above equations is with modulus k'. For brevity we will always regard the elliptic functions as being with modulus k unless the argument is η when it should be understood that the modulus is k'. This should not cause confusion in what follows as we will be doing few manipulations involving Jacobi elliptic functions with respect to the complementary modulus.

We will need to expand $\operatorname{sn} \zeta$, $\operatorname{cn} \zeta$ and $\operatorname{dn} \zeta$ for small values of the argument. We find

$$\operatorname{sn} \zeta = \zeta - \frac{1}{6}(1+k^2)\zeta^3 + O\left(\zeta^5\right), \tag{3.A.40}$$

$$\operatorname{cn} \zeta = 1 - \frac{1}{2}\zeta^2 + O\left(\zeta^4\right), \tag{3.A.41}$$

$$\operatorname{dn} \zeta = 1 - \frac{1}{2}k^2\zeta^2 + O\left(\zeta^4\right). \tag{3.A.42}$$

Finally we note that the $\wp$-function with $\omega_1 = K$ and $\omega_3 = iK'$ may be expressed in terms of the Jacobi functions by

$$\wp(\zeta) = -\frac{1+k^2}{3} + \frac{1}{\operatorname{sn}^2 \zeta} = \frac{\operatorname{cn}^2 \zeta + \frac{1}{3}(1+k'^2)\operatorname{sn}^2 \zeta}{\operatorname{sn}^2 \zeta}. \tag{3.A.43}$$

4. BLACK HOLE UNIQUENESS THEOREMS FOR THE ERNST SOLUTION AND C-METRIC

The question of black hole uniqueness was finally settled in 1983 when Bunting [6] and Mazur [7] independently completed the proof of uniqueness of Kerr-Newman solution which represents a rotating black hole in an asymptotically flat spacetime. Carter had given a thorough treatment of the result leaving aside the final step in his review article [21] some years earlier. Recently it has become of interest to study spacetimes that are not asymptotically flat. In particular the asymptotically Melvin solutions are of some theoretical importance as the Melvin solution models a cosmological magnetic field.

If we adopt a path integral approach to Euclidean quantum gravity, we might consider semi-classical processes mediated by instantons: i.e., exact regular solutions to the classical equations of motion. These solutions are expected to dominate the path integral under certain circumstances and give us an insight into some non-perturbative aspects of the full quantum theory. In this chapter we will be investigating the uniqueness of the C-metric and Ernst solution that we discussed in Chap. 3. It is important to ascertain the uniqueness of the saddle point in the path integral. Our results will prove that there is only one saddle point that contributes to the path integral and we may draw the conclusion that it will give the dominant contribution. This removes one possible objection to the argument that topology change is an essential feature of quantum gravity.

The uniqueness theorems we will be presenting for the C-metric and Ernst solutions are schematically identical to the proof of the uniqueness theorem for the Kerr-Newman black hole. However, the devil is in the details. The most difficult complication arises because of the presence of another horizon: the acceleration horizon. The boundary conditions are then given on five distinct regions: two horizons, two sections of the axis and at infinity. Infinity will be represented as a single point on the boundary after a suitable transformation of coordinates. The other portions of the boundary form a rectangle. The fact that not all rectangles are conformally homeomorphic will be the major complicating factor. Contrast this situation with what happens in the Kerr-Newman uniqueness theorem. In this case there are four parts to the boundary, this is represented by a semi-infinite rectangle, the non-existent fourth side mathematically describes arbitrarily large distances. By a simple scale and an appropriate translation any two such rectangles may be made to coincide.

The uniqueness theorems work by comparing two solutions defined on the same domain, and this is why it is important that the two domains should be conformally homeomorphic, one then uses a suitable divergence identity to prove uniqueness. Establishing a suitable expression with a positive divergence was the obstacle that prevented the uniqueness theorem for the Einstein-Maxwell theory from being proved soon after the corresponding result was proved in Einstein's theory. In Sect. 4.5 we will present a new proof of the positivity of the divergence, tackling the problem with the understanding we gained from Sect. 2.4.

We will be making extended use of the theory of Riemann surfaces in our deliberations. Riemann surface theory is a valuable asset when it comes to investigating the introduction of Weyl coordinates, a necessary step in the theorem. We have already seen in Chap. 3 how effective the application of Riemann surface theory and elliptic function theory was to the description of the C-metric and Ernst solution. We will be making use of these functions once again when it comes to presenting the appropriate boundary conditions to cause the vanishing of the boundary integral arising from applying Stokes' theorem to the divergence identity that we have established.

Our investigations begin in Sect. 4.1 with a summary of the hypotheses that we shall be assuming about any candidate spacetime, and the field equations they satisfy. In Sect. 4.2 we present a detailed derivation of the Generalized Papapetrou Theorem allowing us to introduce t and ϕ coordinates associated the Killing vectors corresponding to invariance under time translations and axial symmetry. This is most efficiently done by making good use of the algebra of differential forms.

Having found two 'standard' coordinates we introduce Weyl coordinates in Sect. 4.3, verifying the fact that the field equations demand that ρ, the norm of the Killing bivector, is harmonic on the two dimensional orbit space. In this section we make use of Riemann surface theory and apply the Riemann Mapping Theorem to do most of the hard work. The following section, Sect. 4.4, shows how one of the metric functions, specifically the conformal factor on the orbit space present after introducing Weyl coordinates, is uniquely determined (subject to an asymptotic boundary condition determining an overall scale) once all the other fields have been found. The remaining field equations being independent of the conformal factor.

Our new proof of the positivity of the relevant divergence for the Einstein-Maxwell theory is presented in Sect. 4.5. It exploits the derivation of the Bergmann metric as an induced metric from a complex manifold ultimately arising from the embedding of a hyperboloid in $\mathbb{C}^3$ with a Minkowskian metric.

To complete the uniqueness proofs we provide the relevant boundary conditions that will make the boundary integral derived in the previous section vanish. These conditions are spelt out in Sect. 4.6.

In Sect. 4.7 we discuss the relevance of our result to the semi-classical process of black hole monopole pair creation. As we have mentioned the uniqueness of the instantons that might mediate such a process is an important issue. Finally we summarize our progress and draw some conclusions in Sect. 4.8.

4.1 Hypotheses

In this section we set down in detail the hypotheses will be assuming in order to establish our uniqueness result. The main differences with the Kerr-Newman black hole uniqueness theorem occur in the boundary conditions and the overall horizon structure we will be assuming. It turns out that the different horizon structure makes proving a uniqueness result much more difficult and necessitates the use of elliptic functions and integrals.

Below we present the hypotheses we will be using for the rest of this chapter:

- Axisymmetry: There exists a Killing vector m such that $\mathcal{L}_m \boldsymbol{g} = 0$, and $\mathcal{L}_m \boldsymbol{F} = 0$

which generates a one-parameter group of isometries whose orbits are closed space-like curves.

- Stationarity: There exists a Killing vector K such that $\mathcal{L}_K \boldsymbol{g} = 0$, and $\mathcal{L}_K \boldsymbol{F} = 0$ which generates a one-parameter group of isometries which acts freely and whose orbits near infinity are timelike curves.
- Commutivity: $[K, m] = 0$.
- Source-free Maxwell equations $\mathbf{d}\boldsymbol{F} = 0$ and $\boldsymbol{\delta F} = 0$ together with the Einstein equations $R_{ab} = 8\pi T_{ab}$ where

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right). \tag{4.1.1}$$

- The domain of outer communication is connected and simply-connected.
- The solution has the same horizon structure as the Ernst Solution.
- For the Ernst solution uniqueness result we assume the solution is asymptotically Melvin's Magnetic Universe, whereas we assume asymptotic flatness when we come to prove the uniqueness of the C-metric.
- Boundary conditions (See Sect. 4.6).

4.2 The Generalized Papapetrou Theorem

Following Carter [20, 21], we shall prove that there exist coordinates t and ϕ defined globally on the domain of outer communication so that the metric takes a diagonal form with the Killing vectors $K = \partial/\partial t$ and $m = \partial/\partial\phi$. In addition we will prove that the electromagnetic field tensor can be derived from a vector potential satisfying the appropriate circularity and invariance conditions.

Our starting point is to make the remark that for a Killing vector K, the Laplacian $-(\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})$ acting on $\boldsymbol{k}$, reduces to $\boldsymbol{\delta}\mathbf{d}\boldsymbol{k} = 2\,\boldsymbol{R}(\boldsymbol{k})$. Here $\boldsymbol{R}(\boldsymbol{k}) = R_{ab}K^a\boldsymbol{e}^b$ is the Ricci form with respect to $\boldsymbol{k}$. We calculate

$$\begin{aligned} \boldsymbol{\delta}(\boldsymbol{k} \wedge \mathbf{d}\boldsymbol{k}) &= -\mathcal{L}_K(\mathbf{d}\boldsymbol{k}) - \boldsymbol{k} \wedge \boldsymbol{\delta}\mathbf{d}\boldsymbol{k} \\ &= -2\,\boldsymbol{k} \wedge \boldsymbol{R}(\boldsymbol{k})\,. \end{aligned} \tag{4.2.1}$$

Hence

$$\begin{aligned} \boldsymbol{\delta}(\boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}\boldsymbol{k}) &= \mathcal{L}_m(\boldsymbol{k} \wedge \mathbf{d}\boldsymbol{k}) + \boldsymbol{m} \wedge \boldsymbol{\delta}(\boldsymbol{k} \wedge \mathbf{d}\boldsymbol{k}) \\ &= 2\,\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{R}(\boldsymbol{k})\,. \end{aligned} \tag{4.2.2}$$

Now the energy-momentum form of the electromagnetic field with respect to $\boldsymbol{k}$ is given by the formula:

$$\boldsymbol{T}(\boldsymbol{k}) = \frac{1}{4\pi}\left(*(i_K\boldsymbol{F} \wedge *\boldsymbol{F}) - \frac{1}{2}*(\boldsymbol{F} \wedge *\boldsymbol{F})\boldsymbol{k} \right)\,. \tag{4.2.3}$$

We shall observe how Maxwell's equations, $\mathbf{d}\boldsymbol{F} = 0$ and $\boldsymbol{\delta}\boldsymbol{F} = 0$, imply the conservation equation, $\boldsymbol{\delta}\boldsymbol{T}(\boldsymbol{k}) = 0$:

$$\begin{aligned} 4\pi\boldsymbol{\delta}\boldsymbol{T}(\boldsymbol{k}) &= -*\mathbf{d}i_K\boldsymbol{F} \wedge *\boldsymbol{F} - \frac{1}{2}\boldsymbol{\delta}\left(\boldsymbol{k} \wedge *\left(\boldsymbol{F} \wedge *\boldsymbol{F}\right)\right) \\ &= -*\mathbf{d}i_K\boldsymbol{F} \wedge *\boldsymbol{F} + \frac{1}{2}\left[\mathcal{L}_K*\left(\boldsymbol{F} \wedge *\boldsymbol{F}\right) + \boldsymbol{k} \wedge \boldsymbol{\delta}*\left(\boldsymbol{F} \wedge *\boldsymbol{F}\right)\right] \\ &= -*\left(\mathcal{L}_K\boldsymbol{F} \wedge *\boldsymbol{F}\right) + \frac{1}{2}*\mathcal{L}_K\left(\boldsymbol{F} \wedge *\boldsymbol{F}\right) = 0. \end{aligned} \tag{4.2.4}$$

Let us now calculate $\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{F}$. Using $\mathcal{L}_K \boldsymbol{F} = \mathcal{L}_m \boldsymbol{F} = 0$ and $\boldsymbol{\delta F} = 0$ we have

$$\begin{aligned} \boldsymbol{\delta}(\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{F}) &= -\mathcal{L}_K(\boldsymbol{m} \wedge \boldsymbol{F}) - \boldsymbol{k} \wedge \boldsymbol{\delta}(\boldsymbol{m} \wedge \boldsymbol{F}) \\ &= \boldsymbol{k} \wedge \mathcal{L}_m \boldsymbol{F} + \boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{\delta F} = 0. \end{aligned} \tag{4.2.5}$$

Hence $\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{F} = c\boldsymbol{\eta}$, for some constant c and $\boldsymbol{\eta}$ the volume form. The boundary condition that $\boldsymbol{m} \to 0$ as one approaches the axis requires $c = 0$, thus proving

$$\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{F} = 0. \tag{4.2.6}$$

Another way to express this is $i_K i_m {*\boldsymbol{F}} = 0$. We will also need to examine the analogous quantity $i_K i_m \boldsymbol{F}$. We have

$$\mathbf{d} i_K i_m \boldsymbol{F} = \mathcal{L}_K i_m \boldsymbol{F} - i_K \mathcal{L}_m \boldsymbol{F} + i_K i_m \mathbf{d}\boldsymbol{F} = 0. \tag{4.2.7}$$

Using the axis-boundary condition again we see that

$$i_K i_m \boldsymbol{F} = 0. \tag{4.2.8}$$

Now use the fact that $\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{T}(\boldsymbol{k}) = -{*i_K i_m {*\boldsymbol{T}(\boldsymbol{k})}}$, but

$$\begin{aligned} 4\pi\, i_m i_K {*\boldsymbol{T}(\boldsymbol{k})} &= i_m(-i_K \boldsymbol{F} \wedge i_K {*\boldsymbol{F}}) \\ &= i_K i_m \boldsymbol{F} \wedge i_K {*\boldsymbol{F}} - i_K \boldsymbol{F} \wedge i_K i_m {*\boldsymbol{F}} = 0. \end{aligned} \tag{4.2.9}$$

That is to say $\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{T}(\boldsymbol{k}) = 0$. Einstein's equation, then proves from Eq. (4.2.2) that $\boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}\boldsymbol{k} = c_k \boldsymbol{\eta}$ and $\boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}\boldsymbol{m} = c_m \boldsymbol{\eta}$. The constants c_k and c_m are then seen be zero by another application of the boundary condition for $\boldsymbol{m}$ on the axis. Let us proceed to define 1-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ by the following:

$$\begin{aligned} \boldsymbol{\alpha} &= i_m\left(\frac{\boldsymbol{k}\wedge\boldsymbol{m}}{\rho^2}\right) \\ \boldsymbol{\beta} &= -i_K\left(\frac{\boldsymbol{k}\wedge\boldsymbol{m}}{\rho^2}\right) \end{aligned} \tag{4.2.10}$$

where

$$\rho^2 = i_K i_m(\boldsymbol{k}\wedge\boldsymbol{m}). \tag{4.2.11}$$

As before, ρ is the norm of the Killing bivector. Notice that by construction we have

$$\begin{aligned} \mathcal{L}_K\boldsymbol{\alpha} &= \mathcal{L}_m\boldsymbol{\alpha} = 0, \\ \mathcal{L}_K\boldsymbol{\beta} &= \mathcal{L}_m\boldsymbol{\beta} = 0, \end{aligned} \tag{4.2.12}$$

and also that

$$\begin{aligned} i_K\boldsymbol{\alpha} &= 1, \qquad & i_m\boldsymbol{\alpha} &= 0, \\ i_K\boldsymbol{\beta} &= 0, \qquad & i_m\boldsymbol{\beta} &= 1. \end{aligned} \tag{4.2.13}$$

Together these imply

$$i_K\mathbf{d}\boldsymbol{\alpha} = i_m\mathbf{d}\boldsymbol{\alpha} = i_K\mathbf{d}\boldsymbol{\beta} = i_m\mathbf{d}\boldsymbol{\beta} = 0. \tag{4.2.14}$$

The integrability conditions we have established may be rewritten as

$$\boldsymbol{k}\wedge\boldsymbol{m}\wedge\mathbf{d}\boldsymbol{\alpha} = 0 \qquad \text{and} \qquad \boldsymbol{k}\wedge\boldsymbol{m}\wedge\mathbf{d}\boldsymbol{\beta} = 0. \tag{4.2.15}$$

Evaluating

$$i_K i_m(\boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}\boldsymbol{\alpha}) \qquad \text{and} \qquad i_K i_m(\boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}\boldsymbol{\beta}) \tag{4.2.16}$$

we find

$$\rho^2 \mathbf{d}\boldsymbol{\alpha} = 0 \qquad \text{and} \qquad \rho^2 \mathbf{d}\boldsymbol{\beta} = 0. \tag{4.2.17}$$

Thus we may write

$$\boldsymbol{\alpha} = \mathbf{d}t \qquad \text{and} \qquad \boldsymbol{\beta} = \mathbf{d}\phi \tag{4.2.18}$$

in the Domain of Outer Communication, which we have assumed is simply-connected. Summarizing, we have shown the existence of coordinates t and ϕ satisfying:

$$\begin{aligned} \boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}t = 0, \qquad i_K \mathbf{d}t = 1, \qquad i_m \mathbf{d}t = 0, \\ \boldsymbol{k} \wedge \boldsymbol{m} \wedge \mathbf{d}\phi = 0, \qquad i_K \mathbf{d}\phi = 0, \qquad i_m \mathbf{d}\phi = 1. \end{aligned} \tag{4.2.19}$$

Turning to the electromagnetic field, Eq. (4.2.6) implies that $\boldsymbol{F}$ takes the form

$$\boldsymbol{F} = \boldsymbol{\alpha} \wedge \boldsymbol{\gamma} + \boldsymbol{\beta} \wedge \boldsymbol{\epsilon}. \tag{4.2.20}$$

Making the replacements:

$$\begin{aligned} \boldsymbol{\gamma} &\mapsto \boldsymbol{\gamma} - i_K\boldsymbol{\gamma}\,\boldsymbol{\alpha} - i_m\boldsymbol{\gamma}\,\boldsymbol{\beta}, \\ \boldsymbol{\epsilon} &\mapsto \boldsymbol{\epsilon} - i_K\boldsymbol{\epsilon}\,\boldsymbol{\alpha} - i_m\boldsymbol{\epsilon}\,\boldsymbol{\beta}, \end{aligned} \tag{4.2.21}$$

we see that $\boldsymbol{F}$ changes to

$$\boldsymbol{F} \mapsto \boldsymbol{F} + (-i_m\boldsymbol{\gamma} + i_K\boldsymbol{\epsilon})\boldsymbol{\alpha} \wedge \boldsymbol{\beta}. \tag{4.2.22}$$

However, Eq. (4.2.8) implies

$$i_K i_m \boldsymbol{F} = -i_m\boldsymbol{\gamma} + i_K\boldsymbol{\epsilon} = 0. \tag{4.2.23}$$

Thus, we may assume without loss of generality that

$$i_K\boldsymbol{\gamma} = 0, \qquad i_m\boldsymbol{\gamma} = 0,$$
$$i_K\boldsymbol{\epsilon} = 0, \qquad i_m\boldsymbol{\epsilon} = 0. \tag{4.2.24}$$

Maxwell's equation $\mathbf{d}\boldsymbol{F} = 0$ and the invariance of $\boldsymbol{F}$ under the action of the isometries generated by the Killing vectors reduce to the pair of equations:

$$\mathbf{d}i_K\boldsymbol{F} = 0, \qquad \text{and} \qquad \mathbf{d}i_m\boldsymbol{F} = 0. \tag{4.2.25}$$

Hence we may introduce electrostatic potentials according to

$$i_K\boldsymbol{F} \;\; = \boldsymbol{\gamma} \;\; = -\mathbf{d}\Phi, \tag{4.2.26}$$
$$i_m\boldsymbol{F} \;\; = \boldsymbol{\epsilon} \;\; = -\mathbf{d}\Psi. \tag{4.2.27}$$

with the potential function Φ for the electric field and Ψ for the magnetic field. It is now a simple matter to define an electromagnetic vector potential $\boldsymbol{A}$ with $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$ by setting

$$\boldsymbol{A} = \Phi\boldsymbol{\alpha} + \Psi\boldsymbol{\beta}. \tag{4.2.28}$$

It is now straightforward to verify that this vector potential satisfies the circularity and invariance conditions:

$$\boldsymbol{k} \wedge \boldsymbol{m} \wedge \boldsymbol{A} = 0 \qquad \text{and} \qquad \mathcal{L}_K\boldsymbol{A} = \mathcal{L}_m\boldsymbol{A} = 0. \tag{4.2.29}$$

In the next section we will look at how to find a set of coordinates that cover the entire Domain of Outer Communication. This involves using the t and ϕ coordinates we have just found together with the quantity ρ and its harmonic conjugate which shall denote by z.

4.3 Global Coordinates on the Domain of Outer Communication

In contrast to Carter's proof of the uniqueness for the Kerr-Newman black hole we will be exploiting the theory of Riemann surfaces to justify the introduction of Weyl coordinates on the Domain of Outer Communication. Previously this step in the uniqueness theorems has been done using Morse theory, however results in Morse theory rely heavily on complex variable methods and one should not be too surprised that the application of Riemann surface theory can successfully be used to prove the result we need. We have already seen how useful Riemann surface theory is when we discussed the C-metric and Ernst solution in Sects. 3.4 and 3.5.

Recall that in Sect. 3.1 we looked at Weyl solutions and introduced the (ρ, z) coordinate system. At the time we merely stated that these quantities provided a coordinate system. In the following sections we will take a more critical look at this introduction, and establish a set of globally defined coordinates that will be useful in establishing the uniqueness theorem we are going to prove.

There is a natural induced two-dimensional metric on the space of orbits of the two-parameter isometry group generated by the Killing vectors K and m. Define $\mathcal{M}_{\mathrm{II}}$ as the space of generic orbits (i.e., two-dimensional orbits) of the isometry group acting on the Domain of Outer Communication. We remark that the fixed point set of the isometry group generated by $\partial/\partial\phi$ is a closed subset of the spacetime. We call this set the *axis*. Notice too that $\mathcal{M}_{\mathrm{II}}$ is open, connected and non-empty. It is contained in the Hausdorff topological space consisting of all orbits of the isometry group acting on the spacetime. It is therefore non-compact (a compact subset of a Hausdorff space is closed, but if $\mathcal{M}_{\mathrm{II}}$ were both open and closed then it must be equal to the entire Hausdorff space, as the space of all orbits is connected, and therefore the axis would have to be empty which is not the case). Let the induced metric on $\mathcal{M}_{\mathrm{II}}$ be written

$$\widetilde{\boldsymbol{g}}_{\mathrm{II}} = \widetilde{g}_{\mathrm{II}\alpha\beta}\mathbf{d}x^{\alpha} \otimes \mathbf{d}x^{\beta}. \tag{4.3.1}$$

Since any two-dimensional metric is conformally flat, we can introduce orthonormal

1-forms $\boldsymbol{E}^1, \boldsymbol{E}^2$ such that

$$\widetilde{\boldsymbol{g}}_{\text{II}} = \Sigma \boldsymbol{g}_{\text{II}} = \Sigma(\boldsymbol{E}^1 \otimes \boldsymbol{E}^1 + \boldsymbol{E}^2 \otimes \boldsymbol{E}^2) \tag{4.3.2}$$

and $\Sigma(p) \neq 0$ for $p \in \mathcal{M}_{\text{II}}$ and where $\boldsymbol{g}_{\text{II}}$ is flat. Take p a base point in $\mathcal{M}_{\text{II}}$. Since $(\mathcal{M}_{\text{II}}, \boldsymbol{g}_{\text{II}})$ is flat and simply connected, its holonomy is trivial, and we may parallely transport the 1-forms $\boldsymbol{E}^1$ and $\boldsymbol{E}^2$ to all other points in $\mathcal{M}_{\text{II}}$ using

$$\mathbf{d}\boldsymbol{E}^\alpha = 0. \tag{4.3.3}$$

Now, as the fundamental group $\pi_1(\mathcal{M}_{\text{II}}) = \{\mathbb{1}\}$ we deduce that there exists scalars u, v such that

$$\boldsymbol{E}^1 = \mathbf{d}u \quad \text{and} \quad \boldsymbol{E}^2 = \mathbf{d}v. \tag{4.3.4}$$

Combining u and v into a complex quantity $\zeta = u + iv$ we see that we have a complex-valued function on the manifold $\mathcal{M}_{\text{II}}$, which need not be injective. However if $q \in \mathcal{M}_{\text{II}}$ then there is an open neighbourhood U of q such that $\zeta|_U : U \to \zeta(U)$ is one to one and hence $\mathcal{M}_{\text{II}}$ is a Riemann surface.

The quantities u and v do not necessarily constitute a coordinate system for the space $\mathcal{M}_{\text{II}}$ as the map ζ on $\mathcal{M}_{\text{II}}$ fails to be injective in general. However we will show that ρ and its harmonic conjugate are better behaved in this respect.

Let us consider the Einstein field equation for ρ. We will assume the metric has been put in the form

$$\boldsymbol{g} = -V\mathbf{d}t \otimes \mathbf{d}t + W(\mathbf{d}\phi \otimes \mathbf{d}t + \mathbf{d}t \otimes \mathbf{d}\phi) + X\mathbf{d}\phi \otimes \mathbf{d}\phi + \widetilde{\boldsymbol{g}}_{\text{II}}. \tag{4.3.5}$$

Explicitly we have $\rho^2 = XV + W^2$. Define

$$(h_{AB}) = \begin{pmatrix} -V & W \\ W & X \end{pmatrix} \qquad \text{and} \qquad \left(h^{AB}\right) = \frac{1}{\rho^2}\begin{pmatrix} -X & W \\ W & V \end{pmatrix}, \tag{4.3.6}$$

then $\rho^2 = -\det(h_{AB})$. Using Eq. (A.30) we find

$$ {}^4R_{AB}h^{AB} = -\frac{1}{2\rho}\nabla^\alpha\left(\rho h^{AB}\nabla_\alpha h_{AB}\right) = -\frac{1}{\rho}\nabla^2\rho, \tag{4.3.7}$$

where A and B refer to the t and ϕ coordinates whilst the covariant derivatives are with respect to the induced metric on the orbit space. Defining

$$ E_\alpha = F_{t\alpha} \qquad \text{and} \qquad B_\alpha = F_{\phi\alpha} \tag{4.3.8}$$

we have

$$ {}^4R_{tt} = 2\boldsymbol{E.E} + \tfrac{1}{2}VF^2, \tag{4.3.9}$$

$$ {}^4R_{t\phi} = 2\boldsymbol{E.B} - \tfrac{1}{2}WF^2, \tag{4.3.10}$$

$$ {}^4R_{\phi\phi} = 2\boldsymbol{B.B} - \tfrac{1}{2}XF^2, \tag{4.3.11}$$

where we have set

$$ F^2 = 2(-X\boldsymbol{E.E} + 2W\boldsymbol{E.B} + V\boldsymbol{B.B})\rho^{-2}. \tag{4.3.12}$$

Evaluating

$$ -\frac{1}{\rho}\nabla^2\rho = \frac{1}{\rho^2}\left(-{}^4R_{tt}X + 2\,{}^4R_{t\phi}W + {}^4R_{\phi\phi}V\right) = 0. \tag{4.3.13}$$

So we have shown that ρ is harmonic. Now any harmonic map may be written as the real or imaginary part of an analytic function, we therefore choose to write

$$ f(\zeta) = z(\zeta) - i\rho(\zeta); \qquad f \text{ analytic}, \tag{4.3.14}$$

so that $z(\zeta)$ is determined up to a constant by integrating the Cauchy-Riemann equations.

We are now in a position to apply the Riemann Mapping Theorem:

The Riemann Mapping Theorem. *Any simply-connected Riemann Surface is conformally homeomorphic to either*

(i) The Riemann Sphere $\mathbb{C}^{\infty}$,

(ii) The complex plane $\mathbb{C}$ *or*

(iii) The unit disc Δ.

We remarked earlier that $\mathcal{M}_{\mathrm{II}}$ is not compact, so $\mathcal{M}_{\mathrm{II}}$ in not conformally homeomorphic to the Riemann Sphere. It is easy to see that $\mathcal{M}_{\mathrm{II}}$ is not conformally homeomorphic to $\mathbb{C}$ either. For suppose it were, consider the function

$$\phi(\zeta) = \frac{1}{f(\zeta) - i} \, , \tag{4.3.15}$$

as $\rho > 0$ on $\mathcal{M}_{\mathrm{II}}$ this is a bounded entire function and hence by Liouville's theorem ϕ (and hence f) must be constant. So we are led to

$$\mathcal{M}_{\mathrm{II}} \cong \Delta. \tag{4.3.16}$$

We may assume from now on that ζ takes values on the unit disc, Δ. Next we make use of the asymptotically Melvin nature of the spacetime (the following also holds for asymptotically flat solutions):

In coordinates where the point at infinity has a neighbourhood conformally homeomorphic to the half-disc, with the point at infinity its centre, the function f should have a simple pole.

This follows easily by expanding the Melvin solution near infinity. We therefore map the unit disc to the lower half-plane by means of a Möbius transformation, we will be able to extend f to the real axis, where it takes purely real values. Next we apply Schwarz reflection in this axis to analytically extend the map to the entire Riemann Sphere. Having defined f on the Riemann Sphere allows us to make use of the Valency Theorem.

We consider the pre-image of infinity to work out the valency of the map, we have already remarked that f only has a simple pole and so by an application of the Valency Theorem the map f must be univalent, i.e., injective. Hence we have established that the coordinates (ρ, z) provide a diffeomorphism from $\mathcal{M}_{\mathrm{II}}$ to the space $\rho > 0$, and may indeed be employed as a coordinate system for the spacetime:

$$\boldsymbol{g} = -V\mathbf{d}t \otimes \mathbf{d}t + W(\mathbf{d}\phi \otimes \mathbf{d}t + \mathbf{d}t \otimes \mathbf{d}\phi) + X\mathbf{d}\phi \otimes \mathbf{d}\phi + \Sigma(\mathbf{d}\rho \otimes \mathbf{d}\rho + \mathbf{d}z \otimes \mathbf{d}z) \tag{4.3.17}$$

$$\boldsymbol{A} = \Phi\mathbf{d}t + \Psi\mathbf{d}\phi. \tag{4.3.18}$$

4.4 Determination of the Conformal Factor

We will show in this section that the conformal factor Σ decouples from the other equations, and for a general harmonic mapping of the type we are discussing can be found through quadrature, provided the harmonic mapping equations are themselves satisfied. In order to see this we make the dimensional reduction from three dimensions to two. We suppose that we have already made a dimensional reduction from four dimensions to three by exploiting the angular Killing vector, introducing Ernst potentials appropriately. The effective Lagrangian then takes the form:

$$\mathcal{L} = \sqrt{|\gamma|}\left({}^{3}R - 2\gamma^{ij}G_{AB}\frac{\partial\phi^A}{\partial x^i}\frac{\partial\phi^B}{\partial x^j}\right). \tag{4.4.1}$$

where the three metric is given by:

$$\boldsymbol{\gamma} = -\rho^2\mathbf{d}t \otimes \mathbf{d}t + \Sigma\left(\mathbf{d}\rho \otimes \mathbf{d}\rho + \mathbf{d}z \otimes \mathbf{d}z\right). \tag{4.4.2}$$

Performing the dimensional reduction on the Killing vector $\partial/\partial t$, using $\nabla^2\rho = 0$ and dropping a total divergence we find the effective Lagrangian is

$$\mathcal{L} = \sqrt{g_{\text{II}}}\, g_{\text{II}}^{\alpha\beta} \left(\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial x^\alpha} \frac{\partial \rho}{\partial x^\beta} - 2\rho\, G_{AB} \frac{\partial \phi^A}{\partial x^\alpha} \frac{\partial \phi^B}{\partial x^\beta} \right). \tag{4.4.3}$$

We have also discarded a term proportional to the Gauss curvature of the two dimensional metric. This term makes no contribution to the Einstein equations (the two dimensional Einstein tensor being trivial) nor does it contribute to the harmonic mapping equations. The Einstein equations, derived from variations with respect to the metric $\boldsymbol{g}_{\text{II}}$ are easily derived:

$$\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial z} = 4\rho\, G_{AB} \frac{\partial \phi^A}{\partial z} \frac{\partial \phi^B}{\partial \rho}, \tag{4.4.4}$$

$$\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial \rho} = 2\rho\, G_{AB} \left(\frac{\partial \phi^A}{\partial \rho} \frac{\partial \phi^B}{\partial \rho} - \frac{\partial \phi^A}{\partial z} \frac{\partial \phi^B}{\partial z} \right). \tag{4.4.5}$$

Variations with respect to the ϕ^A yield the harmonic mapping equation:

$$\nabla . \left(\rho\, G_{AB} \nabla \phi^B \right) - \frac{\rho}{2} \frac{\partial G_{BC}}{\partial \phi^A} \nabla \phi^B . \nabla \phi^C = 0. \tag{4.4.6}$$

This equation can be re-written as

$$\nabla . \left(\rho\, G_{AB} \nabla \phi^B \right) - \rho\, \Gamma^D_{AC} G_{BD} \nabla \phi^B . \nabla \phi^C = 0. \tag{4.4.7}$$

where Γ^D_{AC} is the Christoffel symbol derived from the metric $\boldsymbol{G}$ on the target space.

Multiplying Eq. (4.4.6) by $\partial\phi^A/\partial z$ leads to

$$\begin{aligned}\frac{\partial}{\partial z}\left(\rho\, G_{AB}\right)\left[\frac{\partial\phi^A}{\partial\rho}\frac{\partial\phi^B}{\partial\rho}-\frac{\partial\phi^A}{\partial z}\frac{\partial\phi^B}{\partial z}\right] &= 2\frac{\partial}{\partial\rho}\left(\rho\, G_{AB}\right)\frac{\partial\phi^A}{\partial z}\frac{\partial\phi^B}{\partial\rho}\\ &\quad + 2\rho\, G_{AB}\left[\frac{\partial^2\phi^B}{\partial\rho^2}+\frac{\partial^2\phi^B}{\partial z^2}\right]\frac{\partial\phi^A}{\partial z}. \qquad (4.4.8)\end{aligned}$$

It is now a simple matter to calculate the integrability condition for Σ:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial z}\frac{\partial}{\partial\rho}\log\Sigma-\frac{\partial}{\partial\rho}\frac{\partial}{\partial z}\log\Sigma\right) &= \frac{\partial}{\partial z}\left(\rho\, G_{AB}\right)\left[\frac{\partial\phi^A}{\partial\rho}\frac{\partial\phi^B}{\partial\rho}-\frac{\partial\phi^A}{\partial z}\frac{\partial\phi^B}{\partial z}\right]\\ &\quad - 2\frac{\partial}{\partial\rho}\left(\rho\, G_{AB}\right)\frac{\partial\phi^A}{\partial z}\frac{\partial\phi^B}{\partial\rho}\\ &\quad - 2\rho\, G_{AB}\left[\frac{\partial^2\phi^B}{\partial\rho^2}+\frac{\partial^2\phi^B}{\partial z^2}\right]\frac{\partial\phi^A}{\partial z}=0. \qquad (4.4.9)\end{aligned}$$

We have shown that Σ may be found once the harmonic mapping problem is solved. Finally we remark that the overall scale of Σ is determined by the asymptotic conditions.

4.5 The Divergence Identity.

In this section we give a new proof of the positivity of the electromagnetic generalization of Robinson's identity. This result proved rather elusive when the uniqueness theorems were first developed, being unsolved for nearly ten years, and Carter believed that only through an understanding of the underlying structure could progress be made. In contrast to the proofs given by Bunting and Mazur we do not lean too heavily on the sigma-model formalism but rather use the complex variable embedding of a hyperboloid in complex Minkowski space given in Sect. 2.4.

Recall that in that section the Poincaré and Bergmann metrics were given by the projection of complex rays in $\mathbb{C}^n$ with metric

$$\langle u,v\rangle = u^{\dagger}\eta v. \qquad (4.5.1)$$

With the parameterization given by Eq. (2.4.15), the electromagnetic Lagrangian takes the form

$$\mathcal{L} = \frac{\|(\nabla z)^{\perp}\|^2}{\|z\|^2}. \tag{4.5.2}$$

We are using cylindrical polar coordinates in $\mathbb{R}^3$ for the gradient operator above and have denoted the component of a quantity A orthogonal to z by $A^{\perp}$ where

$$A^{\perp} = A - \frac{\langle z, A\rangle z}{\|z\|^2}. \tag{4.5.3}$$

It should be noticed that there is some gauge freedom in the above Lagrangian; specifically the Lagrangian is unchanged if we multiply z by an arbitrary complex function. This just corresponds to the construction of the Bergmann metric as a projection. The Lagrangian gives rise to the following equation of motion:

$$\nabla(\nabla z)^{\perp} = \left(\frac{-1}{\|z\|^2}\right)\left(\|(\nabla z)^{\perp}\|^2 + \langle z, \nabla z\rangle(\nabla z)^{\perp}\right) \tag{4.5.4}$$

This implies the expression:

$$(\nabla^2 z)^{\perp} = 0. \qquad \text{i.e.,} \qquad \nabla^2 z = \left(\frac{1}{\|z\|^2}\right)\langle z, \nabla^2 z\rangle z. \tag{4.5.5}$$

As yet we have not made use of our gauge freedom. To begin with we shall use the freedom we have to normalize z so that $\|z\|^2 = -1$. In addition we still have the freedom to multiply z by an arbitrary phase. At any point we can exploit this freedom to set $\langle z, \nabla z\rangle = 0$. However, its derivative will not vanish in general. The normalization we have imposed implies $\nabla^2\|z\|^2 = 0$. Consequently we have,

$$\langle z, \nabla^2 z\rangle + \langle \nabla^2 z, z\rangle = -2\|\nabla z\|^2 = -2\|(\nabla z)^{\perp}\|^2. \tag{4.5.6}$$

The last equality coming from the phase gauge condition. Henceforth we will always be imposing these two conditions and therefore $\nabla z = (\nabla z)^{\perp}$.

The divergence identity comes from examining the Laplacian of

$$S = -\|z_1 \wedge z_2\|^2, \tag{4.5.7}$$

where we have extended the inner product to the exterior algebra in the standard manner. The fields z_1 and z_2 are assumed to obey both the field equation and the gauge conditions. We might notice that S is invariant under arbitrary changes in phase of z_1 and z_2. For the moment we point out that the imposition of our phase gauge condition merely serves to make our calculations simpler: the expansion of S does not depend on the parallel component of ∇z.

Before we perform the calculation we make the useful observation,

$$\begin{aligned} \|z_1 \wedge z_2\|^2 &= 1 - |\langle z_1, z_2\rangle|^2 \\ &= -\|z_1^{\perp_2}\|^2 \leq 0. \end{aligned} \tag{4.5.8}$$

Where $z_1^{\perp_2}$ is orthogonal to z_2 and being orthogonal to a timelike vector is spacelike.

Evaluating $\nabla^2 S$ we find,

$$\begin{aligned} \nabla^2 S &= -\langle \nabla^2 z_1 \wedge z_2 + 2\nabla z_1 \wedge \nabla z_2 + z_1 \wedge \nabla^2 z_2, z_1 \wedge z_2\rangle \\ &\quad - \langle z_1 \wedge z_2, \nabla^2 z_1 \wedge z_2 + 2\nabla z_1 \wedge \nabla z_2 + z_1 \wedge \nabla^2 z_2\rangle \\ &\quad - 2\|\nabla(z_1 \wedge z_2)\|^2. \end{aligned} \tag{4.5.9}$$

Making use of Eq. (4.5.5) we find

$$\begin{aligned}
\nabla^2 S &= -2\|z_1 \wedge z_2\|^2(\|\nabla z_1\|^2 + \|\nabla z_2\|^2) - 2\|\nabla(z_1 \wedge z_2)\|^2 \\
&\quad - 2\langle \nabla z_1 \wedge \nabla z_2, z_1 \wedge z_2\rangle - 2\langle z_1 \wedge z_2, \nabla z_1 \wedge \nabla z_2\rangle \\
&= 2|\langle z_1, \nabla z_2\rangle + \langle \nabla z_1, z_2\rangle|^2 \\
&\quad +2|\langle z_1, z_2\rangle|^2(\|\nabla z_1\|^2 + \|\nabla z_2\|^2) \\
&\quad +2\langle z_1, z_2\rangle\langle \nabla z_2, \nabla z_1\rangle + 2\langle z_2, z_1\rangle\langle \nabla z_1, \nabla z_2\rangle. \qquad (4.5.10)
\end{aligned}$$

Next we define $\Omega = \nabla(z_1 \wedge z_2)$ and evaluate the norm of the following quantities,

$$\langle z_1, \Omega\rangle = -(\nabla z_2 + \langle z_1, z_2\rangle \nabla z_1 + \langle z_1, \nabla z_2\rangle z_1) \qquad (4.5.11)$$

and

$$\langle z_2, \Omega\rangle = \nabla z_1 + \langle z_2, z_1\rangle \nabla z_2 + \langle z_2, \nabla z_1\rangle z_2. \qquad (4.5.12)$$

Notice that by construction each is spacelike, being orthogonal to the timelike vectors z_1 and z_2 respectively. We find that

$$\begin{aligned}
\|\langle z_1, \Omega\rangle\|^2 + \|\langle z_2, \Omega\rangle\|^2 &= \left(1 + |\langle z_1, z_2\rangle|^2\right)\left(\|\nabla z_1\|^2 + \|\nabla z_2\|^2\right) \\
&\quad + |\langle z_1, \nabla z_2\rangle|^2 + |\langle z_2, \nabla z_1\rangle|^2 \\
&\quad + 2\langle z_1, z_2\rangle\langle \nabla z_2, \nabla z_1\rangle + 2\langle z_2, z_1\rangle\langle \nabla z_1, \nabla z_2\rangle. \qquad (4.5.13)
\end{aligned}$$

Hence

$$\begin{aligned}\nabla^2 S &= \|\langle z_1, \Omega\rangle\|^2 + \|\langle z_2, \Omega\rangle\|^2 + |\nabla\langle z_1, z_2\rangle|^2 \\ &\quad + \left(|\langle z_1, z_2\rangle|^2 - 1\right)\left(\|\nabla z_1\|^2 + \|\nabla z_2\|^2\right) \\ &\quad + \langle \nabla z_1, z_2\rangle\langle z_1, \nabla z_2\rangle + \langle \nabla z_2, z_1\rangle\langle z_2, \nabla z_1\rangle. \end{aligned} \tag{4.5.14}$$

It only remains to notice that

$$\begin{aligned}|\langle \nabla z_1, z_2\rangle\langle z_1, \nabla z_2\rangle + \langle \nabla z_2, z_1\rangle\langle z_2, \nabla z_1\rangle| &\leq 2|\langle \nabla z_1, z_2\rangle|\,|\langle z_1, \nabla z_2\rangle| \\ &\leq 2\|z_1^{\perp_2}\|\,\|z_2^{\perp_1}\|\,\|\nabla z_1\|\,\|\nabla z_2\| \\ &\leq \left(|\langle z_1, z_2\rangle|^2 - 1\right)\left(\|\nabla z_1\|^2 + \|\nabla z_2\|^2\right)\ . \end{aligned} \tag{4.5.15}$$

We have made use of the Cauchy-Schwarz inequality on the positive-definite subspaces orthogonal to z_1 and to z_2 together with the AM-GM inequality. Putting all this together we have therefore shown that

$$\nabla^2 S \geq 0. \tag{4.5.16}$$

We have equality if and only if $\|z_1 \wedge z_2\|$ is constant. In particular if z_1 and z_2 agree up to a phase anywhere then the constant is zero.

Returning to the problem at hand we use the Ernst parameterization

$$z = \frac{1}{2\sqrt{X}} \begin{pmatrix} 1-\epsilon \\ 1+\epsilon \\ 2\psi \end{pmatrix} \tag{4.5.17}$$

with the Ernst potentials derived from the angular Killing vector.

$$\epsilon = -X - |\psi|^2 + iY, \tag{4.5.18}$$

$$\psi \;\; = \;\; E + iB. \qquad (4.5.19)$$

The condition $\|z_1 \wedge z_2\| = 0$ becomes

$$\frac{\hat{X}^2 + 2(X_1 + X_2)\left(\hat{E}^2 + \hat{B}^2\right) + \left(\hat{E}^2 + \hat{B}^2\right)^2 + \left(\hat{Y} + 2E_1B_2 - 2B_1E_2\right)^2}{4X_1X_2} = 0. \qquad (4.5.20)$$

where we have used the abbreviation $\hat{A} = A_2 - A_1$. Accordingly $X_1 = X_2$, $Y_1 = Y_2$, $E_1 = E_2$ and $B_1 = B_2$, which is to say the solution is unique.

We will make use of the positivity of the 3-dimensional Laplacian of S in exactly the same way we use Green's identity to prove the uniqueness of solutions to Laplace's equation. It is convenient to express the 3-dimensional Laplacian in terms of the two dimensional metric on the space $\mathcal{M}_{\text{II}}$. As the angular coordinate is ignorable we have then that

$$\nabla.(\rho \nabla S) \geq 0. \qquad (4.5.21)$$

Integrating over $\mathcal{M}_{\text{II}}$ and applying Stokes' theorem

$$\int_{\partial \mathcal{M}_{\text{II}}} \rho * \mathbf{d} S \geq 0, \qquad (4.5.22)$$

with equality if and only if S is constant.

4.6 Boundary Conditions

In this section we present appropriate boundary condition that will be sufficient to make

$$\int_{\partial \mathcal{M}_{\text{II}}} \rho * \mathbf{d} S = 0. \qquad (4.6.1)$$

The boundary conditions for the Ernst solution and the C-metric are presented seperately due to their different behaviour near infinity. Provided then that a candidate solution obeys these conditions and the horizon structure coincides with that of the

C-metric, we may deduce that our candidate solution is described mathematically by either an appropriate Ernst Solution or C-metric.

Having introduced Weyl coordinates to describe the candidate solution we may evaluate k the modulus of the elliptic functions by Eq. (3.4.8) from which we can construct K and K' by Eqs. (3.A.26) and (3.A.27). We may then use Eq. (3.4.15) with $2L^2 = z_A - z_S$ to relate (χ, η) to the coordinates (ρ, z) that we may assume the candidate spacetime metric is expressed with respect to. Once we have expressed the solution with respect to these new coordinates we may use the analysis in Sect. 3.6 to select an appropriate Ernst Solution to act as the other solution in the uniqueness proof. The vanishing of the boundary integral will then allow us to conclude that the two solutions are identical.

In these coordinates the integral expression becomes

$$\int_{\partial\mathcal{M}_{\mathrm{II}}} \rho \left(d\chi \frac{\partial}{\partial\eta} - d\eta \frac{\partial}{\partial\chi} \right) S = 0, \tag{4.6.2}$$

where from the last section:

$$S = \frac{\hat{X}^2 + 2(X_1 + X_2)(\hat{E}^2 + \hat{B}^2) + (\hat{E}^2 + \hat{B}^2)^2 + (\hat{Y} + 2E_1B_2 - 2B_1E_2)^2}{4X_1X_2} \ . \tag{4.6.3}$$

4.6.1 Boundary Conditions for the Ernst Solution Uniqueness Theorem

We will impose boundary conditions to make the integrand vanish. On the axis $\chi = 0$ we impose

$$\frac{1}{X}\frac{\partial X}{\partial\chi} = \frac{2}{\chi} + O(1); \tag{4.6.4}$$

$$\frac{\partial X}{\partial\eta} = O\left(\chi^{1/2}\right); \tag{4.6.5}$$

$$B = O\left(\chi^{3/2}\right); \tag{4.6.6}$$

$$\frac{\partial B}{\partial\chi} = O\left(\chi^{1/2}\right); \tag{4.6.7}$$

$$\frac{\partial B}{\partial \eta} = O\left(\chi^{1/2}\right); \tag{4.6.8}$$

$$E = O\left(\chi^{3/2}\right); \tag{4.6.9}$$

$$\frac{\partial E}{\partial \chi} = O\left(\chi^{3/2}\right); \tag{4.6.10}$$

$$\frac{\partial E}{\partial \eta} = O\left(\chi^{1/2}\right); \tag{4.6.11}$$

$$Y = O\left(\chi^{5/2}\right); \tag{4.6.12}$$

$$\frac{\partial Y}{\partial \chi} + 2\left(E\frac{\partial B}{\partial \chi} - B\frac{\partial E}{\partial \chi}\right) = O\left(\chi^{3/2}\right); \tag{4.6.13}$$

$$\frac{\partial Y}{\partial \eta} + 2\left(E\frac{\partial B}{\partial \eta} - B\frac{\partial E}{\partial \eta}\right) = O\left(\chi^{1/2}\right). \tag{4.6.14}$$

On the other axis we insist

$$\frac{1}{X}\frac{\partial X}{\partial u} = \frac{2}{u} + O(1); \tag{4.6.15}$$

$$\frac{\partial X}{\partial \eta} = O\left(u^{1/2}\right); \tag{4.6.16}$$

$$B = \frac{2}{B_0 + 8D/q} + O\left(u^{3/2}\right); \tag{4.6.17}$$

$$\frac{\partial B}{\partial u} = O\left(u^{1/2}\right); \tag{4.6.18}$$

$$\frac{\partial B}{\partial \eta} = O\left(u^{1/2}\right); \tag{4.6.19}$$

$$E = O\left(u^{3/2}\right); \tag{4.6.20}$$

$$\frac{\partial E}{\partial u} = O\left(u^{1/2}\right); \tag{4.6.21}$$

$$\frac{\partial E}{\partial \eta} = O\left(u^{1/2}\right); \tag{4.6.22}$$

$$Y = O\left(u^{5/2}\right) \tag{4.6.23}$$

$$\frac{\partial Y}{\partial u}+2\left(E\frac{\partial B}{\partial u}-B\frac{\partial E}{\partial u}\right) = O\left(u^{3/2}\right); \quad (4.6.24)$$

$$\frac{\partial Y}{\partial \eta}+2\left(E\frac{\partial B}{\partial \eta}-B\frac{\partial E}{\partial \eta}\right) = O\left(u^{1/2}\right). \quad (4.6.25)$$

with $u = K - \chi$. The condition on the fields as one approaches infinity is given by

$$X = \frac{4}{{B_0}^4 L^2 \sin^2\theta}\frac{1}{R}+O\left(\frac{1}{R^2}\right); \quad (4.6.26)$$

$$\frac{1}{X}\frac{\partial X}{\partial R} = -\frac{1}{R}+O\left(\frac{1}{R^2}\right); \quad (4.6.27)$$

$$\frac{\partial X}{\partial \theta} = O\left(\frac{1}{R}\right); \quad (4.6.28)$$

$$B = \frac{2}{B_0}-\frac{2}{{B_0}^3 L^2 \sin^2\theta}\frac{1}{R}+O\left(\frac{1}{R^2}\right); \quad (4.6.29)$$

$$\frac{\partial B}{\partial R} = \frac{2}{{B_0}^3 L^2 \sin^2\theta}\frac{1}{R^2}+O\left(\frac{1}{R^3}\right); \quad (4.6.30)$$

$$\frac{\partial B}{\partial \theta} = O\left(\frac{1}{R}\right); \quad (4.6.31)$$

$$E = O\left(\frac{1}{R^2}\right); \quad (4.6.32)$$

$$\frac{\partial E}{\partial R} = O\left(\frac{1}{R^3}\right); \quad (4.6.33)$$

$$\frac{\partial E}{\partial \theta} = O\left(\frac{1}{R}\right); \quad (4.6.34)$$

$$\frac{\partial Y}{\partial R}+2\left(E\frac{\partial B}{\partial R}-B\frac{\partial E}{\partial R}\right) = O\left(\frac{1}{R^4}\right); \quad (4.6.35)$$

$$\frac{\partial Y}{\partial \theta}+2\left(E\frac{\partial B}{\partial \theta}-B\frac{\partial E}{\partial \theta}\right) = O\left(\frac{1}{R}\right). \quad (4.6.36)$$

Near infinity we have set $\chi = R^{-1/2}\sin\theta$ and $\eta = R^{-1/2}\cos\theta$.

The boundary conditions on the horizons are particularly simple, we require the fields

(X, Y, E, B) to be regular and that $X > 0$ except where the axis and horizon meet. As $X > 0$ on this section of the boundary S will be well-behaved and hence

$$\int_0^K \rho \, d\chi \frac{\partial S}{\partial \eta} = 0 \tag{4.6.37}$$

as a result of $\rho = 0$.

4.6.2 Boundary Conditions for the C-metric Uniqueness Theorem

The boundary conditions will need to impose for the C-metric uniqueness result differ from those we required for the Ernst solution (with $B_0 = 0$) only in the condition at infinity. We require

$$X = 4L^2 \sin^2\theta \, R + O(1); \tag{4.6.38}$$

$$\frac{1}{X}\frac{\partial X}{\partial R} = \frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{4.6.39}$$

$$\frac{\partial X}{\partial \theta} = O(R); \tag{4.6.40}$$

$$B = O\left(\frac{1}{R}\right); \tag{4.6.41}$$

$$\frac{\partial B}{\partial R} = O\left(\frac{1}{R^2}\right); \tag{4.6.42}$$

$$\frac{\partial B}{\partial \theta} = O\left(\frac{1}{R}\right); \tag{4.6.43}$$

$$E = O\left(\frac{1}{R^2}\right); \tag{4.6.44}$$

$$\frac{\partial E}{\partial R} = O\left(\frac{1}{R^3}\right); \tag{4.6.45}$$

$$\frac{\partial E}{\partial \theta} = O\left(\frac{1}{R}\right); \tag{4.6.46}$$

$$\frac{\partial Y}{\partial R} + 2\left(E\frac{\partial B}{\partial R} - B\frac{\partial E}{\partial R}\right) = O\left(\frac{1}{R^4}\right); \tag{4.6.47}$$

$$\frac{\partial Y}{\partial \theta} + 2\left(E\frac{\partial B}{\partial \theta} - B\frac{\partial E}{\partial \theta}\right) = O\left(\frac{1}{R}\right). \tag{4.6.48}$$

These condition are sufficient to cause the appropriate boundary integral to vanish and allow us to deduce the uniqueness of the C-metric. We might remark that the nodal singularity that runs along one or both parts of the axis in this solution causes us no problem once we pass to the space of orbits $\mathcal{M}_{\mathrm{II}}$. Strictly we need to require that the range of the angular coordinate of our C-metric solution be chosen match that of our candidate solution at some point on the axis. The same remark may be made for the Ernst solution uniqueness theorem proved in the previous subsection.

4.7 Black Hole Monopole Pair Creation

We have concerned ourselves with the C-metric and its generalizations to include an external magnetic field. This solution has been of much recent interest. The path integral approach to Euclidean quantum gravity relies on the evaluation of amplitudes in the form

$$\int D\boldsymbol{g}D\boldsymbol{A}e^{-S[\boldsymbol{g},\boldsymbol{A}]}. \tag{4.7.1}$$

Here we set $\hbar = 1$. The path integral goes from the initial state to the final state over all possible paths. The quantity S is the classical action. We cannot perform such integrals. Indeed, it is highly non-trivial to even give meaning to the measures involved in its definition. However, abandoning any pretense at mathematical rigour, we say that the dominant contribution to such an integral is given by analogy with Laplace's method: We look for solutions which satisfy the classical variational principle:

$$\delta S = 0, \tag{4.7.2}$$

i.e., classical Riemannian solutions to the theory. In the real process, the Riemannian section is joined onto a Lorentzian section at a moment of time symmetry where all the momenta vanish. The solution tunnels from the imaginary time formulation to the real time solution. Thus black hole monopoles may be pair created by starting from a suitably strong electromagnetic field as represented by the Melvin solution Eq. (3.3.6). We need a Riemannian solution to Einstein's equations that might represent this process. The Ernst solution can be Wick rotated ($t \mapsto it$) to provide just such a solution, this is *The Melvin-Ernst Instanton* (Gibbons [22], Garfinkle and Strominger [23]). The solution will however acquire a nodal singularity unless we give the t-coordinate a periodicity given by 2π divided by the surface gravity of the horizons. In general the surface gravity of each horizon is different.

The requirement that the two surface gravities be equal reflects the fact that the heat-bath any accelerating observer sees must be in thermal equilibrium with the Hawking radiation from the black hole. In a sense then, it is an extra quantum mechanical condition on the stability of the classical solution, if the temperatures of the acceleration horizon and the black hole horizon where different we would not expect the solution to remain in an equilibrium state. This condition puts a restriction on the polynomial G,

$$G'(x_2) + G'(x_3) = 0. \tag{4.7.3}$$

This may be satisfied if

$$m = g, \tag{4.7.4}$$

and hence

$$G(x) = 1 - x^2(1 + \widetilde{m}x)^2. \tag{4.7.5}$$

This solution has topology $S^2 \times S^2 \setminus \{pt\}$, the topology of the $t =$ constant sections is $S^2 \times S^1 \setminus \{pt\}$ in comparison that of the Melvin solution is $\mathbb{R}^4$.

The quantization of charge now quantizes the mass, typical monopole masses can only be created with extreme magnetic fields, fields for which the Einstein-Maxwell theory is not appropriate to describe. However, the lesson to be learnt from this solution is that topology change must be taken into account in any reasonable theory of quantum gravity, as it is inconsistent not to do so.

4.8 Summary and Conclusion

We have studied the problem of extending the black hole uniqueness proofs to cover accelerating black holes as represented by the C-metric. In addition, we have considered the case where the acceleration has a physical motivating force in the form of a cosmological magnetic field; this situation being modelled by the Ernst Solution. By understanding these solutions to Einstein-Maxwell theory we have constructed a new set of coordinates that turn out to be intimately connected with the theory of elliptic functions and integrals. At first sight this appears to be a troublesome complication, however the elliptic functions are naturally defined on a one-parameter set of rectangles that in some sense are as natural as defining trigonometric functions on a range of 0 to 2π. The uniqueness proof makes good use of these standard rectangles and ultimately the divergence integral that finishes off the proof is over the boundary of one of them.

We also showed how the use of Riemann surface theory assists us to prove the validity of introducing Weyl coordinates in the Domain of Outer Communication. We are fortunate in that the Riemann Mapping Theorem for Riemann surfaces does much of the hard work. We also made good use of the Valency Theorem for compact Riemann surfaces, this allowed us to avoid using the Morse theory that is often employed to prove this step in the uniqueness theorems.

After showing how to determine the conformal factor for the induced two-dimensional metric for any sigma-model, we presented a new proof of the positivity of the divergence required to finish off the uniqueness results. This made use of the construction of the Bergmann metric from Sect. 2.4. It contrasts in style with both existing proofs due to Bunting [6] and Mazur [7]. Given the original difficulty of establishing the result (it was unproved for almost a decade) it is pleasing to present a new proof tackling the problem from a new angle.

We then discussed the boundary conditions required to make the appropriate boundary integral vanish. Fortunately the boundary conditions are as good as one could hope. They are able to distinguish between different Ernst solutions (as the must!) and yet they are not too restrictive. The asymptotically Melvin nature of these solutions we consider uniquely determines the cosmological magnetic field parameter, B_0 at infinity.

The condition on the axis determine what we have called D/q. The boundary conditions place no other consistency requirements. The known parameters then determine the other parameters: mass, m, charge g and acceleration A for the solution.

Finally we discussed the semi-classical process of black hole monopole pair creation. The uniqueness theorems we have established in the chapter foils a possible objection to the interpretation of these instantons as mediating a topology changing semi-classical process. By showing that the instanton is unique we rule out the possibility that the dominant contribution to the path integral be given by another exact solution. Had another solution existed we would have needed to ask which classical action were largest and possibly which contour we would have to take. The latter question being particularly difficult to formulate rigorously.

5. SUPERSTRING BLACK HOLE UNIQUENESS THEOREMS

5.1 Introduction

We now extend the black hole uniqueness theorems to the Superstring and $N = 4$ Supergravity theories. In order to make progress we will need to impose staticity rather than merely stationarity of the solutions, and naturally require the invariance of the dilaton under the action of the isometries generated by the Killing vectors. In addition we will only consider the case where the axionic field has been set equal to zero. This is consistent if we assume the electric and magnetic components are actually derived from two separate $U(1)$ gauge fields. The essential point to notice in our proof is that the effective Lagrangian in such a theory (2.7.2) is equal to that of two copies of that which we find for pure gravity. We will need to verify that the Weyl coordinate system may be introduced as before and then make use of Robinson's identity to establish the uniqueness result.

Firstly we will establish the uniqueness of a class of black holes obtained by performing the Double Ehlers' transform of Sect. 2.7.1 to a spherically symmetric solution found by Gibbons [13]. These solutions are asymptotically Melvin's Stringy Universe, it thus generalizes the result of Hiscock [24] for the Einstein-Maxwell theory. We could equally apply the theory to asymptotically flat solutions but one might feel that the uniqueness of such solutions should be proved under less stringent hypotheses, in particular we note that Masood-ul-Alam has already proved the uniqueness of an asymptotically flat black hole solution in these theories [25].

Secondly, we return to the Ernst solution and the C-metric, or rather their stringy variants and proceed to prove a theorem establishing their uniqueness. The solutions found here represent a generalization of those discussed by Dowker *et al.* [26], and reduce to them when the Double Ehlers' transform has equal parameters. It might be noted that they do not agree with those previously proposed by Ross [27].

In Sect. 5.2 we introduce the spherically symmetric solution in string theory that is the analogue of the Riessner-Nordstrøm black hole. We then perform a double Ehlers' transform to generate a new solution that will be the object of our uniqueness theorem. In the following section, Sect. 5.3, we carefully state the hypotheses we need to prove the theorem and justify the introduction of Weyl coordinates by proving that the norm of the Killing bivector is a harmonic function on the relevant orbit space.

In Sect. 5.4 we explain how Robinson's identity for the pure gravity can be exploited to give us a tool for establishing a uniqueness theorem in string theory and $N = 4$ supergravity subject to the hypotheses laid out in Sect. 5.3. We then complete the proof of our theorem by presenting sufficient boundary conditions to make the appropriate boundary integral vanish. These conditions are laid out in Sect. 5.5.

Having demonstrated how we may establish a uniqueness theorem in these theories we go on to apply our methods to the Stringy C-metric and Stringy Ernst solution. The Stringy C-metric is that found by Dowker *et al.* [26]. We apply the double Ehlers' transformation to derive the Stringy Ernst solution. As in Chap. 3 and Chap. 4 we transform coordinates to ones which have a strong relationship to the elliptic functions and integrals that we used in the last chapter. This is set out in Sect. 5.6. Then in Sect. 5.7 we write down the relevant boundary conditions to complete the uniqueness theorem for these solutions. Finally in the conclusion, Sect. 5.8, we make a few comments on the difficulties in generalizing the result.

5.2 The Class of Solutions

Our starting point is the spherically symmetric solution found by Gibbons [13]:

$$\boldsymbol{g} = -\left(1 - \frac{2M}{R}\right)\mathbf{d}t \otimes \mathbf{d}t + \left(1 - \frac{2M}{R}\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r + r\left(r - \frac{Q^2}{M}\right)\mathbf{d}\boldsymbol{\Omega}^2. \qquad (5.2.1)$$

The electromagnetic field and dilaton are given by

$$\boldsymbol{A} = \frac{Q}{r}\mathbf{d}t, \tag{5.2.2}$$

$$e^{2\phi} = 1 - \frac{Q^2}{Mr} \tag{5.2.3}$$

where we write ϕ for the dilaton field and φ for the angular coordinate.

Now apply the Double Ehlers' Transformation associated with the angular Killing vector $\partial/\partial\varphi$. The transformations are given by Eqs. (2.7.8) to (2.7.11).

The solution given above, Eqs. (5.2.1) to (5.2.2) have potentials:

$$X = r\left(r - \frac{Q^2}{M}\right)\sin^2\theta, \tag{5.2.4}$$

$$\psi_+ = 0, \tag{5.2.5}$$

$$\psi_- = \sqrt{2}Q\cos\theta, \tag{5.2.6}$$

together with (5.2.3). In consequence it is a simple matter to write down the transformed metric and fields:

$$\begin{aligned}\boldsymbol{g} = \Lambda\Theta\left(-\left(1-\frac{2M}{r}\right)\mathbf{d}t\otimes\mathbf{d}t + \left(1-\frac{2M}{r}\right)^{-1}\mathbf{d}r\otimes\mathbf{d}r + r\left(r-\frac{Q^2}{M}\right)\mathbf{d}\theta\otimes\mathbf{d}\theta\right)\\ + \frac{r}{\Lambda\Theta}\left(r-\frac{Q^2}{M}\right)\sin^2\theta\mathbf{d}\varphi\otimes\mathbf{d}\varphi,\end{aligned} \tag{5.2.7}$$

where

$$\Lambda = 1 + \beta^2\left(r - \frac{Q^2}{M}\right)^2\sin^2\theta \qquad \text{and} \qquad \Theta = 1 + \gamma^2 r^2\sin^2\theta. \tag{5.2.8}$$

The new dilaton and potentials are given by

$$\begin{aligned} e^{2\phi} &= \left(1 - \frac{Q^2}{M}\right)\frac{\Theta}{\Lambda}\,, && (5.2.9) \\ \psi_+ &= \frac{\beta}{\Lambda}\left(r - \frac{Q^2}{M}\right)^2 \sin^2\theta, && (5.2.10) \\ \psi_- &= \frac{1}{\Theta}\left[\sqrt{2}Q\cos\theta + \gamma\left(r^2\sin^2\theta + 2Q^2\cos^2\theta\right)\right]. && (5.2.11) \end{aligned}$$

5.3 The Hypotheses

Just as we did for the Einstein-Maxwell theory we will list the hypotheses we will need to prove our uniqueness theorems:

- Axisymmetry: There exists a Killing vector m such that $\mathcal{L}_m\boldsymbol{g} = 0$, $\mathcal{L}_m\boldsymbol{F} = 0$ and $\mathcal{L}_m\phi = 0$ which generates a one-parameter group of isometries whose orbits are closed spacelike curves.
- Staticity: There exists a hypersurface orthogonal Killing vector field K such that $\mathcal{L}_K\boldsymbol{g} = 0$, $\mathcal{L}_K\boldsymbol{F} = 0$ and $\mathcal{L}_K\phi$ which generates a one-parameter group of isometries which acts freely and whose orbits near infinity are timelike curves.
- Commutivity: $[K, m] = 0$.
- Source-free Maxwell equations $\mathbf{d}\boldsymbol{F} = 0$ and $\boldsymbol{\delta}\left(e^{-2\phi}\boldsymbol{F}\right) = 0$ together with the Einstein equations $R_{ab} = 2\nabla_a\phi\nabla_b\phi + 8\pi e^{-2\phi}T^{(F)}_{ab}$ where

$$T^{(F)}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right). \qquad (5.3.1)$$

- The domain of outer communication is connected and simply-connected.
- The solution contains a single black hole.

- The solution is asymptotically the Stringy Melvin Universe.
- Boundary conditions (See section 5.5).

We remark that the Generalized Papapetrou theorem of Sect. 4.2 goes through with a few very minor changes to take account of the modified Einstein and Maxwell relations. In particular the invariance of the dilaton field under the symmetries reads

$$i_K \mathbf{d}\phi = 0 \qquad \text{and} \qquad i_m \mathbf{d}\phi = 0. \tag{5.3.2}$$

Accordingly it does not contribute to $\boldsymbol{T}(\boldsymbol{k})$, which only changes by a factor of $e^{-2\phi}$. In addition the Staticity condition means that the cross term in the metric vanishes, i.e., $W = 0$.

The next step is to introduce Weyl coordinates. We repeat the calculation that ρ is a harmonic function on the space of orbits. Explicitly we have $\rho^2 = XV$. Defining as before

$$(h_{AB}) = \begin{pmatrix} -V & 0 \\ 0 & X \end{pmatrix} \qquad \text{and} \qquad \left(h^{AB}\right) = \frac{1}{\rho^2}\begin{pmatrix} -X & 0 \\ 0 & V \end{pmatrix}, \tag{5.3.3}$$

we need to calculate

$${}^4R_{AB}h^{AB} = -\frac{1}{2\rho}\nabla^\alpha\left(\rho h^{AB}\nabla_\alpha h_{AB}\right) = -\frac{1}{\rho}\nabla^2\rho. \tag{5.3.4}$$

Again A and B refer to the t and φ coordinates whilst the covariant derivatives are with respect to the induced metric on the two-dimensional orbit space. Defining

$$E_\alpha = F_{t\alpha} \qquad \text{and} \qquad B_\alpha = F_{\varphi\alpha} \tag{5.3.5}$$

we have

$${}^4R_{tt} = e^{-2\phi}\left(2\boldsymbol{E.E} + \tfrac{1}{2}VF^2\right), \tag{5.3.6}$$

$${}^4R_{\varphi\varphi} = e^{-2\phi}\left(2\boldsymbol{B.B} - \tfrac{1}{2}XF^2\right). \tag{5.3.7}$$

where

$$F^2 = 2(-X\boldsymbol{E}.\boldsymbol{E} + V\boldsymbol{B}.\boldsymbol{B})\rho^{-2}. \tag{5.3.8}$$

Notice that the invariance of ϕ means that $\partial\phi/\partial t = 0$ and $\partial\phi/\partial\varphi = 0$, and that therefore $\nabla_A\phi\nabla_B\phi$ makes no contribution to ${}^4R_{AB}$. The result is

$$-\frac{1}{\rho}\nabla^2\rho = \frac{1}{\rho^2}\left(-{}^4R_{tt}X + {}^4R_{\varphi\varphi}V\right) = 0. \tag{5.3.9}$$

Thus ρ is harmonic and we may go on to introduce its harmonic conjugate in just the same manner as we did in the previous chapter.

5.4 The Divergence Identity

We recall at this point our discussion in Sect. 2.7 and in particular that the effective two dimensional Lagrangian arising from string theory and $N = 4$ Supergravity takes the form

$$\mathcal{L} = \rho\sqrt{|\gamma|}\left[\frac{|\nabla X_+|^2 + |\nabla\psi_+|^2}{X_+^2} + \frac{|\nabla X_-|^2 + |\nabla\psi_-|^2}{X_-^2}\right] \tag{5.4.1}$$

where

$$X_+^2 = Xe^{2\phi} \qquad \text{and} \qquad X_-^2 = Xe^{-2\phi}. \tag{5.4.2}$$

Each term in the above Lagrangian is a copy of the Lagrangian for pure gravity and in consequence we may thus use Robinson's identity [28],

$$\nabla.\left(\rho\nabla\left(\frac{\hat{X_+}^2 + \hat{\psi_+}^2}{X_+^{(1)}X_+^{(2)}} + \frac{\hat{X_-}^2 + \hat{\psi_-}^2}{X_-^{(1)}X_-^{(2)}}\right)\right)$$
$$= F(X_+^{(1)}, X_+^{(2)}, \psi_+^{(1)}, \psi_+^{(2)}) + F(X_-^{(1)}, X_-^{(2)}, \psi_-^{(1)}, \psi_-^{(2)}) \geq 0, \tag{5.4.3}$$

where $F(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)})$ is defined by

$$F(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)}) = \frac{\rho}{X^{(1)}X^{(2)}}\left[\frac{\hat{Y}\nabla Y^{(1)}}{X^{(1)}} + \frac{X^{(1)}\nabla X^{(2)}}{X^{(2)}} - \nabla X^{(1)}\right]^2$$
$$+ \frac{\rho}{X^{(1)}X^{(2)}}\left[\frac{\hat{Y}\nabla Y^{(2)}}{X^{(2)}} - \frac{X^{(2)}\nabla X^{(1)}}{X^{(1)}} + \nabla X^{(2)}\right]^2$$
$$+ \frac{\rho}{2X^{(1)}X^{(2)}}\left[\left(\frac{\nabla Y^{(2)}}{X^{(2)}} - \frac{\nabla Y^{(1)}}{X^{(1)}}\right)(X^{(1)} + X^{(2)}) - \left(\frac{\nabla X^{(2)}}{X^{(2)}} + \frac{\nabla X^{(1)}}{X^{(1)}}\right)\hat{Y}\right]^2$$
$$+ \frac{\rho}{2X^{(1)}X^{(2)}}\left[\left(\frac{\nabla Y^{(2)}}{X^{(2)}} + \frac{\nabla Y^{(1)}}{X^{(1)}}\right)\hat{X} - \left(\frac{\nabla X^{(2)}}{X^{(2)}} + \frac{\nabla X^{(1)}}{X^{(1)}}\right)\hat{Y}\right]^2. \qquad (5.4.4)$$

As before we have defined $\hat{A} = A_2 - A_1$ etc. It is now evident that we may use this divergence identity to provide us with the key tool in establishing a black hole uniqueness theorem. To complete the proof we will want to change coordinates, and impose suitable boundary conditions to make the relevant boundary integral vanish. We make the change of coordinates:

$$\rho = r\sin\theta, \qquad (5.4.5)$$

$$z = r\cos\theta. \qquad (5.4.6)$$

The value of r runs from M to infinity (we adjust the additive constant to z to make the horizon run from $-M \leq z \leq M$). The overall scaling of ρ and z is made such that asymptotically r becomes the radial coordinate of the Stringy Melvin Universe, i.e.,

$$\boldsymbol{g} \sim A\rho^4(-\mathbf{d}t \otimes \mathbf{d}t + \mathbf{d}\rho \otimes \mathbf{d}\rho + \mathbf{d}z \otimes \mathbf{d}z) + \frac{1}{A\rho^2}\mathbf{d}\varphi \otimes \mathbf{d}\varphi. \qquad (5.4.7)$$

with φ taking values in $[0, 2\pi)$. It is worth remarking that we cannot rescale the coordinates and parameters and retain this form whilst leaving the range of φ unchanged, except for the trivial instance of multiplying the coordinates by -1.

The two dimensional domain we work on is the semi-infinite rectangle, $r > M$ and $-\pi/2 \leq \theta \leq \pi/2$, and the boundary integral we require to vanish is given by

$$\int r\cos\theta \left(r d\theta \frac{\partial}{\partial r} - \frac{dr}{r}\frac{\partial}{\partial \theta} \right) \left(\frac{\hat{X_+}^2 + \hat{\psi}_+^2}{X_+^{(1)} X_+^{(2)}} + \frac{\hat{X_-}^2 + \hat{\psi}_-^2}{X_-^{(1)} X_-^{(2)}} \right) = 0. \tag{5.4.8}$$

5.5 Boundary Conditions

We now need to impose suitable boundary conditions to make the boundary integral vanish. The following prove to be sufficient. At infinity we require

$$X_+ = \frac{1}{\beta^2 \sin\theta}\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{5.5.1}$$

$$\frac{1}{X_+}\frac{\partial X_+}{\partial r} = -\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{5.5.2}$$

$$\psi_+ = \frac{1}{\beta} - \frac{1}{\beta^3 \sin^2\theta}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right); \tag{5.5.3}$$

$$\frac{\partial \psi_+}{\partial r} = \frac{2}{\beta^3 \sin^2\theta}\frac{1}{r^3} + O\left(\frac{1}{r^4}\right); \tag{5.5.4}$$

$$X_- = \frac{1}{\gamma^2 \sin\theta}\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{5.5.5}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial r} = -\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{5.5.6}$$

$$\psi_- = \frac{1}{\gamma} - \frac{1 + \sqrt{2}\gamma Q\cos\theta}{\gamma^3 \sin^2\theta}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right); \tag{5.5.7}$$

$$\frac{\partial \psi_-}{\partial r} = \frac{2\left(1 + \sqrt{2}\gamma Q\cos\theta\right)}{\gamma^3 \sin^2\theta}\frac{1}{r^3} + O\left(\frac{1}{r^4}\right). \tag{5.5.8}$$

On the axes we require (setting $\mu = \sin\theta$)

$$\frac{1}{X_+}\frac{\partial X_+}{\partial \mu} = \frac{-\mu}{1-\mu^2} + O(1); \tag{5.5.9}$$

$$\frac{\partial X_+}{\partial r} = O\left(\left(1-\mu^2\right)^{1/2}\right); \tag{5.5.10}$$

$$\psi_+ = O(1-\mu^2); \tag{5.5.11}$$

$$\frac{\partial \psi_+}{\partial \mu} = O(1); \tag{5.5.12}$$

$$\frac{\partial \psi_+}{\partial r} = O(1-\mu^2); \tag{5.5.13}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial \mu} = \frac{-\mu}{1-\mu^2} + O(1); \tag{5.5.14}$$

$$\frac{\partial X_-}{\partial r} = O\left(\left(1-\mu^2\right)^{1/2}\right); \tag{5.5.15}$$

$$\psi_- = \frac{(\sqrt{2}\mu + 2\gamma Q)Q}{1+2\gamma^2 Q^2} + O(1-\mu^2); \tag{5.5.16}$$

$$\frac{\partial \psi_-}{\partial \mu} = O(1); \tag{5.5.17}$$

$$\frac{\partial \psi_-}{\partial r} = O(1-\mu^2); \tag{5.5.18}$$

where the boundaries correspond to $\mu = \pm 1$. On the horizon we require regularity of X_+, X_-, ψ_+ and ψ_-. These conditions are sufficient to make the boundary integral vanish and hence establish our uniqueness result.

5.6 Uniqueness Theorems for the Stringy C-metric and Stringy-Ernst Solution

In Chap. 4 we proved the uniqueness of both the C-metric and the Ernst solution. In this section we exploit the techniques developed there together with the string uniqueness formalism we have just been using to show that given any Stringy C-metric or Stringy Ernst solution then the boundary conditions uniquely specify the solution. Our philosophy here is slightly less ambitious than for Einstein-Maxwell theory; in the latter case we took the position that any candidate solution that resembled the Ernst solution at infinity was indeed an Ernst solution provided one of the quantities determined on the boundary was greater than a critical value. Here we assume we have an Ernst solution that does satisfy the boundary conditions and prove that no other solution can have the same boundary conditions.

Our starting point is the Dilaton C-metric found by Dowker *et al.* [26],

$$\boldsymbol{g} = \frac{1}{A^2(x-y)^2}\left[F(x)G(y)\mathbf{d}t\otimes\mathbf{d}t + \frac{F(y)\mathbf{d}x\otimes\mathbf{d}x}{G(y)} - \frac{F(x)\mathbf{d}y\otimes\mathbf{d}y}{G(y)} + F(y)G(x)\mathbf{d}\varphi\otimes\mathbf{d}\varphi\right], \qquad (5.6.1)$$

where

$$e^{-2\phi} = \frac{F(y)}{F(x)}\,, \qquad (5.6.2)$$

$$\boldsymbol{A} = \sqrt{\frac{r_+r_-}{2}}(x-x_2)\mathbf{d}\varphi, \qquad (5.6.3)$$

$$F(\xi) = 1 + r_-A\xi\,, \qquad (5.6.4)$$

$$G(\xi) = 1 - \xi^2 - r_+A\xi^3. \qquad (5.6.5)$$

We have labelled the roots of $G(x)$ as $x_3 < x_2 < x_1$ with $x_1 > 0$. The quantity x_4 corresponds to setting $F(x) = 0$, for which we assume $x_4 < x_3$ so as to represent an inner horizon for the black hole.

It is advantageous to represent this solution in terms of the Jacobi elliptic functions. We transform to new coordinates using

$$\frac{\chi}{M} = \int_{x_2}^{x}\frac{d\xi}{\sqrt{F(\xi)G(\xi)}} \qquad \text{and} \qquad \frac{\eta}{M} = \int_{y}^{x_2}\frac{d\xi}{\sqrt{-F(\xi)G(\xi)}}. \qquad (5.6.6)$$

with $M = \sqrt{e_1 - e_3}$ where $e_i = \wp(\omega_i)$ and ω_i being a half period as we had in Appendix 3.A. The appropriate invariants of the $\wp$-function are given by

$$g_2 = \frac{1 + 3A^2 r_-^2 - 9A^2 r_+ r_-}{12}, \tag{5.6.7}$$

$$g_3 = \frac{2 - 27A^2 r_+^2 - 18A^2 r_-^2 + 27A^2 r_+ r_- + 27A^4 r_+ r_-^3}{432}. \tag{5.6.8}$$

Writing the metric as

$$\boldsymbol{g} = -V\mathbf{d}t \otimes \mathbf{d}t + X\mathbf{d}\phi \otimes \mathbf{d}\phi + \Sigma\left(\mathbf{d}\chi \otimes \mathbf{d}\chi + \mathbf{d}\eta \otimes \mathbf{d}\eta\right), \tag{5.6.9}$$

we find:

$$X = \frac{4L^2\left(1 - D\,\mathrm{sn}^2\,\eta\right)\left(1 - E\,\mathrm{sn}^2\,\eta\right)\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi}{\left(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi\right)\left(\mathrm{cn}^2\,\eta + E\,\mathrm{sn}^2\,\eta\right)\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}; \tag{5.6.10}$$

$$V = \frac{4L^2\left(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi\right)\left(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi\right)\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\eta\,\mathrm{dn}^2\,\eta}{\left(1 - D\,\mathrm{sn}^2\,\eta\right)\left(1 - E\,\mathrm{sn}^2\,\eta\right)\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}; \tag{5.6.11}$$

$$\Sigma = \frac{16H^2L^2\left(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi\right)\left(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi\right)\left(1 - D\,\mathrm{sn}^2\,\eta\right)^2\left(1 - E\,\mathrm{sn}^2\,\eta\right)}{\kappa^2\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}. \tag{5.6.12}$$

We have written $M = AL$ together with

$$\kappa = \left.\frac{d(F(\xi)G(\xi))}{d\xi}\right|_{\xi=x_2}, \qquad D = \frac{1+k'^2}{3} - \frac{1}{24M^2}\left.\frac{d^2(F(\xi)G(\xi))}{d\xi^2}\right|_{\xi=x_2} \tag{5.6.13}$$

and

$$E = D + \frac{r_- A\kappa}{4M^2 H}, \qquad H = 1 + Ar_- x_2. \tag{5.6.14}$$

As before the quantity ρ is given by

$$\rho = \frac{4L^2\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,\mathrm{sn}\,\eta\,\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta}{\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}. \tag{5.6.15}$$

Thus once again we have $z - i\rho = 2L^2\wp(\chi + i\eta)$. The dilaton and vector potential are given by the expressions

$$e^{-2\phi} = \frac{(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)\,(1 - E\,\mathrm{sn}^2\eta)}{(\mathrm{cn}^2\chi + E\,\mathrm{sn}^2\chi)\,(1 - D\,\mathrm{sn}^2\chi)}\,; \tag{5.6.16}$$

$$\boldsymbol{A} = \frac{QD\,\mathrm{sn}^2\chi\mathbf{d}\varphi}{4\,(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)}\,; \qquad Q = \frac{\kappa\sqrt{r_+r_-}}{\sqrt{2}A^3L^2D}. \tag{5.6.17}$$

Performing the transformations Eqs. (2.7.8) to (2.7.11) we arrive at the metric of interest. The new metric and fields we have derived are:

$$X = \frac{4L^2\,(1 - D\,\mathrm{sn}^2\eta)\,(1 - E\,\mathrm{sn}^2\eta)\,\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi}{\Lambda\Theta\,(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)\,(\mathrm{cn}^2\eta + E\,\mathrm{sn}^2\eta)\,(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi)^2}\,; \tag{5.6.18}$$

$$V = \frac{4L^2\Lambda\Theta\,(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)\,(\mathrm{cn}^2\chi + E\,\mathrm{sn}^2\chi)\,\mathrm{sn}^2\eta\,\mathrm{cn}^2\eta\,\mathrm{dn}^2\eta}{(1 - D\,\mathrm{sn}^2\eta)\,(1 - E\,\mathrm{sn}^2\eta)\,(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi)^2}; \tag{5.6.19}$$

$$\Sigma = \frac{16H^2L^2\Lambda\Theta\,(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)\,(\mathrm{cn}^2\chi + E\,\mathrm{sn}^2\chi)\,(1 - D\,\mathrm{sn}^2\eta)^2\,(1 - E\,\mathrm{sn}^2\eta)}{\kappa^2\,(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi)^2}; \tag{5.6.20}$$

with

$$\Lambda = 1 + \beta^2\left\{\frac{4L^2\,\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi\,(1 - D\,\mathrm{sn}^2\eta)^2}{(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)^2\,(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi)^2} + \frac{Q^2D^2\,\mathrm{sn}^4\chi}{16\,(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)}\right\}; \tag{5.6.21}$$

$$\Theta = 1 + \frac{4\gamma^2L^2\,\mathrm{sn}^2\chi\,\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi\,(1 - E\,\mathrm{sn}^2\eta)^2}{(\mathrm{cn}^2\chi + E\,\mathrm{sn}^2\chi)^2\,(\mathrm{sn}^2\chi + \mathrm{sn}^2\eta\,\mathrm{cn}^2\chi)^2}\,. \tag{5.6.22}$$

The dilaton is given by

$$e^{-2\phi} = \frac{\Lambda\,(\mathrm{cn}^2\chi + D\,\mathrm{sn}^2\chi)\,(1 - E\,\mathrm{sn}^2\eta)}{\Theta\,(\mathrm{cn}^2\chi + E\,\mathrm{sn}^2\chi)\,(1 - D\,\mathrm{sn}^2\chi)}\,. \tag{5.6.23}$$

We record the values of the quantities $X_\pm$ and the potentials $\psi_\pm$:

$$X_+ = \frac{2L\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,(1-D\,\mathrm{sn}^2\,\eta)}{\Lambda\,(\mathrm{cn}^2\,\chi+D\,\mathrm{sn}^2\,\chi)\,(\mathrm{sn}^2\,\chi+\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)}; \tag{5.6.24}$$

$$X_- = \frac{2L\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,(1-E\,\mathrm{sn}^2\,\eta)}{\Theta\,(\mathrm{cn}^2\,\chi+E\,\mathrm{sn}^2\,\chi)\,(\mathrm{sn}^2\,\chi+\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)}; \tag{5.6.25}$$

$$\begin{aligned}\psi_+ = {} & \frac{1}{\Lambda}\left\{\frac{QD\,\mathrm{sn}^2\,\chi}{4\,(\mathrm{cn}^2\,\chi+D\,\mathrm{sn}^2\,\chi)}\right.\\ & \left.+\beta\left[\frac{4L^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi\,(1-D\,\mathrm{sn}^2\,\eta)^2}{(\mathrm{cn}^2\,\chi+D\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi+\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}+\frac{Q^2D^2\,\mathrm{sn}^4\,\chi}{16\,(\mathrm{cn}^2\,\chi+D\,\mathrm{sn}^2\,\chi)}\right]\right\};\end{aligned} \tag{5.6.26}$$

$$\psi_- = \frac{4\gamma L^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi\,(1-E\,\mathrm{sn}^2\,\eta)^2}{\Theta\,(\mathrm{cn}^2\,\chi+E\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi+\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}. \tag{5.6.27}$$

We will be interested in the behaviour of the fields as one takes the limits $\chi \to 0$, $u \to 0$ with $u = K - \chi$ and $R \to \infty$. The appropriate boundary conditions we need to make the boundary integral vanish are presented in the next section.

5.7 Boundary Conditions for the Stringy Ernst Solution and C-Metric

In order to complete the proof of the uniqueness for the Stringy Ernst solution and Stringy C-metric it only remains to write down a set of boundary conditions that will make the boundary integral vanish. It is fairly simple to verify that the conditions given in the following two subsections are sufficient for this purpose.

5.7.1 Boundary Conditions for the Stringy Ernst Solution Uniqueness Theorem

To start with we will require all the fields to be regular (and in addition for X_+ and X_- to not vanish) as one approaches the acceleration and event horizons. Near the axis

$\chi = 0$ we demand

$$\frac{1}{X_+}\frac{\partial X_+}{\partial \chi} = \frac{1}{\chi} + O(1); \tag{5.7.1}$$

$$\frac{\partial X_+}{\partial \eta} = O(\chi); \tag{5.7.2}$$

$$\psi_+ = O\left(\chi^2\right); \tag{5.7.3}$$

$$\frac{\partial \psi_+}{\partial \chi} = O(\chi); \tag{5.7.4}$$

$$\frac{\partial \psi_+}{\partial \eta} = O(\chi); \tag{5.7.5}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial \chi} = \frac{1}{\chi} + O(1); \tag{5.7.6}$$

$$\frac{\partial X_-}{\partial \eta} = O(\chi); \tag{5.7.7}$$

$$\psi_- = O\left(\chi^2\right); \tag{5.7.8}$$

$$\frac{\partial \psi_-}{\partial \chi} = O(\chi); \tag{5.7.9}$$

$$\frac{\partial \psi_-}{\partial \eta} = O(\chi). \tag{5.7.10}$$

For the other axis we will require

$$\frac{1}{X_+}\frac{\partial X_+}{\partial u} = \frac{1}{u} + O(1); \tag{5.7.11}$$

$$\frac{\partial X_+}{\partial \eta} = O(u); \tag{5.7.12}$$

$$\psi_+ = \frac{Q\left(4 + \beta Q\right)}{16 + \beta^2 Q^2} + O\left(u^2\right); \tag{5.7.13}$$

$$\frac{\partial \psi_+}{\partial u} = O\left(u^2\right); \tag{5.7.14}$$

$$\frac{\partial \psi_+}{\partial \eta} = O(u); \tag{5.7.15}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial u} = \frac{1}{u} + O(1); \tag{5.7.16}$$

$$\frac{\partial X_-}{\partial \eta} = O\left(u\right); \tag{5.7.17}$$

$$\psi_- = O\left(u^2\right); \tag{5.7.18}$$

$$\frac{\partial \psi_-}{\partial u} = O\left(u\right); \tag{5.7.19}$$

$$\frac{\partial \psi_-}{\partial \eta} = O\left(u\right). \tag{5.7.20}$$

Whilst as $R \to \infty$ with $\chi = R^{-1/2}\sin\theta$ and $\eta = R^{-1/2}\cos\theta$ we will demand

$$X_+ = \frac{1}{2\beta^2 L\sin\theta}\,\frac{1}{R^{1/2}} + O\left(\frac{1}{R^{3/2}}\right); \tag{5.7.21}$$

$$\frac{1}{X_+}\frac{\partial X_+}{\partial R} = -\frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{5.7.22}$$

$$\frac{\partial X_+}{\partial \theta} = O\left(\frac{1}{R^{1/2}}\right); \tag{5.7.23}$$

$$\psi_+ = \frac{1}{\beta} - \frac{1}{4\beta^3 L^2\sin^2\theta}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{5.7.24}$$

$$\frac{\partial \psi_+}{\partial R} = \frac{1}{4\beta^3 L^2\sin^2\theta}\,\frac{1}{R^2} + O\left(\frac{1}{R^3}\right); \tag{5.7.25}$$

$$\frac{\partial \psi_+}{\partial \theta} = O\left(\frac{1}{R}\right); \tag{5.7.26}$$

$$X_- = \frac{1}{2\gamma^2 L\sin\theta}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{5.7.27}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial R} = -\frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{5.7.28}$$

$$\frac{\partial X_-}{\partial \theta} = O\left(\frac{1}{R^{1/2}}\right); \tag{5.7.29}$$

$$\psi_- = \frac{1}{\gamma} - \frac{1}{4\gamma^3 L^2\sin^2\theta}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{5.7.30}$$

$$\frac{\partial \psi_-}{\partial R} = \frac{1}{4\sqrt{2}\gamma^3 L^2 \sin^2\theta}\frac{1}{R^2} + O\left(\frac{1}{R^3}\right); \tag{5.7.31}$$

$$\frac{\partial \psi_-}{\partial \theta} = O\left(\frac{1}{R^{1/2}}\right). \tag{5.7.32}$$

These boundary conditions are sufficient to establish the uniqueness of the Stringy Ernst solutions. For good measure we also present the boundary conditions for the Stringy C-metric problem.

5.7.2 Boundary Conditions for the Stringy C-Metric Uniqueness Theorem

The appropriate conditions are as follows. Near $\chi = 0$ we will insist

$$\frac{1}{X_+}\frac{\partial X_+}{\partial \chi} = \frac{1}{\chi} + O(1); \tag{5.7.33}$$

$$\frac{\partial X_+}{\partial \eta} = O(\chi); \tag{5.7.34}$$

$$\psi_+ = O\left(\chi^2\right); \tag{5.7.35}$$

$$\frac{\partial \psi_+}{\partial \chi} = O(\chi); \tag{5.7.36}$$

$$\frac{\partial \psi_+}{\partial \eta} = O(\chi); \tag{5.7.37}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial \chi} = \frac{1}{\chi} + O(1); \tag{5.7.38}$$

$$\frac{\partial X_-}{\partial \eta} = O(\chi); \tag{5.7.39}$$

$$\psi_- = O\left(\chi^2\right); \tag{5.7.40}$$

$$\frac{\partial \psi_-}{\partial \chi} = O(\chi); \tag{5.7.41}$$

$$\frac{\partial \psi_-}{\partial \eta} = O(\chi). \tag{5.7.42}$$

For the other axis we will require

$$\frac{1}{X_+}\frac{\partial X_+}{\partial u} = \frac{1}{u}+O(1); \qquad (5.7.43)$$

$$\frac{\partial X_+}{\partial \eta} = O\left(u\right); \qquad (5.7.44)$$

$$\psi_+ = \frac{Q}{2\sqrt{2}}+O\left(u^2\right); \qquad (5.7.45)$$

$$\frac{\partial \psi_+}{\partial u} = O\left(u\right); \qquad (5.7.46)$$

$$\frac{\partial \psi_+}{\partial \eta} = O\left(u\right); \qquad (5.7.47)$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial u} = \frac{1}{u}+O(1); \qquad (5.7.48)$$

$$\frac{\partial X_-}{\partial \eta} = O\left(u\right); \qquad (5.7.49)$$

$$\psi_- = O\left(u^2\right); \qquad (5.7.50)$$

$$\frac{\partial \psi_-}{\partial u} = O\left(u\right); \qquad (5.7.51)$$

$$\frac{\partial \psi_-}{\partial \eta} = O\left(u\right). \qquad (5.7.52)$$

Whilst as $R\to\infty$ with $\chi = R^{-1/2}\sin\theta$ and $\eta = R^{-1/2}\cos\theta$ we will demand

$$X_+ = 2L\sin\theta\, R^{1/2}+O\left(1\right); \qquad (5.7.53)$$

$$\frac{1}{X_+}\frac{\partial X_+}{\partial R} = \frac{1}{2R}+O\left(\frac{1}{R^2}\right); \qquad (5.7.54)$$

$$\frac{\partial X_+}{\partial \theta} = O\left(R^{1/2}\right); \qquad (5.7.55)$$

$$\psi_+ = \frac{QD\sin^2\theta}{2\sqrt{2}}\,\frac{1}{R}+O\left(\frac{1}{R^2}\right); \qquad (5.7.56)$$

$$\frac{\partial \psi_+}{\partial R} = -\frac{QD\sin^2\theta}{2\sqrt{2}}\,\frac{1}{R^2} + O\left(\frac{1}{R^3}\right); \tag{5.7.57}$$

$$\frac{\partial \psi_+}{\partial \theta} = O\left(\frac{1}{R}\right); \tag{5.7.58}$$

$$X_- = 2L\sin\theta\, R^{1/2} + O\left(1\right); \tag{5.7.59}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial R} = \frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{5.7.60}$$

$$\frac{\partial X_-}{\partial \theta} = O\left(R^{1/2}\right); \tag{5.7.61}$$

$$\psi_- = O\left(\frac{1}{R^2}\right); \tag{5.7.62}$$

$$\frac{\partial \psi_-}{\partial R} = O\left(\frac{1}{R^3}\right); \tag{5.7.63}$$

$$\frac{\partial \psi_-}{\partial \theta} = O\left(\frac{1}{R}\right). \tag{5.7.64}$$

5.8 Conclusion

We have been able to prove the uniqueness of two classes of asymptotically Melvin black holes. We would hope that the formalism developed in this chapter to prove the uniqueness of our class of black holes could be used to prove the uniqueness of other classes of static solutions in these theories. We would also like to have a formalism that incorporates the possibility of rotation and includes the axionic field, however it seems likely that such an extension would not be straightforward. The crux of the uniqueness proof is the establishing of the positivity of a suitable divergence. It turned out that for the static truncation of string theory that we considered the Lagrangian split into two separate copies of that for pure gravity. Consequently we could simply add together two copies of the relevant divergence identity (Robinson's identity) to furnish us with an expression that we could use in our black hole uniqueness investigations. If we include rotation or an axionic field the Lagrangian will not decompose so easily, and we would need to deal with it as a whole. This is problematical as the target space of the harmonic map possesses (at least) two timelike directions. Unfortunately this prohibits a simple application of the Mazur construction or a suitable analogue of the new construction presented in Sect. 4.5. It seems that Bunting's approach may be the best way forward under these circumstances relying, as it does, more heavily on the negative curvature of the target space metric than on its particular form as an $SU(1,2)/S\left(U(1)\times U(2)\right)$ symmetric space harmonic mapping system.

6. ANTI-GRAVITY BOUNDS

6.1 Introduction

In theories of gravity in four spacetime dimensions in which, in addition to the graviton, there are additional massless boson fields of spin zero and spin one the long range inverse square law attraction produced by the graviton and scalar or pseudo-scalar particles can to some extent be compensated by the long range repulsion produced by the spin one vector fields. However if one insists that there are no naked singularities and that the sources, if there are any, satisfy appropriate conditions, one typically finds that the repulsive forces can at best exactly compensate the attractive forces to produce a state of equipoise; they can never overwhelm the attractive forces altogether.

This phenomenon is sometimes referred to as *anti-gravity* and it arises in various theories, including supergravity and Kaluza-Klein theories. It is possible to investigate anti-gravity using some ideas from supersymmetry and supergravity even though the theories one may be interested in are not necessarily supergravity theories. Using a generalization [29] of Witten's proof of the positive mass theorem [30] it is possible to show that the absence of systems which are repulsive is a general phenomenon because the total mass of an isolated system is bounded below, in suitable units, by the magnitude of any of its central charges. A state of anti-gravity may occur if this bound is attained. In supersymmetric theories this state is typically supersymmetric.

In this chapter we place what seem to be different limits on the ADM mass M, scalar charge Σ and electromagnetics charges Q and P of an isolated system which also guarantees that it attracts at large distances from a different view-point. The idea is to adapt the technique exploited by Penrose, Sorkin and Woolgar to prove a version of the Positive Mass Theorem [31]. These authors establish a close link between the attractive properties of an isolated source (i.e., positivity of the total ADM mass M) and the non-negativity of the Ricci tensor R_{ab} of the four dimensional spacetime metric g_{ab} contracted with the tangent vector of a null geodesic. The non-negativity of the Ricci tensor is sufficient to establish a focussing property of null geodesics which plays an essential rôle in the proof.

The strategy to be adopted here is to consider not null geodesics but rather the timelike paths of particles of mass m, scalar charge gm and electric charge sm, in a background metric g_{ab}, scalar field ϕ and vector field $\boldsymbol{A}$. These timelike paths may be regarded as the projection of null geodesics moving in a suitable auxiliary five dimensional metric g_{AB}, $A = 1, 2, 3, 4, 5$ of signature $+ + + - +$ with Killing field $\partial/\partial x^5$. It then turns out that as long as the Ricci tensor, ${}^5R_{AB}$ of the auxiliary five dimensional metric ${}^5g_{AB}$ is non-negative when contracted with a null five-vector V^A then the total mass M, scalar charge Σ and electric charge Q must satisfy the *anti-gravity bound*

$$M - g\Sigma \geq s|Q| \ . \tag{6.1.1}$$

The result for a magnetic charge is given by considering the duality transformation:

$$\phi \mapsto -\phi \qquad \text{and} \qquad \boldsymbol{F} \mapsto e^{-2\phi} {*\boldsymbol{F}}. \tag{6.1.2}$$

This has the effect of swapping the electric and magnetic charges, and reversing the sign of Σ; hence the anti-gravity bound is given by

$$M + g\Sigma \geq s|P|. \tag{6.1.3}$$

The non-negative Ricci condition, which is the five dimensional null convergence condition used by Hawking and Ellis [32], will be satisfied as long as:

$$g^2 \leq 3 \ ; \quad s^2 \geq 2(g^2 - 1) \ ; \quad s^2 \leq 1 + g^2 \ . \tag{6.1.4}$$

Thus if $0 \leq g^2 \leq 1$ it suffices that

$$0 \leq s \leq \sqrt{1+g^2} \ , \tag{6.1.5}$$

if $1 \leq g^2 \leq 3$ it suffices that

$$\sqrt{2(g^2-1)} \leq s \leq \sqrt{1+g^2} \tag{6.1.6}$$

while if $g^2 > 3$ the non-negative Ricci condition will not in general be satisfied.

The structure of this chapter is as follows: In Sect. 6.2 we present the Lagrangian, *ansatz* and field equations for the class of theories including dilaton we will be studying. We identify some Nöther currents and comment on the electrodynamics of the system in terms of effective permittivities and permeabilities. In the next section, Sect. 6.3, we embed our four dimensional spacetime in an auxiliary five dimensional spacetime, and use Hamilton-Jacobi methods to solve for the null geodesics. We mirror the arguments of Penrose, Sorkin and Woolgar and establish our new inequality subject to the conditions set forth in Sect. 6.4 and 6.5. In Sect. 6.6 we compare our new inequality with the spherically symmetric black hole solutions in these theories, whilst in Sect. 6.7 we make the comparison with the analogues of the Papapetrou-Majumdar solutions that saturate our new bound. Before drawing our conclusions in Sect. 6.9 we contrast our anti-gravity bound with that previously discovered by Scherk. This is done in Sect. 6.8.

The new inequality Eq. (6.1.1) is the same as that obtained using the spinorial technique for pure Einstein-Maxwell theory but differs from it if there are scalar fields. Recall that in the absence of scalar fields (i.e., if $g = 0$) and in the absence of sources then the Bogomol'nyi bound [29] is

$$M \geq \sqrt{Q^2+P^2} \tag{6.1.7}$$

and saturation implies that the background is supersymmetric in that it admits Killing spinors when thought of as a solution of the $N = 2$ supergravity theory.

If sources are present then, as pointed out by Sparling and Moreschi [33] a straightforward modification of the spinorial argument shows that if the sources have a local energy density $T_{\bar{4}\bar{4}}$ to charge density $|J_{\bar{4}}|$ bounded by

$$T_{\bar{4}\bar{4}}/|J_{\bar{4}}| \geq s \tag{6.1.8}$$

with $0 \leq s \leq 1$ then

$$M \geq s\sqrt{Q^2 + P^2} \; . \tag{6.1.9}$$

If on the other hand the scalar fields are present but we stick to the case that there are no sources, i.e., if $g \neq 0$ then the spinorial technique [29] gives the bound

$$M \geq \frac{\sqrt{Q^2 + P^2}}{\sqrt{1 + g^2}} \; . \tag{6.1.10}$$

Finally if sources are present which satisfy

$$T_{\bar{4}\bar{4}}/|J_{\bar{4}}| \geq s e^{g\phi} \tag{6.1.11}$$

then the spinorial argument yields (6.1.9) as long as:

$$0 \leq s \leq \frac{1}{\sqrt{1 + g^2}} \; . \tag{6.1.12}$$

It seems clear therefore that since Eq. (6.1.1) and Eq. (6.1.10) do not coincide our new inequality is giving us some independent information from that provided by the spinorial method.

6.2 The Four Dimensional Field Equations

We shall consider theories, possibly with sources, whose field equations are derivable from an action of the form

$$\int \sqrt{-g}\, d^4x \left(R - e^{-2g\phi} F_{ab}F^{ab} - 2g^{ab}\nabla_a\phi\nabla_b\phi\right) \tag{6.2.1}$$

where $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$ and g is a dimensionless constant which takes different values for different theories. For example $g = \sqrt{3}$ for standard Kaluza-Klein theory, $g = 1/\sqrt{3}$ corresponds to dimensionally reduced Einstein-Maxwell theory from five to four dimensions [29], and $g = 1$ corresponds to a truncation of $N = 4$ supergravity theory and is related to superstring theory. Of course $g = 0$ corresponds to Einstein-Maxwell theory. Note that we are using signature $+ + + -$. An isolated system has mass M defined in the usual way and electric charge Q, magnetic charge P and scalar charge Σ defined by

$$A \sim \frac{Q}{r}\mathbf{d}t + P\cos\theta\mathbf{d}\varphi \ , \tag{6.2.2}$$

$$\phi \sim \frac{\Sigma}{r}. \tag{6.2.3}$$

Two such systems with masses M_1 and M_2, scalar charges Σ_1 and Σ_2, electric charges Q_1 and Q_2 and magnetic charges P_1 and P_2 will experience a net attraction of

$$(M_1M_2 + \Sigma_1\Sigma_2 - Q_1Q_2 - P_1P_2)\,\frac{1}{r^2} \ . \tag{6.2.4}$$

The field equations obtained from varying the action are (with appropriate additional sources T_{ab}, J and J_a)

$$R_{ab} = 2e^{-2g\phi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) + 2\nabla_a\phi\nabla_b\phi + 2\left(T_{ab} - \frac{1}{2}g_{ab}T_c{}^c\right), \tag{6.2.5}$$

$$\nabla_a\left(e^{-2g\phi}F_b{}^a\right) = J_b \tag{6.2.6}$$

and

$$\nabla_a(\nabla^a\phi + ge^{-2g\phi}F^{ab}A_b) = \nabla^2\phi + \frac{g}{2}e^{-2g\phi}F_{ab}F^{ab} = J \ . \tag{6.2.7}$$

The quantities $e^{-2g\phi}F^{ab}$ and $(\nabla^a\phi + ge^{-2g\phi}F^{ab}A_b)$ are, in the absence of the additional sources, the conserved Nöther currents associated with the transformations

$$\boldsymbol{A} \mapsto \boldsymbol{A} + \delta \boldsymbol{A} \tag{6.2.8}$$

and

$$\phi \mapsto \phi + \delta\phi, \tag{6.2.9}$$

$$\boldsymbol{A} \mapsto \boldsymbol{A} e^{g\delta\phi} \tag{6.2.10}$$

respectively.

From the point of view of the Maxwell field the dilaton field behaves as a dielectric constant

$$\epsilon = e^{-2g\phi} \tag{6.2.11}$$

and magnetic permeability

$$\mu = e^{2g\phi}. \tag{6.2.12}$$

Note that the product $\epsilon\mu$ is unity so the local speed of light is still one, which is consistent with local Lorentz invariance. However now because empty space behaves in the presence of a dilaton field a little like a material medium one must distinguish the electric field strength $E_i = F_{4i}$ from the divergence-free electric displacement $D_i = \epsilon E_i = e^{-2g\phi} E_i$ and divergence-free magnetic induction $B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$ from the magnetic field strength $H_i = \mu^{-1} B_i = \frac{1}{2} e^{-2g\phi} \epsilon_{ijk} F^{jk}$. The contribution of the Maxwell field to the local energy density is thus given by:

$$\tfrac{1}{2}\left(\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H}\right) \ . \tag{6.2.13}$$

We will be investigating flat space dilaton-electrodynamics solutions in Chap. 7.

6.3 The Five Dimensional Metric

Let us now consider the metric defined on $\mathcal{M}_5 = \mathbb{R} \times \mathcal{M}_4$, where $\mathcal{M}_4$ is the physical spacetime manifold.

$$\boldsymbol{g}^{(5)} = e^{2\nu}(\mathbf{d}x^5 + s\boldsymbol{A}) \otimes (\mathbf{d}x^5 + s\boldsymbol{A}) + e^{2\chi} g_{ab} \mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta \qquad (6.3.1)$$

where ν and χ are scalar fields to be specified later.

Thus the five dimensional metric is a twisted warped product. Note that in contrast to the standard Kaulza-Klein approach we are *not* identifying the fifth coordinate x^5. Moreover, we will insist that $e^{2\nu}$ and $e^{2\chi}$ never vanish so that both the five dimensional metric g_{AB} and the four dimensional metric g_{ab} are regular outside any event horizons. In this way we exclude possible counter-examples involving metrics which while regular in five dimensions are not regular as four dimensional metrics [34, 35].

Geodesics of the five dimensional metric may be found by consideration of the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2} g_{AB} \dot{x}^A \dot{x}^B \qquad (6.3.2)$$

where dot denotes $d/d\lambda$ and λ is any parameter along the geodesic. We will find it convenient to use standard Hamilton-Jacobi theory to investigate the relationship between the geodesics and the Hamilton-Jacobi function $\mathcal{S}$. We may write down momenta conjugate to the coordinates x^A and construct a Hamiltonian density. Set

$$p_A = \frac{\partial \mathcal{L}}{\partial \dot{x}^A} = g_{AB} \dot{x}^B \qquad (6.3.3)$$

and

$$\mathcal{H}(x^A, p_B) = \tfrac{1}{2} g^{AB} p_A p_B \ . \qquad (6.3.4)$$

The coordinates (x^A, p_B) are local coordinates on phase space, i.e., the cotangent bundle $T^*(\mathcal{M}_5)$. We find the Hamilton-Jacobi equation for this system by replacing the conjugate momenta with the gradient of some scalar $\mathcal{S}$, the resulting quantity is set equal to a constant as $\mathcal{H}$ is a first integral of the motion (it is zero for null geodesics and negative for timelike ones). Let this constant be $-m^2$. We have

$$\mathcal{H}(x^A, \nabla_B \mathcal{S}) = -\tfrac{1}{2}m^2, \tag{6.3.5}$$

i.e.,

$$g^{AB}\frac{\partial \mathcal{S}}{\partial x^A}\frac{\partial \mathcal{S}}{\partial x^B} = -m^2 \ . \tag{6.3.6}$$

Any solution of Eq. (6.3.6) gives rise to a set of geodesics by setting

$$p_A = \nabla_A \mathcal{S} \tag{6.3.7}$$

to be the tangent vector field to the geodesics, furthermore if $m > 0$ then

$$\frac{d\mathcal{S}}{d\lambda} = \nabla_A \mathcal{S}\frac{dx^A}{d\lambda} = -m^2. \tag{6.3.8}$$

Thus showing locally the Hamilton-Jacobi function $\mathcal{S}$ to be proportional to the proper time measured along the family of geodesics from some spacelike hypersurface. Conversely one may start from a spacelike hypersurface and construct a family of surfaces by propagating the initial surface a given distance along the geodesics defined by the surface normal. This should really be regarded as a construction in the total space of the cotangent bundle (i.e., in phase space) as we can lift geodesics from the manifold without concerning ourselves with the possibility that the family of geodesics may intersect. The relation to phase space is rather interesting in this respect. A congruence of geodesics is lifted to a four dimensional Lagrangian submanifold in phase space which is the natural place to consider the geodesics, the projection onto the manifold may result in geodesics with intersections and our surfaces of constant $\mathcal{S}$ may be contain caustics. Indeed, the formation of caustics plays an essential rôle in the proof of our result.

For the null case ($m = 0$) we have no notion of proper time, so we take any affine parameter. Notice that Eq. (6.3.6) is now invariant under any diffeomorphism, $f : \mathbb{R} \to \mathbb{R}$, which corresponds to changing the affine parameterization. Since x^5 is ignorable we have the result

$$p_5 = \frac{\partial \mathcal{S}}{\partial x^5} = \epsilon \qquad \text{constant}. \tag{6.3.9}$$

The Hamilton-Jacobi function then takes the form:

$$\mathcal{S} = \epsilon x^5 + S \tag{6.3.10}$$

we find that S satisfies the four dimensional Hamilton-Jacobi equation

$$g^{ab}\left(\frac{\partial S}{\partial x^a} - s\epsilon A_a\right)\left(\frac{\partial S}{\partial x^b} - s\epsilon A_b\right) = -\epsilon^2 e^{2(\chi-\nu)} \tag{6.3.11}$$

while Eq. (6.3.3) with Eq. (6.3.6) becomes

$$\frac{dx^5}{d\lambda} = \epsilon e^{-2\nu} - s e^{-2\chi} g^{ab}\left(\frac{\partial S}{\partial x^a} - s\epsilon A_a\right) A_b \tag{6.3.12}$$

and

$$\frac{dx^a}{d\lambda} = g^{ab} e^{-2\chi}\left(\frac{\partial S}{\partial x^b} - s\epsilon A_b\right). \tag{6.3.13}$$

The interpretation of these equations is that the projection of null geodesics into four dimensions gives the world line of a particle of mass

$$m = \epsilon \tag{6.3.14}$$

and charge

$$q = s\epsilon \tag{6.3.15}$$

with a coupling to the scalar field $(\chi - \nu)$.

If one now considers a four-metric g_{ab} which, in a quasi-Cartesian coordinate system, to order $1/r$ is given by

$$\boldsymbol{g} \sim -\left(1 - \frac{2M}{r}\right)\mathbf{d}t \otimes \mathbf{d}t + \left(1 + \frac{2M}{r}\right)(\mathbf{d}x \otimes \mathbf{d}x + \mathbf{d}y \otimes \mathbf{d}y + \mathbf{d}z \otimes \mathbf{d}z), \tag{6.3.16}$$

$$r^2 = x^2 + y^2 + z^2. \tag{6.3.17}$$

with (again to order $1/r$)

$$\boldsymbol{A} \sim \frac{Q}{r}\mathbf{d}t + P\cos\theta\mathbf{d}\varphi \tag{6.3.18}$$

and

$$\chi \sim \frac{\chi_0}{r} \quad , \quad \nu \sim \frac{\nu_0}{r} \ . \tag{6.3.19}$$

Notice that the magnetic component to the vector potential is of order $1/r$ if we transform to a quasi-cartesian coordinate system.

We wish to solve the Hamilton-Jacobi equation (6.3.6) at large impact parameter for a light ray in the five dimensional metric, one readily finds that the solution (upto an overall scale is given by)

$$S = -t + x^5\cos\gamma + z\sin\gamma + 2\hat{M}\operatorname{cosec}\gamma\log\left(\frac{r+z}{b}\right) + \mathcal{E} \tag{6.3.20}$$

where the coordinates have been chosen so that the tangent vector to the null geodesic is asymptotic to K,

$$K = \frac{\partial}{\partial t} + \sin\gamma\frac{\partial}{\partial z} + \cos\gamma\frac{\partial}{\partial x^5} \tag{6.3.21}$$

with $\sin\gamma \neq 0$. The impact parameter, b, is given by

$$b^2 = x^2 + y^2 \tag{6.3.22}$$

and $\mathcal{E}$ is small in the sense that $\partial_b\mathcal{E}, \partial_z\mathcal{E}, \partial_t\mathcal{E}, \partial_5\mathcal{E} = O(1/b)$, $K\mathcal{E} = O(1/b^2)$. The value of $\hat{M}$ is given by

$$2\hat{M} = 2M - \cos^2\gamma(M + \chi_0 - \nu_0) + sQ\cos\gamma \ . \tag{6.3.23}$$

Let us calculate τ the 'time of flight' as defined by Penrose, Sorkin and Woolgar. We are interested in the Shapiro time delay/advance at large impact parameter. One evaluates the change in S along a finite section of the geodesic from z_0 to z_1 say, at impact parameter b. We are only interested in the dependence on b and so one subtracts off a similar contribution at a fixed (large) value of the impact parameter, which we will call b_0. Letting z_0 and z_1 tend to the initial and final endpoints of the geodesic (so

$z_0 \to -\infty$ and $z_1 \to \infty$ if $\sin\gamma > 0$ whilst if $\sin\gamma < 0$, $z_0 \to \infty$ and $z_1 \to -\infty$) one finds

$$\tau = -4\hat{M}|\mathrm{cosec}\,\gamma|\log\left(\frac{b}{b_0}\right) + O(1)\ , \tag{6.3.24}$$

where $O(1)$ signifies bounded as $b \to \infty$. The conclusion is that for $\hat{M} < 0$ the time of flight can be made arbitrarily large by venturing to large enough impact parameter. In other words, there exist a fastest null curve ζ, in the sense that it minimizes τ.

On the other hand the theorem proved in [31] shows, provided that any singularities are inside event horizons, that ζ may be taken to be geodesic, and more especially it lies on the boundary of the causal future of some point on the appropriate generator of past null infinity, $\mathcal{I}^-$. This theorem implies that it cannot have any conjugate points as no geodesic with conjugate points can stay on the boundary of the causal future of some point for more than a finite affine length. We conclude that $\hat{M} \geq 0$, for all $\sin\gamma \neq 0$, provided that every null geodesic develops conjugate points (and hence gives a contradiction to the construction of such a fastest geodesic). The property that every null geodesic develops conjugate points is implied by the null convergence condition in the following sections, together with an appeal to the genericity condition, $V_{[A}{}^5R_{B]CD[E}V_{F]}V^CV^D \neq 0$ somewhere on every null geodesic with tangent vector V^A [32]. The purpose of this condition is to force a congruence of null rays to start to converge, i.e., the expansion can be made negative and hence by Raychaudhuri's equation for null geodesics we can conclude the existence of conjugate points if we assume the non-negativity of the Ricci tensor. In this chapter we shall be considering spacetimes that obey the genericity condition. The null convergence condition guarantees the existence of conjugate points and has been employed particularly in relationship to the singularity theorems of Penrose, Hawking and others. We shall relate the quantities χ and ν as follows:

$$\nu = -2\chi\ , \tag{6.3.25}$$

$$3\nu = -2g\phi \tag{6.3.26}$$

in order that we shall have a non-negative Ricci tensor. By looking at the inequality

$\hat{M} \geq 0$ when we take the limits $\gamma \to 0, \pi$ one establishes

$$M - g\Sigma \geq |sQ| \ . \tag{6.3.27}$$

6.4 The Five Dimensional Ricci Tensor

In what follows we shall use an orthonormal frame

$$\boldsymbol{E}^A = \left(\boldsymbol{E}^5, e^{\chi}\boldsymbol{e}^a\right) \tag{6.4.1}$$

where $\boldsymbol{e}^a$ is an orthonormal frame with respect to the metric g_{ab}. We shall denote by 5 the $\boldsymbol{E}^5$ component and by a the component with respect to $\boldsymbol{e}^a$. In an orthonormal frame and with such an understanding the components of the five dimensional Ricci tensor ${}^5R_{AB}$ of the metric g_{AB} are given by Eqs. (A.23) to (A.25):

$$\begin{aligned}{}^5R_{ab} = {}^4R_{ab} - 2\nabla_a\nabla_b\chi - g_{ab}\nabla^2\chi + 2\nabla_a\chi\nabla_b\chi \\ - 2g_{ab}\nabla_c\chi\nabla^c\chi - \nabla_a\nu\nabla_b\nu - \nabla_a\nabla_b\nu + \nabla_a\nu\nabla_b\chi + \nabla_a\chi\nabla_b\nu - g_{ab}\nabla_c\chi\nabla^c\nu \\ - \frac{s^2}{2}e^{2\nu-2\chi}F_{ac}F_b{}^c \ ,\end{aligned} \tag{6.4.2}$$

$${}^5R_{5a} = \frac{1}{2}se^{-2\nu-2\chi}\nabla_b\left(e^{3\nu}F_a{}^b\right) \ , \tag{6.4.3}$$

$${}^5R_{55} = \frac{s^2}{4}e^{2\nu-4\chi}F_{ab}F^{ab} - e^{-2\chi}\left(\nabla^2\nu + \nabla_a\nu\nabla^a\nu + 2\nabla_a\chi\nabla^a\nu\right) \ , \tag{6.4.4}$$

If one compares Eqs. (6.4.2) to (6.4.4) with the field equations (6.2.5) to (6.2.7) it becomes clear that a good choice (and possibly the only choice, if one is to eliminate the second derivatives over whose sign one would otherwise have no control) is

$$3\nu = -2g\phi \ , \tag{6.4.5}$$

$$\nu + 2\chi = 0 \ . \tag{6.4.6}$$

This gives

$$M - g\Sigma \geq s|Q| \ . \tag{6.4.7}$$

Given these choices for ν and χ the components of the five dimensional Ricci tensor are given by

$$^5R_{a5} = \frac{1}{2} s e^{-\nu} J_a \ , \tag{6.4.8}$$

$$^5R_{55} = e^{4\nu} F_{ab} F^{ab} \left(\frac{s^2}{4} - \frac{g^2}{3} \right) + \frac{2g}{3} e^{\nu} J, \tag{6.4.9}$$

and

$$\begin{aligned} ^5R_{ab} = {} & 2\left(1 - \frac{g^2}{3}\right) \nabla_a\phi\nabla_b\phi + 2e^{3\nu}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \\ & - \frac{s^2}{2} e^{3\nu} F_{ac}F_b{}^c + \frac{g^2}{6} e^{3\nu} g_{ab} F_{cd}F^{cd} + 2\left(T_{ab} - \frac{1}{2} g_{ab} T_c{}^c\right) \ . \end{aligned} \tag{6.4.10}$$

As a check we note that the standard five dimensional theory without additional sources corresponds to the values

$$g^2 = 3 \tag{6.4.11}$$

and

$$s = 2 \ . \tag{6.4.12}$$

In this case we have

$$^5R_{AB} = 0 \ . \tag{6.4.13}$$

6.5 The Null Convergence Condition in Five Dimensions

We must now calculate

$$^5R_{AB} \frac{dx^A}{d\lambda} \frac{dx^B}{d\lambda} \tag{6.5.1}$$

and find under what conditions it is non-negative. The lightlike vector $V^A = dx^A/d\lambda$ has components

$$\boldsymbol{V} = V_5 \boldsymbol{E}^5 + V_a \boldsymbol{e}^a \tag{6.5.2}$$

so that

$$(V_5)^2 = -e^{-2\chi} g^{ab} V_a V_b \ . \tag{6.5.3}$$

Consequently we have

$$\begin{aligned} {}^5R_{AB}V^AV^B &= 2e^{2\nu}\left(1-\frac{g^2}{3}\right)(V^a\nabla_a\phi)^2 \\ &+ e^{\nu}(V_5)^2\left[\left(1-g^2+\frac{s^2}{2}\right)\mathbf{B}.\mathbf{H} + (1+g^2-s^2)\,\mathbf{E}.\mathbf{D}\right] \\ &+ 2e^{2\nu}\left(T_{ab}-\frac{1}{2}g_{ab}T_c{}^c\right)V^aV^b + \frac{2}{3}(V_5)^2e^{\nu}gJ + sV_5V^aJ_a \end{aligned} \tag{6.5.4}$$

where we define

$$E_b = \frac{e^{-\chi}V^a}{|V_5|}F_{ab} \,, \tag{6.5.5}$$

$$B_a = \frac{1}{2}\epsilon_{abcd}\frac{e^{-\chi}V^b}{|V_5|}F^{cd} \ . \tag{6.5.6}$$

and

$$\mathbf{D} = \epsilon\mathbf{E} = e^{-2g\phi}\mathbf{E} \ , \tag{6.5.7}$$

$$\mathbf{H} = \mu^{-1}\mathbf{B} = e^{-2g\phi}\mathbf{B} \ . \tag{6.5.8}$$

Hence, Eq. (6.5.1) will be non-negative provided we impose the conditions,

$$s^2 \leq 1+g^2 \qquad \text{and} \qquad s^2 \geq 2(g^2-1) \ , \tag{6.5.9}$$

and the sources satisfy

$$J \geq 0 \,, \tag{6.5.10}$$

$$\left(T_{ab}-\frac{1}{2}g_{ab}T_c{}^c\right)\hat{V}^a\hat{V}^b \geq e^{g\phi}\left|\frac{s\hat{V}^aJ_a}{2}\right| \ . \tag{6.5.11}$$

Here $\hat{V}$ is a unit timelike vector in four dimensions. These conditions are sufficient for the theorem to hold, though all we require is the non-negativity of the right hand side of Eq. (6.5.4). We remark that Eq. (6.5.11) relates the gravitating energy (in the sense that it is the source for the Ricci tensor and hence it appears as the appropriate density in the Poisson equation in the Newtonian limit) to the local energy density of the vector field, and should be compared to Eq. (6.1.11) which is the appropriate condition for the spinorial technique to apply. We might remark that for a pressure-free fluid the two equations (6.1.11) and (6.5.11) coincide.

6.6 Comparison with Black Hole Solutions

It is interesting to compare our inequalities with the explicit spherically symmetric solutions of the field equations.

The four-metric is given by

$$\boldsymbol{g} = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{(1-g^2)/(1+g^2)} \mathbf{d}t\otimes\mathbf{d}t$$

$$+\left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{(g^2-1)/(1+g^2)} \mathbf{d}r\otimes\mathbf{d}r + r^2\left(1-\frac{r_-}{r}\right)^{2g^2/(1+g^2)} \mathbf{d}\boldsymbol{\Omega}^2 \quad (6.6.1)$$

where

$$\mathbf{d}\boldsymbol{\Omega}^2 = \mathbf{d}\theta\otimes\mathbf{d}\theta + \sin^2\theta\,\mathbf{d}\phi\otimes\mathbf{d}\phi \quad (6.6.2)$$

with dilaton field

$$e^{\phi} = \left(1-\frac{r_-}{r}\right)^{g/(1+g^2)} \quad (6.6.3)$$

and Maxwell field

$$\boldsymbol{F} = e^{2g\phi}\left(1-\frac{r_-}{r}\right)^{-2g^2/(1+g^2)} \frac{Q\mathbf{d}t\wedge\mathbf{d}r}{r^2}\ . \quad (6.6.4)$$

We have

$$M = \frac{1}{2}\left(r_+ + r_-\frac{1-g^2}{1+g^2}\right), \quad (6.6.5)$$

$$|Q| = \frac{\sqrt{r_+r_-}}{\sqrt{1+g^2}} \quad (6.6.6)$$

and

$$\Sigma = -\frac{gr_-}{1+g^2} \ . \tag{6.6.7}$$

Whence

$$M - g\Sigma = \tfrac{1}{2}(r_+ + r_-) \geq \sqrt{r_+ r_-} = \sqrt{1+g^2}|Q| \tag{6.6.8}$$

which is consistent with our general result. Note however that in obtaining Eq. (6.6.8) for this example we have not needed to restrict the coupling constant g to be less than $\sqrt{3}$. Thus it seems that while $g^2 \leq 3$ is a sufficient condition for the validity of our inequality Eq. (6.1.1) it may not be a necessary condition. This opens up the possibility that one might be able to extend our proof beyond the the case $g^2 \leq 3$ by using some other judiciously chosen metric.

6.7 The Extreme Case

Let us consider the analogues of Papapetrou-Majumdar solution in the theory we have been considering. Gibbons [13] has found the appropriate form of the metric:

$$\boldsymbol{g} = -H^{-2/(1+g^2)}\mathbf{d}t \otimes \mathbf{d}t + H^{2/(1+g^2)}\left(\mathbf{d}x \otimes \mathbf{d}x + \mathbf{d}y \otimes \mathbf{d}y + \mathbf{d}z \otimes \mathbf{d}z\right) \ , \tag{6.7.1}$$

with H a harmonic function in Euclidean space with Cartesian coordinates x, y and z. The dilaton and vector potential are given by

$$e^{\phi} = H^{-g/(1+g^2)} \tag{6.7.2}$$

and

$$\boldsymbol{A} = \frac{\mathbf{d}t}{H\sqrt{1+g^2}} \tag{6.7.3}$$

We wish to relate this solution to a five dimensional metric:

$$\boldsymbol{g}^{(5)} = H^{\alpha}\left(\mathbf{d}u + \frac{s\mathbf{d}t}{H\sqrt{1+g^2}}\right) \otimes \left(\mathbf{d}u + \frac{s\mathbf{d}t}{H\sqrt{1+g^2}}\right)$$
$$+ H^{\beta}\left(-H^{-2/(1+g^2)}\mathbf{d}t \otimes \mathbf{d}t + H^{2/(1+g^2)}\left(\mathbf{d}x \otimes \mathbf{d}x + \mathbf{d}y \otimes \mathbf{d}y + \mathbf{d}z \otimes \mathbf{d}z\right)\right) . \tag{6.7.4}$$

We write $u = x^5 - t$ in Eq. (6.7.4), as the gauge potential in Eq. (6.7.3) does not conform to our usual gauge choice. We demand $H(\mathbf{x}) \to 1$ as $|\mathbf{x}| \to \infty$ as a condition in order to have an asymptotically flat solution in standard quasi-Cartesian coordinates. The choice of α, β and s will be made in such a way that the resulting five dimensional metric possesses a lightlike Killing vector $\partial/\partial t$. We are therefore led to the conditions:

$$s = \sqrt{1+g^2}\,, \tag{6.7.5}$$

$$\alpha - \beta = \frac{2g^2}{1+g^2}\,. \tag{6.7.6}$$

On the other hand, comparing Eq. (6.7.4) with our general *ansatz* for the metric we write

$$H^{\alpha} = e^{2\nu} = e^{-4g\phi/3} = H^{4g^2/3(1+g^2)}\,. \tag{6.7.7}$$

Hence from Eq. (6.7.6) we deduce that

$$\alpha + 2\beta = 0\,, \tag{6.7.8}$$

which is a necessary relationship if we are to eliminate second order derivatives of the dilaton from the expansion of the Ricci tensor in Sect. 6.4, and hence to impose the null convergence condition on the Ricci tensor. The lightlike Killing vector K has components

$$K_a = H^{\alpha-1}\nabla_a u\,, \tag{6.7.9}$$

and hence is hypersurface orthogonal. The relevant hypersurfaces being those of constant u.

We now wish to show that Eq. (6.7.1) saturates the bound, in the sense that $\hat{M} \to 0$ as $\gamma \to 0$. Consider the case $\gamma = 0$ exactly. The corresponding null geodesics are null generators of the null surface $u =$ constant. The associated exact solution of the Hamilton-Jacobi equation is given (upto scalar multiplication and the addition of a constant) by

$$S = u = -t + x^5, \tag{6.7.10}$$

independent of the value of the impact parameter. If this is compared with the $\gamma \to 0$ limit of Eq. (6.3.20), one concludes that (6.7.1) is a solution that saturates the bound, i.e., $\hat{M} = 0$ with $\gamma = 0$ so that

$$M - g\Sigma = -sQ\,. \tag{6.7.11}$$

Equation (6.7.11) may also be shown by direct computation. Set

$$H = 1 + \sum_{i=1}^{n} \frac{\mu_i}{|\mathbf{x} - \mathbf{x}_i|}\ , \tag{6.7.12}$$

representing n isolated black holes of 'strengths' $(\mu_i)_{i=1}^n$ at locations $(\mathbf{x}_i)_{i=1}^n$ in equilibrium. One calculates the mass, scalar and electric charges:

$$M = \frac{1}{1+g^2} \sum_{i=1}^{n} \mu_i\ , \tag{6.7.13}$$

$$Q = \frac{-1}{\sqrt{1+g^2}} \sum_{i=1}^{n} \mu_i \tag{6.7.14}$$

and

$$\Sigma = \frac{-g}{1+g^2} \sum_{i=1}^{n} \mu_i\ . \tag{6.7.15}$$

Together with $s = \sqrt{1+g^2}$, we verify that Eq. (6.7.11) is indeed satisfied.

6.8 An Analogue of Scherk's Anti-gravity Condition

We have found the result that

$$M - g\Sigma \geq s|Q|. \tag{6.8.1}$$

By considering the duality transformation in these theories we find in addition

$$M + g\Sigma \geq s|P|. \tag{6.8.2}$$

Squaring and adding we have

$$M^2 + g^2\Sigma^2 - \tfrac{1}{2}s^2(Q^2 + P^2) \geq 0. \tag{6.8.3}$$

For $g = 1$ we recover Scherk's anti-gravity condition [36]:

$$M^2 + \Sigma^2 - Q^2 - P^2 \geq 0. \tag{6.8.4}$$

The other two special cases correspond to dimensionally reduced Einstein-Maxwell theory ($g = 1/\sqrt{3}$) and standard five to four dimensional Kaluza-Klein ($g = \sqrt{3}$) theory. They give the conditions

$$M^2 + \frac{1}{3}\Sigma^2 - \frac{2}{3}(Q^2 + P^2) \geq 0, \qquad g = \frac{1}{\sqrt{3}}; \tag{6.8.5}$$

and

$$M^2 + 3\Sigma^2 - 2(Q^2 + P^2) \geq 0, \qquad g = \sqrt{3}. \tag{6.8.6}$$

To get an idea of the bound Eq. (6.8.3) we can draw an ellipse in the $M - \Sigma$ plane at constant $Q^2 + P^2$, and compare with the circle given by Eq. (6.8.4). Fig. 6.1 show the allowable regions defined by the anti-gravity and Scherk bounds for $g < 1$ and $g > 1$.

We may see the bounds Eqs. (6.8.1) and (6.8.2) simultaneously satisfied by looking at the multi-black hole solutions in the $g = 1$ theory given by [13]:

$$\boldsymbol{g} = -\frac{1}{H_1H_2}\mathbf{d}t \otimes \mathbf{d}t + H_1H_2\left(\mathbf{d}x \otimes \mathbf{d}x + \mathbf{d}y \otimes \mathbf{d}y + \mathbf{d}z \otimes \mathbf{d}z\right) \tag{6.8.7}$$

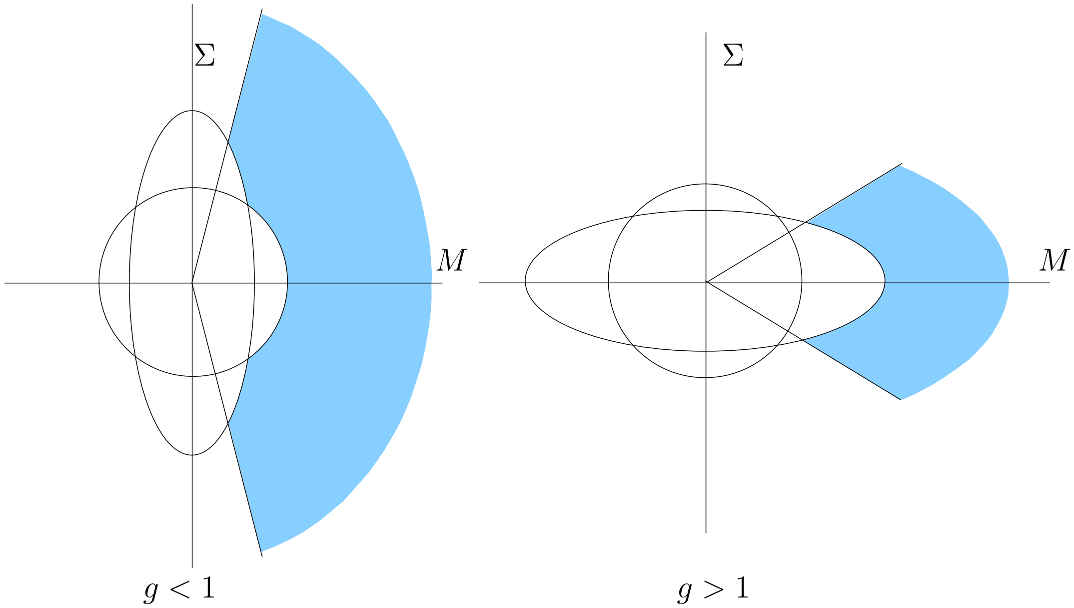

Fig. 6.1: *The anti-gravity bound defines an ellipse in the $M - \Sigma$ plane whereas the Scherk bound is a circle. In addition the condition $M \pm g\Sigma \geq 0$ means any solution must lie in the coloured regions indicated.*

and

$$e^{2\sigma} = \frac{H_1}{H_2}; \tag{6.8.8}$$

$$-i_K \boldsymbol{F} = \mathbf{d}H_1; \tag{6.8.9}$$

$$e^{-2\phi} i_K {*\boldsymbol{F}} = \mathbf{d}H_2; \tag{6.8.10}$$

where

$$H_1 = 1 + \sum_{i=1}^{n} \frac{\mu_i}{|\mathbf{x} - \mathbf{x}_i|} \qquad \text{and} \qquad H_2 = 1 + \sum_{i=1}^{n} \frac{\nu_i}{|\mathbf{x} - \mathbf{x}_i|}, \tag{6.8.11}$$

and $K = \partial/\partial t$. We require that none of the electric nor magnetic strengths vanish, i.e.,

μ_i, $\nu_i \neq 0$. The mass and charges are then given by

$$M = \frac{1}{2}\sum_{i=1}^{n}(\mu_i + \nu_i); \tag{6.8.12}$$

$$\Sigma = \frac{1}{2}\sum_{i=1}^{n}(\nu_i - \mu_i); \tag{6.8.13}$$

$$Q = \frac{1}{\sqrt{2}}\sum_{i=1}^{n}\mu_i; \tag{6.8.14}$$

$$P = \frac{1}{\sqrt{2}}\sum_{i=1}^{n}\nu_i. \tag{6.8.15}$$

Using $g = 1$ and $s = \sqrt{2}$ we see that Eqs. (6.8.1) and (6.8.2) both hold.

6.9 Summary and Conclusion

In this chapter we have considered four dimensional spacetimes, Maxwell field $\boldsymbol{A}$ and dilaton field ϕ with associated five dimensional metric:

$$\boldsymbol{g}^{(5)} = e^{-4g\phi/3}\left(\mathbf{d}x^5 + s\boldsymbol{A}\right) \otimes \left(\mathbf{d}x^5 + s\boldsymbol{A}\right) + e^{2g\phi/3}g_{ab}\mathbf{d}x^a \otimes \mathbf{d}x^b \tag{6.9.1}$$

where g and s are dimensionless constants. Assuming that any singularities are contained within an event horizon and that the five dimensional Ricci tensor ${}^5R_{AB}$ obeys the non-negative Ricci condition:

$${}^5R_{AB}V^AV^B \geq 0 \tag{6.9.2}$$

for all null five-vectors V^A, we have shown that the mass, electric and scalar charges satisfy the relationship:

$$M - g\Sigma \geq s|Q| \; . \tag{6.9.3}$$

If we impose the Einstein equations and the equations of motion for the dilaton and Maxwell fields arising from the Lagrangian Eq. (6.2.1), we find that Eq. (6.9.2) may be satisfied provided the additional sources J, J_a and T_{ab} obey

$$J \geq 0\,, \tag{6.9.4}$$

$$\left(T_{ab} - \frac{1}{2} g_{ab} T_c{}^c\right) \hat{V}^a \hat{V}^b \geq e^{g\phi} \left| \frac{s \hat{V}^a J_a}{2} \right| \tag{6.9.5}$$

for all timelike unit four-vectors $\hat{V}^a$ and the quantities s and g satisfy the following:

$$2(g^2 - 1) \leq s^2 \leq 1 + g^2, \tag{6.9.6}$$

for which we require $g \leq \sqrt{3}$.

Our proof is valid in the full non-linear theory but it is illuminating to see how it follows almost trivially in the linearized theory. In De Donder gauge linear theory gives

$$-\frac{1}{2} \nabla_C \nabla^C h_{AB} = {}^5R_{AB} \tag{6.9.7}$$

where $g_{AB} = \eta_{AB} + h_{AB}$. If the metric is independent of t and x^5 we have from Eq. (6.9.7) near infinity

$$h_{AB} = \frac{h^0_{AB}}{r} + O\left(\frac{1}{r^2}\right) \tag{6.9.8}$$

where

$$h^0_{AB} = \frac{1}{2\pi} \int_{\mathbb{R}^3} {}^5R_{AB}\, dx^3\ . \tag{6.9.9}$$

At the linear level the null convergence condition on ${}^5R_{AB}$ implies that h^0_{AB} is non-negative when contracted with any constant lightlike vector. Because

$$\left(h^0_{AB}\right) = \begin{pmatrix} \left(2M + \frac{2g\Sigma}{3}\right) \mathbb{1} & & \\ & 2M - \frac{2g\Sigma}{3} & sQ \\ & sQ & \frac{-4g\Sigma}{3} \end{pmatrix} \tag{6.9.10}$$

where the order of rows and columns in Eq. (6.9.10) is $(i, 4, 5)$, inequality Eq. (6.9.3) follows by considering the five-vector $(V^A) = (\mathbf{0}, 1, \pm 1)$. If the metric is independent of x^5 but depends on time t we expect the same conclusion to follow from linear theory. However we need not check this point explicitly since our stronger non-linear argument does not require the metric to be time-independent.

7. DILATON VORTEX SOLUTIONS IN FLAT SPACE

7.1 Introduction

There has been considerable interest recently in 4-dimensional Dilaton-Einstein-Maxwell and Dilaton-Einstein-Yang-Mills theory. In the abelian case single and multi static and stationary black hole solutions with electric and magnetic charges have been extensively studied. Another class of interesting solutions describe magnetic fields with or without black holes. For Einstein-Maxwell theory these are based on the Melvin solution, which we discussed in Sect. 3.3. This represents a sort of super-massive cosmic string [37, 38]. The Melvin solution may be generalized to include a coupling to the dilaton [39]. Recently Maki and Shiraishi [40] have obtained some interesting time-dependent solutions with a dilaton potential.

In this chapter, in order to gain some physical insight into dilaton-electrodynamics and its non-abelian generalization, we will study the simpler flat-space version in which the effects of gravity are ignored. The action is

$$\int d^4x \left(-e^{-2g\phi} F_{ab}F^{ab} - 2\eta^{ab}\nabla_a\phi\nabla_b\phi\right) \tag{7.1.1}$$

where $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$ and the dimensionful quantity g in the action can be changed by a suitable rescaling of the variables and so its numerical value (as long as it does not vanish) has no physical significance in the purely classical theory which we study here, henceforth we will put $g = 1$. The field $\boldsymbol{F}$ may be abelian or non-abelian. In the latter case there remains, again in the classical theory, sufficient freedom to scale the Yang-Mills connection $\boldsymbol{A}$ so that $\boldsymbol{F} = \mathbf{d}\boldsymbol{A} + \boldsymbol{A}\wedge\boldsymbol{A}$. Note that had we taken the above limit of the gravitational Lagrangian expressed in string conformal gauge (i.e., in terms of the metric $e^{2\phi}g_{ab}$) we would have obtained a different action from Eq. (7.1.1).

The flat space version Eq. (7.1.1) has been studied by Lavrelashvili and Maison [41] and also by Bizon [42] who have obtained sphaleron type solutions in which the Yang-Mills field is confined by the attractive forces exerted by the dilaton which replaces the attractive forces due to gravity in the Bartnik-McKinnon solution [43]. In the abelian case it is easy to obtain the general static spherically symmetric purely magnetic solution representing a Dirac monopole coupled to the dilaton. Bizon has pointed out a special case of the spherically symmetric Dirac monopoles may be generalized to give multi-Dirac-monopole solutions. These are Bogomol'nyi type solutions and may be regarded as a limiting case of the multi black hole solutions as we shall show in section 7.4. It is also completely straightforward to obtain plane wave solutions in which the dilaton and the photon are travelling parallel to one another.

In Sect. 7.2 we discuss the properties of dilaton electrodynamics and interpret the dilaton field as effectively polarizing the vacuum. We will give some examples illustrating this interpretation from the gravitational theory and the Higgs field for the BPS monopole. We will write down the field equations in Sect. 7.3 and by making an appropriate *ansatz* derive some cosmic string-like solutions. We will also exhibit a solution that may be regarded as a domain wall in this theory. In the following section, Sect. 7.4, we will investigate the Bogomol'nyi and spherically symmetric monopole solutions. We will show that the Bogomol'nyi solutions can be derived by a limiting procedure from the gravitational black hole solutions. Finally, in Sect. 7.5 we investigate how the solutions we have found alter when we suppose the dilaton is massive. We do this by understanding the relevant field equations qualitively and verify our assertions numerically.

7.2 Permeabilities and Permittivities

It follows immediately from Eq. (7.1.1) that the equations of motion for the field $\boldsymbol{F}$ in the presence of the dilaton field are those for a medium in which the electric permittivity ϵ is given by

$$\epsilon = e^{-2\phi} \tag{7.2.1}$$

and the magnetic permeability μ is given by

$$\mu = e^{2\phi}. \tag{7.2.2}$$

The product $\epsilon\mu$ is unity and so the velocity of light remains one everywhere. With this interpretation we have that regions of spacetime for which $\phi < 0$ are diamagnetic while regions with $\phi > 0$ are paramagnetic. One does not usually encounter permittivities ϵ which are less than unity. In the non-abelian theory e^{ϕ} plays the rôle of spacetime dependent gauge coupling constant and in string perturbation theory its expectation value plays the rôle of a variable coupling constant. Thus weak coupling corresponds to a diamagnetic phase and strong coupling to a paramagnetic phase. The action (7.1.1) is invariant under the simultaneous change of the sign of the dilaton field ϕ and the replacement of the Maxwell field $\boldsymbol{F}$ by $e^{-2\phi}{*\boldsymbol{F}}$. This symmetry therefore interchanges the weak and the strong coupling phases.

The field equation derived from the varying of ϕ in the static case is

$$\nabla^2\phi = -e^{-2\phi}\left(\mathbf{B}^2 - \mathbf{E}^2\right), \tag{7.2.3}$$

where $\mathbf{E}$ and $\mathbf{B}$ have their usual meaning. It follows from Eq. (7.2.3) that ϕ can have no minimum in a purely magnetic field and no maximum in a purely electric field. Thus if ϕ is taken to be zero at infinity then magnetic regions tend to be paramagnetically polarized ($\mu > 1$) and electric regions tend to be dielectrically polarized ($\mu < 1$). Intuitively magnetic flux ($\int \mathbf{B}.\mathbf{dS}$) tends to get self-trapped in strong coupling domains and electric flux ($\int \mathbf{D}.\mathbf{dS}$) in weak coupling domains, where $\mathbf{D} = \epsilon\mathbf{E}$ is the electric displacement and (for later use) $\mathbf{H} = \mu^{-1}\mathbf{B}$ is the magnetic induction. These observations are borne out by the particular solutions mentioned above. Thus for the Maison-Lavrelashvili-Bizon sphalerons μ has a maximum at the centre and decreases monotonically to unity at infinity. For electrically charged black hole solutions μ decreases monotonically inwards from unity at infinity. The appropriate solution is given by:

$$\boldsymbol{g} = -\left(1 - \frac{r_+}{r}\right)\mathbf{d}t \otimes \mathbf{d}t + \left(1 - \frac{r_+}{r}\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r + r^2\left(1 - \frac{r_-}{r}\right)\mathbf{d\Omega}^2 \tag{7.2.4}$$

where

$$\mathbf{d\Omega}^2 = \mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta \mathbf{d}\phi \otimes \mathbf{d}\phi \tag{7.2.5}$$

with dilaton field

$$e^{2\phi} = 1 - \frac{r_-}{r} \tag{7.2.6}$$

and Maxwell field

$$\boldsymbol{F} = e^{2\phi}\left(1 - \frac{r_-}{r}\right)\frac{\sqrt{r_+ r_-}\mathbf{d}t \wedge \mathbf{d}r}{\sqrt{2}\,r^2}\ . \tag{7.2.7}$$

In the extreme case ϕ becomes infinitely large and negative as one approaches the horizon. From the string point of view the infinitely long throat is a weak coupling region. In the magnetically charged case the opposite is true. There is a parallel here with monopoles in Yang-Mills theory and vortices in the abelian Higgs theory. In the case of $SU(2)$ Yang-Mills theory with the Higgs field in the adjoint representation one may take the components of the Higgs field as a triplet of permittivities. The parallel is not completely precise but it is the case for the 't Hooft-Polyakov monopole that the associated permeabilities monotonically increase as one moves radially inwards. As a special case of this we notice that the BPS monopole has a Higgs field which can be written as

$$\phi^a = \left(\alpha \coth \alpha r - \frac{1}{r}\right)\frac{x^a}{r}. \tag{7.2.8}$$

Similarly for the Nileson-Olesen vortex one may think of the magnetic flux as being confined inside a core of high permeability where the Higgs field has a smaller magnitude than it does at infinity. We shall see similar features arising for dilaton electrodynamics.

The paramagnetic behaviour described above and the existence of the dilaton-Melvin solution strongly suggest that there may be non-singular static cosmic string type solutions in which a finite amount of flux is trapped. This is indeed the case, as we shall show in the next section.

7.3 Dilaton Cosmic Strings and Domain Walls

In the static case the dilatonic Maxwell equations are readily seen to be satisfied if the magnetic induction $\mathbf{H} = (0, 0, H)$, where H is a constant so long as the dilaton satisfies the two-dimensional Liouville equation:

$$\nabla^2\phi = -H^2 e^{2\phi} \ . \tag{7.3.1}$$

Following Liouville [44], the general solution of Eq. (7.3.1) is

$$\mu = e^{2\phi} = \frac{4f'(\zeta)g'(\overline{\zeta})}{H^2\left(1+f(\zeta)g(\overline{\zeta})\right)^2} \quad , \tag{7.3.2}$$

where f and g are locally holomorphic functions and $\zeta = x+iy$. If ϕ is real then

$$\overline{g\left(\,\overline{\zeta}\,\right)}/f(\zeta) \tag{7.3.3}$$

must be a real valued holomorphic function of ζ and hence constant (by, for instance, the Open Mapping Theorem). With no loss of generality this constant may be taken to be unity and therefore the solution we require is

$$\mu = e^{2\phi} = \frac{4\left|f'(\zeta)\right|^2}{H^2\left(1+\left|f(\zeta)\right|^2\right)^2} \quad . \tag{7.3.4}$$

Note that f and $1/f$ give the same solution ϕ. We remark that Eq. (7.3.1) is precisely the equation one finds when one calculates the Gauss curvature of a sphere of radius H^{-1} where the metric is written in a conformally flat form with conformal factor $e^{2\phi}$. The function f then just corresponds to the freedom to make complex diffeomorphisms.

Choosing different functions f gives different types of solution. For example choosing $f(\zeta) = \zeta$ gives a cylindrically symmetric solution with finite total magnetic flux:

$$\Phi = \int_{\mathbb{R}^2} \mathbf{B}.\mathbf{dS} = \frac{4\pi}{H} \quad . \tag{7.3.5}$$

This solution may be obtained as limit of the dilaton-Melvin solution.

The magnetic contribution to the total energy per unit length of our solution is $\frac{1}{2}\Phi H = 2\pi$ which is independent of the magnetic field H. The dilaton however contributes a logarithmically divergent energy per unit length because of its logarithmic dependence on the radius. In this respect our solution resembles a global rather than a local string. However it should be pointed out that the solution, in common with its gravitational version, breaks neither electromagnetic gauge invariance nor any compact internal symmetry.

In addition to the single string solution there are multi string solutions. Choosing a rational function of Brouwer degree k, with the ratio of two polynomials of order p and q, gives a solution with finite total magnetic flux

$$\Phi = k\frac{4\pi}{H} \quad . \tag{7.3.6}$$

Note that the permeability decreases to zero as $(x^2+y^2)^{-(|p-q|+1)}$ at infinity. Thus the weak coupling region at infinity is strongly diamagnetic and confines the magnetic flux Φ. The multi-string solutions are not axisymmetric. In flat-space Maxwell theory the only regular solution is the uniform magnetic field which is necessarily axisymmetric. When gravity is included this goes over into the Melvin solution which is also has axial symmetry. If one insists that the metric be boost-invariant then the axisymmetry and hence uniqueness, follow by a version of Birkhoff's theorem [45]. However the proof given in [45] does not go through in the presence of a scalar field. This suggests that there may exist static non-axisymmetric multi-dilaton-Melvin solutions.

Another interesting solution arises if we take $f(\zeta) = \exp(\zeta)$ then

$$\mu = e^{2\phi} = \frac{1}{H^2}\frac{1}{\cosh^2 x} \quad . \tag{7.3.7}$$

This solution describes a sheet or membrane confining an amount of flux per unit length of $2/H$. It may be thought of as a sort of domain wall separating two weak coupling domains.

7.4 Monopoles and Bogomol'nyi Solutions

In this section we shall consider a general static magnetic solution. Since

$$\nabla \times \mathbf{H} = \mathbf{0} \ , \tag{7.4.1}$$

we may locally introduce a magnetic potential χ by

$$\mathbf{H} = \nabla \chi \ . \tag{7.4.2}$$

If we make the *ansatz*

$$e^{-\phi} = \chi \tag{7.4.3}$$

then all the equations will be satisfied if in addition

$$\nabla^2 e^{\phi} = 0 \ . \tag{7.4.4}$$

In addition to these Bogomol'nyi solutions it is easy to find the general spherically symmetric monopole solution. If one insists that ϕ does not blow up at finite non-zero radius or at infinity one finds that

$$e^{\phi} = \frac{1}{\alpha} \sinh \left[\alpha P \left(\frac{1}{r} + \frac{1}{b} \right) \right] \tag{7.4.5}$$

and

$$\mathbf{B} = \frac{P}{r^3} \mathbf{r}, \tag{7.4.6}$$

where the constant of integration b is chosen so that $\phi = 0$ at infinity and P is the total magnetic charge. Just as in the case of magnetic black holes and 't Hooft-Polyakov monopoles we find that the magnetic permeability increases monotonically inwards. The Bogomol'nyi solution (7.4.4) is obtained from the general solution Eqs. (7.4.5) and (7.4.6) by letting α go to zero.

7.4.1 The limit of the gravitational multi black hole solution

The solutions (7.4.4) are the same as those mentioned by Bizon and may be obtained from the multi black hole solutions by a limiting procedure. To see this we start from the duality transformed solution, Eqs. (6.7.1) to (6.7.3) to obtain

$$\boldsymbol{g} = -F^{-2/(1+g^2)}\mathbf{d}t \otimes \mathbf{d}t + F^{2/(1+g^2)}\left(\mathbf{d}x \otimes \mathbf{d}x + \mathbf{d}y \otimes \mathbf{d}y + \mathbf{d}z \otimes \mathbf{d}z\right) , \tag{7.4.7}$$

with F a harmonic function in Euclidean space with Cartesian coordinates x, y and z. The dilaton and Maxwell field are given by

$$e^{g\Phi} = e^{g\Phi_0} F^{g^2/(1+g^2)} \tag{7.4.8}$$

and

$$\boldsymbol{F} = e^{2g\Phi} e^{-g\Phi_0} \frac{\sqrt{1+g^2}}{2(1-g^2)} \epsilon^{ijk} \nabla_k F^{(1-g^2)/(1+g^2)} \mathbf{d}x^i \wedge \mathbf{d}x^j . \tag{7.4.9}$$

Here the tensor ϵ^{ijk} is that associated with the standard Euclidean metric on $\mathbb{R}^3$. The dilaton fields are related by $\phi = g\Phi$. Thus

$$\mathbf{H} = -\frac{\sqrt{1+g^2}}{1-g^2} e^{g\Phi_0} \nabla F^{(1-g^2)/(1+g^2)} . \tag{7.4.10}$$

We take

$$F = \frac{1}{\sqrt{g}\,\chi} \qquad \text{with} \qquad \nabla^2 \chi^{-1} = 0, \tag{7.4.11}$$

and

$$e^{g\Phi_0} = \sqrt{g}. \tag{7.4.12}$$

Now take the limit $g \to \infty$. This gives us the flat space solution Eqs. (7.4.2) to (7.4.4).

7.5 Massive Dilatons

It is widely believed by string theorists that the dilaton acquires a mass due to non-perturbative effects connected with the breaking of supersymmetry. It is therefore of interest to ask what the effect of a mass term for the field ϕ might have on our solutions. In this section we shall simply add in a mass term 'by hand'; specifically we will add to the Lagrangian in Eq. (7.1.1) a term $-\frac{1}{2}m^2\phi^2$. One checks that the work above goes through as long as one replaces ∇^2 by $\nabla^2 - m^2$ in Eqs. (7.2.3) and (7.3.1). However the solution Eq. (7.3.2) is no longer valid and it seems the new version of (7.3.1) is not easily expressed as an analytical solution. We therefore analyse this system using qualitive arguments and some numerical work. The circularly symmetric solutions may be treated by regarding the radial coordinate r as a fictitious time variable. The equation for ϕ becomes that of a particle subject to a time dependent frictional force and moving in a potential $U(\phi)$ defined by

$$U(\phi) = -\tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}H^2 e^{2\phi} \ . \tag{7.5.1}$$

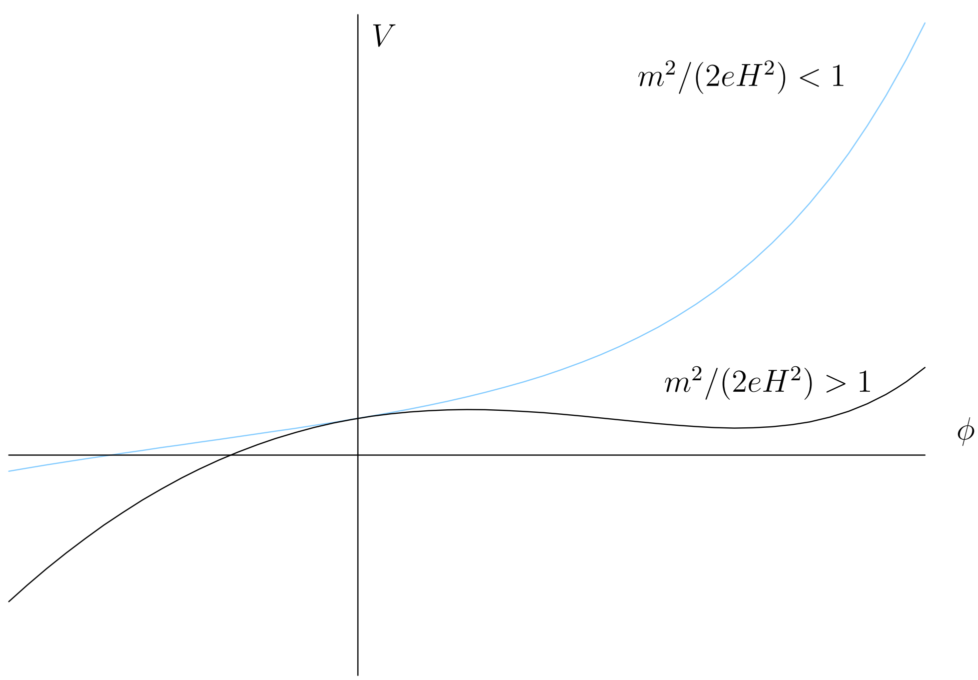

Fig. 7.1: *Graph of the potential function $V = U(\phi)$ given by Eq. (7.5.1)*

Regularity at the origin implies that the radial derivative of ϕ vanishes there. A solution exists for each value of ϕ at the origin. If $m = 0$ one has those solutions given in Sect. 7.3 in which ϕ decreases monotonically to minus infinity at infinity as

$$\phi \sim -2\log r. \tag{7.5.2}$$

For non-vanishing mass m the behaviour depends on the ratio $m^2/(2eH^2)$ where e is the base of natural logarithms. Fig. 7.1 shows the graph of the potential for two different values of this parameter. If $0 < m^2/(2eH^2) < 1$ the potential $U(\phi)$ is a monotonic increasing function of ϕ and for all values of $\phi(0)$, ϕ decreases monotonically with r and tends to minus infinity at infinity

$$\phi \sim -c\exp(mr) \tag{7.5.3}$$

where c is a positive constant. The behaviour is illustrated by the graph drawn with the coloured line in Fig. 7.1.

However if $m^2/(2eH^2)$ is greater than unity then the potential $U(\phi)$ has a local minimum and maximum (as illustrated by the graph drawn with the solid line). The behaviour of the solutions depends upon $\phi(0)$. If $\phi(0)$ is positive and sufficiently large then ϕ decreases monotonically to minus infinity as before. However if $\phi(0)$ lies in a finite interval bounded below by the smaller solution x of the equation:

$$x = \frac{H^2}{m^2}e^{2x} \tag{7.5.4}$$

then the solutions oscillate about the minimum with an amplitude which decreases to zero as r tends to infinity. Fig. 7.2 illustrates a numerical solution to the modified potential problem that exhibits this behaviour. Finally if $\phi(0) < x$ the solutions decrease monotonically to minus infinity.

Thus for given magnetic field H there always exist solutions with finite total flux. If however the mass is large enough, one has solutions in which ϕ tends to a minimum value, dependent upon H, of the potential $U(\phi)$.

We may repeat these calculations for the monopole solutions of Sect. 7.4 in the case

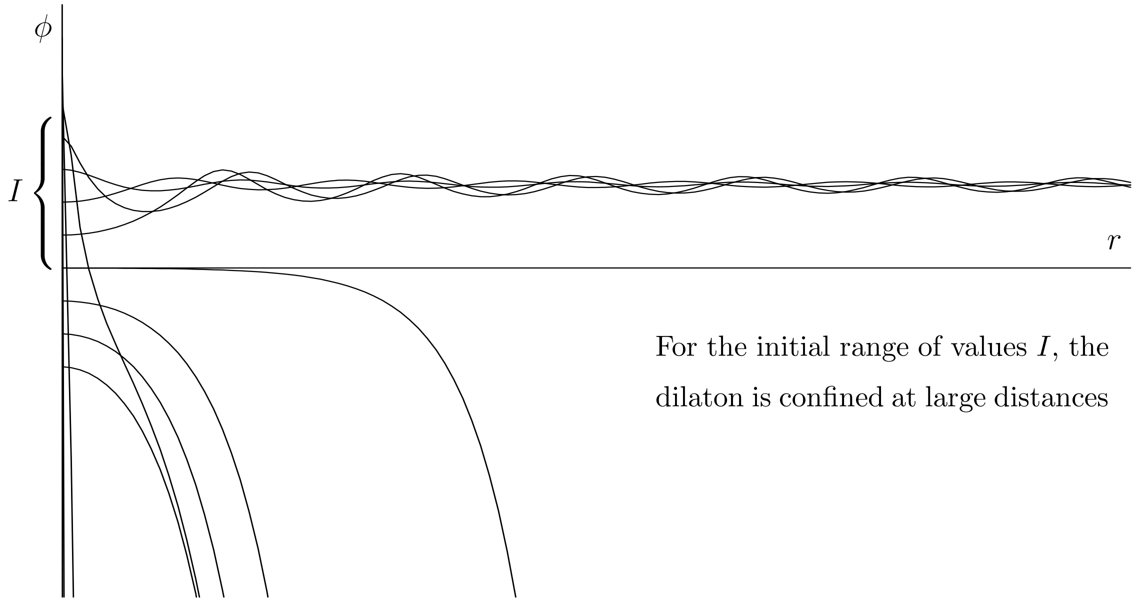

Fig. 7.2: *Solutions for the dilaton field $\phi(r)$ in the massive case.*

of a massive dilaton. Doing so we find that the qualitative behaviour of the dilaton is very similar to that in the massless case.

One might imagine a cosmological scenario in which the dilaton is initially massless at some high temperature and acquires a mass during a cosmological phase transition at a lower temperature. If cosmic strings of the type we have described confining a finite flux were initially present and the mass were large enough it seems from our calculations that provided $\phi(0)$ took suitable values the flux would become unconfined. If this were true it might have important consequences for magnetic monopoles. If flux was confined by strings at early times then one might expect magnetic monopoles, of the sort described in Sect. 7.4, to be found at the ends of flux tubes. These flux tubes should pull the monopoles together and cause their rapid annihilation. At late times magnetic fields would become unconfined. In this way one might have a natural solution to the monopole problem. Clearly more work needs to be done to establish whether this picture is really viable.

It is interesting to note that dilaton electrodynamics with an effective mass term has already been invoked [46] to account for a possible primordial magnetic field. It would be interesting to investigate the relation between that work and the monopoles and vortices that we have been discussing.

Part II

TIME MACHINES AND UNITARITY

8. RESTORATION OF UNITARITY IN CHRONOLOGY VIOLATING SPACETIMES

8.1 Introduction

Various studies [47, 48, 49] of perturbative interacting quantum field theory in the presence of a compact region of closed timelike curves (CTC's) have concluded that the evolution from initial states in the far past of the CTC's to final states in their far future fails to be unitary, in contrast with the situation for free fields [47, 50, 51]. The same conclusion has also been reached non-perturbatively for a model quantum field theory [52]. This presents many problems for the usual Hilbert space framework of quantum theory: as we describe in Sect. 8.2, the Schrödinger and Heisenberg pictures are inequivalent and ambiguities arise in assigning probabilities to events occurring before [48], or spacelike separated from [53], the region of non-unitary evolution.

The main reaction to these difficulties has been to abandon the Hilbert space formulation in favour of a sum over histories approach such as the generalized quantum mechanics of Gell-Mann and Hartle (see, e.g., [54]). In particular, Hartle [55] has addressed the issue of non-unitary evolutions in generalized quantum mechanics. Nonetheless, it is of interest to see if the Hilbert space approach can be 'repaired' by restoring unitarity. Anderson [56] has proposed that this be done as follows. Suppose a non-unitary evolution operator X is defined on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot \mid \cdot \rangle$. We assume that X is bounded with bounded inverse. Anderson defines a new inner product $\langle \cdot \mid \cdot \rangle'$ on $\mathcal{H}$ by $\langle \psi \mid \varphi \rangle' = \langle X^{-1}\psi \mid X^{-1}\varphi \rangle$, and denotes $\mathcal{H}$ equipped with the new inner product as $\mathcal{H}'$. Regarded as a map from $\mathcal{H}$ to $\mathcal{H}'$, X is clearly unitary. The essence of Anderson's proposal is to restore unitarity by regarding X in this way. Of course, one also needs to be able to represent observables as self-adjoint operators on both Hilbert spaces; Anderson has shown how this may be done by establishing a correspondence (depending on the evolution) between self-adjoint operators on $\mathcal{H}$ and those on $\mathcal{H}'$. When only one non-unitary evolution is considered, this proposal is equivalent to remaining in the Hilbert space $\mathcal{H}$ and replacing X by $U_X = (XX^*)^{-1/2}X$, i.e., the unitary part of X in the sense of the polar decomposition [57].

A curious feature of Anderson's proposal emerges when one considers the composition of two or more consecutive periods of non-unitary evolution [58]. If an evolution Y is followed by X, one might expect that the combined evolution would be represented by the composition of the unitary parts, i.e., $U_X U_Y$. However, this does not generally agree with the unitary part of the composition, U_{XY}, and so there would be an ambiguity depending on whether one thought of the full evolution as a one-stage or two-stage process. Anderson's response to this is to argue that the second evolution should be treated in a different way, essentially (as we show in Sect. 8.3) by replacing X by the unitary part of $X(YY^*)^{1/2}$. This removes the ambiguity mentioned above, but has the undesirable feature that the treatment of the second evolution depends on the first. In Sect. 8.3, we will show that this leads to an operational problem for observers living in a universe containing CTC regions.

It is therefore prudent to seek other means by which unitarity can be restored. We will be investigating a method of unitarity restoration using the mathematical technique of *unitary dilations*. This is motivated by the simple geometric observation that any linear transformation of the real line is the projection of an orthogonal transformation (called an *orthogonal dilation* of the original mapping) in a larger (possibly indefinite) inner product space. To see this, note that any linear contraction on the line may be regarded as the projection of a rotation in the plane: the contraction in length along the x-axis, say, being balanced by a growth in the y-component. Similarly, a linear dilation on the line may be regarded as the projection of a Lorentz boost in two dimensional Minkowski space. This observation may be extended to operators on Hilbert spaces: it was shown by Sz.-Nagy [59] that any contraction (i.e., an operator X such that $\|X\psi\| \leq \|\psi\|$ for all ψ) has a unitary dilation acting on a larger Hilbert space. The theory was subsequently extended to non-contractive operators by Davis [60] at the cost of introducing indefinite inner product spaces.

Put concisely, starting with a non-unitary evolution X, we pass to a unitary dilation of X, mapping between enlarged inner product spaces whose inner product may (possibly generically) be indefinite. The signature of the inner product is determined by the operator norm $\|X\|$ of X: if $\|X\| \leq 1$, the enlarged inner product spaces are Hilbert spaces, whilst for $\|X\| > 1$, they are indefinite inner product spaces (Krein spaces). Within the context of the unitary dilation proposal, it is therefore important to determine $\|X\|$ for any given CTC evolution operator.

Essentially, the unitary dilation proposal performs the minimal book-keeping required to restore unitarity by asserting the presence of a hidden component of the wave-function, which is naturally associated with the CTC region. These 'extra dimensions' are not accessible to experiments conducted outside the CTC region, but provide somewhere for particles to hide from view, whilst maintaining global unitarity. We will see that our proposal thereby circumvents the problems associated with non-unitary evolutions mentioned above.

Of course, it is a moot point whether or not one should require a unitary evolution of quantum fields in the presence of CTC's; one might prefer a more radical approach such as that advocated by Hartle [55], in Chap. 9 we will be investigating the Quantum Field Theory directly within the formalism of the *Quantum Initial Value Problem*. However for the moment our philosophy is to determine the extent to which the conventional formalism of quantum theory can be repaired.

We shall begin in Sect. 8.2 by describing the implications of non-unitarity for the Hilbert space formulation of quantum mechanics and then give a rigorous description of Anderson's proposal in Sect. 8.3, where we also note the operational problem mentioned above. In Sect. 8.4, we introduce the unitary dilaton proposal for unitarity restoration, and show how composition may be treated within this context in Sect. 8.5. In Sect. 8.6, we conclude by discussing the physical significance of our proposal. There are two appendices: Appendix 8.A contains the proof of two results required in the text, whilst Appendix 8.B describes yet another proposal for unitarity restoration based on tensor products. However, this proposal (in contrast to that advocated by Anderson, and the unitary dilation proposal) fails to remove the ambiguity noted by Jacobson [53].

8.2 Non-Unitary Quantum Mechanics

As we mentioned previously, a non-unitary evolution raises many problems for the standard formalism and interpretation of quantum theory, some of which we now discuss.

Firstly, the usual equivalence of the Schrödinger and Heisenberg pictures is lost. Given an evolution X of states and an observable A, we would naturally define the evolved observable A' so that for all initial states ψ, the expectation value of A' in state ψ equals the expectation of A in the evolved state $X\psi$. Explicitly, we require

$$\frac{\langle \psi \,|\, A'\psi \rangle}{\langle \psi \,|\, \psi \rangle} = \frac{\langle X\psi \,|\, AX\psi \rangle}{\langle X\psi \,|\, X\psi \rangle} \tag{8.2.1}$$

for all ψ in the Hilbert space $\mathcal{H}$. If X is unitary up to a scale (i.e., $X^*X = XX^* = \lambda \mathbb{1}$, $\lambda \in \mathbb{R}^+$), then Eq. (8.2.1) is uniquely solved by the Heisenberg evolution $A' = X^{-1}AX$. On the other hand, if X is not unitary up to scale, then there is no operator A' satisfying (8.2.1) unless A is a scalar multiple of the identity.

Proof: Defining $f(\psi)$ to equal the RHS of (8.2.1), and taking ψ and φ to be any orthonormal vectors, we note that linearity of A' entails

$$f(\psi) + f(\varphi) = f(\psi + \varphi) + f(\psi - \varphi), \tag{8.2.2}$$

whilst linearity of A implies

$$\begin{aligned} f(\psi)\|X\psi\|^2 + f(\varphi)\|X\varphi\|^2 \;\; = \;\; & \frac{1}{2}\left\{ f(\psi + \varphi)\|X(\psi + \varphi)\|^2 \right. \\ & \left. + f(\psi - \varphi)\|X(\psi - \varphi)\|^2 \right\}. \end{aligned} \tag{8.2.3}$$

Multiplying φ by a phase to ensure that $\langle X\psi \mid X\varphi \rangle$ is imaginary (and hence that $\|X(\psi \pm \varphi)\|^2 = \|X\psi\|^2 + \|X\varphi\|^2$), we combine these relations to obtain

$$(f(\psi) - f(\varphi))(\|X\psi\|^2 - \|X\varphi\|^2) = 0, \tag{8.2.4}$$

which is clearly insensitive to the phase of φ and therefore holds for all orthonormal vectors ψ and φ. If X is not unitary up to scale, we choose φ and ψ so that $\|X\psi\| \neq \|X\varphi\|$. Thus $f(\psi) = f(\varphi) = F$ for some F. It follows that $f(\chi) = F$ for all χ orthogonal to span $\{\psi, \varphi\}$ (because $\|X\chi\|$ cannot equal both $\|X\psi\|$ and $\|X\varphi\|$) and hence for all $\chi \in \mathcal{H}$. Thus we have proved that A is a scalar multiple of the identity. $\square$

We have seen how the conventional equivalence of the Schrödinger and Heisenberg pictures is radically broken. If there are evolved states, there are no evolved operators, and *vice versa*. In addition, the Heisenberg picture places restrictions on the class of allowed observables. In order to preserve the canonical commutation relations, we take the evolution to be $A \mapsto X^{-1}AX$; however, we also want to preserve self-adjointness of observables under evolution. Combining these two requirements, we conclude that A must commute with XX^* and therefore with $(XX^*)^{1/2}$ – the non-unitary part of the evolution in the sense of the polar decomposition. Thus, the claim attributed to Dirac [61] that 'Heisenberg mechanics is the good mechanics' carries the price of a restricted class of observables when the evolution is non-unitary.

A second problem with non-unitary evolutions, noted by Jacobson [53] (see also Hartle's elaboration [55]) is that one cannot assign unambiguous values to expectation values of operators localized in regions spacelike separated from the CTC region. Let $\mathcal{R}$ be a compact region spacelike separated from the CTC's, and which is contained in two spacelike hypersurfaces σ_+ and σ_-, such that σ_- passes to the past of the CTC's and σ_+ to their future. If A is an observable which is localized within $\mathcal{R}$, one can measure its expectation value with respect to the wavefunction on either spacelike surface. In order for these values to agree, Eq. (8.2.1) must hold with $A' = A$. If X is unitary up to scale, this is satisfied by any observable which commutes with X – in particular by all observables localized in $\mathcal{R}$. However, if X is not unitary up to scale, our arguments above show that there is no observable (other than multiples of the identity) for which unambiguous expectation values may be calculated. Jacobson concludes that a breakdown of unitarity implies a breakdown of causality.

Thirdly, Friedman, Papastamatiou and Simon [48] have pointed out related problems with the assignment of probabilities for events occurring before the region of CTC's.

They consider a microscopic system which interacts momentarily with a measuring device before the CTC region and which is decoupled from it thereafter. The microscopic system passes through the CTC region, whilst the measuring device does not. However, the probability that a certain outcome is observed on the measuring device depends on whether it is observed before or after the microscopic system passes through the CTC's. This is at variance with the Copenhagen interpretation of quantum theory.

8.3 The Anderson Proposal

We begin by giving a rigorous description of Anderson's proposal [56]. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot \mid \cdot\rangle$ and suppose that the non-unitary evolution operator $X : \mathcal{H} \to \mathcal{H}$ is bounded with bounded inverse. We now define a quadratic form on $\mathcal{H}$ by

$$q(\psi, \varphi) = \langle X^{-1}\psi \,|\, X^{-1}\varphi\rangle, \tag{8.3.1}$$

which (because $(X^{-1})^* X^{-1}$ is positive and X and X^{-1} are bounded) defines a positive definite inner product on $\mathcal{H}$ whose associated norm is complete. Replacing $\langle\cdot \mid \cdot\rangle$ by this inner product, we obtain a new Hilbert space which we denote by $\mathcal{H}'$. Because $\mathcal{H}'$ coincides with $\mathcal{H}$ as a vector space, there is an identification mapping $\imath : \mathcal{H} \to \mathcal{H}'$ which maps $\psi \in \mathcal{H}$ to $\psi \in \mathcal{H}'$. The inner product of $\mathcal{H}'$ is

$$\langle\psi \,|\, \varphi\rangle' = \langle X^{-1}\imath^{-1}\psi \,|\, X^{-1}\imath^{-1}\varphi\rangle, \tag{8.3.2}$$

for $\psi, \varphi \in \mathcal{H}'$. The identification mapping is present because X^{-1} is not, strictly speaking, defined on $\mathcal{H}'$. As a minor abuse of notation, one can omit these mappings provided that one takes care of which inner product and adjoint are used in any manipulations. This is the approach adopted by Anderson. The advantage of writing in the identifications is that one cannot lose track of the domain or range of any operator, and adjoints automatically take care of themselves.

From Eq. (8.3.2), it is clear that $\imath X : \mathcal{H} \to \mathcal{H}'$ (i.e., 'X regarded as a map from $\mathcal{H}$ to $\mathcal{H}'$') is unitary – the non-unitarity of X is cancelled by that of $\imath$. Anderson therefore adopts $\imath X$ as the correct unitary evolution: in the Schrödinger picture, an initial state $\psi \in \mathcal{H}$ is evolved unitarily to $\imath X\psi \in \mathcal{H}'$.

The next component in Anderson's proposal concerns observables. Given an observable (e.g., momentum or position) represented as a self-adjoint operator A on $\mathcal{H}$, one needs to know how this observable is represented on $\mathcal{H}'$ in order to evolve expectation values in the Schrödinger picture. At first, one might imagine that A should be carried over directly using the identification mapping to form $A' = \imath A \imath^{-1}$. However, this idea fails because $\imath A \imath^{-1}$ is not self-adjoint in $\mathcal{H}'$ unless A commutes with XX^*: an unacceptable restriction on the class of observables. Instead, Anderson proposes that A' should be defined by

$$A' = \imath R_X A R_X^{-1} \imath^{-1} \tag{8.3.3}$$

where $R_X = (XX^*)^{1/2}$ is self-adjoint and positive on $\mathcal{H}$. The operator $\imath R_X$ is easily seen to be unitary as the unitarity of $\imath X$ implies $\imath^* \imath = (XX^*)^{-1}$, and it then follows that A' is self-adjoint on $\mathcal{H}'$. With this definition, the expectation value of A in a (normalized) state ψ evolves as

$$\begin{aligned} \langle \psi \,|\, A\psi \rangle \mapsto \langle \imath X\psi \,|\, A' \imath X \psi \rangle' &= \langle \imath X \psi \,|\, \imath R_X A R_X^{-1} X \psi \rangle \\ &= \langle R_X^{-1} X \psi \,|\, A R_X^{-1} X \psi \rangle \\ &= \langle U_X \psi \,|\, A U_X \psi \rangle, \end{aligned} \tag{8.3.4}$$

where $U_X = R_X^{-1} X$ is the unitary part of X in the sense of the polar decomposition [57].

So far, it appears that Anderson's proposal is equivalent to Schrödinger picture evolution using U_X in the original Hilbert space, or Heisenberg evolution $A \mapsto U_X^{-1} A U_X$. However, one must be careful with this statement when one considers the composition of two consecutive periods of evolution, say Y followed by X. We take both operators to be maps of $\mathcal{H}$ to itself, as required by Anderson [58, 62]. Proceeding naïvely, we encounter the following problem: taking the unitary parts and composing, we obtain $U_X U_Y$, whilst composing and taking the unitary part (i.e., considering the evolution as a whole, rather than as a two stage process) we find U_{XY}. For consistency, we would require that these evolutions should be equal up to a complex phase λ. As we will demonstrate in Appendix 8.A, this is possible if and only if X^*X commutes with YY^* and $\lambda = 1$. Composition would therefore fail in general.

In response to this, Anderson has proposed that composition be treated as follows [58]. Suppose $Y : \mathcal{H} \to \mathcal{H}$ is the first non-unitary evolution, and apply Anderson's proposal to form a Hilbert space $\mathcal{H}'$ and an identification map $\jmath : \mathcal{H} \to \mathcal{H}'$ so that $\jmath Y$ is unitary. The next step is to form the 'push-forward' X' of the operator XR_Y to $\mathcal{H}'$, which is defined by

$$X' = \jmath R_Y (XR_Y) R_Y^{-1} \jmath^{-1} = \jmath R_Y X \jmath^{-1}. \tag{8.3.5}$$

X' is decomposed as $R_{X'} U_{X'}$ in $\mathcal{H}'$, and $U_{X'}$ is 'pulled back' to $\mathcal{H}$ as $\widetilde{U_{X'}} = R_Y^{-1} \jmath^{-1} U_{X'} \jmath R_Y$. Anderson states that the correct composition law is to form the product $\widetilde{U_{X'}} U_Y$. In fact, we can simplify this slightly, because

$$\widetilde{U_{X'}} = R_Y^{-1} \jmath^{-1} U_{X'} \jmath R_Y = U_{R_Y^{-1} \jmath^{-1} X' \jmath R_Y} = U_{XR_Y} \tag{8.3.6}$$

where we have used the fact that $U_{VXW} = VU_X W$ if V and W are unitary. Thus we can eliminate $\mathcal{H}'$ from the discussion, and the composition rule is essentially to replace the second evolution by U_{XR_Y} rather than U_X. This is certainly consistent: for $U_{XR_Y} = U_{XYU_Y^{-1}} = U_{XY} U_Y^{-1}$, and so $U_{XR_Y} U_Y = U_{XY}$.

However, although this prescription is consistent, it has the drawback that one must know about the first non-unitary evolution in order to treat the second correctly (i.e., one must use U_{XR_Y} rather than U_X). More generally, it is easy to see that, given n consecutive evolutions $X_1, \ldots, X_n$, one should replace each X_r by $U_{X_r R_{X_{r-1} \cdots X_2 X_1}}$ for $r \geq 1$, so one needs to know about all previous evolutions at each step.

This gives rise to the following operational problem: suppose two observers, A and B live in a universe with two isolated compact CTC regions corresponding to evolutions Y and X respectively. Suppose that A knows about both evolutions, but B only knows about X. Thus, according to Anderson's proposal, A replaces these evolutions by U_Y and U_{XR_Y} respectively. But B replaces X by U_X, which differs from U_{XR_Y} unless X^*X

commutes with YY^* (as a corollary of the Theorem in Appendix 8.A). *The two observers treat the second evolution in different ways and will therefore compute different values for expectation values of physical observables in the final state.* This shows that, in Anderson's proposal, it is necessary to know about all non-unitary evolutions in one's past in order to treat non-unitary evolutions in one's future correctly.

For completeness, let us see how this composition law appears in the formulation of Anderson's proposal in which one modifies the Hilbert space inner product. Again we start with the evolution Y, and form the identification map $\jmath : \mathcal{H} \to \mathcal{H}'$. In addition, we can treat the combined evolution $Z = XY$ using Anderson's proposal to form a Hilbert space $\mathcal{H}''$ and identification map $\mathrm{k} : \mathcal{H} \to \mathcal{H}''$, such that $\mathrm{k}Z$ is unitary. The wavefunction is evolved from $\mathcal{H}$ to $\mathcal{H}'$ using $\jmath Y$, and from $\mathcal{H}$ to $\mathcal{H}''$ using $\mathrm{k}Z$. Thus it evolves from $\mathcal{H}'$ to $\mathcal{H}''$ under $\mathrm{k}Z(\jmath Y)^{-1} = \imath\jmath X \jmath^{-1}$, where $\imath = \mathrm{k}\jmath^{-1}$ is clearly the identification mapping between $\mathcal{H}'$ and $\mathcal{H}''$. This evolution, which is forced upon us by the requirement that the wavefunction be evolved consistently, is exactly what arises from Anderson's proposal applied to the operator $\jmath X \jmath^{-1}$ in $\mathcal{H}'$. One might expect that observables would be transformed from $\mathcal{H}'$ to $\mathcal{H}''$ using the rule (8.3.3) applied to this evolution. However, we will now show that this is not the case.

An observable A on $\mathcal{H}$ is represented as the self-adjoint operator $A' = \jmath R_Y A R_Y^{-1} \jmath^{-1}$ on $\mathcal{H}'$, and by $A'' = \mathrm{k} R_Z A R_Z^{-1} \mathrm{k}^{-1}$ on $\mathcal{H}''$. Thus, the transformation between A' and A'' is

$$A'' = \mathrm{k} R_Z R_Y^{-1} \jmath^{-1} A' \jmath R_Y R_Z^{-1} \mathrm{k}^{-1}. \tag{8.3.7}$$

Let us note that this is *not* the transformation law which follows from a naïve application of Anderson's proposal to $\jmath X \jmath^{-1}$ in $\mathcal{H}'$, which would be of form

$$A'' = \imath R_W A' R_W^{-1} \imath^{-1} \tag{8.3.8}$$

with $W = \jmath X \jmath^{-1}$. Indeed, the expression (8.3.7) cannot generally be put into this form for any W. For suppose that there exists some W such that Eqs. (8.3.7) and (8.3.8) are equivalent for all self-adjoint A'. Then $R_W = \lambda \jmath R_Z R_Y^{-1} \jmath^{-1}$ for some $\lambda \in \mathbb{C}$ which may be re-written as $\jmath^{-1} R_W (\jmath^{-1})^* = \lambda R_Z R_Y$ using the unitarity of $\jmath R_Y$. The LHS is self-adjoint, so the lemma in Appendix 8.A entails that ZZ^* and YY^* must commute, which is a non-trivial condition on X and Y when both are non-unitary. Hence in general, the transformation (8.3.7) is not of the form (8.3.8).

Thus, for consistency to be maintained, the transformation rule for observables between $\mathcal{H}'$ and $\mathcal{H}''$ takes a different form from that which holds between $\mathcal{H}$ and $\mathcal{H}'$ or $\mathcal{H}''$. This is a highly undesirable feature of Anderson's proposal. We now look at an alternative method of unitarity restoration that does not suffer from this drawback, however it pays the price of introducing indefinite *(Krein)* inner product spaces.

8.4 The Unitary Dilation Proposal

We begin by describing the theory of unitary dilations [59, 60, 63]. Let $\mathcal{H}_1, \ldots, \mathcal{H}_4$ be Hilbert spaces and let X be a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then an operator $\hat{X}$ from $\mathcal{H}_1 \oplus \mathcal{H}_3$ to $\mathcal{H}_2 \oplus \mathcal{H}_4$ is called a *dilation* of X if $X = P_{\mathcal{H}_2}\hat{X}|_{\mathcal{H}_1}$ where $P_{\mathcal{H}_2}$ is the orthogonal projection onto $\mathcal{H}_2$. In block matrix form, $\hat{X}$ takes form

$$\hat{X} = \begin{pmatrix} X & P \\ Q & R \end{pmatrix}. \tag{8.4.1}$$

Given $X : \mathcal{H}_1 \to \mathcal{H}_2$, we may construct a unitary dilation of X as follows. Firstly, its departure from unitarity may be quantified with the operators $M_1 = \mathbb{1} - XX^*$ and $M_2 = \mathbb{1} - X^*X$. As a consequence of the spectral theorem, we have the intertwining relations

$$X^*f(M_1) = f(M_2)X^*; \qquad Xf(M_2) = f(M_1)X \tag{8.4.2}$$

for any continuous Borel function f. The closures of the images of M_1 and M_2 are denoted $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

We now define $\mathcal{K}_i = \mathcal{H}_i \oplus \mathcal{M}_i$ for $i = 1, 2$, equipped with the (possibly indefinite) inner product $[\cdot,\cdot]_{\mathcal{K}_i}$ given by

$$\left[\begin{pmatrix} \varphi \\ \Phi \end{pmatrix}, \begin{pmatrix} \psi \\ \Psi \end{pmatrix} \right]_{\mathcal{K}_i} = \langle \varphi | \psi \rangle + \langle \Phi | \operatorname{sgn} M_i \Psi \rangle, \tag{8.4.3}$$

where the inner products on the right are taken in $\mathcal{H}$ and $\operatorname{sgn} M_i = |M_i|^{-1} M_i$ where $|M_i| = (M_i^* M_i)^{1/2}$. It is easy to show that $\operatorname{sgn} M_i$ is positive if $\|X\| \leq 1$, in which case $[\cdot,\cdot]_{\mathcal{K}_i}$ is positive definite; however, for $\|X\| > 1$, the inner products above are indefinite, and $\mathcal{K}_1$ and $\mathcal{K}_2$ are *Krein spaces* (for details on the theory of operators in indefinite inner product spaces, see the monographs [64, 65]). It is important to remember that the $\mathcal{K}_i$ also have a positive definite inner product from their original definition as a direct sum of Hilbert spaces. Thus a bounded linear operator A from $\mathcal{K}_1$ to $\mathcal{K}_2$ has two adjoints: the Hilbert space adjoint A^*, and the Krein space adjoint, which we denote $A^\dagger$. It is a simple exercise to show that $A^\dagger$ is given by

$$A^\dagger = J_1 A^* J_2, \tag{8.4.4}$$

where the operators J_i defined on $\mathcal{K}_i$ are unitary involutions given by $J_i = \mathbb{1}_{\mathcal{H}_i} \oplus \operatorname{sgn}(M_i)$.

Next, we define a dilation $\hat{X} : \mathcal{K}_1 \to \mathcal{K}_2$ of X by

$$\hat{X} = \begin{pmatrix} X & -\operatorname{sgn}(M_1)|M_1|^{1/2} \\ |M_2|^{1/2} & X^*|_{\mathcal{M}_1} \end{pmatrix}, \tag{8.4.5}$$

which has adjoint $\hat{X}^\dagger$ given by Eq. (8.4.4) as

$$\hat{X}^\dagger = \begin{pmatrix} X^* & \operatorname{sgn}(M_2)|M_2|^{1/2} \\ -|M_1|^{1/2} & \operatorname{sgn}(M_1)X|_{\mathcal{M}_2}\operatorname{sgn}(M_2) \end{pmatrix}. \tag{8.4.6}$$

It is then just a simple matter of computation using the intertwining relations to show that $\hat{X}^\dagger \hat{X} = \mathbb{1}_{\mathcal{K}_1}$ and $\hat{X}\hat{X}^\dagger = \mathbb{1}_{\mathcal{K}_2}$. $\hat{X}$ is therefore a unitary dilation of X.

The construction we have given is not unique in providing a unitary dilation. For suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are Krein spaces, and that $U_i : \mathcal{M}_i \to \mathcal{N}_i$ are unitary (with respect to the indefinite inner products). Then

$$\widetilde{X} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_2 \end{pmatrix} \hat{X} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_1^\dagger \end{pmatrix} \tag{8.4.7}$$

is also a unitary dilation of X, mapping between $\mathcal{H} \oplus \mathcal{N}_1$ and $\mathcal{H} \oplus \mathcal{N}_2$. Because this just amounts to a redefinition of the auxiliary spaces, it carries no additional physical significance. One may show that all other unitary dilations of X require the addition of larger auxiliary spaces than the $\mathcal{M}_i$ (for example, one could dilate $\hat{X}$ further). Thus $\hat{X}$ is the minimal unitary dilation of X up to unitary equivalence of the above form.

Having described the general theory, let us now apply it to the case of interest. For simplicity, we assume that the Hilbert spaces of initial and final states are identical, so $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. We also assume that the evolution operator X is bounded with bounded inverse. If the initial hypersurface contains regions which are causally separate from the CTC region, we assume that X has been normalized to be unitary on states localized in such regions. We point out that such exterior regions may not exist – even if the CTC region is itself compact. Consider, for example, a spacetime that is asymptotically (the universal cover of) anti-de Sitter space. In such a spacetime, hypersurfaces sufficiently far to the future and far to the past of the CTC region will be entirely contained within the CTC region's light cone and there will be no exterior region on which to set up our normalization. We may normalize the evolution operator on hypersurfaces for which an exterior region may be identified and extend arbitrarily to those surfaces where no such region exists. Indeed, it is entirely possible that every point in spacetime is contained in the light cone of the CTC region; in this case we give up any attempt to find a 'physical' normalization for the evolution operator.

The spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as above. Note that we have the polar decomposition $X = (XX^*)^{1/2}U$, where U is a *unitary* operator because X is invertible. As a consequence of the intertwining relations, we have

$$UM_2 = M_1U \tag{8.4.8}$$

and hence that $\mathcal{M}_1 = U\mathcal{M}_2$. Thus the $\mathcal{M}_i$ are isomorphic as Hilbert spaces. Moreover, U is also unitary with respect to the indefinite inner products on the auxiliary spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, which follows from the identity $U\mathrm{sgn}\,(M_2) = \mathrm{sgn}\,(M_1)U$. We can therefore use the freedom provided by Eq. (8.4.7) to arrange that the same auxiliary space is used both before and after the evolution.

The Unitary Dilation Proposal is the following. Given a non-unitary evolution X, there exists an (indefinite) auxiliary space $\mathcal{M}$ (isomorphic to the $\mathcal{M}_i$) and a unitary dilation $\widetilde{X} : \mathcal{K} \to \mathcal{K}$ of X, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{M}$. We regard this as describing the full physics of the situation: on $\mathcal{K}$, the evolution is unitary, whilst its restriction to the original Hilbert space $\mathcal{H}$ yields the non-unitary operator X. The auxiliary space $\mathcal{M}$ represents degrees of freedom localized within the CTC region, not directly accessible to experiments outside. However indirectly, we can infer their presence by analysing X.

Observables are defined as follows. Given any self-adjoint operator A on $\mathcal{H}$, we define the corresponding observable on $\mathcal{K}$:

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.4.9}$$

The form of $\widetilde{A}$ is chosen to prevent the internal degrees of freedom being probed from outside.

Let us point out that many features of this proposal can only be determined in the context of a particular evolution X and therefore a particular CTC spacetime. There are, however, various model independent features of our proposal, which we discuss below.

Predictability Because the initial state involves degrees of freedom not present on the initial hypersurface (i.e., the component of the wavefunction in $\mathcal{M}$), it is clear that – as far as physical measurements are concerned – there is some loss of predictability in the final state. This problem can be circumvented by the requirement that the initial state should have no component in $\mathcal{M}$. However at an operational level this may not be the case.

Expectation Values Let us examine the evolution of the expectation value of $\widetilde{A}$. On the premise that the initial state has no component in $\mathcal{M}$ and takes the vector form

$(\psi, 0)^T$, the initial expectation value of $\widetilde{A}$ is

$$\frac{\left[\begin{pmatrix}\psi\\0\end{pmatrix}, \widetilde{A}\begin{pmatrix}\psi\\0\end{pmatrix}\right]_{\mathcal{K}_2}}{\left[\begin{pmatrix}\psi\\0\end{pmatrix}, \begin{pmatrix}\psi\\0\end{pmatrix}\right]_{\mathcal{K}_2}} = \frac{\langle\psi\,|\,A\psi\rangle}{\langle\psi\,|\,\psi\rangle}, \tag{8.4.10}$$

i.e., the expectation value of A in state ψ. After evolution, the expectation value is

$$\frac{\left[\widetilde{X}\begin{pmatrix}\psi\\0\end{pmatrix}, \widetilde{A}\widetilde{X}\begin{pmatrix}\psi\\0\end{pmatrix}\right]_{\mathcal{K}_2}}{\left[\widetilde{X}\begin{pmatrix}\psi\\0\end{pmatrix}, \widetilde{X}\begin{pmatrix}\psi\\0\end{pmatrix}\right]_{\mathcal{K}_2}} = \frac{\langle X\psi\,|\,AX\psi\rangle}{\langle\psi\,|\,\psi\rangle}. \tag{8.4.11}$$

It is important to note that both denominators are equal to $\|\psi\|^2$ (because the full evolution is unitary) – this removes many of the problems encountered in Sect. 8.2.

In particular, let us return to the problem noted by Jacobson [53], writing $\mathcal{R}$ for the region spacelike separated from the CTC region, and taking X to be the evolution from states on σ_- to states on σ_+. We assume (as in [53]) that X acts as the identity on $\mathcal{H}_{\mathcal{R}}$, the subspace of states supported in $\mathcal{R}$. Any local observable associated with $\mathcal{R}$ should vanish on the orthogonal complement of $\mathcal{H}_{\mathcal{R}}$ in $\mathcal{H}$: accordingly, it follows that $X^*AX = A$, and hence that the expectation value is independent of the choice of hypersurface (σ_+ or σ_-) on which it is computed. Thus Jacobson's ambiguity is avoided for all local observables associated with regions spacelike separated from the causality-violating region. More generally, it is avoided for all observables A such that $A = X^*AX$. This is satisfied if the image of A is contained in $\mathcal{U} = \ker M_1 \cap \ker M_2 \subseteq \mathcal{H}$ and A commutes with the restriction $X|_{\mathcal{U}}$ of X to $\mathcal{U}$.

In addition, the breakdown of the Copenhagen interpretation noted in [48] is avoided as a direct consequence of the unitarity of $\widetilde{X}$.

Time Reversal Let us suppose the existence of an antiunitary involution T (i.e., an antilinear involution obeying $\langle T\psi \mid T\varphi\rangle = \langle\varphi \mid \psi\rangle)$ on $\mathcal{H}$ implementing time reversal. The *time reverse* $X_{\rm rev}$ of X is given by $X_{\rm rev} = TXT^{-1}$; X is said to be *time reversible* if $X_{\rm rev} = X^{-1}$. We would like to understand how the time reversal properties of $\hat{X}$ are related to those of X. For convenience we will work in terms of $\hat{X}$; the discussion may be rephrased in terms of $\widetilde{X}$ by inserting suitable unitary operators between the $\mathcal{M}_i$ and $\mathcal{M}$. With these definitions it is then a simple matter to check that $T\mathcal{M}_i = \mathcal{M}_i^{\rm rev}$.

First, we must define the time reversal of $\hat{X}$. The natural definition is

$$(\hat{X})_{\rm rev} = \begin{pmatrix} T & 0 \\ 0 & T|_{\mathcal{M}_2} \end{pmatrix} \hat{X} \begin{pmatrix} T^{-1} & 0 \\ 0 & (T|_{\mathcal{M}_1})^{-1} \end{pmatrix}, \tag{8.4.12}$$

which entails that time reversal and dilation commute in the sense that $(\hat{X})_{\rm rev} = \widehat{X_{\rm rev}}$. However, because dilation and inversion do not commute (i.e., $(\hat{X})^{-1} \neq \widehat{X^{-1}}$) unless X is unitary, we find that a time reversible evolution X does not generally yield a time reversible dilation:

$$(\hat{X})_{\rm rev} = \widehat{X_{\rm rev}} = \widehat{X^{-1}} \neq (\hat{X})^{-1}. \tag{8.4.13}$$

Thus if X is non-unitary and time reversible, then $\hat{X}$ is not time reversible. On the other hand, suppose that $\hat{X}$ is time reversible. Then $\widehat{X_{\rm rev}} = \widehat{X^*}$ from which it follows that X would obey the modified reversal property $X_{\rm rev} = X^*$. It would be interesting to determine, for concrete CTC models, whether X obeys the usual time reversal property $X_{\rm rev} = X^{-1}$ or the modified property $X_{\rm rev} = X^*$ (of course it might not obey either property).

To summarize this section, we have seen how unitarity can be restored using the method of unitary dilations, thereby removing the problems associated with non-unitary evolutions. Any observable on $\mathcal{H}$ defines an observable in this proposal.

8.5 Composition

We have described how a single non-unitary evolution may be dilated to a unitary evolution between enlarged inner product spaces. In what sense does our proposal respect the composition of two (or more) non-unitary evolutions?

Let us consider two evolutions X and Y on $\mathcal{H}$ and their composition XY. We define the M_i and $\mathcal{M}_i$ as before and introduce $N_1 = \mathbb{1} - YY^*$, $N_2 = \mathbb{1} - Y^*Y$ and $\mathcal{N}_i = \overline{\mathrm{Im}\, N_i}$ to be the closure of the image of N_i for $i = 1, 2$. As before, we can construct dilations $\hat{X}$ and $\hat{Y}$. However, because $\hat{X} : \mathcal{H} \oplus \mathcal{M}_1 \to \mathcal{H} \oplus \mathcal{M}_2$ and $\hat{Y} : \mathcal{H} \oplus \mathcal{N}_1 \to \mathcal{H} \oplus \mathcal{N}_2$, it is not immediately apparent how the dilations may be composed. The solution is to dilate both $\hat{X}$ and $\hat{Y}$ further, as follows: $\check{Y} : \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \to \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_2$ is given by

$$\check{Y} = \begin{pmatrix} Y & 0 & -\mathrm{sgn}\, N_1 |N_1|^{1/2} \\ 0 & \mathbb{1}_{\mathcal{M}_1} & 0 \\ |N_2|^{1/2} & 0 & Y^*|_{\mathcal{N}_1} \end{pmatrix}, \tag{8.5.1}$$

and $\check{X} : \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_2 \to \mathcal{H} \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$ is given by

$$\check{X} = \begin{pmatrix} X & -\mathrm{sgn}\, M_1 |M_1|^{1/2} & 0 \\ |M_2|^{1/2} & X^*|_{\mathcal{M}_1} & 0 \\ 0 & 0 & \mathbb{1}_{\mathcal{N}_2} \end{pmatrix}. \tag{8.5.2}$$

The product $\check{X}\check{Y}$ is given by

$$\check{X}\check{Y} = \begin{pmatrix} XY & -\mathrm{sgn}\, M_1 |M_1|^{1/2} & -X \mathrm{sgn}\, N_1 |N_1|^{1/2} \\ |M_2|^{1/2} Y & X^*|_{\mathcal{M}_1} & -|M_2|^{1/2} \mathrm{sgn}\, N_1 |N_1|^{1/2} \\ |N_2|^{1/2} & 0 & Y^*|_{\mathcal{N}_1} \end{pmatrix}, \tag{8.5.3}$$

and is a unitary dilation of XY, mapping from $\mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_1$ to $\mathcal{H} \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$.

This state of affairs is quite natural: we have argued that each CTC region carries with it its own auxiliary space (isomorphic to the $\mathcal{M}_i$ and the $\mathcal{N}_i$); one would therefore expect that the combined evolution should be associated with the direct sum of these auxiliary spaces. However, in order to show how our proposal respects composition, we need to show how the product $\check{X}\check{Y}$ is related to the dilation $\widehat{XY}$ arising from the prescription (8.4.5). To this end, we introduce $P_1 = \mathbb{1} - XYY^*X^*$, $P_2 = \mathbb{1} - Y^*X^*XY$

and $\mathcal{P}_i = \overline{\operatorname{Im} P_i}$. Note that

$$P_1 = M_1 + XN_1X^* \quad \text{and} \quad P_2 = N_2 + Y^*M_2Y. \tag{8.5.4}$$

Now let

$$Q_1 = \begin{pmatrix} |M_1|^{1/2} \\ |N_1|^{1/2}X \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} |M_2|^{1/2}Y \\ |N_2|^{1/2} \end{pmatrix}, \tag{8.5.5}$$

and define U_i $(i = 1, 2)$ on $\operatorname{Im}|P_i|^{1/2} \subseteq \mathcal{P}_i$ by $U_i = Q_i|P_i|^{-1/2}$. The U_i are easily seen to be isometries (with respect to the appropriate inner products) from their domains into $\mathcal{M}_i \oplus \mathcal{N}_i$ such that $Q_i|_{\overline{\operatorname{Im} P_i}} = U_i|P_i|^{1/2}$. Provided that $\mathcal{Q}_i = \overline{Q_i\overline{\operatorname{Im} P_i}}$ is orthocomplemented in $\mathcal{M}_i \oplus \mathcal{N}_i$ (in the indefinite inner product), we may then verify that

$$\mathrm{P}_{\mathcal{H}\oplus\mathcal{Q}_2}\check{X}\check{Y}|_{\mathcal{H}\oplus\mathcal{Q}_1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} XY & -\operatorname{sgn} P_1|P_1|^{1/2} \\ |P_2|^{1/2} & (XY)^*|_{\mathcal{P}_1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_1^\dagger \end{pmatrix}, \tag{8.5.6}$$

where $\mathrm{P}_{\mathcal{H}\oplus\mathcal{Q}_2}$ is the orthogonal projection onto $\mathcal{H} \oplus \mathcal{Q}_2$. Thus $\check{X}\check{Y}$ is a dilation of an operator isometrically equivalent to $\widehat{XY}$. The isometries act non-trivially only on the auxiliary spaces and have no physical significance. The extra dimensions introduced by the dilation are also to be expected because the combined evolution $Z = XY$ may be factorized in many different ways; hence the two individual evolutions carry more information than their combination.

The assumption that the $\mathcal{Q}_i$ are orthocomplemented is easily verified if the operators U_i are bounded, for in this case, they may be extended to unitary operators on the whole of $\mathcal{P}_i$. Then $\mathcal{Q}_i$ is the unitary image of a Krein space and is orthocomplemented by Theorem VI.3.8 in [64]. We note that U_1 is bounded if there exists K such that $\|P_1\psi\| < \epsilon$ only if $\|M_1\psi\| + \|N_1X\psi\| < K\epsilon$ for all sufficiently small $\epsilon > 0$. Similarly, U_2 is bounded if $\|P_1\psi\| < \epsilon$ only if $\|M_2Y\psi\| + \|N_2\psi\| < K\epsilon$ for all sufficiently small $\epsilon > 0$.

As a particular instance of the above, we consider the case where Y is unitary. The N_i therefore vanish and the $\mathcal{N}_i$ are trivial; in addition, $P_1 = M_1$ and $P_2 = Y^*M_2Y$. The operator $\check{Y}$ is

$$\check{Y} = \begin{pmatrix} Y & 0 \\ 0 & \mathbb{1}_{\mathcal{M}_1} \end{pmatrix} \tag{8.5.7}$$

and $\check{X} = \hat{X}$. The combined evolution is thus

$$\check{X}\check{Y} = \hat{X} \begin{pmatrix} Y & 0 \\ 0 & \mathbb{1}_{\mathcal{M}_1} \end{pmatrix} \tag{8.5.8}$$

which is unitarily equivalent to $\widehat{XY}$ in the sense that

$$\hat{X} \begin{pmatrix} Y & 0 \\ 0 & \mathbb{1}_{\mathcal{M}_1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & Y \end{pmatrix} \widehat{XY}. \tag{8.5.9}$$

We emphasize that the first factor on the RHS has no physical significance and is merely concerned with mapping the auxiliary spaces $\mathcal{P}_2$ to $\mathcal{M}_2$ in a natural way.

To conclude this section, we make three comments. Firstly, note that if A belongs to the class of observables which avoid the Jacobson ambiguity for each CTC region individually, then it also avoids this ambiguity for the combined evolution; for if $A = X^*AX = Y^*AY$, then certainly $A = Y^*X^*AXY$. Thus there is no 'multiple Jacobson ambiguity'. Secondly, in this proposal one does not need to know the past history of the universe in order to evolve forward from any given time, because the auxiliary degrees of freedom associated with one CTC region are essentially passive 'spectators' during the evolution associated with any other such region. This is in contrast with the composition rule proposed by Anderson [58]. Thirdly, one might ask [62] what would happen if the non-unitary evolution was continuous rather than occurring in discrete steps. This question could be tackled using a suitable generalization of the theory of unitary dilations of semi-groups discussed by Davies [66].

8.6 Conclusion

We have examined Anderson's proposal [56] for restoring unitarity to quantum evolution in CTC spacetimes, and noted an operational problem arising when one considers the composition of two or more non-unitary evolutions. Instead, we have investigated a new method for the restoration of unitarity, based on the theory of unitary dilations, which respects composition under certain reasonable conditions. Because unitarity is restored on the full inner product space, problems associated with non-unitary evolutions such as Jacobson's ambiguity are avoided.

Our philosophy here has been to regard the non-unitarity of X as a signal that the full physics (and a unitary evolution) is being played out on a larger state space than is observed. This bears some resemblance to the situation in special relativity, where time dilation signals that one must pass to spacetime (and an indefinite metric) in order to restore an orthogonal transformation between reference frames. (Indeed, the Lorentz boost in two dimensional Minkowski space is precisely an orthogonal dilation of the time dilation effect).

For the case of interest, the physical picture is that the auxiliary space $\mathcal{M}$ corresponds to degrees of freedom within the CTC region. Non-unitarity of the evolution signals that a particle cannot pass through the CTC region unscathed: part of the initial state becomes trapped in the auxiliary space corresponding to the CTC's. A similar conclusion is espoused by three of the authors of [67].

In the case in which X has norm less than or equal to unity (so that the full space $\mathcal{K}$ has a positive definite inner product), this effect has a relatively simple interpretation. Namely, there is a non-zero probability that an incident particle will never emerge from the CTC region. To see how this can occur, we note that computations of the propagator (see particularly [52]) proceed by requiring consistency of the evolution round the CTC's. It seems that part of the incident state becomes trapped in order to achieve this consistency.

On the other hand, perturbative calculations in $\lambda\phi^4$ theory by Boulware [47] suggest that $\|X\|$ could well exceed unity. In this case, $\mathcal{K}$ is an indefinite Krein space, and it would apparently be possible that the 'probability' of the particle escaping from the CTC region could be greater than one. In principle, one might try to avoid this by seeking natural positive definite subspaces of the initial and final Krein spaces. The obvious choice would be to take the initial Hilbert space to be $\mathcal{H}$ and the final Hilbert space to be the image of $\mathcal{H}$ under $\widetilde{X}$. However, this may lead to some problems in defining observables on the final Hilbert space. If one decides to face the problem directly (which seems preferable), one would be forced to conclude that CTCs are incompatible with the twin requirements of unitarity *and* a Hilbert space structure. The initial and final state spaces would naturally be Krein spaces. This would not be entirely unexpected: studies of quantum mechanics on the 'spinning cone' spacetime [68] have concluded that the inner product becomes indefinite precisely inside the region of CTC's. 'Probabilities' greater than unity would denote the breakdown of the theory in a manner analogous to the Klein paradox (see the extensive discussion in the monograph of Fulling [69]) in which strong electrostatic fields force the Klein-Gordon inner product to be indefinite. In our case, it is *the geometry* of spacetime which leads us to an indefinite inner product. We expect that particle creation would occur in this case, as it does in the usual Klein paradox.

The Klein paradox can be resolved by treating the electromagnetic field as a dynamic field, rather than as a fixed external field. Particles are created in a burst as the field collapses (unless it is maintained by some external agency). In our case it seems reasonable that, in the context of a full quantum theory of gravity, a burst of particle creation occurs and the CTC region collapses. This is essentially the content of Hawking's Chronology Protection Conjecture [70]. Thus the emergence of Krein spaces in our proposal may be interpreted as a signal for the instability of the CTC spacetime.

Finally, our treatment has been entirely in terms of states and operators; it would be interesting to see how it translates into density matrices and the language of generalized quantum mechanics [54].

8.A Proof of Theorem

In this appendix, we prove the following two results:

Theorem *Suppose X and Y are bounded with bounded inverses. Then $U_{XY} = \lambda U_X U_Y$ if and only if X^*X commutes with YY^* and $\lambda = 1$.*

Proof: Starting with the sufficiency, we note that $Z = (X^*)^{-1}(X^*X)^{1/2}(YY^*)^{-1/2}X^{-1}$ is positive and squares to give $(XYY^*X^*)^{-1}$ (using the commutation property). It follows that Z is equal to the unique positive square root of $(XYY^*X^*)^{-1}$ and hence that

$$U_{XY} = (XYY^*X^*)^{-1/2}XY = (X^*)^{-1}(X^*X)^{1/2}(YY^*)^{-1/2}Y. \tag{8.A.1}$$

Using the fact that $(X^*)^{-1}(X^*X)^{1/2} = U_X$, we have proved sufficiency.

To demonstrate necessity, we note that $U_{XY} = \lambda U_X U_Y$ only if

$$X^*(XYY^*X^*)^{-1/2}X = \lambda(X^*X)^{1/2}(YY^*)^{-1/2}. \tag{8.A.2}$$

It follows that the RHS must be self-adjoint and positive. We now apply the following Lemma:

Lemma *Suppose that A and B are bounded with bounded inverses and self-adjoint, and suppose that αAB is self-adjoint and positive for some $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Then $\alpha \in \mathbb{R}$ and A and B commute.*

Proof: Because αAB is self-adjoint, we have

$$\alpha AB = \overline{\alpha} BA. \tag{8.A.3}$$

Now note that

$$\begin{aligned} \overline{\alpha}(\alpha AB - z)^{-1} &= \overline{\alpha}(\overline{\alpha}BA - z)^{-1} \\ &= \alpha B(\alpha AB - z\alpha/\overline{\alpha})^{-1}B^{-1}. \end{aligned} \tag{8.A.4}$$

Because αAB has non-empty spectrum on the positive real axis and because the resolvent $(\alpha AB - z)^{-1}$ is an analytic operator valued function of z in $\mathbb{C} \setminus \mathbb{R}^+$, we conclude that

$\alpha/\overline{\alpha}$ must be real and positive. Accordingly, $\alpha \in \mathbb{R}$ and Eq. (8.A.3) implies that A and B commute. □

In our case, this implies that $\lambda = \pm 1$ and that X^*X commutes with YY^*. Moreover, because the two square roots on the RHS of Eq. (8.A.2) are positive and commute, we conclude that $\lambda = 1$ in order that the RHS be positive. □

8.B An Alternative Method to Restore Unitarity

Here, we consider another possible method for the restoration of unitarity which, however, suffers from problems related to Jacobson's ambiguity. Instead of focussing on direct sums of Hilbert spaces, this proposal uses tensor products and always maintains a positive definite inner product. We start with $X : \mathcal{H} \to \mathcal{H}$, bounded with bounded inverse and non-unitary as before, and define a new Hilbert space $\mathcal{H}_X = (\mathbb{1} \otimes X)\Sigma$, where $\Sigma \subseteq \mathcal{H} \otimes \mathcal{H}$ is the closure of the space of finite linear combinations of terms of form $\psi \otimes \psi$ for $\psi \in \mathcal{H}$. Similarly, we define $\mathcal{H}_{X^{-1}} = (\mathbb{1} \otimes X^{-1})\Sigma$. Now define the operator $\widetilde{X} = X \otimes X^{-1}$ restricted to $\mathcal{H}_X$. Clearly, $\widetilde{X}(\psi \otimes X\psi) = \varphi \otimes X^{-1}\varphi$ where $\varphi = X\psi$, and so $\widetilde{X} : \mathcal{H}_X \to \mathcal{H}_{X^{-1}}$. Moreover,

$$\begin{aligned} \langle \widetilde{X}(\psi \otimes X\psi) \,|\, \widetilde{X}(\varphi \otimes X\varphi) \rangle &= \langle X\psi \otimes \psi \,|\, X\varphi \otimes \varphi \rangle \\ &= \langle X\psi \,|\, X\varphi \rangle \langle \psi \,|\, \varphi \rangle \\ &= \langle \psi \otimes X\psi \,|\, \varphi \otimes X\varphi \rangle \end{aligned} \tag{8.B.1}$$

and therefore $\widetilde{X}$ is a unitary operator from $\mathcal{H}_X$ to $\mathcal{H}_{X^{-1}}$.

Let us examine the structure of this proposal in more detail. First, there is a natural transposition operation $\mathcal{T}$ on $\mathcal{H} \otimes \mathcal{H}$: $\mathcal{T}(\varphi \otimes \psi) = \psi \otimes \varphi$. It is easy to see that $\widetilde{X}$ is the restriction of $\mathcal{T}$ to $\mathcal{H}_X$: hence all the information about X is encoded into the definition of $\mathcal{H}_X$. We will want to know whether we have lost any information in this process. Suppose $\mathcal{H}_X = \mathcal{H}_Y$ for two distinct operators X and Y. Then $\mathbb{1} \otimes Z$ is a bounded invertible linear map (though not necessarily unitary) of Σ onto itself, where $Z = X^{-1}Y$. Because $\mathcal{T}$ restricts to the identity on Σ, we require $\psi \otimes Z\psi = Z\psi \otimes \psi$ for each $\psi \in \mathcal{H}$. Taking an inner product with $\varphi \otimes \psi$ for some φ, we obtain

$$\langle \varphi \,|\, \psi \rangle \langle \psi \,|\, Z\psi \rangle = \langle \varphi \,|\, Z\psi \rangle \langle \psi \,|\, \psi \rangle. \tag{8.B.2}$$

Because φ is arbitrary, ψ is therefore an eigenvector of Z. But ψ was also arbitrary and therefore $Z = \lambda\mathbb{1}$ for some constant $\lambda \in \mathbb{C}\backslash\{0\}$. Thus $Y = \lambda X$, so this construction loses exactly one scalar degree of freedom. Effectively, we have lost the (scalar) operator norm $\|X\|$ of X, but no other information.

We have therefore restored unitarity at the price of introducing a second Hilbert space and correlations between the two. The evolution on the large space is unitary. This fits well with the picture of acausal interaction between the initial space and the CTC region in its future. The physical interpretation is as follows: the 'time machine' contains a copy of the external universe, which evolves backwards in time, starting with the final state of the quantum fields and ending with their initial state. It is impossible to prepare the initial state of the CTC region independently from the initial state of the exterior quantum fields.

However, problems arise when observables are defined. Here, observables on the initial space are naturally defined to be self-adjoint operators on $\mathcal{H} \otimes \mathcal{H}$ with $\mathcal{H}_X$ as an invariant subspace (observables on the final space would have $\mathcal{H}_{X^{-1}}$ invariant). An operator of form $A \otimes B$ maps $\mathcal{H}_X$ to itself only if $B = XAX^{-1}$; combining this with the requirement of self-adjointness, one finds that A must commute with X^*X and its powers. Thus this proposal places restrictions on the class of allowed observables.

The requirement that $\mathcal{H}_X$ be an invariant subspace for all observables was adopted so that our space of initial states is invariant under the unitary groups generated by observables (e.g. translations). If we relax this, and define observables to be self-adjoint operators on $\mathcal{H} \otimes \mathcal{H}$, it appears that $A \otimes \mathbb{1}$ corresponds naturally to the operator A on $\mathcal{H}$. However, this suffers from the ambiguity pointed out by Jacobson [53].

9. THE QUANTUM INITIAL VALUE PROBLEM FOR CTC MODELS

9.1 Introduction

Spacetimes containing closed timelike curves (CTC's) provide an intriguing environment for the formulation of both classical and quantum physics. Because the present is influenced by both the past and the future, neither existence nor uniqueness is guaranteed *a priori* for solutions to initial value problems for particles and fields on such spacetimes; these issues underlie many of the apparent paradoxes associated with time travel. In this chapter, we attempt to gain insight into the initial value problem for a class of nonlinear differential equations (which may be regarded as toy field theories) on 'spacetimes' of a type introduced by Politzer [52]. These spacetimes are defined by taking the Cartesian product of a number of discrete points (representing space) with the real line (time) and then imposing certain identifications to introduce CTC's. The simple nature of these models removes many technical problems and allows us to pursue the analysis to its end.

Previous studies of classical initial value problems on chronology violating spacetimes have mostly focussed on linear fields [71, 72, 73, 49, 51] and billiard ball models [73, 74, 75, 76]. Deutsch [77] has also studied examples of classical computational networks with chronology violating components. For linear fields, it turns out that one can formulate a well posed initial value problem under certain conditions. Friedman and Morris [71, 72] have rigorously proved existence and partial uniqueness results for massless fields propagating on a class of smooth static wormhole spacetimes with data specified at past null infinity. In addition, they have conjectured that the initial value problem is well posed for asymptotically flat spacetimes with a compact nonchronal region whose past and future regions are globally hyperbolic whenever the problem is well posed in the geometric optics limit. It is much easier to prove existence and uniqueness for linear fields on certain non-smooth chronology violating spacetimes [49, 51].

In the billiard ball case, Echeverria *et al.* [74] and (for similar and more elaborate systems) Novikov [75] have shown that the initial value problem is often ill posed in the sense that the evolution is not unique; moreover, Rama and Sen [76] have given similar examples in which there appears to be no global self-consistent solution for certain initial data.

One would expect that nonlinear fields should interpolate between the behaviour exhibited by linear fields on one hand and billiard balls (as representing a strongly non-linear interaction) on the other. We will show that this is indeed the case for our class of models: we prove that the initial value problem is well posed for arbitrary data specified before the nonchronal region in both the linear and weakly nonlinear case, but uniqueness (though not existence) fails in the strongly nonlinear regime. In addition to these analytical results, we give an explicit example to demonstrate the lack of uniqueness for a particular value of 'coupling strength'. We also show that the evolution from the past of the nonchronal region to its future preserves the symplectic structure.

The loss of uniqueness for interacting systems on chronology violating spacetimes entails that classical physics loses its predictive power. Various authors have expressed the hope that quantum dynamics on such spacetimes might be better behaved than its classical counterpart, with attention focussing on spacetimes possessing both initial and final chronal regions. Friedman *et al.* [50] considered linear quantum fields and showed that, provided the classical initial value problem is well posed, the quantum evolution between spacelike surfaces in the initial and final chronal regions is unitary; a conclusion borne out by Boulware [47] in a Gott space example (see also [52, 49, 51] for related results). However, the situation is very different for interacting fields. Both Boulware [47] and Friedman *et al.* [48] found that the S-matrix between the initial and final chronal regions fails to be unitary in perturbative $\lambda\phi^4$ theory. Politzer also obtained similar perturbative results in quantum mechanics [49] and also some nonperturbative results in exactly soluble models [52] in which nonunitarity also arises. It is also worth pointing out that some interacting systems do have unitary quantum theories [49]. In

Chap. 10 we will be focussing on the issue of particle creation in a lattice spacetime as a model of a local interacting quantum theory with a Thirring-type interaction. It turns out that this system is one of those that has a unitary quantum theory.

The breakdown of unitarity raises many problems for the probability interpretation of quantum theory; in particular, ambiguities arise in assigning probabilities to the outcomes of measurements conducted before [48] or spacelike separated from [53] the nonchronal region. There have been various reactions to these problems. Firstly, Hartle [55] has discussed how nonunitary evolutions can be accommodated within the framework of generalized quantum mechanics, and a similar proposal has also been advanced by Friedman *et al.* [48]. A second approach has been to 'repair' the theory by modifying the evolution to yield a unitary theory [56, 3]. This was our approach in Chap. 8. Thirdly, Hawking [78] has argued that one should expect loss of quantum coherence in the presence of CTC's and that the evolution should be specified by means of a superscattering operator (i.e., a linear mapping from initial to final density matrices) which moreover would not factorize into a unitary S-matrix and its adjoint. From this viewpoint, the quantity computed using the usual rules for the S-matrix is not the physically relevant quantity and its nonunitarity is irrelevant. Rather, one should compute the matrix elements of the superscattering operator. Deutsch [77] has also advocated a density matrix formalism in the context of quantum computational networks (see also [52]). However, this prescription turns out to be nonlinear in the initial density matrix [79].

For the most part, the quantization method employed in discussions of chronology violation has been based on path integrals in which one sums over all consistent trajectories or field configurations. We follow an operator approach based on the *Quantum Initial Value Problem* (QIVP). Namely, we seek operator valued solutions to the equation of motion and any consistency conditions arising from the CTC's, with initial data specified before the nonchronal region and forming a representation of the canonical (anti)commutation relations. If there exists a unique solution with this initial data, and the evolved data to the future of the nonchronal region also represents the commutation relations, then we say that the QIVP is well posed.

We will show that the QIVP is well posed for all linear models in our class of interest with both Bose and Fermi statistics. The corresponding quantum theory is unitary and agrees with that derived by path integral methods. In the interacting case, we prove the remarkable fact that (with normal ordering) the QIVP always has a unique solution and describe how this solution may be constructed. To obtain more specific results, we consider the cases in which ‘space’ consists of either 2 or 3 points. In the 2-point model, we find that the unique solution to the QIVP satisfies the CCR/CAR’s to the future of the nonchronal region (so the QIVP is well posed) and that in consequence the resulting quantum theory is *unitary*. This contrasts strongly with the corresponding path integral result (generalizing that of Politzer [52]) in which the evolution is found to be nonunitary. In consequence, and because the path integral also employs normal ordering, we conclude that the self-consistent path integral evolution does not generally correspond to a solution of the equation of motion. Given the different starting points of the two approaches this is not entirely surprising.

In the 3-point model, we show that the QIVP is ill posed for both Bose and Fermi statistics because the evolved data does not satisfy the CCR/CAR’s to the future of the nonchronal region. The corresponding quantum theory is therefore not unitary. We then discuss the nature of this evolution in the fermionic case in order to determine whether or not it can be described by means of a superscattering operator. To do this it is necessary to translate our results from the Heisenberg picture to the Schrödinger picture. Although there is no unique translation prescription (as a consequence of the violation of the CAR’s), we are nonetheless able to show that *no* Schrödinger picture evolution consistent with the QIVP solution can factorize into the product of an operator and its adjoint, lending support to one element of Hawking’s position [78]. However, it also transpires that no such Schrödinger picture evolution can be described by a superscattering matrix as it must either *increase* the trace of density matrices or map them to non-positive operators. In this sense, the loss of unitarity in our model is much more radical than envisaged by Hawking.

We also study the classical limit of our quantum theory. One might imagine that this limit would fail when the classical theory is non-unique; however, this is not the case. It appears that there are bands of ‘coupling strength’ for which the limit does exist even when there are many classical solutions. Within the convergence bands, our quantum theory resolves the classical non-uniqueness. We will examine some numerical evidence to exemplify this behaviour. These bands continue to appear as the coupling strength is increased indefinitely, though they become narrower. In addition to these bands where the classical limit picks out a unique classical solution, there are other values of the coupling strength where the classical limit exists but does not correspond to a solution of the classical equations, rather these solutions correspond to the superposition of solutions that obey the CTC boundary conditions only after a finite number of traversals around the CTC region. We shall call these *winding number N trajectories*, where N is the number of times the quantum particle must traverse the wormhole in order to achieve consistency. In addition to this sort of behaviour there are other values of the coupling strength for which it is arguable that *no* classical solution is physically relevant.

We also consider the effect of altering the operator ordering used and find that the solutions to the QIVP can become non-unique for large quantum numbers and non-normal operator ordering. We study a 1-parameter family of operator orderings for the 3-point model and show that the resulting quantum theories are all nonunitary.

This chapter is structured as follows. We describe first our class of chronology violating models in Sect. 9.2 and then study the classical initial value problem for both linear and nonlinear fields in Sect. 9.3. Next, in Sect. 9.4 we discuss the quantum initial value problem for our models in the absence of CTC's and demonstrate its equivalence with canonical quantization. This serves to fix our notation and definitions for Sect. 9.5 in which we uniquely solve the QIVP with CTC's present, and discuss the 2- and 3-point models, showing that the CCR/CAR's are violated in the 3-point case. This nonunitary evolution is investigated in Sect. 9.6 and is shown not to be described by a superscattering operator. Sect. 9.7 treats the classical limit, whilst Sect. 9.8 contains a brief discussion of the effect of operator ordering on our results. In Sect. 9.9, we review the self-consistent path integral formalism, extending and in one instance correcting the treatment given by Politzer [52]. We use this formalism to compute the general (unitary) evolution for the free models, obtaining agreement with the QIVP. For the 2- and 3-point interacting models we show that the QIVP and path integral differ. We comment on this and other issues in the Conclusion (Sect. 9.10). There are five Appendices.

Appendix 9.A reproduces our treatment of the free classical evolution using the methods of Goldwirth *et al.* [51], whilst Appendix 9.B gives a derivation of the quantum evolution of the free models using the formalism of Politzer [52], rather than the more direct method employed in the text. In Appendix 9.C, we present the details of a calculation which shows that the CCR/CAR's are violated in the 3-point interacting model and in Appendices 9.D and 9.E we prove some rigorous elementary estimates on the Poisson distribution which are technicalities necessary for our discussion of the classical limit together with an analysis of two iterative sequences important in our deliberations.

9.2 A Class of Chronology Violating Models

In this section, we describe a class of nonlinear differential equations on 'spacetimes' in which 'space' consists of finitely many discrete points. By making identifications in these spacetimes, we introduce CTC's and obtain spacetime models generalizing that studied by Politzer [52]. These identifications are implemented in the field theory by imposing certain boundary conditions which place constraints on the theory.

Let $\mathcal{S}$ be a finite collection of points $\mathcal{S} = \{z_\alpha \mid \alpha = 1, \ldots, s\}$ for some $s \geq 2$, and define spacetime to be the Cartesian product $\mathcal{S} \times \mathbb{R}$. We define $\mathfrak{H}$ to be the Hilbert space of complex-valued functions on $\mathcal{S}$ with inner product $\langle f \mid g \rangle = \sum_{z \in \mathcal{S}} \overline{f(z)} g(z)$. This space has vectors v_α as an orthonormal basis, where we define $v_\alpha(z_\beta) = \delta_{\alpha\beta}$. With respect to this basis, we may write functions in $\mathfrak{H}$ as s-dimensional complex vectors, so that $\langle f \,|\, g \rangle = f^\dagger g = \overline{f_\alpha} g_\alpha$, where we sum over the repeated index.

We will study model field theories derived from Lagrangians of form

$$\mathcal{L} = \frac{i}{2}(\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) - \psi^\dagger W \psi - \frac{\lambda}{2}(\psi^\dagger \psi)^2, \tag{9.2.1}$$

where $\psi(t) \in \mathfrak{H}$, W is a self-adjoint positive operator on $\mathfrak{H}$ and $\lambda \in \mathbb{R}^+$. The corresponding field equation is

$$\dot{\psi} = -iW\psi - i\lambda(\psi^\dagger \psi)\psi, \tag{9.2.2}$$

which conserves the quantity $\psi^\dagger\psi$, and therefore reduces to the *linear* equation

$$\dot{\psi} = -iW\psi - i\lambda\psi^\dagger(0)\psi(0)\psi, \tag{9.2.3}$$

once the initial data $\psi(0)$ is specified. Thus the unique solution to Eq. (9.2.2) with this data is

$$\psi(t) = e^{-i\lambda t\psi^\dagger(0)\psi(0)}e^{-iWt}\psi(0). \tag{9.2.4}$$

The configuration space variables ψ_α have conjugate momenta $i\psi_\alpha^\dagger$ (naïvely, one might expect the momenta to be $\frac{1}{2}i\psi_\alpha^\dagger$. However, the Lagrangian (9.2.1) is a second class constrained system and the correct momenta may be obtained using Dirac brackets [80].) and Eq. (9.2.2) may be written in the Hamiltonian form

$$\dot{\psi}_\alpha = \frac{\partial h}{\partial(i\psi_\alpha^\dagger)}, \tag{9.2.5}$$

with Hamiltonian

$$h(\psi, i\psi^\dagger) = \psi_\alpha^\dagger W_{\alpha\beta}\psi_\beta + \frac{\lambda}{2}\psi_\alpha^\dagger\psi_\beta^\dagger\psi_\beta\psi_\alpha. \tag{9.2.6}$$

To introduce CTC's we partition $\mathcal{S}$ into two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ containing s_1 and s_2 elements respectively, with $s_1+s_2 = s$ and $s_2 \leq s_1$, and make pointwise identifications of $\mathcal{S}_2 \times \{T^+\}$ with $\mathcal{S}_2 \times \{0^-\}$ and $\mathcal{S}_2 \times \{T^-\}$ with $\mathcal{S}_2 \times \{0^+\}$ for some $T > 0$. This idealises wormholes linking the lower surface of $\mathcal{S}_2$ at $t = 0$ with the upper surface of $\mathcal{S}_2$ at $t = T$, and the upper surface of $\mathcal{S}_2$ at $t = 0$ with the lower surface of $\mathcal{S}_2$ at $t = T$. Note that 0^- and 0^+ (and correspondingly T^- and T^+) are regarded as distinct topological points for this purpose.

The partition of $\mathcal{S}$ induces a partition of the basis vectors v_α into the sets $e_1, \ldots, e_{s_1}$ and $f_1, \ldots, f_{s_2}$ whose respective spans are denoted $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Clearly, we have that $\dim\mathfrak{H}_2 \leq \dim\mathfrak{H}_1$. We will also write the projection of $\psi \in \mathfrak{H}$ onto $\mathfrak{H}_i$ $(i = 1, 2)$ as ψ_i. In the classical field theory, the identifications are implemented by the imposition of the boundary conditions

$$\psi_2(T^-) = A\psi_2(0^+) \quad \text{and} \quad \psi_2(T^+) = B\psi_2(0^-), \tag{9.2.7}$$

where A and B are unitary maps of $\mathfrak{H}_2$ to itself, corresponding to the evolution through the wormholes (cf. Goldwirth *et al.* [51]). Politzer [52] takes $A = B = \mathbb{1}$. The rôle of these boundary conditions is simply to ensure that the evolution around the wormholes is consistent. We require $\psi_1(t)$ to be everywhere continuous, although $\dot{\psi}_1(t)$ may be discontinuous at $t = 0, T$. Thus (9.2.2) is suspended at these points.

Except in the special cases in which $\mathcal{S}$ consists either of 2 points or 3 points arranged in a ring, the interaction term in (9.2.1) is not a nearest neighbour interaction and is therefore rather unsatisfactory as a model field theory. We will therefore restrict our discussion of specific interacting models to these cases. In Chap. 10 we will discuss a lattice Thirring model and its continuum limit, this is a model of a local field theory. The results presented in that chapter lend support to the idea that knowledge about such point spacetime models carries over (at least for the simple interacting models we have been discussing) to the continuum limit.

9.3 The Classical Initial Value Problem

In this section, we examine the behaviour of the classical field equation (9.2.2) subject to the CTC boundary conditions (9.2.7). For a generic class of W and T, we show that the free field initial value problem is well posed for data specified before the nonchronal region. We then examine the nonlinear theory and show that (generically) solutions exist for all initial data specified before the nonchronal region; moreover, this solution is unique in the case of 'weak coupling', but fails to be unique for 'strong coupling'.

To define the class of generic W and T, we decompose the operator e^{-iWT} (which implements the free evolution between $t = 0^+$ and $t = T^-$) in the block form

$$e^{-iWT} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{9.3.1}$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. The *generic case* is defined to be the case in which the norm $\|S\|$ of S is *strictly* less than unity. (Note that $\|S\| \leq 1$ because e^{-iWT} is unitary.) Equivalently, we require that Q should be an injection from $\mathfrak{H}_2$ into

$\mathfrak{H}_1$ so that Q has a left inverse K, (i.e., such that $KQ = \mathbb{1}|_{\mathfrak{H}_2}$) which is uniquely specified if we require it to annihilate the orthogonal complement of $\mathrm{Im}\, Q$. This requirement is the reason for our restriction that $\dim \mathfrak{H}_2 \leq \dim \mathfrak{H}_1$: otherwise, Q would necessarily have nontrivial kernel. The generic case corresponds to the situation expected in physically realistic field theories in which wavepackets spread out so that some proportion of any wave emerging from the wormhole at $t = 0^+$ manages to avoid reentering it at $t = T^-$.

9.3.1 Free Case

We show now that, in the generic case with $\lambda = 0$, the equation of motion (9.2.2) with boundary conditions (9.2.7) constitutes a well posed initial value problem for arbitrary data $\psi \in \mathfrak{H}$ specified at $t = 0^-$ (and therefore for any $t < 0$). In fact, we will only need the weaker condition that $A - S$ be invertible on $\mathfrak{H}_2$.

The evolution between $t = 0^+$ and $t = T^-$ is given simply by the operator e^{-iWT}; accordingly, given data at $t = 0^-$, the problem reduces to the study of the evolution between $t = 0^-$ and $t = 0^+$. Because $\psi_1(t)$ is required to be continuous at $t = 0$, it remains to determine $\psi_2(0^+)$ in terms of $\psi(0^-)$. The only constraint on $\psi_2(0^+)$ is that the CTC boundary conditions be satisfied, i.e., that $\psi_2(T^-) = A\psi_2(0^+)$. From Eq. (9.3.1) we have $\psi_2(T^-) = R\psi_1(0) + S\psi_2(0^+)$ so, provided $A - S$ is invertible, $\psi_2(0^+)$ is uniquely specified as

$$\psi_2(0^+) = (A - S)^{-1} R\psi_1(0). \tag{9.3.2}$$

For $0 < t < T$, the solution is thus

$$\psi(t) = e^{-iWt} \begin{pmatrix} \psi_1(0) \\ (A - S)^{-1} R\psi_1(0) \end{pmatrix}, \tag{9.3.3}$$

and in particular, we obtain

$$\psi_1(T) = M\psi_1(0), \tag{9.3.4}$$

where the matrix M is

$$M = P + Q(A - S)^{-1} R. \tag{9.3.5}$$

The matrix M is unitary, as we now show. Let $\psi \in \mathfrak{H}_1$ and use the unitarity of e^{-iWT} to compute

$$\begin{aligned} \|\psi\|^2 + \|(A-S)^{-1}R\psi\|^2 &= \left\| e^{-iWT} \begin{pmatrix} \psi \\ (A-S)^{-1}R\psi \end{pmatrix} \right\|^2 \\ &= \|(P + Q(A-S)^{-1}R)\psi\|^2 + \|(\mathbb{1} + S(A-S)^{-1})R\psi\|^2 \\ &= \|M\psi\|^2 + \|A(A-S)^{-1}R\psi\|^2. \end{aligned} \tag{9.3.6}$$

By unitarity of A, we now have $\|M\psi\|^2 = \|\psi\|^2$ and conclude that M is unitary.

Thus we have shown that there is a unique classical solution for each choice of initial data at $t = 0^-$ and that the full classical evolution from $t = 0^-$ to $t = T^+$ is

$$\psi(T^+) = \begin{pmatrix} M & 0 \\ 0 & B \end{pmatrix} \psi(0^-). \tag{9.3.7}$$

Moreover, the solution is clearly continuous in the initial data, so we conclude that this initial value problem is well posed for data specified in the past of the CTC region on surfaces of constant t.

The situation is different for data specified between $t = 0^+$ and $t = T^-$. Here, the initial value problem is well posed only for a subclass of data satisfying certain consistency requirements. For example, data specified at $t = 0^+$ must obey Eq. (9.3.2). This phenomenon has been noted before in various situations [73, 51, 81]; it arises because the CTC's introduce constraints on the dynamics and has important implications for the quantum theory. Note that one may nonetheless specify the data at any given point freely: it is always possible to choose the remaining initial data so as to satisfy the consistency requirements. Thus our system has a 'benignity' property analogous to those discussed in [73, 81]. Related to this phenomenon is the fact that the classical evolution is *nonunitary* between $t = 0^-$ and 0^+ and between $t = T^-$ and T^+. To see this, take any initial data with $\psi_1(0) = 0$: at $t = 0^-$, the initial data has norm $\|\psi_2(0^-)\|$; for $0 < t < T$ the solution vanishes identically; and finally, at $t = T^+$, the solution again has norm

$\|\psi_2(0^-)\|$.

Finally, it is instructive to see how this classical evolution may be derived using the path integral methods of Goldwirth *et al.* [51]. This is described in Appendix 9.A.

9.3.2 Interacting Case

We now consider the full interacting classical field theory in the generic case. We will show: (i) there exists at least one solution for arbitrary initial data; (ii) there is a weak coupling regime in which there is a *unique* solution; and (iii) there is a strong coupling regime in which there exist many distinct solutions for each choice of initial data.

In the absence of CTC boundary conditions, the solution is given by (9.2.4). We write $a = \psi_1(0)$ and $b = \psi_2(0^+)$ and implement the CTC boundary conditions by requiring b to satisfy

$$Ab = e^{-i\lambda T(a^\dagger a + b^\dagger b)}(Ra + Sb), \tag{9.3.8}$$

for given a.

To study the solutions to this equation, we first note that it implies $\|b\| = \|Ra + Sb\|$ and hence, by the unitarity of e^{-iWT}, that $\|a\| = \|Pa + Qb\|$. In the generic case (in which Q has left inverse K, which will be uniquely determined if we require K to annihilate the subspace orthogonal to $\operatorname{Im} Q$) any solution b must therefore take the form

$$b = K(U - P)a, \tag{9.3.9}$$

for some unitary U on $\mathfrak{H}_1$. Substituting back into Eq. (9.3.8), and rearranging, we find that b solves (9.3.8) if and only if

$$KUa = Kf(U)a, \tag{9.3.10}$$

where $f(U) = P + Q(Ae^{i\eta(U)} - S)^{-1}R$ and

$$\eta(U) = \lambda T a^\dagger \left\{ \mathbb{1} + (U - P)^\dagger K^\dagger K(U - P) \right\} a. \tag{9.3.11}$$

Because $\eta(U)$ is real-valued, $f(U)$ is a unitary operator on $\mathfrak{H}_2$.

Clearly, any solution to the fixed point equation $U = f(U)$ necessarily yields a solution to Eq. (9.3.8); moreover, any such U must take the form $U(z) = P + Q(zA - S)^{-1}R$ for some z on the unit circle. (Note that this expression is always well-defined because $\|S\| < 1$.) Thus the problem of existence reduces to finding fixed points of the equation $z = e^{i\eta(U(z))}$ on the unit circle. Now

$$\eta(U(z)) = \lambda T \left(\|a\|^2 + \|(zA - S)^{-1}Ra\|^2\right), \tag{9.3.12}$$

which is a continuous single-valued function from the unit circle to the real line; thus $e^{i\eta(U(z))}$ is a mapping of the unit circle to itself with vanishing Brouwer degree. Accordingly, for each choice of initial data $a \in \mathfrak{H}_1$ there exists at least one fixed point of f and thus at least one solution to Eq. (9.3.8), so we have proved the claim (i) above.

To establish claim (ii), we write the RHS of Eq. (9.3.8) as $Ag(b)$ where $g : \mathfrak{H}_2 \to \mathfrak{H}_2$ and consider the fixed point problem $b = g(b)$ on the ball $\mathcal{B} = \{b \in \mathfrak{H}_2 \mid \|b\| \leq r_0\|a\|\}$, where $r_0 = \|K\|(1+\|P\|)$. This ball contains all solutions to Eq. (9.3.8) as a consequence of Eq. (9.3.9). For any b_1, b_2 in $\mathcal{B}$, we have

$$\begin{aligned} \|g(b_1) - g(b_2)\| &= \left\| \left(1 - e^{i\lambda T(\|b_1\|^2 - \|b_2\|^2)}\right)(Ra + Sb_1) \right. \\ &\qquad \left. + e^{i\lambda T(\|b_1\|^2 - \|b_2\|^2)} S(b_1 - b_2) \right\| \\ &\leq \left(c_0 \lambda T \|a\|^2 + \|S\|\right) \|b_1 - b_2\|, \end{aligned} \tag{9.3.13}$$

in which we have used the elementary estimates $|1 - e^{i\alpha}| = 2|\sin\alpha/2| \leq |\alpha|$ and $|\|b_1\| - \|b_2\|| \leq \|b_1 - b_2\|$, and $c_0 = 2r_0(\|R\| + \|S\|r_0)$ is a positive real constant depending only on P, Q, R and S. In the generic case, for $\lambda T\|a\|^2 < c_0^{-1}(1 - \|S\|)$ (i.e., weak coupling), $g|_{\mathcal{B}}$ is a strict contraction (which need not map $\mathcal{B}$ to itself) and standard contraction mapping arguments now imply that there can be at most one fixed point in $\mathcal{B}$. Putting this together with (i) and using the fact that all fixed points of g must lie in

$\mathcal{B}$, we have proved (ii).

Finally, to prove (iii) we note that if $\lambda T\|Ra\|^2 \gg 1$ (i.e., strong coupling) then the fixed point problem $z = e^{i\eta(U(z))}$ described above has many solutions on the unit circle; moreover, because $Ra \neq 0$, these solutions must correspond to distinct values of b and hence of $\psi_1(T)$.

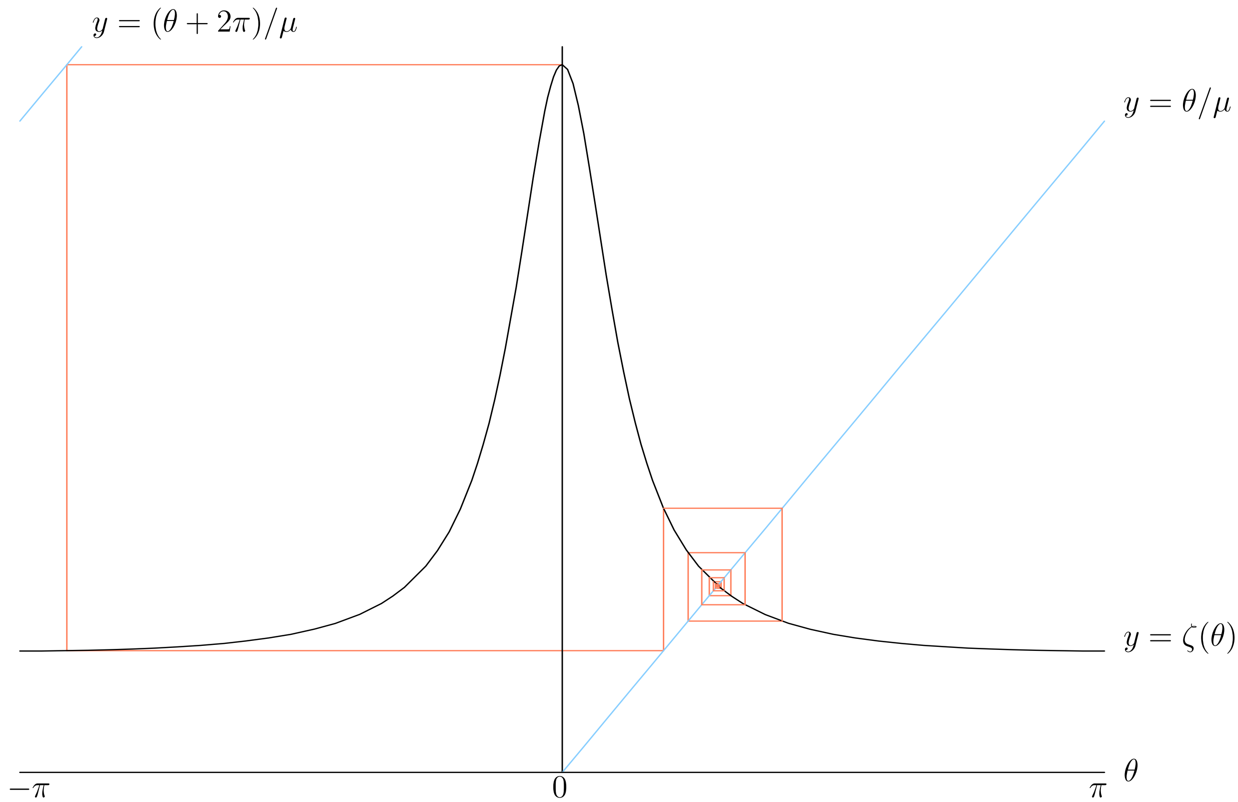

Fig. 9.1: *Graphical solution of Eq. (9.3.14) for* $\mu = 0.5$ *showing that there is a unique solution of the classical solution for this coupling strength.*

In Figs. 9.1 and 9.2, we explicitly show how non-uniqueness arises in a model with two spatial points and $P = -Q = R = S = 1/\sqrt{2}$ with $A = \mathbb{1}$. For this model the classical solutions are in one-to-one correspondence with the solutions of the fixed point equation $z = e^{i\eta(U(z))}$ on the unit circle, because K and $U(z)$ are scalars. Writing $z = e^{i\theta}$, this becomes

$$\zeta(\theta) = (\theta + 2k\pi)/\mu, \tag{9.3.14}$$

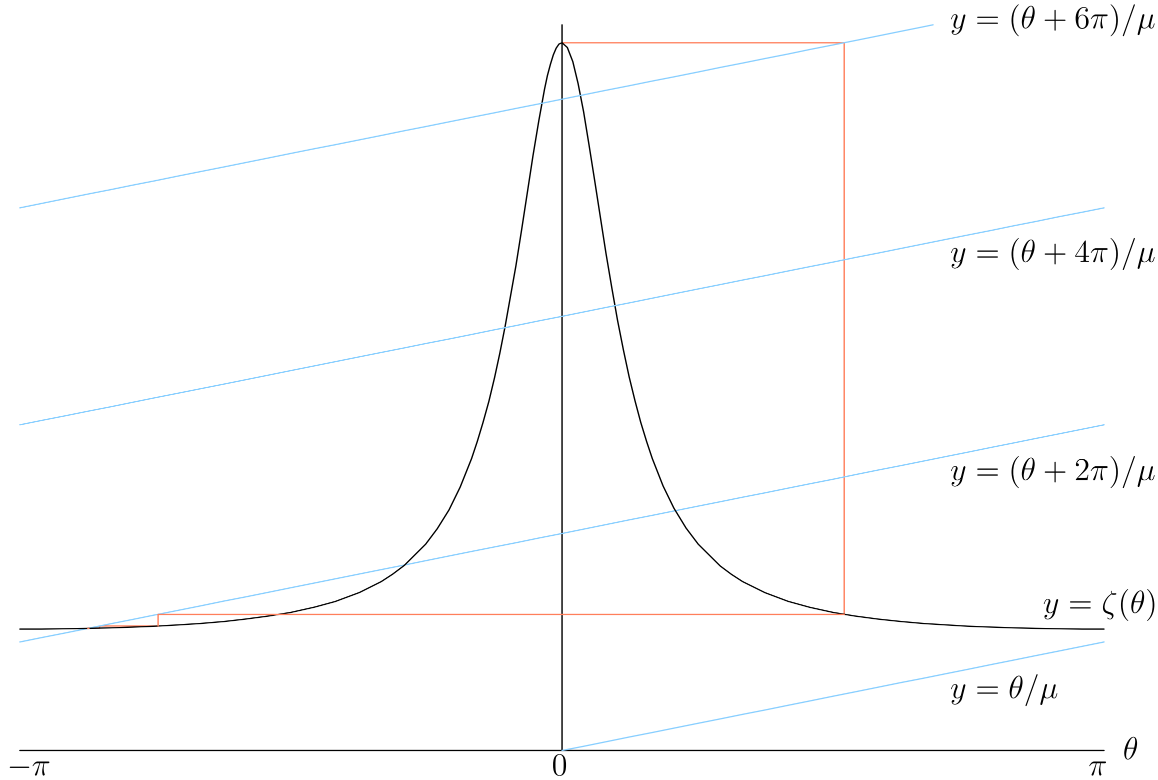

Fig. 9.2: *Graphical solution of Eq. (9.3.14) for $\mu = 3.0$ showing that there are 7 solutions for this coupling strength.*

for $k \in \mathbb{Z}$ where $\zeta(\theta)$ is defined by the expression

$$\zeta(\theta) = 1 + \left|\sqrt{2}e^{i\theta} - 1\right|^{-2} \tag{9.3.15}$$

and we have written $\mu = \lambda T|a|^2$ for the 'coupling strength'. The fixed point equation may be solved graphically by plotting (9.3.14) and (9.3.15) on the same diagram, for $-\pi < \theta \leq \pi$ and looking for intersections. Fig. 9.1 shows the appropriate plots for $\mu = 0.5$, from which it is clear that there is a unique solution, whilst Fig. 9.2 corresponds to the case $\mu = 3.0$ where there are seven solutions. The iteration of the map illustrated by the red lines turns out to be important for our discussion of the classical limit to the model. We will return to these diagrams when we discuss this point in detail in Sect. 9.7.

In order to get an idea of how the number of classical solutions varies with the

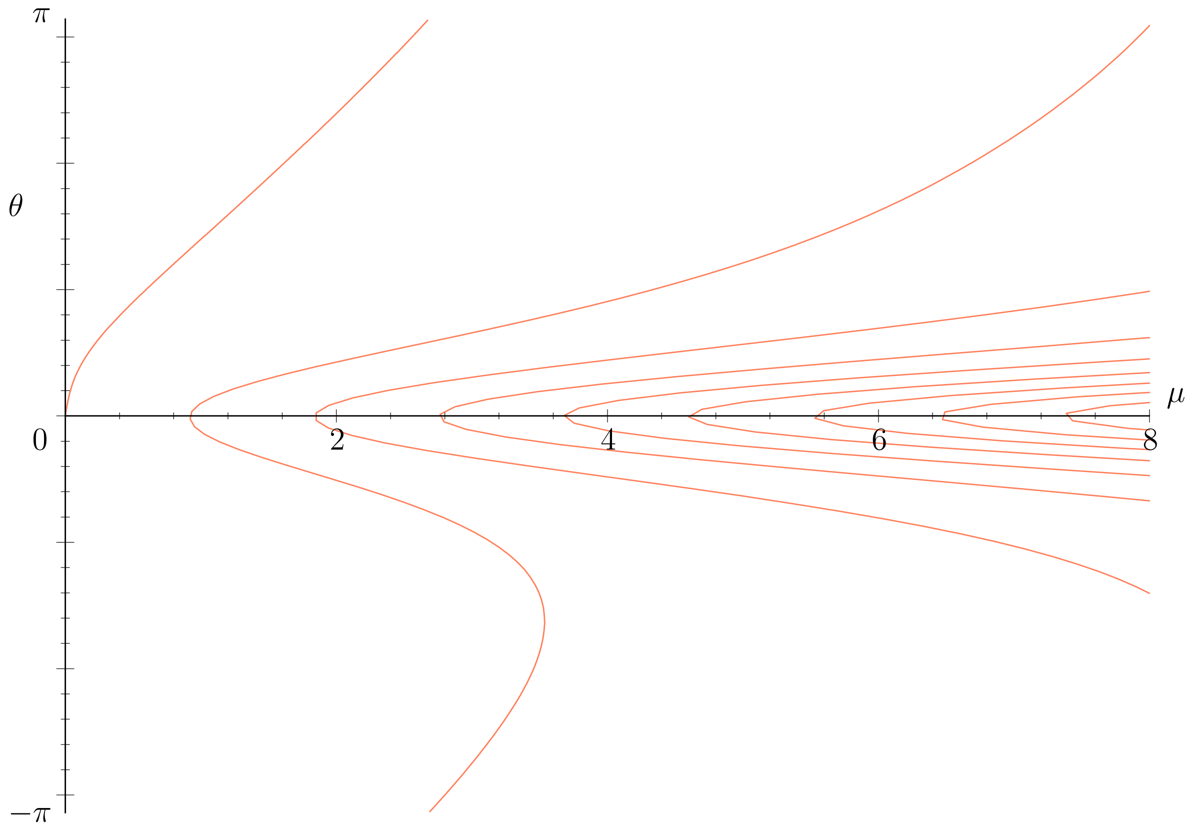

Fig. 9.3: *Diagram showing the fixed points of the classical equations plotted against the coupling strength.*

coupling strength μ, we may plot a diagram of μ against θ, where θ solves the fixed point equation, Eq. (9.3.14). It can be seen from Fig. 9.3 that for small values, i.e., less than approximately $\pi(2-\sqrt{2})/2 \approx 0.9202$ there is a unique classical solution. As μ increases there are an increasing number of fixed point solutions.

9.3.3 Preservation of the Symplectic Structure

Except at $t = 0$ and T, the classical dynamics is generated by the Hamiltonian h, and therefore preserves the symplectic structure on phase space in the initial and final chronal regions. Owing to the CTC boundary conditions, it is not clear that the evolution from initial to final chronal regions also preserves the symplectic structure. Here, we express the classical evolution in phase space language and prove that the evolution from initial to final chronal regions is implemented by a symplectic transformation.

The classical phase space is $\Gamma = \mathbb{C}^s$ with symplectic structure given by the 2-form $\mathbf{\Omega} = -i\mathbf{d}\psi^\dagger \wedge \mathbf{d}\psi = -i\mathbf{d}\overline{\psi}_\alpha \wedge \mathbf{d}\psi_\alpha$. In the usual way, functions on Γ are regarded as functions of independent variables ψ and $\psi^\dagger$. A symplectic transformation ξ is a diffeomorphism of Γ which preserves $\mathbf{\Omega}$, i.e., $\xi^*\mathbf{\Omega} = \mathbf{\Omega}$, where $\xi^*\mathbf{\Omega}$ is the pull-back of $\mathbf{\Omega}$ by ξ. This is equivalent (see e.g., §40 in [82]) to the Dirac bracket relation

$$\{f \circ \xi, g \circ \xi\}_{D,x} = \{f, g\}_{D,\xi(x)}\,, \tag{9.3.16}$$

where the Dirac bracket of two functions on Γ is defined by

$$\{f, g\}_{D,x} = \left.\left(\frac{\partial f}{\partial \psi_\gamma}\frac{\partial g}{\partial i\overline{\psi}_\gamma} - \frac{\partial g}{\partial \psi_\gamma}\frac{\partial f}{\partial i\overline{\psi}_\gamma}\right)\right|_x . \tag{9.3.17}$$

Under a Hamiltonian evolution the symplectic structure is preserved by virtue of Hamilton's equations: $d\mathbf{\Omega}/dt = \mathbf{d}^2 h = 0$. Corresponding to the field decomposition $\psi = (\psi_1, \psi_2)$, we have $\Gamma = \Gamma_1 \times \Gamma_2$, and associated natural projections $\pi_k : \Gamma \to \Gamma_k$. Then $\mathbf{\Omega}$ can be expressed as

$$\mathbf{\Omega} = \pi_1^*\mathbf{\Omega}_1 + \pi_2^*\mathbf{\Omega}_2, \tag{9.3.18}$$

where, for $k = 1, 2$, $\mathbf{\Omega}_k = -i\mathbf{d}\psi_k^\dagger \wedge \mathbf{d}\psi_k$ is the symplectic form on Γ_k. Any unitary matrix U on $\mathfrak{H}_2$ defines a corresponding natural symplectic transformation of Γ_2, which we denote χ_U. In addition, for $t \in \mathbb{R}$, $\tau_t = \exp tI\mathbf{d}h$ is the evolution generated by the Hamiltonian h, where I is the canonical isomorphism between 1-forms and vector fields on Γ specified by $\mathbf{\Omega}(I\boldsymbol{\omega}, \cdot) = \boldsymbol{\omega}(\cdot)$. We have $\tau_t^*\mathbf{\Omega} = \mathbf{\Omega}$ for all t.

With these definitions, the diffeomorphism η implementing evolution from $t = 0^-$ to $t = T^+$ is $\eta = (\kappa, \chi_B)$, where B is the unitary matrix appearing in the CTC boundary conditions and κ is a mapping from an open set $U \subset \Gamma_1$ into Γ_1 defined as follows. First, we define a differentiable map $\sigma : U \to \Gamma$ as a solution to the equations

$$\pi_1 \circ \sigma = \mathbb{1}_1, \quad \pi_2 \circ \tau_T \circ \sigma = \chi_A \circ \pi_2 \circ \sigma, \tag{9.3.19}$$

which express the classical consistency requirement Eq. (9.3.8). Then κ is defined by

$$\kappa = \pi_1 \circ \tau_T \circ \sigma. \tag{9.3.20}$$

In general, there will be many possible choices for κ reflecting the non-uniqueness of the classical evolution. For any such choice, the relation $\kappa^* \mathbf{\Omega}_1 = \mathbf{\Omega}_1$ can be proved using the composition rule $(f \circ g)^* = g^* f^*$ as

$$\begin{aligned} \kappa^* \mathbf{\Omega}_1 &= \sigma^* \tau_T^* \pi_1^* \mathbf{\Omega}_1 \\ &= \sigma^* \tau_T^* (\mathbf{\Omega} - \pi_2^* \mathbf{\Omega}_2) \\ &= \sigma^* \mathbf{\Omega} - \sigma^* \tau_T^* \pi_2^* \mathbf{\Omega}_2 \\ &= \sigma^* (\pi_1^* \mathbf{\Omega}_1 + \pi_2^* \mathbf{\Omega}_2) - \sigma^* \pi_2^* \chi_A^* \mathbf{\Omega}_2 \\ &= \mathbf{\Omega}_1. \end{aligned} \tag{9.3.21}$$

Thus, because $\chi_B^* \mathbf{\Omega}_2 = \mathbf{\Omega}_2$, we conclude that $\eta = (\kappa, \chi_B)$ preserves $\mathbf{\Omega}$.

In terms of Dirac brackets, writing $\psi(T^+) = \eta(\psi, i\psi^\dagger)$, we have proved

$$\left\{ \psi_\alpha(T^+), \psi_\beta(T^+) \right\}_D = 0, \tag{9.3.22}$$

and

$$\left\{ \psi_\alpha(T^+), \overline{\psi}_\beta(T^+) \right\}_D = -i\delta_{\alpha\beta}. \tag{9.3.23}$$

Note that the evolution between $t = 0^-$ and $t = 0^+$ (and similarly between $t = T^-$ and $t = T^+$) is *not* symplectic in general.

9.4 The Quantum Initial Value Problem In the Absence of CTC's

In order to prepare for our discussion of chronology violating models, it is useful to show how a study of the QIVP reproduces the results of canonical quantization for Eq. (9.2.2) in the absence of CTC's. We first discuss the case of Fermi statistics to avoid the operator domain technicalities of the bosonic case.

The canonical approach starts by identifying the classical canonical coordinates ψ_α and $i\psi_\alpha^\dagger$ and the classical Hamiltonian $h(\psi_\alpha, i\psi_\alpha^\dagger)$ defined in Eq. (9.2.6). A Hilbert space $\mathfrak{F}$ is then constructed on which bounded operators $\Psi_1, \ldots, \Psi_s$ represent the CAR's for s degrees of freedom – that is, $\{\Psi_\alpha, \Psi_\beta\} = 0$ and $\{\Psi_\alpha, \Psi_\beta^\dagger\} = \delta_{\alpha\beta}$ for all α, β. The quantized (normal ordered) Hamiltonian H is defined as a (bounded) self-adjoint operator on $\mathfrak{F}$ by substituting Ψ_α for ψ_α in the RHS of Eq. (9.2.6) using its literal ordering. The quantum evolution generated by H evolves a general operator A from time 0 to t by

$$A(t) = e^{iHt} A e^{-iHt}, \tag{9.4.1}$$

and the evolved operator therefore satisfies the Heisenberg equation of motion

$$\dot{A}(t) = i[H, A(t)]. \tag{9.4.2}$$

Thus, by virtue of the CAR's, $\Psi_\alpha(t) = e^{iHt}\Psi_\alpha e^{-iHt}$ solves the original equation of motion (9.2.2) as an operator differential equation with initial data $\Psi_\alpha(0) = \Psi_\alpha$. Moreover, the CAR's are necessarily preserved by this evolution.

It is possible to reproduce these results from a slightly different angle, namely by treating Eq. (9.2.2) as an operator differential equation and considering the *Quantum Initial Value Problem* (QIVP). Given initial data Ψ_α representing the CAR's, we say that the QIVP is well posed if there exists a unique operator-valued solution $\Psi_\alpha(t)$ to Eq. (9.2.2) with $\Psi_\alpha(0) = \Psi_\alpha$ and the evolution preserves the CAR's. To show that this is indeed the case, we note that for *arbitrary* initial data given as bounded operators on $\mathfrak{F}$, Eq. (9.2.2) has the unique solution

$$\Psi_\alpha(t) = e^{-i\lambda t \Psi_\gamma^\dagger \Psi_\gamma} \left(e^{-iWt}\right)_{\alpha\beta} \Psi_\beta. \tag{9.4.3}$$

The proof of uniqueness closely parallels the analogous argument for the classical differential equation. One may check that this evolution preserves the CAR's either by explicit computation or by noting that the above solution must agree (by uniqueness) with that obtained from the canonical approach. Thus the QIVP for Eq. (9.2.2) is well posed in the fermionic case.

Of course, it is not usually advantageous to consider the QIVP directly because it is rare that the equation of motion may be solved in closed form for general operator-valued initial data. However, for the chronology violating models considered here, it will not always be possible to assume that the initial data is a representation of the canonical (anti)commutation relations and therefore the canonical method is no longer guaranteed to yield solutions to the equation of motion Eq. (9.2.2). In these situations, we must therefore employ the more general setting of the QIVP.

In the bosonic case, of necessity we encounter unbounded operators and therefore must proceed more carefully. We now describe the technicalities required in order to generalize the foregoing to this case.[1]

Definition *Let $\mathcal{D}$ be dense in Hilbert space $\mathfrak{F}$, and let $\Psi_1(t), \ldots, \Psi_s(t)$ be closed operator-valued functions on $\mathbb{R}$ such that $\mathcal{D}$ is a core for each $\Psi_\alpha(t)$ and is invariant under the $\Psi_\alpha(t)$ and $\Psi^\dagger_\alpha(t)$. Then the $\Psi_\alpha(t)$ are said to be a solution to Eq. (9.2.2) on $\mathcal{D}$ if each $\Psi_\alpha(t)$ is strongly differentiable with respect to t on $\mathcal{D}$ with derivative*

$$-iW_{\alpha\beta}\Psi_\beta(t) - i\lambda\Psi^\dagger_\gamma(t)\Psi_\gamma(t)\Psi_\alpha(t). \tag{9.4.4}$$

Note that this definition extends that used above for the bounded case.

Definition *The closed operators $\Psi_1, \ldots, \Psi_s$ are said to represent the CCR's on $\mathfrak{F}$ if they*

[1] An algebraic subspace $\mathcal{D}$ of $\mathfrak{F}$ contained in $D(A)$ is a *core* for a closed operator A if A is the closure of its restriction to $\mathcal{D}$. A densely defined operator A is *essentially self-adjoint* if its closure is self-adjoint, and an operator-valued function $A(t)$ is *strongly differentiable* with respect to t on $\mathcal{D}$ with derivative $B(t)$ if $\mathcal{D}$ is contained in $D(B(t))$ and $D(A(\tau))$ for all τ in some neighbourhood of t and $\|(\epsilon^{-1}(A(t+\epsilon) - A(t)) - B(t))f\| \to 0$ as $\epsilon \to 0$ for all $f \in \mathcal{D}$.

have a common dense domain $\mathcal{X}$ invariant under both the Ψ_α and the $\Psi_\alpha^\dagger$ with

$$[\Psi_\alpha, \Psi_\beta]f = [\Psi_\alpha^\dagger, \Psi_\beta^\dagger]f = 0 \tag{9.4.5}$$

and

$$[\Psi_\alpha, \Psi_\beta^\dagger]f = \delta_{\alpha\beta} f \tag{9.4.6}$$

for all $f \in \mathcal{X}$ and such that $\Psi_\alpha^\dagger \Psi_\alpha$ (summing over the repeated index) is decomposable on $\mathcal{X}$. That is, there exists a projection-valued measure[2] P_Ω on $\mathbb{R}$ such that $\mathcal{X}$ contains $\mathcal{D}_0 = \bigcup_{\mu \geq 0} P_{[-\mu,\mu]}\mathfrak{F}$ and $\Psi_\alpha^\dagger \Psi_\alpha f = \int_{\mathbb{R}} \mu \, dP_\mu f$ for all $f \in \mathcal{X}$.

The reason for the technical requirement of decomposability is that it guarantees [83] that all such representations of the CCR's are equivalent up to unitary equivalence and multiplicity (i.e., the conclusion of von Neumann's theorem holds).

The canonical quantization of Eq. (9.2.2) proceeds as follows. Suppose that operators Ψ_α represent the CCR's on Hilbert space $\mathfrak{F}$ with dense invariant domain $\mathcal{X}$, and let $\mathcal{D}_0 \subset \mathcal{X}$ be defined as above. The quantum Hamiltonian may be defined on $\mathcal{D}_0$ by substituting the operators Ψ_α into the RHS of Eq. (9.2.6) to yield an essentially self-adjoint operator whose closure is denoted by H. Moreover, $\mathcal{D}_0$ is easily seen to be invariant under e^{iHt} for $t \in \mathbb{R}$. Thus, the evolved operators $\Psi_\alpha(t)$ defined by

$$\Psi_\alpha(t) = e^{iHt} \Psi_\alpha e^{-iHt} \tag{9.4.7}$$

are strongly differentiable with respect to t on $\mathcal{D}_0$ with derivative $ie^{iHt}[H, \Psi_\alpha]e^{-iHt}$ and the CCR's may then be used (on $\mathcal{D}_0$) to conclude that the $\Psi_\alpha(t)$ solve Eq. (9.2.2) on $\mathcal{D}_0$ in the sense defined above.

As in the fermionic case, we may reproduce these results by studying the QIVP. The situation for general initial data is summarized by the following:

Proposition *Let $\mathfrak{F}$ be a Hilbert space and $\mathcal{D} \subseteq \mathfrak{F}$ be dense. Suppose further that Ψ_α, ($\alpha = 1, \ldots, s$) are closed (possibly unbounded) operators on $\mathfrak{F}$ such that*

[2] For a treatment of projection-valued measures and unbounded operators see [57]

(i) $\mathcal{D}$ is a core for each Ψ_α and a domain of essential self-adjointness for $\Psi_\gamma^\dagger \Psi_\gamma$

(ii) $\mathcal{D}$ is invariant under Ψ_α, $\Psi_\alpha^\dagger$ and $e^{-i\lambda t \Psi_\gamma^\dagger \Psi_\gamma}$ for all $t \in \mathbb{R}$.

Then the operators $\Psi_\alpha(t)$ defined as the closure of $e^{-i\lambda t \Psi_\gamma^\dagger \Psi_\gamma} \left(e^{-iWt}\right)_{\alpha\beta} \Psi_\beta$ on $\mathcal{D}$ constitute the unique solution to Eq. (9.2.2) on $\mathcal{D}$ with initial data Ψ_α.

An immediate corollary of this is that if the Ψ_α represent the CCR's on $\mathfrak{F}$ and the domain $\mathcal{D}_0$ is defined as above, then the QIVP for Eq. (9.2.2) is well posed on $\mathcal{D}_0$.

9.5 The Quantum Initial Value Problem for Chronology Violating Models

9.5.1 General Formalism

We now analyse the quantum initial value problem for Eq. (9.2.2) in the presence of CTC's, beginning with the case of the CAR's. Suppose that the operators Ψ_α ($\alpha = 1, \ldots, s$) provide a representation of the CAR's on Hilbert space $\mathfrak{F}$. We specify these operators as the initial data for the QIVP at time $t = 0^-$. Writing Ψ_1 and Ψ_2 to denote those operators associated with $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, we therefore seek operators $\Psi_2(0^+)$ such that the evolution between $t = 0^+$ and T^- obeys the consistency requirement $\Psi_2(T^-) = A\Psi_2(0^+)$. Denoting $\Psi_1 = a$, $\Psi_2(0^+) = b$ we therefore require b to satisfy

$$Ab = e^{-i\lambda T(a^\dagger a + b^\dagger b)} \left(Ra + Sb\right). \tag{9.5.1}$$

Remarkably, and in contrast to the situation for the classical theory, it turns out that this specifies b uniquely in the generic case as we now show.

We first construct a solution to Eq. (9.5.1) and then prove its uniqueness. For $z \in \mathbb{C}$, let $N(z)$ be the matrix-valued function of z defined by $N(z) = (zA - S)^{-1}R$, which is analytic in an open neighbourhood of the unit circle in the generic case. Then for any unitary operator V on Hilbert space $\mathfrak{K}$, we may use the (Dunford) functional calculus (see e.g., pp. 556-577 of [84]) to define $N(V)$ as a matrix of bounded operators on $\mathfrak{K}$.

Using this notation, Eq. (9.5.1) may be rewritten in the form

$$b = N(e^{i\lambda T(a^\dagger a + b^\dagger b)})a. \tag{9.5.2}$$

Next, let $\mathfrak{F}_r$ be the eigenspace of $a^\dagger a$ with eigenvalue r and decompose $\mathfrak{F} = \bigoplus_r \mathfrak{F}_r$. We emphasize that $a^\dagger a$ is not the total particle number on $\mathcal{S}$ at $t = 0^-$, but rather the particle number on $\mathcal{S}_1$. Thus, for example, $\mathfrak{F}_0$ is not 1-dimensional, but consists of all states at $t = 0^-$ with no $\mathcal{S}_1$-particles. We now define unitary operators U_r on the $\mathfrak{F}_r$ by the recurrence relation

$$U_{r+1} = \exp i\lambda T a^\dagger \left(\mathbb{1} + N(U_r)^\dagger N(U_r)\right) a, \tag{9.5.3}$$

with $U_0 = \mathbb{1}$. Denoting $U = \bigoplus_r U_r$, it is easy to see that Eq. (9.5.1) is solved by

$$b = N(U)a, \tag{9.5.4}$$

by comparing with Eq. (9.5.2) and using the fact that each component of a maps $\mathfrak{F}_{r+1}$ to $\mathfrak{F}_r$ and annihilates $\mathfrak{F}_0$.

We now prove that (9.5.4) is the unique solution to Eq. (9.5.1). Suppose that $b = (b_1, \ldots, b_{s_2})^T$ solve Eq. (9.5.1), and write $U = e^{i\lambda T(a^\dagger a + b^\dagger b)}$. Because $N(U)$ is a matrix of bounded operators, Eq. (9.5.2) implies that b annihilate $\mathfrak{F}_0$. Accordingly, U leaves $\mathfrak{F}_0$ invariant and $U|_{\mathfrak{F}_0} = \mathbb{1}$. Now suppose inductively that U leaves $\mathfrak{F}_r$ invariant for some $r \geq 0$. Provided that r is not the largest eigenvalue of $a^\dagger a$, Eq. (9.5.2) and its adjoint imply that b maps $\mathfrak{F}_{r+1}$ to $\mathfrak{F}_r$ and $b^\dagger$ maps $\mathfrak{F}_r$ to $\mathfrak{F}_{r+1}$. Accordingly $a^\dagger a + b^\dagger b$ and thus U leave $\mathfrak{F}_{r+1}$ invariant. Hence by induction, we find that each $\mathfrak{F}_r$ is an invariant subspace for U, so we may write $U = \bigoplus_r U_r$ with each U_r unitary on $\mathfrak{F}_r$. It is then easy to see that the U_r must satisfy the recurrence relation (9.5.3) with $U_0 = U|_{\mathfrak{F}_0} = \mathbb{1}$. We have therefore completed the proof of uniqueness.

Finally, we note that this solution is representation independent in the following sense. Suppose that Ψ_α form a Fock representation of the CAR's, and let b be the unique solution to Eq. (9.5.1) on $\mathfrak{F}$. By the Jordan–Wigner theorem (see e.g., [85]), an arbitrary representation Ψ'_α on $\mathfrak{F}'$ takes the form $\Psi'_\alpha = U^{-1}(\Psi_\alpha \otimes \mathbb{1})U$, where $U : \mathfrak{F}' \to \mathfrak{F} \otimes \mathfrak{N}$ is unitary and $\mathfrak{N}$ is an auxiliary Hilbert space. Then the unique solution to the analogue of Eq. (9.5.1) on $\mathfrak{F}'$ is $b'_i = U^{-1}(b_i \otimes \mathbb{1})U$.

In the CCR case, certain domain questions must be addressed. We suppose that the Ψ_α are a representation of the CCR's on $\mathfrak{F}$ with common invariant domain $\mathcal{X}$ and define $\mathcal{D}_0 \subset \mathcal{X}$ as in Sect. 9.4. An important property of this domain is that $\mathfrak{F}_r \subset \mathcal{D}_0$ for all r, where $\mathfrak{F}_r$ is again defined as the eigenspace of $a^\dagger a$ with eigenvalue r. Then it is easy to see that the same construction as used in the CAR case yields a solution to Eq. (9.5.1) on $\mathcal{D}_0$; moreover, one may show that it is the unique solution such that $\mathcal{D}_0$ is a core for each b_i, and is independent of representation in the same sense as in the CAR case.

Once the unique solution to Eq. (9.5.1) has been obtained (for either CAR's or CCR's) we may substitute back to find

$$a(T) = e^{-i\lambda T(a^\dagger a + b^\dagger b)} \left(Pa + Qb\right), \tag{9.5.5}$$

and check to see whether or not this evolution preserves the CCR/CAR's and is therefore unitary. We will analyse various cases of this problem in the following subsections.

9.5.2 Free Fields

Here $\lambda = 0$ and Eq. (9.5.2) immediately yields the unique solution

$$b = (A - S)^{-1} Ra. \tag{9.5.6}$$

Substituting, we find that the evolution is given by

$$\begin{aligned} \Psi_1(T) &= M\Psi_1(0) \\ \Psi_2(T^+) &= B\Psi_2(0^-), \end{aligned} \tag{9.5.7}$$

where $M = P + Q(A - S)^{-1}R$ is unitary. Note that one obtains the same result for both Bose and Fermi statistics. This evolution is easily seen to preserve the CCR/CAR's; there is therefore a unitary X on $\mathfrak{F}$ such that

$$\Psi_\alpha(T^+) = X^\dagger \Psi_\alpha(0^-) X. \tag{9.5.8}$$

An interesting feature of the above is that the operators b_i are linearly dependent on the a_i. Thus the components of $\Psi(0^+)$ *do not* form a representation of the CCR/CAR's for $s_1 + s_2$ degrees of freedom. In effect the system is reduced to only s_1 degrees of freedom, reflecting the fact that the CTC's place s_2 constraints on the system. Accordingly, the evolution between $t = 0^-$ and $t = 0^+$ is nonunitary, although unitarity is restored at $t = T^+$. In addition, we see that it is not legitimate to employ canonical methods to evolve the quantum field in the nonchronal region (if one intends to solve the equation of motion Eq. (9.2.2)) because the data at $t = 0^+$ does not obey the CCR/CAR's.

As a final check on our result in this case, and on the loss of degrees of freedom, let us quantize by the familiar method of obtaining classical mode solutions. Let $e_i(t)$ (respectively, $f_j(t)$) be the classical solution to the free equation of motion with initial data $e_i(0^-) = e_i$ ($f_j(0^-) = f_j$) where the basis vectors e_i and f_j were defined in Sect. 9.2. We write the quantum field $\Psi(t)$ as

$$\Psi(t) = a_i e_i(t) + b_j f_j(t), \tag{9.5.9}$$

where the a_i and b_j form a representation of the CCR/CAR's on Hilbert space $\mathfrak{F}$. The components Ψ_α of the field are obtained by taking the inner product with v_α. The time evolution of the a_i and b_i is defined by re-expressing the field as

$$\Psi(t) = a_i(t) e_i + b_j(t) f_j, \tag{9.5.10}$$

which leads quickly to the above unitary evolution from 0^- to T^+ using the results of Sect. 9.3. In the nonchronal region, however, $f_j(t)$ vanishes and so $\Psi(t) = a_i e_i(t)$ and the reduction to s_1 degrees of freedom is explicit.

9.5.3 Interacting Fields

Here, we consider three simple examples. Model 1 is a system with two spatial points and yields a unitary theory for both Fermi and Bose statistics. Model 2 is a system with three spatial points. We study this theory for Fermi statistics and show that the resulting theory is *not unitary*. For simplicity we work in the appropriate Fock representations and take A and B to be the identity.

Model 1 Our set of spatial points is $\mathcal{S} = \{z_1, z_2\}$, and $\mathcal{S}_i = \{z_i\}$ for $i = 1, 2$. Thus W is a 2×2 matrix and P, Q, R, S are scalars.

Fermi statistics The Hilbert space $\mathfrak{F}$ for two fermionic degrees of freedom is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^2$. The unique solution to Eq. (9.5.1) is

$$b = (1 - S)^{-1} R a, \tag{9.5.11}$$

as is easily verified using the fact that $e^{i\kappa a^\dagger a} a = a$. Substituting back, we obtain

$$a(T) = \left(P + Q(1 - S)^{-1} R\right) a, \tag{9.5.12}$$

which is identical to the unitary free evolution obtained in the previous subsection. This contrasts with the generically nonunitary evolution obtained by Politzer [52] for this model using the self-consistent path integral – see Sect. 9.9.

Bose Statistics Here, $\mathfrak{F} = \ell^2 \otimes \ell^2$ (where ℓ^2 is the Hilbert space of square summable sequences) and the unique solution to Eq. (9.5.1) takes the form

$$b = f(a^\dagger a) a, \tag{9.5.13}$$

where $f : \mathbb{N} \to \mathbb{C}$ is defined by $f(n) = \langle n | b | n+1 \rangle$, thus

$$f(n+1) = (e^{i\lambda T(n+1)(1+|f(n)|^2)} - S)^{-1} R, \tag{9.5.14}$$

with $f(0) = (1 - S)^{-1} R$.

Thus the evolution of a is given by

$$\begin{aligned} a(T) &= e^{-i\lambda T a^\dagger(1+|f(a^\dagger a)|^2)a}\left(P+Qf(a^\dagger a)\right)a \\ &= e^{-ig(a^\dagger a)}\left(P+Q(e^{ig(a^\dagger a)}-S)^{-1}R\right)a, \end{aligned} \tag{9.5.15}$$

where g is a real-valued function on $\mathbb{N}$ defined by $g(0)=0$ and $g(n)=\lambda Tn(1+|f(n-1)|^2)$ for $n\geq 1$. This may be rewritten as

$$a(T) = X^\dagger a X, \tag{9.5.16}$$

with $X = e^{-ih(a^\dagger a)}$ and $h(n)$ defined by $h(0)=0$ and

$$e^{-ih(n+1)} = e^{-ih(n)}e^{-ig(n)}\left[P+Q\left(e^{ig(n)}-S\right)^{-1}R\right]. \tag{9.5.17}$$

The left hand side is always of unit modulus, so $h(n)$ is real-valued and the operator X is unitary. Thus the evolution from $t=0^-$ to $t=T^+$ is again unitary. We note that this theory agrees with the corresponding free theory on $\mathfrak{F}_1$ (though the theories differ on $\mathfrak{F}_r$ for $r\geq 2$).

Model 2 In this example, our set of spatial points $\mathcal{S}=\{z_1,z_2,z_3\}$, is partitioned into $\mathcal{S}_1=\{z_1,z_2\}$ and $\mathcal{S}_2=\{z_3\}$. The matrix W is now a 3×3 self-adjoint, positive matrix, and the block decomposition of e^{-iWT} yields a 2×2 matrix P, a 2-dimensional column vector $Q=(Q_1,Q_2)^T$, a 2-dimensional row vector $R=(R_1,R_2)$ and a scalar S.

Fermi statistics The Fock space is $\mathfrak{F}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. Given operators a_1 and a_2 at $t=0$, we seek an operator b such that

$$b = e^{-i\lambda T(a^\dagger a+b^\dagger b)}\left(Ra+Sb\right). \tag{9.5.18}$$

Using the results above, the unique solution to this equation is

$$b = (e^{i\lambda T a^\dagger a}-S)^{-1}Ra, \tag{9.5.19}$$

as may easily be checked by decomposing $\mathfrak{F} = \mathfrak{F}_0 \oplus \mathfrak{F}_1 \oplus \mathfrak{F}_2$ with $\mathfrak{F}_r$ the eigenspace of $a^\dagger a$ with eigenvalue r.

Substituting, we find that

$$\begin{pmatrix} a_1(T) \\ a_2(T) \end{pmatrix} = e^{-i\lambda T(a^\dagger a + b^\dagger b)} \left[P + Q(e^{i\lambda T a^\dagger a} - S)^{-1} R \right] \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{9.5.20}$$

In Appendix 9.C, we show that

$$\begin{aligned} \langle 0 | (a_i(T) a_j(T) + a_i(T) a_j(T)) a_2^\dagger a_1^\dagger | 0 \rangle \;&=\\ -e^{-i\lambda T} F \det P &\begin{pmatrix} 2Q_1 \overline{Q}_2 & |Q_2|^2 - |Q_1|^2 \\ |Q_2|^2 - |Q_1|^2 & -2Q_2 \overline{Q}_1 \end{pmatrix}_{ij}, \end{aligned} \tag{9.5.21}$$

where

$$F = \frac{1}{\overline{S}} \left[\frac{1}{e^{i\lambda T} - S} - \frac{e^{-i\lambda \alpha(S) T}}{1 - S} \right] + \frac{e^{-i\lambda \alpha(S) T} - 1}{1 - |S|^2}, \tag{9.5.22}$$

and $\alpha(S)$ is defined by

$$\alpha(S) = \frac{\|R\|^2}{|1 - S|^2} = \frac{1 - |S|^2}{|1 - S|^2}. \tag{9.5.23}$$

Thus, except in the free case or for very carefully tuned parameters the CAR's are necessarily violated and the evolution is therefore nonunitary. Note that the coefficient of F in Eq. (9.5.21) vanishes for all T if and only if W is block diagonal with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ (in which case $|S| = 1$ and we are no longer in the generic case).

We point out that $a(T) = Ma$ on the space $\mathfrak{F}_1$ and $a(T) = Na$ on the space $\mathfrak{F}_2$ where the unitary matrix $M = P + Q(1 - S)^{-1} R$ and N is also a 2×2 unitary matrix given by $N = (P + Q(e^{i\lambda T} - S)^{-1} R) U$, where U is another 2×2 unitary defined by

$$U a|_{\mathfrak{F}_2} = e^{-i\lambda T(a^\dagger a + b^\dagger b)} a|_{\mathfrak{F}_2}, \tag{9.5.24}$$

(which makes sense because the exponential preserves $\mathfrak{F}_1$). The precise form of N will not concern us; however, we note that $M \neq e^{i\theta} N$ for any θ, because $a_1(T)$ and $a_2(T)$

fail to anticommute.

Bose statistics The Fock space is $\ell^2 \otimes \ell^2 \otimes \ell^2$ and the unique solution to Eq. (9.5.1) is

$$b = f(d^\dagger d, c^\dagger c)c, \tag{9.5.25}$$

where we make the (special) unitary transformation:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\|R\|} \begin{pmatrix} R_1 & R_2 \\ -\overline{R}_2 & \overline{R}_1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{9.5.26}$$

and $f(m, n)$ satisfies

$$f(m, n+1) = \left(e^{i\lambda T[m+(n+1)(1+|f(m,n)|^2)]} - S\right)^{-1} \|R\|, \tag{9.5.27}$$

with $f(m, 0) = (e^{i\lambda Tm} - S)^{-1}\|R\|$. Substituting back to determine $a_i(T)$, we show in Appendix 9.C that

$$\langle 0 \,|\, (a_i(T)a_j(T) - a_j(T)a_i(T))d^\dagger c^\dagger \,|\, 0\rangle = -Fe^{-i\lambda T}\|R\|^2 \det P \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ij}, \tag{9.5.28}$$

with F given by Eq. (9.5.22). This should be compared with Eq. (9.5.21). Thus the evolution fails to be unitary on $\mathfrak{F}_2$.

9.6 Discussion of the Nonunitary Evolution

In the previous section, we showed that Model 2 was subject to a nonunitary evolution for both Bose and Fermi statistics. In this section, we discuss this evolution in more depth in the fermionic case. Recall that the Fock space $\mathfrak{F}$ is 8-dimensional, and that the operators $\Psi_\alpha(0^-)$ ($\alpha = 1, 2, 3$) represent the CAR's for three degrees of freedom on $\mathfrak{F}$. Writing $\Psi_{1,i}$ for the operators associated with points in $\mathcal{S}_1$, and Ψ_2 for the operator associated with the single element of $\mathcal{S}_2$, we write $\Psi_{1,i}(0^-) = a_i$ for $i = 1, 2$. The Heisenberg evolution $\Psi_\alpha(0^-) \mapsto \Psi_\alpha(T^+)$ is such that $\Psi_{1,i}(T^+) = a_i(T)$ and $\Psi_2(T^+) = \Psi_2(T^-)$. Our principal results in this section are, firstly, that the Heisenberg picture evolution cannot be expressed in either of the forms

$$\Psi_\alpha(T^+) = X^{-1}\Psi_\alpha(0^-)X, \tag{9.6.1}$$

or

$$\Psi_\alpha(T^+) = X^\dagger\Psi_\alpha(0^-)X, \tag{9.6.2}$$

for some operator X on $\mathfrak{F}$; secondly, that the Heisenberg picture evolution does not admit an equivalent Schrödinger picture description in terms of a superscattering operator. In addition, we will discuss the problem of extending the evolution from that of the $\Psi_\alpha(0^-)$ to arbitrary operators on $\mathfrak{F}$.

Firstly, then, we show that the Heisenberg picture evolution cannot be expressed in either of the forms Eq. (9.6.1) or (9.6.2). The form Eq. (9.6.1) is clearly impossible because it would entail $\{a_1(T), a_2(T)\} = 0$, and we may dispose of Eq. (9.6.2) as follows. The explicit form of the $a_i(T)$ given above shows that any such operator X would necessarily preserve the subspaces $\mathfrak{F}_0, \mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\mathfrak{F}$; moreover, because

$$a(T)|_{\mathfrak{F}_1} = Ma|_{\mathfrak{F}_1}, \tag{9.6.3}$$

where M is unitary, we conclude that $X|_{\mathfrak{F}_1}$ is unitary up to scale. Then it suffices to note that

$$\begin{aligned}\{a_1(T), a_2(T)\}\,|11\rangle &= X^\dagger(a_1XX^\dagger a_2 \\ &\quad + a_2XX^\dagger a_1)X\,|11\rangle\end{aligned} \tag{9.6.4}$$

which vanishes because X preserves $\mathfrak{F}_2$ and $X|_{\mathfrak{F}_1}$ is unitary up to scale. Accordingly, we cannot cast the evolution into either of the special forms Eq. (9.6.1) or (9.6.2).

Secondly, we show that the Heisenberg picture evolution cannot be described by a superscattering operator. Recall that a superscattering operator on the state space of a (separable) Hilbert space $\mathfrak{F}$ is a linear mapping \$ of the trace class operators $\mathcal{T}(\mathfrak{F})$ on $\mathfrak{F}$ such that if $\rho \in \mathcal{T}(\mathfrak{F})$ is a positive operator (For our purposes, a 'positive operator' means one which is non-negative definite) of unit trace, then $\$\rho$ is also a positive element of $\mathcal{T}(\mathfrak{F})$ with unit trace. Thus, \$ is a linear mapping of density matrices to density matrices, which need not preserve purity. If a superscattering operator \$ describes the Schrödinger picture evolution of a system, then the Heisenberg picture evolution is given by the linear mapping $\$'$ of the bounded operators $\mathfrak{L}(\mathfrak{F})$ on $\mathfrak{F}$, defined by

$$\mathrm{Tr}\,\rho(\$'Z) = \mathrm{Tr}\,(\$\rho)Z, \tag{9.6.5}$$

for all $\rho \in \mathcal{T}(\mathfrak{F})$ and $Z \in \mathfrak{L}(\mathfrak{F})$. In fact, $\$'$ is the dual mapping to \$ under the natural identification of $\mathfrak{L}(\mathfrak{F})$ with the dual space of $\mathcal{T}(\mathfrak{F})$.

The dual mapping $\$'$ possesses three easily established properties: (i) $\$'\mathbb{1} = \mathbb{1}$; (ii) $(\$'Z)^\dagger = \$'(Z^\dagger)$ for all Z; and (iii) $\$'$ is positive in the sense that $\$'Z$ is a positive operator whenever Z is. If one writes the superscattering operator using the index notation (e.g., [78])

$$(\$\rho)^A{}_B = \$^A{}_{BC}{}^D \rho^C{}_D, \tag{9.6.6}$$

then $\$'$ may be written as

$$(\$'Z)^D{}_C = \$^A{}_{BC}{}^D Z^B{}_A. \tag{9.6.7}$$

Returning to our case of interest, we now show that there is no superscattering operator \$ for which the Heisenberg evolution can be written as

$$\Psi_\alpha(T^+) = \$'\Psi_\alpha(0^-). \tag{9.6.8}$$

Define $Z = \alpha\mathbb{1} + a^\dagger w + wa^\dagger$ for some $w \in \mathbb{C}^2$. We will write the reduced (four dimensional) Fock space $\mathfrak{F}' = \mathfrak{F}'_0 \otimes \mathfrak{F}'_1 \otimes \mathfrak{F}'_2$ for the space built up using the $a^\dagger$ operators (i.e., $\mathfrak{F}_r \cong \mathfrak{F}'_r \otimes \mathbb{C}^2$). We remark that b evolves trivially on account of the boundary condition $B = \mathbb{1}$. In what follows it is useful to define the two component row vector

$|\mathbf{1}\rangle = a^\dagger |0\rangle$ and the anti-hermitian matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9.6.9}$$

Notice that we have that

$$a \mapsto a(T) = \begin{cases} 0 & \text{on } \mathfrak{F}'_0 \\ Ma & \text{on } \mathfrak{F}'_1 \\ Na & \text{on } \mathfrak{F}'_2. \end{cases} \tag{9.6.10}$$

Thus,

$$\$'Z\,|0\rangle = \alpha\,|0\rangle + |\mathbf{1}\rangle u \tag{9.6.11}$$

$$\$'Z\,|\mathbf{1}\rangle = u^\dagger\,|0\rangle + \alpha\,|\mathbf{1}\rangle + (Jv)^T\,|11\rangle \tag{9.6.12}$$

$$\$'Z\,|11\rangle = |\mathbf{1}\rangle J\overline{v} + \alpha\,|11\rangle. \tag{9.6.13}$$

Where $u = M^\dagger w$ and $v = N^\dagger w$. We investigate the condition for the evolution to be positive definite. To do this we examine the eigenvalues of the matrix of $\$'Z$ with respect to the basis $\{|0\rangle, |\mathbf{1}\rangle, |11\rangle\}$:

$$\begin{pmatrix} \alpha & \overline{u}_1 & \overline{u}_2 & 0 \\ u_1 & \alpha & 0 & -\overline{v}_2 \\ u_2 & 0 & \alpha & \overline{v}_1 \\ 0 & -v_2 & v_1 & \alpha \end{pmatrix}. \tag{9.6.14}$$

The eigenvalues μ obey the equation,

$$(\mu-\alpha)^4 - (\mu-\alpha)^2(u^\dagger u + v^\dagger v) + |u^\dagger v|^2 = 0. \tag{9.6.15}$$

Using the unitarity of M and N we find that

$$\mu = \alpha \pm \sqrt{\|w\|^2 \pm \Delta}, \tag{9.6.16}$$

where $\Delta^2 = \|w\|^4 - |w^\dagger M N^{-1} w|^2$. The two $\pm$ signs are independent. We notice that Z is positive definite (corresponding to substituting $M = N = \mathbb{1}$) when $\alpha > \|w\|$. The evolved operator is positive definite whenever $\alpha > \sqrt{\|w\|^2 + \Delta}$.

Clearly then, as we have shown that M and N are not proportional we can find $w \in \mathbb{C}^2$ such that $|w^\dagger M N^{-1} w| < \|w\|^2$, and hence have $\Delta > 0$. Choosing α so that

$$\|w\| < \alpha < \sqrt{\|w\|^2 + \Delta} \tag{9.6.17}$$

gives a positive definite operator Z whose image under the dual of the superscattering operator, $\$'$, fails to be. Accordingly $\$'$ violates property (iii) above and therefore cannot be the dual of a superscattering operator.

Next, we consider the Heisenberg evolution itself in more detail. It is worth pointing out that we have not by any means obtained the full Heisenberg picture evolution; at present we know the evolution of only a 3-dimensional subspace (spanned by the $\Psi_\alpha(0^-)$) of the 64-dimensional space $\mathfrak{L}(\mathfrak{F})$ of linear operators on the 8-dimensional Hilbert space $\mathfrak{F}$. Owing to our results above, various natural strategies for extending this evolution to the whole of $\mathfrak{L}(\mathfrak{F})$ are denied to us: the evolution cannot be extended as a $*$-homomorphism (i.e., mapping any polynomial in the $\Psi_\alpha(0^-)$ to the corresponding polynomial in the $\Psi_\alpha(T^+)$) because the CAR's are violated; we cannot write $Z \mapsto X^{-1} Z X$ or $Z \mapsto X^\dagger Z X$ because of our observations above, nor can we write $Z \mapsto \$' Z$ for some superscattering operator $\$$.

It therefore seems that there is no natural extension of our evolution to $\mathfrak{L}(\mathfrak{F})$. As a concrete illustration of this type of behaviour, let us consider an example with one fermionic degree of freedom. Define the operator a on $\mathbb{C}^2$ by

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{9.6.18}$$

and suppose an evolution is given such that $\mathbb{1} \mapsto \mathbb{1}$, $a \mapsto \mu a$ and $a^\dagger \mapsto \mu a^\dagger$, where $0 \le \mu < 1$. It turns out that there are at least two choices for the evolution of $a^\dagger a$ consistent with a superscattering operator description. The first is that $a^\dagger a \mapsto a^\dagger a$, corresponding to a superscattering operator \$ with action

$$\$ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & 1-\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \mu\beta \\ \mu\overline{\beta} & 1-\alpha \end{pmatrix}, \tag{9.6.19}$$

on the state space of $\mathbb{C}^2$, whilst the second is $a^\dagger a \mapsto aa^\dagger$ and corresponds to the superscattering operator $\pounds$ with action

$$\pounds \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & 1-\alpha \end{pmatrix} = \begin{pmatrix} 1-\alpha & \mu\beta \\ \mu\overline{\beta} & \alpha \end{pmatrix}. \tag{9.6.20}$$

To conclude this section, we note that the failure of positivity which showed the nonexistence of a superscattering operator can be traded for a loss of the trace preserving property: by allowing $\mathbb{1} \mapsto \kappa\mathbb{1}$ for $\kappa \geq \sqrt{2}$, any positive Z of the form discussed above is mapped to a positive operator. One might therefore attempt to extend this in some way to a positive evolution on the whole of $\mathfrak{L}(\mathfrak{F})$ (which can be done if the evolution on $\mathbb{1}, a_i(0), a_i(0)^\dagger$ is *completely positive* – see Theorem 1.2.3 in Arveson [86]) thereby obtaining (by duality) a Schrödinger picture evolution possessing all the properties of a superscattering matrix except the preservation of trace. Rather than allowing individual probabilities to be negative with total probability equal to unity, we would now have positive probabilities with a total in excess of unity. It would be tempting to rescale this total to remove this problem, but that would amount to rescaling $a_i(T)$, for which there is no obvious justification.

9.7 The Classical Limit

With the normal ordering used above, we have shown that the quantum theory is uniquely determined in the generic case for all values of the coupling constant λ. On the other hand, we have also seen that the classical theory fails to be unique in the strong coupling regime. It is therefore interesting to determine the extent to which the classical theory may be regarded as a limit of the quantum theory.

We consider Model 1 with Bose statistics. Reintroducing the units of action by replacing a and b by $\sqrt{\hbar}\,a$ and $\sqrt{\hbar}\,b$ respectively, the consistency requirement Eq. (9.5.1) becomes

$$b = e^{-i\hbar\lambda T(a^\dagger a+b^\dagger b)}\left(Ra+Sb\right), \tag{9.7.1}$$

in which a and $a^\dagger$ obey the CCR's $[a,a^\dagger]=1$. The unique solution to this is given by $b=f(\hbar\lambda T;a^\dagger a)a$, where

$$f(\nu;n+1) = (e^{i\nu(n+1)(1+|f(\nu;n)|^2)}-S)^{-1}R, \tag{9.7.2}$$

with $f(\nu;0)=(1-S)^{-1}R$ for all ν.

The classical limit is found by taking the expectation of the relevant quantum mechanical operator in an appropriately defined coherent state, then letting $\hbar\to 0$, [87]. We shall form the coherent state associated with the classical quantity $\psi_{\rm cl}$ by writing

$$|\,\psi_{\rm cl}\rangle = e^{-|\psi_{\rm cl}|^2/2\hbar}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{\psi_{\rm cl}}{\sqrt{\hbar}}\right)^n|n\rangle = e^{-|x|^2/2}\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n!}}|n\rangle, \tag{9.7.3}$$

where

$$|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle \qquad \text{and} \qquad x=\frac{\psi_{\rm cl}}{\sqrt{\hbar}}. \tag{9.7.4}$$

Let us write $\chi=|x|$. The classical limit of a function $G=g\left(a^\dagger a\right)a$ is then easily found to be

$$\langle\psi_{\rm cl}\,|\,G\,|\,\psi_{\rm cl}\rangle = xe^{-\chi^2}\sum_{n=0}^{\infty}\frac{\chi^{2n}}{n!}g(n). \tag{9.7.5}$$

Notice that if we set $\gamma=\chi^2$ then we may interpret this as the expectation of a function in the Poisson distribution with parameter γ. Observe that from Eq. (9.7.5) we have that

$$\langle\psi_{\rm cl}\,|\,\Psi_1(0)\,|\,\psi_{\rm cl}\rangle = \sqrt{\hbar}\langle\psi_{\rm cl}\,|\,a\,|\,\psi_{\rm cl}\rangle = \psi_{\rm cl} \qquad \text{for all } \hbar. \tag{9.7.6}$$

In order to understand the evolution of $\Psi(t)$ between $t=0$ and T we will need to use the iterative scheme Eq. (9.7.2). We recast this into a map on the circle, by setting

$$f(\nu;n) = (\zeta_n - S)^{-1}R, \tag{9.7.7}$$

so that we have

$$\zeta_n = \exp\left\{ i\mu \left(\frac{n}{\chi^2}\right)\left(1 + \frac{|R|^2}{|\zeta_{n-1} - S|^2}\right)\right\}, \qquad \mu = \lambda T |\psi_{\rm cl}|^2. \tag{9.7.8}$$

Now let us investigate the limiting procedure $\hbar \to 0$, or equivalently, $\chi \to \infty$. As we noted previously the expectation of some function can be interpreted in terms of a Poisson distribution with mean and variance χ^2. In the limit as $\chi \to \infty$ the distribution becomes Normal. To show this we write $n = \chi^2 + y\chi$. Then we have

$$\mathrm{Prob}(N \in [n, n+1)) = \frac{e^{-\chi^2}\chi^{2n}}{n!}. \tag{9.7.9}$$

Using Stirling's approximation and $\Delta y = 1/\chi$, we find

$$\log \mathrm{Prob}(N \in [n, n+1)) = -y^2/2 - \log\sqrt{2\pi} + \log \Delta y + O\left(\chi^{-1}\right), \tag{9.7.10}$$

so that

$$\mathrm{Prob}(N \in [A, B]) = \int_{y_A}^{y_B} \frac{e^{-y^2/2}\, dy}{\sqrt{2\pi}} + O\left(\frac{1}{\chi}\right), \tag{9.7.11}$$

with $y_A = (A - \chi^2)/\chi$ etc. The integrand on LHS of Eq. (9.7.11) is a standard $N(0, 1)$ probability distribution. The important point to notice is that for any expectation value of a bounded function, the dominant contribution to the final value comes from a band around $y = 0$ of a width proportional to some multiple of the standard deviation. In terms of the original Poisson distribution then, the dominant contribution arises from those terms centred on $n = \chi^2$ and of a width proportional to χ.

The limit as a Normal distribution provides a useful picture of what is happening. However it seems preferable when considering the expectation value of functions that do not have a well defined continuum limit to work with the Poisson distribution itself. In appendix 9.D we provide some elementary bounds on the probabilities associated with the regions away from the central band about $n = \chi^2$. If we wish to evaluate the expectation value of a function which is uniformly bounded $g(n, x) < C$ then given $\epsilon > 0$

there exists $\alpha > 0$ such that (for all sufficiently large χ) we have that

$$\left| \sum_{0}^{\infty} \frac{e^{-\chi^2}\chi^{2n}}{n!} g(n,x) - \sum_{\chi^2-\alpha\chi}^{\chi^2+\alpha\chi} \frac{e^{-\chi^2}\chi^{2n}}{n!} g(n,x) \right| < \epsilon. \tag{9.7.12}$$

In order to evaluate $\langle \psi_{\rm cl} \,|\, \Psi_2(0^+) \,|\, \psi_{\rm cl} \rangle$ and the other analogues of the quantum system we return to the iterative scheme given by Eq. (9.7.8). In terms of the variable y it reads;

$$\zeta_n = \exp\left\{ i\mu \left(1 + \frac{y}{\chi}\right) \left(1 + \frac{|R|^2}{|\zeta_{n-1} - S|^2}\right) \right\}, \tag{9.7.13}$$

where we are only going to be interested in $-\alpha < y < \alpha$, and large values of χ (this means we are interested in a large number of iterates, specifically $2\alpha\chi$ of them). Clearly for large χ the iterative scheme may be regarded as a perturbation on the χ-independent iterative relation

$$z_{n+1} = \exp\left\{ i\mu \left(1 + \frac{|R|^2}{|z_n - S|^2}\right) \right\}. \tag{9.7.14}$$

It seems therefore that the behaviour of Eq. (9.7.14) is crucial. We begin the iteration by setting

$$z_{\chi^2-\alpha\chi} = \zeta_{\chi^2-\alpha\chi}\,. \tag{9.7.15}$$

In Appendix 9.E we prove the following theorem:

Theorem. *Suppose we have an sequence of complex values defined by* $w_n = f_x\left(w_{n-1}; n\right)$ *where*

$$f_x(z;n) = \exp\left\{ i\mu \left(1 - \frac{\alpha}{\chi} + \frac{n}{\chi^2}\right) \left(1 + \frac{R^2}{|z - S|^2}\right) \right\} \tag{9.7.16}$$

and a further iterative sequence $z_n = f\left(z_{n-1}\right)$ *with*

$$f(z) = \exp\left\{ i\mu \left(1 + \frac{R^2}{|z - S|^2}\right) \right\}. \tag{9.7.17}$$

Furthermore, let us suppose $w_0 = z_0$*. If* z_n *tends to a stable limit cycle* $\{Z_0, \ldots, Z_{N-1}\}$ *as* $n \to \infty$*, then given* $\epsilon > 0$*, there exists* X *such that for all* $\chi > X$*,*

$$|z_n - w_n| < \epsilon \qquad \textit{for all } n < 2\alpha\chi. \tag{9.7.18}$$

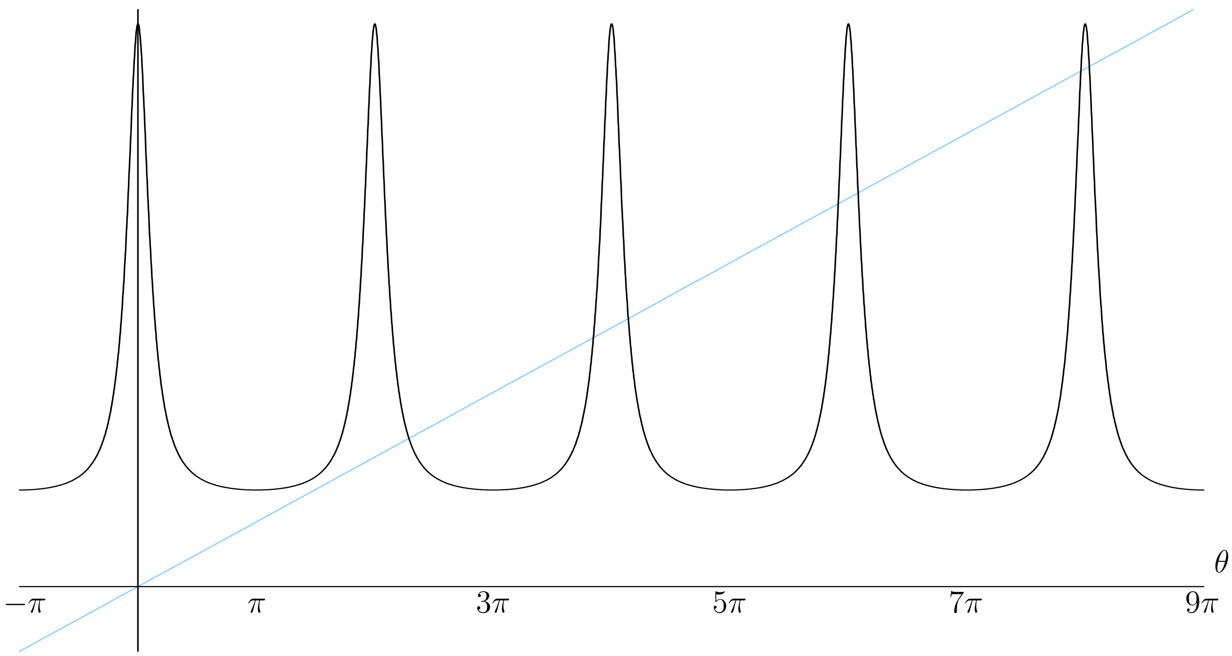

Fig. 9.4: *Graphical representation of the iteration scheme. Notice that there cannot be any exceptional orbits.*

Provided that the iteration scheme defined by the z_n has a limit cycle then the expectation of the ζ_n sequence is governed by that of the z_n. There is one final subtlety: by defining $w_n = \zeta_{n+\chi^2-\alpha\chi}$, we make the initial condition $z_0 = w_0$ depend on χ. We may now make use of the fact that $g(\theta)$ has negative Schwarzian derivative, i.e.,

$$D_S\, g(\theta) = \frac{g'''(\theta)}{g'(\theta)} - \frac{3}{2}\left[\frac{g''(\theta)}{g'(\theta)}\right]^2 < 0 \tag{9.7.19}$$

where

$$e^{ig(\theta)} = f(e^{i\theta}). \tag{9.7.20}$$

A result due to Singer (see, for example Proposition 4.2 in [88]) says that every stable (i.e., attractor) limit cycle which is non-exceptional arises from considering the orbits of the critical points. We shall regard θ as running from $-\pi$ to $(2k+1)\pi$, for some integer k chosen so that $g([-\pi,(2k+1)\pi]) \subseteq [-\pi,(2k+1)\pi]$, see Fig. 9.4. As all maxima, and all minima are mapped to the same value after a single iteration it is clear that there are at most two such cycles.

Let us now consider the exceptional orbits. The exceptional orbits (if they exist) for an iteration g defined on $[-\pi,(2k+1)\pi]$ are defined to be either

- A fixed point α of g, where g is increasing on $[\alpha,(2k+1)\pi]$ and $g(\theta) < \theta$ for all $\theta \in (\alpha,(2k+1)\pi)$
- A fixed point α of g, where g is decreasing on $[-\pi,\alpha]$ and $g(\theta) > \theta$ for all $\theta \in (-\pi,\alpha)$
- A cycle $\{\beta, g(\beta)\}$, where $\beta < g(\beta)$ with g decreasing on $[-\pi,\beta]$ and $[g(\beta),(2k{+}1)\pi]$, and $g^2(\theta) > \theta$ for all $\theta \in (-\pi,\beta)$.

In particular, the first pair and the last one are mutually exclusive.

As g is not monotonic between $-\pi$ and the first fixed point and is decreasing as we approach $\theta = (2k+1)\pi$, we conclude that none of the cases can arise. There can thus be at most two stable periodic orbits; frequently there will be only be one, in which case the set of points outside its basin of attraction has Lesbegue measure zero, see [88].

Let us now suppose that there is a single attractive orbit. It's basin of attraction is dense, so given any starting point for the iteration $w_0 = \zeta_{\chi^2-\alpha\chi}$ we can find a value of z_0 arbitrarily close for which its iterates converge to the limit cycle. This will prove to be sufficient to show that the classical limit exists. Put another way if it should turn out that for any particular value of χ the initial condition $z_0 = w_0(\chi)$ leads to an unstable orbit (e.g., we might by very carefully tuning of the parameters hit an unstable fixed point), then we may increase χ slightly. As $w_0(\chi)$ is continuous in χ and is nowhere constant, it follows that this results in a small change in z_0. Using the fact that the basin of attraction of the limit cycle is dense, we can arrange matters so that the perturbed starting condition once again leads to the stable orbit.

Next we introduce a lemma:

Lemma. *If $g(n_r)$ is a convergent subsequence of $g(n)$ converging to g then*

$$\lim_{\chi\to\infty}\sum_{r=0}^{\infty}\frac{e^{-\chi^2}\chi^{2n_r}}{n_r!}g(n_r) = \lim_{\chi\to\infty} g\sum_{r=0}^{\infty}\frac{e^{-\chi^2}\chi^{2n_r}}{n_r!}. \tag{9.7.21}$$

Proof: As $g(n_r)$ converges to g we know that given $\epsilon > 0$ there exists R such that for all $r > R$ we have $|g(n_r) - g| < \epsilon/2$.

We also know that the $g(n_r)$ are bounded. So we have $|g(n_r) - g| < G$ for all r. Consider then

$$\left|\sum_{r=0}^{R}\frac{e^{-\chi^2}\chi^{2n_r}}{n_r!}\left(g(n_r)-g\right)\right| < G\left|\sum_{r=0}^{R}\frac{e^{-\chi^2}\chi^{2n_r}}{n_r!}\right| \tag{9.7.22}$$

but the RHS is a finite sum of terms that tend to zero as $\chi \to \infty$ so there exists χ_0 such that for all $\chi > \chi_0$ the RHS of Eq. (9.7.22) is less than $\epsilon/2$. Note too that

$$\left|\sum_{r=R+1}^{\infty}\frac{e^{-\chi^2}\chi^{2n_r}}{n_r!}\left(g(n_r)-g\right)\right| < \frac{\epsilon}{2}\sum_{r=R+1}^{\infty}\frac{e^{-\chi^2}\chi^{2n_r}}{n_r!} \leq \frac{\epsilon}{2}. \tag{9.7.23}$$

Putting the two together and using the triangle inequality completes the proof. □

As a special case of this suppose $g(n)$ is a sequence of iterates converging to a periodic orbit of length N, let $G_0, \ldots, G_{N-1}$ be the limit cycle, so that $g(Nr+k)$ converges to G_k. Then

$$\lim_{\chi\to\infty}\sum_{r=0}^{\infty}\frac{e^{-\chi^2}\chi^{2(Nr+k)}}{(Nr+k)!}g(Nr+k) = \lim_{\chi\to\infty}\frac{G_k}{N}\sum_{s=0}^{N-1}\omega_N^{-sk}e^{\chi^2(\omega_N^s-1)} = \frac{G_k}{N} \tag{9.7.24}$$

where we have written $\omega_N = e^{2\pi i/N}$. We may therefore establish that if the $g(n_r)$ form an iteration sequence converging to a limit cycle the classical limit is given by

$$\frac{1}{N}\sum_{k=0}^{N-1}G_k, \tag{9.7.25}$$

i.e., the arithmetic mean of the limit cycle.

Let us apply our results to work out the classical limit of $\langle\psi_{\rm cl}\,|\,\Psi_2(0^+)\,|\,\psi_{\rm cl}\rangle$. We shall suppose the iteration scheme Eq. (9.7.14) for z_n converges to a limit cycle $Z_0,\ldots,Z_{N-1}$. It is clear then that $f(n)$ converges to a cycle $F_0,\ldots,F_{N-1}$ where

$$F_k = (Z_k - S)^{-1}R. \tag{9.7.26}$$

Thus,

$$\langle\psi_{\rm cl}\,|\,\Psi_2(0^+)\,|\,\psi_{\rm cl}\rangle = \sqrt{\hbar}\langle\psi_{\rm cl}\,|\,b\,|\,\psi_{\rm cl}\rangle \to \frac{1}{N}\sum_{k=0}^{N-1}(Z_k - S)^{-1}R\psi_{\rm cl}. \tag{9.7.27}$$

The evolution of the a-operators is perhaps more significant. We find

$$\langle\psi_{\rm cl}\,|\,\Psi_1(T)\,|\,\psi_{\rm cl}\rangle = \sqrt{\hbar}\langle\psi_{\rm cl}\,|\,a(T)\,|\,\psi_{\rm cl}\rangle \to \frac{1}{N}\sum_{k=0}^{N-1}Z_k^{-1}(P + Q(Z_k - S)^{-1}R)\psi_{\rm cl}. \tag{9.7.28}$$

We may interpret Eqs. (9.7.27) and (9.7.28) as being the linear superposition of solutions that do not themselves satisfy the CTC boundary conditions (unless $N = 1$). Instead the quantum theory yields a classical limit that corresponds to the superposition of solutions that only obey the boundary conditions after a finite number of traversals around the CTC. We will call such a solution *a winding number N trajectory*. The possibility of winding numbers greater than one is a surprising and fascinating result of our analysis.

So far we have restricted our attention to the case where there has been a periodic limit cycle. There are values of the coupling strength where no such cycle exists. In this case the iterative scheme does not appear to give us a well-defined classical limit, though the system is not easily treated using analytical techniques. Numerical studies seem to support our interpretation, but are not wholly conclusive.

In order to understand the behaviour of our system as we vary the coupling strength it is useful to plot a bifurcation diagram. Fig. 9.5 shows the bifurcation diagram for $P = -Q = R = S = 1/\sqrt{2}$. Its classical counterpart, Fig. 9.3 has been superposed in red. We notice that for the quantum system there are bands of unique solutions corresponding to a classical solution, as well as period doubling points where the classical

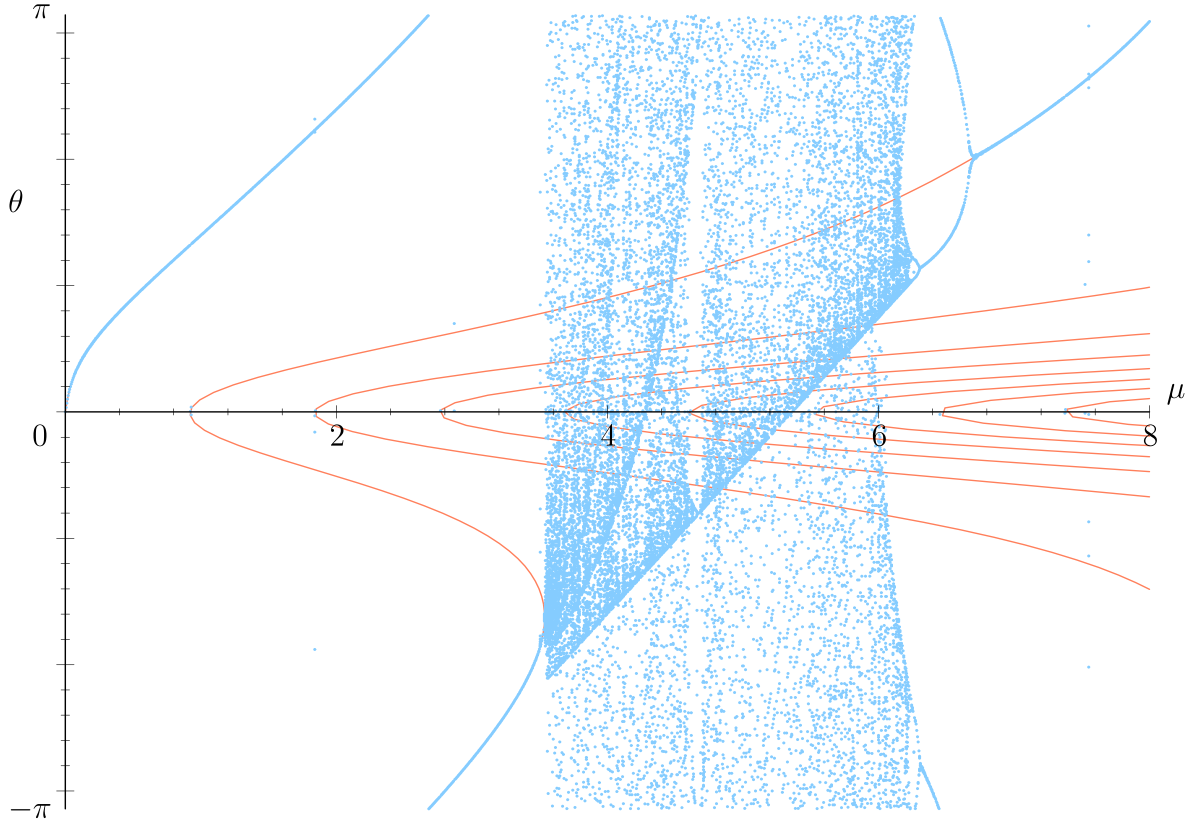

Fig. 9.5: *Bifurcation Diagram relevant for the Classical limit.*

limits becomes a superposition of states together with a band where there appear to be chaotic orbits. These we have suggested correspond to no classical limit.

To understand the diagram we make a few observations. In Fig. 9.1 and Fig. 9.2 we plot the iterative function $y = \zeta(\theta)$ from Eq. (9.3.15) and the line $y = (\theta + 2\pi k)/\mu$, for $k \in \mathbb{Z}$. The points of intersection correspond to possible classical solutions. We see that in the quantum theory it is the iterates of $\mu\zeta$ that are important.

We observe that $\zeta(\theta)$ can have at most two attractive fixed points. The attractors can only exist on points on the curve where the gradient is less than $1/\mu$. The two regions being centred around $\theta = \pi$ (and $-\pi$) and around $\theta = 0$. Taking μ from zero we see that the blue line is initially vertical and there is an attractor at $\theta = 0$. As μ increases there is a unique fixed point which turns out to always be an attractor (see Fig 9.1) until we reach the first critical value. At this point new fixed points (i.e., classical solutions) occur however they generally do not lie in an attractor region and are repellors. In

Fig. 9.2 we see how the quantum theory picks out a unique classical solution from seven possibilities by selecting the one that is an attractor.

Increasing μ further (i.e., decreasing the slope of the blue line) will cause the fixed point attractor to become a point of tangency of the line to the curve. Immediately after there are no attractive fixed points as indicated on the bifurcation diagram. We have a band of what appears to be chaotic orbits (though they might be periodic with long periods). A classical limit resumes when we have a fixed attractor of an Nth iterate giving the winding number N trajectories described previously. After some period undoublings we find a unique attractor and a unique classical solution obeying the classical equations.

We point out that as μ increases the range of values where an attractor can lie becomes ever smaller. In this way we notice that the values where there is a unique classical solution picked out by the quantum theory occur in bands that become narrower as μ increases, i.e., as the coupling becomes stronger. Finally we note that the behaviour we have been describing is heavily dependent on the operator ordering we have employed, as we shall show in the next section.

9.8 Operator Ordering

So far, we have worked with a single choice of operator ordering, namely the literal ordering of Eq. (9.2.2) which corresponds to normal ordering of the quantized Hamiltonian. In this section, we briefly discuss the effect of allowing alternative orderings in which Eq. (9.2.2) is ordered as

$$\dot{\psi} = -iW\psi - i\alpha\lambda(\psi^\dagger\psi)\psi - i(1-\alpha)\lambda\psi(\psi^\dagger\psi), \tag{9.8.1}$$

for $\alpha \in [0, 1]$. The foregoing treatment is the case $\alpha = 1$.

Consider Model 1 for Bose statistics. The analogue of Eq. (9.5.1) is

$$b = e^{-i\alpha\lambda T(a^\dagger a + b^\dagger b)}(Ra + Sb)e^{-i(1-\alpha)\lambda T(a^\dagger a + b^\dagger b)}, \tag{9.8.2}$$

(we have set $A = \mathbb{1}$ for simplicity). Making the *ansatz* $b = f(a^\dagger a)a$, we find that f satisfies

$$f(0) = (R + Sf(0))\, e^{-i\lambda T(1-\alpha)(1+|f(0)|^2)}, \tag{9.8.3}$$

and

$$f(n+1) = e^{-i\lambda T(n+1)\alpha(1+|f(n)|^2)}\,(R + Sf(n+1))\, e^{-i\lambda T(n+2)(1-\alpha)(1+|f(n+1)|^2)}, \tag{9.8.4}$$

for $n \geq 0$. The case $\alpha = 1$ was treated in Sect. 9.5 and uniquely determines $f(n+1)$ in terms of $f(n)$ for each n. However, the case $\alpha = 0$ is rather different and is described by

$$f(n) = e^{-i\lambda T\hbar(n+1)(1+|f(n)|^2)}\,(R + Sf(n))\,, \tag{9.8.5}$$

where we have written $\hbar$ explicitly. It is easy to recast this into the form of the *classical* consistency requirement Eq. (9.3.8) and it follows that $f(n)$ is uniquely determined for small quantum numbers $n\hbar \ll (\lambda T)^{-1}$ but not for $n\hbar \gg (\lambda T)^{-1}$, i.e., classical non-uniqueness re-emerges at high quantum numbers. There are therefore many functions $f(n)$ solving Eq. (9.8.5), each one of which corresponds to a different 'branch' of the quantum theory. Most of these branches do not possess a classical limit. However, in contrast to the situation for normal ordering, *every* classical solution will arise as the classical limit of some branch of the quantum theory.

It would be interesting if the nonunitarity of Model 2 could be removed by a suitable ordering prescription. In Appendix 9.C, we investigate this for orderings of form (9.8.1) with the *ansatz* $b = f(d^\dagger d, c^\dagger c)c$ with c and d given by Eq. (9.5.26). For all $0 \leq \alpha \leq 1$, we find that the (anti)commutation relations are violated for generic values of the parameters.

9.9 Self-Consistent Path Integral

9.9.1 General Formalism

In this section, we compare the results obtained from the QIVP with those obtained using the self-consistent path integral developed by Thorne and collaborators [73, 74, 89] and employed by Politzer [52]. To establish our notation, we briefly review the quantization of our system by path integral methods in the absence of CTC's. Starting with the bosonic case, it is convenient to use the holomorphic representation (see, e.g., [90]) in which the Hilbert space $\mathfrak{F}$ is the space of analytic functions $f(\overline{c_1}, \ldots, \overline{c_s})$ with inner product

$$\langle f \,|\, g \rangle = \int \mathcal{D}c^\dagger \mathcal{D}c \, e^{-c^\dagger c} \overline{f(c^\dagger)} g(c^\dagger), \tag{9.9.1}$$

where we write $c^\dagger$ to denote $(\overline{c_1}, \ldots, \overline{c_s})$ and the measure is

$$\mathcal{D}c^\dagger \mathcal{D}c = \prod_j \frac{d\overline{c_j} dc_j}{2\pi i}. \tag{9.9.2}$$

The Hilbert space $\mathfrak{F}$ carries a (Fock) representation of the CCR's in which $c_j^\dagger$ acts as multiplication by $\overline{c_j}$ and c_j as $\partial / \partial \overline{c_j}$. Operators on $\mathfrak{F}$ are described by their kernels:

$$(Af)(c^\dagger) = \int \mathcal{D}c'^\dagger \mathcal{D}c' \, e^{-c'^\dagger c'} A(c^\dagger; c') f(c'^\dagger). \tag{9.9.3}$$

In particular, if K is a $s \times s$ matrix, then the mapping $f(c^\dagger) \mapsto f(c^\dagger K)$ has kernel $\exp c^\dagger K c'$.

Starting with the (normal ordered) quantized bosonic Hamiltonian H on $\mathfrak{F}$, one may obtain the kernel for $U = e^{-iHt}$ in the form

$$U_t(c^\dagger; c') = \int \prod_{t'} \mathcal{D}\gamma(t')^\dagger \mathcal{D}\gamma(t') \, \exp\left\{ \frac{1}{2}(\gamma^\dagger(t)\gamma(t) + \gamma^\dagger(0)\gamma(0)) + iS[\gamma] \right\}, \tag{9.9.4}$$

where the action functional $S[\gamma]$ is defined in terms of the classical Hamiltonian (9.2.6) by

$$S[\gamma] = \int_0^t \left(\frac{i}{2} (\gamma(t')^\dagger \dot{\gamma}(t') - \dot{\gamma}(t')^\dagger \gamma(t')) - H(\gamma(t'), i\gamma(t')^\dagger) \right) dt', \tag{9.9.5}$$

and the paths $\gamma(t')$ are subject to the boundary conditions $\gamma^\dagger(t) = c^\dagger$ and $\gamma(0) = c'$. In the free case, for example, one may evaluate the path integral explicitly to give

$$U_t(c^\dagger; c') = \exp c^\dagger e^{-iWt} c'. \tag{9.9.6}$$

One may develop the path integral treatment for Fermi statistics in a parallel fashion [90] by replacing the integration variables by Grassmann numbers and $\mathcal{D}c\mathcal{D}c^\dagger$ by Berezin measure. Again, the resulting kernel has the action of e^{-iHt} on $\mathfrak{F}$, where H is now the fermionic normal ordered quantized Hamiltonian.

A natural generalization of this to enable the treatment of chronology violating systems is the *self-consistent path integral* [52, 73, 74, 89]. Instead of integrating over all field configurations with $\gamma(0) = c'$ and $\gamma^\dagger(T) = c^\dagger$ to form the kernel $U_T(c^\dagger, c')$, the self-consistent path integral prescription requires that one should restrict the class of field configurations to those obeying the self-consistency requirements imposed by any CTC's present (here, the boundary conditions (9.2.7)). To implement this, we first decompose $\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$, where $\mathfrak{F}_1$ is the space of analytic functions in variables $\overline{a_1}, \ldots, \overline{a_{s_1}}$, and $\mathfrak{F}_2$ is the space of analytic functions in $\overline{b_1}, \ldots, \overline{b_{s_2}}$. The (self-consistent) evolution kernel from $t = 0^-$ to $t = T^+$ can then be written in the form

$$X(a^\dagger, b^\dagger; a', b') = \mathcal{N} e^{b^\dagger B b'} \widetilde{U_T}(a^\dagger; a'). \tag{9.9.7}$$

Here, $\mathcal{N}$ is a normalization constant and the factor $e^{b^\dagger B b'}$ implements the boundary condition $\psi_2(T^+) = B\psi_2(0^-)$ while $\widetilde{U_T}$ is given by the same path integral as U_T but taken over all field configurations with $\gamma(0) = (a', b')$, $\gamma^\dagger(T) = (a, Ab')^\dagger$ for any b'. As noted by Politzer [52], $\widetilde{U_T}(a^\dagger; a')$ may be obtained from $U_T(a^\dagger, b^\dagger; a', b')$ by setting $b = Ab'$ and integrating over all possibilities in the Hilbert space measure of $\mathfrak{F}_2$, that is,

$$\widetilde{U_T}(a^\dagger; a') = \int \mathcal{D}b^\dagger \mathcal{D}b \, e^{-b^\dagger b} U_T(a^\dagger, b^\dagger A^\dagger; a', b), \tag{9.9.8}$$

which may be rewritten in the form

$$\widetilde{U_T}(a^\dagger;a') = \int \mathcal{D}b^\dagger \mathcal{D}b \int \mathcal{D}c^\dagger \mathcal{D}c\, e^{-b^\dagger b - c^\dagger c} e^{b^\dagger A^\dagger c} U_T(a^\dagger, c^\dagger; a', b). \tag{9.9.9}$$

Thus, by expanding $e^{b^\dagger A^\dagger c}$ as

$$e^{b^\dagger A^\dagger c} = \prod_{i=1}^{s_2} \sum_{n_i=0}^{\infty} \frac{(A^\dagger c)_i^{n_i} (b_i^\dagger)^{n_i}}{n_i!}, \tag{9.9.10}$$

we obtain the matrix element $\langle \mathbf{m} | \widetilde{U_T} | \mathbf{m}' \rangle$ in the form

$$\langle \mathbf{m} | \widetilde{U_T} | \mathbf{m}' \rangle = \sum_{\mathbf{n}} \langle \mathbf{m}; \widetilde{\mathbf{n}} | U_T | \mathbf{m}'; \mathbf{n} \rangle, \tag{9.9.11}$$

where the vector $|\, \mathbf{m} \rangle \in \mathfrak{F}_1$ is the function $\prod_i (m_i!)^{-1/2} \overline{a_i}^{m_i}$, and the vector $|\, \widetilde{\mathbf{n}} \rangle \in \mathfrak{F}_2$ is the function $\prod_i (n_i!)^{-1/2} \overline{(A^\dagger b)_i}^{n_i}$. We refer to Eq. (9.9.11), which is a generalization of the expression given by Politzer [52] as the *partial trace definition* of the self-consistent path integral.

The fermionic case follows a similar pattern, when one replaces the integration variables by Grassmann numbers and uses Berezin measure; the main difference lies in the partial trace definition. Starting from the analogue of (9.9.9), we expand $e^{b^\dagger A^\dagger c}$ as

$$\begin{aligned} e^{b^\dagger A^\dagger c} &= \prod_{i=1}^{s_2} \left(1 + b_i^\dagger (A^\dagger c)_i\right) \\ &= \sum_{\mathbf{n}} (-1)^n \prod_i (A^\dagger c)_i^{n_i} (b_i^\dagger)^{n_i}, \end{aligned} \tag{9.9.12}$$

where $n = \sum_i n_i$, and therefore obtain

$$\langle \mathbf{m} | \widetilde{U_T} | \mathbf{m}' \rangle = \sum_{\mathbf{n}} (-1)^n \langle \mathbf{m}; \widetilde{\mathbf{n}} | U_T | \mathbf{m}'; \mathbf{n} \rangle, \tag{9.9.13}$$

under the assumption that the Grassmann number $b_i^\dagger$ commutes with the kernel of $U_T(a^\dagger, c^\dagger; a', b)$, which holds if H conserves particle number (as it does in our case of interest). The factor of $(-1)^n$ was omitted by Politzer [52]; it arises because terms of the form $b_i^\dagger (A^\dagger c)_i$ coming from $e^{b^\dagger A^\dagger c}$ must be rearranged in order to move the $b_i^\dagger$'s into the ket and the $(A^\dagger c)_i$'s into the bra of the matrix element $\langle \mathbf{m}; \widetilde{\mathbf{n}} \,|\, U_T \,|\, \mathbf{m}'; \mathbf{n} \rangle$. In Appendix 9.B, we will see how, for free fields, these factors ensure that the evolution computed from (9.9.13) agrees with that obtained directly from the path integral, and also with that obtained from the QIVP.

9.9.2 Free Fields

Whilst one can use the partial trace definition to compute the quantum evolution X for free fields (see Appendix 9.B), it is easier to evaluate the path integral directly, using the fact that the kernel of the free evolution is given by

$$U_T(c^\dagger; c') = \exp c^\dagger e^{-iWT} c'. \tag{9.9.14}$$

Writing e^{-iWT} in the block form (9.3.1) as above, we obtain

$$\begin{aligned} \widetilde{U_T}(a^\dagger; a') \;=\; & \int \mathcal{D}b^\dagger \mathcal{D}b \, \exp\left\{ -b^\dagger (\mathbb{1} - A^\dagger S) b + a^\dagger P a' \right. \\ & \left. + a^\dagger Q b + b^\dagger A^\dagger R a' \right\}, \end{aligned} \tag{9.9.15}$$

which may be evaluated to give

$$\widetilde{U_T}(a^\dagger; a') = (\det(\mathbb{1} - A^\dagger S))^{-1} \exp a^\dagger M a', \tag{9.9.16}$$

where $M = (P + Q(A - S)^{-1} R)$. In the generic case, the convergence of the path integral is guaranteed because $\|A^\dagger S\| < 1$ and so $\mathbb{1} - A^\dagger S$ has positive hermitian part.

Noting that $V(a^\dagger; a') = \exp a^\dagger M a'$ is the unitary kernel, because M is unitary, we conclude that the unitary kernel obtained from the self-consistent path integral is

$$X(a^\dagger, b^\dagger; a', b') = \exp\left\{a^\dagger M a' + b^\dagger B b'\right\}, \tag{9.9.17}$$

whose corresponding operator X acts on annihilation operators a_i and b_i according to

$$X^\dagger a_i X = M_{ij} a_j, \qquad X^\dagger b_i X = B_{ij} b_j. \tag{9.9.18}$$

Moreover, the normalization constant is given by $\mathcal{N} = \det(\mathbb{1} - A^\dagger S)$.

In the fermionic case, the path integral may be evaluated explicitly to obtain a unitary evolution with the action (9.9.18) on annihilation operators and normalization constant $\mathcal{N} = \det(\mathbb{1} - A^\dagger S)^{-1}$.

Thus in both cases, we have obtained agreement with the QIVP evolution. Moreover, we have given a general proof of the unitarity of free field evolution using the self-consistent path integral; previously this had only been established in a particular case [52].

9.9.3 An Interacting Model

We study Model 1 of Sect. 9.5 for both Bose and Fermi statistics, employing the partial trace definition, and choosing the normalization constant so that $\langle 0;0 \mid X \mid 0;0\rangle = 1$, which is reasonable because the Hamiltonian H is particle-number preserving. In the fermionic case, we obtain

$$\begin{aligned}
\langle 0 | \widetilde{U_T} | 0\rangle &= \langle 00 | e^{-iHT} | 00\rangle - \langle 01 | e^{-iHT} | 01\rangle \\
&= 1 - S \\
\langle 1 | \widetilde{U_T} | 1\rangle &= \langle 10 | e^{-iHT} | 10\rangle - \langle 11 | e^{-iHT} | 11\rangle \\
&= P - (PS - RQ)e^{-i\lambda T},
\end{aligned} \tag{9.9.19}$$

from which it follows that the evolution from $t = 0^-$ to $t = T^+$ is given by

$$\langle m; n \,|\, X \,|\, m'; n' \rangle = \delta_{nn'} \delta_{mm'} f(m), \tag{9.9.20}$$

where

$$f(m) = \begin{cases} 1 & m = 0 \\ \dfrac{P - (PS - RQ)e^{-i\lambda T}}{1 - S} & m = 1. \end{cases} \tag{9.9.21}$$

Thus X is nonunitary in general, which is essentially the result obtained by Politzer [52] in special cases, modulo some changes of sign owing to the factors of $(-1)^n$ discussed above. Except when $\lambda T/(2\pi) \in \mathbb{Z}$ this differs from the unitary evolution obtained from the QIVP.

In the bosonic case, we have

$$\langle m \,|\, \widetilde{U_T} \,|\, m' \rangle = \sum_{n=0}^{\infty} \frac{e^{-i\lambda T(m+n)(m+n-1)/2}}{(m!n!)^{1/2}} \langle 00 \,|\, (Ra + Sb)^n (Pa + Qb)^m \,|\, m'n \rangle, \tag{9.9.22}$$

and therefore conclude that $\langle mn \,|\, X \,|\, m'n' \rangle = \delta_{mm'} \delta_{nn'} f(m)$ with $f(0) = 1$ and

$$f(m) = \frac{\displaystyle\sum_{n=0}^{\infty} e^{-i\lambda T(m+n)(m+n-1)/2} \sum_{r=r_0}^{n} \binom{n}{r} \binom{m}{n-r} (RQ)^{n-r} P^{m+r-n} S^r}{\displaystyle\sum_n S^n e^{-i\lambda T n(n-1)/2}}, \tag{9.9.23}$$

with $r_0 = \max\{n - m, 0\}$. One may show that X fails to be unitary in general. Again, it clearly differs from the unitary evolution obtained from the QIVP.

9.10 Conclusion

We have analysed in detail the classical and quantum behaviour of a class of nonlinear chronology violating systems. Classically, we found that unique solutions exist for all choices of initial data in the linear and weak-coupling regimes, whilst the solutions become non-unique in the strong-coupling regime. This confirms the expectation that the behaviour of nonlinear fields interpolates between that of classical linear fields and hard-sphere mechanics. Quantum mechanically, we have shown that one can make sense of the quantum initial value problem for chronology violating systems; moreover, (at least with a natural choice of operator ordering) the quantum dynamics is unique for all values of the coupling constant. We have also exhibited examples in which this evolution does not preserve the (anti)commutation relations; it seems highly likely that this is the general situation. Moreover, the nonunitary evolution cannot be described by a superscattering operator – the loss of unitarity is more radical than previously thought, e.g., by Hawking [78].

We have also compared our quantum evolution with that computed using the self-consistent path integral, and found that they *do not* agree. This is not surprising, because the equivalence of these approaches for non-chronology violating systems relies on the existence of a foliation by Cauchy surfaces and there is no *a priori* reason to expect the equivalence to persist in the presence of CTC's. In this regard it is interesting that the QIVP and self-consistent path integral *are* nonetheless equivalent for linear fields. To some extent, it is a matter of taste which approach one prefers. For the models considered the QIVP approach has two main advantages. Firstly, we have found circumstances (e.g., Model 1 in Sect. 9.5.3) in which one obtains a unitary theory from the QIVP but not from the path integral. Secondly, the effect of the CTC's in our models is to introduce constraints which lead to a nontrivial geometric structure in the classical phase space. This might lead one to suspect that the quantization of this system requires more than just a restriction of the class of allowed histories, and that the path integral measure should also be modified (a similar comment has also been made in [56]). A hint of this appears in the treatment of linear fields, in which the propagator obtained from the self-consistent path integral must be rescaled by a factor of $\det(\mathbb{1} - A^\dagger S)^{\pm 1}$. It is plausible that in the linear case, the required modification to the path integral measure reduces to rescaling by this constant factor, but that for the nonlinear case the modification is nontrivial. At present it is not clear exactly how the path integral should be modified; on the other hand it is clear that the QIVP does correctly implement the CTC constraints and remains close to the spirit of the classical treatment. In Sect. 9.5 we noted that there was an effective reduction in the number of degrees of freedom between times where the

CTC is in existence. It seems likely that the class of paths one needs to sum over in a path integral approach must be altered to take greater account of this phenomenon.

The relationship between the unique quantum theory and the non-unique classical theory is intriguing. We have seen that there exist ranges of the coupling strength in which the quantum theory has a classical limit which selects precisely one of the many classical solutions, other ranges in which no classical limit exists and still other ranges where a classical limit exists but does not correspond to any of the classical solutions.

Finally, it is curious that the classical symplectic structure can be preserved for systems which do not preserve the quantum commutation relations. It is tempting to wonder whether there is a way of quantizing these models so that unitarity is preserved. Our uniqueness result for the QIVP rules this out within a Hilbert space context (at least with normal operator ordering) but it is possible that the situation might be different for the QIVP on an indefinite (Krein) inner product space in which irreducible non-Fock representations of the CCR's exist for even a single degree of freedom [91]. The motivation for studying Krein spaces would be that the loss of physical degrees of freedom in the nonchronal region might be equivalent to the addition of unphysical states with negative norm-squared.

9.A Path Integral Approach to the Free Classical Evolution

In this Appendix, we show how the classical evolution derived in Sect. 9.3.1 may be reproduced using a method due to Goldwirth *et al.* [51] and based on path integrals. (Goldwirth *et al.* regarded the classical wave equation as the first quantization of an underlying particle mechanics.) The central idea is to sum the propagators for all possible trajectories through the CTC region. We will use this method to determine the propagator between $t = 0^-$ and $t = T^+$, essentially repeating the calculation of [51] in our (slightly simpler) notation.

The block matrix decomposition Eq. (9.3.1) suggests that we break the problem into four parts, evaluating the propagators from $\mathcal{S}_i$ at $t = 0^-$ to $\mathcal{S}_j$ at $t = T^+$ separately for each $i, j = 1, 2$. Note that a particle on $\mathcal{S}_2$ at $t = 0^-$ must enter the wormhole there and re-emerge on $\mathcal{S}_2$ at $t = T^+$. Thus the $\mathcal{S}_2 \to \mathcal{S}_2$ propagator equals B, whilst that for $\mathcal{S}_2 \to \mathcal{S}_1$ vanishes. In addition, the propagator $\mathcal{S}_1 \to \mathcal{S}_2$ also vanishes by the time reverse of this argument. It remains to compute the propagator for $\mathcal{S}_1 \to \mathcal{S}_1$. In this case, there are countably many possible trajectories. The particle can either go directly to $\mathcal{S}_1$ with propagator P, or it can enter the CTC region to arrive at $\mathcal{S}_2$ at $t = T^-$ (propagator R), pass through the wormhole to $\mathcal{S}_2$ at $t = 0^+$ (propagator A^{-1}), execute n circuits of the CTC's (propagator $(A^{-1}S)^n$) and finally travel from $\mathcal{S}_2$ at $t = 0^+$ to $\mathcal{S}_1$ at $t = T^+$ (propagator Q). The combined propagator for this trajectory is $Q(A^{-1}S)^n A^{-1}R$; summing over all possible winding numbers and the direct trajectory, we obtain the total propagator

$$
\begin{aligned}
M &= P + Q\left(\sum_{n=0}^{\infty}(A^{-1}S)^n\right)A^{-1}R \\
&= P + Q(A-S)^{-1}R, \qquad (9.A.1)
\end{aligned}
$$

which agrees with the result obtained in Sect. 9.3.1.

9.B Partial Trace Formalism for Free Fields

In this Appendix, we derive the evolution operator for free field models in the presence of CTC's using the partial trace formulation of the self-consistent path integral.

We consider a general free theory whose Fock space is built using creation operators $a_1^\dagger, \ldots, a_{s_1}^\dagger$ and $b_1^\dagger, \ldots, b_{s_2}^\dagger$, acting on vacuum $|\,\mathbf{0};\mathbf{0}\rangle$. The a_i and b_i obey the CCR/CAR's.

The basis elements are written[3]

$$|\,\mathbf{m};\mathbf{n}\rangle = \prod_{i=1}^{s_1}(m_i!)^{-1/2}(a_i^\dagger)^{m_i}\prod_{i=1}^{s_2}(n_i!)^{-1/2}(b_i^\dagger)^{n_i}\,|\,\mathbf{0};\mathbf{0}\rangle, \tag{9.B.1}$$

and we write $m=\sum m_i$, $n=\sum n_i$ etc. It will also be convenient to define an alternative basis $|\,\mathbf{m};\widetilde{\mathbf{n}}\rangle$ by

$$|\,\mathbf{m};\widetilde{\mathbf{n}}\rangle = \prod_{i=1}^{s_1}(m_i!)^{-1/2}(a_i^\dagger)^{m_i}\prod_{i=1}^{s_2}(\widetilde{n}_i!)^{-1/2}((A^\dagger b)_i^\dagger)^{\widetilde{n}_i}\,|\,\mathbf{0};\mathbf{0}\rangle. \tag{9.B.2}$$

Suppose the evolution U on Fock space is unitary and such that

$$\begin{aligned} a(T) &= U^\dagger a U = Pa + Qb \\ b(T) &= U^\dagger b U = Ra + Sb, \end{aligned} \tag{9.B.3}$$

where the matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{9.B.4}$$

is unitary. Note that this entails that U preserves the total particle number $\sum a^\dagger a+\sum b^\dagger b$.

We now specialize to the bosonic case. From Sect. 9.9.1, the evolution operator X has matrix elements given by

$$\langle \mathbf{m};\mathbf{n}\,|\,X\,|\,\mathbf{m}';\mathbf{n}'\rangle = \mathcal{N}_{\rm b}\delta_{\mathbf{n}\mathbf{n}'}\sum_{\mathbf{n}'}\langle \mathbf{m};\widetilde{\mathbf{n}}'\,|\,U\,|\,\mathbf{m}';\mathbf{n}'\rangle, \tag{9.B.5}$$

where $\mathcal{N}_{\rm b}$ is a normalization constant, chosen to ensure that $\langle \mathbf{0}\,|\,X\,|\,\mathbf{0}\rangle = 1$ (as it should be for any free theory). This allows us to evaluate $\mathcal{N}_{\rm b}$ explicitly, because the matrix

[3] We note in passing that the basis used in Eq. (2) of Ref. [52] for fermionic systems is not properly anticommuting.

element $\langle \mathbf{0}; \widetilde{\mathbf{n}} | U | \mathbf{0}; \mathbf{n} \rangle$ is

$$\langle \mathbf{0}; \widetilde{\mathbf{n}} | U | \mathbf{0}; \mathbf{n} \rangle = \langle \mathbf{0}; \mathbf{0} | \prod_i \frac{(A^\dagger b)_i^{\widetilde{n}_i}}{(\widetilde{n}_i!)^{1/2}} U | \mathbf{0}; \mathbf{n} \rangle = \langle \mathbf{0}; \mathbf{0} | U \prod_i \frac{(A^\dagger b(T))_i^{\widetilde{n}_i}}{(\widetilde{n}_i!)^{1/2}} | \mathbf{0}; \mathbf{n} \rangle, \qquad (9.B.6)$$

and is therefore equal to the coefficient of $\prod_i x_i^{n_i}$ in the expansion of $\prod_i (\sum_j (A^\dagger S)_{ij} x_j)^{n_i}$. Here we have used the fact that U preserves the vacuum. The generating function for these coefficients, $G(x_1, \ldots, x_{s_2})$, can be found in §66 of [92], and is given by

$$G(x_1, \ldots, x_{s_2}) = \frac{(-1)^{s_2} (x_1 x_2 \ldots x_{s_2})^{-1}}{\det(A^\dagger S - \operatorname{diag}(x_1^{-1}, x_2^{-1}, \ldots, x_{s_2}^{-1}))}. \qquad (9.B.7)$$

The sum over all $\mathbf{n}$ of these matrix elements is obtained simply by evaluating the generating function with all x_i equal to unity. Thus we obtain

$$\mathcal{N}_{\mathrm{b}} = \det(\mathbb{1} - A^\dagger S). \qquad (9.B.8)$$

Next, we claim that

$$X^{-1} a X = M a, \qquad (9.B.9)$$

where $M = P + Q(A - S)^{-1} R$ is unitary. Together with the trivial evolution $X^{-1} b X = B b$, this shows that X is unitary. Moreover, this is the free evolution derived in various ways in the body of the previous sections.

To establish (9.B.9), we first note that

$$\begin{aligned} \sum_{\mathbf{n}} \langle \mathbf{m}; \widetilde{\mathbf{n}} | U b | \mathbf{m}'; \mathbf{n} \rangle &= \sum_{\mathbf{n}} \langle \mathbf{m}; \widetilde{\mathbf{n}} | (A^\dagger b) U | \mathbf{m}'; \mathbf{n} \rangle \\ &= \sum_{\mathbf{n}} \langle \mathbf{m}; \widetilde{\mathbf{n}} | U A^\dagger (R a + S b) | \mathbf{m}'; \mathbf{n} \rangle, \end{aligned} \qquad (9.B.10)$$

where the first step follows by relabelling the sum over n_i. Collecting terms in the b_i

and rearranging, we have

$$\sum_{\mathbf{n}}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,Ub\,|\,\mathbf{m}';\mathbf{n}\rangle = \sum_{\mathbf{n}}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,U(A-S)^{-1}Ra\,|\,\mathbf{m}';\mathbf{n}\rangle, \tag{9.B.11}$$

and hence

$$\begin{aligned}\sum_{\mathbf{n}}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,aU\,|\,\mathbf{m}';\mathbf{n}\rangle &= \sum_{\mathbf{n}}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,U(Pa+Qb)\,|\,\mathbf{m}';\mathbf{n}\rangle \\ &= \sum_{\mathbf{n}}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,UMa\,|\,\mathbf{m}';\mathbf{n}\rangle, \end{aligned}\tag{9.B.12}$$

where $M = P + Q(A-S)^{-1}R$. Thus we have $aX = XMa$ as required.

In the fermionic case, we define the operator X by

$$\langle \mathbf{m};\mathbf{n}\,|\,X\,|\,\mathbf{m}';\mathbf{n}'\rangle = \mathcal{N}_{\mathrm{f}}\delta_{\mathbf{n}\mathbf{n}'}\sum_{\mathbf{n}'}(-1)^{n'}\langle \mathbf{m};\widetilde{\mathbf{n}}'\,|\,U\,|\,\mathbf{m}';\mathbf{n}'\rangle, \tag{9.B.13}$$

where $\mathcal{N}_{\mathrm{f}}$ is chosen to ensure that $\langle \mathbf{0}\,|\,X\,|\,\mathbf{0}\rangle = 1$. The factor of $(-1)^{n'}$ is necessary in order to obtain agreement with the canonical theory. To see this, note that the first step in (9.B.10) is not valid in the fermionic case, due to the anticommutation relations satisfied by the a_i and b_i and the definition (9.B.1). Instead, the corresponding result is

$$\sum_{\mathbf{n}}(-1)^{n(m+m')}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,Ub\,|\,\mathbf{m}';\mathbf{n}\rangle = \sum_{\mathbf{n}}(-1)^{n(m+m')}\langle \mathbf{m};\widetilde{\mathbf{n}}\,|\,(A^\dagger b)U\,|\,\mathbf{m}';\mathbf{n}\rangle, \tag{9.B.14}$$

in which the factors of $(-1)^m$ and $(-1)^{m'}$ arise from anticommuting b_i past the string of creation operators for $|\,\mathbf{m}\rangle$ and $|\,\mathbf{m}'\rangle$ respectively. We may replace $(-1)^{n(m+m')}$ by $(-1)^n$ because U preserves the total particle number and therefore the summands can be nonzero only when $m' = m+1$.

Exactly analogous arguments to those for the bosonic case then show that Eq. (9.B.9) holds, and that X is unitary. Thus we have obtained agreement with the canonical theory.

The constant $\mathcal{N}_{\mathrm{f}}$ is easily evaluated once it has been expressed in the form

$$\mathcal{N}_{\mathrm{f}}^{-1} = \langle \mathbf{0};\mathbf{0}\,|\left[\bigwedge^{s_2}(\mathbb{1}-A^{\dagger}S)b_{s_2}\dots b_1\right] b_1^{\dagger}\dots b_{s_2}^{\dagger}\,|\,\mathbf{0};\mathbf{0}\rangle, \tag{9.B.15}$$

for then one may use the exterior algebra definition of the determinant to conclude that

$$\mathcal{N}_{\mathrm{f}} = \left[\det(\mathbb{1}-A^{\dagger}S)\right]^{-1}. \tag{9.B.16}$$

To establish Eq. (9.B.15), we write its RHS as $\mathcal{N}^{-1}$ and expand the exterior power to obtain

$$\mathcal{N}^{-1} = \sum_{\mathbf{n}} \langle \mathbf{0};\mathbf{0}\,|\,(-1)^n c_{s_2}^{(n_{s_2})}\dots c_1^{(n_1)} b_1^{\dagger}\dots b_{s_2}^{\dagger}\,|\,\mathbf{0};\mathbf{0}\rangle, \tag{9.B.17}$$

where $c_i^{(n_i)}$ is defined to be equal to b_i if $n_i = 0$ or $(A^{\dagger}Sb)_i$ if $n_i = 1$. Next, move the leftmost $c_i^{(n_i)}$ with $n_i = 0$ rightwards using the anticommutation relations until it sits next to $b_i^{\dagger}$, at which point the $b_i b_i^{\dagger}$ combination may be removed by a further application of the CAR's. Repeating the process until all $c_i^{(0)}$'s have been removed, one eventually finds

$$\mathcal{N}^{-1} = \sum_{\mathbf{n}} (-1)^n \langle \mathbf{0};\mathbf{0}\,|\,(A^{\dagger}Sb)_i^{n_i}\,|\,\mathbf{0};\mathbf{n}\rangle, \tag{9.B.18}$$

which is easily shown to be equal to $\sum_{\mathbf{n}}(-1)^n\langle \mathbf{0};\widetilde{\mathbf{n}}\,|\,U\,|\,\mathbf{0};\mathbf{n}\rangle = \mathcal{N}_{\mathrm{f}}^{-1}$, thus verifying our claim.

9.C Violation of CCR/CAR's in the 3-Point Model

In this appendix we present the details of the calculation leading to Eqs. (9.5.21) and (9.5.28) and the statements made at the end of Sect. 9.8. We consider the 1-parameter family of operator orderings labelled by $\alpha \in [0,1]$ discussed in Sect. 9.8 for which

$$\Psi(t) = e^{-i\lambda T\alpha\Psi(0)^{\dagger}\Psi(0)}\left(e^{-iWT}\Psi(0)\right)e^{-i\lambda T(1-\alpha)\Psi(0)^{\dagger}\Psi(0)}, \tag{9.C.1}$$

and consider Bose and Fermi statistics simultaneously, seeking solutions of the form $b = f(d^\dagger d, c^\dagger c)c$ where

$$\begin{pmatrix} c \\ d \end{pmatrix} = U_R \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \text{with} \qquad U_R = \frac{1}{\|R\|}\begin{pmatrix} R_1 & R_2 \\ -\overline{R}_2 & \overline{R}_1 \end{pmatrix} \tag{9.C.2}$$

obey the same commutation relations as a_1 and a_2. We remark that the norm of R can be regarded as the operator norm of a linear map or as the $\mathbb{C}^2$ norm of a vector quantity (as they coincide).

Applying b to elements of form $(d^\dagger)^m c^\dagger \,|\, 0\rangle$, we obtain the consistency requirement (from $b(T) = b(0)$):

$$f(m,0) = (\|R\| + Sf(m,0))\, e^{-i\lambda T[m+(1-\alpha)(1+|f(m,0)|^2)]}, \tag{9.C.3}$$

and applying b to elements of form $(d^\dagger)^m (c^\dagger)^{n+2} |\, 0\rangle$ for $m, n \geq 0$, we obtain the recursion relation

$$f(m,n+1) = (\|R\| + Sf(m,n+1))\, e^{-i\lambda T[m+\alpha(n+1)(1+|f(m,n)|^2)+(1-\alpha)(n+2)(1+|f(m,n+1)|^2)]}. \tag{9.C.4}$$

These equations have solutions. To see this take the norm of both sides of Eq. (9.C.4). We find that we may write

$$f(m,n+1) = \frac{\overline{S} + e^{i\theta_{n+1}}}{\|R\|}. \tag{9.C.5}$$

If we now substitute back we have

$$e^{i\theta_{n+1}} = \frac{1 + Se^{i\theta_{n+1}}}{1 + \overline{S}e^{-i\theta_{n+1}}} e^{-i\lambda T[m+\alpha(n+1)(1+|f(m,n)|^2)+(1-\alpha)(n+2)(1+|f(m,n+1)|^2)]} \tag{9.C.6}$$

Now notice that

$$\mathrm{Re}\left(\frac{1 + Se^{i\theta}}{1 + \overline{S}e^{-i\theta}}\right) \geq 0 \tag{9.C.7}$$

because $|S| < 1$ and hence this map from the circle to the circle is of Brouwer degree zero. So too is the mapping that gives the phase in Eq. (9.C.4). Hence there exists a solution to (9.C.4) as any map from the circle to the circle of Brouwer degree zero has a fixed point.

We now compute the (anti)commutation quantities

$$(m_{\mathrm{f}})_{ij} = \langle 0 |\, (a_i(T)a_j(T) + a_j(T)a_i(T))\, d^\dagger c^\dagger \,| 0 \rangle \qquad \text{for Fermions,} \tag{9.C.8}$$

and

$$(m_{\mathrm{b}})_{ij} = \langle 0 |\, (a_i(T)a_j(T) - a_j(T)a_i(T))\, d^\dagger c^\dagger \,| 0 \rangle \qquad \text{for Bosons.} \tag{9.C.9}$$

We shall define the matrices

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9.C.10}$$

and use the unitarity of e^{-iWT} to deduce

$$PR^\dagger = -\overline{S}Q, \tag{9.C.11}$$

$$P^T\overline{Q} = -\overline{S}R^T \tag{9.C.12}$$

together with $\|Q\| = \|R\|$. We will also need to define the unitary matrices

$$U_R = \frac{1}{\|R\|}\begin{pmatrix} R \\ -\overline{R}J \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} e^{-i\lambda T|f(0,0)|^2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{9.C.13}$$

The vector space of antisymmetric matrices is one dimensional and spanned by the matrix J, this immediately leads to the results $AJA^T = (\det A)J$ and $A - A^T = -\mathrm{Tr}\,(AJ)J$ for any 2×2 complex matrix A. Geometrically J is a rotation by $\pi/2$ and therefore for any vector $x \in \mathbb{C}^2$ we have $x^TJx = 0$. As a special case of the first relation we find $U_RJ = J\overline{U_R}$.

We will need to compute quantities of the form

$$\begin{aligned} m &= e^{-i\omega}\langle 0\,|\,u^T a\, e^{-i\lambda T c^\dagger |f(d^\dagger d, c^\dagger c)|^2 c}\, v^T a\, d^\dagger c^\dagger\,|\,0\rangle \\ &= e^{-i\omega}\langle 0\,|\,u^T U_R^\dagger U U_R a\, v^T a\, d^\dagger c^\dagger\,|\,0\rangle \end{aligned} \tag{9.C.14}$$

with $\omega = \lambda T[1+(1-\alpha)(|f(1,0)|^2+2)]$. For Fermions we find this is

$$m = -e^{-i\omega} u^T U_R^\dagger U U_R J v = -e^{-i\omega} u^T U_R^\dagger U J \overline{U_R} v \tag{9.C.15}$$

whilst for Bosons we have

$$m = e^{-i\omega} u^T U_R^\dagger U K \overline{U_R} v. \tag{9.C.16}$$

Let us define M by

$$\langle 0\,|\,M a = \langle 0\,|\,(Pa + Qb) = \langle 0\,|\left(P + \frac{QRf(0,0)}{\|R\|}\right) a \tag{9.C.17}$$

Using the unitarity of e^{-iWT}, i.e., Eq. (9.C.11) we see

$$M = \left(\mathbb{1} - s_0 Q Q^\dagger\right) P, \qquad s_0 = \frac{f(0,0)}{\overline{S}\|R\|}. \tag{9.C.18}$$

Similarly, let N be defined by

$$N = \left(\mathbb{1} - s_1 Q Q^\dagger\right) P, \qquad s_1 = \frac{f(1,0)}{\overline{S}\|R\|}. \tag{9.C.19}$$

It will be useful to evaluate the quantity we shall call $P_R = P U_R^\dagger$:

$$P_R = \frac{1}{\|R\|}\left(\, P R^\dagger \quad P J R^T \,\right). \tag{9.C.20}$$

Now observe that $PJR^T = -PJP^T\overline{Q}/\overline{S} = -\det P\, J\overline{Q}/\overline{S}$. This calculation together with Eq. (9.C.11) gives

$$P_R = \frac{-1}{\|R\|} \left(\overline{S} Q \quad \det P \, J \overline{Q} / \overline{S} \right). \tag{9.C.21}$$

Fermionic case: We are now in a position to evaluate the matrix $m_{\rm f}$.

$$\begin{aligned} u^T m_{\rm f} v &= e^{-i\omega} \langle 0 | \, u^T M a \, e^{-i\lambda T c^\dagger |f(d^\dagger d, c^\dagger c)|^2 c} \, v^T N a \, d^\dagger c^\dagger \, | 0 \rangle \\ &\qquad + e^{-i\omega} \langle 0 | \, v^T M a \, e^{-i\lambda T c^\dagger |f(d^\dagger d, c^\dagger c)|^2 c} \, u^T N a \, d^\dagger c^\dagger \, | 0 \rangle \\ &= -u^T e^{-i\omega} \left[\left(\mathbb{1} - s_0 Q Q^\dagger \right) P_R U J P_R^T \left(\mathbb{1} - s_1 \overline{Q} Q^T \right) \right. \\ &\qquad \left. - \left(\mathbb{1} - s_1 Q Q^\dagger \right) P_R J U P_R^T \left(\mathbb{1} - s_0 \overline{Q} Q^T \right) \right] v. \end{aligned} \tag{9.C.22}$$

It is useful to calculate $W = P_R U J P_R^T$.

$$\begin{aligned} W &= \frac{1}{\|R\|^2} \left(\overline{S} Q \quad \det P \, J \overline{Q} / \overline{S} \right) \begin{pmatrix} \det P \, e^{-i\lambda T |f(0,0)|^2} Q^\dagger J / \overline{S} \\ \overline{S} Q^T \end{pmatrix} \\ &= \frac{\det P}{\|R\|^2} \left(Q Q^\dagger J e^{-i\lambda T |f(0,0)|^2} + J \overline{Q} Q^T \right). \end{aligned} \tag{9.C.23}$$

Note that

$$W \overline{Q} Q^T = \det P \, J \overline{Q} Q^T \overline{Q} Q^T = \det P \, J \overline{Q} Q^T, \tag{9.C.24}$$

$$Q Q^\dagger W = \det P \, e^{-i\lambda T |f(0,0)|^2} Q Q^\dagger J, \tag{9.C.25}$$

$$Q Q^\dagger W \overline{Q} Q^T = 0. \tag{9.C.26}$$

The matrix $m_{\rm f}$ is given by

$$m_{\rm f} = -e^{-i\omega} \left[W + W^T - s_0 \left(Q Q^\dagger W + W^T \overline{Q} Q^T \right) - s_1 \left(\overline{Q} Q^T W + W^T Q Q^\dagger \right) \right]. \tag{9.C.27}$$

Finally we find

$$
\begin{aligned}
m_{\mathrm{f}} &= -\det P\, e^{-i\omega} F \left(QQ^\dagger J - J\overline{Q}Q^T\right) \\
&= -\det P\, e^{-i\omega} F \begin{pmatrix} 2Q_1\overline{Q}_2 & |Q_2|^2 - |Q_1|^2 \\ |Q_2|^2 - |Q_1|^2 & -2Q_2\overline{Q}_1 \end{pmatrix}
\end{aligned}
\tag{9.C.28}
$$

where we have defined

$$
F = \frac{e^{-i\lambda T|f(0,0)|^2} - 1}{\|R\|^2} + \frac{f(1,0) - f(0,0)e^{-i\lambda T|f(0,0)|^2}}{\overline{S}\|R\|}. \tag{9.C.29}
$$

Bosonic case: We now present the analogous calculation for the Bosonic system. The matrix m_{b} is clearly antisymmetric and therefore $m_{\mathrm{b}} = -\mathrm{Tr}\,(m_{\mathrm{b}}J)\,J$. We have

$$
\begin{aligned}
m_{\mathrm{b}} &= e^{-i\omega}\left[\left(\mathbb{1} - s_0 QQ^\dagger\right) P_R U K P_R^T \left(\mathbb{1} - s_1\overline{Q}Q^T\right)\right. \\
&\qquad \left. - \left(\mathbb{1} - s_1 QQ^\dagger\right) P_R K U P_R^T \left(\mathbb{1} - s_0\overline{Q}Q^T\right)\right].
\end{aligned}
\tag{9.C.30}
$$

Let $X = P_R U K P_R^T$, so that

$$
\begin{aligned}
X &= \frac{1}{\|R\|^2}\left(\,\overline{S}Q \quad \det P\, J\overline{Q}/\overline{S}\,\right) \begin{pmatrix} -\det P\, e^{-i\lambda T|f(0,0)|^2} Q^\dagger J/\overline{S} \\ \overline{S}Q^T \end{pmatrix} \\
&= \frac{1}{\|R\|^2}\det P\left(J\overline{Q}Q^T - QQ^\dagger J e^{-i\lambda T|f(0,0)|^2}\right).
\end{aligned}
\tag{9.C.31}
$$

Therefore,

$$
QQ^\dagger X = -\det P\, e^{-i\lambda T|f(0,0)|^2} QQ^\dagger J, \tag{9.C.32}
$$

$$
X\overline{Q}Q^T = \det P\, J\overline{Q}Q^T, \tag{9.C.33}
$$

$$
QQ^\dagger X\overline{Q}Q^T = 0. \tag{9.C.34}
$$

Now

$$\begin{aligned} m_{\rm b} &= e^{-i\omega}\left[X - X^T - s_0\left(QQ^\dagger X - X^T\overline{Q}Q\right) - s_1\left(X\overline{Q}Q^T - QQ^\dagger X^T\right)\right] \\ &= -e^{-i\omega}\,\mathrm{Tr}\left(XJ - s_0 QQ^\dagger XJ - s_1 X\overline{Q}Q^T J\right)J. \end{aligned} \tag{9.C.35}$$

Evaluating the traces:

$$\mathrm{Tr}\,(XJ) = \det P\left(e^{-i\lambda T|f(0,0)|^2} - 1\right), \tag{9.C.36}$$

$$\mathrm{Tr}\,\left(QQ^\dagger XJ\right) = \det P\,\|R\|^2 e^{-i\lambda T|f(0,0)|^2}, \tag{9.C.37}$$

$$\mathrm{Tr}\,\left(X\overline{Q}Q^T J\right) = -\det P\,\|R\|^2. \tag{9.C.38}$$

Thus we find

$$m_{\rm b} = -e^{-i\omega}\det P\,F\|R\|^2 J, \tag{9.C.39}$$

where F is given by Eq. (9.C.29).

Finally, one should also check that the expression for F, i.e., Eq. (9.C.29) does not vanish. For $\lambda T \ll 1$, one may prove this by perturbing about the free solution to obtain $f(0,0)$ and $f(1,0)$ to second order in λT if $S \notin \mathbb{R}$. If S is real, one needs to go to third order.

9.D Estimates on the Poisson Distribution

In this appendix we place some estimates on the Poisson distribution introduced in Sect. 9.7. We shall prove that in the limit we are interested in (corresponding to the classical limit $\hbar \to 0$) contributions coming from all but a central band around the mean of the distribution have a vanishingly small effect in the classical limit.

To start with we write the probability distribution as P_n and use Stirling's approximation:

$$P_n = \frac{e^{-\chi^2}\chi^{2n}}{n!} \le \frac{1}{\sqrt{2\pi}}\,\frac{\chi^{2n}e^{n-\chi^2}}{n^{n+1/2}}. \tag{9.D.1}$$

We set $n = \chi^2 + \gamma\chi$ with $\gamma > 0$ to obtain

$$P_n \leq \frac{1}{\sqrt{2\pi}} \frac{(1+\gamma/\chi)^{-(\chi^2+\gamma\chi)} e^{\gamma\chi}}{\chi\sqrt{1+\gamma/\chi}}, \tag{9.D.2}$$

where we restrict γ to run from $\alpha > 1$ to $\chi/2$. Using the inequality

$$\log(1+\gamma/\chi) \geq \gamma/\chi - \gamma^2/2\chi^2, \tag{9.D.3}$$

leads to

$$-(\chi^2+\gamma\chi)\log(1+\gamma/\chi) \leq -\gamma\chi - \gamma^2/2 + \gamma^3/2\chi \leq -\gamma\chi - \gamma^2/4. \tag{9.D.4}$$

thus,

$$P_n \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\gamma^2/4}}{\chi\sqrt{1+\gamma/\chi}} \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\gamma/4}}{\chi}. \tag{9.D.5}$$

Hence

$$\sum_{\gamma\chi=\alpha\chi}^{\chi^2/2-1} P_n \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\alpha/4}}{\chi\,(1-e^{-1/4\chi})} \leq \frac{1}{3}\sqrt{\frac{8}{\pi}}\, e^{-\alpha/4}. \qquad \text{when} \quad \chi > \frac{1}{2}. \tag{9.D.6}$$

Importantly this result is independent of χ. Next we examine the behaviour of P_n for $n \geq \chi^2/2$ and show that the probability can be made arbitrarily small be taking χ sufficiently large enough. We set $n = 3\chi^2/2 + m$, then

$$\begin{aligned} P_n &= \frac{e^{-\chi^2}\left(\chi^2\right)^{3\chi^2/2}}{(3\chi^2/2)!}\left(\frac{\chi^2}{3\chi^2/2+1}\right)\cdots\left(\frac{\chi^2}{3\chi^2/2+m}\right) \\ &\leq \frac{e^{-\chi^2}\left(\chi^2\right)^{3\chi^2/2}}{(3\chi^2/2)!}\left(\frac{2}{3}\right)^m, \end{aligned} \tag{9.D.7}$$

and hence,

$$\sum_{m=0}^{\infty} P_{3\chi^2/2+m} \leq \frac{3e^{-\chi^2}\chi^{3\chi^2}}{(3\chi^2/2)!}. \tag{9.D.8}$$

Notice that as $\chi \to \infty$ the RHS of Eq. (9.D.8) tends to zero.

So far we have shown that by choosing χ and α sufficiently large enough we can make the Prob$(N > \chi^2 + \alpha\chi)$ arbitrarily small (where $\alpha > 1$ is independent of χ). We now proceed to prove the same is true for Prob$(N < \chi^2 - \alpha\chi)$. Firstly we consider $n = \chi^2 - \gamma\chi$ for $\alpha \leq \gamma \leq 1/2$. We have

$$P_n \leq \frac{1}{\sqrt{2\pi}} \frac{(1-\gamma/\chi)^{-(\chi^2-\gamma\chi)} e^{-\gamma\chi}}{\chi\sqrt{1-\gamma/\chi}}. \tag{9.D.9}$$

Now we use the inequality

$$-\log(1-\gamma/\chi) \leq \gamma/\chi + \gamma^2/2\chi^2 + \gamma^3(8\log 2 - 5)/\chi^3 \tag{9.D.10}$$

for $0 \leq \gamma/\chi \leq 1/2$. This inequality comes from noticing that the LHS of Eq. (9.D.10) is increasing and by considering the power series expansion. Evaluating

$$-(\chi^2 - \gamma\chi)\log(1-\gamma/\chi) \leq \gamma\chi - \gamma^2/2 + (16\log 2 - 11)\gamma^3/2\chi \leq \gamma\chi - \beta\gamma^2 \tag{9.D.11}$$

where $\beta = (13 - 16\log 2)/4 > 0$. Therefore,

$$P_n \leq \frac{1}{\sqrt{\pi}} \frac{e^{-\beta\gamma^2}}{\chi} \leq \frac{1}{\sqrt{\pi}} \frac{e^{-\beta\gamma}}{\chi}. \tag{9.D.12}$$

and hence

$$\sum_{\gamma\chi=\alpha\chi}^{\chi^2/2} P_n \leq \frac{1}{\sqrt{\pi}} \frac{e^{-\beta\alpha}}{\chi(1-e^{-\beta/\chi})} \leq \frac{4}{3\sqrt{\pi}} \frac{e^{-\beta\alpha}}{\beta} \qquad \text{when } \chi > 2\beta. \tag{9.D.13}$$

Again this is independent of χ, provided for instance $\chi > 1$. We now need to take care of the remaining terms where $0 \leq n < \chi^2/2$. As P_n is increasing on this range we have

$$P_n \leq \frac{e^{-\chi^2}\chi^{\chi^2}}{(\chi^2/2)!}. \tag{9.D.14}$$

Therefore

$$\sum_{n=0}^{\chi^2/2-1} P_n \leq \frac{e^{-\chi^2}\chi^{\chi^2+2}}{2(\chi^2/2)!}. \tag{9.D.15}$$

Notice that the RHS tends to zero as $\chi \to \infty$. In summary then, given $\epsilon > 0$ we can find $\alpha > 1$ such that

$$\max\left\{\frac{1}{3}\sqrt{\frac{8}{\pi}}\, e^{-\alpha/4}, \frac{4}{3\sqrt{\pi}}\frac{e^{-\beta\alpha}}{\beta}\right\} < \frac{\epsilon}{4}, \tag{9.D.16}$$

then for all $\chi > \max\{\chi_0, 1\}$ where

$$\max\left\{\frac{3e^{-\chi_0^2}\chi_0^{3\chi_0^2}}{(3\chi_0^2/2)!}, \frac{e^{-\chi_0^2}\chi_0^{\chi_0^2+2}}{2(\chi_0^2/2)!}\right\} < \frac{\epsilon}{4} \tag{9.D.17}$$

we will have $\mathrm{Prob}(N \notin [\chi^2 - \alpha\chi, \chi^2 + \alpha\chi]) < \epsilon$, proving our result.

9.E Analysis of the Iteration Sequence

We present here the detailed analysis of the statement made in Sect. 9.7 concerning the validity of replacing one sequence of eigenvalues defined by Eq. (9.7.8) by those defined by Eq. (9.7.14). To summarize rigorously our approach we state and prove the following theorem.

Theorem. *Suppose we have an sequence of complex values defined by* $w_n = f_x(w_{n-1}; n)$ *where*

$$f_x(z; n) = \exp\left\{i\mu\left(1 - \frac{\alpha}{\chi} + \frac{n}{\chi^2}\right)\left(1 + \frac{R^2}{|z - S|^2}\right)\right\} \tag{9.E.1}$$

and a further iterative sequence $z_n = f(z_{n-1})$ *with*

$$f(z) = \exp\left\{i\mu\left(1 + \frac{R^2}{|z - S|^2}\right)\right\}. \tag{9.E.2}$$

Furthermore, let us suppose $w_0 = z_0$*. If* z_n *tends to a stable limit cycle* $\{Z_0, \ldots, Z_{N-1}\}$ *as* $n \to \infty$*, then given* $\epsilon > 0$*, there exists* X *such that for all* $\chi > X$*,*

$$|z_n - w_n| < \epsilon \qquad \text{for all } n < 2\alpha\chi. \tag{9.E.3}$$

Proof: We must make some bounds on the difference between the iterative schemes. It will be necessary to consider the N-th iteratives of f_x and f. As the limit cycle consists of some attractor fixed points of f^N. To start with we shall need an estimate of how much of an error we make by using f rather than f_x with the same starting point, therefore calculate

$$\begin{aligned} |f_x(z) - f(z)| &= 2\sin\left\{\frac{\mu}{2}\left(\frac{n}{\chi^2} - \frac{\alpha}{\chi}\right)\left(1 + \frac{R^2}{|z-S|^2}\right)\right\} \\ &\le \frac{2\alpha\mu}{1-|S|}\frac{1}{\chi} = \frac{A}{\chi}. \end{aligned} \tag{9.E.4}$$

We have made use of the condition $n < 2\alpha\chi$ in deriving Eq. (9.E.4). Let us now bound the derivative of f_x with respect to z. Notice that f_x is not analytic, and the derivative is that in the tangential direction on the circle. We find that these derivatives are uniformly bounded:

$$\begin{aligned} |f_x'(z)| \le \left|\left(\frac{-2\mu S R^2}{|z-S|^4}\right)\left(1 - \frac{\alpha}{\chi} + \frac{n}{\chi^2}\right)\right| &\le \frac{2\mu\left(1+|S|\right)|S|}{\left(1-|S|\right)^3}\left(1 + \frac{\alpha}{\chi}\right) \\ &\le \frac{8\mu}{\left(1-|S|\right)^3} = B, \end{aligned} \tag{9.E.5}$$

assuming $\chi > \alpha$. We may now make use of the complex mean value theorem to show

$$\begin{aligned} |f_x^n(z) - f^n(z)| &\le B\left|f_x^{n-1}(z) - f^{n-1}(z)\right| + \frac{A}{\chi} \\ &\le \frac{A(1-B^n)}{1-B}\frac{1}{\chi}. \end{aligned} \tag{9.E.6}$$

It is now expedient to introduce the N-th iterates: $F(z) = f^N(z)$ and $F_x(z) = f_x^N(z)$.

We will need to examine $F'_x(z)$. Notice that

$$F'_x(z) = \prod_{k=0}^{N-1} f'_x\left(f_x^k(z)\right). \tag{9.E.7}$$

Let us now introduce $h = 1/\chi$. It will be useful to differentiate $F'_x(z)$ with respect to h:

$$\begin{aligned} \left|\frac{\partial F'_x(z)}{\partial h}\right| &\leq \frac{2N\mu B^{N-1}\alpha|S|(1+|S|)}{(1-|S|)^3}\left(1+\frac{4\mu}{1-|S|}\right) \\ &\leq \alpha N B^N\left(1+\frac{4\mu}{1-|S|}\right) = C. \end{aligned} \tag{9.E.8}$$

We know that since $\{Z_0, \ldots, Z_{N-1}\}$ is a limit cycle that for each $k = 0, \ldots, N-1$ we have $F^R(z_k) \to Z_k$ as $R \to \infty$. We also know that the derivative $F'(Z_k)$ (actually independent of k) is less than unity in magnitude because the cycle is an attractor. Let us now define $K = [1 + |F'(Z_k)|]/2$, require

$$\chi > \frac{2C}{1-K} \tag{9.E.9}$$

and let Δ be the intersection of the unit circle with the disc around Z_k of radius $\delta = \min\{\delta_0, \epsilon/2\}$, where for all z with $|z - Z_k| < \delta_0$ we have $|F'(z)| < K$. It now follows by an application of the mean value theorem (regarding $F'_x(z)$ as a function of h) that for any $c \in \Delta$,

$$|F'_x(c)| \leq |F'(c)| + \frac{C}{\chi} < K + \frac{C}{\chi} < \frac{1+K}{2} = K' < 1. \tag{9.E.10}$$

As a consequence we find this entails

$$|F_x(v_1) - F_x(v_2)| \leq K'\,|v_1 - v_2| \quad \text{for all } v_1, v_2 \in \Delta. \tag{9.E.11}$$

We are now in a position to prove the result. It is necessary to use the fact that

$F^R(z_k) \to Z_k$ as $R \to \infty$, so there exists R_0 such that for all $R > R_0 - 1$

$$|z_{NR+k} - Z_k| < \delta(1 - K')/6. \tag{9.E.12}$$

we shall choose χ sufficiently large so that

$$|z_n - w_n| < \delta(1 - K')/6 \qquad \text{for } R \le R_0 + 1 \tag{9.E.13}$$

and $n = NR + k$. We can achieve Eq. (9.E.13) by letting

$$\chi > \frac{6A\left(1 - B^{(R_0+2)N}\right)}{\delta(1 - B)(1 - K')} \tag{9.E.14}$$

and using Eq. (9.E.6). Next we inductively assume each w_{NR+k} is in Δ for all $R \ge R_0 - 1$. The conditions above with $R = R_0 - 1$ implies that the induction starts, since

$$\begin{aligned} \left|w_{N(R_0-1)+k} - Z_k\right| &\le \left|w_{N(R_0-1)+k} - z_{N(R_0-1)+k}\right| + \left|z_{N(R_0-1)+k} - Z_k\right| \\ &\le \delta(1 - K')/3 < \delta, \end{aligned} \tag{9.E.15}$$

and therefore $w_{N(R_0-1)+k} \in \Delta$. Next we notice that Eq. (9.E.11) entails

$$|w_{n+N} - w_n| \le K' \, |w_n - w_{n-N}| \qquad \text{for} \qquad n > NR_0. \tag{9.E.16}$$

As a consequence,

$$\begin{aligned} |w_{NR+k} - w_{n-N}| &< \frac{1}{1 - K'} \, |w_n - w_{n-N}| \\ &< \frac{1}{1 - K'} \left(|w_n - z_n| + |z_n - Z_k| + |Z_k - z_{n-N}| + |z_{n-N} - w_{n-N}|\right) \\ &< 2\delta/3, \end{aligned} \tag{9.E.17}$$

we have set $n = NR_0 + k$ in the above. We may now verify the inductive hypothesis:

$$\begin{aligned} |w_{NR+k} - Z_k| &\leq |w_{NR+k} - w_{n-N}| + |w_{n-N} - z_{n-N}| + |z_{n-N} - Z_k| \\ &< 2\delta/3 + \delta(1-K')/3 < \delta, \end{aligned} \tag{9.E.18}$$

i.e., $w_{NR+k} \in \Delta$. Finally we establish

$$|w_{NR+k} - z_{NR+k}| \leq |w_{NR+k} - Z_k| + |z_{NR+k} - Z_k| < \epsilon. \tag{9.E.19}$$

Therefore, for all $n < 2\alpha\chi$ we have found that, provided χ is sufficiently large,

$$|z_n - w_n| < \epsilon. \tag{9.E.20}$$

10. CHRONOLOGY VIOLATION IN A MASSLESS THIRRING MODEL

Chronology violating spacetimes provide us with an arena whereby we may compare different formalisms of quantum mechanics. As we have seen in Chap. 9 path integral methods and operator differential equation approaches yield inequivalent evolutions. In order to study the quantum theory further we will be looking at a two dimensional lattice spacetime model with a massless two component spinor field obeying a Thirring-type interaction. In contrast to the continuum model of this system, we shall not be working from a Lagrangian, but rather postulate the field equations directly. Naturally the Thirring model has a well-defined Lagrangian, but this does not easily translate to the lattice model we discuss here.

We shall be trying to answer a few questions that arise from the QIVP of Chap. 9. It is interesting to determine to what extent, if any, the results of the self-consistent path integral and QIVP formalisms differ in a concrete model. Furthermore, we shall investigate the question of particle creation in this model. It has been argued [70] that one finds an infinite amount of particle creation when CTC's form, thereby rendering doubtful the underlying assumptions taken in the calculation. One should probably regard such a result as an inconsistency of the quantum CTC model. It will also be of interest to see how closely the lattice spacetime model discussed here resembles its continuum limit. In the previous chapter we considered some rather simplified spacetimes, consisting of the Cartesian product of a finite number of points, representing space, and a line with suitable identifications representing time and the CTC's.

The structure of the chapter is as follows. In the next section we describe the 'Baby Thirring Model', a simplified version Thirring's model [93, 94, 95, 96] that we shall be working with. In Sect. 10.2 we explain how the system of CTC's effectively acts as a permutation on the incoming particles. Sect. 10.3 is devoted to presenting the unitary S-matrix for the system even in the presence of the chronology violating region. In Sect. 10.4 we go through the computation of the evolution operator in the partial trace approach to the self-consistent path integral, and find it coincides with that found directly from elementary considerations within the QIVP approach. We begin to discuss the issue of particle production in Sect. 10.5 completing the calculation for the non-interacting lattice model in Sect. 10.6, with the operator ordering we will impose, the interaction plays no rôle in the calculation. Finally, in Sect. 10.7, we compare our results with the analogous computation for the continuum limit of the model we have been discussing. We discover the same ultraviolet divergence as for the lattice model and we draw our conclusions.

10.1 The Baby Thirring Model

We shall work on a lattice spacetime $\mathbb{Z}^2$ and implement a chronology violating region by identifying $n+1$ points labelled $-n/2 \ldots n/2$ with points a distance m away vertically (i.e., in the time direction). If n is even the lattice points take integer coordinates, but if n is odd we work with a lattice comprising integer plus one half values. Denoting $t_0 = n/2$ and $t_1 = m + n/2$ we make the identification such that the points with coordinates $(x,t) = (s, t_0^-)$ are identified with (s, t_1^+) and the identification of (s, t_0^+) with (s, t_1^-) where $-n/2 \leq s \leq n/2$. Later we will be investigating the continuum Thirring Model directly. The purpose of this somewhat curious labelling of the lattice is so that the lightlike coordinates $t+x$ and $t-x$ intersect the boundary of the CTC identifications, as illustrated in Fig. 10.1.

The approach we adopt is to assume the quantum theory is described as a QIVP. This means that we shall not need to concern ourselves with any renormalization. Given an initial data set we propagate that solution according to the evolution presented below. As we have not dealt with problems of renormalization care must be taken in regarding this theory as the standard Thirring model applied to a lattice. This is not a problem though as our results are illustrative of the techniques of the QIVP formalism and not of the Thirring model itself. So what we are actually considering is a ‘Baby Thirring

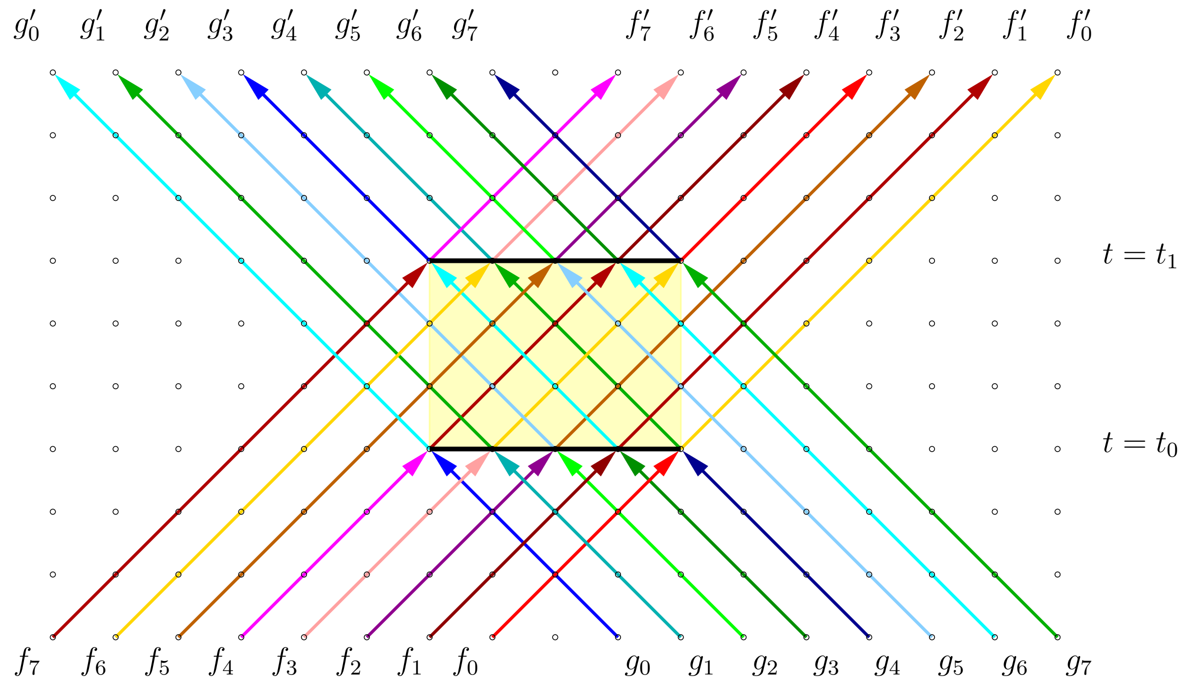

Fig. 10.1: *Diagram of Time Machine identifications on a Lattice Model Spacetime with parameters* $(n, m) = (4, 3)$.

Model' simplified to illustrate the important features due to the presence of CTC's and not the complexities of renormalization.

The fundamental quantum field is a two-component massless spinor $\psi = (\psi_1, \psi_2)$. We shall impose the following evolution rule:

$$\begin{aligned} \psi_1(x, t+1) &= \psi_1(x-1, t) \exp i\lambda \left(|\psi_2(x,t)|^2 + |\psi_2(x+1,t)|^2\right) \\ \psi_2(x, t+1) &= \psi_2(x+1, t) \exp i\lambda \left(|\psi_1(x,t)|^2 + |\psi_1(x-1,t)|^2\right) \end{aligned} \qquad (10.1.1)$$

This rule has a nice graphical interpretation: ψ_1 particles move rightwards (at the 'speed of light') picking up a phase from every ψ_2 they encounter, whilst ψ_2 particles move leftwards picking up phases from encounters with ψ_1's.

The general solution to these equations is

$$\begin{aligned} \psi_1(x,t) &= f(v)\exp i\lambda \sum_{-\infty}^{u} |g(u')|^2 \\ \psi_2(x,t) &= g(u)\exp i\lambda \sum_{-\infty}^{v} |f(v')|^2 \end{aligned} \qquad (10.1.2)$$

where f and g are arbitrary functions, and $u = t + x$, $v = t - x$.

10.2 The Permutation of Rays

As shown in Fig. 10.1, rays travelling in our model spacetime fall into four classes, those that never intersect the time machine, those that are pushed forward by the time machine (e.g., $f_0, \ldots, f_4$) and two classes which travel backwards in time, and loop around a number of times, depending on the precise ray this can happen q times (e.g., f_5) or $q+1$ times (f_6, f_7), where q is some integer. The time machine acts as a permutation on the incoming rays. In order to make this permutation explicit we define a quotient and remainder as follows:

$$n = qm + R, \qquad R \in [0, m-1]. \qquad (10.2.1)$$

Define $\sigma : s \mapsto s'$ by

$$\sigma(s) = s' = \begin{cases} s + (q+1)m & s \in [0, R] \\ s + qm & s \in [R+1, m-1] \\ s - m & s \in [m, n+m] \\ s & \text{otherwise.} \end{cases} \qquad (10.2.2)$$

The time machine permutation of the right-moving rays is then given by

$$f'(s) = \begin{cases} f(s')\, e^{i\lambda(Q_L+M_0)} & s \in [0, R] \\ f(s')\, e^{i\lambda(Q_L+M_1)} & s \in [R+1, m-1] \\ f(s')\, e^{i\lambda(Q_L+M_2)} & s \in [m, n+m] \\ f(s')\, e^{i\lambda Q_L} & \text{otherwise.} \end{cases} \tag{10.2.3}$$

We have set

$$M_0 = -\sum_{r=0}^{n} :g^\dagger(r)g(r): + (2q+1)\sum_{r=n+1}^{(q+1)m-1} :g^\dagger(r)g(r): + 2(q+1)\sum_{r=(q+1)m}^{n+m} :g^\dagger(r)g(r): \, , \tag{10.2.4}$$

$$M_1 = -\sum_{r=0}^{n} :g^\dagger(r)g(r): + 2q\sum_{r=n+1}^{(q+1)m-1} :g^\dagger(r)g(r): + (2q+1)\sum_{r=(q+1)m}^{n+m} :g^\dagger(r)g(r): \tag{10.2.5}$$

and

$$M_2 = -\sum_{r=n+1}^{n+m} :g^\dagger(r)g(r): \, . \tag{10.2.6}$$

For completeness, the corresponding result for the left-moving particles is seen to be

$$g'(s) = \begin{cases} g(s')\, e^{i\lambda(Q_R+L_0)} & s \in [0, R] \\ g(s')\, e^{i\lambda(Q_R+L_1)} & s \in [R+1, m-1] \\ g(s')\, e^{i\lambda(Q_R+L_2)} & s \in [m, n+m] \\ g(s')\, e^{i\lambda Q_R} & \text{otherwise,} \end{cases} \tag{10.2.7}$$

where now,

$$L_0 = -\sum_{r=0}^{n} :f^\dagger(r)f(r): + (2q+1)\sum_{r=n+1}^{(q+1)m-1} :f^\dagger(r)f(r): + 2(q+1)\sum_{r=(q+1)m}^{n+m} :f^\dagger(r)f(r): \, , \tag{10.2.8}$$

$$L_1 = -\sum_{r=0}^{n} :f^\dagger(r)f(r): + 2q\sum_{r=n+1}^{(q+1)m-1} :f^\dagger(r)f(r): + (2q+1)\sum_{r=(q+1)m}^{n+m} :f^\dagger(r)f(r): \tag{10.2.9}$$

and

$$L_2 = -\sum_{r=n+1}^{n+m} :f^\dagger(r)f(r): \, . \tag{10.2.10}$$

10.3 The S-Matrix

In contrast to the 3-point model we discussed in Chap. 9, our lattice Thirring model has a well-defined S-matrix. This is easily calculated and is given by

$$S = Ve^{i\lambda W}, \tag{10.3.1}$$

where

$$\begin{aligned} W = Q_L Q_R - \sum_{\substack{0\le s\le n \\ n+1\le t\le n+m}} \left[:f^\dagger(s)f(s)g^\dagger(t)g(t): + :g^\dagger(s)g(s)f^\dagger(t)f(t): \right] \\ + 2q\sum_{\substack{n+1\le s\le n+m \\ n+1\le t\le n+m}} :f^\dagger(s)f(s)g^\dagger(t)g(t): \\ + \sum_{\substack{(q+1)m\le s\le n+m \\ n+1\le t\le n+m}} \left[:f^\dagger(s)f(s)g^\dagger(t)g(t): + :g^\dagger(s)g(s)f^\dagger(t)f(t): \right], \end{aligned} \tag{10.3.2}$$

V implements the free case: $V^\dagger f(s)V = f(s')$ and

$$Q_R = \sum_{-\infty}^{\infty} :f(s)^\dagger f(s): , \qquad Q_L = \sum_{-\infty}^{\infty} :g(s)^\dagger g(s): \tag{10.3.3}$$

are in the absence of the time machine the conserved left and right charges. The important point to notice is that even in the presence of the time machine the far past to far future evolution is unitary.

10.4 Comparison with the Path Integral

We may compare the results of the QIVP formalism to that of the *self-consistent path integral*. In order to calculate the evolution operator between t_0^- and t_1^+ we will need to work out the evolution between these times when there are no identifications. We note that

$$\psi_1(x, t_0) = f(n/2 - x) \exp i\lambda \sum_{s=-\infty}^{n/2+x} :g^\dagger(s) g(s): , \quad (10.4.1)$$

$$\psi_2(x, t_0) = g(n/2 + x) \exp i\lambda \sum_{s=-\infty}^{n/2-x} :f^\dagger(s) f(s): . \quad (10.4.2)$$

The evolved fields are

$$\psi_1(x, t_1) = f(n/2 + m - x) \exp i\lambda \sum_{s=-\infty}^{n/2+m+x} :g^\dagger(s) g(s): , \quad (10.4.3)$$

$$\psi_2(x, t_1) = g(n/2 + m + x) \exp i\lambda \sum_{s=-\infty}^{n/2+m-x} :f^\dagger(s) f(s): . \quad (10.4.4)$$

Now define a Fock space at $t = t_0$ using annihilation operators $a_i(x) = \psi_i(x, t_0)$. We write the basis vectors as $|\,\mathbf{m}, \mathbf{n}\rangle$, where $\mathbf{m}$ describes the a_1 degrees of freedom, and $\mathbf{n}$ those of a_2. We therefore have

$$\psi_1(x, t_1) = a_1(x - m) \exp i\lambda \sum_{s=n/2+1-m}^{n/2+m} :g^\dagger(x + s) g(x + s): , \quad (10.4.5)$$

$$\psi_2(x, t_1) = a_2(x + m) \exp i\lambda \sum_{s=n/2-m}^{n/2-1+m} :f^\dagger(x + s) f(x + s): . \quad (10.4.6)$$

This evolution is implemented by the unitary propagator

$$U = T \exp i\lambda \sum_{x=-\infty}^{\infty} \left\{ a_1^\dagger(x) a_1(x) \sum_{s=1}^{2m} a_2^\dagger(x + s) a_2(x + s) \right\} \quad (10.4.7)$$

so that $\psi(x,t_1) = U^\dagger \psi(x,t_0) U$. Here T is a translation operator with the action

$$T^\dagger a_1(x) T = a_1(x-m); \qquad T^\dagger a_2(x) T = a_2(x+m). \tag{10.4.8}$$

The matrix elements of U are therefore given by

$$\begin{aligned} \langle \mathbf{m};\mathbf{n}|\,U\,|\mathbf{m}';\mathbf{n}'\rangle &= \langle \mathbf{m};\mathbf{n}|\,T\,|\mathbf{m}';\mathbf{n}'\rangle \exp i\lambda \sum_{i=\infty}^{\infty}\left\{ m'_i \sum_{j=1}^{2m} n'_{i+j}\right\} \\ &= \left(\prod_{k=-\infty}^{\infty} \delta_{m_k m'_{k-m}} \delta_{n'_k n_{k-m}}\right) \exp i\lambda \sum_{i=\infty}^{\infty}\left\{ m'_i \sum_{j=1}^{2m} n'_{i+j}\right\}. \end{aligned} \tag{10.4.9}$$

Next we trace over those states that obey the CTC boundary condition $a_i(t_0) = a_i(t_1)$:

$$\mathrm{Tr}\,\langle \mathbf{m};\mathbf{n}|\,U\,|\mathbf{m}';\mathbf{n}'\rangle = \sum_{n'_{-n/2}} \sum_{m'_{-n/2}} \sum_{n'_{-n/2+1}} \sum_{m'_{-n/2+1}} \cdots \sum_{n'_{n/2}} \sum_{m'_{n/2}} \left(\prod_{i=-n/2}^{n/2} \delta_{m_i m'_i}\delta_{n_i n'_i}\right)\left(\prod_{k=-\infty}^{\infty} \delta_{m_k m'_{k-m}} \delta_{n'_k n_{k-m}}\right) \exp i\lambda \sum_{i=\infty}^{\infty}\left\{ m'_i \sum_{j=1}^{2m} n'_{i+j}\right\}. \tag{10.4.10}$$

This has the effect of setting

$$m'_i = m_{[(i+n/2) \bmod m]-m-n/2} \qquad -n/2 \le i \le m+n/2, \tag{10.4.11}$$

$$n'_i = n_{m+n/2-[(n/2-i) \bmod m]} \qquad -m-n/2 \le i \le n/2. \tag{10.4.12}$$

The final step is to implement the boundary conditions $a_i(t_0^-) = a_i(t_1^+)$. Thus we find

$$\langle \mathbf{m};\mathbf{n}\,|\,X\,|\,\mathbf{m}';\mathbf{n}'\rangle = \left(\prod_{i=-n/2}^{m+n/2} \delta_{m_{\alpha(i)}m'_i}\delta_{n_{\beta(i)}n'_i}\right)\left(\prod_{k\notin[-n/2,m+n/2]} \delta_{m_{k+m}m'_k}\delta_{n'_k n_{k-m}}\right) \times \exp i\lambda \sum_{i=-\infty}^{\infty}\left\{ m'_i \sum_{j=1}^{2m} n'_{i+j}\right\}, \quad (10.4.13)$$

where we have defined

$$\alpha(x) = m + n/2 - \sigma^{-1}(n/2 - x), \quad (10.4.14)$$

$$\beta(x) = \sigma^{-1}(x + n/2) - n/2 - m \quad (10.4.15)$$

and the values of m'_i and n'_i are given by Eqs. (10.4.11) and (10.4.12) in the appropriate ranges when one evaluates the sum. This result coincides with the QIVP formulation. To evaluate these matrix elements in this formalism we use the graphical form of the evolution rule. That is to say we trace the paths of particles passing through the time machine adjusting its phase by a factor dependent on those rays it crosses. Put another way, the evolution Eq. (10.4.13) is a solution to the operator valued equations of motion. This contrasts with what we found in Chap. 9 when we discussed the 2- and 3-point models.

10.5 The Field Expansion

So far we have concerned ourselves with the question of whether not our model has a unitary quantum theory and the comparison of the QIVP approach to the self-consistent path integral formalism. We now go on to look at another question: that of possible particle creation due to the CTC identifications we have made. To this end we begin with the IN and OUT fields described in accordance with the standard decomposition,

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ish} \left(a_{_{\rm IN}}(h)\theta(h) + b^{\dagger}_{_{\rm IN}}(-h)\theta(-h) \right) \, dh \, ; \tag{10.5.1}$$

$$a_{_{\rm IN}}(k)\theta(k) + b^{\dagger}_{_{\rm IN}}(-k)\theta(-k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{isk} f(s); \tag{10.5.2}$$

$$g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ish} \left(a_{_{\rm IN}}(-h)\theta(h) + b^{\dagger}_{_{\rm IN}}(h)\theta(-h) \right) \, dh \, ; \tag{10.5.3}$$

$$a_{_{\rm IN}}(k)\theta(-k) + b^{\dagger}_{_{\rm IN}}(-k)\theta(k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-isk} g(s); \tag{10.5.4}$$

$$a_{_{\rm IN}}(-k)\theta(k) + b^{\dagger}_{_{\rm IN}}(k)\theta(-k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{isk} g(s); \tag{10.5.5}$$

$$a_{_{\rm IN}}(-k)\theta(-k) + b^{\dagger}_{_{\rm IN}}(k)\theta(k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-isk} f(s) \ . \tag{10.5.6}$$

The $a^{\dagger}_{_{\rm IN}}(k)$ operator creating a quantum of momentum k in the IN state. The $b^{\dagger}_{_{\rm IN}}(k)$ operator creates the corresponding anti-particle. For right-movers, $k > 0$,

$$a_{_{\rm OUT}}(k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{isk} f'(s) \tag{10.5.7}$$

$$\begin{aligned} &= a_{_{\rm IN}}(k) e^{i\lambda Q_L} + \frac{1}{\sqrt{2\pi}} \left\{ \sum_{s=0}^{R} e^{isk} \left(f(s+(q+1)m) e^{i\lambda M_0} - f(s) \right) \right. \\ &\quad + \sum_{s=0}^{m-R-2} e^{i(s+R+1)k} \left(f(s+n+1) e^{i\lambda M_1} - f(s+R+1) \right) \\ &\quad \left. + \sum_{s=0}^{n} e^{i(s+m)k} \left(f(s) e^{i\lambda M_2} - f(s+m) \right) \right\} e^{i\lambda Q_L} \end{aligned} \tag{10.5.8}$$

and the analogous statement for left-movers:

$$a_{_{\rm OUT}}(-k) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{isk} g'(s) \tag{10.5.9}$$

$$= a_{\text{IN}}(-k)e^{i\lambda Q_R} + \frac{1}{\sqrt{2\pi}}\left\{\sum_{s=0}^{R} e^{isk}\left(g(s+(q+1)m)e^{i\lambda L_0} - g(s)\right)\right.$$

$$+\sum_{s=0}^{m-R-2} e^{i(s+R+1)k}\left(g(s+n+1)e^{i\lambda L_1} - g(s+R+1)\right)$$

$$\left. + \sum_{s=0}^{n} e^{i(s+m)k}\left(g(s)e^{i\lambda L_2} - g(s+m)\right)\right\} e^{i\lambda Q_R}. \qquad (10.5.10)$$

In this equation (and those subsequent) it may be that $R = m - 1$ in which case the summation from 0 to $m - R - 2$ is defined to be zero.

We may expand the corresponding OUT annihilation and creation operators in terms of the IN state, for $k > 0$,

$$a_{\text{OUT}}(k)e^{-i\lambda Q_L}|0\rangle = \frac{1}{2\pi}\int_0^{\pi}\left\{\sum_{s=0}^{R} e^{isk}\left(e^{i(s+(q+1)m)h} - e^{ish}\right)\right.$$

$$+\sum_{s=0}^{m-R-2} e^{i(s+R+1)k}\left(e^{i(s+n+1)h} - e^{i(s+R+1)h}\right)$$

$$\left.+\sum_{s=0}^{n} e^{i(s+m)k}\left(e^{ish} - e^{i(s+m)h}\right)\right\} dh\; b^{\dagger}_{\text{IN}}(h)|0\rangle \qquad (10.5.11)$$

and the left-moving particles obey

$$a_{\text{OUT}}(-k)e^{-i\lambda Q_R}|0\rangle = \frac{1}{2\pi}\int_0^{\pi}\left\{\sum_{s=0}^{R} e^{isk}\left(e^{i(s+(q+1)m)h} - e^{ish}\right)\right.$$

$$+\sum_{s=0}^{m-R-2} e^{i(s+R+1)k}\left(e^{i(s+n+1)h} - e^{i(s+R+1)h}\right)$$

$$\left.+\sum_{s=0}^{n} e^{i(s+m)k}\left(e^{ish} - e^{i(s+m)h}\right)\right\} dh\; b^{\dagger}_{\text{IN}}(-h)|0\rangle. \qquad (10.5.12)$$

It is useful for us to define the following function:

$$
\begin{aligned}
G(h,k) \;=\; & \sum_{s=0}^{R} e^{isk}\left(e^{i(s+(q+1)m)h} - e^{ish}\right) \\
& + \sum_{s=0}^{m-R-2} e^{i(s+R+1)k}\left(e^{i(s+n+1)h} - e^{i(s+R+1)h}\right) \\
& + \sum_{s=0}^{n} e^{i(s+m)k}\left(e^{ish} - e^{i(s+m)h}\right) \;. \qquad (10.5.13)
\end{aligned}
$$

Then the expected total particle number of $a_{\rm OUT}$-particles produced from the vacuum IN-state, i.e., the particle creation due to the system of CTC's is given by

$$
N \;=\; \int_{-\pi}^{\pi} \langle 0 \,|\, a^\dagger_{\rm OUT}(k) a_{\rm OUT}(k) \,|\, 0\rangle \, dk \,. \qquad (10.5.14)
$$

$$
\;=\; \frac{1}{2\pi^2}\int_0^{\pi} dk \int_0^{\pi} dh \,|G(h,k)|^2 \;. \qquad (10.5.15)
$$

In the next section we will perform this calculation.

10.6 Non Interacting Lattice Model Calculation

To calculate N, the total particle number, we have initially to find the modulus squared of $G(h,k)$. To do this it is advantageous to define new variables,

$$
\rho = m(h+k), \qquad \rho\theta = mh, \qquad R+1 = mr \qquad (10.6.1)
$$

and

$$
\sigma = m(2\pi - h - k), \qquad \sigma\phi = m(\pi - h). \qquad (10.6.2)
$$

After performing the sum over s in Eq. (10.5.13), we may use the reciprocity relation: $G(h,k) = G(k,h)$ to expand the exponentials as

$$4\sin^2\frac{\rho}{2m}\,|G(h,k)|^2 = I_1(\rho,\theta) + I_1(\rho,1-\theta). \tag{10.6.3}$$

We will also need the result

$$4\sin^2\frac{\sigma}{2m}\,|G(\pi-h,\pi-k)|^2 = I_2(\sigma,\phi) + I_2(\sigma,1-\phi) \tag{10.6.4}$$

where we have defined

$$\begin{aligned} I_1(\rho,\theta) &= 2\cos(\theta-r)\rho - 4\cos\theta\rho + 2\cos(\theta+q+r)\rho \\ &+ 2\cos((q+2)\theta-1)\rho - 2\cos((q+2)\theta+r-1)\rho + 2\cos(q\theta+1)\rho - 2\cos(q\theta+r)\rho \\ &+ 2\cos((q+1)\theta+r)\rho - 4\cos(q+1)\theta\rho + 2\cos((q+1)\theta+r-1)\rho \\ &+ 4 - \cos(1-r)\rho - \cos r\rho - \cos(q+r)\rho - \cos(q+r+1)\rho \end{aligned} \tag{10.6.5}$$

and

$$\begin{aligned} I_2(\sigma,\phi) &= (-1)^m\,[2\cos(\phi-r)\sigma - 4\cos\phi\sigma + 2\cos(\phi+q+r)\sigma] \\ &+ (-1)^{qm}\,[2\cos((q+2)\phi-1)\sigma - 2\cos((q+2)\phi+r-1)\sigma \\ &\qquad + 2\cos(q\phi+1)\sigma - 2\cos(q\phi+r)\sigma] \\ &+ (-1)^{(q+1)m}\,[2\cos((q+1)\phi+r)\sigma - 4\cos(q+1)\phi\sigma + 2\cos((q+1)\phi+r-1)\sigma] \\ &+ 4 - \cos(1-r)\sigma - \cos r\sigma - \cos(q+r)\sigma - \cos(q+r+1)\sigma\ . \end{aligned} \tag{10.6.6}$$

The region of integration in the (h,k) space is divided into two triangular regions with $0 \le \rho \le m\pi$, $0 \le \theta \le 1$ and $0 \le \sigma \le m\pi$, $0 \le \phi \le 1$. Writing ρ and θ as integration variables in place of σ and ϕ in the second of these integrals one finds that the expression for N is then given by

$$N = \frac{1}{2\pi^2 m^2} \int_0^{m\pi} \frac{\rho\, d\rho}{\sin^2(\rho/2m)} J(\rho) \tag{10.6.7}$$

where we define

$$J(\rho) = \frac{1}{2} \int_0^1 d\theta \, \left(I_1(\rho,\theta) + I_2(\rho,\theta) \right). \tag{10.6.8}$$

Evaluating the integral $J(\rho)$, we end up with ($q \neq 0$),

$$\begin{aligned} J(\rho) = {} & \frac{(1+(-1)^m)}{\rho} \left[\sin(1-r)\rho + \sin r\rho - 2\sin\rho + \sin(q+r+1)\rho - \sin(q+r)\rho\right] \\ & + \frac{(1+(-1)^{qm})}{\rho} \left[\frac{\sin(q+1)\rho}{q+2} + \frac{\sin\rho}{q+2} - \frac{\sin(q+r+1)\rho}{q+2} - \frac{\sin(1-r)\rho}{q+2} \right. \\ & \qquad \left. + \frac{\sin(q+1)\rho}{q} - \frac{\sin\rho}{q} - \frac{\sin(q+r)\rho}{q} + \frac{\sin r\rho}{q} \right] \\ & + \frac{\left(1+(-1)^{(q+1)m}\right)}{\rho} \left[\frac{\sin(q+r+1)\rho}{q+1} - \frac{\sin r\rho}{q+1} \right. \\ & \qquad \left. - 2\frac{\sin(q+1)\rho}{q+1} + \frac{\sin(q+r)\rho}{q+1} + \frac{\sin(1-r)\rho}{q+1} \right] \\ & + 4 - \cos(1-r)\rho - \cos r\rho - \cos(q+r)\rho - \cos(q+r+1)\rho\,. \end{aligned} \tag{10.6.9}$$

Using $\sin x \leq x$ for $x \geq 0$, we may put a lower bound on N:

$$N \geq \frac{1}{2\pi^2} \int_0^{m\pi} \frac{d\rho}{\rho} J(\rho)\,. \tag{10.6.10}$$

We shall define the coefficients α_i, β_i and γ_j by

$$J(\rho) = \sum_i \beta_i \frac{\sin\alpha_i\rho - \alpha_i\rho}{\rho} + \sum_j (1 - \cos\gamma_j\rho) \tag{10.6.11}$$

where we have used the fact that $\sum \alpha_i\beta_i = 0$. This integral is seen to be convergent.

After an integration by parts we find

$$N \geq \frac{1}{2\pi^2} \int_0^{m\pi} \left(\sum_i \alpha_i \beta_i \frac{\cos \alpha_i \rho - 1}{\rho} + \sum_j \frac{1 - \cos \gamma_j \rho}{\rho} \right) d\rho \ . \tag{10.6.12}$$

The integral may now be expressed in terms of the Cosine Integral. We use the result that (see, for instance, Arfken [97])

$$\int_0^x \frac{1 - \cos t}{t} \, dt = \gamma + \log x - Ci(x) \ . \tag{10.6.13}$$

The quantity γ is the Euler-Mascheroni constant, and $Ci(x)$ is the Cosine integral defined by

$$Ci(x) = -\int_x^\infty \frac{\cos t \, dt}{t} \ . \tag{10.6.14}$$

Finally we have the result

$$N \geq \frac{1}{2\pi^2} \left(-\sum_{\alpha_i \neq 0} \beta_i \left[\alpha_i \log \alpha_i + Ci(\alpha_i m\pi) \right] \right.$$
$$\left. + 4\gamma + 4 \log m + \sum_{\gamma_j \neq 0} \left[\log \gamma_j \pi - Ci(\gamma_j m\pi) \right] \right) \ . \tag{10.6.15}$$

The right hand side diverges as $(2 \log m)/\pi^2$ as m becomes large. This demonstrates that as we approach the continuum limit we should expect an ultraviolet divergence in the total particle creation. This is significant as it implies that the stress-energy tensor will also diverge (if we only knew that the total particle number were infinite, we might be saved from a physical catastrophe if the distribution of momenta was such that it was suitably skewed towards low momenta. We take the opportunity at the point to mention that the restriction that $q \neq 0$ is actually superfluous, the result holds also for $q = 0$. This may be checked explicitly or by taking a suitable limit as $q \to 0$ (neglecting at this point in the calculation that q is supposed to be integral).

10.7 The Continuum Massless Thirring Model

In this section we tackle the question of particle creation in the continuum version of the massless Thirring Model we have been working with. As we may notice from the lattice model it turns out that the interaction plays no part in the creation rate (with normal ordering). This would appear to be due to the very special nature of the interaction in Thirring's Model. Left-movers only interact with right-movers and *vice versa* the form of the interaction such that it brings about an altering of the phase of the particle concerned.

In an analogous manner to the discrete example we have just examined we define the time machine by making identifications of (s, t_0^-) with (s, t_1^+) and of (s, t_0^+) with (s, t_1^-) where $-n/2 \leq s \leq n/2$ where $t_0 = n/2$ and $t_1 = n/2 + m$. The quotient and remainder are defined by

$$n = (q + r)m \tag{10.7.1}$$

with q an integer and $r \in [0, 1)$. The permutation that the identifications induce on the right moving rays is given by

$$f'(s) = \begin{cases} f(s + (q+1)m)\, e^{i\lambda(Q_L + M_0)} & s \in [0, rm] \\ f(s + qm)\, e^{i\lambda(Q_L + M_1)} & s \in (rm, m) \\ f(s - m)\, e^{i\lambda(Q_L + M_2)} & s \in [m, n+m] \\ f(s)\, e^{i\lambda Q_L} & \text{otherwise.} \end{cases} \tag{10.7.2}$$

where

$$M_0 = -\int_0^n :g^\dagger(r)g(r):\, dr + (2q+1)\int_n^{(q+1)m} :g^\dagger(r)g(r):\, dr + 2(q+1)\int_{(q+1)m}^{n+m} :g^\dagger(r)g(r):\, dr, \tag{10.7.3}$$

$$M_1 = -\int_0^n :g^\dagger(r)g(r):\, dr + 2q\int_n^{(q+1)m} :g^\dagger(r)g(r):\, dr + (2q+1)\int_{(q+1)m}^{n+m} :g^\dagger(r)g(r):\, dr, \tag{10.7.4}$$

$$M_2 = -\int_n^{n+m} :g^\dagger(r)g(r):\, dr \tag{10.7.5}$$

and

$$Q_L = \int_{-\infty}^{\infty} :g^\dagger(r)g(r):\, dr. \tag{10.7.6}$$

The corresponding result for the left-moving particles is readily seen to be

$$g'(s) = \begin{cases} g(s+(q+1)m)\, e^{i\lambda(Q_R+L_0)} & s \in [0, rm] \\ g(s+qm)\, e^{i\lambda(Q_R+L_1)} & s \in (rm, m) \\ g(s-m)\, e^{i\lambda(Q_R+L_2)} & s \in [m, n+m] \\ g(s)\, e^{i\lambda Q_R} & \text{otherwise,} \end{cases} \tag{10.7.7}$$

where now,

$$L_0 = -\int_0^n :f^\dagger(r)f(r):\, dr + (2q+1)\int_n^{(q+1)m} :f^\dagger(r)f(r):\, dr + 2(q+1)\int_{(q+1)m}^{n+m} :f^\dagger(r)f(r):\, dr, \tag{10.7.8}$$

$$L_1 = -\int_0^n :f^\dagger(r)f(r):\, dr + 2q\int_n^{(q+1)m} :f^\dagger(r)f(r):\, dr + (2q+1)\int_{(q+1)m}^{n+m} :f^\dagger(r)f(r):\, dr, \tag{10.7.9}$$

$$L_2 = -\int_n^{n+m} :f^\dagger(r)f(r):\, dr \tag{10.7.10}$$

and

$$Q_R = \int_{-\infty}^{\infty} :f^\dagger(r)f(r):\, dr. \tag{10.7.11}$$

The S-Matrix for this system, $S = Ve^{i\lambda W}$ is then given by direct analogy of Eq. (10.3.2).

$$\begin{aligned} W = Q_L Q_R &- \int_0^n ds \int_n^{n+m} dt\, \left[:f^\dagger(s)f(s)g^\dagger(t)g(t): + :g^\dagger(s)g(s)f^\dagger(t)f(t):\right] \\ &+ 2q\int_n^{n+m} ds \int_n^{n+m} dt :f^\dagger(s)f(s)g^\dagger(t)g(t): \\ &+ \int_{(q+1)m}^{n+m} ds \int_n^{n+m} dt\, \left[:f^\dagger(s)f(s)g^\dagger(t)g(t): + :g^\dagger(s)g(s)f^\dagger(t)f(t):\right] \end{aligned} \tag{10.7.12}$$

and V again implements the free case: $V^\dagger f(s) V = f(s')$. We remark that this too is a unitary evolution from the IN region to the OUT region.

The f and g fields are expanded in terms of annihilation and creation operators by

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ish} \left(a_{\text{IN}}(h)\theta(h) + b^\dagger_{\text{IN}}(-h)\theta(-h)\right)\, dh\, ; \tag{10.7.13}$$

$$a_{\text{IN}}(k)\theta(k) + b^\dagger_{\text{IN}}(-k)\theta(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isk} f(s)\, ds\,; \tag{10.7.14}$$

$$g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ish} \left(a_{\text{IN}}(-h)\theta(h) + b^\dagger_{\text{IN}}(h)\theta(-h) \right) dh \tag{10.7.15}$$

and

$$a_{\text{IN}}(k)\theta(-k) + b^\dagger_{\text{IN}}(-k)\theta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isk} f(s)\, ds\,. \tag{10.7.16}$$

Proceeding as before, we form expressions for the right-movers in the OUT region, i.e., for $k > 0$,

$$a_{\text{OUT}}(k) = a_{\text{IN}}(k) + \frac{1}{\sqrt{2\pi}} \int_0^{rm} e^{isk} \left(f(s + (q+1)m) - f(s) \right) ds$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{rm}^{m} e^{isk} \left(f(s + qm) - f(s) \right) ds \,+ \frac{1}{\sqrt{2\pi}} \int_{m}^{n+m} e^{isk} \left(f(s - m) - f(s) \right) ds\,. \tag{10.7.17}$$

Similarly for the left-movers:

$$a_{\text{OUT}}(-k) = a_{\text{IN}}(-k) + \frac{1}{\sqrt{2\pi}} \int_0^{rm} e^{isk} \left(g(s + (q+1)m) - g(s) \right) ds$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{rm}^{m} e^{isk} \left(g(s + qm) - g(s) \right) ds \,+ \frac{1}{\sqrt{2\pi}} \int_{m}^{n+m} e^{isk} \left(g(s - m) - g(s) \right) ds\,. \tag{10.7.18}$$

Now define the function

$$G(h,k) \;=\; \int_0^{rm} \left(e^{i(q+1)mh} - 1 \right) e^{is(h+k)}\, ds \,+ \int_{rm}^{m} \left(e^{iqmh} - 1 \right) e^{is(h+k)}\, ds$$
$$+ \int_{m}^{n+m} \left(e^{-imh} - 1 \right) e^{is(h+k)}\, ds\,. \tag{10.7.19}$$

With this definition we have

$$a_{\text{OUT}}(k)\,|\,0\rangle = \frac{1}{2\pi}\int_0^\infty dh\, G(h,k) b^\dagger_{\text{IN}}(h)\,|\,0\rangle, \tag{10.7.20}$$

$$a_{\text{OUT}}(-k)\,|\,0\rangle = \frac{1}{2\pi}\int_0^\infty dh\, G(h,k) b^\dagger_{\text{IN}}(-h)\,|\,0\rangle \tag{10.7.21}$$

and

$$N = \frac{1}{2\pi^2}\int_0^\infty dh \int_0^\infty dk\, |G(h,k)|^2\,. \tag{10.7.22}$$

We now proceed to calculate N. We find that

$$\frac{\rho^2}{m^2}|G(h,k)|^2 = I_1(\rho,\theta) + I_1(\rho, 1-\theta), \tag{10.7.23}$$

using the definitions in Eqs. (10.6.1) and (10.6.5). Define

$$K(\rho) = \int_0^1 I_1(\rho,\theta)\, d\theta, \tag{10.7.24}$$

so that

$$\begin{aligned} K(\rho) = {} & \frac{2}{\rho}\left[\sin(1-r)\rho + \sin r\rho - 2\sin\rho + \sin(q+r+1)\rho - \sin(q+r)\rho\right] \\ & + \frac{2}{\rho}\left[\frac{\sin(q+1)\rho}{q+2} + \frac{\sin\rho}{q+2} - \frac{\sin(q+r+1)\rho}{q+2} - \frac{\sin(1-r)\rho}{q+2}\right. \\ & \left. + \frac{\sin(q+1)\rho}{q} - \frac{\sin\rho}{q} - \frac{\sin(q+r)\rho}{q} + \frac{\sin r\rho}{q}\right] \\ & + \frac{2}{\rho}\left[\frac{\sin(q+r+1)\rho}{q+1} - \frac{\sin r\rho}{q+1} - 2\frac{\sin(q+1)\rho}{q+1} + \frac{\sin(q+r)\rho}{q+1} + \frac{\sin(1-r)\rho}{q+1}\right] \\ & + 4 - \cos(1-r)\rho - \cos r\rho - \cos(q+r)\rho - \cos(q+r+1)\rho\,. \end{aligned} \tag{10.7.25}$$

As before we write this equation in terms of coefficients α_i, β_i and γ_j,

$$K(\rho) = \sum_i \beta_i \frac{\sin\alpha_i\rho - \alpha_i\rho}{\rho} + \sum_j (1-\cos\gamma_j\rho)\,. \tag{10.7.26}$$

The expected total particle number in the OUT region is then given by the $\Lambda \to \infty$ limit of

$$
\begin{aligned}
N &= \frac{1}{2\pi^2}\int_0^{\Lambda} \frac{d\rho}{\rho}\, K(\rho) &\quad (10.7.27)\\
&= \frac{1}{2\pi^2}\Bigg(-\sum_{\alpha_i \neq 0} \beta_i \left[\alpha_i \log \alpha_i + Ci(\alpha_i \Lambda)\right] \\
&\qquad + 4\gamma + 4\log\Lambda + \sum_{\gamma_j \neq 0} \left[\log \gamma_j \pi - Ci(\gamma_j \Lambda)\right]\Bigg) \\
&\sim \frac{2}{\pi^2}\log\Lambda. &\quad (10.7.28)
\end{aligned}
$$

Thus we find that the continuum system is actually the limit of the lattice model, at least in respect to the calculations of the S-Matrix and particle creation. This is reassuring, as we have studied lattice and point-space spacetimes in order to gain some insight into more realistic systems. For the continuum we have found the same ultraviolet divergence of the momentum distribution as for the lattice calculation. This behaviour lends support to Hawking's Chronology Protection Conjecture [70], and suggests that the back-reaction on the metric, and hence on the causal structure is important. One might hope that in a physical spacetime, that at the point when the spacetime is about to develop a causality violating region, the back-reaction would be sufficient to prevent this happening, or that an event horizon would form so that any attempt to probe the causality violating region would be unsuccessful.

APPENDIX

A. DIMENSIONAL REDUCTION FORMULAE

In this appendix we derive some useful formulae which we need periodically throughout the main text. The dimensional reduction from four to three dimensions in our discussion of internal symmetries and solution generating techniques, and from four to two for our analysis of the black hole uniqueness result we discussed together with the dimensional reduction from five to four dimensions when we looked at applying the techniques of Penrose, Sorkin and Woolgar [31] can all be examined in the general scenario of a reduction from D dimensions to n which we present here.

Our starting point is the D-dimensional metric

$$\boldsymbol{g}^{(D)} = e^{2\nu}\left(\mathbf{d}x^D + \boldsymbol{A}\right) \otimes \left(\mathbf{d}x^D + \boldsymbol{A}\right) + e^{2\chi}\eta_{ab}\boldsymbol{E}^a \otimes \boldsymbol{E}^b, \tag{A.1}$$

where $\partial/\partial x^D$ is a Killing vector, and $\boldsymbol{A}$ has no component in the x^D direction. The norm of the Killing vector, $e^{2\nu}$ is here written as expressly positive; this turns out to be an unnecessary restriction and our final formulae are valid under the substitution $\nu \mapsto \nu + i\pi/2$. Let us define $\underline{\boldsymbol{\omega}} = e^{\chi}\underline{\boldsymbol{E}}$ and $\boldsymbol{\omega}^D = e^{\nu}\boldsymbol{\beta}$ with $\boldsymbol{\beta} = \mathbf{d}x^D + \boldsymbol{A}$. Here, the vector $\underline{\boldsymbol{E}}$ is a vector of orthonormal 1-forms with respect to the $(D-1)$-dimensional metric η_{ab}, whilst $\{\boldsymbol{\omega}^D, \underline{\boldsymbol{\omega}}_i\}$ is an orthonormal basis with respect to the D-dimensional metric $\boldsymbol{g}^{(D)}$.

The torsion-free condition on the covariant derivative of the $(D-1)$-metric implies

$$D\underline{\boldsymbol{E}} = 0 \qquad \text{i.e.,} \qquad \mathbf{d}\underline{\boldsymbol{E}} + \Gamma \wedge \underline{\boldsymbol{E}} = 0. \tag{A.2}$$

Where Γ is the $SO(s, D-1-s)$ Lie algebra valued connection 1-form for the metric η_{ab}

of signature s. Similarly the torsion-free condition on the D-metric derivative implies:

$$\begin{pmatrix} \mathbf{d}\underline{\boldsymbol{\omega}} \\ \mathbf{d}\boldsymbol{\omega}^D \end{pmatrix} + \begin{pmatrix} \Gamma + B & \underline{c} \\ -\underline{c}^T & 0 \end{pmatrix} \wedge \begin{pmatrix} \underline{\boldsymbol{\omega}} \\ \boldsymbol{\omega}^D \end{pmatrix} = 0. \tag{A.3}$$

We now try to find B and $\underline{c}$ with $B^T = -B$. We have

$$\mathbf{d}\boldsymbol{\omega}^D = \mathbf{d}\nu \wedge \boldsymbol{\omega}^D + e^\nu \mathbf{d}\boldsymbol{A}. \tag{A.4}$$

Let us define $\boldsymbol{F} = \mathbf{d}\boldsymbol{A} = \frac{1}{2}\underline{\boldsymbol{E}}^T \wedge F\underline{\boldsymbol{E}}$ and set $\mathbf{d}\nu = \underline{v}^T\underline{\boldsymbol{E}}$. Then we may write

$$\mathbf{d}\boldsymbol{\omega}^D = \left(-\underline{v}^T\boldsymbol{\omega}^D + \frac{1}{2}e^\nu \underline{\boldsymbol{E}}^T F\right) \wedge \underline{\boldsymbol{E}}\ . \tag{A.5}$$

We also have, $\mathbf{d}\boldsymbol{\omega}^D = \underline{c}^T e^\chi \wedge \underline{\boldsymbol{E}}$. Therefore let

$$\underline{c}^T = -\underline{v}^T e^{-\chi}\boldsymbol{\omega}^D + \frac{1}{2}e^{\nu-\chi}\underline{\boldsymbol{E}}^T F, \tag{A.6}$$

i.e.,

$$\underline{c} = -\underline{v}\, e^{-\chi}\boldsymbol{\omega}^D - \frac{1}{2}e^{\nu-\chi}F\underline{\boldsymbol{E}}. \tag{A.7}$$

In addition one has from Eq. (A.3),

$$e^\chi \mathbf{d}\chi \wedge \underline{\boldsymbol{E}} + e^\chi B \wedge \underline{\boldsymbol{E}} + \underline{c} \wedge \boldsymbol{\omega}^D = 0. \tag{A.8}$$

It is useful to find to coefficients of $\mathbf{d}\chi$ with respect to the $(D-1)$-dimensional basis, $\underline{\boldsymbol{E}}$. Let us then define $\mathbf{d}\chi = \underline{X}^T\underline{\boldsymbol{E}}$,

$$\left(B - \underline{\boldsymbol{E}}\,\underline{X}^T + \underline{X}\,\underline{\boldsymbol{E}}^T + \frac{1}{2}e^{\nu-2\chi}\boldsymbol{\omega}^D F\right) \wedge \underline{\boldsymbol{E}} = 0. \tag{A.9}$$

Hence we may determine the matrix valued 1-form B:

$$B = \underline{\boldsymbol{E}}\,\underline{X}^T - \underline{X}\,\underline{\boldsymbol{E}}^T - \frac{1}{2}e^{\nu-2\chi}\boldsymbol{\omega}^D F. \tag{A.10}$$

We are now in a position to relate the curvature 2-forms of the respective metrics,

$$ {}^{D}\boldsymbol{\Omega} = \begin{pmatrix} {}^{D-1}\boldsymbol{\Omega} + DB + B\wedge B - \underline{c}\wedge\underline{c}^{T} & D\underline{c} + B\wedge\underline{c} \\ -D\underline{c}^{T} - \underline{c}^{T}\wedge B & 0 \end{pmatrix}. \tag{A.11} $$

We proceed to evaluate this in terms of the derivatives of B and c. Using

$$ \begin{aligned} DB = &-D\underline{X}\wedge\underline{\boldsymbol{E}}^{T} - \underline{\boldsymbol{E}}\wedge D\underline{X}^{T} - \frac{1}{2}D\left(e^{\nu-2\chi}F\right)\wedge\boldsymbol{\omega}^{D} \\ &+ \frac{1}{2}e^{\nu-2\chi}F\underline{v}^{T}\underline{\boldsymbol{\omega}}\wedge\underline{\boldsymbol{E}} - \frac{1}{4}e^{2\nu-2\chi}F\underline{\boldsymbol{E}}^{T}\wedge F\underline{\boldsymbol{E}} \end{aligned} \tag{A.12} $$

and

$$ \begin{aligned} B\wedge B = &\,\underline{X}\,\underline{\boldsymbol{E}}^{T}\wedge\underline{X}\,\underline{\boldsymbol{E}}^{T} + \underline{\boldsymbol{E}}\,\underline{X}^{T}\wedge\underline{\boldsymbol{E}}\,\underline{X}^{T} - \underline{\boldsymbol{E}}\,\underline{X}^{T}\underline{X}\wedge\underline{\boldsymbol{E}}^{T} + \frac{1}{2}e^{\nu-2\chi}\underline{X}\,\underline{\boldsymbol{E}}^{T}F\wedge\boldsymbol{\omega}^{D} \\ &- \frac{1}{2}e^{\nu-2\chi}\underline{\boldsymbol{E}}\,\underline{X}^{T}F\wedge\boldsymbol{\omega}^{D} - \frac{1}{2}e^{\nu-2\chi}F\underline{X}\,\underline{\boldsymbol{E}}^{T}\wedge\boldsymbol{\omega}^{D} + \frac{1}{2}e^{\nu-2\chi}F\underline{\boldsymbol{E}}\,\underline{X}^{T}\wedge\boldsymbol{\omega}^{D} \end{aligned} \tag{A.13} $$

together with the formulae

$$ \underline{c}\wedge\underline{c}^{T} = \frac{1}{2}e^{\nu-2\chi}\underline{v}\,\underline{\boldsymbol{E}}^{T}F\wedge\boldsymbol{\omega}^{D} + \frac{1}{2}e^{\nu-2\chi}F\underline{\boldsymbol{E}}\,v^{T}\wedge\boldsymbol{\omega}^{D} - \frac{1}{4}e^{2\nu-2\chi}F\underline{\boldsymbol{E}}\wedge\underline{\boldsymbol{E}}^{T}F \tag{A.14} $$

and,

$$ D\underline{c} = -D\left(e^{-\chi}\underline{v}\right)\wedge\boldsymbol{\omega}^{D} - \frac{1}{2}D\left(e^{\nu-\chi}F\right)\wedge\underline{\boldsymbol{E}} + \underline{v}\,\underline{v}^{T}e^{-\chi}\boldsymbol{\omega}^{D}\wedge\underline{\boldsymbol{E}} - \frac{1}{2}e^{\nu-\chi}\underline{v}\,\underline{\boldsymbol{E}}^{T}F\wedge\underline{\boldsymbol{E}}\ . \tag{A.15} $$

Finally we need to calculate the quantity

$$ \begin{aligned} B\wedge\underline{c} = &\, e^{-\chi}\underline{X}\,\underline{\boldsymbol{E}}^{T}\underline{v}\wedge\boldsymbol{\omega}^{D} + \frac{1}{2}e^{\nu-\chi}\underline{X}\,\underline{\boldsymbol{E}}^{T}\wedge F\underline{\boldsymbol{E}} - e^{-\chi}\underline{\boldsymbol{E}}\,\underline{X}^{T}\underline{v}\wedge\boldsymbol{\omega}^{D} \\ &- \frac{1}{2}e^{\nu-\chi}\underline{\boldsymbol{E}}\,\underline{X}^{T}F\underline{\boldsymbol{E}} - \frac{1}{4}e^{2\nu-3\chi}F^{2}\underline{\boldsymbol{E}}\wedge\boldsymbol{\omega}^{D}. \end{aligned} \tag{A.16} $$

We may now unravel the Riemann tensor, after some careful book-keeping, we find

$$\begin{aligned}
\frac{1}{2}\,{}^{D}R^{a}{}_{bAB}\boldsymbol{\omega}^{A}\wedge\boldsymbol{\omega}^{B} &= \frac{1}{2}\,{}^{D-1}R^{a}{}_{bcd}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} - \delta^{a}{}_{c}D_{d}D_{b}\chi\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} - \eta_{bd}D_{c}D^{a}\chi\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} \\
&- \frac{1}{2}D_{c}\left(e^{\nu-2\chi}F^{a}{}_{b}\right)\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} + \frac{1}{2}e^{\nu-2\chi}F^{a}{}_{b}D_{d}\chi\boldsymbol{\omega}^{D}\wedge\boldsymbol{E}^{d} - \frac{1}{4}e^{2\nu-2\chi}F^{a}{}_{b}F_{cd}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} \\
&+ \delta^{a}{}_{c}D_{b}\chi D_{d}\chi\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} - \delta^{a}{}_{c}\eta_{bd}\left(D\chi\right)^{2}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} - \frac{1}{2}e^{\nu-2\chi}\delta^{a}{}_{c}D_{d}\chi F^{d}{}_{b}\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} \\
&+ \eta_{bd}D^{a}\chi D_{c}\chi\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} + \frac{1}{2}e^{\nu-2\chi}D^{a}\chi F_{cb}\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} - \frac{1}{2}e^{\nu-2\chi}D_{b}\chi F^{a}{}_{d}\boldsymbol{\omega}^{D}\wedge\boldsymbol{E}^{d} \\
&+ \frac{1}{2}e^{\nu-2\chi}F^{ac}D_{c}\chi\eta_{bd}\boldsymbol{\omega}^{D}\wedge\boldsymbol{E}^{d} + \frac{1}{4}e^{2\nu-2\chi}F^{a}{}_{c}F_{db}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} \\
&- \frac{1}{2}e^{\nu-2\chi}F_{cb}D^{a}\nu\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} - \frac{1}{2}e^{\nu-2\chi}F^{a}{}_{c}D_{b}\nu\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D}, \qquad \text{(A.17)}
\end{aligned}$$

and also

$$\begin{aligned}
\frac{1}{2}{}^{D}R^{a}{}_{DAB}\boldsymbol{\omega}^{A}\wedge\boldsymbol{\omega}^{B} &= -D_{c}\left(e^{-\chi}D^{a}\nu\right)\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} + e^{-\chi}D^{a}\nu D_{d}\nu\boldsymbol{\omega}^{D}\wedge\boldsymbol{E}^{d} \\
&- \frac{1}{2}e^{\nu-\chi}D^{a}\nu F_{cd}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} - \frac{1}{2}D_{c}\left(e^{\nu-\chi}F^{a}{}_{d}\right)\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} \\
&- e^{-\chi}\delta^{a}{}_{c}D_{d}\chi D^{d}\nu\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} - \frac{1}{2}e^{\nu-\chi}\delta^{a}{}_{c}D_{b}\chi F^{b}{}_{d}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d} + e^{-\chi}D^{a}\chi D_{c}\nu\boldsymbol{E}^{c}\wedge\boldsymbol{\omega}^{D} \\
&+ \frac{1}{4}e^{2\nu-3\chi}F^{a}{}_{b}F^{b}{}_{d}\boldsymbol{\omega}^{D}\wedge\boldsymbol{E}^{d} + \frac{1}{2}e^{\nu-\chi}D^{a}\chi F_{cd}\boldsymbol{E}^{c}\wedge\boldsymbol{E}^{d}. \qquad \text{(A.18)}
\end{aligned}$$

Let us now pick out the components of the Riemann tensor (in the $\boldsymbol{\beta}, \boldsymbol{E}^{a}$ coordinate system), we find

$$
\begin{aligned}
{}^{D}R^{a}{}_{bcd} = e^{-2\chi}\Big[& {}^{D-1}R^{a}{}_{bcd} - \delta^{a}{}_{c}\nabla_{d}\nabla_{b}\chi + \delta^{a}{}_{d}\nabla_{c}\nabla_{b}\chi - \eta_{bd}\nabla_{c}\nabla^{a}\chi + \eta_{bc}\nabla_{d}\nabla^{a}\chi \\
& + \delta^{a}{}_{c}\nabla_{b}\chi\nabla_{d}\chi - \delta^{a}{}_{d}\nabla_{b}\chi\nabla_{c}\chi - \delta^{a}{}_{c}\eta_{bd}\left(\nabla\chi\right)^{2} + \delta^{a}{}_{d}\eta_{bc}\left(\nabla\chi\right)^{2} \\
& + \eta_{bd}\nabla^{a}\chi\nabla_{c}\chi - \eta_{bc}\nabla^{a}\chi\nabla_{d}\chi - \frac{1}{2}e^{2\nu-2\chi}F^{a}{}_{b}F_{cd} \\
& + \frac{1}{4}e^{2\nu-2\chi}F^{a}{}_{c}F_{db} - \frac{1}{4}e^{2\nu-2\chi}F^{a}{}_{d}F_{cb}\Big] \,,
\end{aligned}
\tag{A.19}
$$

with

$$
\begin{aligned}
{}^{D}R^{a}{}_{bDd} = {} & e^{\nu-2\chi}\nabla_{d}\left(e^{\nu-\chi}F^{a}{}_{b}\right) + e^{2\nu-3\chi}\delta^{a}{}_{d}\nabla_{c}\chi F^{c}{}_{b} - e^{2\nu-3\chi}\nabla^{a}\chi F_{db} - e^{2\nu-3\chi}\nabla_{b}\chi F^{a}{}_{d} \\
& + e^{2\nu-3\chi}\eta_{bd}\nabla_{c}\chi F^{ac} + e^{2\nu-3\chi}\nabla^{a}\nu F_{db} + e^{2\nu-3\chi}\nabla_{b}\nu F^{a}{}_{d} \,,
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
{}^{D}R^{a}{}_{Dcd} = {} & e^{2\nu-3\chi}\nabla^{a}\chi F_{cd} - \frac{1}{2}e^{\nu-2\chi}\nabla_{c}\left(e^{\nu-\chi}F^{a}{}_{d}\right) + \frac{1}{2}e^{\nu-2\chi}\nabla_{d}\left(e^{\nu-\chi}F^{a}{}_{c}\right) \\
& - \frac{1}{2}e^{2\nu-3\chi}\delta^{a}{}_{c}\nabla_{b}\chi F^{b}{}_{d} + \frac{1}{2}e^{2\nu-3\chi}\delta^{a}{}_{d}\nabla_{b}\chi F^{b}{}_{c} - e^{2\nu-3\chi}\nabla^{a}\nu F_{cd} \,.
\end{aligned}
\tag{A.21}
$$

and

$$
\begin{aligned}
{}^{D}R^{a}{}_{DcD} = {} & -e^{2\nu-\chi}\nabla_{c}\left(e^{-\chi}\nabla^{a}\nu\right) - e^{2\nu-2\chi}\nabla^{a}\nu\nabla_{c}\nu - e^{2\nu-2\chi}\delta^{a}{}_{c}\nabla_{d}\chi\nabla^{d}\nu \\
& + e^{2\nu-2\chi}\nabla^{a}\chi\nabla_{c}\nu - \frac{1}{4}e^{4\nu-4\chi}F^{ab}F_{bc} \,.
\end{aligned}
\tag{A.22}
$$

Thus the Ricci tensors may be evaluated easily, the results of the contraction are presented below:

$$
\begin{aligned}
{}^{D}R_{bd} = e^{-2\chi}\left({}^{D-1}R_{bd}\right) - (D-3)e^{-2\chi}\nabla_b\nabla_d\chi - e^{-2\chi}\eta_{bd}\nabla^2\chi + (D-3)e^{-2\chi}\nabla_b\chi\nabla_d\chi \\
- (D-3)e^{-2\chi}\eta_{bd}\left(\nabla\chi\right)^2 - e^{-2\chi}\nabla_b\nabla_d\nu + e^{-2\chi}\nabla_b\nu\nabla_d\chi - e^{-2\chi}\nabla_b\nu\nabla_d\nu \\
- e^{-2\chi}\eta_{bd}\nabla_c\chi\nabla^c\nu + e^{-2\chi}\nabla_b\chi\nabla_d\nu - \frac{1}{2}e^{2\nu-4\chi}F^c{}_bF_{cd}\ .
\end{aligned}
\tag{A.23}
$$

The cross term is

$$
{}^{D}R_{Dd} = \frac{1}{2}e^{-\nu-(D-2)\chi}\nabla_a\left(e^{3\nu+(D-5)\chi}F^a{}_d\right) \tag{A.24}
$$

and the R_{DD} component is given by

$$
{}^{D}R_{DD} = \frac{1}{4}e^{4\nu-4\chi}F_{ab}F^{ab} - e^{2\nu-2\chi}\left(\nabla^2\nu + \left(\nabla\nu\right)^2 + (D-3)\nabla^a\chi\nabla_a\nu\right)\ . \tag{A.25}
$$

A particularly useful result for our consideration of dimensional reduction from four to three or from five to four dimensions is the relationship between the Ricci scalars; one finds

$$
\begin{aligned}
{}^{D}R = e^{-2\chi}\Big[{}^{D-1}R - (D-2)(D-3)\left(\nabla\chi\right)^2 - \frac{1}{4}e^{2\nu-2\chi}F_{ab}F^{ab} \\
- 2\left((D-2)\nabla^2\chi + \nabla^2\nu + \left(\nabla\nu\right)^2 + (D-3)\nabla_a\chi\nabla^a\nu\right)\Big]\ .
\end{aligned}
\tag{A.26}
$$

Now that we have been able to relate the Ricci tensors and scalars during the dimensional reduction from D to $D-1$ dimensions we can go on to apply the procedure repeatedly to derive the dimensional reduction from D to n dimensions on a sequence of Killing vectors of the higher dimensional metric. There are important uses of dimensional reduction where we weaken this condition (in particular it implies that the topology of the fibred space consists of products of $\mathbb{R}$ and S^1), however we will not be needing a more general reduction scheme in our discussions. We therefore consider the metric

$$\boldsymbol{g}^{(D)} = h_{AB}\left(\mathbf{d}x^A + \boldsymbol{A}^A\right) \otimes \left(\mathbf{d}x^B + \boldsymbol{A}^B\right) + e^{2\chi}\eta_{ab}\boldsymbol{E}^a \otimes \boldsymbol{E}^b \ . \tag{A.27}$$

With nothing depending on the x^A coordinates. Here, the metric $\eta_{ab}\boldsymbol{E}^a \otimes \boldsymbol{E}^b$ is n-dimensional. Diagonalizing h_{AB}, we may use the results just established to successively reduce the dimension. One finds that, after much algebra, the Ricci tensor and scalar are given by the following expressions:

$$\begin{aligned} {}^D R_{bd} = {} & e^{-2\chi}\left({}^n R_{bd}\right) + (D-2)e^{-\chi}\nabla_b\nabla_d e^{-\chi} - \frac{1}{4}e^{-2\chi}\nabla_b\left(e^{2\chi}h^{AB}\right)\nabla_d\left(e^{-2\chi}h_{AB}\right) \\ & - \frac{1}{2}h^{AB}\nabla_b\nabla_d\left(e^{-2\chi}h_{AB}\right) - \frac{1}{\sqrt{|h|}}\eta_{bd}e^{-n\chi}\nabla^a\left(\sqrt{|h|}\,e^{n\chi}e^{-2\chi}\nabla_a\chi\right) \\ & - \frac{1}{2}e^{-4\chi}h_{AB}F^A{}_{ab}F^{B\,a}{}_d \ , \end{aligned} \tag{A.28}$$

$${}^D R_{Ab} = \frac{1}{2\sqrt{|h|}}e^{-n\chi}e^{\chi}\nabla_a\left(\sqrt{|h|}\,e^{n\chi}e^{-4\chi}h_{AB}F^{B\,a}{}_b\right) \tag{A.29}$$

and

$$\begin{aligned} {}^D R_{AB} = {} & -\frac{1}{2\sqrt{|h|}}h_{AC}e^{-n\chi}\nabla^a\left(\sqrt{|h|}\,e^{n\chi}e^{-2\chi}h^{CD}\nabla_a h_{BD}\right) \\ & + \frac{1}{4}e^{-4\chi}h_{AC}h_{BD}F^C{}_{ab}F^{D\,ab} \ . \end{aligned} \tag{A.30}$$

Finally the Ricci Scalar is given by:

$$\begin{aligned} {}^D R = {} & e^{-2\chi}\left({}^n R\right) + (D-2)e^{-\chi}\nabla^2 e^{-\chi} - \frac{1}{4}e^{-2\chi}\nabla^a\left(e^{2\chi}h^{AB}\right)\nabla_a\left(e^{-2\chi}h_{AB}\right) \\ & - \frac{1}{2}h^{AB}\nabla^2\left(e^{-2\chi}h_{AB}\right) - \frac{n}{\sqrt{|h|}}e^{-n\chi}\nabla^a\left(\sqrt{|h|}\,e^{n\chi}e^{-2\chi}\nabla_a\chi\right) \\ & - \frac{1}{2\sqrt{|h|}}e^{-n\chi}\nabla^a\left(\sqrt{|h|}\,e^{n\chi}e^{-2\chi}h^{AB}\nabla_a h_{AB}\right) - \frac{1}{4}e^{-4\chi}h_{AB}F^A{}_{ab}F^{B\,ab} \ . \end{aligned} \tag{A.31}$$

with $h = \det(h_{AB})$.

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