# Conformal Higher Spins and Scattering Amplitudes

A thesis presented for the degree of
Doctor of Philosophy
by

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ABSTRACT: This thesis presents results pertaining to scattering amplitudes in Conformal Higher Spin (CHS) theory, most of which was published in [1–3].

CHS theory contains Maxwell theory, Conformal Gravity and generalises them for higher spin. After briefly introducing the general field of Higher Spins, we therefore discuss Conformal Gravity as a warm up. Since it is a 4-derivative theory, it contains more on-shell states than just the usual 2-derivative Einstein Gravitons. Some of these states are found to be admissible for scattering and lead to finite expressions for amplitudes. We compute three point tree-level amplitudes scattering all possible states. We give a formula which captures these amplitudes using twistor spinors.

We then define CHS theory and its symmetries. We descirbe how it is obtained as the logarithmically divergent part of the partition function for a free scalar coupled to general spin background sources. We characterise its scattering states and proceed to present a series of amplitude computations.

We first compute four-point amplitudes for an external scalar interacting with the full tower of CHS fields. These amplitudes need a natural prescription for summing over that infinite tower of fields. Doing so in a way that is compatible with CHS symmetry leads to vanishing amplitudes.

We then present similar amplitudes in pure CHS theory where the external legs are 2-derivative spin 1 and 2 CHS modes. Once again, these amplitudes are trivial. As the theory is conformal, it has a natural description in the language of twistor-spinors and we give a formula for three-point tree level amplitudes of *all* states, including those which are not associated with 2 derivative equations of motion.

Finally we look at the theory in curved spacetime, where its quadratic sector is non-diagonal. We compute some of these terms and their contributions to the conformal anomaly $c$-coefficient.

## Declaration of Authorship

Unless otherwise referenced, the work presented in this thesis is my own - Simon Nakach.

# Contents

# Chapter 1

# Introduction

The importance of the role which Symmetry has played in the development of Theoretical Physics is hard to overstate. Describing *any* physical system is a task made easier by the presence of symmetry. [1] But the role of Physics is not just to describe the Universe, it is also to understand it. In this endeavour, Symmetries have been a beacon guiding our way for the last century. As an example, it has led us to write down the Standard Model Lagrangian - a theory symmetric under SU(3) $\times$ SU(2) $\times$ U(1). Remarkably, this was done long before its gauge bosons - which are direct representations of that very symmetry group - were detected.

A natural concept which arises from understanding the symmetry of the Universe is spin. As we will see, spin is nothing but a classification of possible quantum mechanical wave equations which are consistent with the global symmetries of the Universe. In this thesis, we will be concerned with *higher spin theories*. These contain fields of all possible spins as well as infinite dimensional symmetry algebras.

This seems a fairly natural thing to do - after all if symmetry is so important, why not look for theories with as much of it as possible? For this reason, one might want to promote the space-time symmetry used to build the higher spin algebra to also be conformal . The result of this is the theory of Conformal Higher Spins (CHS) [4,5], which will be the main focus of this thesis.

There are, of course, many more reasons to study this theory - from its improved UV finiteness, its important role in the AdS/CFT correspondence to the

[1] It is a fact well-understood by all Physics students that computations are significantly less painful when cows are taken to be spherical - and preferably in a vacuum.

alluring possibility of vanishing conformal anomalies. There are also many difficulties and obstructions, as the definition of consistent higher spin interacting theories is not a trivial task.

In this thesis, we will begin by giving an overview of the field of Higher Spin Theory. This will by no means be complete or extensive, but it should be able to motivate the interest in these theories, provide some context, and give the interested reader relevant references. The rest of the thesis will focus on building the knowledge required to present the work done by the author - most of which was published in [1–3].

For most of this paper we will assume $d = 4$, though we may comment on more general dimensions at times. We focus on this case not only for the usual natural reason, but also because 4 dimensions appears to be special for CHS theory - for instance its lower spin truncation is given by Conformal Gravity and Maxwell theory. In what follows, we will take the signature of the Minkowski metric to be $(-,+,+,+)$. Any non-standard notation used is introduced in Appendix A.1.

## 1.1 Higher Spin Theories

We begin with an introduction to the topic of Higher Spin theories. We will outline how they arise from basic considerations, the theoretical difficulties they face, and give an impression of how these theories are built. Later on, this will allow us to place them in the context of AdS/CFT, and eventually make ground with the specific theory of Conformal Higher Spins.

### 1.1.1 Spin and wave equations

Spin is one of the most fundamental properties that fields can have. Let us highlight how it naturally arises in Quantum Field Theory as a consequence of marrying Special Relativity with Quantum Mechanics - a more comprehensive description can be found in [6–8] . The former stipulates that Physics is the same in all inertial frames, while the latter describes microscopic physics as a Hilbert Space with states whose time evolution is described by a wave equation. Merging these concepts, we have a Hilbert space $\mathcal{H}$, which contains states $|\psi\rangle$ and we can define new states by simply going to a new inertial frame, $|\psi'\rangle \equiv U(L)|\psi\rangle$. Here, $L$ is an element of the isometry group of the spacetime, and $U$ is a representation of that group. In a flat spacetime, isometries are spacetime translations, spatial rotations and boosts, which form the Poincaré group, ISO(3,1). Among the $U(L)$ are Heisenberg picture wave operators, since they contain time translation transformations. One is then led to classify all irreducible unitary representations of ISO(3,1), and associate a wave equation with each one. This is called the *Bargmann-Wigner programme* [9].

This classification was done by Wigner [10]. One starts with the Poincaré algebra spanned by the Lorentz generators $M_{\mu\nu}$ and the translation generators $P_\mu$, which satisfy the algebra:

$$i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho} \tag{1.1}$$

$$i[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \tag{1.2}$$

$$i[P_\mu, P_\nu] = 0 \tag{1.3}$$

Since $P_m$ commutes with itself, it is particularly interesting to consider the operator $P_\mu P^\mu$. This is a Casimir operator, meaning that it commutes with all the generators of the algebra, and it leads us to express the states in a momentum eigenbasis:

$$P^\mu|\psi_{p,\sigma}\rangle = p^\mu|\psi_{p,\sigma}\rangle\,, \tag{1.4}$$

where $\sigma$ defines other possible quantum numbers the state could have. The states are then classified in terms of a real number, $m$, which is related to the eigenvalue of our Casimir, $p^2$. There are three distinct cases:

- $p^2 = m^2$: corresponding to particles with negative masses. These states are unphysical and we will ignore them.
- $p^2 = 0$: corresponding to massless particles.
- $p^2 = -m^2$: corresponding to massive particles.

One must then find the maximal subgroup of ISO$(3,1)$ which leaves this eigenbasis invariant. This is done most easily by going to a convenient frame; we summarise the results:

- For massive states, we go to the frame where $p^\mu = (m, 0, 0, 0)$. The little group is $SO(3)$.

- For massless states, we go to the frame where $p^\mu = (p, 0, 0, p)$. For physical states [2] , the little group is SO(2).

Finally, we need to write down the irreducible representations of SO(3) and SO(2). These are labelled by a single integer, often called spin [3] . A field in the spin $s$ representation of SO(3) can be written as a totally symmetric tensor $\phi^{i_1 \cdots i_s}$, with vanishing traces: $\phi^{i_3 \cdots i_s i}{}_i = 0$ where $i_j = (1, 2, 3)$. For irreducible representations of SO(2), this is exactly the same except that the indices would only run in $(1, 2)$.

We have a classification of all the (physical) representations of the Poincaré group. The task is to associate each of them with a covariant relativistic differential equation. We will sketch how this is done for the massive case, and state the result for the massless case as it is more subtle.

For a spin $s$ massive representation, the first step is to write down a symmetric traceless tensor of SO(3, 1), $\phi^{\mu_1 \cdots \mu_s}$. We then write the equation:

$$(\Box - m^2)\phi_{\mu_1, \ldots, \mu_s} = 0\,, \tag{1.5}$$

which when translated to momentum space assigns the particle's mass, ie. it determines the value of the quadratic Casimir operator. Next, we impose the transversality condition:

$$\partial_{\mu_1}\phi^{\mu_1 \cdots \mu_s} = 0\,. \tag{1.6}$$

When translated to momentum space, this equation ensures that only the spatial components of $\phi^{\mu_1 \cdots \mu_s}$ are non-trivial in the rest frame. This restricts our symmetric traceless tensors of SO(3, 1) to symmetric tensors of SO(3). Finally, the SO(3, 1) traceless condition becomes restricted to SO(3) tracelessness.

Massless tensors are more subtle, so we refer the reader to the reviews [7,8] for details. We simply state the equation for free massless particles of spin $s$,

[2]The massless case is slightly subtle. It turns out that the little group for the massless case is actually ISO(2).One then needs to repeat the procedure we started with ISO(3, 1). Studying the Casimir of ISO(2) leads once again to two separate cases. One case leads to an SO(2) little group. The other depends on a continuous parameter, often called "continuous spin" which we do not observe in Physics.

[3]In $d$ dimensions, the little groups of massive and massless particles are given by SO$(d-1)$ and SO$(d-2)$ respectively. Since the representations of SO$(n)$ are characterised by $[\frac{n}{2}]$ integers, for $d > 4$ more numbers are necessary. Still, the term spin is kept to refer to symmetric representations - which can be described by only one number - in analogy with the $d = 4$ case.

known as the Fronsdal equations [11]:

$$\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\varphi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1}\partial^{\nu}\varphi_{\mu_2\ldots\mu_s)\nu} + \partial_{(\mu_1}\partial_{\mu_2}\varphi_{\mu_3\ldots\mu_s)\nu}{}^{\nu} = 0\,. \tag{1.7}$$

where the fields satsify a slightly unusual double tracelessness condition:

$$\varphi_{\mu_1\ldots\mu_{s-4}\nu}{}^{\nu}{}_{\rho}{}^{\rho} = 0 \tag{1.8}$$

It is appropriate to note at this point that these equations reduce to the Maxwell equations for the case of $s = 1$ and the Einstein equations perturbed around flat space for $s = 2$. These equations are invariant under "generalised linearised diffeomorphisms", which is to say under the variation:

$$\delta h_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}\,, \tag{1.9}$$

with the added condition that the gauge parameter be traceless:

$$\epsilon_{\mu_1\ldots\mu_{s-2}\mu}{}^{\mu} = 0\,. \tag{1.10}$$

This has been a lightning review of the concept of spin and its relevance in the Bargman-Wigner program. We should remark that this applies to *free* theories only. In order to introduce interactions, in general one wants to write down a Lagrangian leading to the equations obtained this way and deform it.
The takeaway here is that the concept of spin arises naturally when considering QFT and investigating the role of spacetime isometries. As such, classifying and understanding theories of arbitrary spin is an important avenue of research.

### 1.1.2 Why no Higher Spin?

Despite this assertion, theories with fields higher than 2 were not studied for a long time - it is only relatively recently that there is considerable interest in the field. Why?

The main reason is that there are a number of no-go theorems which prohibit interacting massless Higher Spin theories from existing. A comprehensive review of these can be found in [12]. We merely give here a quick description of these theorems, and comment on how they can be circumvented.

**van Dam-Veltman-Zakharov (vDVZ) discontinuity**

Most of the interest in HS theory is concentrated on massless particles, but let us briefly comment on massive theories first. Massive hadrons of higher spins are observed experimentally, though they are bound states. Fundamental massive particles are problematic due to the so-called vDVZ discontinuity [13–15]. In general, massive theories of spin $s$ fields have more degrees of freedom than their massless counterpart. For example, in the case of spin 2, the massive graviton has 5 degrees of freedom, while its massless cousin has only 2. We therefore expect, in the limit of $m \to 0$, that the massive theory will have some of its polarizations decouple so that we obtain the correct massless limit.
When the theories are interacting, this is not necessarily observed to be the case, ie. the massless limit is wrong. For the case of spin 2, this may be attributed to the way that the theory is linearised [16] but the issue remains for massive higher spin fields.

**Weinberg Low Energy Theorem**

This theorem [17] gives a powerful formula from quite general considerations. Start with a generic amplitude containing $N$ particles of spin $s_i$ and external momentum $p_i$ for $i = (1, \dots, N)$. Furthermore, the amplitude has an external massless particle of spin $s$ and momentum $q$. By looking at the limit $q \to 0$ and demanding that it is invariant under the variation (1.9), we can obtain the constraint:

$$\sum_{i=1}^{N} g_i^{(s)} p_i^{\mu_1} \dots p_i^{\mu_{s-1}} = 0\,, \tag{1.11}$$

where $g_i^{(s)}$ designates the parameter of the minimal coupling cubic vertex between one particle of spin $s$, and two particles of spin $s_i$.
In particular, if we take the case of $s = 2$, and use momentum conservation, we obtain the necessity that

$$g_i^{(2)} = g_j^{(2)} \quad \forall\, i, j \tag{1.12}$$

which is also known as the weak equivalence principle: it implies that gravity has the same minimal coupling strength to all types of matter.
On its own, this is not prohibitive for higher spin particles, but it has dire implications when combined with Aragone-Deser Obstruction.

#### Minimal Coupling arguments

Here we highlight certain arguments that prevent HS particles from having minimal coupling to gravity in flat space. The first argument comes from Weinberg-Witten theorem [18] and its generalisation by Porrati [19]. These respectively forbid the existence of Lorentz and gauge invariant stress energy tensors for massless particles of spin $s > 1$ and any particles of spin $s > 2$ to couple to gravitons in flat space.
There are also the Aragone-Deser arguments [20, 21]. These start from the idea that minimal coupling is done by replacing partial derivatives with covariant derivatives. The latter famously do not commute, and when one checks gauge invariance of Higher Spin actions, ie. invariance under the covariant form of (1.9), curvature terms will arise which cannot be compensated for.
Finally, in [22], Metsaev gave an important bound for cubic vertices involving HS particles. For a vertex involving particles of spin $s_1, s_2, s_3$, then if $s_1 \leq s_2 \leq s_3$, the number of derivatives in the vertex, $n$, is bound by:

$$s_3 + s_2 - s_1 \leq n \leq s_3 + s_2 + s_1 \tag{1.13}$$

which means that if there is at least one HS particle of spin $s > 2$, the number of derivatives is greater than 2, ie. there can be no minimal coupling to gravity.

#### Porrati Arguments

This argument, [19], combines the low energy and minimal coupling arguments . First we notice that in nature, matter is known to couple minimally to gravity. But when combined with the weak equivalence principle, this means that the graviton coupling constant $g_i^{(2)}$ is non zero for *any* particles coupled to our system. In other words, any higher spin particles interacting with lower spin particles in our system *must* be minimally coupled to gravity.
However, as we said, many arguments exist prohibiting that very coupling. The conclusion is that higher spin particles cannot be coupled to any lower spin particles at all.

#### Coleman-Mandula

Finally we mention the famous Coleman-Mandula theorem [23] (see also [24] for the case of supersymmetry). This theorem looks at scattering amplitudes of fields with finite numbers of types of fields and limits the types of symmetries that are possible. It finds that the most general symmetries an interacting theory can have are:

- Poincaré symmetry
- A SUSY or superconformal extension of Poincaré
- An internal symmetry group (ie. in direct product with the rest of the symmetries).

This precludes interacting theories from having symmetries of the type (1.9).

### 1.1.3 Ways around

We now list and comment on the various ways that one can get around these theorems. As usual, one simply needs to evaluate the assumptions made on those arguments, and exploit them. Here we simply list and briefly comment on the holes in the no-go theorems. For a more exhaustive list as well as more details, we refer the reader to [12].

- **Finite Particle number**: This applies to the Coleman-Mandula theorem in particular. The assumption of finite number of fields is not a good one in HS theories, as they always need to contain infinite sets of fields to be consistent, as we shall shortly see.
- **Flat-Space**: Many of these arguments rely on the S-matrix of flat space. In the presence of a non-vanishing cosmological constant $\Lambda$, these arguments no longer apply as the definition of an S-matrix is either altered or ambiguous (depending on the sign of $\Lambda$). In AdS for example, higher spin theories may therefore have non-trivial interactions.
- **Lorentz covariance**: Many of the arguments we have made assume a manifest Lorentz covariant approach which allows them to make guesses on the form of certain terms. However, light-cone approaches (see [25] ) exist.
- **Unitarity** The arguments above assume that the theory has 2-derivatives and is Unitary. As such, higher derivative theories can escape them.

### 1.1.4 Interacting Massless Higher Spin Theories

Among the ways of avoiding the no-go theorems, perhaps the most successful has been to go to AdS space where the S-matrix becomes ill defined, and most of the theorems are circumvented. This yielded to the successful creation of consistent cubic vertices for HS particles in AdS space in cubic vertices [26]. Later, Vasiliev developed a fully interacting and consistent theory of massless higher spins, [27–30][4]. The construction was first done in 4 dimensions but generalised to higher dimensions in [34, 35].

In this subsection, we aim to highlight a few features of Vasiliev theory and discuss some of the ideas behind it, without going into too much technical detail. We will do so somewhat heuristically and highlight how everything generalises in the literature at the end. This subsection is mainly based on the notes [33].

One natural starting point for understanding the construction of Vasiliev theory is to consider spin 2, and look at the frame-like formulation of Einstein Gravity. This means understanding Einstein gravity as a theory of gauged isomorphisms. The idea is simple: we make the isometries of the space-time local, associate a gauge field to each of the generators then attempt to write down a gauge invariant equation of motion reproducing Einstein gravity that we know and love.

In this section, we will introduce the formulation before sketching how one can generalise it to obtain a simple theory of interacting massless higher spins. Finally, we will highlight features of Vasiliev theory, its current state in the field as well as open problems.

### Spin 2

If we are in $D \equiv d + 1$ dimensional space-time, one starts with the einbein field and the spin connection $e_\mu^a$ and $w_\mu^{ab}$ [5] and turn them into one-form gauge fields

$$e^a = e_\mu^a dx^\mu \tag{1.14}$$

$$w^{ab} = w_\mu^{ab} dx^\mu \tag{1.15}$$

with $a, b \in (0, \ldots, D-1)$. The gauge fields $w_{ab}$ are associated with the Lorentz boost generators $M^{ab}$ while $e_a$ are associated with the translation generators $P^a$.

[4] The reader may also see [8, 31–33] for reviews.

[5] In this chapter only, we use Greek indices $(\mu, \nu, \ldots)$ as space-time indices and Latin indices $(a, b, \ldots)$ as frame indices. From Chapter 2 onwards, we will always use Latin indices as spacetime indices.

Their representations in the $\mathrm{SO}(D-1,1)$ algebra are given via the following Young diagrams:

$$e_a \in \boxed{\phantom{x}}_{\mathrm{SO}(D-1,1)} \qquad , \qquad w_{ab} \in \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array}_{\mathrm{SO}(D-1,1)} \quad . \tag{1.16}$$

Roughly speaking, $e^a$ contains information about metric fluctuations (ie. it contains the degrees of freedom of the spin 2 Fronsdal field $\varphi_{\mu\nu}$) while $w^{ab}$ contains redundant information that can be used to gauge torsion away. This will be fixed later, when we introduce an equation of motion for our spin 2 theory.

Next, we need to write down an algebra for the generators. There are several possibilities, each of them corresponding to a different background isometry. The most relevant for us, as hinted by the previous section, is the one corresponding to an $\mathrm{AdS}_D$ background, see (A.16) - (A.17) [6].

The two generators can be grouped into the algebra of $\mathrm{SO}(D-1,2)$ as in appendix A.2. Consequently, we can consider the single gauge field $W$:

$$W_2 = w_{AB}T^{AB} = e_a P^a + w_{ab}M^{ab} \tag{1.17}$$

where $(A,B) = (0,1,\ldots,D+1)$. These generators are in the adjoint representation of $\mathrm{SO}(D-1,2)$ which is a two row one column Young diagram:

$$w_{AB} \in \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array}_{\mathrm{SO}(D-1,2)} \tag{1.18}$$

Finally, we can form a curvature from W and write down a flatness condition for it:

$$dW_2 + W_2 \wedge W_2 = 0 \tag{1.19}$$

which is invariant under the gauge variation:

$$\delta W_2 = d\epsilon_2 + [W_2, \epsilon_2] \tag{1.20}$$

[6]In principle, one could also consider a Minkowski or de-Sitter (dS) isometries corresponding to the algebra $\mathrm{ISO}(D-1,1)$ and $\mathrm{SO}(D,1)$ respectively, but we focus on AdS to avoid no-go theorems, and because of its relation to CFTs.

where $\epsilon_2 = \epsilon_a P^a + \epsilon_{ab} M^{ab}$ contains the local gauge parameters associated with local Lorentz transformations and local translation. Two pieces of information are contained within (1.19). The first one is a constraint on the torsion tensor, which allows us to solve for $w^{ab}$ in terms of $e^a$ as we discussed earlier:

$$T^a = de^a + w^a{}_b \wedge e^b = 0\,. \tag{1.21}$$

The second condition constrains the Riemann curvature to be that of the AdS background, as expected.

### Higher Spins

The development of an interacting higher spin theory in AdS$_D$ follows the same general steps but uses higher rank symmetric gauge fields. The correct system of fields to write down is:

$$\begin{aligned} e^{a_1 \dots a_{s-1}} \\ w^{a_1 \dots a_{s-1}, b_1} \\ \vdots \\ w^{a_1 \dots a_{s-1}, b_1 \dots b_{s-1}} \end{aligned}$$

where the symmetry properties of these fields can be read off their SO$(D-1,1)$ Young Tableaux:

$$e_{a_1 \dots a_{s-1}} \in \overbrace{\boxed{\phantom{x}} \cdots \boxed{\phantom{x}}}^{s-1}{}_{\mathrm{SO}(D-1,1)} \quad , \quad w^{a_1 \dots a_{s-1}, b_1 \dots b_r} \in \overbrace{\underbrace{\begin{array}{|c|c|c|}\hline \phantom{x} & \cdots & \phantom{x} \\ \hline \phantom{x} & \cdots & \phantom{x} \\ \hline \end{array}}_{r} \cdots \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \end{array}}^{s-1}{}_{\mathrm{SO}(D-1,1)}$$

Similarly to the spin 2 case, the field $e^{a_1 \dots a_s}$ contains the information carried by the spin $s$ Fronsdal field, as well as extra degrees of freedom which are removed by fixing the rest of the gauge fields.

Once again, all these gauge fields can be combined into one field of the

$SO(D-1,2)$ algebra, $w^{A_1\dots A_{s-1},B_1\dots B_{s-1}}$ belonging to the two- row Young diagram:

$$w^{A_1\dots A_{s-1},B_1\dots B_{s-1}} \in \overbrace{\begin{array}{|c|c|c|}\hline \phantom{x} & \cdots & \phantom{x} \\ \hline \phantom{x} & \cdots & \phantom{x} \\ \hline \end{array}}^{s-1}{}_{SO(D-1,2)}$$

A crucial point is that each of these fields are associated to a generator $T^{A_1\dots A_{s-1},B_1\dots B_{s-1}}$ in the same representation. All fields are then combined into one master-field, W:

$$W \equiv \sum_{s=2} w_{A_1\dots A_{s-1},B_1\dots B_{s-1},} T^{A_1\dots A_{s-1},B_1\dots B_{s-1}} \tag{1.22}$$

These newfangled generators have a Lie bracket determined by the $SO(D-1,2)$ algebra. This allows us to say something important about the general structure of higher spins algebras. Using the schematic notation of $T_s$ to denote the generator associated with spin $s$ fields, the commutator between two generators of spin is given by: [7]

$$[T_s, T_{s'}] \sim T_{s+s'-2} + T_{s+s'-4} + \dots T_{|s-s'|-2} \tag{1.23}$$

If we consider this expression for $s = s' = 3$, we see that $T_4$ must be included in the algebra. This in turn implies the existence of $T_5$ and $T_6$, and so on. So one can see that as soon as a field of spin $s > 2$ is included in the spectrum, one must necessarily include an *infinite* tower of spin. There two such algebras: one containing all integer of spin $s \geq 1$ and its truncation to all even spins.

Finally, we can impose a curvature equation, analogous to the flatness equation (1.19):

$$dW + W \wedge W = 0 \tag{1.24}$$

which is invariant under $\delta W = d\epsilon + [W, \epsilon]$. We have thus obtained a set of gauge-invariant linear equations for higher spin fields. Note that a linearisation of (1.24) yields relations between the various fields $e^{a_1\dots a_{s-1}}$ and $w^{a_1\dots a_{s-1},b_1}$ much in the same way that the flatness condition (1.19) gave us (1.21) for spin 2 case.

[7] This can be gleaned from considering the product of two-row Young tableaux and restricting to terms with the correct symmetries for example, see [36]. Later we give a slightly different heuristic explanation from the perspective of CFTs.

#### Vasiliev Theory

The steps outlined above are roughly those followed in the construction of interactive massless higher spins by Vasiliev. The actual construction has a few more ingredients. First, it involves another master field, included for the theory to contain a scalar particle. Second, the higher spin algebra must be realised. This can be done in terms of auxiliary oscillators. Finally, the space of oscillators is actually doubled, a trick necessary to include consistent interactions. These steps are beyond the scope of this thesis, so we refer the reader to the reviews [31–33, 37]. We end this section by summarising and highlighting some of the features of these interacting Massless higher spin theories.

- In 4 dimensions, these theories necessarily include an infinite number of fields, each of which appears only once. There exists a consistent truncation where only fields of even spins feature.

- These theories are defined in AdS space-times and therefore contain one parameter: $\Lambda$, the curvature of the space.

- The interactions involving higher spin fields include higher derivatives. This means they are non-analytic in the curvature of spacetime. To illustrate this, we remember the result of [22] used earlier and write a generic cubic vertex with $s_1 \leq s_2 \leq s_3$:

  $$\mathcal{L}_{s_1 s_2 s_3} = \Lambda^{\frac{s_1 - s_2 - s_3 - d/2 + 3}{2}} \partial^{-s_1 + s_2 + s_3} h_{s_1} h_{s_2} h_{s_3} + \dots \tag{1.25}$$

  where the dots represent terms with more derivatives and lower powers of the compensating $\Lambda$. We see that if any higher spin fields are involved, negative powers of $\Lambda$ will be present. This is a manifestation of the earlier no-go theorems - such interactions could not exist in flat space.

- This construction is done at the level of the equations of motion only. So far, no known Lagrangian form exists [8]. This is major obstruction to quantizing the theory.

- String theory also boasts an infinite spectrum of particles states. Those have masses arranged in Regge trajectories and parametrised by the constant $\alpha'$. It is expected [9] that the tensionless limit of String theory, if it exists, could

[8] A few proposals have been made, see [38] or [39] for more recent work.

[9] See, for example, [31].

be described by a higher spin theory. The idea is that the large amount of symmetry present in String Theory would actually a broken phase of an even larger Higher Spin symmetry group.

## 1.2 AdS/CFT

AdS/CFT plays an important role in the field of Higher Spin Theories, and is a major reason for the current interest [40–42][10] .

This correspondence, states that specific gravity theories in $d+1$ dimensions in an AdS background are dual to Conformal Field Theories (CFTs) living on the $d-$dimensional boundary of the AdS space.
The most famous correspondence exists between type IIB superstring theory in the $AdS_5 \times S_5$ background as dual to $\mathcal{N} = 4$ Super Yang-Mills in 4 dimensions. When we say that the two theories are equivalent, several things are meant:

- The isometries of the global symmetry group on the gravity side is the same as the (bosonic) symmetry group of the CFT side. For instance, we've mentioned before that the isometry group of $AdS_{d+1}$ is SO$(d,2)$. This is also exactly the conformal algebra in $d$ dimensions, see Appendix A.2.
- There is a dictionary that can match observables and objects from one theory to another (eg. fields in AdS correspond to certain operators on the CFT side).
- There is a relation between the partition functions of the gravity theory and that of the CFT, $Z_{bulk} \sim Z_{CFT}$ once certain identifications have been made. In particular, at the boundary, we have the relation:

$$Z_{bulk}[\phi_0] = \langle e^{\int \mathcal{O}\phi_0} \rangle_{CFT}\,, \tag{1.26}$$

where $\phi_0$ designates the boundary value of the fields of the bulk gravity theory, while $\mathcal{O}$ are the operators which are dual to those fields (ie. they act as sources on the boundary).

As we saw, Vasiliev theory naturally prefers to live in backgrounds like AdS. Furthermore, it necessarily contains gravity, since there is no consistent truncation of the Higher Spin algebras which excludes spin 2 fields. A natural question

[10] We cite [43–45] as an incomplete list of reviews and introductions to the topic.

is the whether there is a duality between Vasiliev theory and some CFT. This was answered in the positive in [46]. There, it was found that there is a duality between Free O(N)/U(N) vector models and Vasiliev theory. Let us introduce the vector model briefly, and sketch how it is related to Vasiliev theory.

### 1.2.1 Free $\mathrm{O}(N)/\mathrm{U}(N)$ vector model

As the name implies, the vector model corresponds to a theory of $N$ scalar fields in the vector representation of a global $\mathrm{O}(N)$ or $\mathrm{U}(N)$ group. Since it is more general, we will focus on the latter case which requires the scalars to be complex. In $d$ dimensions, the action of the theory is then simply:

$$S_{CFT,0} = \int \mathrm{d}^d x \, \partial_\mu \phi_i^* \, \partial^\mu \phi^i \tag{1.27}$$

where $i = 1 \ldots N$ correspond to the group index. The equation of motion is then:

$$\Box\phi = 0 \,. \tag{1.28}$$

This theory is indeed invariant under the unitary transformations $\phi_i \to U_i{}^j \phi_j$. However, it also benefits from many more symmetries, including conformal symmetries. To see this, we seek currents which are conserved on the equations of motion. In this section we focus on currents which are bilinear in $\phi^i$. This is because they have a clearer interpretation in the context of AdS/CFT, as will be explained later. The first one is given by:

$$J_1^\mu = i\xi \left( \phi^i \partial^\mu \phi_i^* - \phi_i^* \partial^\mu \phi^i \right) \tag{1.29}$$

where we have included the infinitesimal gauge parameter $\xi$ for reasons that will soon be obvious, and the subscript 1 reflects the fact that the current contains 1 derivative.

Noether's theorem implies that this conserved current is due to a continuous symmetry of the action. The infinitesimal symmetry in question is: [11]

$$\delta_1 \phi_i = i\xi \phi_i \,, \tag{1.30}$$

[11] Following Noether's theorem, the conserved current allows us to find the symmetry from whence it arises, by constructing a conserved charge, $Q_1 \equiv \int \mathrm{d}^{d-1} x \, J^0$. The charge then generates the symmetry via $\delta\phi = [Q_1, \phi]$.

which is simply generated by the operator $T_1 \equiv i\mathbb{1}$. [12]

Looking at (1.29), it is possible to see that there exist more conserved currents containing higher derivatives. For instance, given the traceless energy-momentum tensor:

$$T^{\mu\nu} = 2\partial^{(\mu}\phi^*\partial^{\nu)}\phi - \tfrac{1}{2}\eta^{\mu\nu}\partial_\rho\phi^*\partial^\rho\phi - \tfrac{1}{2}\phi^*\partial^\mu\partial^\nu\phi - \tfrac{1}{2}\phi\partial^\mu\partial^\nu\phi^* \tag{1.31}$$

$$T^\mu{}_\mu = \partial_\mu T^{\mu\nu} = 0 \tag{1.32}$$

we can construct a conserved current with the conformal Killing vector $\xi_\mu$:

$$J_2^\mu \equiv T^{\mu\nu}\xi_\nu\,, \qquad\qquad \partial_\mu J_2^\mu = 0\,, \tag{1.33}$$

with:

$$\partial^{(\mu}\xi^{\nu)} + \frac{2}{d}\eta^{\mu\nu}\partial^\rho\xi_\rho = 0\,. \tag{1.34}$$

This current actually arises due to the conformal invariance of the free scalar action. Infinitesimally, the symmetries act as:

$$\delta_2\phi_i = \xi^\mu\partial_\mu\phi_i \tag{1.35}$$

and are generated by $P^\mu, M^{\mu\nu}, D, K^\mu$, the generators of the conformal algebra. In fact these can be arranged into one generator of the $\mathrm{SO}(d-2,2)$ algebra, $M^{AB}$ with $A, B \in (-1, 0, \ldots, d-1)$ - See Appendix A.2 for details . This generator is antisymmetric, which means its representation in Young diagram is:

$$M^{AB} \in \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array}_{\,\mathrm{SO}(D-1,2)} \tag{1.36}$$

At this point, one can see how to extend this procedure. We can keep building a tower of traceless conserved currents with $s$ derivatives. Schematically, these look like (see [47, 48]):

$$J^{\mu_1\cdots\mu_s} \sim i^s \left(\sum_k \partial^{(\mu_1\cdots\mu_k}\phi_i^*\,\partial^{\mu_{k+1}\cdots\mu_s)}\,\phi^i\right) - \textit{traces} \tag{1.37}$$

[12]Note that exponentiating this generator allows one to see that this symmetry corresponds to the global $\mathrm{U}(1)$ phase shift $\phi \to e^{i\xi}\phi$.

$$J_\mu{}^{\mu\mu_3\cdots\mu_s} = 0 = \partial_\mu J^{\mu\mu_2\cdots\mu_s}\,. \tag{1.38}$$

Then, given the rank $s-1$ Killing tensors satisfying:

$$\partial^{(\mu_1}\xi^{\mu_2\cdots\mu_s)} + \frac{s-1}{d+2s-4}\eta^{(\mu_1\mu_2}\,\partial_\mu\,\xi^{\mu_3\cdots\mu_s)\mu} = 0\,, \tag{1.39}$$

we get a conserved current $J_s^\mu \equiv J^{\mu\mu_2\cdots\mu_s}\xi_{\mu_2\ldots\mu_s}$, which gives rise to symmetries of the form

$$\delta_s\phi_i = \xi^{\mu_1\cdots\mu_{s-1}}\partial_{\mu_1}\ldots\partial_{\mu_{s-1}}\phi_i\,. \tag{1.40}$$

These are generated by order $s-1$ differential operators.

As it turns out, these operators can be classified into two-row Young Diagrams of length $s-1$, just like the ones we saw in the construction of massless higher spin algebras earlier.

This opens up another avenue to illustrate the infinite dimensionality of these algebras. Indeed, the action of the spin 2 generators is linear in partial derivatives. One can see that a commutators of these two operators must be at most linear in derivatives as well (the second order term vanishes by antisymmetry). However as soon as we include operators of order 2, their commutators generate operators of order 3, and so on.

We conclude this subsection by pointing out that we could have take real scalar fields with a global $O(N)$ algebra instead. The discussion would have stayed the same, except that we would have only had currents of even order in derivatives, ie. generators of even spin.

### 1.2.2 Relation between vector model and Vasiliev theory

Having introduced some features of both Vasiliev theory and vector models, we can notice some similarities. Most glaring at this point is the manifestation of the same higher spin algebra - namely the one spanned by generator in the two-row Young tableaux $\mathrm{SO}(d,2)$ tensors. Beyond that, the duality between the two models has been made more precise, let us list how.

- There is a duality between the fields of the bulk Vasiliev theory, and primary single-trace $\mathrm{U}(N)$ invariant operators of the free vector model. Here single trace implies that there is only one sum over $N$ [13]. This pertains to our currents $J^{\mu_1\cdots\mu_s}$ which , by virtue of being conserved, then match to massless fields in the bulk. Those are precisely the ones we have in the "maximal" Vasiliev theory, containing all integer spins.

- There is an additional single-trace invariant operator, $J_0 \equiv \phi_i\phi^i$. This corresponds to a scalar field in the bulk, though it is not massless. Instead, it has the mass of a conformally-coupled scalar, $m^2 = -2d(d-2)\Lambda$.

- If the vector model has O(N) symmetry instead, the same discussion applies, but the dual theory is the "minimal" Vasiliev theory containing only even spins.

- Correlation functions of the single-trace operators in the vector model correspond to Witten diagrams in AdS$_{d+1}$. Expressions for such correlation functions can be found in [49–52]. Three-point functions were used to reconstruct cubic vertices in for a certain class of Vasiliev theory in [53].

- The dimensionless coupling of the bulk higher spin theories correspond to $g_v \sim \frac{1}{N^{1/2}}$. That is to say, we can write the bulk theory's partition function as:
$$Z_{bulk} = \int D\phi\, e^{-NS_{bulk}}\,, \tag{1.41}$$
where $N$ plays the role of the coupling (ie. $S_{bulk}$ has no $N$ dependence ).

- One-loop checks of this duality exist. From (1.41), the effective action for the bulk theory can be written as:
$$\Gamma_{bulk} \equiv -\log Z_{bulk} = NS + \Gamma_1 + \frac{1}{N}\Gamma_2 + \dots \tag{1.42}$$
where the first term subsumes classical contributions to the partition function, while the other terms are quantum corrections. In the free CFT, the corresponding quantity one can explicitly compute is the free energy $F_{CFT} \equiv -\log Z_{CFT}$ and find that it is strictly proportional to $N$. Together with the expectation that $Z_{bulk} \sim Z_{CFT}$, this implies that the quantum contributions in the bulk theory should all vanish. Explicit computations for a free massless higher spin theory, or a Vasiliev theory in the bulk have shown that $\Gamma_1 = 0$ indeed [54,55] [14].

[13] Multi-trace operators correspond to multi-particle states in the bulk theory.

[14] For a summary discussing quantum corrections in higher spin theories see [56]. For computations in more general higher spin theories see [57].

## 1.3 Conformal Higher Spin Theories

We are now led to an interesting observation. There actually exists a higher spin theory living on the boundary - even though that boundary is flat.
Indeed, on the CFT side, we can introduce new fields, $h_{\mu_1\cdots\mu_s}$ which enter in the action via:

$$S_{CFT} = \int \mathrm{d}^d x\, \phi^* \Box \phi - \sum_s \frac{1}{s!} J^{\mu_1\cdots\mu_s} h_{\mu_1\cdots\mu_s}\,. \tag{1.43}$$

These "shadow" fields have dimension $2-s$ and act as sources for our conserved operators. Following the AdS/CFT relation between partition functions, these fields also have the interpretation as the Dirichlet boundary conditions for the bulk higher spin fields $\phi_s$:

$$\int D\phi_s\, e^{-S_{bulk}}\Bigg|_{\phi_s(z=0)=h_s} = \langle e^{-S_{CFT}} \rangle_{CFT} \tag{1.44}$$

It turns out that performing these integrals allows us to define a theory of Conformal Higher Spins (CHS). As these theories are the main subject of this thesis, we will introduce them in more detail in Chapter 3. For now we just state some of the attractive features that motivate studying the theory.

CHS theory is a fully interacting higher spin theory that can be realised in flat space - it avoids no-go theorems by having higher derivative kinetic terms and being non-unitary. This last fact is of course problematic if one wants to quantize the theory, but it also inherits nice UV properties as a result. What's more, due to the AdS/CFT relation we sketched above, one can use CHS theory to better understand massless higher spins in AdS. Finally, the theory is rife with non-trivial cancellations due to its large symmetry group. This will be a one of the focal points of this thesis.

## 1.4 Thesis Outline

The spin 2 truncation of CHS theory is given by the relatively well understood theory of Conformal Gravity. This will be the focus of Chapter 2, where we will introduce the theory, and study its scattering amplitudes. In effect, Conformal Gravity will be used as a toy-model to study some of the features of CHS theory and its amplitudes: it will allow us to study some of the peculiarities of higher derivative on-shell states, and look at the question of defining a scattering amplitudes for higher derivative theory. We will also cover some of the work presented in [3] and introduce the twistor formalism which allows us to write down a formula capturing all possible tree level 3-point scattering amplitudes in the theory.

Chapter 3 will be our introduction to CHS theory. There, we will look at CHS symmetry by studying the maximal symmetry of the Laplace operator. We will then show how to induce the full CHS theory by looking at the logarithmic divergence of a scalar field coupled to background CHS fields through traceless bilinear currents. Next, we explicitly compute the relevant sectors of the theory in preparation for amplitude computations. Finally, we will categorise the on-shell scattering states by generalising the twistor formulation of Conformal Gravity introduced the previous chapter.

Chapter 4 will contain the bulk of the computation done in [1]. That is to say, we'll be looking at scattering scalar fields coupled to CHS theory, and observing some interesting cancellations. This will give us the opportunity to observe first-hand the action of the large symmetry group on observables. We will also discuss 1-loop amplitudes in this context.

Chapter 5 pertains mainly to [2], which looks at tree-level scattering in pure CHS theory. Once again, non-trivial cancellations will be observed. We will also briefly discuss the latter half of [3], that is to say look at a twistor formula capturing *all* scattering amplitudes of the theory, including higher derivative modes.

Chapter 6 moves the discussion to curved space, where we present some previously unpublished results. More precisely, we discuss how in curved spaces, the CHS symmetry is deformed to become nonlinear, which introduces some non-diagonal terms in the action. We compute these deformations, which allow us to infer the form of these non-diagonal terms, and compute some new non-diagonal contributions to the theory's conformal anomaly.

Finally, Chapter 7 will be the conclusion. There, we will summarise the results presented here and give some parting remarks regarding the outlook of the field.

The appendices found at the end are split as follows. Appendix A will give some conventions and explain the notation, and include general background expositions that are not immediately tied to the main topics of the thesis.
Appendix B includes some useful tools and introduction to the formalisms that we make use of.
Finally, C is reserved for more technically involved computational details of the later sections.

# Chapter 2

# Conformal Gravity

Einstein's action for gravity (with a cosmological constant) is both remarkable and elegant:

$$S_E = \int \sqrt{|g|}\,(R - 2\Lambda)\,. \tag{2.1}$$

Despite its importance and successes, we know that it is inconsistent at the quantum level. In particular, its UV properties are problematic as in $d > 2$ dimensions it is non-renormalisable.

One is then led to look for alternatives. One such possibility is to look for theories that are described by curvature invariants, like Einstein gravity, but potentially with higher derivatives. For instance with the addition of $R^{ab}R_{ab}$ or $R^2$ to (2.1). As it turns out, adding these terms makes the theory renormalisable [58],[15].

The cost for studying higher derivative theories, is that they are non unitary in general. Despite this, they remain very interesting. One such theory of particular interest is Conformal Gravity, also known as Weyl Gravity. It boasts conformal invariance which means it has close ties with the action of twistor string theory, as explored in [60]. It has soft UV properties and is renormalisable [58]. There has also been some phenomenological interest in the theory, though when it is considered in conjunction with Einstein gravity, [61]. Furthermore, in [62], it is shown that at the classical level, Weyl gravity contains classical information of Einstein gravity. More precisely, by studying Conformal gravity in AdS and picking the correct boundary conditions, at the classical level is equivalent to Einstein Gravity. [16].

[15]For a review on adding quadratic curvature terms to Einstein gravity, see [59].

[16]Note that including interactions with matter problematic, as the ghost modes of conformal gravity may then lead to problems despite the boundary conditions, as claimed in [63].

For us, one final reason to study it is that Conformal Gravity provides the low-spin truncation of Conformal Higher Spin theories. In some sense, the conformal group is the maximal bosonic spacetime group of symmetry for a given dimension, and Conformal Gravity is the gauging of that group [64]. As such, it is a relatively well understood cornerstone between the rest of CHS and its large symmetry group. In particular, we will later be able to generalise straightforwardly certain results we obtain in this chapter to the case of CHS theory.

Much of this chapter is closely based on the parts of [3] which deal with Conformal Gravity; we will proceed as follows. Our first section in this chapter will focus on a toy scalar model to see how one might approach theories of higher derivatives. We will then introduce the action of conformal gravity, comment on its symmetries and its propagating degrees of freedom. Finally, we will compute some of its scattering amplitudes and make some remarks on the structures that appear.

## 2.1 Higher Derivative Scalar Models

In this section, we will be studying the higher derivative scalar action

$$\mathcal{L}_1 = -\frac{1}{2}\varphi\,\Box^2\varphi \tag{2.2}$$

as well as an interactive extension. In this simple model, some of the features and peculiarities of higher derivative theories can already be explored.

The first comment one should make is about the propagator of (2.2) which goes as $\tilde{D}_F \sim k^{-4}$ instead of the usual $k^{-2}$ behaviour. This is a boon for renormalisability: the superficial degree of divergence of an interacting version of the theory will be drastically improved. However this is also an issue, since the theory becomes non-unitary, which will be the topic of the first subsection. There, we will briefly (and incompletely) comment on the unitarity of higher derivative theories. We will then go through the process of trying to define the theory's "scattering amplitudes". In order to do so, we will first explore the option of expressing it as an ordinary derivative theory, before redefining what we mean by a scattering amplitude. Finally, we will give a few quick examples of such amplitudes in an interacting version of the model.

### 2.1.1 Unitarity

When doing canonical analysis on a theory of the likes of (2.2), we find that it is non-unitary. Indeed, Ostrogradsky showed - under some assumptions - that the Hamiltonian of such theories will contain a term linear in at least one of the canonical momenta: $H \sim P_i Q_j + \dots$. This allows it to reach arbitrarily low energies for large negative values of the canonical momentum $P_i$ [65]. This reflects the presence of "ghosts", ie. degrees of freedom which carry negative energy. This is disastrous for a theory: if any interactions are present, it becomes entropically favored to produce these ghosts and the system is unstable.

The theories we study in this thesis are non-unitary. While we make no claims to the contrary, it is interesting to note that *not all* higher derivative theories are non-unitary. We quickly mention here some of the ideas on this subject in the literature.

First of all, Ostrogradsky's theorem assumes that the theory is *non-degenerate*, meaning that the higher time derivative can be expressed as a function of canonical momenta. Degenerate theories (such as $f(R)$ theories) can thus avoid the instability. This is related to the fact that they are actually related to $2^{\text{nd}}$ derivative theories by field redefinitions, making them stable (see eg. [66]).
A different possibility is to make a system stable by adding appropriate constraints to it, see [63].
Another avenue is to consider alternative quantisations of these higher-derivative theories, see [67–71].
More recently, complexification of higher derivative theories as a means to restore unitarity has been considered [72].
Finally, in [73–75], the mechanism by which certain higher derivative theories become unitary is explored, namely when these are considered as non-analytical Wick rotations of their Euclidean versions.

### 2.1.2 Relation to Ordinary Derivative theory

Let us attempt to highlight one of the ways one can approach higher derivative theories - namely the attempt to define our higher derivative theory as a

simple limit of a theory with 2 derivatives.
The motivation for trying this is as follows: the Lagrangian (2.2) can be seen as the $\epsilon \to 0$ limit of:

$$\mathcal{L}_2 = -\frac{1}{2}\varphi(\Box + \epsilon^2)\Box\varphi\,. \tag{2.3}$$

In (2.2) and (2.3), the scalar field has mass dimension $[\varphi] = 0$ which means that $\epsilon$ has mass dimension $[\epsilon] = 1$. Now, we notice that the propagator goes as $\tilde{D}_F \sim \frac{1}{k^2(k^2-\epsilon^2)}$ which can be separated $\tilde{D}_F \sim -\frac{1}{\epsilon^2}\left(\frac{1}{k^2} - \frac{1}{k^2-\epsilon^2}\right)$. This seems to indicate that we can express $\mathcal{L}_2$ - and consequently $\mathcal{L}_1$ - as a 2-derivative theory of two different fields. The fact that both of these fractions have opposite sign indicates that one of these fields will be ghost-like.

Let us see how that is done. We introduce the auxiliary field $\varphi'$ with mass dimension $[\varphi'] = 1$ and write:

$$\mathcal{L}_2' = -\epsilon^2\frac{1}{2}\varphi\Box\varphi - \epsilon\varphi'\Box\varphi + \frac{1}{2}\epsilon^2(\varphi')^2\,. \tag{2.4}$$

The algebraic equation for $\varphi'$ is

$$\varphi' = \epsilon^{-1}\Box\varphi\,, \tag{2.5}$$

which upon substitution in (2.4) gives us (2.3), so the two actions are equivalent. Now we can diagonalise $\mathcal{L}_2'$ through the field redefinitions:

$$\varphi \to \epsilon^{-1}(\alpha + \beta) \qquad\qquad \varphi' \to -\beta\,, \tag{2.6}$$

where $\alpha$ and $\beta$ are scalar fields of dimension $[\alpha] = [\beta] = 1$. We then obtain:

$$\mathcal{L}_2' \to -\frac{1}{2}\alpha\Box\alpha + \frac{1}{2}\beta\left(\Box + \epsilon^2\right)\beta\,. \tag{2.7}$$

We have thus obtained an action for a massive and massless 2-derivative scalar field. What's more, notice that one of the two fields has the wrong sign for the kinetic term, and is therefore ghost-like, as expected [17].

This treatment highlights an important fact: going back to $\mathcal{L}_1$ requires taking the limit $\epsilon \to 0$. This limit clearly does not commute with the redefinition (2.6); in other words, the original theory (2.2) and the diagonal theory (2.7) are not an-

[17] One could explore the option of Wick rotating one of the fields, ie. $\beta \to i\beta$, but introducing interactions is then problematic, and this will in general not be enough to restore unitarity.

alytically connected.[18] Actually, considering the case of vector fields or higher spin fields is even more troublesome, as the inclusion of a mass term necessarily breaks some gauge invariance resulting in mismatching number of degrees of freedom. As was studied in [77], one could simply keep the form of (2.4), ie. formulate higher derivative theories as an ordinary theory along with some auxiliary fields to be integrated out later, without diagonalising. However, for the purposes of computing on-shell amplitudes scattering ghost modes, this analysis is problematic.
Instead, we will stick with the Lagrangian (2.2), and modify the way we compute scattering amplitudes.

### 2.1.3 Solution to equations of motion

Loosely speaking, scattering amplitudes correspond to taking well-separated free states of the theory, and bringing them closer so that they may interact. Before properly defining that, let us first study the free solutions to our particular theory.

The equation of motion for (2.2) is simply $\Box^2\varphi = 0$. A basis for the solutions is given by:

$$\varphi_0 = (A + (n \cdot x)B)\, e^{ik\cdot x} \qquad\qquad k^2 = 0\,, \tag{2.8}$$

where $A$ parametrises typical plane-wave modes which also satisfy the 2 derivative equation $\Box\varphi_0 = 0$, while $B$ parametrise growing " ghost" modes, first found for the Pais-Uhlenbeck 4-derivative oscillator [78]. Here, $n^a$ corresponds to a vector such that $n \cdot k \neq 0$, though the particular form of $n^a$ does not matter. Indeed, (2.8) is merely a basis for solutions of the equation of motion - a full solution includes an integration over arbitrary wave modes. As such, different choices of $n^a$ do not correspond to different solutions, since a complete solution corresponds to arbitrary linear combinations of (2.8) [60,76].

The fact that the ghost modes grow uncontrollably near the boundary is a manifestation of the lack of unitarity we were talking about in the previous section. Still, we would like to consider the formal object of a scattering amplitude of these modes. Clearly, the usual LSZ prescription isn't enough for that. One possibility is to modify it. One could repackage (2.8) by writing $\varphi_0 = \exp(iN(k)\cdot x)$,

[18] See [76] for a treatment of this limit.

with $N(k)^a \equiv k^a + \alpha\, n^a$ representing a "shifted" momentum. The expressions are then identical after Taylor expanding to first order in $\alpha$. This allows us to use LSZ with a crucial difference: a typical scattering momentum space amplitude looks like $A(p_1, \dots p_n)\delta^{(4)}(\sum_i p_i^2)$ . Here the amplitude will instead be replaced by $\tilde{A}(N(p_1), \dots N(p_n))\delta^{(4)}(\sum_i N(p_i)^2)$. Upon Taylor expanding, the coefficient of $\sum_k \epsilon_k$ would give the scattering amplitude with growing modes on the corresponding legs and non-growing modes on the others. Note that in general, such amplitudes may not follow momentum conservation since they will be proportional to derivatives of the momentum conserving delta function [19].

In general, this approach is rather messy. Furthermore when we generalise this discussion to the case of conformal gravitons, some of the subtleties arising from gauge invariance are harder to see. In conformal field theory or in AdS where the notion of "asymptotic states" is not well defined, the prescription is ambiguous. As such, we won't pursue this idea further. Instead, we will choose to study the on-shell effective action, which is well defined.

### 2.1.4 On-shell action

The "on-shell effective action" is a useful object which is equivalent to the theory's S-matrix for typical fields [79,80]. What's more, it remains a well-defined object for higher-derivative studies, so we will be employing it in our computations. First, let us introduce it heuristically using a generic model for a theory of field(s) $\Phi$. We start with the action

$$S[\Phi] = \int \mathrm{d}^d x \, \left(\mathcal{L}_0[\Phi] + \mathcal{L}_{int}[\Phi]\right) \tag{2.9}$$

$$\mathcal{L}_0[\Phi_0] = \Phi D_{kin} \Phi \,. \tag{2.10}$$

where $D_{kin}$ is an invertible kinetic operator and $\mathcal{L}_{int}$ contains the interactions. We denote the solutions of the *free* equations of motion $\Phi_0$, and the solutions of the full equations of motion $\Phi^{cl}$:

$$\left.\frac{\delta \mathcal{L}_0}{\delta \Phi}\right|_{\Phi=\Phi_0} = 0\,, \qquad\qquad \left.\frac{\delta\left(\mathcal{L}_0 + \mathcal{L}_{int}\right)}{\delta \Phi}\right|_{\Phi=\Phi^{cl}} = 0\,. \tag{2.11}$$

[19]Alternatively, this can be seen by rewriting the x-dependence of (2.8) in terms of derivative with respect to the momentum, which when integrated by parts may spoil momentum conservation.

It is possible to parametrise the latter solutions by the former:

$$\Phi^{cl} = \Phi_0 + D_{kin}^{-1} \frac{\delta \mathcal{L}_{int}}{\delta \Phi} \Bigg|_{\Phi = \Phi^{cl}(\Phi_0)} , \tag{2.12}$$

here, $D_{kin}^{-1}$ is the inverse operator of $D_{kin}$ defined by its green function, $G$, via $D_{kin}^{-1}\Psi(x) \equiv \int \mathrm{d}y\, G(x-y)\Psi(y)$. This equation is implicitly solved iteratively order by order in $\Phi_0$

In this language, the generating function for the connected S-matrix is given by:

$$\mathcal{S}[\Phi_0] = W[J] \Bigg|_{J=\phi_0 D_{kin}} \tag{2.13}$$

Since this corresponds to amputating legs, and multiplying the amplitude with the free solutions. This is equivalent to the LSZ reduction formula for the usual setup, but does not require us to define asymptotic states.
It turns out that the following relation holds:

$$e^{\frac{i}{\hbar}\mathcal{S}[\Phi_0]} \simeq \int D\Phi\, e^{\frac{i}{\hbar}S[\Phi^{cl}+\Phi]} \tag{2.14}$$

where the integration is over fields that decrease at infinity and "$\simeq$" means that the expression holds up to some boundary terms which we will ignore in our discussion [20].
Expanding this expression in orders of $\hbar$ leads us to the conclusion that *the tree-level S-matrix of the theory is simply given by the multi-linear part of the on-shell action parametrised by the free fields, $\Phi_0$* [21]. In other words,

$$\mathcal{S}^{tree}[\Phi_0] \simeq S[\Phi^{cl}] \tag{2.15}$$

[20] These boundary terms arise due to the fact that $\Phi_0$ do not decrease sufficiently fast at infinity and there is therefore an ambiguity between different equivalent actions if one integrates by parts. In the end, including these modifications will only change the overall constant of the amplitude, but not the steps of the computation [79, 80].

[21] Actually, it turns out that there is a relationship between the full S-matrix and the effective action: $\mathcal{S}[\Phi_0] = \Gamma[\Phi^*]$ where $\Phi^*$ is the solution to the "quantum equation of motion", $\frac{\delta\Gamma[\Phi]}{\delta\Phi}\Big|_{\Phi^*} = 0$. For our purposes, the tree-level relation will be enough.

### 2.1.5 Examples: 3 and 4-point amplitudes

Let us illustrate this with the example of our higher derivative scalar along with the interaction Lagrangian:

$$\mathcal{L}_0 = -\frac{1}{2}\varphi\Box^2\varphi \qquad\qquad \mathcal{L}_{int} = -\frac{g}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 \tag{2.16}$$

The free fields satisfy:

$$\Box^2\varphi_0 = 0\,, \tag{2.17}$$

while the classical fields $\varphi^{cl}$ satisfy:

$$\Box^2\varphi^{cl} = -\frac{g}{2}\left(\varphi^{cl}\right)^2 - \frac{\lambda}{3!}\left(\varphi^{cl}\right)^3. \tag{2.18}$$

These can be expanded in an infinite series of the free fields:

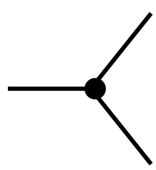
$$\varphi^{cl} = \varphi_0 - \frac{g}{2}\Box^{-2}\left(\varphi_0\right)^2 + \frac{g^2}{2}\varphi_0\Box^{-2}\left(\varphi_0\right)^2 - \frac{\lambda}{3!}\Box^{-2}\left(\varphi_0\right)^3 + \mathcal{O}(\varphi_0^4)\,. \tag{2.19}$$

Now, following (2.15), we can easily see that the 3-point amplitude is can be obtained from:

$$\begin{aligned} S_3^{tree}[\varphi_0] &\simeq \int \mathrm{d}^4x\,\left(\varphi^{cl}\Box^2\varphi^{cl} - \frac{g}{3!}\left(\varphi^{cl}\right)^3\right)\Bigg|_{\varphi_0^3} \\ &= \int \mathrm{d}^4x\,\frac{g}{3!}\left(\varphi_0\right)^3 \end{aligned} \tag{2.20}$$

where the last line is obtained after properly taking the boundary term into account.

In order to obtain the amplitude, one needs to label the legs by subbing $\varphi_0 = \sum_i^3 \epsilon_i\phi_{0,i}$ and look at the coefficient of $\epsilon_1\epsilon_2\epsilon_3$. After plugging in the plane wave solution (2.8), the different terms give us the amplitudes of various ghost and plane-wave modes. For instance, the term corresponding to scattering 3 plane-wave modes would give us $A(p_1, p_2, p_3) = g$. This matches what one would have obtained by using the usual Feynman rules and computing the diagram:

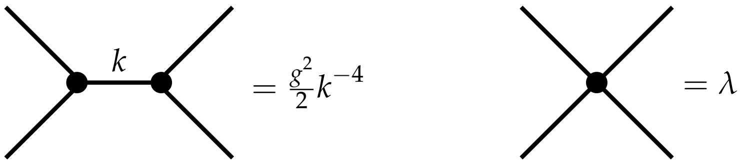


(2.21)

and amputated the external legs. What about the 4-point amplitude? The relevant part of the on-shell action is:

$$\mathcal{S}_4^{tree}[\varphi_0] \simeq \int \mathrm{d}^4x \, \frac{g^2}{4} \varphi_0^2 \Box^{-2} (\varphi_0)^2 - \frac{\lambda}{4!} (\varphi_0)^4 \tag{2.22}$$

The first term corresponds to the usual exchange diagrams and will lead to the s, t, u channels, while the second term is a contact diagram.




To summarise, we have studied a simple scalar higher-derivative model. Its non-unitarity is apparent already at the level of equations of motion, and we have thus adopted the rather formal approach of studying its on-shell equations of motion. In the case where we use the plane-wave solutions, this gives us a well-defined amplitude, like the LSZ formula would. As such, we can say that those solutions are admissible, and safely use them in scattering computations [2,81]. What this model didn't display was the interplay between higher derivatives and gauge symmetry, as well as the presence of interactions containing derivatives. When studying models of fields with spin $s > 0$, these features become important. This leads to finding more admissible states, which must be carefully defined, as we will see next section.

## 2.2 Conformal Gravity

We are now ready to introduce the main topic of this chapter. The conformal gravity action in 4 dimensions is given by:

$$S_W = \int \sqrt{|g|} \, C^{abcd} C_{abcd} \tag{2.23}$$

where $C^{abcd}$ is the Weyl tensor, defined in appendix A.3. Being quadratic in curvature, it has the soft UV properties alluded to before, and it is also the unique quadratic curvature action which is invariant under conformal transformations: $g_{ab}(x) \to \mathrm{O}(x) g_{ab}(x)$[22].

[22]Note that this is no longer true if it is added to the Einstein action (2.1).

For most of this chapter, we can instead use a slightly different formulation of the action:

$$S_{CG} = \int \sqrt{|g|} \left( R_{ab}^2 - \frac{1}{3} R^2 \right) . \tag{2.24}$$

Indeed, the Gauss-Bonnet term:

$$G = \int \sqrt{|g|} \left( R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2 \right) \tag{2.25}$$

is a topological invariant, which implies the equivalence of (2.23) and (2.24) up to boundary terms.

Let us now study the theory's equations of motion in order to characterise its scattering states.

### 2.2.1 Equations of Motion

The equations of motion arising from (2.23) are:

$$\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_W}{\delta g_{ab}} \equiv B^{ab} = 0 \, , \tag{2.26}$$

where $B^{ab}$ is the Bach tensor (see appendix (A.32)), which can be written as

$$B_{ab} = \left( D^c D^d + \frac{1}{2} R^{cd} \right) W_{acdb} \, . \tag{2.27}$$

Note that it is a tensor of $4^{\text{th}}$ order in derivatives. Completely analogously to the case we saw last section, certain solutions of 2-derivative equations solve (2.26). Indeed, for an Einstein manifold satisfying $R_{ab} = g_{ab}\, \alpha$, with $\alpha$ a constant, we can see that (2.27) vanishes.

In this chapter we will be interested in solutions which are perturbed around flat space. To this end, let us chose the metric to be

$$g_{ab} = \eta_{ab} + h_{ab} \tag{2.28}$$

Infinitesimal diffeomorphisms and metric rescaling transformations are then given by:

$$\delta_\epsilon h_{ab} = \partial_a \epsilon_b + \partial_b \epsilon_a \, , \tag{2.29}$$

$$\delta_\alpha h_{ab} = \eta_{ab}\alpha\,. \tag{2.30}$$

Taking this into account, we write the equations of motions, which can be obtained by looking at the linear perturbation of the Bach tensor:

$$B^{(1)}_{ab} \propto \Box^2 h_{ab} + \partial_a V_b + \partial_b V_a - \frac{1}{2}\eta_{ab}\partial^c V_c = 0\,, \tag{2.31}$$

where we defined:

$$V_a \equiv \frac{1}{3}\partial_a{}^{bc}h_{bc} - \Box\partial^b h_{ba}\,, \qquad . \tag{2.32}$$

As an aside, one can indeed compare this to the linearised Einstein equations,

$$G^{(1)ab} \propto \partial^{ab}h^c{}_c - \partial^a{}_c h^{bc} - \partial^b{}_c h^{ac} + \partial^c{}_c h^{ab} = 0 \tag{2.33}$$

and explicitly show that (2.33) is contained in (2.31) [23]. This will become clearer by picking a gauge.

The next step is to give the solutions to (2.31). This approach will explicitly allow us to see how many degrees of freedom our theory carries. The discussion will follow along the lines of section 2.1.3, except that here the presence of gauge symmetries (2.29)-(2.30) complicates things. As such, we will first have to pick a gauge, then solve the equations of motion.

We first pick a traceless gauge, $h^a{}_a = 0$ using the algebraic gauge variation (2.30). This modifies (2.29) to become:

$$\delta h_{ab} = \partial_a \epsilon_b + \partial_b \epsilon_a - \frac{1}{2}\eta_{ab}\,\partial^c\epsilon_c\,, \tag{2.34}$$

meaning we must fix this gauge freedom to define propagating solutions of (2.26).

### 2.2.2 Fixing the gauge

#### Conformal Gauge

This subsection will follow along the lines of the analysis presented in of [82]. We pick what we will call the "conformal gauge":

$$V^a = 0 \tag{2.35}$$

[23]In other words, it is possible to add a suitable linear combination of (2.33) and its derivatives to (2.31) to make the LHS vanish.

where $V^a$ was defined in (2.32). The reason for this choice is that the equations of motion (2.26) now reduce to simply: [24]

$$\Box^2 h_{ab} = 0\,. \tag{2.36}$$

Similarly to the scalar case, this has the solution:

$$h_{ab}(x) = \left(A_{ab} + B_{ab}(n\cdot x)\right) e^{ik\cdot x}\,, \qquad\qquad k^2 = 0\,, \tag{2.37}$$

where once again, $n\cdot k \neq 0$ , $A_{ab}$ and $B_{ab}$ are symmetric traceless polarization tensors. They are further constrained by (2.35):

$$(n\cdot k)B_{cb}k^b - \tfrac{1}{4}ik^a k^b A_{ab}k_c = 0\,. \tag{2.38}$$

At this stage, we note that we haven't fully fixed the gauge. The equations of motion are left unchanged by the shift:

$$\delta h_{ab} = \partial_a \xi_b + \partial_b \xi_a - \frac{1}{2}\eta_{ab}\partial_c \xi^c\,, \tag{2.39}$$

so long as $\Box^2 \xi_a = 0$. This $\xi^a$ is of the form:

$$\xi_a(x) = \left(u_a + v_a(n\cdot x)\right) e^{ik\cdot x}\,. \tag{2.40}$$

where $u_a$ and $v_a$ are just constant vectors. Combining (2.40) and (2.37), we see that the polarization tensors are fixed up to the transformations:

$$\delta A^{ab} = iu^a k^b + iu^b k^a - \tfrac{1}{2}ig^{ab}u^c k_c + v^a n^b + v^b n^a - \tfrac{1}{2}g^{ab}v^c n_c \tag{2.41}$$

$$\delta B^{ab} = iv^a k^b + iv^b k^a - \tfrac{1}{2}ig^{ab}v^c k_c\,. \tag{2.42}$$

Let us pause and count degrees of freedom. We started with the $2\times 10 - 2 = 18$ degrees of freedom of the traceless tensors $A_{ab}$ and $B_{ab}$. The condition (2.38) removes 4 of those, and the choice of $u^a$ and $v^a$ will allow us to remove $2\times 4 = 8$ more. So in total we have $18 - 12 = 6$ physical degrees of freedom. This matches the counting done in [83, 84].

One question remains: what type of fields do these degrees of freedom cor-

[24] As a quick aside, we mention that using the diffeomorphism invariance of Einstein gravity, one may pick the gauge $V'_a \equiv \partial_b h^{ba} - \frac{1}{2}\partial^a h^b{}_b = 0$, which reduces (2.33) to $\Box h_{ab} = 0$. This makes manifest our earlier assertion: though this graviton is not traceless, it clearly solves eq. (2.36).

respond to? In order to answer that, it is most convenient to go to a particular basis, and fix the remnant gauge. Then, it becomes useful to switch to a helicity basis, so that each degree of freedom is associated with a particle of definite helicity. This is done in appendix C.1. In the end, we can show that $A^{ab}$ can be decomposed into 4 helicity modes: $\pm 2$ and $\pm 1$, while $B^{ab}$ contains 2 modes of helicity $\pm 2$. So it seems we have recovered the usual Einstein oscillatory modes of helicity $\pm 2$ and gained four more of helicity $\pm 2$ and $\pm 1$, only two of which are growing.

At this stage, one might be tempted to draw the conclusion that the helicity $\pm 1$ ghost modes correspond to valid scattering states, and move on to computing amplitudes. However, one needs to be more careful: it turns out that this decomposition is *not* gauge-invariant. Let us see this explicitly in a different gauge.

**TT gauge**

We now choose the traceless transverse (TT) gauge:

$$\partial^a h_{ab} = 0 \qquad\qquad h^a{}_a = 0 \tag{2.43}$$

Once again this leads to the equations of motion (2.36) except that this time the conditions on the polarization tensor are

$$k^a\,\mathrm{B}_{ab} = 0\,, \qquad\qquad i\,k^a\,\mathrm{A}_{ab} + n^a\,\mathrm{B}_{ab} = 0\,. \tag{2.44}$$

which accounts for 8 conditions. Once again, there is some residual gauge left, of the form (2.39), but this time $\xi^a$ must also satisfy

$$\Box\xi_a + \frac{1}{2}\partial_{ab}\xi^b = 0\,, \tag{2.45}$$

in order to preserve (2.43). In terms of the polarization tensors, this translates to a variation parametrised by a single constant vector $u^a$:

$$\begin{aligned} \delta\mathrm{A}_{ab} &= i\Big(u_a k_b + u_b k_a - \frac{2}{5}\eta_{ab} u\cdot k\Big) - \frac{i\,u\cdot k}{5\,n\cdot k}\left(k_a n_b + k_b n_a\right)\,, \\ \delta\mathrm{B}_{ab} &= \frac{2}{5}\frac{u\cdot k}{n\cdot k}k_a k_b\,. \end{aligned} \tag{2.46}$$

The DoF counting gives the same result: $2\times 9 - 2\times 4 - 4 = 6$.

The remnant gauge fixing is done explicitly in Appendix C.1, and one finds that the decomposition is different: we still have two sets of spin $\pm 2$ helicities, one oscillatory and one growing, but this time the spin 1 modes are mixed: they have a purely oscillatory part and a growing part. In other words, it is possible to isolate the helicity $\pm 1$ modes contained in $\eta_{ab}$ in a generic frame as:

$$h_{ab} \sim \left( (1 - 2i\, n \cdot x) S^{(\pm)}_{ab} + (1 + 2i\, n \cdot x) \tilde{S}_{ab} \right) e^{ik \cdot x} \,, \tag{2.47}$$

where $S_{ab}$ and $\tilde{S}_{ab}$ are constant matrices of helicity $\pm 1$ which depend only on $k^a$ and $n^a$.

This may seem worrying, but it is actually somewhat expected: the field $h_{ab}$ is not a gauge-invariant quantity, so there is no reason that its helicity decomposition should be as well. This seems problematic for the our study of amplitudes: which modes are oscillatory and suitable for scattering, and which modes are not?

To answer this question, we must look at the gauge-invariant quantity associated with $h_{ab}$, namely the linearised curvature [25]. If we pick, say, the conformal gauge and compute $C^{(1)}_{abcd}$ using the relevant solution for $h_{ab}$, it splits into a purely oscillatory part and a growing part:

$$C^{(1)}_{abcd} = M_{abcd} + x^e N_{abcde} \,.$$

Crucially, we find that the helicity $\pm 1$ modes are only contained within $M_{abcd}$ while the growing part, $N_{abcde}$ is solely composed of the growing helicity $\pm 2$ ghosts. Since this is a gauge-invariant statement, we can now definitively say that there are two states of admissible modes: the two-derivative Einstein gravitons and the helicity $\pm 1$ modes.

### 2.2.3 Two-derivative formulation of Conformal Gravity

Another perspective comes from reformulating the theory as a 2-derivative theory, much like we did for the scalar field in section 2.1.2. We start with the modified action (2.24). One way to reduce it to a two-derivative theory is to sim-

[25] Note that only the curvature $C^a{}_{bcd}$ is invariant under conformal transformations, while for instance $C_{abcd}$ is not. However, at the linearised level and around flat-space $C^{(1)a}{}_{bcd} = \eta^{ae} C^{(1)}_{ebcd} = \ldots$, so $C^{(1)}_{abcd}$ is a gauge invariant quantity.

ply introduce an auxiliary tensor $\varphi_{ab}$ and rewrite the action as

$$\hat{\mathcal{L}}_W = -\sqrt{|g|}\left(\varphi^{ab}R_{ab} - \frac{1}{2}\varphi^a{}_a R + \frac{1}{4}\varphi_{ab}\varphi^{ab} - \frac{1}{4}\varphi^a{}_a\varphi^b{}_b\right) \tag{2.48}$$

which leads back to (2.24) upon integration of $\varphi_{ab}$. It turns out to be more illuminating to build Weyl gravity as a gauged theory of the conformal group SO(2,4) [85], to get [77]:

$$\mathcal{L}'_W = -\sqrt{|g|}\left[\varphi^{ab}\hat{G}_{ab} + \tfrac{1}{4}(\varphi^{ab}\varphi_{ab} - \varphi^a{}_a\varphi^b{}_b) + \tfrac{1}{4}F^{ab}F_{ab}\right]. \tag{2.49}$$

with:

$$\hat{G}_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R + D_{(a}\mathrm{b}_{b)} + \frac{1}{2}\mathrm{b}_a\mathrm{b}_b - g_{ab}(D^c\mathrm{b}_c - \tfrac{1}{4}\mathrm{b}^c\mathrm{b}_c) \tag{2.50}$$

$$F_{ab} = \partial_a b_b - \partial_b b_a\,, \tag{2.51}$$

where $D_a$ is the covariant derivative with respect to $g_{ab}$. Here, $\varphi_{ab}$ and $b_a$ are the gauge fields associated to special conformal transformations and dilatation respectively. This action has the usual Weyl invariance of the metric, but is also invariant under:

$$\delta\varphi_{ab} = 2D_{(a}\zeta_{b)} + 2\mathrm{b}_{(a}\zeta_{b)} - g_{ab}\mathrm{b}^c\zeta_c\,, \qquad \delta\mathrm{b}_a = \partial_a\lambda - \zeta_a\,. \tag{2.52}$$

Already, we can see that the field $b_a$ has a Stuckelberg-type symmetry and can be fully gauged away. This brings $\mathcal{L}'_W$ into the form of $\hat{\mathcal{L}}_W$ , in (2.48), thus ensuring that we are dealing with an equivalent action. However an interesting observation can be made by picking a different gauge. If we linearise the action around a flat background like in the previous section, we find the equations of motion to be:

$$\varphi_{ab} = -2(\hat{G}_{ab} - \tfrac{1}{3}g_{ab}\hat{G}^c{}_c) = -2(R_{ab} - \tfrac{1}{6}g_{ab}R) - 2\partial_{(a}\mathrm{b}_{b)} + \ldots\,, \tag{2.53}$$

$$\Delta_2\varphi_{ab} + \ldots = 0\,, \tag{2.54}$$

$$\partial^a F_{ab} + \partial^a\varphi_{ab} - \partial_b\varphi^c c + \ldots = 0\,. \tag{2.55}$$

where the dots indicate terms of higher orders in the fields, $\Delta_2$ is the linearized Einstein operator: $G_{ab} = \frac{1}{2}\Delta_2 h_{ab} + O(h^2)$ , so that $\Delta_2 = -\Box + \cdots$ ( see appendix A.3 for the relevant linearised curvatures). We then use the Weyl and diffeomorphism invariance of the theory, as well as (2.52) to fix the TT gauge on $h_{ab}$ and the harmonic gauge on $\varphi_{ab}$:

$$\partial^a h_{ab} = 0\,, \quad h^a{}_a = 0\,; \qquad \partial^a \varphi_{ab} = \frac{1}{2}\partial_b \varphi^c{}_c\,, \tag{2.56}$$

Plugging this in (2.53)-(2.55) and taking traces and derivatives of those, we get the extra conditions:

$$\Box h_{ab} = \varphi_{ab} + 2\partial_{(a} b_{b)}\,; \qquad \Box \varphi_{ab} = 0\,, \qquad \varphi^a{}_a + 2\partial^a b_a = 0\,; \qquad \Box b_a = 0\,, \tag{2.57}$$

Finally, on-shell it is possible to use the remnant gauge invariance of $\varphi_{ab}$ to fix $\varphi^a{}_a = 0$ much like in the usual treatment of Einstein perturbations (see eg. [86]). This then sets $\partial^a b_a = 0$.

From these equations, we see that the field $\varphi_{ab}$ describes a massless purely oscillating field while $b_a$ is a massless vector. As for $h_{ab}$, it satisfies $\Box^2 h_{ab} = 0$ so it can be expanded in terms of an oscillatory and growing part, like in (2.37). However, the growing part is related to the $\xi$ gauge-invariant combination of $\varphi_{ab} + 2\partial_{(a} b_{b)}$. Overall, this means that we have $2+2+2=6$ DoFs, as predicted. The admissible states are contained by the oscillatory part of $h_{ab}$ and $b_a$.

The attractive feature of this representation is that the action is that it gives a nice off-shell and non-linear separation of the theory's degrees of freedom. This allows us to predict what certain 3-point amplitudes will look like. Indeed, we can see that the "mixed" scattering consisting of amplitudes $A(b,h,h)$ of two Einstein gravitons and one massless vector field $b_a$ will vanish - regardless of helicity. Furthermore, $A(b,b,h)$ will correspond to the cubic amplitudes obtained in Einstein-Maxwell theory: the only possible contributions will come from the linearisation of the term $\sqrt{|g|} F_{ab} F_{cd}\, g^{ac} g^{bd}$.
It remains to be seen whether similar arguments may be extended to the mixed amplitudes of 4-points and higher.

## 2.3 3-point amplitudes in Conformal Gravity.

We would now like to compute scattering amplitudes in the theory. It is known that at tree-level scattering amplitudes of Conformal Gravity with Einstein modes on the external legs are all trivial, [1,2,62,81,87] [26] , but the question of the helicity $\pm 1$ modes remain. We will compute these, and focus on three-point functions as they are considerably simpler due to the absence of any internal propagators on internal legs [27].

When we fully fixed a gauge earlier, we had to choose a particular frame. However, when we have to consider several on-shell momenta and corresponding polarization tensors at once, this becomes clunky. Furthermore, it can be shown that three point amplitudes for massless theories are trivial for real momenta - we will have to look at complexified momenta. Since we are in $d = 4$ and computing the scattering of massive states, we are therefore naturally led to employ the spinor helicity formalism (see appendix B.1 for a brief review). In the next subsection, we translate the earlier results for polarization tensors and linearised curvatures to the spinor helicity language. This will then be used to compute the scattering amplitudes and express them in a convenient way.

### 2.3.1 Polarization Data in Spinor Helicity Formalism

In 4 dimensions, the Weyl tensor decomposes into a self-dual (SD) and anti self-dual sector (ASD). The spinor helicity formalism makes this manifest:

$$C_{abcd} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}}\,\Psi_{\alpha\beta\gamma\delta} + \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\,\widetilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \tag{2.58}$$

where above, we understand that the equality means after translating to the language of $SL(2,\mathbb{C})$ spinor indices. Hence, $\widetilde{\Psi}$ and $\Psi$ are the SD and ASD Weyl tensors respectively.

The free equations of motion are relatively simple:

$$\partial^{\alpha\dot{\alpha}}\,\partial^{\beta\dot{\beta}}\,\widetilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0\,, \qquad \partial^{\alpha\dot{\alpha}}\,\partial^{\beta\dot{\beta}}\,\Psi_{\alpha\beta\gamma\delta} = 0\,, \tag{2.59}$$

where we now understand $\widetilde{\Psi}$ and $\Psi$ to mean the *linearised* curvatures. Guided by

[26] In particular, amplitudes scattering the Einstein modes in flat space were computed in [88] where they started with the twistor string theory description of [60]. This is related to non-minimal supergravity which includes coupling to a dimension 0 scalar via, schematically $\phi\Box^2\phi + (1 + k\,\phi + ...)C^2 + ....$ The tree-level amplitudes then receives only contributions from the scalar exchange, implying that the pure Conformal Gravity amplitude is trivial.

[27] In the language of the on-shell action, this statement can be translated as "absence of non local operators like $\Box^{-2}$".

the previous section, in order to solve these, we write :

$$h_{ab} = \varepsilon_{ab}\, e^{\mathrm{i}k\cdot x}\,. \tag{2.60}$$

We now seek the various $\varepsilon_{\alpha\dot\alpha\beta\dot\beta}$ which can solve these equations. To do so, we introduce the auxiliary spinors $a^{\alpha}$, $\tilde a^{\dot\alpha}$, which must satisfy $[\tilde a\tilde\lambda], \langle a\lambda\rangle \neq 0$. This condition actually fixes the auxiliary spinors up to a scale. In fact, we will impose the normalisations:

$$\langle a\,\lambda\rangle = 1 = [\tilde a\,\tilde\lambda] \tag{2.61}$$

which fixes $\tilde a$ and $a$ to have little group weight $-1$ and $+1$ respectively.

Finally, we are able to write down solutions to (2.59), expecting to get the same spectrum that we derived in the covariant formalism before. Indeed, the most obvious solutions are the usual Einstein helicity $\pm 2$ modes:

$$\varepsilon^{(-2)}_{\alpha\dot\alpha\beta\dot\beta} = \kappa\,\lambda_\alpha\lambda_\beta\,\tilde a_{\dot\alpha}\tilde a_{\dot\beta}\,, \qquad \varepsilon^{(+2)}_{\alpha\dot\alpha\beta\dot\beta} = \kappa\,\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}\,a_\alpha a_\beta\,, \tag{2.62}$$

where $\kappa$ is a constant of proportionality. One can straightforwardly show that the linearised curvature tensors obtained using these polarizations are:

$$\Psi^{(-2)}_{\alpha\beta\gamma\delta} = \kappa\,\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\delta\,e^{\mathrm{i}\,k\cdot x}\,, \qquad \widetilde\Psi^{(+2)}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \kappa\,\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}\tilde\lambda_{\dot\gamma}\tilde\lambda_{\dot\delta}\,e^{\mathrm{i}\,k\cdot x}\,. \tag{2.63}$$

From this one can see that $\kappa$ must have conformal dimension $[\kappa] = -1$. [28] This actually arises because these objects are the familiar Einstein linearised curvature: they satisfy the linearised *Einstein* equations:

$$\partial^{\alpha\dot\alpha}\,\widetilde\Psi_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 0\,, \qquad \partial^{\alpha\dot\alpha}\,\Psi_{\alpha\beta\gamma\delta} = 0\,, \tag{2.64}$$

which are lower in derivatives. Note that this dimensionful constant is not intrinsic to Conformal Gravity, it is merely introduced by selecting lower derivative solutions. In [62], where it is shown explicitly how to obtain classical Einstein solutions from Weyl solutions, the constant $\kappa$ is explicitly related to Newton's constant: $\kappa = \sqrt{8\pi G_N}$. Note that above the dependence of the auxiliary spinors drops out - they play the role of pure gauge in this sector of the theory. Further-

[28] This can be seen as follows. Momentum spinors have mass dimension $[\lambda] = [\tilde\lambda] = \frac{1}{2}$ by definition. Since the metric in spinor coordinates is given by $\mathrm{d}s^2 = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\mathrm{d}x^{\alpha\dot\alpha}\mathrm{d}x^{\beta\dot\beta}$ meaning that the Levi-Civita symbols scale with weight $\frac{1}{2}$. Finally, the Weyl tensor has weight $[C] = +2$, so that $[\widetilde\Psi] = [\Psi] = +1$ Under a rescaling with weight $+1$.

more, these linearised curvatures satisfy the linearised Einstein equations (2.64).

Next, we express the helicity $\pm 1$ modes through the polarization tensors:

$$\varepsilon^{(-1)}_{\alpha\dot{\alpha}\beta\dot{\beta}} = \lambda_{(\alpha} a_{\beta)}\, \tilde{a}_{\dot{\alpha}} \tilde{a}_{\dot{\beta}}\,, \qquad \varepsilon^{(+1)}_{\alpha\dot{\alpha}\beta\dot{\beta}} = \tilde{\lambda}_{(\dot{\alpha}} \tilde{a}_{\dot{\beta})}\, a_{\alpha} a_{\beta}\,. \tag{2.65}$$

these, in turn, give rise to the following linearised curvatures:

$$\Psi^{(-1)}_{\alpha\beta\gamma\delta} = a_{(\alpha}\lambda_{\beta}\lambda_{\gamma}\lambda_{\delta)}\, e^{\mathrm{i}\,k\cdot x}\,, \qquad \widetilde{\Psi}^{(+1)}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \tilde{a}_{(\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}\tilde{\lambda}_{\dot{\gamma}}\tilde{\lambda}_{\dot{\delta})}\, e^{\mathrm{i}\,k\cdot x}\,, \tag{2.66}$$

where we note that the auxiliary spinors do not vanish - they are not pure gauge here.

Finally, there is one other type of polarization tensor that solves (2.59): these are the growing helicity $\pm 2$ modes. Introducing a further set of auxiliary spinor $\beta, \tilde{\beta}$ normalised as $a, \tilde{a}$:

$$\langle \beta\, \lambda \rangle = 1 = [\tilde{\beta}\, \tilde{\lambda}]\,, \tag{2.67}$$

we write:

$$\varepsilon^{(-2)}_{\alpha\dot{\alpha}\beta\dot{\beta}} = \tilde{\beta}^{\dot{\gamma}} x_{\dot{\gamma}(\alpha}\lambda_{\beta)}\, \tilde{a}_{\dot{\alpha}}\tilde{a}_{\dot{\beta}}\,, \qquad \varepsilon^{(+2)}_{\alpha\dot{\alpha}\beta\dot{\beta}} = \beta^{\gamma} x_{\gamma(\dot{\alpha}}\tilde{\lambda}_{\dot{\beta})}\, a_{\alpha} a_{\beta}\,, \tag{2.68}$$

which lead to the growing curvatures:

$$\Psi^{\mathrm{g}(+2)}_{\alpha\beta\gamma\delta} = \lambda_{(\alpha}\lambda_{\beta}\lambda_{\gamma}\, x_{\delta)}{}^{\dot{\alpha}}\tilde{\beta}_{\dot{\alpha}}\, e^{\mathrm{i}\,k\cdot x}\,, \qquad \tilde{\Psi}^{\mathrm{g}(-2)}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \tilde{\lambda}_{(\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}\tilde{\lambda}_{\dot{\gamma}}\, x_{\dot{\delta})}{}^{\alpha}\beta_{\alpha}\, e^{\mathrm{i}\,k\cdot x}\,. \tag{2.69}$$

This concludes the list of independent polarization tensors, and their associated linearised curvatures which solve eq. (2.59). Writing these down so far required us to make correct guesses using our understanding of the types of solutions from the covariant formalism, but we will soon see a way to derive them more inductively from the twistor formalism.

### 2.3.2 Relation to double copy description

We now address, as an aside, the relation between the formulation above, and a possible *double copy* description of Conformal Gravity. More precisely, in [89], is shown certain "non-minimal" conformal gravities can have their amplitudes constructed from the tensor product of kinematic operators from two different spin 1 gauge theories [90,91]. These gauge theories are Yang-Mills, and a higher derivative version of it, whose kinematic term schematically looks like $(DF)^2$.

On the gravity side, we note that "non-minimal" conformal gravity is different from the one we study (it contains additional scalars whose couplings to the graviton is controlled by arbitrary functions cf. [4, 60, 92–94]) but at the linearised level the two are identical. This translates to the fact that we can see a manifestation of this double copy in the polarization data of our theory. Indeed, the Yang-Mills polarization tensors are simply given by:

$$\epsilon^{(-1)}_{\alpha\dot{\alpha}} = \lambda_\alpha \, \tilde{a}_{\dot{\alpha}} \,, \qquad \epsilon^{(+1)}_{\alpha\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \, a_\alpha \,. \tag{2.70}$$

One can then simply see that taking a symmetric product of these polarization tensors lead to the Einstein polarizations - eg. $\epsilon^{(-2)}_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon^{(-1)}_{\alpha\dot{\alpha}}\epsilon^{(-1)}_{\beta\dot{\beta}}$. Taking a closer look at the four-derivative theory considered in [89], its equations of motion are:

$$\Box \partial_a F^{ab} = 0 \,, \tag{2.71}$$

which is clearly solved by the gluons (2.70). Once again, there exist more solutions: there are helicity $\pm 1$ growing modes: $A^{\mathrm{g}}_a \sim (n \cdot x) \mathrm{B}_a \, e^{\mathrm{i}k\cdot x}$, which are non-admissible states. There is also an oscillatory scalar mode, whose polarization is given by :

$$e^{(0)}_{\alpha\dot{\alpha}} = a_\alpha \, \tilde{a}_{\dot{\alpha}} \,. \tag{2.72}$$

One can see that taking symmetric products of these spin 0 modes with the earlier modes (2.70), we can recover the Conformal Gravity helicity $\pm 1$ oscillatory polarizations (2.65).

### 2.3.3 Conformal Gravity 3-Point amplitudes

We now have all the ingredients to compute the 3-point function. Taking the definition of (2.15), to compute a particular amplitude, all we need is to simply compute the cubic part of the action and plug in the appropriate solutions (2.65) and (2.62).

We first note that 3-point functions have a constraint: only $\overline{\text{MHV}}$ or MHV amplitudes can be non vanishing - that is to say amplitudes involving the scattering of two positive and one negative helicity particles or two negative and one positive. Furthermore, due to 3-point special kinematics one can show that $\overline{\text{MHV}}$ (respectively MHV ) amplitudes must be a function of only dotted (undotted) spinors [95, 96]. Below, we will focus on $\overline{\text{MHV}}$, knowing that MHV amplitudes can easily be obtained via conjugation (ie. swapping angle and square brackets).

We now state the results of computing the $\overline{\text{MHV}}$ on-shell cubic action using the various external legs polarizations.

**3 Einstein Gravitons**

$$A_3(1^-, 2^+, 3^+) = 0\,. \tag{2.73}$$

The amplitude is trivial. This is a known result and a nice sanity check. [1, 2, 62, 81, 87].

**1 spin 1 ghost and 2 gravitons**

$$A_3(a_1, 2^+, 3^+) = \kappa^2 \, \frac{[2\,3]^5}{[1\,2]\,[3\,1]} \, \delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i\Big) = 0\,. \tag{2.74}$$

The second equality is implied by crossing symmetry: it can be seen that $A_3(a_1, 2^+, 3^+) = -A_3(a_1, 3^+, 2^+)$ implying that it actually vanishes.

**2 spin 1 ghost and 2 gravitons**

$$A_3(a_1, \tilde{a}_2, 3^+) = \kappa \, \frac{[2\,3]^4 \, \langle a_1\,1\rangle \, [\tilde{a}_2\,2]}{[1\,2]^2} \, \delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i\Big)\,. \tag{2.75}$$

This is the first amplitude which is truly non-vanishing. In fact, it is further simplified when we impose the normalisations (2.61):

$$A_3(a_1, \tilde{a}_2, 3^+) = \kappa \, \frac{[2\,3]^4}{[1\,2]^2} \, \delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i\Big)\,. \tag{2.76}$$

The kinematics of this amplitude match what is expected for the scattering of particles of helicity $(-1, +1, +2)$ from [96]. Furthermore,2.76 is proportional to an Einstein-Maxwell scattering amplitude, as we predicted in section 2.2.3.

### Other amplitudes

Other amplitudes involving admissible states are found to vanish. We could, formally, compute other "amplitudes" using the modes of (2.68). Generically, these amplitudes were computed in [88] in the context of twistor-string theory. We note that there exists a degenerate configuration leading to a perfectly finite expression, namely when we scatter one growing ghost and two Einstein modes:

$$A_3(\tilde{\beta}_1, 2^+, 3^+) = \kappa^2 \, [1\,\tilde{\beta}_1] \, \frac{[2\,3]^6}{[1\,2]^2 \, [3\,1]^2} \, \delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i\Big) \, . \qquad (2.77)$$

This amplitude was first found in [97]. Despite the existence of this amplitude, the growing ghosts are still unphysical: should they be included in the set of admissible scattering states they *will* lead to amplitudes which aren't well defined, even if a few degenerate finite configurations exist.

Finally, we note that it is possible to obtain the MHV analogue of (2.76) by simple conjugation:

$$A_3(a_1, \tilde{a}_2, 3^+) = \kappa \, \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle^2} \, \delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i\Big) \, . \qquad (2.78)$$

## 2.4 Twistor-spinor representation and Scattering Formula

In the previous section, we successfully computed amplitudes involving *any* external states. However, the computation of each of those amplitudes requires the evaluation of the on-shell action where we have fixed the polarization tensors to have the correct form by hand. One could ask whether it is possible to obtain a compact expression, which depends on some generalised notion of polarization tensors, and which captures all the amplitudes we've computed above.

As it turns out, there is such a formula, and it can be obtained by enhancing the spinor-helicity formalism and using "twistor-spinors" [29].

[29] This is similar to the way one can use spinor helicity to describe massive states by including indices of the little group [98–100].

The usefulness of the twistor formalism ultimately descends from the fact that conformal symmetry acts linearly in twistor space. In other words, there is a connection, known as the Cartan connection (or local twistor connection), which acts covariantly on sectinos of the local twistor bundle [101–103]. Consequently, it is a natural language to study *any* conformally invariant field theory, and as such the results we will obtain for CG in this way will later naturally generalise to CHS theory.

First we will see how one can repackage the linearised Bach equations (2.59) by introducing a twistor-spinor object, before showing how to obtain the various on-shell states that we had before. Finally we will reach the convenient formula eqs. (2.117) and (2.118).

### 2.4.1 Twistor-Spinors and Lower Derivative formulation

We now introduce twistor indices, $A, B, C$ which are equivalent to spinor indices of $SL(4, \mathbb{C})$ [104,105]. We split the four values of these twistor indices into $SL(2, \mathbb{C})$ spinor indices of opposite chirality according to:

$$T_A(x) = \begin{pmatrix} \tilde{t}_{\dot{\alpha}}(x) \\ t^{\alpha}(x) \end{pmatrix}, \qquad S^A(x) = \begin{pmatrix} \tilde{s}^{\dot{\alpha}}(x) \\ s_{\alpha}(x) \end{pmatrix}. \tag{2.79}$$

A "twistor-spinor" object is then a tensor carrying both twistor indices and spinor indices. It follows the constraint that its traces vanish:

$$T_{A\beta} = \begin{pmatrix} \tilde{t}_{\dot{\alpha}\beta} \\ t^{\alpha}{}_{\beta} \end{pmatrix}, \qquad t^{\alpha}{}_{\alpha} = 0\,. \tag{2.80}$$

As we said before, there is a Cartan connection which acts covariantly on these tensors. It is given by:

$$\mathcal{D}_{\alpha\dot{\alpha}} = D_{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}}\,, \tag{2.81}$$

where $D_{\alpha\dot{\alpha}}$ is the usual Levi-Civita connection and $\mathcal{A}$ is a 1-form which takes values in the (complexified) conformal algebra $\mathfrak{sl}(4, \mathbb{C})$. This "potential" is given by:

$$(\mathcal{A}_{\alpha\dot{\alpha}})^B{}_C = \begin{pmatrix} 0 & \delta^{\gamma}_{\alpha}\,\delta^{\dot{\beta}}_{\dot{\alpha}} \\ -P_{\alpha\dot{\alpha}\beta\dot{\gamma}} & 0 \end{pmatrix}, \tag{2.82}$$

where $P_{\alpha\dot{\alpha}\gamma\dot{\beta}}$ is the Schouten tensor given in terms of the trace free Ricci curvature $\Phi_{\alpha\beta\dot{\alpha}\dot{\gamma}}$ and the scalar curvature $\Lambda$:

$$P_{\alpha\dot{\alpha}\beta\dot{\gamma}} \equiv \Phi_{\alpha\beta\dot{\alpha}\dot{\gamma}} - \Lambda\, \epsilon_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\gamma}} \,. \tag{2.83}$$

The action of the connection on a twistor object can be deduced according to

$$\mathcal{D}_{\alpha\dot{\alpha}} T^B = D_{\alpha\dot{\alpha}} T^B + (\mathcal{A}_{\alpha\dot{\alpha}})^B{}_C\, T^C \,, \qquad \mathcal{D}_{\alpha\dot{\alpha}} S_B = D_{\alpha\dot{\alpha}} S_B + (\mathcal{A}_{\alpha\dot{\alpha}})_B{}^C\, S_C \,, \tag{2.84}$$

and the fact that it follows the Leibniz rule [105].

Conformal Gravity can be described as a gauging of conformal symmetry - it is for this reason that the language of twistors is appropriate here: twistor indices are in spinorial representations of the Conformal Group. Furthermore, the Cartan connection is conformally invariant (unlike the Levi-Civita connection). This can be seen by showing that the curvature built from this connections:

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = (\mathcal{F}_{\alpha\dot{\alpha}\beta\dot{\beta}})^C{}_D = \begin{pmatrix} \epsilon_{\alpha\beta}\, \widetilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\delta}}{}^{\dot{\gamma}} & 0 \\ (\epsilon_{\beta\alpha} \nabla_{\gamma}{}^{\dot{\rho}} \widetilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\delta}\dot{\rho}} + \epsilon_{\dot{\beta}\dot{\alpha}} \nabla_{\dot{\delta}}{}^{\rho} \Psi_{\alpha\beta\gamma\rho}) & \epsilon_{\dot{\beta}\dot{\alpha}}\, \Psi_{\alpha\beta\gamma}{}^{\delta} \end{pmatrix} . \tag{2.85}$$

is *covariant* with respect to conformal transformations [106]. Note that this property will extend to the case of CHS theory, which is also conformally invariant.

Another reason to use twistors is that they allow us to reduce the second-order spinor equations (2.59) into first-order ones on a twistor-tensor [107, 108]. To see this, we consider the following twistor-spinor:

$$\Gamma_{A\beta\gamma\delta} = \begin{pmatrix} \gamma_{\dot{\alpha}\beta\gamma\delta} \\ \Psi^{\alpha}{}_{\beta\gamma\delta} \end{pmatrix} , \qquad \Gamma_{A\beta\gamma\delta} = \Gamma_{A(\beta\gamma\delta)} \,. \tag{2.86}$$

Enforcing the tracelessness condition of (2.80), we obtain:

$$\Psi^{\alpha}{}_{\alpha\gamma\delta} = 0 \quad \Rightarrow \quad \Psi_{\alpha\beta\gamma\delta} = \Psi_{(\alpha\beta\gamma\delta)} \,. \tag{2.87}$$

We will consider $\Gamma_{A\beta\gamma\delta}$ to be a field living in flat Minkowski space. We then use the Cartan connection of (2.81) to impose a Maxwell-like equation of motion on it:

$$\mathcal{D}^{\beta\dot{\beta}}\, \Gamma_{A\beta\gamma\delta} = 0 \,. \tag{2.88}$$

which, in flat space, gives us the system of equations:

$$\partial^{\beta\dot{\beta}}\,\gamma_{\dot{\alpha}\beta\gamma\delta} = 0\,, \qquad \partial^{\beta\dot{\beta}}\,\Psi^{\alpha}{}_{\beta\gamma\delta} - \gamma^{\dot{\beta}\alpha}{}_{\gamma\delta} = 0\,. \tag{2.89}$$

The latter equation is an algebraic equation for $\gamma^{\dot{\beta}\alpha}{}_{\gamma\delta}$, which can be substituted back into the first to obtain:

$$\partial^{\alpha\dot{\alpha}}\,\partial^{\beta\dot{\beta}}\,\Psi_{\alpha\beta\gamma\delta} = 0\,, \tag{2.90}$$

as in (2.59). A conjugate operator can be introduced to obtain the positive helicity sector:

$$\widetilde{\Gamma}^{A}{}_{\dot{\beta}\dot{\gamma}\dot{\delta}} = \begin{pmatrix} \widetilde{\Psi}^{\dot{\alpha}}{}_{\dot{\beta}\dot{\gamma}\dot{\delta}} \\ \tilde{\gamma}_{\alpha\dot{\beta}\dot{\gamma}\dot{\delta}} \end{pmatrix}, \qquad \widetilde{\Psi}^{\dot{\alpha}}{}_{\dot{\alpha}\dot{\gamma}\dot{\delta}} = 0\,, \tag{2.91}$$

with the same equation of motion:

$$\mathcal{D}^{\beta\dot{\beta}}\,\widetilde{\Gamma}^{A}{}_{\dot{\beta}\dot{\gamma}\dot{\delta}} = \begin{pmatrix} \partial^{\beta\dot{\beta}}\widetilde{\Psi}^{\dot{\alpha}}{}_{\dot{\beta}\dot{\gamma}\dot{\delta}} - \tilde{\gamma}^{\beta\dot{\alpha}}{}_{\dot{\gamma}\dot{\delta}} \\ \partial^{\beta\dot{\beta}}\tilde{\gamma}_{\alpha\dot{\beta}\dot{\gamma}\dot{\delta}} \end{pmatrix} = 0\,. \tag{2.92}$$

Once again, these lead back to (2.59). In spirit, this procedure is similar to the one we highlighted in section 2.2.3 for the covariant formalism.

### 2.4.2 Momentum eigenstates

We will now go to momentum state and find solutions of equations (2.88) and (2.92). Focusing first on the negative helicity sector, we can write

$$\Gamma_{A\beta\gamma\delta} = B_A\,\lambda_\beta\,\lambda_\gamma\,\lambda_\delta\,e^{\mathrm{i}\,k\cdot x}\,, \tag{2.93}$$

where, once again, $\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\alpha}}$ are the spinors related to the null momentum $k^a$. One can show that $\lambda_\beta\,\lambda_\gamma\,\lambda_\delta\,e^{\mathrm{i}\,k\cdot x}$ corresponds to a helicity $-\frac{3}{2}$ Rarita Schwinger field - the helicity can be easily counted from little group scaling. Since $\Gamma_{A\beta\gamma\delta}$ solves the linearised ASD Bach equations, it must contain a helicity $-2$ state, meaning that $B_A = (\tilde{B}_{\dot{\alpha}}, B^\alpha)$ is a helicity lowering operator. Furthermore, since the curvature has mass dimension $[C_{abcd}] = [\Psi_{\alpha\beta\gamma\delta}] = 2$ and using (2.89), we can determine that its components have mass dimensions:

$$[\tilde{B}_{\dot{\alpha}}] = \frac{1}{2}\,, \qquad [B^\alpha] = -\frac{1}{2}\,, \tag{2.94}$$

There exists a further condition on our twistor-spinor. Indeed, from its twistor-space construction, one finds that it needs to satisfy (see [60,87]):

$$C^A\,\Gamma_{A\beta\gamma\delta} = 0\,. \tag{2.95}$$

where $C^A$ is a differential operator on the on-shell momenta:

$$C^A \equiv \left(-\mathrm{i}\frac{\partial}{\partial\tilde{\lambda}_{\dot{\alpha}}},\,\lambda_\alpha\right)\,. \tag{2.96}$$

If we make the assumptions

$$\frac{\partial B^\alpha}{\partial\tilde{\lambda}_{\dot{\beta}}} = 0\,, \qquad \frac{\partial}{\partial x^{\beta\dot{\beta}}}\tilde{\lambda}^{\dot{\alpha}}\tilde{B}_{\dot{\alpha}} = 0\,, \tag{2.97}$$

the constraint (2.95) becomes a simple PDE in on-shell momentum space:

$$\frac{\partial\tilde{B}_{\dot{\alpha}}}{\partial\tilde{\lambda}_{\dot{\alpha}}} + \mathrm{i}\,\lambda_\alpha\,B^\alpha = 0\,. \tag{2.98}$$

equations (2.97) then become:

$$\frac{\partial\tilde{B}_{\dot{\alpha}}}{\partial\tilde{\lambda}_{\dot{\alpha}}} + \mathrm{i}\,\lambda_\alpha\,B^\alpha = 0\,. \tag{2.99}$$

The solution in terms of $\tilde{B}_{\dot{\alpha}}$ is:

$$\tilde{B}_{\dot{\alpha}} = \frac{\partial B}{\partial\tilde{\lambda}^{\dot{\alpha}}} - \frac{\mathrm{i}}{2}\,\tilde{\lambda}_{\dot{\alpha}}\lambda_\alpha\,B^\alpha\,. \tag{2.100}$$

This means that there are *three* degrees of freedom, carried by the spinors: $\{B,\,B^\alpha\}$. This is exactly what we expected from the negative helicity sector of the theory. We now need to construct three distinct solutions for $\{B,\,B^\alpha\}$, which obey the equations of motion.

Let us start by finding the Einstein modes. As stated before, these satisfy the reduced equation $\partial^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} = 0$. As before, this introduces the dimension $-1$ parameter, $\kappa = \sqrt{8\pi G_{\mathrm{N}}}$ [30]. The Einstein solution has helicity $-2$, we must have that $B^a \propto \lambda^\alpha$. Furthermore, since $B^\alpha$ has mass dimension $-\frac{1}{2}$, we can fix

[30] Note that technically there exists another dimensionful parameter: Conformal Gravity, being conformally invariant, does not distinguish between Minkowski and (A)dS. This means that when selecting Einstein solutions, we should also consider $\Lambda$, the cosmological constant. However, the value of this constant is $\Lambda = 0$ for a Minkowski background.

$B^\alpha = \kappa\,\lambda^\alpha$. On the other hand, $B$ has dimension $+1$, but there are no combinations of available parameters that allow us to reach that. We must therefore set $B = 0$. This concludes our search for the first solution:

$$\text{Einstein:} \qquad \{B,\, B^\alpha\} = \left\{0,\, \kappa\,\lambda^\alpha\right\}\,, \quad B_A = \kappa \begin{pmatrix} 0 \\ \lambda^\alpha \end{pmatrix}, \tag{2.101}$$

which gives us the same linearised curvature as in (2.63).

$$\Psi^{(-2)}_{\alpha\beta\gamma\delta} = \kappa\,\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\delta\, e^{\mathrm{i}\,k\cdot x}\,. \tag{2.102}$$

We now look for the helicity $-1$ modes. As before, we introduce a spinor $a^\alpha$ such that $\langle a\lambda\rangle \neq 0$ which has mass dimension $-\frac{1}{2}$. Once again balancing dimensions leads us to write the solution:

$$\text{Spin-1:} \qquad \{B,\, B^\alpha\} = \left\{0,\, a^\alpha\right\}\,, \quad B_A = \begin{pmatrix} -\frac{\mathrm{i}}{2}\,\tilde{\lambda}_{\dot\alpha}\,\langle a\,\lambda\rangle \\ a^\alpha \end{pmatrix}. \tag{2.103}$$

which leads to the linearised curvature of (2.104):

$$\Psi^{(-1)}_{\alpha\beta\gamma\delta} = a_{(\alpha}\lambda_\beta\lambda_\gamma\lambda_{\delta)}\, e^{\mathrm{i}\,k\cdot x} \tag{2.104}$$

From our earlier analysis, we know that this concludes the hunt for oscillatory solutions. For completeness, we include the final solution. This one necessitates the presence of another spinor $\tilde{\beta}_{\dot\alpha}$ with mass dimension $+\frac{1}{2}$. We then have the growing solution:

$$\text{Growing:} \qquad \{B,\, B^\alpha\} = \left\{[\tilde{\lambda}\,\tilde{\beta}],\, x^{\alpha\dot\beta}\tilde{\beta}_{\dot\beta}\right\}\,, \quad B_A = \begin{pmatrix} \tilde{\beta}_{\dot\alpha} - \frac{\mathrm{i}}{2}\langle\lambda|x|\tilde{\beta}]\,\tilde{\lambda}_{\dot\alpha} \\ x^{\alpha\dot\beta}\tilde{\beta}_{\dot\beta} \end{pmatrix}, \tag{2.105}$$

which yields the solution of (2.69):

$$\Psi^{\mathrm{g}(-2)}_{\alpha\beta\gamma\delta} = \lambda_{(\alpha}\lambda_\beta\lambda_\gamma\, x_{\delta)}{}^{\dot\alpha}\,\tilde{\beta}_{\dot\alpha}\, e^{\mathrm{i}\,k\cdot x}\,, \tag{2.106}$$

We can now construct the positive helicity sector in very much the same way. We introduce the twistor spinor

$$\widetilde{\Gamma}^A{}_{\dot\beta\dot\gamma\dot\delta} = A^A\,\tilde{\lambda}_{\dot\beta}\tilde{\lambda}_{\dot\gamma}\tilde{\lambda}_{\dot\delta}\,e^{\mathrm{i}\,k\cdot x}\,, \qquad A^A = (\tilde{A}^{\dot\alpha},\, A_\alpha)\,. \tag{2.107}$$

where $A^A$ is a helicity raising operator. Once again the field obeys a geometric constraint:

$$\widetilde{C}_A\,\widetilde{\Gamma}^A{}_{\dot\beta\dot\gamma\dot\delta} = 0\,, \qquad\qquad \widetilde{C}_A \equiv \left(\tilde\lambda_{\dot\alpha},\, -\mathrm{i}\frac{\partial}{\partial\lambda_\alpha}\right)\,. \tag{2.108}$$

Making similar assumptions to (2.99), we get the condition

$$\mathrm{i}\,\tilde\lambda_{\dot\alpha}\,\tilde A^{\dot\alpha} + \frac{\partial A_\alpha}{\partial\lambda_\alpha} = 0\,, \tag{2.109}$$

which are solved by:

$$A_\alpha = \frac{\partial\tilde A}{\partial\lambda^\alpha} - \mathrm{i}\,\frac{\lambda_\alpha}{2}\,\tilde\lambda_{\dot\alpha}\,\tilde A^{\dot\alpha}\,. \tag{2.110}$$

Once again, there are three degrees of freedom contained in $\{\tilde A,\, \tilde A^{\dot\alpha}\}$. The mass dimensions of these fields are then :

$$[\tilde A^{\dot\alpha}] = -\frac{1}{2}\,, \qquad [A_\alpha] = \frac{1}{2}\,, \qquad [\tilde A] = 1\,. \tag{2.111}$$

We introduce conjugate versions of the auxiliary spinors $a^\alpha, \tilde\beta^{\dot a}$ with the following dimensions:

$$[\beta_\alpha] = \frac{1}{2}\,, \qquad [\tilde a^{\dot\alpha}] = -\frac{1}{2}\,, \tag{2.112}$$

This allows us to finally find the conjugate versions of eqs. (2.101), (2.103) and (2.105):

$$\text{Einstein:} \qquad \{\tilde A,\, \tilde A^{\dot\alpha}\} = \left\{0,\, \kappa\,\tilde\lambda^{\dot\alpha}\right\}\,, \quad A^A = \kappa\begin{pmatrix}\tilde\lambda^{\dot\alpha}\\ 0\end{pmatrix}\,, \tag{2.113}$$

$$\text{Spin-1:} \qquad \{\tilde A,\, \tilde A^{\dot\alpha}\} = \left\{0,\, \tilde a^{\dot\alpha}\right\}\,, \quad A^A = \begin{pmatrix}\tilde a^{\dot\alpha}\\ -\frac{\mathrm{i}}{2}\lambda_\alpha\,[\tilde a\,\tilde\lambda]\end{pmatrix}\,. \tag{2.114}$$

$$\text{Growing:} \qquad \{\tilde A,\, \tilde A^{\dot\alpha}\} = \left\{\langle\lambda\,\beta\rangle,\, x^{\gamma\dot\alpha}\beta_\gamma\right\}\,, \quad A^A = \begin{pmatrix}x^{\gamma\dot\alpha}\beta_\gamma\\ \beta_\alpha - \frac{\mathrm{i}}{2}\,\langle\beta|x|\tilde\lambda]\,\lambda_\alpha\end{pmatrix}\,. \tag{2.115}$$

which lead to the linearised curvatures:

$$\widetilde{\Psi}^{(+2)}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \kappa\,\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}\tilde\lambda_{\dot\gamma}\tilde\lambda_{\dot\delta}\,e^{\mathrm{i}\,k\cdot x}\,, \qquad \widetilde{\Psi}^{(+1)}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \tilde a_{(\dot\alpha}\tilde\lambda_{\dot\beta}\tilde\lambda_{\dot\gamma}\tilde\lambda_{\dot\delta)}\,e^{\mathrm{i}\,k\cdot x}\,, \tag{2.116}$$

$$\widetilde{\Psi}^{\mathrm{g}(+2)}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \tilde\lambda_{(\dot\alpha}\tilde\lambda_{\dot\beta}\tilde\lambda_{\dot\gamma}\,x^\alpha{}_{\dot\delta)}\beta_\alpha\,e^{\mathrm{i}\,k\cdot x}\,.$$

Once again, we re-iterate that the auxiliary spinors $a_\alpha, \tilde a_{\dot\alpha}, \beta_\alpha, \tilde\beta_{\dot\alpha}$ *do not* contain additional data, as the conditions:

$$\langle a\,\lambda\rangle \neq 0\,, \quad \langle \beta\,\lambda\rangle \neq 0\,, \quad [\tilde{a}\,\tilde{\lambda}] \neq 0\,, \quad [\tilde{\beta}\,\tilde{\lambda}] \neq 0\,,$$

restrict them up to a scale. In general, we simply fix the normalisations (2.61) and (2.94).

This concludes our reformulation of the equations of motion and polarization states in terms of twistor spinors. With all the ingredients in place, we can write down the three-point amplitudes.

### 2.4.3 Twistor formula for three-point amplitude

Now that we have a way to encode all types of external polarization states, we would like a formula that describes the three point amplitudes we computed earlier. We expect it to explicitly depend on the twistor-valued objects $A^A, B^A$ and their conjugates, as well as on-shell momenta. Furthermore since we know that the only-non vanishing amplitudes are either $\overline{\text{MHV}}$ or MHV, we can deduce that they will involve either two $A^A$ twistors and one $B^A$, or just one $A^A$ and two $B^A$ twistors respectively.

Such a formula can indeed be derived from the twistor formulation of Conformal gravity [87, 109, 110]. Here we write the result for $\overline{\text{MHV}}$ amplitudes:

$$A_3 = \Big[A_2\cdot\widetilde{C}_3\Big(B_1\cdot A_3\,\frac{[2\,3]^4}{[1\,2]\,[3\,1]^2}\Big) + A_3\cdot\widetilde{C}_2\Big(B_1\cdot A_2\,\frac{[2\,3]^4}{[1\,2]^2\,[3\,1]}\Big)\Big]\,\delta^{(4)}\Big(\sum_{i=1}^{3}\lambda_i\tilde{\lambda}_i\Big)\,. \tag{2.117}$$

Here, $\tilde{C}$ is defined by (2.108) and acts on everything to its right - including the delta function. Plugging in the various solutions for $A^A$ and $B^A$ gives us the same result as the previous amplitudes eqs. (2.75) and (2.76).

The helicity conjugate formula for obtaining MHV amplitudes is given by

$$A_3 = \Big[B_2\cdot C_3\Big(A_1\cdot B_3\,\frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\,\langle 3\,1\rangle^2}\Big) + B_3\cdot C_2\Big(A_1\cdot B_3\,\frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle^2\,\langle 3\,1\rangle}\Big)\Big]\,\delta^{(4)}\Big(\sum_{i=1}^{3}\lambda_i\tilde{\lambda}_i\Big)\,, \tag{2.118}$$

where again, $C_i^A$ is defined in (2.96) and acts on everything to its right.

One thing to note is that (2.118) is valid for *any* scattering states, including growing ones. Once again, those will be ill-defined in general, although some degenerate configurations may be finite. In order to actually compute them, we need to change any x-dependence in the polarization modes in terms of momentum derivatives. For instance, for $\overline{\text{MHV}}$ growing modes, we would re-write (2.105) as:

$$B_A^{\mathrm{g}} \to \begin{pmatrix} \tilde{\beta}_{\dot{\alpha}} + \frac{\tilde{\beta}_{\dot{\beta}}}{2}\,\tilde{\lambda}_{\dot{\alpha}}\,\frac{\partial}{\partial\tilde{\lambda}_{\dot{\beta}}} \\ -\mathrm{i}\,\tilde{\beta}_{\dot{\beta}}\,\frac{\partial}{\partial k_{\alpha\dot{\beta}}} \end{pmatrix}. \tag{2.119}$$

In particular, this formula recovers the strange amplitude of (2.77).

### 2.4.4 Scattering in an AdS background

We finish this chapter by taking a look at what the amplitudes we computed would look like for non-vanishing cosmological constant $\Lambda \neq 0$.

For dS spacetimes, where $\Lambda > 0$, the notion of a scattering amplitude is ambiguous (cf. [111–113]). We will instead look at $\text{AdS}_4$ where a tree-level scattering amplitude can also be defined in terms of a multi-linear piece of the on-shell action, where the solutions obey particular boundary conditions [31]. Since we are computing amplitudes in a conformally invariant theory, they are related (at tree-level) to amplitudes of half of Minkowski spacetime [114]. This means that AdS amplitudes are the same as the flat-space ones up to boundary conditions.
Indeed, though Minkowski and AdS are related, they have a distinct boundary. In particular, boundary terms that we drop when using (2.24) instead of (2.23) cannot be ignored in AdS.

Furthermore, we must modify our notion of admissible states in AdS. Indeed, in Minkowski we declared that only states leading to amplitudes that do not violate conservation of momentum were physical. However, in AdS, full momentum conservation is usually broken since there is no global space-like Killing vector. Instead, AdS amplitudes have momentum conservation in the directions parallel to the AdS boundary, while transverse momenta have a singularity (cf. [113–116]). So in AdS, we will relax our notion of admissible states to mean the ones which lead to an on-shell action with momentum conservation *up*

[31] At tree-level, one can relate such AdS amplitudes to the in-in formalism of dS by analytic continuation [113].

*to AdS isometries.*

Another difference we hinted at earlier will come during the analysis of polarization states. In section 2.4.2 we made use of the fact that we only had one non-zero dimensionful quantity. This leads to slightly different polarization states.

Let us introduce the $\mathrm{AdS}_4$ metric: [32]

$$\delta s^2 = \frac{\delta x_{\alpha\dot\alpha}\,\delta x^{\alpha\dot\alpha}}{(1+\Lambda\,x^2)^2}\,, \tag{2.120}$$

The utility of using these coordinates (2.120) is that the flat $\Lambda \to 0$ limit is smooth. In these coordinates, the boundary of $\mathrm{AdS}_4$ is given by the 3-sphere satisfying $1+\Lambda x^2 = 0$ in the affine Minkowski space charted by $x^{\alpha\dot\alpha}$. If we take the flat limit, this hypersurface approaches the conformal boundary of Minkowski space, $\mathscr{I}$.

Here, the linearised Bach equations are just the covariant form of (2.59):

$$D^{\alpha\dot\alpha}D^{\beta\dot\beta}\Psi_{\alpha\beta\gamma\delta} = 0 = D^{\alpha\dot\alpha}D^{\beta\dot\beta}\widetilde{\Psi}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\,. \tag{2.121}$$

where $D^{\alpha\dot\alpha}$ is now the Levi-Civita connection of (2.120).
These equations can be obtained - as before - by introducing the Cartan connection and using it to impose an equation on twistor spinors. In $\mathrm{AdS}_4$, is is given by:

$$\mathcal{D}_{\alpha\dot\alpha} = D_{\alpha\dot\alpha} + (\mathcal{A}_{\alpha\dot\alpha})^B{}_C = D_{\alpha\dot\alpha} + \begin{pmatrix} 0 & \delta^\gamma_\alpha\,\delta^{\dot\beta}_{\dot\alpha} \\ \Lambda\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\gamma} & 0 \end{pmatrix}\,, \tag{2.122}$$

and we impose $\mathcal{D}^{\beta\dot\beta}\Gamma_{A\beta\gamma\delta} = \mathcal{D}^{\beta\dot\beta}\tilde\Gamma^A{}_{\dot\beta\dot\gamma\dot\delta} = 0$. It is useful to rewrite these equations in affine Minkowski coordinates (cf. [106]). For the negative helicity sector one finds

$$\partial^{\beta\dot\beta}\gamma_{\dot\alpha\beta\gamma\delta} + \frac{2\,\Lambda}{1+\Lambda\,x^2}\left(x^\beta{}_{\dot\alpha}\,\gamma^{\dot\beta}{}_{\beta\gamma\delta} - x^{\beta\dot\beta}\,\gamma_{\dot\alpha\beta\gamma\delta}\right) = 0\,, \tag{2.123}$$

$$\partial^{\beta\dot\beta}\Psi^\alpha{}_{\beta\gamma\delta} - \frac{2\,\Lambda}{1+\Lambda\,x^2}\,x^{\beta\dot\beta}\,\Psi^\alpha{}_{\beta\gamma\delta} = \frac{\gamma^{\dot\beta\alpha}{}_{\gamma\delta}}{1+\Lambda\,x^2}\,, \tag{2.124}$$

[32] This unusual form of the $\mathrm{AdS}_4$ metric is obtained by the analytic continuation $\Lambda \to -\Lambda$ of the standard $S^4$ metric, and a rescaling of coordinates to take into account the fact that $|\Lambda| = 3R^{-2}$ instead of $\frac{1}{4}R^{-2}$ in terms of the radius $R$ of the sphere.

where the positive helicity sector is found by conjugation.

Once again, we wish to find the independent solutions of the twistor-spinor

$$\Gamma_{A\beta\gamma\delta} = B_A\, \lambda_\beta \lambda_\gamma \lambda_\delta\, e^{i\,k\cdot x}\,, \tag{2.125}$$

ie. the different solutions for the helicity lowering operator $B_A$. The Einstein and spin 1 mode can be found by deforming the flat space solutions eqs. (2.101) and (2.103) respectively to give:

$$\text{Einstein:} \qquad B_A = \kappa \begin{pmatrix} 2\Lambda\, x_{\beta\dot\alpha}\, \lambda^\beta \\ \lambda^\alpha \end{pmatrix}. \tag{2.126}$$

(2.103)

$$\text{Spin-1:} \qquad B_A = \begin{pmatrix} \frac{\mathrm{i}}{2}\tilde\lambda_{\dot\alpha}\, \langle \lambda\, a\rangle - \Lambda\, x^{\beta\dot\alpha}\, a_\beta \\ a^\alpha \end{pmatrix}, \tag{2.127}$$

where $a^\alpha$ is the same auxiliary spinor. The other spin 2 modes are a little different, and cannot be found by deforming the flat space (2.105) modes. Instead, solving the equations of motion (2.123) and (2.124), we find:

$$\text{Spin-2:} \qquad B_A = \Lambda\, \kappa \begin{pmatrix} x^\beta{}_{\dot\alpha}\, \lambda_\beta \\ \frac{\mathrm{i}}{2}\, x^2\, \lambda^\alpha \end{pmatrix}. \tag{2.128}$$

The physical interpretation of this last mode is different in AdS: on the boundary, we have that $x^2 = -1/\Lambda$, so saying that it is growing is erroneous - despite the quadratic dependence on $x$. What is more, the curvature associated with it is finite [33].
This can be taken as further indication that the flat space limit is singular - otherwise, the spin 2 modes (2.128) would have the flat-space growing modes (2.106) as a limit.

Using this basis of AdS states, one should now be able to consider the "scattering" amplitude. This analysis has yet to be done, but one can conjecture that the flat-space amplitudes of (2.117) and (2.118) still hold up - up to some contributions coming from the boundary. This claim is made in light of [62, 118] which indicates that if one takes Neumann boundary conditions of CG on AdS, we should get an amplitude proportional to Einstein Gravity amplitudes, with

[33] These spin 2 modes together with (2.127) correspond to the partially-massless graviton which only has scalar gauge invariance [117].

an overall factor of $\Lambda$.

By interpreting the $x$-dependence in the polarization twistors (2.126) as momentum derivatives, we get that Einstein polarizations are encoded by:

$$A^A = \kappa \begin{pmatrix} \tilde{\lambda}^{\dot\alpha} \\ -2\mathrm{i}\,\Lambda\,\frac{\partial}{\partial\lambda^\alpha} \end{pmatrix}, \qquad B_A = \kappa \begin{pmatrix} -2\mathrm{i}\,\Lambda\,\frac{\partial}{\partial\tilde{\lambda}^{\dot\alpha}} \\ \lambda^\alpha \end{pmatrix}. \tag{2.129}$$

These can then be plugged into the amplitude formulae (2.117) and (2.118). For the $\overline{\text{MHV}}$ case, one gets:

$$\begin{aligned} A_3^\Lambda(1^-,2^+,3^+) = -2\mathrm{i}\,\Lambda\,\kappa^3 \Bigg[ &\left( [2\,3] - 2\Lambda \left\langle \frac{\partial}{\partial\lambda_3}\,\frac{\partial}{\partial\lambda_2} \right\rangle \right) \frac{[2\,3]^5}{[1\,2]^2\,[3\,1]^2} \\ &+ \left( [3\,2] - 2\Lambda \left\langle \frac{\partial}{\partial\lambda_2}\,\frac{\partial}{\partial\lambda_3} \right\rangle \right) \frac{[2\,3]^5}{[1\,2]^2\,[3\,1]^2} \Bigg] \,\delta^{(4)}\Big(\sum_{i=1}^{3} \lambda_i\tilde{\lambda}_i\Big), \end{aligned} \tag{2.130}$$

where we've made all derivative operators explicit. Note that since the square brackets contain only dotted spinors, these derivatives act solely on the delta function. Introducing the notation:

$$\left\langle \frac{\partial}{\partial\lambda_3}\,\frac{\partial}{\partial\lambda_2} \right\rangle = \frac{[3\,2]}{2}\,\frac{\partial}{\partial K_{\alpha\dot\alpha}}\,\frac{\partial}{\partial K^{\alpha\dot\alpha}}, \qquad K^{\alpha\dot\alpha} \equiv (\lambda_1\,\tilde\lambda_1 + \lambda_2\,\tilde\lambda_2 + \lambda_3\,\tilde\lambda_3)^{\alpha\dot\alpha}, \tag{2.131}$$

and the "momentum wave operator":

$$\Box_K \equiv \frac{\partial}{\partial K_{\alpha\dot\alpha}}\,\frac{\partial}{\partial K^{\alpha\dot\alpha}}, \tag{2.132}$$

we can rewrite (2.130) in the compact form:

$$A_3^\Lambda(1^-,2^+,3^+) = -4\mathrm{i}\,\Lambda\,\kappa^3\,\frac{[2\,3]^6}{[1\,2]^2\,[3\,1]^2}\,(1+\Lambda\Box_K)\,\delta^{(4)}\Big(\sum_{i=1}^{3}\lambda_i\tilde\lambda_i\Big). \tag{2.133}$$

Two remarks are in order. The first is that this amplitude is exactly proportional to the Einstein Gravity $\overline{\text{MHV}}$ three-point amplitude. The second is that momentum conservation is broken by the operator $(1+\Lambda\Box_K)$. The second term breaks momentum conservation, but this is as expected for an AdS amplitude.

The takeaway from this is that when tree-level Conformal Gravity amplitudes vanish in flat-space they do so by being zero times the corresponding Einstein Gravity amplitude, as one might expect from the fact that both theories are

related.
More precisely, the statement is that a rescaled version of (2.133):

$$\frac{\mathrm{i}\, A_3^{\Lambda}(1^-,2^+,3^+)}{4\,\kappa^2\,\Lambda} = \frac{[2\,3]^6}{[1\,2]^2\,[3\,1]^2}\,(1+\Lambda\Box_K)\,\delta^{(4)}\Big(\sum_{i=1}^{3}\lambda_i\tilde{\lambda}_i\Big)\,, \tag{2.134}$$

gives exactly the $\overline{\text{MHV}}$ bulk contribution of Einstein Gravity in $\text{AdS}_4$ [119, 120]. This ignores boundary contributions, but in the flat-space limit these contributions decouple, while the bulk terms have a smooth limit [121]. For completeness, we include the corresponding formula for MHV amplitude, which can be obtained by simply conjugating (2.133) :

$$A_3^{\Lambda}(1^+,2^-,3^-) = -4\mathrm{i}\,\Lambda\,\kappa^3\,\frac{\langle 2\,3\rangle^6}{\langle 1\,2\rangle^2\,\langle 3\,1\rangle^2}\,(1+\Lambda\Box_K)\,\delta^{(4)}\Big(\sum_{i=1}^{3}\lambda_i\tilde{\lambda}_i\Big)\,. \tag{2.135}$$

# Chapter 3

# Conformal Higher Spin Theory

This chapter is dedicated to introducing Conformal Higher Spin Theory [2, 4, 5, 77, 81, 122–136]. The idea is simple: to write down a conformally-invariant action for fields of arbitrary spins. This was first done at the free level by Fradkin and Tseytlin in [4]. Later , it was studied at the cubic level by Fradkin and Linetsky [122] [34]. Finally, Segal introduced a consistent fully interacting action in [5].

Let us give a taster of the free action. We will focus on the case of $d = 4$, although much of what we say has a generalisation for generic $d$. The quadratic action of Conformal Higher Spins can be written simply [4] :

$$S_{2,CHS} = \frac{n_s}{2} \sum_{s=0}^{\infty} \int \mathrm{d}^d x \, h_{a(s)} P^{a(s),b(s)} h_{b(s)} \,, \tag{3.1}$$

where $n_s$ are numerical constants, $h_{a(s)}$ is the spin $s$ CHS field, and $P^{a(s),b(s)}$ is an operator of order $2s$ in partial derivatives which is traceless and transverse:

$$P^{a(s),b(s)} = P^{b(s),a(s)} \,, \qquad \partial_{a_1} P^{a_1 a(s-1),b(s)} = 0 = \eta_{a_1 a_2} P^{a_1 a_2 a(s-2),b(s)} \,. \tag{3.2}$$

This action is invariant under:

$$\delta h_{a_1 \cdots a_s} = \partial_{(a_1} \varepsilon_{a_2 \cdots a_s)} + \eta_{(a_1 a_2} \alpha_{a_3 \cdots a_s)} \,, \tag{3.3}$$

which are generalisations of diffeomorphisms and Weyl rescalings respectively.

As it turns out, it is possible to extend this theory to be fully interacting (much like Conformal Gravity). This theory is rife with interesting features and

[34] See [123] for a supersymmetric generalisation.

questions. For one, it is the gauge theory of an infinite dimensional algebra [34, 137] . Each generator is associated with a conformal killing tensor - the ones we mentioned in Section 1.2. This means that CHS fields therefore have a close relation with massless higher spin fields via AdS/CFT: they are those fields restricted to the conformal boundary, with a Dirichlet condition. Another point of view is that the conformal fields are the sources for the higher derivative scalar currents.

As we will explain later this chapter, the fully interacting CHS theory can be obtained by integrating out the scalar fields in flat space, and keep the logarithmically divergent term. We thus obtain a local action which schematically looks like:

$$\begin{aligned} S[h] &\sim \log\det\left(\partial^2 + \sum_s h_s J_s\right)\Big|_{\log\Lambda} \\ &\sim \sum_s \int d^4x \left(h_s \partial^{2s} h_s + \partial^{s_1+s_2+s_3-2} h_{s_1} h_{s_2} h_{s_3} + \partial^{s_1+s_2+s_3+s_4-4} h_{s_1} h_{s_2} h_{s_3} h_{s_4} + \dots\right) . \end{aligned} \tag{3.4}$$

An important point about this theory is that each term has a definite derivative structure. This can be understood by dimensional analysis: the free scalar fields have mass dimension $[\phi] = 1$ (in $d = 4$), so that the currents $J_s$ have dimension $2 + s$. This means that the CHS fields have the “shadow dimension” $[h_s] = 2 - s$. Since CHS theory has no dimensionful parameter and is local, this implies the derivative structure of (3.4). This definiteness in the action leads to tremendous simplifications later on.

The large symmetry group of the theory appears to be related to hidden simplicity in the theory. We will explore this in the following chapters, but we briefly mention that many observables seem to non-trivially vanish. For instance, the contribution to the free theory’s 1-loop partition function in flat space is non-trivial, but when summing all the contributions, it vanishes [55]. This mechanism seems to imply the vanishing of the conformal anomaly’s $a$ and $c$-coefficient [55, 57, 127, 134, 135]. In the context of AdS/CFT, this agrees with the vanishing of massless higher spin 1-loop partition function [57].

Crucially, this vanishing requires one to properly define how the sum over infinite spin contributions is taken. In other words, one needs to introduce a regularisation which is consistent with the underlying CHS symmetry of the theory. It turns out that tree-level amplitudes vanish after summation. These computations (first done in [1,2], and the later part of [3].) will be the subject of Chapters 4 and 5.

In this chapter, we will first start by studying the action of the free U(N) scalar field, and its full symmetries. This will allow us to discover the full CHS algebra. Then, we will see how one can obtain a (unique) local action invariant under this symmetry by integrating out those scalars and looking at the logarithmic divergence, as in (3.4). The resulting action contains Maxwell theory and Conformal Gravity as a lower spin truncation, which we will verify perturbatively. Finally, we will explicitly compute certain sectors of this action, which will be relevant for use in later chapters.

## 3.1 Inducing Global CHS symmetry from a U(N) scalar

In this section we will be "inducing" the CHS action of [5]. Much of the discussion is based on the references [126, 138]. Let us start with a theory of $N$ free complex scalars:

$$S_{\text{free}}[\phi] = \int \mathrm{d}^4x\, \phi_i^* \,\Box \phi_i\,, \qquad\qquad i = 1,\ldots,N\,, \tag{3.5}$$

We will find it useful to express this in the operator formalism:

$$S_{\text{free}}[\phi] = \langle\phi|\, \hat{P}^2\, |\phi\rangle\,, \tag{3.6}$$

where $\phi_i(x) = \langle x|\phi\rangle$ and $\hat{P}_a = i\,\partial_a$ and the sum over $i$ is implied by the bra-ket inner product. Now we look for the most generic transformations which leave (3.6) invariant. They are simply given by:

$$\delta\,|\phi\rangle = \hat{O}^{-1}\,|\phi\rangle\,, \tag{3.7}$$

where the operator $\hat{O}$ is generated by the Hermtian generators:

$$\hat{O} = e^{\frac{1}{2}(\hat{A}+i\hat{E})}\,. \tag{3.8}$$

In order to leave (3.6) invariant, they must satisfy:

$$\hat{P}^2\,\hat{A} = -\hat{A}^\dagger\,\hat{P}^2\,, \qquad\qquad \hat{P}^2\,\hat{E} = \hat{E}^\dagger\,\hat{P}^2\,. \tag{3.9}$$

This is trivially satisfied for transformations of the form: [35]

$$\hat{E} = \hat{R}\,\hat{P}^2\,, \qquad \hat{R} = -\hat{R}^\dagger \tag{3.10}$$

In conclusion, we must consider symmetries generated by $\hat{A}$ and $\hat{E}$ satisfying (3.9) modulo the trivial ones (3.10).

In order to study this we go to the Wigner representation reviewed in B.2. The idea is to associate with any operator $\hat{M}$ a symbol $m(x,p)$ which takes values in position space and is a polynomial in powers of momentum. Operator action is replaced by the "Moyal star-product", (cf (B.11)) . The transformation (3.7) then takes the form:

$$\delta\phi(x) = \big(\mathfrak{e}(x,p) + \mathrm{i}\,\mathfrak{a}(x,p)\big) * \phi(x) \tag{3.11}$$

$$= e^{-\frac{i}{2}\partial_{x_2}\cdot\partial_p}\,\big(\mathfrak{e}(x_1,p) + \mathrm{i}\,\mathfrak{a}(x_1,p)\big)\;\phi(x_2)\Big|_{\substack{x_1=x_2=x\\ p=0}}\,, \tag{3.12}$$

where $\mathfrak{e}$ and $\mathfrak{a}$ are the symbol associated with $\hat{E}$ and $\hat{A}$ respectively. The conditions (3.9) and (3.10) then become:

$$p\cdot\partial_x\,\mathfrak{e} - (p^2+\partial_x^2)\mathfrak{a} = 0\,, \tag{3.13}$$

defined up to:

$$(\mathfrak{e},\mathfrak{a}) \sim (\mathfrak{e},\mathfrak{a}) + \Big((p^2+\partial_x^2)\,r\,,\,p\cdot\partial_x\,r\Big)\,. \tag{3.14}$$

The algebra is then determined by the Moyal commutator:

$$\Big[\,(\mathfrak{e}_1\,,\,\mathfrak{a}_1)\,,(\mathfrak{e}_2\,,\,\mathfrak{a}_2)\Big] = \big(\,[\,\mathfrak{e}_1 \overset{*}{,} \mathfrak{e}_2\,] - [\,\mathfrak{a}_1 \overset{*}{,} \mathfrak{a}_2\,]\,,\,[\,\mathfrak{e}_1 \overset{*}{,} \mathfrak{a}_2\,] + [\,\mathfrak{a}_1 \overset{*}{,} \mathfrak{e}_2\,]\,\big)\,. \tag{3.15}$$

This is actually the global CHS algebra. In particular, one can check that it contains the conformal algebra (given in (A.13)- (A.15)) realised by (3.15) and :

$$P_a = (p_a,0)\,, \qquad M_{ab} = (x_{[a}\,p_{b]},0)\,, \qquad K_a = (x_a\,x\cdot p, x_a)\,, \qquad D = (x\cdot p,1)\,, \tag{3.16}$$

which is the maximal finite subalgebra. Among other generators, we can find

[35] This is related to "Zilch symmetries". Indeed, the action is trivially invariant under $\delta\phi^i = C^{ij}\frac{\delta S}{\delta\phi_J}$ for $C^{ij} = -C^{ji}$.

*hypertranslations*

$$\mathsf{P}_{a_1 \ldots a_r} \equiv p_{\{a_1} \ldots p_{a_r\}}\,, \tag{3.17}$$

where $\{\ldots\}$ indicates the subtraction of all traces. These will play an important role in constraining amplitudes later on.

We now turn our attention towards *gauging* this global symmetry. As such, we will add a generic operator $\hat{H}$ to (3.6), so that

$$S[\phi, \mathfrak{h}] = \langle\phi|\,\hat{G}\,|\phi\rangle\,, \qquad\qquad \hat{G} \equiv \hat{P}^2 - \hat{H}\,, \tag{3.18}$$

where we pre-emptively wrote down the dependence on $\mathfrak{h}$, the Weyl symbol of $\hat{H}$. We now ask what are the most general symmetries of this action, with the understanding that the operator $\hat{H}$ itself will vary, reminiscent of the usual Noether coupling procedure. The variation is given by (cf. (3.19)):

$$\delta\,|\phi\rangle = \hat{O}^{-1}\,|\phi\rangle\,, \qquad\qquad \delta\hat{G} = \hat{O}^{\dagger}\hat{G}\hat{O}\,, \tag{3.19}$$

which leaves the action invariant as long as, infinitesimally:

$$\delta_{\hat{E}}\hat{H} = \frac{i}{2}\,[\hat{G}, \hat{E}] \qquad\qquad \delta_{\hat{A}}\hat{H} = \frac{1}{2}\,\{\hat{G}, \hat{A}\}\,. \tag{3.20}$$

Next, we notice that we can re-write the coupling in (3.18) as a trace:

$$S_{int} = \langle\phi|\,\hat{H}\,|\phi\rangle \;= \mathrm{Tr}\,\big[|\phi\rangle\langle\phi|\hat{H}\big]\,. \tag{3.21}$$

By finding the Weyl symbol of the operator $|\phi\rangle\langle\phi|$, and using the trace identities derived in Appendix B.2, one can express the interaction (3.21) as (see (B.20) and (B.21)):

$$S_{int}[\phi, \mathfrak{h}] = \int \mathrm{d}^d x\, \mathfrak{J}(x, \partial_u)\mathfrak{h}(x, u)\Big|_{u=0} \tag{3.22}$$

where the $\mathfrak{J}(x, u)$ is the traceful current generator given by:

$$\mathfrak{J}(x, u) = \phi_i^*(x + \frac{\mathrm{i}}{2}u)\,\phi_i(x - \frac{\mathrm{i}}{2}u)\,. \tag{3.23}$$

Expanding this in terms of components, we can write

$$S_{int}[\phi, \mathfrak{h}] = \sum_{s=0}^{\infty} \frac{1}{s!} \int \mathrm{d}^d x \, \mathfrak{J}_{a(s)} \mathfrak{h}^{a(s)} \tag{3.24}$$

$$\mathfrak{J}_{a(s)} = \left(\frac{\mathrm{i}}{2}\right)^s \sum_{n=0}^{s} (-1)^n \binom{s}{n} \partial_{a(n)} \phi_i^* \partial_{a(s-n)} \phi_i \, . \tag{3.25}$$

The gauge invariance expressed in (3.20) now becomes, in terms of Weyl symbols:

$$\delta_{\mathfrak{e}} \, \mathfrak{h}(x,u) = (u \cdot \partial_x) \, \mathfrak{e}(x,u) - \frac{i}{2} \Big[ \mathfrak{h}(x,u) \stackrel{\star}{,} \mathfrak{e}(x,u) \Big] \, , \tag{3.26}$$

$$\delta_{\mathfrak{a}} \, \mathfrak{h}(x,u) = \left( u^2 - \frac{1}{4} \partial_x^2 \right) \mathfrak{e}(x,u) - \frac{1}{2} \Big\{ \mathfrak{h}(x,u) \stackrel{\star}{,} \mathfrak{a}(x,u) \Big\} \, . \tag{3.27}$$

Expanding this in terms of components reveals that the transformations are similar to those generated by $\epsilon$ and $\alpha$ in (3.3) - though the latter is modified: $\delta_{\mathfrak{a}} \mathfrak{h}_{a(s)} \sim \eta_{\alpha(2)} \mathfrak{a}_{a(s-2)} + k_s \, \mathfrak{a}_{a(s)}$ where $k_s$ is some numerical constant.

So, to recap, we are studying a set of $N$ complex scalars coupled to a general background field, and we found that its action is given by:

$$S = \int \mathrm{d}^d x \, \left( \phi_i^* \Box \phi_i - \sum_{s=0}^{\infty} \frac{1}{s!} \mathfrak{J}_{a(s)} \mathfrak{h}^{a(s)} \right) \tag{3.28}$$

The equations of motion are simply $\Box \phi_i \approx 0$ where "$\approx$ " is used to designate equations which hold on-shell. One can easily check that the current generator satisfies the on-shell relations:

$$(u \cdot \partial_x) \mathfrak{J}(x,u) \approx 0 \, , \qquad \left( \partial_u^2 + \frac{1}{4} \Box \right) \mathfrak{J}(x,u) \approx 0 \, , \tag{3.29}$$

which relates to the fact that (3.28) invariant *on-shell* under the field-independent part of transformations (3.26) and (3.27). This becomes an off-shell invariance when we consider the full transformations of CHS fields (3.26) and (3.27) accompanied by (3.12).

The second equation in (3.29) looks almost like an on-shell tracelessness condition, which is reminiscent of the traceless conserved currents we mentioned in the context of AdS/CFT in section 1.2. Furthermore, we saw that the conformal algebra was a subgroup of the CHS global algebra, which further indicates that we should be able to relate the currents generated by $\mathfrak{J}(x,u)$ to traceless-conserved currents.

This can indeed be done by introducing the $d$-dimensional projector $\Pi_d(u,\partial_x)$ (cf [126]):

$$\Pi_d(u,\partial_x) = \sum_{n=0}^{\infty} \frac{1}{n!\,(-u\cdot\partial_u - \frac{d-5}{2})_n} \Big(\frac{u^2\,\partial_x^2 - (u\cdot\partial_x)^2}{16}\Big)^n\,, \tag{3.30}$$

where $(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}$ is the Pochhammer symbol. We can then define a new set of currents $J(x,u) = \Pi_d(u,\partial_x)\mathfrak{J}(x,u)$, where the operator $\Pi_d(u,\partial_x)$ ensures that on-shell:

$$(u\cdot\partial_x)J(x,u) \approx 0\,, \qquad\qquad \partial_u^2 J(x,u) \approx 0\,, \tag{3.31}$$

ie. it projects tensors onto their totally symmetric traceless transverse part. This leads us to define the "undressed" CHS fields, (henceforth just CHS fields) via the generating function $h(x,u) \equiv \Pi_d^{-1}(u,\partial_x)\mathfrak{h}(x,u)$. From now on we will consider $S[\phi,h]$ whose gauge symmetry is expressed in terms of $\epsilon(x,u)$ and $\alpha(x,u)$, which are related to $\mathfrak{e}(x,u)\mathfrak{a}(x,u)$ via:

$$\begin{aligned}
\mathfrak{e}(x,u) &= \Pi_{d+2}(\partial_u,\partial_x)\,\epsilon(x,u) + (\partial_x\cdot\partial_u)\,\Pi_{d+2}(\partial_u,\partial_x)\,\frac{1}{2(d-1)+4\,u\cdot\partial_u}\,\alpha(x,u)\,,\\
\mathfrak{a}(x,u) &= \Pi_{d+4}(\partial_u,\partial_x)\,\alpha(x,u)\,.
\end{aligned} \tag{3.32}$$

In this basis, the statement is now that the action is invariant on-shell under the field independent parts of the CHS transformation: [36]

$$\delta_\epsilon h(x,u) = (u\cdot\partial_x)\epsilon(x,u)\,, \qquad\qquad \delta_\alpha h(x,u) = u^2\alpha(x,u)\,, \tag{3.33}$$

When expressed in terms of components, this corresponds exactly to (3.3). Finally, we write the action of the scalar coupled to the undressed fields $\phi_i$:

$$S[\phi,h] = \phi_i^*\Box\phi_i - \sum_{s=0}^{\infty}\frac{1}{s!}\int d^dx\, J^{a(s)}\,h_{a(s)}\,. \tag{3.34}$$

where the currents are traceless and conserved on-shell.[37] Explicitly, they are

[36] For off-shell invariance, one needs to take (3.12),(3.26) and (3.27) and relate them to the parameters $\alpha$ and $\epsilon$ via (3.32).

[37] Note that these are the same currents mentioned earlier in Section 1.2.1.

given by:

$$J_{a(s)}(x) = \frac{i^s\, 2^s\, s!\, s!}{(2s)!} \sum_{k=0}^{s} \binom{s}{k} \binom{\frac{s+k-1}{2}}{s} G^{(k)}_{a(s)}\,, \tag{3.35}$$

$$G^{(k)}_{a(s)} = \Big[(\partial - \partial')_{a(k)} (\partial + \partial')_{a(s-k)} \phi_i(x)\, \phi_i^*(x')\Big]_{x=x'}\,. \tag{3.36}$$

## 3.2 CHS action as an effective action

We've now seen how to couple a general U(N) scalar field to an arbitrary-background, as well as study the action of the symmetry. Now we will look at how to actually define the CHS action.
We note at this stage that in [5] Segal originally defined his action in rather more formal term, by taking the trace of the positive eigenspace of the operator $\hat{H}$. For us, it turns out to be operationally much simpler to obtain the theory as an *induced effective action*.

More precisely, CHS action is obtained by regularising (3.34) then integrating out the scalars $\phi_i$. The coefficient of the logarithmic divergent term - which exists only for even dimensions - is then the only coefficient which is invariant under the CHS symmetries. This is closely related to the fact that the conformal anomaly of a theory is itself Weyl invariant. To see this, consider the partition function of some theory in a background $h$. If we introduce an energy cut-off $\Lambda$, and integrate out the other fields, we get, (for even dimensions) :

$$Z[h] \sim Z_{fin}[h] + \log(\Lambda) a_{d/2}[h] + \sum_{n \neq d/2} \Lambda^{d-2n} a_n[h]\,, \tag{3.37}$$

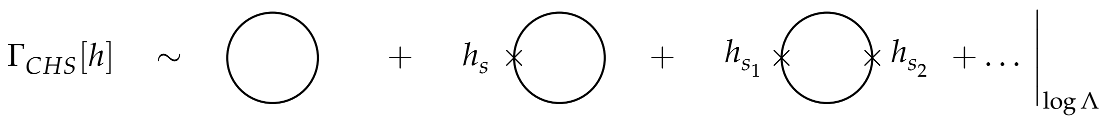
where $a_n$ are known as *Seeley* coefficients. Since the regulator sets a maximum scale in the theory, any scale transformations must vary $\Lambda$ as well. As such, if $Z[h]$ is classically scale invariant, only $a_{d/2}[h]$ can be scale invariant. [38] More generally, in [126] it is shown that the coefficient of the logarithmic term is the only one which is invariant under the full CHS symmetry generated by $\hat{E}$ and $\hat{A}$. Furthermore, in that paper one can find the expression for the Seeley coefficient $a_{d/2}$, ie. for $S_{CHS}$.

[38] Indeed, $Z_{fin}[h]$ must vary to compensate the variation of $\log\Lambda$, while $a_n[h]$ must vary to compensate the transformation of $\Lambda^{d-2n}$.

In fact, while in Chapter 4 we use some of the functional expressions given in [126] - which were obtained using heat kernel methods, for Chapter 5 we will want to use more explicit expressions whose spin structure is apparent and which are in a gauge-fixed form, ready for computing amplitudes. We will now see how to get such explicit expressions.

## 3.3 Structure of the CHS action

In this section, we compute certain terms of the CHS action through Feynman diagram techniques. The action (3.34) contains no self-interactions between the scalars. As such, when we integrate them out, the partition function is 1-loop finite. Diagrammatically, this means that we can represent $Z[h]$ as:




$$\sim \quad S_{0,CHS}[h] \quad + \quad S_{1,CHS}[h] \quad + \quad S_{2,CHS}[h] \quad + \dots \tag{3.38}$$

where the complex scalars run in the loop, and the $\times$ represent insertions CHS fields of arbitrary spins. By computing the logarithmic divergence of each of these individual terms, we can compute explicitly certain sectors of the CHS action.
One way to do this is to think of the various diagrams terms in (3.38) as the coefficients of the coinciding-point limit of the correlators $\langle J_{s_1}(x_1)...J_{s_n}(x_n)\rangle|_{x_i\to x}$. This can be done by using differential regularisation (see for example [139] ). Instead, we will use dimensional regularisation by going to $d = 4-\epsilon$ dimensions. The logarithmic divergence then appears as a $1/\epsilon$ pole. More precisely, we get the CHS action as:

$$\Gamma_{CHS}[h] = \frac{N}{(4\,\pi)^2\,\epsilon}\,S_{CHS}[h] + \text{finite}\,. \tag{3.39}$$

Note that in doing so, the action comes with the coupling constant $N$, which is the number of scalars running in the loop. For large $N$, we can use perturbation theory.
The usefulness of this method is that we can pick out exactly which sectors of the CHS action we want by simply computing the loops with the relevant number of CHS field insertions. Furthermore, great simplification arises when we take those fields to be already in the TT gauge.

### 3.3.1 Lower Spin Sector

Before we compute the diagrams in (3.38), let us look more closely at the lower spin ($s = 0, 1, 2$) truncation of CHS theory. As we've claimed before, it is equivalent to the action of of a conformal scalar + Maxwell vectors + Conformal gravity. More precisely, we obtain the same action but in a different basis of fields. We start with the lower spin currents of (3.36) which are given by:

$$\begin{aligned} J = \phi_i \phi_i^*, \qquad J_a = \tfrac{i}{2}\,(\phi_i^* \partial_a \phi_i - \phi_i\, \partial_a \phi_i^*), \\ J_{ab} = \tfrac{1}{3}\left[\partial_a \phi_i^*\, \partial_b \phi_i + \partial_b \phi_i^*\, \partial_a \phi_i - \tfrac{1}{2}(\phi_i^* \partial_a \partial_b \phi_i + \phi_i\, \partial_a \partial_b \phi_i^*)\right]. \end{aligned} \tag{3.40}$$

Assuming that the fields $h_a$ and $h_{ab}$ are transverse and traceless (which is true on the shell of $\phi_i$), the scalar-CHS action takes the form:

$$\mathcal{L} = \partial_a \phi_i^* \partial^a \phi_i + h_0 \phi_i^* \phi_i + i h^a \phi_i^* \partial_a \phi_i + \frac{1}{2} h^{ab} \partial_a \phi_i^* \partial_b \phi_i \tag{3.41}$$

where we've integrated by parts. This action must be compared with the more familiar $U(1)$ and Weyl invariant action of a scalar conformally coupled to the metric $g_{ab} = \eta_{ab} + h'_{ab}$ and the vector field $h'_a$:

$$I = \int d^4x \sqrt{|g|}\Big[-g^{ab}(\partial_a - \tfrac{\mathrm{i}}{2} h'_a)\phi_i^* (\partial_b + \tfrac{\mathrm{i}}{2} h'_b)\phi_i + (h'_0 - \tfrac{1}{6} R)\phi_i^* \phi_i\Big]. \tag{3.42}$$

The two actions (3.41) and (3.42) are related by a non-linear field redefinition:

$$h'_0 = h_0 + \tfrac{1}{4} h_a h^a + \tfrac{1}{96}(\partial_c h_{ab} \partial^c h^{ab} + 2 h_{ab} \Box h^{ab} + 2 \partial_c h_{ab} \partial^a h^{cb}) + \dots, \tag{3.43}$$
$$h'_a = h_a + \tfrac{1}{2} h_{ab} h^b + \tfrac{1}{4} h_{ab} h^{bc} h_c + \dots, \qquad h'_{ab} = \tfrac{1}{2} h_{ab} + \tfrac{1}{4} h_{ac} h^c_b - \tfrac{1}{16} \eta_{ab} h^{cd} h_{cd} + \dots,$$

so in particular we can see that the scalar CHS field $h_0$ subsumes many of the standard higher point couplings, like the vector scalar $h_a h^a h_0^2$ vertex.

It is known that if one integrates out the scalars $\phi_i$ in (3.42), the coefficient of the logarithmic divergence is $N$ times the action [140]: [39]

$$S[h'_0, h'_1, h'_2] = \int d^4x \sqrt{|g|}\left(h_0'^2 - \tfrac{1}{24} F'_{ab} F'^{ab} + \tfrac{1}{60} C_{abcd} C^{abcd}\right), \tag{3.44}$$

where $F'_{ab} = \partial_a h'_b - \partial_b h'_a$ and $C$ is the Weyl tensor for $g_{ab}$. One can obtain knowledge of the lower-spin vertex structure in CHS theory in the basis of the fields $h_s$

[39] This is the bosonic sector of $\mathcal{N} = 1$ conformal supergravity where $h'_0$ is the auxiliary field.

by using (3.44). For example, the vector scalar sector is given by:

$$S[h_0, h_1] = \int d^4x \left[ (h_0 + \tfrac{1}{4} h_a h^a)^2 - \tfrac{1}{24} F_{ab} F^{ab} \right] . \tag{3.45}$$

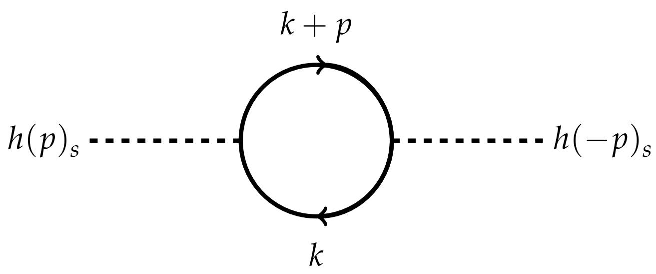
If we studied the scattering of particles of spin 1, this sector of the action would not contribute: the scalar $h_0$ has an algebraic equation of motion $h_0 = -\frac{1}{4} h_a h^a + \ldots$ where the dots are contributions of higher spins. Diagrammatically it implies that the 4 point contact term is exactly cancelled by the exchange of an $h_0$ particle. This means that we can already expect any non-trivial contributions to such amplitudes to come from CHS exchanges. We now compute the vertices that will be used for later chapters.

### 3.3.2 Quadratic Sector

We will now compute the quartic part of the CHS action, ie. $S_{2,CHS}$ from the schematic expression (3.38). This will help us compute scattering amplitudes later down the line, and we will want to do so using the TT gauge; this gauge can be set using the linearised symmetries (3.3). [40] Using this, we can "integrate by parts" the interactions in (3.34) easily. The result is that the momentum space vertex which multiplies $h_s(p' - p)\phi^*(-p')\phi(p)$ is given by:

$$V_{a(s)}(p) = \tfrac{1}{s!} \, p_{a_1} \ldots p_{a_s} \, . \tag{3.46}$$

Computing the relevant loop diagram with two CHS fields insertions:




(3.47)

we find that

$$S_{2,CHS}[h] = \sum_{s=0}^{\infty} \frac{1}{2^s \, (2s+1)!} \int d^4x \, h_{a(s)} \, \Box^s \, h^{a(s)} \, . \tag{3.48}$$

[40] Note that massless higher spins - whose free sector is given by the Fronsdal action - do not have the algebraic part of the symmetry in (3.3), so one cannot make the field traceless *off-shell*. This is one of the reasons computations in CHS theory are simpler.

which is the expected TT gauge form of (3.1) with the constant

$$n_s = \frac{1}{2^{s-1}(2s+1)!}\,. \tag{3.49}$$

Inverting this, the momentum space propagator is then given by

$$D^{a(s)}_{b(s)}(p) = \tfrac{1}{n_s}\tfrac{1}{(p^2)^s}\,\mathcal{P}^{a_1\cdots a_s}_{b_1\cdots b_s}(p)\,, \tag{3.50}$$

where $\mathcal{P}$ is the TT projector ( see Appendix A.4 for its explicit form).

Note that this matches the expression obtained in $d$ dimensions in [126] in terms of generating functions (see A.1):

$$S_{2,CHS}[h] = N\int d^dp\; G(X,Y)\,h(p,u_1)\,h(-p,u_2)\Big|_{u_i=0}\,. \tag{3.51}$$

Here $G(X,Y)$ is an operator on $u_1, u_2$ defined by:

$$G(X,Y) = \sum_{s=0}^{\infty}\frac{\Gamma(\frac{d-3}{2})}{2^{4s}\,\Gamma(s+\frac{d-3}{2})\,\Gamma(s+\frac{d-1}{2})}\,C_s^{(\frac{d-3}{2})}\Big(\frac{X}{\sqrt{Y}}\Big)\,Y^{\frac{s}{2}}\,, \tag{3.52}$$

where $C_s^{(\lambda)}(z)$ is the Gegenbauer polynomial and $X$ and $Y$ are given by:

$$\begin{aligned} X &= p^2\,\partial_{u_1}\cdot\partial_{u_2} - p\cdot\partial_{u_1}\,p\cdot\partial_{u_2}\,,\\ Y &= \Big[(p\cdot\partial_{u_1})^2 - p^2\,\partial^2_{u_1}\Big]\Big[(p\cdot\partial_{u_2})^2 - p^2\,\partial^2_{u_2}\Big]\,. \end{aligned} \tag{3.53}$$

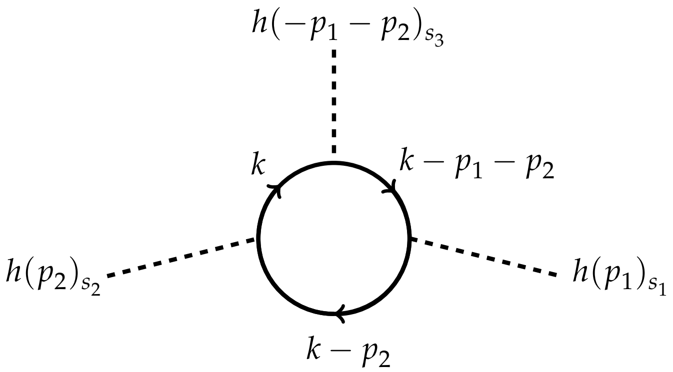
If we take this expression in the TT gauge, (3.52) simplifies drastically, since the operator $Y$ always contains traces or divergences, we can drop any monomial containing $Y$. If we go to $d=4$ the Gegenbauer polynomial further simplifies to a Legendre polynomial, and finally we obtain

$$G(X,Y) = \sum_{s=0}^{\infty}\frac{\Gamma(\frac{1}{2})}{2^{3s}\,\Gamma(s+\frac{3}{2})}\,\frac{X^s}{s!} + \mathcal{O}(Y)\,. \tag{3.54}$$

which leads to (3.48) when plugged in (3.51).

### 3.3.3 Cubic Sector

Next, the cubic $h_{s_1}h_{s_2}h_{s_3}$ term is given by the UV divergent term in the diagram:




(3.55)

Note that each spin $s$ vertex comes with $s$ derivatives. Since the action (3.34) has parity invariance, this implies that the sum of the spins $s_1+s_2+s_3$ must be even. In Chapter 5, we will be interested in computing CHS scattering amplitudes where we have on-shell lower spin particles on the external legs. As such, we will need to compute cubic vertices with $h_0$, $h_a$, $h_{ab}$ on the external legs, and we can make the simplification that $\Box h_a = 0 = \Box h_{ab}$. As explained in Chapter 2 there are more admissible scattering states for the spin 2 fields, but we shall treat those differently. Here we write down the relevant vertices, while the details of the loop-diagram computation will be given in Appendix C.2

Firstly, one can check by dimensional analysis that the 0-0-$s$ vertices are all trivial. Indeed, the fields have mass dimension $[h_s]=2-s$, and the relevant diagram (4.55) implies there are $s$ derivatives in the vertex. It is easy to check that it is impossible to write down a local 0-0-$s$ vertex without a dimensionful parameter - which the theory lacks.

Next, we give the 1-1-$s$ vertex. As we said, it is non-zero only for even $s$ and the relevant interaction term is found to be:

$$
\begin{aligned}
S_3[h_1,h_1,h_s] = \tfrac{(-1)^{s/2}}{(s+2)!}\int d^4x \Big[&\partial_{c(s)}h_a h^a h^{c(s)} - 2h_a\,\partial^a{}_{c(s-1)}\,h_b\,h^{bc(s-1)} \\
&- \partial^{c(s-2)}{}_d\,h^a\partial^d h^b h_{abc(s-2)} - \tfrac{s}{2}\partial^{c(s-2)}\Box h^a h^b h_{abc(s-2)} - \tfrac{s}{2}\partial^{c(s-2)}h^a\Box h^b h_{abc(s-2)}\Big]\,,
\end{aligned}
\tag{3.56}
$$

We now give the 2-2-$s$ vertex in momentum space. It is defined by

$\mathrm{V}_{a_1,a_2,b_1b_2,c(s)} = V_{a_1a_2,b_1b_2,c(s)}(p_1,p_2) + V_{a_1a_2,b_1b_2,c(s)}(p_2,p_1)$, with: [41]

$$
\begin{aligned}
&V_{a_1a_2,b_1b_2,c(s)}(p_1,p_2) = \tfrac{1}{8\,(s+4)!}\Big[-\sum_{\neq a',b\neq b'} \eta_{ab}\, p_{2a'}\, p_{1b'}\,(p_1)_{c(s)} \\
&+2\sum_{a\neq a'} p_{2a'}\, p_{1b_1}\, p_{1b_2}\, \eta_{ac_1} p_{1c_2}\cdots p_{1c_s} - 2\sum_{b\neq b'} p_{1b'}\, p_{2a_1}\, p_{2a_2}\, \eta_{bc_1} p_{1c_2}\cdots p_{1c_s}\Big] \\
&-\tfrac{p_1\cdot p_2}{16\,(s+4)!}\Big\{2(\eta_{a_1b_1}\eta_{a_2b_2}+\eta_{a_1b_2}\eta_{a_2b_1})\,(p_1)_{c(s)} \\
&-4(\,p_{1b_1}\eta_{a_1b_2}\eta_{a_2c_1} - p_{2a_1}\eta_{a_2b_1}\eta_{b_2c_1} + \text{sym } a_{1,2}, b_{1,2})\, p_{1c_2}\cdots p_{1c_s} \\
&+\Big[6\,(\eta_{a_1c_1}\eta_{a_2c_2}+\eta_{a_1c_2}\eta_{a_2c_1})\, p_{1b_1}\, p_{1b_2} + 6\,(\eta_{b_1c_1}\eta_{b_2c_2}+\eta_{b_1c_2}\eta_{b_2c_1})\, p_{2a_1}\, p_{2a_2} \\
&-\sum_{a\neq a',b\neq b'} 4\,(\eta_{ac_1}\eta_{bc_2}+\eta_{ac_2}\eta_{bc_1})\, p_{2a'}\, p_{1b'}\Big]\, p_{1c_3}\cdots p_{1c_s}\Big\} \\
&+\tfrac{(p_1\cdot p_2)^2}{8\,(s+4)!}\Big\{\sum_{a\neq a',b\neq b'} (\eta_{ac_1}\eta_{bc_2}\eta_{a'b'}+\eta_{ac_2}\eta_{bc_1}\eta_{a'b'})\, p_{1c_3}\cdots p_{1c_s} \\
&-(p_{1b_1}\,\eta_{a_1c_1}\eta_{a_2c_2}\eta_{b_2c_3} - p_{2a_1}\,\eta_{a_2c_1}\eta_{b_1c_2}\eta_{b_2c_3} + \text{sym } c_{1,2,3})\, p_{1c_4}\cdots p_{1c_s}\Big\} \\
&-\tfrac{(p_1\cdot p_2)^3}{32\,(s+4)!}\,(\eta_{a_1c_1}\eta_{a_2c_2}\eta_{b_1c_3}\eta_{b_2c_4} + \text{sym } c_{1,2,3,4})\; p_{1c_5}\cdots p_{1c_s}\,,
\end{aligned}
\tag{3.57}
$$

where sym $c_{i,j,\ldots}$ means that one is to add the terms required to make the expression symmetric in the indices $(c_i, c_j, \ldots)$.

Next, the 1-0-$s$ vertex appearing in front of $h^a(p_1)\, h_0(p_2)\, h^{c(s)}(-p_1-p_2)$ is:

$$
\mathrm{V}_{a,c(s)}(p_1,p_2) = \tfrac{2}{(s+1)!}\,\eta_{ac_1}\, p_{c_2}\cdots p_{c_s}\,. \tag{3.58}
$$

Here $p$ can be either $p_1$ or $p_2$ since the vertex is symmetric under $p_1 \leftrightarrow p_2$ and $s$ must be odd. Next, keeping the same momentum assignment, the 2-0-$s$ vertex is:

$$
\begin{aligned}
\mathrm{V}_{a_1a_2,c(s)}(p_1,p_2) = &\tfrac{1}{(s+2)!}\Big[-(\eta_{a_1c_1}\, p_{1\,a_2} + \eta_{a_2c_1} p_{1\,a_1})\, p_{1\,c_2}\cdots p_{1\,c_s} \\
&- \tfrac{1}{2} p_1\cdot p_2\,(\eta_{a_1c_1}\eta_{a_2c_2}+\eta_{a_2c_2}\eta_{a_1c_1})\, p_{1\,c_3}\cdots p_{1\,c_s}\Big]\,.
\end{aligned}
\tag{3.59}
$$

[41] Note that this vertex is not appropriate for the $s=2$ case, since the relevant diagram has more symmetry, which needs to be taken into account.

For the 1-2-$s$ vertex multiplying $h_{a_1a_2}(p_1)\,h_b(p_2)\,h_{c(s)}(-p_1-p_2)$, we get:

$$\begin{aligned}
&\mathrm{V}_{a_1a_2,b,c(s)}(p_1,p_2) = \tfrac{1}{(s+3)!}\Big\{(\eta_{a_1b}p_{2a_2}+\eta_{a_2b}p_{2a_1})p_{1c(s)}\\
&+(-\eta_{a_1c_1}p_{2a_2}p_{1b}-\eta_{a_2c_1}p_{2a_1}p_{1b}+2\eta_{bc_1}p_{2a_1}p_{2a_2})\,p_{1c_2}\cdots p_{1c_s}\\
&-(p_1\cdot p_2)\Big[(\eta_{a_1b}\eta_{a_2c_1}+\eta_{a_1c_1}\eta_{a_2b})\,p_{1c_2}\cdots p_{1c_s}\\
&+\Big((\eta_{a_1c_1}\eta_{bc_2}+\eta_{a_1c_2}\eta_{bc_1})\,p_{2a_2}-(\eta_{a_1c_1}\eta_{a_2c_2}+\eta_{a_2c_1}\eta_{a_1c_2})\,p_{1b}\\
&+(\eta_{a_2c_1}\eta_{bc_2}+\eta_{a_2c_2}\eta_{bc_1})\,p_{2a_1}\Big)\,p_{1c_3}\cdots p_{1c_s}\Big]\\
&+\tfrac{1}{3}(p_1\cdot p_2)^2(\eta_{a_1c_1}\eta_{a_2c_2}\eta_{bc_3}+\mathrm{sym}\;c_{1,2,3})\,p_{1c_3}\cdots p_{1c_s}\Big\}\,.
\end{aligned} \tag{3.60}$$

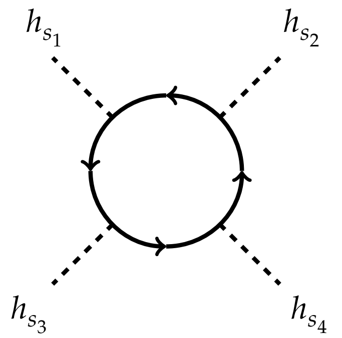
Finally, the lower spin vertices are:

$$\begin{aligned}
S_3[h_1,h_1,h_0] &= \tfrac{1}{2}\int d^4x\,h_ah^ah_0\\
S_3[h_1,h_1,h_2] &= \tfrac{1}{24}\int d^4x\Big[\partial_ch_a\,\partial_\sigma h^ah^{c\sigma}-2\partial_ch_a\,\partial^a\,h_b\,h^{bc}+\partial_\rho h^a\partial^\rho h^bh_{ab}+2h^a\,\Box h^bh_{ab}\Big]\\
S_3[h_0,h_2,h_2] &= \tfrac{1}{48}\int d^4x\,h_0(\partial_ch_{ab}\partial^ch^{ab}+2\partial_ch_{ab}\partial^ah^{cb})\,.
\end{aligned} \tag{3.61}$$

which can all be obtained by expanding the standard scalar Maxwell-Weyl action (3.44), performing the field redefinition (3.43) and imposing the TT gauge.

### 3.3.4 Quartic Sector

We can extend this to the quartic contact terms from computing the divergence of the diagram:




(3.62)

to obtain contact terms. For $s_1=s_2=s_3=s_4=1$, one gets:

$$S_4[h_1,h_1,h_1,h_1]=\tfrac{1}{16}\int d^4x(h_ah^a)^2\,. \tag{3.63}$$

This is in agreement with (3.45). In fact, other contact vertices, such as the 1122 or 2222 vertex can be obtained by expanding the action (3.44) and performing the field redefinitions (3.43).

## 3.4 Twistor Formulation of CHS theory

In this section, we will give some details on how one can represent (the free sector of) CHS theory using the twistor-spinor formalism we introduced for Conformal Gravity in Section 2.4.

To do this, we will look at the "linearised spin $s$ Weyl tensors" in the spinor-helicity formalism, which are just higher spin analogues of the gauge-invariant $\Psi$ and $\widetilde{\Psi}$ curvatures of Conformal Gravity. Indeed, due to the conformal symmetry of CHS theory, a spin $s$ Weyl tensor can be decomposed into an anti-self dual and self-dual sector:

$$C_{a(s)b(s)} = \epsilon_{\dot{\alpha}_1\dot{\beta}_1} \cdots \epsilon_{\dot{\alpha}_s\dot{\beta}_s} \Psi_{\alpha(s)\beta(s)} + \epsilon_{\alpha_1\beta_1} \cdots \epsilon_{\alpha_s\beta_s} \widetilde{\Psi}_{\dot{\alpha}(s)\dot{\beta}(s)} , \tag{3.64}$$

In terms of these, the equations of motion generalise (2.59):

$$\partial^{\alpha(s)\dot{\alpha}(s)} \Psi_{\alpha(s)\beta(s)} = 0 , \qquad \partial^{\beta(s)\dot{\beta}(s)} \widetilde{\Psi}_{\dot{\alpha}(s)\dot{\beta}(s)} = 0 . \tag{3.65}$$

As with Conformal Gravity, we will now aim to count and classify the different solutions to these equations, in order to determine which states are oscillatory (ie. admissible for scattering) and which are growing.

### 3.4.1 Spectrum of CHS theory

If we focus on the ASD sector, the most obvious solution of (3.65) solves the 2-derivative equation (ie. one derivative in spinor-helicity formalism) and is given by the helicity $s$ state:

$$\Psi^{(-s)}_{\alpha(s)\beta(s)} = \kappa_s \, \lambda_{\alpha(s)} \, \lambda_{\beta(s)} \, e^{i k \cdot x} , \tag{3.66}$$

where as in section 2.3.1, we have introduced a constant $\kappa_s$ of mass dimension $s - 1$ by selecting solutions which satisfy lower a derivative equation. These are the modes we have been assuming in the earlier sections of this chapter as "on-shell" states. However, as for Conformal Gravity, there are more. There are solutions given in terms of the helicity $+1$ spinor $a_\alpha$ satisfying $\langle a\,\lambda\rangle = 1$. This gives us another $s-1$ states::

$$\Psi^{(-h)}_{\alpha(s)\beta(s)} = \kappa_h\, a_{(\alpha(s-h)}\, \lambda_{\alpha(h)}\, \lambda_{\beta(s))}\, e^{\mathrm{i}\,k\cdot x}\,, \qquad h = 1,\ldots,s-1\,, \tag{3.67}$$

When adding the positive helicity modes, this means that there are a total of $2s$ helicity states which are purely oscillatory, and can be scattered, as we will see in Section 5.4. In order to match the degree of freedom counting of, for eg. [83, 84], we need to find another $s(s-1)$ states. These will be solutions featuring polynomial growth in $x$. Introducing the helicity $+1$ spinors $\tilde{\beta}_{\dot{\alpha}}$ (which again satisfy a normalisation condition $[\tilde{\beta}\tilde{\lambda}] = 1$), the state with the highest polynomial growth which satisfies (3.65) is given by:

$$\Psi^{\mathrm{g}(-s)}_{\alpha(s)\beta(s)} = \lambda_{(\beta(s)}\, \lambda_{\alpha_1}\, x_{\alpha(s-1))}{}^{\dot{\alpha}(s-1)}\, \tilde{\beta}_{\dot{\alpha}(s-1)}\, e^{\mathrm{i}\,k\cdot x}\,. \tag{3.68}$$

There are more modes of helicity $-s$ which solve the solution with lesser growth. They are given by:

$$\Psi^{\mathrm{g}(-s)}_{\alpha(s)\beta(s)} = \kappa_l \lambda_{(\beta(s)}\, \lambda_{\alpha(l)}\, x_{\alpha(s-l))}{}^{\dot{\alpha}(s-2)}\, \tilde{\beta}_{\dot{\alpha}(s-l)}\, e^{\mathrm{i}\,k\cdot x}\,, \tag{3.69}$$

for $1 \le l \le s-1$, so there are a total of $s-1$ helicity $-s$ modes. This argument can be repeated for helicities $-h$ with $2 \le h < s$; these correspond to the number of ways that one can partition the $s$ undotted spinor indices $\alpha(s)$ among the spinors $\lambda_\alpha, a_\alpha$ and $x_\alpha{}^{\dot{\alpha}}\tilde{\beta}_{\dot{\alpha}}$. Overall, there are a total of $\sum_{h=2}^{s}(h-1) = \frac{s(s-1)}{2}$ growing negative helicity modes, as expected.

To sum up, the CHS spin $s$ fields contain many states with helicities $\pm h$ where h ranges over $1 \le h \le s$. For a given helicity $h$ there is exactly *one* admissible oscillatory state, and $h-1$ growing modes.

### 3.4.2 Twistor formulation of the free fields

We now proceed similarly to Section 2.4, introducing twistor objects which encompass all the scattering states we just described. [42]

For the spin $s$ fields, we introduce the negative chirality twistor-spinor

[42] This is done in more detail in refs. [81, 129].

$\Gamma_{A(s-1)\,\beta(s+1)}$. It obeys the equation:

$$\mathcal{D}^{\beta\dot{\beta}}\,\Gamma_{A(s-1)\,\beta(s+1)} = 0\,, \tag{3.70}$$

where $\mathcal{D}^{\beta\dot{\beta}}$ is the Cartan connection we introduced in (2.81) taken on a Minkowski background. Once again, the component of $\Gamma$ with only negative helcities is identified with $\Psi$:

$$\Gamma_{\alpha(s-1)\,\beta(s+1)} \equiv \Psi_{\alpha(s-1)\beta(s+1)}\,, \tag{3.71}$$

The equations (3.70) are then equivalent to the system:

$$\begin{aligned}
\partial^{\beta\dot{\beta}}\Psi^{\alpha(s-1)}{}_{\beta\gamma(s)} - \Gamma^{\dot{\beta}\alpha(s-1)}{}_{\gamma(s)} &= 0\,,\\
\partial^{\beta\dot{\beta}}\Gamma^{\alpha(s-2)}{}_{\dot{\alpha}\beta\gamma(s)} - \Gamma^{\dot{\beta}\alpha(s-2)}{}_{\dot{\alpha}\gamma(s)} &= 0\,,\\
&\vdots\\
\partial^{\beta\dot{\beta}}\Gamma^{\alpha}{}_{\dot{\alpha}(s-2)\beta\gamma(s)} - \Gamma^{\dot{\beta}\alpha}{}_{\dot{\alpha}(s-2)\gamma(s)} &= 0\,,\\
\partial^{\beta\dot{\beta}}\Gamma_{\dot{\alpha}(s-1)\beta\gamma(s)} &= 0\,,
\end{aligned} \tag{3.72}$$

in analogy with eqs. (2.86)–(2.89). By plugging the top equation into the one below and so on, this is equivalent to the equation for $\Psi$ in (3.65). To solve this, we use a generalisation of the helicity lowering operator of (2.93) to write:

$$\Gamma_{A(s-1)\,\beta(s+1)} = B_{A(s-1)}\,\lambda_{\beta(s+1)}\;e^{\mathrm{i}\,k\cdot x}\,. \tag{3.73}$$

Again, one inherits a condition from the underlying twistor geometry:

$$C^{A_1}\,\Gamma_{A_1 A(s-2)\,\beta(s+1)} = 0\,, \tag{3.74}$$

where $C^A \equiv \left(-\mathrm{i}\frac{\partial}{\partial\tilde{\lambda}_{\dot{\alpha}}},\,\lambda_\alpha\right)$ was originally defined in (2.96). Again, with some mild assumptions similar to (2.97), (3.74) becomes:

$$\frac{\partial}{\partial\tilde{\lambda}_{\dot{\alpha}_1}}B_{\dot{\alpha}_1\,A(s-2)} + \mathrm{i}\,\lambda_{\alpha_1}\,B^{\alpha_1}{}_{A(s-2)} = 0\,. \tag{3.75}$$

Since this equation holds for all values of the $s-2$ twistor indices, this constrains $B_{A(s-1)}$ to be fully determined by the set of negative helicity spinors:

$$B_{A(s-1)} \quad \leftrightarrow \quad \left\{B,\,B^{\alpha},\,B^{\alpha(2)},\,\ldots,\,B^{\alpha(s-1)}\right\}\,, \tag{3.76}$$

of which there are $\frac{s(s+1)}{2}$, as expected. The ASD Weyl curvature is encoded in the highest rank spinor (cf. (3.71),(3.73)):

$$\Psi_{\alpha(s)\beta(s)} = B_{(\alpha(s-1)}\,\lambda_{\alpha_s)}\,\lambda_{\beta(s)}\,e^{\mathrm{i}\,k\cdot x}\,, \tag{3.77}$$

The other components are then given by:

$$B_{\dot\alpha(k)}{}^{\alpha(s-k-1)} = \sum_{|I|=0}^{k}\left(-\frac{\mathrm{i}}{2}\right)^{|I|}\lambda_{\beta_I}\,\tilde\lambda_{(\dot\alpha_I}\,\frac{\partial^{k-|I|}B^{\beta_I\alpha(s-k-1)}}{\partial\tilde\lambda^{\dot\alpha_{k-I})}}\,, \tag{3.78}$$

for $k = 0,\ldots,s-1$. From (3.66),(3.67), we expect $s$ different solutions for $B^{\alpha(s-1)}$. They are given by:

$$B_h^{\alpha(s-1)} = \kappa_h\,a^{(\alpha(s-h)}\,\lambda^{\alpha(h-1))}\,, \qquad \Psi^{(-h)}_{\alpha(s)\beta(s)} = \kappa_h\,a_{(\alpha(s-h)}\,\lambda_{\alpha(h)}\,\lambda_{\beta(s)\,)}\,e^{\mathrm{i}\,k\cdot x}\,, \tag{3.79}$$

with $h = 1,\ldots,s$.

This generalises readily to the positive helicity sector, where we have the twistor-spinor:

$$\widetilde\Gamma^{B}{}_{\dot\alpha(s+1)} = A^{B(s-1)}\,\tilde\lambda_{\dot\alpha(s+1)}\,e^{\mathrm{i}\,k\cdot x}\,, \qquad \mathcal{D}^{\alpha\dot\alpha}\widetilde\Gamma^{B(s-1)}{}_{\dot\alpha(s+1)} = 0\,, \tag{3.80}$$

following the constraint:

$$\mathrm{i}\,\tilde\lambda_{\dot\beta_1}\,A^{\dot\beta_1 B(s-2)} + \frac{\partial}{\partial\lambda_{\beta_1}}A_{\beta_1}{}^{B(s-2)} = 0\,. \tag{3.81}$$

The spinors $A^{B(s-1)}$ are then determined in terms of the spinors:

$$A^{B(s-1)} \quad\leftrightarrow\quad \left\{\tilde A,\,\tilde A^{\dot\beta},\,\tilde A^{\dot\beta(2)},\,\ldots,\,\tilde A^{\dot\beta(s-1)}\right\}, \tag{3.82}$$

through:

$$A_{\beta(k)}{}^{\dot\beta(s-k-1)} = \sum_{|I|=0}^{k}\left(-\frac{\mathrm{i}}{2}\right)^{|I|}\tilde\lambda_{\dot\alpha_I}\,\lambda_{(\beta_I}\,\frac{\partial^{k-|I|}\tilde A^{\dot\alpha_I\dot\beta(s-k-1)}}{\partial\lambda^{\beta_{k-I})}}\,. \tag{3.83}$$

The oscillatory states are then given by:

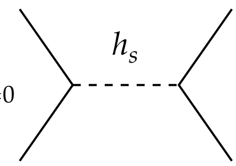
$$\tilde A_h^{\dot\beta(s-1)} = \kappa_h\,\tilde a^{(\dot\beta(s-h)}\,\tilde\lambda^{\dot\beta(h-1))}\,, \qquad \widetilde\Psi^{(h)}_{\dot\alpha(s)\dot\beta(s)} = \kappa_h\,\tilde a_{(\dot\alpha(s-h)}\,\tilde\lambda_{\dot\alpha(h)}\,\tilde\lambda_{\dot\beta(s)\,)}\,e^{\mathrm{i}\,k\cdot x}\,, \tag{3.84}$$

with $h = 1, \ldots, s$. The usefulness of introducing this formalism will reveal itself in In Chapter 5. There, we will give a formula generalising (2.117) and (2.118) for the tree-level three point amplitudes in terms of the twistor-spinors introduced here.

# Chapter 4

# Scalar Amplitudes in CHS theory

Having introduced CHS theory and its symmetries, we now turn our attention to a first series of computations. One of the interesting aspects of Higher Spin theories in general is that they involve an infinite set of fields. This means that certain observables include infinite sums over spins - which must be regularised properly to have meaning. The sums were originally defined in [55], following [54, 57, 127, 128, 134, 135, 141]. One concrete example of such a sum is in $2 \to 2$ scattering amplitudes: the exchange particle can be of any spin, so we must sum over all diagrams.

$\sum_{s=0}^{\infty}$ 


(4.1)

This is analogous to the infinite spin exchanges in the Veneziano amplitude for example. Such as sum was computed for the case of flat space massless higher spins in [138].[43]

In this section, we will compute amplitudes of scalars in a *fully interacting* CHS background in 4 dimensions. One way of introducing this is start with $N+1$ complex scalars $\Phi_A = (\phi_i, \varphi)$. We then proceed to integrate out the $N$ scalars $\phi_i$ and look at the coefficient of $\log \Lambda$ . This will be simply given by $N$ times the CHS action as well as one remaining scalar-CHS coupling given by (3.34) except that the current is built from the $\varphi$ scalar. Since there are no self-interactions among

[43] As stated in section 1.1, it is unclear whether these theories are well-defined in flat space. See [25, 142] for some evidence that one *can* define those theories there. CHS theory on the other hand definitely exists in flat space [5].

the scalars, integrating out $\phi_i$ does not affect the remaining $\varphi$ sector of the original action. Schematically, the action will look like:

$$S[\varphi, h] = \int d^4x \Big[\varphi^* \Box \varphi - \sum_s \frac{1}{s!} h_s\, J_s(\varphi)\Big] + \Gamma_{CHS}[h]\,, \tag{4.2}$$

where, $\Gamma_{CHS}[h]$ is the effective action which includes includes a factor or $N$, and contains the quadratic CHS action $S_{2,CHS}$ given by (3.1), as well as higher order vertices. Again, here $N$ plays the role of the inverse coupling constant: if we rescale the fields $h_s \to N^{-1/2} h_s$, equation (4.2) will look like:

$$S[\varphi, h] = \int d^4x \Big[\varphi^* \Box \varphi + \sum_s \frac{n_s}{2}\, h_s P_{s,s}\, h_s - \frac{1}{\sqrt{N}} \sum_s \frac{1}{s!} h_s\, J_s(\varphi) + N\, \mathcal{O}\left((h_s/\sqrt{N})^3\right)\Big]\,. \tag{4.3}$$

Naturally, we will consider the large $N$ limit, so that perturbation theory is valid.

## 4.1 Scalar scattering tree-level amplitude

Given the action (4.3), we can now start to compute scattering amplitudes of the scalars $\varphi$. Since the only interaction between scalars and CHS fields is given by a $\varphi - \varphi - h_s$ vertex, the first non-trivial diagram of interest will simply be the exchange (4.1) where the solid line represents the scalar $\varphi$ while the dashed line represents a CHS propagator.

### 4.1.1 Spin $s$ Exchange

We first compute an individual term in (4.1), for a generic spin $s$ exchange. We start with the vertex given in (3.48) (where we understand that the currents are bilinear in the single scalar $\varphi$), and look at the correlation function:

$$\begin{aligned} &\langle S_{\text{int}}[\varphi, h]\, S_{\text{int}}[\varphi, h]\,\rangle_0 \\ &= \sum_{s=0}^{\infty} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(s!)^2}\, J^{a(s)}(p) \left\langle h_{a(s)}(p)\, h_{b(s)}(-p) \right\rangle_0 J^{b(s)}(-p)\,, \end{aligned} \tag{4.4}$$

where the subscript 0 means keeping tree level diagrams only, keeping $\varphi$ on the external legs and $J^{a(s)}(p)$ are the Fourier transformed (on-shell traceless conserved) currents. Using the definition of the CHS propagator, (3.50), we have:

$$\left\langle h_{a(s)}(p)\, h^{b(s)}(-p)\right\rangle_0 = \frac{1}{N} D^{a(s)}_{b(s)}(p) = \frac{1}{n_s N}\frac{\mathcal{P}^{b(s)}_{a(s)}(p)}{\left(p^2\right)^s}\,, \tag{4.5}$$

where $n_s = \frac{1}{2^{s-1}(2s+1)!}$ (cf. (3.49) ) and $\mathcal{P}^{b(s)}_{a(s)} = \delta^{(b_1}_{(a_1}\ldots\delta^{b_s)}_{a_s)} + \ldots$ is the projector on totally symmetric traceless transverse space (see Appendix A.4 for its full form). Here we can simply ignore the terms subsumed in the dots. Indeed, they are contracted with currents $J^{a(s)}$ which are traceless and transverse on-shell. Since the external scalar legs *are* on shell, any other terms in the propagator will vanish from these contractions.

If we rewrite (4.4) in terms of the spin sum:

$$\langle S_{\text{int}}[\phi,h]\, S_{\text{int}}[\phi,h]\,\rangle_0 = N^{-1}\sum_{s=0}^{\infty}\frac{1}{n_s}\mathcal{A}_s\,, \tag{4.6}$$

where the spin $s$ contribution is given by:

$$\mathcal{A}_s = \frac{1}{2\,s!}\int\frac{d^4p}{(2\pi)^4}\, J^{a(s)}(p)\,\frac{1}{(p^2)^s}\, J_{a(s)}(-p)\,. \tag{4.7}$$

We can express this in terms of the *traceful* current generators $\mathfrak{J}(x,u)$ of (3.25) by using the projector $\Pi_4$ from (3.30): [44]

$$\mathcal{A}_s = \frac{1}{2\,s!}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{(p^2)^s}\,(\partial_{u_1}\cdot\partial_{u_2})^s\,\Pi_4(u_1,i\,p)\,\mathfrak{J}(p,u_1)\,\mathfrak{J}(-p,u_2)\Big|_{u_i=0}. \tag{4.8}$$

Rewriting the currents $\mathfrak{J}(p,u)$:

$$\begin{aligned}\mathfrak{J}(p,u) &= \int d^4x\,\varphi^*(x+\tfrac{i}{2}\,u)\,\varphi(x-\tfrac{i}{2}\,u)\,e^{-i\,x\cdot p}\\ &= \int\frac{d^4k\,d^4\ell}{(2\pi)^8}\,\varphi^*(k)\,\varphi(\ell)\,e^{u\cdot\frac{k+\ell}{2}}\,(2\pi)^4\,\delta^{(4)}(p+k-\ell)\,,\end{aligned} \tag{4.9}$$

the spin $s$ contribution becomes:

$$\begin{aligned}\mathcal{A}_s &= \frac{1}{2}\int\frac{d^4k_1\,d^4\ell_1\,d^4k_2\,d^4\ell_2}{(2\pi)^{16}}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2-\ell_1-\ell_2)\\ &\qquad\times\varphi^*(k_1)\,\varphi(\ell_1)\,\varphi^*(k_2)\,\varphi(\ell_2)\,A_s(k_1,k_2,\ell_1,\ell_2)\,,\end{aligned} \tag{4.10}$$

[44] In the TT gauge, the dressed and undressed formalism are equivalent, see Appendix C.3.

where $A_s$ is the spin $s$ contribution to the momentum space 4-point amplitude (since it is the coefficient of four on-shell fields, see Section 2.1). It is therefore given by:

$$A_4^{(s)}(k_1,k_2,\ell_1,\ell_2) = \frac{1}{2\,(p^2)^s}\,\frac{(\partial_{u_1}\cdot\partial_{u_2})^s}{s!}\,\Pi_4(u_1,i\,p)\,e^{\frac{1}{2}[u_1\cdot(k_1+\ell_1)+u_2\cdot(k_2+\ell_2)]}\Big|_{u_i=0}\,. \tag{4.11}$$

Using the explicit form of $\Pi_4$, (3.30), and using the Mandelstam variables $(\mathsf{s},\mathsf{t},\mathsf{u})$, we find the $\mathsf{t}$ channel to be:

$$\begin{aligned} A_{\mathsf{t}}^{(s)}(\mathsf{s},\mathsf{t},\mathsf{u}) &= \frac{1}{2(-4)^s\,(-\mathsf{t})}\sum_{n=0}^{[s/2]}\frac{1}{2^{2n}\,n!\,(s-2n)!\,(-s+\frac{1}{2})_n}\left(\frac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\right)^{s-2n} \\ &= -\frac{\mathsf{t}}{2(-8)^s\,(\frac{1}{2})_s}\,P_s\left(\frac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\right), \end{aligned} \tag{4.12}$$

where $P_s(z)$ is the Legendre polynomial. Its appearance is related to the $d=4$ description of the CHS kinetic term we gave in (3.51) and (3.52).

We note that the amplitude is manifestly scale *covariant*, a reflection of the underlying conformal invariance of CHS theory. The total summed amplitude is then:

$$A_{\mathsf{t}}(\mathsf{s},\mathsf{t},\mathsf{u}) = N^{-1}\sum_{s=0}^{\infty}\frac{1}{n_s}\,A_{\mathsf{t}}^{(s)}(\mathsf{s},\mathsf{t},\mathsf{u}) = N^{-1}\,\frac{1}{2}\,F\Big(-\frac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\Big)\,, \tag{4.13}$$

$$F(z) \equiv \sum_{s=0}^{\infty}\frac{1}{2^{3s}\,n_s\,(\frac{1}{2})_s}\,P_s(z)\,. \tag{4.14}$$

Using the expression for $n_s$, this becomes:

$$F(z) = \sum_{s=0}^{\infty}\left(s+\frac{1}{2}\right)P_s(z)\,. \tag{4.15}$$

This sum diverges in general, and it therefore needs to be regularised.

### 4.1.2 Summing over spins

When an infinite sum such as (4.15) presents itself, it needs to be carefully defined in a way that is consistent with the underlying symmetry of the theory. This will guide us later, but for now, let us proceed by introducing a new cut-off prescription. We will use the parameter $w \equiv e^{-\varepsilon}$ with $\varepsilon$ infinitesimal and then define (4.15) as the $w \to 1$ limit:

$$F(z) = \lim_{w \to 1} F(z, w)\,, \qquad F(z, w) \equiv \sum_{s=0}^{\infty} \left(s + \frac{1}{2}\right) w^s\, P_s(z)\,. \tag{4.16}$$

It is useful to rewrite $F(z, w)$ as :

$$F(z, w) = w^{1/2} \frac{d}{dw}\left(w^{1/2} \sum_{s=0}^{\infty} w^s\, P_s(z)\right)\,, \tag{4.17}$$

and use the fact that the right hand side features the generating function for Legendre polynomials:

$$\sum_{s=0}^{\infty} w^s\, P_s(z) = (1 - 2zw + w^2)^{-1/2}\,. \tag{4.18}$$

This allows us to analytically continue $F(z, w)$:

$$F^{\text{reg}}(z, w) = \frac{1}{2} \frac{1 - w^2}{(1 - 2z\,w + w^2)^{\frac{3}{2}}}\,. \tag{4.19}$$

Notice that $F^{\text{reg}}(z, 1)$ happens to vanish for $z \neq 1$, while for $z = 1$ , we get

$$F^{\text{reg}}(1, w) = \frac{1}{2} \frac{1 + w}{(1 - w)^2}\,, \tag{4.20}$$

which diverges for $w \to 1$. In other words, $F^{\text{reg}}(z)$ is to be taken as a distribution with support at $z = 1$. In fact, it is proportional to the Dirac delta function. This can be seen by looking at (4.19) and changing variables $z = x + w,\ \epsilon^2 = 1 - w^2$. We then get $F^{\text{reg}}(x, \epsilon) = \frac{\epsilon^2}{2(x^2+\epsilon^2)^{\frac{3}{2}}}$. As a result, we have the more familiar form: $F^{\text{reg}}(z) = \lim_{\epsilon \to 0} F^{\text{reg}}(x, \epsilon) = \delta^{(4)}(x)$. In other words :

$$F_4^{\text{reg}}(z) = \delta^{(4)}(z - 1)\,. \tag{4.21}$$

The regularisation used above is the same as in [55, 128, 135] for the computation of higher spin partition functions. [45] We also note that this prescription works in arbitrary $d$ [1].

[45] There, for the case of CHS theory in $d$ dimensions, the sum $\sum_{s=0}^{\infty} f_d(s)$ is replaced by $\sum_{s=0}^{\infty} e^{-\varepsilon(s+\alpha_d)} f_d(s)$ where $\alpha_d = \frac{d-3}{2}$. One then takes the limit $\varepsilon \to 0$ and all $\frac{1}{\varepsilon^n}$ poles are dropped.

Another way to regularise this sum is to use the integral representation:

$$P_s(z) = \frac{1}{\pi} \int_0^\pi dx \left( z + \sqrt{z^2 - 1} \cos x \right)^s , \tag{4.22}$$

of the Legendre Polynomial. Performing the sum over the integrand first, we get:

$$\sum_{s=0}^{\infty} (s + \frac{1}{2}) \left( z + \sqrt{z^2 - 1} \cos x \right)^s = \frac{z + 1 + \sqrt{z^2 - 1} \cos x}{2 \, (z - 1 + \sqrt{z^2 - 1} \cos x)^2} . \tag{4.23}$$

where we used analytical continuation, since for $x \in [0, \pi]$ the integrand is always divergent for some values of $z$. The regularised function is now given by:

$$F^{\text{reg}}(z) = \frac{1}{\pi} \int_0^\pi dx \, \frac{z + 1 + \sqrt{z^2 - 1} \cos x}{2 \, (z - 1 + \sqrt{z^2 - 1} \cos x)^2} = \delta(z - 1) , \tag{4.24}$$

recovering the result of (4.21).

### 4.1.3 Total Amplitude

We are now ready to write out explicit results. For the case of $\varphi \, \varphi \to \varphi^* \, \varphi^*$ scattering, the total amplitude contains $\mathsf{t}$ and $\mathsf{u}$ channels which sum to ( cf (4.13) and (4.21), (4.24) ):

$$A_{\varphi \, \varphi \to \varphi^* \, \varphi^*} = \frac{1}{4N} \left[ \delta \left( \frac{\mathsf{s}}{\mathsf{t}} \right) + \delta \left( \frac{\mathsf{s}}{\mathsf{u}} \right) \right] . \tag{4.25}$$

This amplitude looks strange, but we can use kinematics to show that it actually vanishes for all physical momenta. First, we go to the centre of mass frame, for which $\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$. We then introduce the angle $\theta$ via $\cos \theta = \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1||\vec{p}_3|}$. Since all the particles are massless, we have $E_i = |\vec{p}_i|$ which means that:[46]

$$\frac{\mathsf{s}}{\mathsf{t}} = -\frac{1}{\sin^2 \frac{\theta}{2}} , \qquad \frac{\mathsf{s}}{\mathsf{u}} = -\frac{1}{\cos^2 \frac{\theta}{2}} . \tag{4.26}$$

We can therefore conclude that for real $\theta$, the arguments of the delta functions in (4.25) are never zero, so the amplitude actually vanishes:

$$A_{\varphi\varphi \to \varphi^* \varphi^*} = 0 . \tag{4.27}$$

[46] Note that this is ill defined in the collinear limit $p_1^a = \alpha p_2^a$, but that limit requires complex momenta, so its physical interpretation is unclear.

Next, we look at $\varphi\varphi^* \rightarrow \varphi\varphi^*$ scattering. It has $\mathsf{s}$ and $\mathsf{t}$ channel contributions:

$$A_{\varphi\varphi^*\rightarrow\varphi\varphi^*} = \frac{1}{4N}\left[\delta\left(\frac{\mathsf{u}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{u}}{\mathsf{s}}\right)\right] = \frac{1}{4N}\left[\delta\left(\cot^2\frac{\theta}{2}\right) + \delta\left(\cos^2\frac{\theta}{2}\right)\right]. \tag{4.28}$$

The delta functions are only zero for the single value $\theta = \pi$, so omitting that point:

$$A_{\varphi\varphi^*\rightarrow\varphi\varphi^*} = 0\,. \tag{4.29}$$

As a final remark, note that had we considered a real scalar in the original action (4.3) instead of a complex one, we would only have had a coupling between even currents and even spin CHS fields. This would modify the computation to include only even terms in (5.25) and would have lead to (4.24) and lead to:

$$\begin{aligned} A^{(\mathbb{R})}_{\varphi\varphi\rightarrow\varphi\varphi} &= \frac{1}{8N}\left[\delta\left(\frac{\mathsf{u}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{s}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{u}}{\mathsf{s}}\right) + \delta\left(\frac{\mathsf{t}}{\mathsf{s}}\right) + \delta\left(\frac{\mathsf{t}}{\mathsf{u}}\right) + \delta\left(\frac{\mathsf{s}}{\mathsf{u}}\right)\right] \qquad (4.30)\\ &= \frac{1}{8N}\left[\delta\left(\cot^2\frac{\theta}{2}\right) + \delta\left(\csc^2\frac{\theta}{2}\right) + \delta\left(\cos^2\frac{\theta}{2}\right) + \delta\left(\sin^2\frac{\theta}{2}\right) + \delta\left(\tan^2\frac{\theta}{2}\right) + \delta\left(\sec^2\frac{\theta}{2}\right)\right]. \end{aligned}$$

which includes $\mathsf{s}$, $\mathsf{t}$ and $\mathsf{u}$ channel contributions. Once again this amplitude is non-zero only for $\theta = 0, \pi$ and thus excluding these points we get:

$$A^{(\mathbb{R})}_{\varphi\varphi\rightarrow\varphi\varphi} = 0\,. \tag{4.31}$$

Overall, we see that while each individual spin $s$ contribution to the diagram (4.1) is non trivial, the sum appears to be vanishing - after our particular regularisation scheme. We will now argue that this is implied by global CHS symmetry.

## 4.2 Global CHS symmetry and amplitude constraint

We have just observed the vanishing of an *a priori* non-trivial amplitude, which seemed to depend on our regularisation. As mentioned before, said regularisation should be selected to be compatible with the symmetries of the theory.[47]

The regularisation introduced in (4.16) can be obtained by simply introducing the factor $w$ in the quadratic CHS action in the following way (see (3.51) -

[47] This is analogous to the following scenario. Consider compactifying a 5d theory to 4d on a circle. One obtains an infinite sum over Kaluza-Klein modes, which are only manifestly Lorentz covariant in 4 dimensions. However, requiring that the modes have 5d Lorentz symmetry will put constrains on the infinite sum that will ensure results from 5d can be recovered.

(3.52)):

$$S^{\rm reg}_{\rm CHS,2}[h;w] = \int d^4p\ G(w^{-1}\,X, w^{-2}\,Y)\,h(p,u_1)\,h(-p,u_2)\Big|_{u_i=0}\,. \tag{4.32}$$

We then ask that this regularised action still be invariant under the global CHS symmetries introduced in Section 3.1. Let us see how it acts on the correlators of massless scalar fields. Note we will assume here that theory is anomaly free, which seems to be implied by other work in the literature (cf. [55,57,127,134,135,143]). We defer a more in-depth discussion of anomalies to Chapter 6.

To do this, we will show that the vanishing of the amplitude is actually implied by the existence of hypertranslations (cf. eq. (3.17)):

$$\delta\varphi(x) = \varepsilon^{a(r)}\,\partial_{a(r)}\varphi(x)\,. \tag{4.33}$$

where we will take $\varepsilon^{a(r)}$ to be a constant parameter. Note that any traces it contains parametrise the trivial parts of the global symmetry which vanish on-shell; as such we will ignore them. We further refine this parameter to be proportional to $y^{\{a_r}\dots y^{a_r\}} = y^{a_1}\dots y^{a_r} - traces$, where $y^a$ play the role of auxiliary vectors. This implies that the action is also invariant under:

$$\delta\varphi(x) = (e^{y\cdot\partial_x} - e^{-y\cdot\partial_x})\varphi(x) = \varphi(x+y) - \varphi(x-y)\,. \tag{4.34}$$

This invariance applied to scalar four point functions yields:

$$\begin{aligned}&\langle\varphi(x_1+y)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\rangle + \langle\varphi(x_1)\,\varphi(x_2+y)\,\varphi(x_3)\,\varphi(x_4)\rangle\\ &+\langle\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3+y)\,\varphi(x_4)\rangle + \langle\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4+y)\rangle\\ &-(y\leftrightarrow -y) = 0\,,\end{aligned} \tag{4.35}$$

which is in momentum space:

$$\sin(p_{12}\cdot y)\,\sin(p_{13}\cdot y)\,\sin(p_{14}\cdot y)\,\langle\varphi(p_1)\,\varphi(p_2)\,\varphi(p_3)\,\varphi(p_4)\rangle = 0\,, \tag{4.36}$$

where $p_{ij} = \frac{1}{2}(p_i + p_j)$, and we have used trigonometric identities and momentum conservation, $p_1+p_2+p_3+p_4 = 0$.

We now further refine our choice and pick $y^a$ to be:

$$y^a = d_1\,p^a_{12} + d_2\,p^a_{13} + d_3\,p^a_{14}\,, \tag{4.37}$$

where $d_i$ are constants. Plugging this in (C.4), this yields:

$$\sin(\tfrac{1}{4}d_1\,\mathsf{s})\,\sin(\tfrac{1}{4}d_2\,\mathsf{t})\,\sin(\tfrac{1}{4}d_3\,\mathsf{u})\,A_4^{(\mathbb{R})}(\varphi\varphi\to\varphi\varphi)=0\,, \tag{4.38}$$

where we used the on-shell condition $p_i^2=0$. Since the constants $d_i$ are arbitrary, this is equivalent to $\mathsf{s}\,\mathsf{t}\,\mathsf{u}\,A^{(\mathbb{R})}_{\varphi\varphi\to\varphi\varphi}=0$, whose solution is given by the following distribution:

$$A^{(\mathbb{R})}_{\varphi\varphi\to\varphi\varphi}=k_1(\mathsf{t},\mathsf{u})\,\delta(\mathsf{s})+k_2(\mathsf{u},\mathsf{s})\,\delta(\mathsf{t})+k_3(\mathsf{s},\mathsf{t})\,\delta(\mathsf{u})\,, \tag{4.39}$$

where $k_i$ are unfixed functions. Next, since conformal symmetry is contained within the full CHS symmetry group, we expect the amplitude to be invariant under a rescaling of momenta by a constant $\lambda$:

$$A^{(\mathbb{R})}_{\varphi\varphi\to\varphi\varphi}(\lambda^2\,\mathsf{s},\lambda^2\,\mathsf{t},\lambda^2\,\mathsf{u})=A^{(\mathbb{R})}_{\varphi\varphi\to\varphi\varphi}(\mathsf{s},\mathsf{t},\mathsf{u})\,. \tag{4.40}$$

This implies that $k_i$ must be homogeneous functions of degree 1, to compensate. Finally we use crossing symmetry, which implies that $k_i(x,y)=k(x,y)$. We are scattering massless particles we means that we have $\mathsf{s}+\mathsf{t}+\mathsf{u}=0$ which further simplifies the form of $k$. For instance in the first term of (4.39), we have :

$$k(\mathsf{t},\mathsf{u})\delta(\mathsf{s})=k(\mathsf{t},-\mathsf{t}-\mathsf{s})\delta(\mathsf{s})=k(\mathsf{t})\delta(\mathsf{s})\,, \tag{4.41}$$

since the functions are homogeneous of degree 1. We can posit that $k$ is simply a linear function, but this is trivial in (4.41) after using crossing symmetry. The only choice is then $k(\mathsf{t},\mathsf{u})\sim|\mathsf{t}|\delta(\mathsf{s})$. Writing this in manifestly crossing-symmetric way:

$$A^{(\mathbb{R})}_{\varphi\varphi\to\varphi\varphi}(\mathsf{s},\mathsf{t},\mathsf{u})=f\Big[(|\mathsf{t}|+|\mathsf{u}|)\,\delta(\mathsf{s})+(|\mathsf{u}|+|\mathsf{s}|)\,\delta(\mathsf{t})+(|\mathsf{s}|+|\mathsf{t}|)\,\delta(\mathsf{u})\Big], \tag{4.42}$$

where $f$ is an arbitrary overall constant. This form is actually equivalent to the amplitude we computed in (5.42), since we can write $|\mathsf{t}|\delta(\mathsf{s})=\delta(\frac{\mathsf{s}}{\mathsf{t}})$ for example. So once again, we conclude that the amplitude vanishes everywhere except for a measure 0 domain in momentum space.

Finally, we remark that this argument applies at loop-level so long as the symmetry is anomaly-free. Furthermore, it should be no different in the case of complex scalars, whose amplitude also vanishes as we've seen.

We conclude that the prescription of our sum over spins is compatible with the CHS symmetry, which seems to constrain the amplitude to vanish. Next, we look at the 1-loop computation to see if it can bolster our claim that this vanishing holds at loop level.

## 4.3 One-loop Amplitudes

Let us compute the UV divergent part of the four point scalar amplitudes. This is similar to computing the 1-loop correction to the 4 scalar amplitude in Scalar-QED, except that we also have fields of spin $s > 1$ which can run on internal propagators. In Scalar QED, there is a logarithmic divergence, and we expect the same here. The question is whether after a regularised sum over spins such a divergence could cancel.

If we integrate out the CHS fields $h_s$ in the action (4.3), the only possible logarithmic UV divergent local term is proportional to $\int d^4x\, (\varphi^* \varphi)^2$, since the theory has no other dimensionful parameter. This is the term we will compute. Actually, since this term involves no derivatives, we can compute it by simply considering the external $\varphi$ to have zero momenta. The result should still be gauge independent as this simply corresponds to a particular point in an on-shell amplitude. Let us now write down the diagrams which contribute to the four-point amplitude.

**Box diagram**

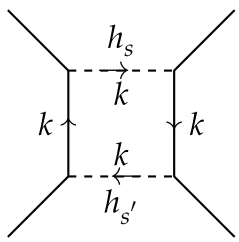



Figure 4.1: Box diagram with vanishing external momenta

The first contribution is given by the box diagram, Fig. 4.1. Since external momenta are taken to vanish, only the internal momentum $k^a$ runs in the loop. Furthermore, recall that the CHS propagators are proportional to the TT propagators, $\mathcal{P}^{a(s)}_{b(s)}(k)$ , which satisfy:

$$k_{a_1}\,\mathcal{P}^{a_1\cdots a_s}_{b_1\cdots b_s}(k) = 0 = \eta_{a_1 a_2}\,\mathcal{P}^{a_1\cdots a_s}_{b_1\cdots b_s}(k)\,. \tag{4.43}$$

For terms with $s, s' \neq 0$, there will be factors of $k^a$ on the numerator coming from the higher derivative coupling in (3.48), and they will necessarily vanish. The only non-trivial term comes from the case $s = s' = 0$, for which the CHS fields are "non-propagating", since they have no derivatives in their kinetic terms. The result is given by:

$$A^{(1)}_{\text{Box}} = \left(\frac{1}{2N}\right)^2 I(\Lambda)\,. \tag{4.44}$$

where $I(\Lambda)$ is the standard UV divergent loop integral:

$$I(\Lambda) = \int^{\Lambda} \frac{d^4k}{(k^2)^2}\,. \tag{4.45}$$

**CHS bubble diagram**

The next diagram that would appear in Scalar QED (in fact the only contributing diagram) is the bubble diagram shown in Fig. 4.2. In CHS theory, we

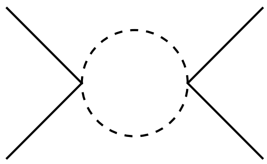


Figure 4.2: Bubble diagram in scalar QED

only have cubic couplings between the CHS fields and the scalars, so at first sight it looks like such a diagram cannot appear. However, it turns out to be the same as the non-1PI diagrams shown in Fig. 4.3, where the $h_0$ lines are shrunk to a point. Indeed, since the spin 0 CHS field is non-dynamical, the usual considera-

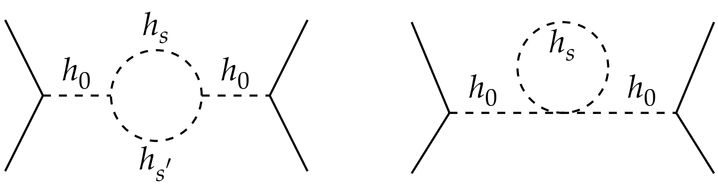



Figure 4.3: Diagrams contributing to $(\phi^*\phi)^2$

tions that we apply to the separation between 1-PI and non-1PI diagrams do not apply. These diagrams do have a contribution, which we'll compute explicitly in section 4.4.

#### Charge renormalisation diagrams

The next diagrams are the ones that would provide a contribution to the cubic coupling (3.48), shown in Fig. (4.4). Similarly to the box diagrams, the only

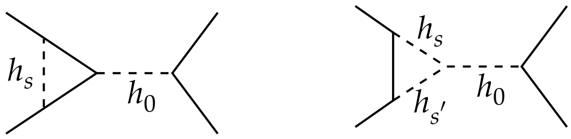



Figure 4.4: Charge renormalisation diagrams

non-trivial terms must have CHS fields of spins $s = s' = 0$ running in the loops. The first diagram yields a contribution given by:

$$A^{(1)}_{\text{charge ren.}} = \left(\frac{1}{2N}\right)^2 I(\Lambda)\,. \tag{4.46}$$

As for the second diagram, it vanishes; indeed there is no $(h_0)^3$ vertex in the CHS action. This is because $h_0$ has mass dimension $[h_0] = 2$ which means such a vertex would have to be non-local.

#### Scalar bubble diagram

The remaining contribution that exists is given by the non 1-PI diagram in Fig 4.5. Its contribution is found to be simply:

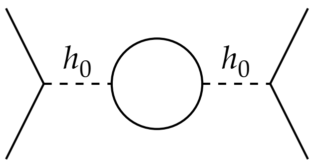



Figure 4.5: Non 1-PI diagram with scalar loop

$$A^{(1)}_{\text{scalar bubble}} = \left(\frac{1}{2N}\right)^2 I(\Lambda)\,, \tag{4.47}$$

### 4.3.1 Equivalent approach: integrating out $h_0$ first

We close this set of computations by highlighting an equivalent approach. The CHS fields $h_0$ in CHS theory can be taken to be auxiliary since it obeys purely algebraic equations of motion. Doing so introduces new interaction vertices between the remaining CHS fields and the scalars $\varphi$. One can see that since the algebraic equation for $h_0$ is of the form:

$$h_0 \sim \varphi^* \varphi + \mathcal{O}(h_{s\geq 1}^2) \,. \tag{4.48}$$

The second term in this equation implies that upon integrating $h_0$ out, we will introduce new quartic and higher order vertices in the CHS fields. However, these terms don't contribute to our 1-loop computation, so we can safely ignore them. The first term is more interesting. First of all, it introduces a quartic self-interaction vertex $(\varphi^*\varphi)^2$. These are exactly equivalent to the diagrams of Fig. 4.1, 4.4 and 4.5 where the internal propagators carry spin 0. Integrating out $h_0$ is then like shrinking those propagators to a point, such that we compute the diagrams of Fig. 4.6. Finally, there will be new interactions between the scalars

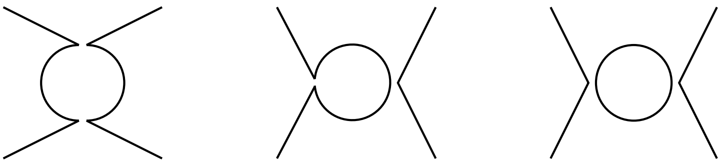


Figure 4.6: One-loop diagrams with $(\phi^*\phi)^2$ vertices (broken or open lines indicate the origin of these diagrams in relation to diagrams in Figs. 4.1, 4.4, 4.5).

and other CHS fields. In particular, there are vertices of the form $h^2(\varphi^*\varphi)$ and $h^2(\varphi^*\varphi)^2$, shown in Fig 4.7. These give new diagrams shown in Fig. 4.8 which

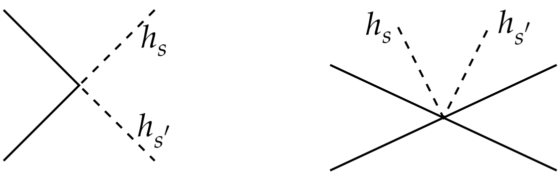



Figure 4.7: Higher order contact vertices

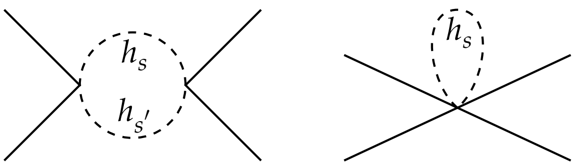



Figure 4.8: One-loop diagrams with $h^2\,\phi^2$ and $h^2\,\phi^4$ vertices

are equivalent to the diagram (4.3) where the spin 0 propagators are shrunk to a point.

In this approach, the contributions to the 1-loop logarithmic divergence are then split into two categories: the ones that arise due to a $\varphi$-loop such as Fig 4.6, and the ones involving $h_s$ loops as in Fig. 4.8 [48]. As such we can rewrite the amplitude as:

$$A^{(1)}_{\text{tot}} = A^{(1)}_{\phi-\text{loop}} + A^{(1)}_{h_s-\text{loop}} \,. \tag{4.49}$$

The scalar contributions can be computed to sum to (cf (4.44),(4.46) and (4.47)):

$$A^{(1)}_{\phi-\text{loop}} = 3\left(\frac{1}{2N}\right)^2 I(\Lambda)\,. \tag{4.50}$$

We note that had we included $N_\varphi$ external scalars instead of just one, this would only have changed (4.47) to become rescaled by a factor of $N_\varphi$, so this alone could not lead the amplitude to vanish. This implies that the scalar loops make CHS symmetry anomalous.

One can therefore ask what would happen if we made $\varphi$ into background fields only, in which case the only contribution to the computation would come from $A^{(1)}_{h_s-\text{loop}}$. This amplitude could indeed vanish after a regularised sum over spins. Let us address this now.

## 4.4 Computing the UV divergence of the one-loop CHS effective action in $h_0$ background

We now go back to the diagram of Fig. 4.3 (or those in 4.8). Remembering that the external legs have zero momentum, we find the contribution to be :

$$c_{\text{CHS}}\left(\frac{1}{2N}\right)^2 I(\Lambda)\int d^4x\,(\phi^*\,\phi)^2\,, \tag{4.51}$$

where $c_{CHS}$ is a coefficient encoding infinitely many contributions from all possible loops. Another way of writing this is that the one-loop effective action of the theory in an external $h_0$ background is (cf. (3.39)):

$$\Gamma_{\text{div}}[h_0] = c_{\text{CHS}}\, I(\Lambda)\int d^4x\,(h_0)^2\,. \tag{4.52}$$

[48]Those may involve ghost contributions as we'll later see.

As noted before, the CHS action is totally fixed and schematically looks like $S_{\rm CHS} = N \int d^4x (h_0^2 + F_{ab}^2 + C_{abcd}^2 + ....)$. As such, if we turn on a spin 2 background in (4.52), we expect to find the same coefficient $c_{CHS} \log \Lambda$ in front of a the linearised square Weyl tensor term $(C_{abcd})^2$. In other words, $c_{CHS}$ should be the same as the $c$-coefficient of the conformal anomaly of CHS theory. The logic for this is as follows. We expect the effective action to be invariant under the full global CHS symmetry. The logarithmic divergence is local, and the only local such theory is CHS theory itself. As such, we expect the coefficient in front of $h_0^2 \log \Lambda$ to be the same as the one in front of $C_{abcd}^2 \log \Lambda$, ie. the conformal anomaly coefficient.

Let us see if evaluating this yields the same result. In order to compute $c_{CHS}$ we split it into a "physical" and a ghost contribution:

$$c_{\rm CHS} = c^{ph}_{\rm CHS} + c^{gh}_{\rm CHS} \,. \tag{4.53}$$

### 4.4.1 Physical field loop contribution

We first compute diagrams which only include CHS fields running in the loops and two $h_0$ insertions. In what follows, we will use the linearised CHS symmetry (3.3) to pick the TT gauge: $p_{a_1} h(p)^{a_1 \dots a_s} = 0 = \eta_{a_1 a_2} h(p)^{a_1 \dots a_s}$ . The contributions are given by the two diagrams of Fig. 4.9. These diagrams involve

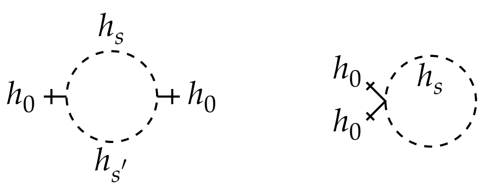
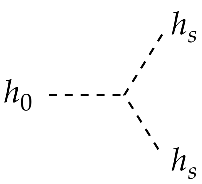
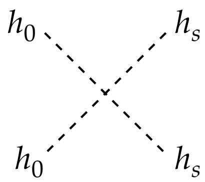



Figure 4.9: CHS effective action in $h_0$ background

two types of vertices: $h_0 h_s h_{s'}$ and ${h_0}^2 {h_s}^2$. In momentum space, these are given by:




$$= N\, h_0(0)\, T_s(k, \partial_{u_1}, \partial_{u_2})\, h_s(k, u_1)\, h_s(-k, u_2)\, \big|_{u_i=0} \,, \tag{4.54}$$




$$= N\,(h_0(0))^2\, Q_s(k,\partial_{u_1},\partial_{u_2})\, h_s(k,u_1)\, h_s(-k,u_2)\,\big|_{u_i=0}\,. \tag{4.55}$$

where $T_s$ and $Q_s$ are (see Appendix C.3 ):

$$\begin{aligned} T_s(k,\partial_{u_1},\partial_{u_2}) \quad &= t_s\,(k^2)^{s-1}\left(\partial_{u_1}\cdot\partial_{u_2}\right)^s + \ldots \\ Q_s(k,\partial_{u_1},\partial_{u_2}) \quad &= q_s\,(k^2)^{s-2}\left(\partial_{u_1}\cdot\partial_{u_2}\right)^s + \ldots\,. \end{aligned} \tag{4.56}$$

Above, the dots mean terms that include traces or divergences of the field, so they drop out in the TT gauge. This explains why we do not need to consider diagrams with vertices of the form ${h_0}^2 h_s h_{s'}$ with $s \neq s'$, since those would involve a trace or divergence contraction.

Using (4.56), the diagrams in 4.9 are given by

$$\begin{aligned} I_1 &= \frac{1}{4}\left(\frac{1}{N\,n_s}\right)^2 \int d^4k\, \frac{N\,T_s(k,\partial_{u_1},\partial_{u_2})\,N\,T_s(k,\partial_{v_1},\partial_{v_2})\,\mathrm{P}_s(k,u_1,v_1)\,\mathrm{P}_s(k,u_2,v_2)}{(k^2)^{2s}}\Bigg|_{u_i=v_i=0}\,, \\ I_2 &= \frac{1}{4}\,\frac{1}{N\,n_s}\int d^4k\,\frac{N\,Q_s(k,\partial_{u_1},\partial_{u_2})\,\mathrm{P}_s(k,u_1,u_2)}{(k^2)^s}\Bigg|_{u_i=0}\,, \end{aligned} \tag{4.57}$$

for the left and right diagrams of 4.9 respectively. There, we've used the propagator (4.5). Expanding in the auxiliary variables and contracting accordingly, one finds:

$$I_1 = \tfrac{1}{4}(2s+1)\left(\tfrac{t_s}{n_s}\right)^2 I(\Lambda)\,, \qquad\qquad I_2 = \tfrac{1}{4}\,(2s+1)\,\tfrac{q_s}{n_s}\,I(\Lambda)\,. \tag{4.58}$$

where the fator of $(2s+1)$ comes from the trace of the TT projector. The constants $t_s$ and $q_s$ can be obtained from expanding the CHS action explicitly. This can be read off from (C.31) to yield:

$$\tfrac{t_s}{n_s} = -4\left(s+\tfrac{1}{2}\right)\,,\quad s\geq 1\,; \qquad\qquad \tfrac{q_s}{n_s} = 8\left(s+\tfrac{1}{2}\right)\left(s-\tfrac{1}{2}\right)\,,\quad s\geq 2\,. \tag{4.59}$$

Finally, this leads to

$$c^{ph}_{\text{CHS}} = 2^3 \sum_{s=1}^{\infty} \left(s+\tfrac{1}{2}\right)^3 + 2^2 \sum_{s=2}^{\infty} \left(s+\tfrac{1}{2}\right)^2 \left(s-\tfrac{1}{2}\right) . \tag{4.60}$$

The sum in this expression is divergent, and therefore needs an appropriate regularisation.

### 4.4.2 Ghost loop contribution

We next compute the contribution arising from ghosts that appear after gauge-fixing the action. Since we are looking at the theory in a constant $h_0$ background, we consider the classical CHS action at quadratic order spins strictly greater than 0. In other words, we look at the part of the CHS action of the form:

$$S_{\text{CHS}} = \int d^4x \, \langle h|K(h_0)|h\rangle \,, \tag{4.61}$$

where $K$ is an $h_0$-dependent kinetic operator, and $\langle\cdot|\cdot\rangle$ stands for the contraction of indices - see Appendix A.1. This action is invariant under:[49]

$$\delta_{\epsilon,\alpha}\, h = u\cdot\partial_x\, \epsilon + \left[u^2 - h_0\, \mathcal{F}(\partial_u,\partial_x)\right] \alpha \,, \tag{4.62}$$

where $\mathcal{F}(\partial_u,\partial_x) = \Pi_d(\partial_u,\partial_x)\, \Pi^{-1}_{d+4}(\partial_u,\partial_x)$, and we will pick the gauge parameters to be traceless and double traceless respectively, ie. $(\partial_u^2\epsilon = 0 = (\partial_u^2)^2\alpha)$.
We now show explicitly how the TT gauge is fixed using (4.62). First, we fix the trace using $\alpha$. Since this part of the transformation is algebraic, it does not introduce a ghost. The remaining symmetry becomes:

$$\delta_\epsilon\, h = T(h_0,\epsilon) = \mathrm{P}_{\mathrm{T}}\left[u\cdot\partial_x - h_0\, \mathcal{G}(\partial_u,\partial_x)\right]\epsilon \,, \tag{4.63}$$

$\mathrm{P}_{\mathrm{T}}$ is the traceless projector and the precise form of $\mathcal{G}(\partial_u,\partial_x)$ is given in Appendix C.4. We then use (4.63) to impose transversality. This gives rise to a non-trivial Jacobian factor in the partition function, which can be represented by a ghost contribution. The relevant computations are done in Appendix C.4 and we give here the main results. This contribution is found to be:

$$S_{gh} = \int d^4x \, \langle \bar{c}|K_{gh}(h_0)|c\rangle \,, \tag{4.64}$$

[49] This is obtained by looking at the field-independent transformation of (3.33), and adding the contribution from $h_0$ that comes from "undressing " (3.26) and (3.27)

$$K_{gh}(h_0) = \partial_x \cdot \partial_u \frac{\delta T(h_0, \epsilon)}{\delta \epsilon} = \partial_x \cdot \partial_u \, \mathrm{P}_T \, [u \cdot \partial - h_0 \, \mathcal{G}(\partial_u, \partial_x)] \, . \tag{4.65}$$

In (4.64), it is possible to perform shift each spin $s$ ghost fields once in order to remove all $h_0$ dependence, as done in Appendix C.4. The conclusion is then that CHS ghosts do not couple to $h_0$ and therefore:

$$c^{gh}_{\text{CHS}} = 0 \, . \tag{4.66}$$

**Summing over spins**

Finally, from (4.52) and (4.53), the total contribution to $c_{CHS}$ is found by adding (4.60) and (4.66) and yields: (4.60),(4.66))

$$c_{\text{CHS}} = -5 + 4 \sum_{s=0}^{\infty} \left[ 3 \left( s + \tfrac{1}{2} \right)^3 - \left( s + \tfrac{1}{2} \right)^2 \right] \, . \tag{4.67}$$

As mentioned earlier, this is expected to be related to the $c$-coefficient of the conformal anomaly, which was found to vanish in [57,127,135] (see also [143]) when the sum is regularised with the cut-off $e^{-(s+\frac{1}{2})\,\epsilon}$. Using this cut-off in (4.67), we find its finite part to be:

$$c_{\text{CHS}}\big|_{fin} = -\frac{407}{80} \, , \tag{4.68}$$

which is not the expected result of 0. The meaning of this remains unclear. A possibility is that there exists some other contribution one must take into account. An indication of that can be obtained from the fact that the spin $s$ contribution to the anomaly coefficients in [55,57,127] are $6^{\text{th}}$ order polynomials in $s$, while, (4.67) is only of cubic order. What's more, each spin $s$ contribution to the logarithmic divergent coefficient of $h_0^2$ and $C^2_{abcd}$ respectively might not match - we only expect the whole sum to match at the end.
Another possibility is that the process of regularising the sum over spins does not commute with the limit of taking the momentum of carried by the external particles to 0. As such, it might be that keeping $h_0$ non-constant yields a different result [50].

[50] Something similar happens in [135] where in AdS, for massless higher spins, one obtains a consistent result after performing the sum *first*, and then sending then removing the UV cut-off.

### 4.4.3 Conclusions

In this chapter we have computed amplitudes of scalar fields coupled to CHS theory. In particular, the four point $\varphi$ amplitude involves a spin $s$ exchange, and one must therefore regularise an infinite sum over spins. Doing so in a way that is consistent with the underlying global symmetry group leads to trivial scattering amplitude. We have given evidence that the very same global symmetry implies the vanishing of these amplitudes.

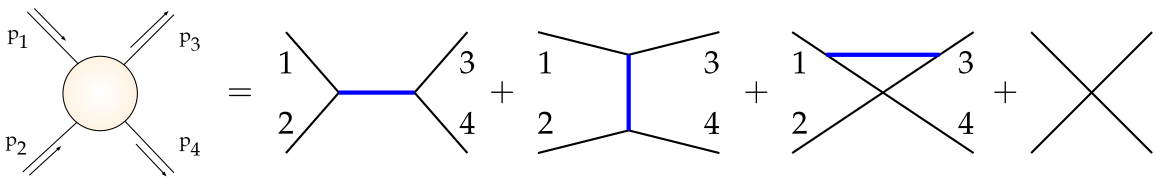
We then looked at the one-loop computation. It was found that two types of diagrams contributed: some involving loops of scalar fields $\varphi$ and others involving a loop of CHS fields. The former are non vanishing, and in fact appear to depend on the number of scalars. This does not violate the aforementioned symmetry arguments; it merely implies that the CHS symmetry is made anomalous by scalar loops. Focusing on the CHS loops instead, we expected it to be the same as the conformal anomaly of CHS theory, and to vanish after a summation over all spins. This does not appear to be the case, implying that the relation between the coefficient $c_{CHS}$ and the conformal anomaly of CHS theory is to be clarified.

In the next chapter, we will compute amplitudes of the CHS fields themselves, with the lower spin particles (Maxwell and Conformal Gravitons) on external legs, though only at tree-level. We will observe a similar action of the CHS global algebra.

# Chapter 5

# Scattering in CHS theory

In this chapter we will be looking at scattering amplitudes in pure CHS theory, in two main parts. The first part, Sections 5.1 - 5.3, will follow the main analysis of [2]. We will make use of the vertices derived in Chapter 3 in order to compute four-point amplitudes where the external legs have 2-derivative lower spin external modes. Note that since the CHS scalar $h_0$ is non-propagating, we exclude it from the scattering states. Given this, the external particles will follow the on-shell relation $p_i^2 = 0$ and we will be in the TT gauge. Introducing the relevant notation, we will want to compute the following process




(5.1)

There, the blue lines correspond to internal propagators with a priori arbitrary spins. For a scattering of helicities and momenta $(\lambda_1, p_1), (\lambda_2, p_2) \to (\lambda_3, p_3), (\lambda_4, p_4)$, we will assume the centre-of-mass frame for which the momenta and the Maxwell polarizations are: [51]

$$
\begin{aligned}
p_1 &= (\omega, 0, 0, \omega), & \varepsilon_1(p_1) &= -\tfrac{1}{\sqrt{2}}\lambda_1\,(0, 1, i\,\lambda_1, 0) \\
p_2 &= (\omega, 0, 0, -\omega), & \varepsilon_2(p_2) &= -\tfrac{1}{\sqrt{2}}\lambda_2\,(0, -1, i\,\lambda_2, 0) \\
p_3 &= (\omega, \omega\,\sin\theta, 0, \omega\,\cos\theta), & [\varepsilon_3(p_3)]^* &= -\tfrac{1}{\sqrt{2}}\lambda_3\,(0, \cos\theta, -i\,\lambda_3, -\sin\theta) \\
p_4 &= (\omega, -\omega\,\sin\theta, 0, -\omega\,\cos\theta), & [\varepsilon_4(p_4)]^* &= -\tfrac{1}{\sqrt{2}}\lambda_4\,(0, -\cos\theta, -i\,\lambda_4, \sin\theta)
\end{aligned}
\tag{5.2}
$$

[51]Note that for the Einstein helicity 2 states, we have $\varepsilon_{ab} = \varepsilon_a \varepsilon_b$. Also note that if the above, the states particles 1 and 2 are in-going and particles 3 and 4 are outgoing - this is why we list the conjugate polarization tensors $[\varepsilon_{3,4}]^*$, see [144, 145].

Here, the Mandelstam variables are:

$$\mathsf{s} = -(p_1 + p_2)^2 = 4\omega^2, \qquad \mathsf{t} = -(p_1 - p_3)^2 = -2\omega^2 (1 - \cos\theta),$$
$$\mathsf{u} = -(p_1 - p_4)^2 = -2\omega^2 (1 + \cos\theta)\,, \qquad \mathsf{s} + \mathsf{t} + \mathsf{u} = 0\,. \tag{5.3}$$

We will first compute all relevant four point amplitudes, which involves having to sum over an infinite number of exchanges. As in Chapter 4, we will have to regularise this, and we see that it will lead to vanishing 4-point amplitudes.

The second part of this chapter will be covered in Section 5.4, whre we will present the work done in the latter part of [3]. There, we will look at three point amplitudes where the external legs are *any* admissible scattering states in the theory by using the twistor-spinor formulation introduced in Section 3.4. The result will not only show more evidence that the 2-derivative states are trivial in CHS theory, but also that there exist non-vanishing finite amplitudes. Furthermore, we will look at the formula for scattering states on AdS, which will enable us to further discuss the relation between CHS theory and massless higher spin theories.

## 5.1 $11 \to 11$ Amplitude

We open by computing an amplitude with only the spin 1 states on the external legs. As we saw earlier in (3.45), the only quadratic contact term for the vector CHS fields is exactly cancelled by the $h_0$ exchange diagram. What is more, the $1-1-s$ vertices only exist for odd $s$. As such, the total amplitude is given by the exchange diagrams in (5.1) with odd spin propagators. Using the vertex (3.56) [52]

$$\begin{aligned} V_{a,b,c(s)}(p,q) = \tfrac{1}{(s+2)!}\Big\{&\eta_{ab}\left[\tfrac{1}{2}\,(p)_{c(s)} + \tfrac{1}{2}\,(q)_{c(s)}\right] \\ &- \tfrac{1}{2}\eta_{ac_1} p_b p_{c_2}\cdots p_{c_s} + \tfrac{1}{2}\eta_{bc_1} q_a p_{c_2}\cdots p_{c_s} - \tfrac{1}{2}\eta_{bc_1} q_a q_{c_2}\cdots q_{c_s} + \tfrac{1}{2}\eta_{ac_1} p_b q_{c_2}\cdots q_{c_s} \\ &- \tfrac{1}{2}\,\eta_{ac_1}\eta_{bc_2}\, p_{c_3}\cdots p_{c_s}\, p\cdot q - \tfrac{1}{2}\eta_{ac_1}\eta_{bc_2}\, q_{c_3}\cdots q_{c_s}\, p\cdot q\Big\}. \end{aligned} \tag{5.4}$$

[52] The momentum assignment is such that $p = p_1$ and $q = p_2$ in the s channel etc. We also enforce $p^2 = q^2 = 0$.

Next, the TT propagator is given by (cf. (3.49),(3.50)):

$$D^{a(s)}_{b(s)}(p) = \tfrac{2^{s-1}(2s+1)!}{(p^2)^s}\, \mathcal{P}^{a(s)}_{b(s)}(p)\,, \tag{5.5}$$

where $\mathcal{P}^{a(s)}_{b(s)}$ is the TT propagator, see Appendix A.4. Putting everything together, the different channels are given by: [53]

$$\begin{aligned} A^{(s)}_{\mathsf{s}} &= 2\,\mathrm{V}(p_1,p_2)\cdot D(p_1+p_2)\cdot 2\,\mathrm{V}(p_3,p_4)\cdot \varepsilon_1\,\varepsilon_2\,\varepsilon_3^*\,\varepsilon_4^*,\\ A^{(s)}_{\mathsf{t}} &= 2\,\mathrm{V}(p_1,p_3)\cdot D(p_1-p_3)\cdot 2\,\mathrm{V}(p_2,p_4)\cdot \varepsilon_1\,\varepsilon_2\,\varepsilon_3^*\,\varepsilon_4^*,\\ A^{(s)}_{\mathsf{u}} &= 2\,\mathrm{V}(p_1,p_4)\cdot D(p_1-p_4)\cdot 2\,\mathrm{V}(p_2,p_3)\cdot \varepsilon_1\,\varepsilon_2\,\varepsilon_3^*\,\varepsilon_4^*\,, \end{aligned} \tag{5.6}$$

Computing these amplitudes, one finds that only MHV or $\overline{\text{MHV}}$ amplitudes are non-zero, as expected. Explicitly, they take the form: [54]

$$\begin{aligned} \pm\pm \to \pm\pm: \qquad & A^{(s)}_{\mathsf{s}} = 0, \qquad A^{(s)}_{\mathsf{t}} = k_s\left(\tfrac{\mathsf{s}}{\mathsf{t}}\right)^s P_s\left(\tfrac{\mathsf{t}}{\mathsf{s}}\right), \qquad A^{(s)}_{\mathsf{u}} = k_s\left(\tfrac{\mathsf{s}}{\mathsf{u}}\right)^s P_s\left(\tfrac{\mathsf{u}}{\mathsf{s}}\right),\\ \pm\mp \to \pm\mp: \qquad & A^{(s)}_{\mathsf{s}} = k_s\left(\tfrac{\mathsf{u}}{\mathsf{s}}\right)^s P_s\left(\tfrac{\mathsf{s}}{\mathsf{u}}\right), \qquad A^{(s)}_{\mathsf{t}} = k_s\left(\tfrac{\mathsf{u}}{\mathsf{t}}\right)^s P_s\left(\tfrac{\mathsf{t}}{\mathsf{u}}\right), \qquad A^{(s)}_{\mathsf{u}} = 0\,, \end{aligned} \tag{5.7}$$

where $k_s$ are constants to be computed and $P_s$ are polynomials. Since the theory is scale invariant and since $h_a$ has mass dimension $[h_a] = 1$, the amplitude should only be a function of Mandelstam variables. This explains the form of (5.7): each amplitude has a momentum factor coming from the internal propagator ($\mathsf{t}^{-s}$ for the t-channel for example), which must be compensated. As such, $P_s(x)$ are polynomials of degree $s-2$. Normalising them with $P_2 = 1$ and $P_{s>2}(-1) = 1$, we compute explicitly:

$$\begin{aligned} P_2(x) &= 1, \qquad\qquad P_4(x) = 28 + 42\,x + 15\,x^2,\\ P_6(x) &= 495 + 1320\,x + 1260\,x^2 + 504\,x^3 + 70\,x^4,\\ P_8(x) &= 8008 + 30030\,x + 45045\,x^2 + 34320\,x^3 + 13860\,x^4 + 2772\,x^5 + 210\,x^6,\\ P_{10}(x) &= 125970 + 604656\,x + 1225224\,x^2 + 1361360\,x^3 + 900900\,x^4 + 360360\,x^5\\ &\quad + 84084\,x^6 + 10296\,x^7 + 495\,x^8\,, \end{aligned} \tag{5.8}$$

[53] Here the factors of 2 arise due to the exchange symmetry of the 2 spin 1 fields, ie. because an interacting term of the form $\phi^n$ gives rise to a factor of $n!$ in Feynman diagrams.

[54] The fact that the s-channel vanishes in the same helicity configuration may be due to the fact that helicity is conserved in 3-point vertices, making that channel trivial. This happens for gravitational interactions, cf. [146].

with the coefficients $k_s$:

$$k_2 = \tfrac{5}{12}, \qquad k_4 = \tfrac{1}{20}, \qquad k_6 = \tfrac{13}{840}, \qquad k_8 = \tfrac{17}{2520}, \qquad k_{10} = \tfrac{7}{1980}. \tag{5.9}$$

These expressions were computed explicitly up to $s = 10$, which allows one to generalise for higher spin (a derivation for this is given in Appendix C.6):

$$\begin{aligned} P_s(x) &= x^{s-2}\, P^{(4,0)}_{s-2}\left(\tfrac{x+2}{x}\right), && (5.10)\\ k_s &= \tfrac{2\,(2\,s+1)}{(s-1)\,s\,(s+1)\,(s+2)}, && (5.11)\end{aligned}$$

where $P^{(a,b)}_n(x)$ are Jacobi polynomials. One can compare this to the results of the previous chapter. In particular, the scalar exchange (4.15) can be expressed as:

$$A^{(s)}_{\mathsf{s}\,\varphi\varphi^*\to\varphi\varphi^*} = (s + \frac{1}{2})\, P^{(0,0)}_s(-1-2x)\,, \qquad x = \tfrac{\mathsf{t}}{\mathsf{s}}\,, \tag{5.12}$$

where $P^{(0,0)}_s(z) = P_s(z)$ is just the Legendre polynomial. One could also consider adding the current coupling (4.3) and computing mixed scalar-vector amplitude to find:

$$A^{(s)}_{\mathsf{s}\,\varphi\varphi^*\to 11} = \mathrm{V}_{\varphi\varphi^* s}(p_1, p_2)\cdot D^{(s)}(p_1+p_2)\cdot 2\,\mathrm{V}(p_3, p_4)\cdot \varepsilon^*_3\,\varepsilon^*_4\,, \tag{5.13}$$

where $\mathrm{V}_{\varphi\varphi^* s}(p_1, p_2)$ is simply the spin $s$ vertex coming from the interaction in (3.34). One finds that the $\pm\pm$ amplitudes are trivial, while the $\pm\mp$ helicity preserving ones are written as:

$$A^{(s)}_{\mathsf{s}\,\varphi\varphi^*\to\pm1\mp1} = k_s\,\frac{\mathsf{t}\,\mathsf{u}}{\mathsf{s}^2}\,Q_s\Big(\frac{\mathsf{t}}{\mathsf{s}}\Big)\,, \tag{5.14}$$

where again, $Q_s$ are homogeneous polynomials of order $s-2$ normalised by $Q_s(-1) = 1$. Once again, we can find a general $s$ expression for these:

$$Q_s(x) = \tfrac{2}{s\,(s-1)}\, P^{(2,2)}_{s-2}(-1-2\,x)\,, \qquad\qquad k_s = s + \tfrac{1}{2}\,, \tag{5.15}$$

### 5.1.1 Spin summation

It is time to sum over spins. For simplicity we will look at the $\pm\pm \to \pm\pm$ amplitudes, but the discussion generalises to the helicity preserving $\pm\mp \to \pm\mp$

case. The total amplitude is:

$$\pm\pm \to \pm\pm : \qquad A^{(s)} = k_s \left[ \left(\tfrac{\mathsf{s}}{\mathsf{t}}\right)^s P_s\left(\tfrac{\mathsf{t}}{\mathsf{s}}\right) + \left(\tfrac{\mathsf{s}}{\mathsf{u}}\right)^s P_s\left(\tfrac{\mathsf{u}}{\mathsf{s}}\right) \right]. \tag{5.16}$$

Since we are scattering massless particles, $\mathsf{u} = -\mathsf{s} - \mathsf{t}$ allowing us to write the expression in terms of the single variable $x \equiv \frac{\mathsf{t}}{\mathsf{s}}$:

$$A^{(s)} = \sigma_s(x) + \sigma_s(-1-x), \qquad \sigma_s(x) \equiv k_s\, x^{-s}\, P_s(x)\,. \tag{5.17}$$

As before, we must introduce a regularisation on the function $\sigma(x)$ to perform the sum. We do this via the introduction of the parameter $z$ and define:

$$\sigma(x) \equiv \lim_{z\to 1} \sigma(x;z)\,, \qquad \sigma(x;z) \equiv \sum_{s=2,4,6,\ldots}^{\infty} \sigma_s(x)\, z^{s-2}\,. \tag{5.18}$$

Stripping away the numerical constant $k_s$, let us study the sum over all $s > 2$ of the similar function:

$$K(x;z) \equiv \sum_{s=2}^{\infty} x^{-s}\, P_s(x)\, z^{s-2}\,. \tag{5.19}$$

Using the generating function form of the Jacobi polynomials $P_{s-2}^{(4,0)}$ (cf. [147] ), we can write:

$$K(x;z) = \tfrac{16}{x^2} \left[\sqrt{z^2 - \tfrac{2z(x+2)}{x} + 1}\right]^{-1} \left[\sqrt{z^2 - \tfrac{2z(x+2)}{x} + 1} - z + 1\right]^{-4}. \tag{5.20}$$

Finally, using the separated form of $k_s$ (cf. (5.11)):

$$k_s = \tfrac{1}{s+2} - \tfrac{1}{s+1} + \tfrac{1}{s-1} - \tfrac{1}{s}\,, \tag{5.21}$$

we can perform the sum by multiplying (5.20) by various powers of $z$, integrating over that, and dividing again by a power of $z$. For instance, from (5.19)

$$z^{-4} \int_0^z \mathrm{d}z'\, z'^3 K(x;z') = \sum_{s=2}^{\infty} \frac{1}{s+2} x^{-s}\, P_s(x)\, z^{s-2}\,. \tag{5.22}$$

In order to restrict the sum to only even spins, we simply look at the combination $\frac{1}{2}\left(K(x;z) + K(x;-z)\right)$. Overall, the $z \to 1$ limit is finite and simple:

$$\sigma(x) = x(x+1)\, \log \tfrac{x+1}{x} - x - \tfrac{1}{2}\,. \tag{5.23}$$

One can check that $\sigma(x) = -\sigma(-1-x)$ , so that the overall summed amplitude is zero:

$$A(x) = \sum_{s=2,4,6,\ldots}^{\infty} A^{(s)}(x) = \sigma(x) + \sigma(-1-x) = 0\,. \tag{5.24}$$

Note that here we made the assumption that $\sigma(x)$ is defined for any $x$ via analytical continuation. Actually, it is real in the interval $x \in [-\infty, -1] \cup [0, \infty]$ but the argument in (5.17) is $x = \frac{\mathsf{t}}{\mathsf{s}} = -\frac{1}{2}(1-\cos\theta) \in [-1, 0]$. One can then consider adding an infinitesimal imaginary part, in which case we find that the amplitude vanishes again, $A(x + i\,0^{\pm}) = 0$ (approaching zero from either side).

At the special kinematic points $x = -1$ and $x = 0$, one may expect some non-trivial delta function contributions (similar to what we observed in Chapter 4.). In Appendix C.5, we show explicitly that this is not the case for $x = -1$ with an independent check.

## 5.2 General structure of CHS exchange amplitudes

At this stage, it is useful to pause and make an important remark. The appearance of Jacobi polynomials in expressions (5.10), (5.12) and (5.15) is actually related to the partial wave expansion of the amplitudes studied by Jacob and Wick in [148] (see also [149, 150]).

There, they show that for a helicity $\lambda_1, \lambda_2 \to \lambda_3, \lambda_4$ transition amplitudes in the c.o.m. frame, one can use the completeness of states relation to express a generic scattering amplitude as a sum over states with mass $M = \sqrt{\mathsf{s}}$ and spin $J$:

$$A_{\{\lambda_i\}}(\mathsf{s}, \theta) = R_{\{\lambda_i\}}(\theta) \sum_{J \geq M} (J + \tfrac{1}{2})\, \mathcal{F}^{(J)}_{\{\lambda_i\}}(\mathsf{s})\, P^{(|\lambda-\mu|, |\lambda+\mu|)}_{J-M}(\cos\theta)\,, \tag{5.25}$$

$$\lambda = \lambda_1 - \lambda_2, \quad \mu = \lambda_3 - \lambda_4, \qquad M = \max(|\lambda|, |\mu|)\,, \tag{5.26}$$

$$R_{\{\lambda_i\}}(\theta) = \big(\cos\tfrac{\theta}{2}\big)^{|\lambda+\mu|} \big(\sin\tfrac{\theta}{2}\big)^{|\lambda-\mu|} = \big(-\tfrac{\mathsf{u}}{\mathsf{s}}\big)^{\frac{1}{2}|\lambda+\mu|} \big(-\tfrac{\mathsf{t}}{\mathsf{s}}\big)^{\frac{1}{2}|\lambda-\mu|}\,, \tag{5.27}$$

where $\{\lambda_i\} = (\lambda_1, \lambda_2; \lambda_3, \lambda_4)$ and the angle $\theta$ is defined via $\cos\theta = 1 + 2\frac{\mathsf{t}}{\mathsf{s}}$ . The appearance of the Jacobi polynomial $P^{(a,b)}_k$ in (5.25) comes from the expression for spherical $d$-functions:

$$d^J_{\lambda\mu}(\theta) = \sqrt{\tfrac{(J+M)!(J-M)!}{(J+N)!(J-N)!}}\, R_{\{\lambda_i\}}(\theta)\, P^{(|\lambda-\mu|, |\lambda+\mu|)}_{J-M}(\cos\theta)\,, . \tag{5.28}$$

where $N = \min(|\lambda|, |\mu|)$. Let us now turn our attention to the case where we scatter massless particles of a scale invariant theory. The dependence on $\mathcal{F}^{(J)}_{\{\lambda_i\}}$ can be inferred from the dimensions $\Delta_i$ of the scattering states:

$$\mathcal{F}^{(J)}_{\{\lambda_i\}}(\mathsf{s}) = F^{(J)}_{\{\lambda_i\}}\,\mathsf{s}^{\Delta}\,, \qquad\qquad \Delta \equiv \frac{1}{2}(4 - \sum_{i=1}^{4}\Delta_i)\,, \tag{5.29}$$

where $F^{(J)}_{\{\lambda_i\}}$ are numerical coefficients related to the dynamics of the theory in question.[55]

The key point is that the spin $J$ term in (5.25) will have the same structure as the s-channel diagram of a CHS amplitude with a spin $s = J$ particle on the internal line. This comes from the fact that a massive particle with mass $m^2 = \mathsf{s}$ is described by a symmetric field field $\phi_{a_1 \ldots a_J}$ satisfying $(-\Box + m^2)\phi_{a_1\ldots a_J} = 0$ which is traceless and transverse (on-shell) (so it has $2J+1$ degrees of freedom). This is exactly what we have in our CHS diagrams, the only difference being the overall $\mathsf{s}$ dependence that arises due to the higher derivatives in the propagator, but this can be controlled by the scale invariance (cf. (5.29)).

Now, the analysis above only applies to the s-channel: since we picked $\mathsf{s}$ to be the c.o.m. it will appear in the propagator of the exchanged CHS field. One can relabel the variables to obtain the other contributions, as we will soon see.

We also note that this identification does not appear to work massless higher spin exchanges discussed in [142] since the theory lacks the algebraic part of CHS symmetry allowing to render the spin $s$ propagator traceless.

We now see how (5.25) can be related to the cases (5.7),(5.12),(5.14). For the $\varphi\varphi^* \to \varphi\varphi^*$ amplitude, $\lambda_i = 0,\ \lambda = \mu = 0,\ M = 0$, leading to the s channel spin $J$ exchange:

$$A^{(J)}_{\mathsf{s}\,0,0;0,0}(\mathsf{s}, \cos\theta) = (J + \tfrac{1}{2})F^{(J)}_{0,0;0,0}\,P^{(0,0)}_J(\cos\theta)\,. \tag{5.30}$$

This matches (5.12) for the identification $J = s$ so long as:

$$F^{(s)}_{0,0;0,0} = 1\,. \tag{5.31}$$

[55]For instance, we just considered scattering the dimension 1 vector fields $h_a$ and scalars $\varphi$, for which $\Delta = 0$.

Next, for $\varphi\varphi^* \to \pm\mp$ (cf. (5.13),(5.14)), we have $\lambda = \lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\lambda_4 = \pm 1$, $\mu = M = \pm 2$, yielding

$$A^{(J)}_{\mathsf{s}\,0,0;\pm1,\mp1}(\mathsf{s},\cos\theta) = (J+\tfrac{1}{2})F^{(J)}_{0,0;\pm1,\mp1}\,\frac{\mathsf{u}\,\mathsf{t}}{\mathsf{s}^2}\,P^{(2,2)}_{J-2}(\cos\theta)\,. \tag{5.32}$$

This matches (5.25),(5.29) for even $J$ and

$$F^{(s)}_{0,0;\pm\mp} = \frac{2}{s\,(s-1)}. \tag{5.33}$$

Finally, for the vector scatterings $\pm1\pm1 \to \pm1\pm1$, there are contributions coming from the $\mathsf{t}$ and $\mathsf{u}$ channels, which can be obtained by relabelling the states and Mandelstam variables. The t-channel of "$X$"-particles $X_1 + \overline{X}_3 \to X_4 + \overline{X}_2$ is the same as the s-channel effective "$Y$"-particles $Y_1 + Y_2 \to Y_3 + Y_4$. So for the exchange in question, for Y particles we have $\lambda_1 = -\lambda_2 = \pm 1$, $\lambda_3 = -\lambda_4 = \pm 1$, $\lambda = \mu = 2$, $M = 2$ allowing us to write (cf. (5.25),(5.29) ):

$$A^{(J)}_{\mathsf{s}\,\pm1,\mp1;\mp1,\pm1} = (J+\tfrac{1}{2})F^{(J)}_{\pm1,\mp1;\mp1,\pm1}\,\frac{\mathsf{u}_Y^2}{\mathsf{s}_Y^2}\,P^{(0,4)}_{J-2}(\cos\theta_Y), \qquad \cos\theta_Y = -1 - 2\,\frac{\mathsf{u}_Y}{\mathsf{s}_Y}\,. \tag{5.34}$$

We can then go to the kinematics of the X particles by relabelling: $\mathsf{s}_Y \to \mathsf{t}$, $\mathsf{t}_Y \to \mathsf{u}$, $\mathsf{u}_Y \to \mathsf{s}$. The t-channel is then given by:

$$A^{(J)}_{\mathsf{t}\,\pm1,\pm1;\pm1,\pm1}(\mathsf{t},\cos\theta) = (J+\tfrac{1}{2})F^{(J)}_{\pm1,\pm1;\pm1,\pm1}\,\frac{\mathsf{s}^2}{\mathsf{t}^2}\,P^{(0,4)}_{J-2}(-1-2\,\frac{\mathsf{s}}{\mathsf{t}})\,. \tag{5.35}$$

This matches the results (5.7),(5.10),(5.11) with $J = s$ and the choice:

$$F^{(s)}_{\pm1,\pm1;\pm1,\pm1} = \frac{k_s}{s+\frac{1}{2}} = \frac{4}{(s-1)\,s\,(s+1)\,(s+2)}\,. \tag{5.36}$$

This can be seen by making use of the identity:

$$\left(\frac{\mathsf{s}}{\mathsf{t}}\right)^s \left(\frac{\mathsf{t}}{\mathsf{s}}\right)^{s-2} P^{(4,0)}_{s-2}(1+2\,\frac{\mathsf{s}}{\mathsf{t}}) = \frac{\mathsf{s}^2}{\mathsf{t}^2}\,P^{(0,4)}_{s-2}(-1-2\,\frac{\mathsf{s}}{\mathsf{t}})\,, \tag{5.37}$$

Following our three examples (5.31),(5.33) and (5.36), one may make a conjecture on the dependence of the functions $F^{(J)}_{\{\lambda_i\}}$ in (5.29) on $J$ to be:

$$F^{(s)}_{\{\lambda_i\}} = \mathrm{k}_{\lambda,\mu}\,\frac{(s-M)!}{(s+N)!}, \qquad N = \min(|\lambda|,|\mu|),\; M = \max(|\lambda|,|\mu|)\,. \tag{5.38}$$

This leads to the correct expression for $F^{(s)}_{\{0\}}$ , $F^{(s)}_{0,0;\pm\mp}$ and $F^{(s)}_{\{\pm1\}}$ from (5.10), (5.33) and (5.36) respectively provided $\mathsf{k}_{\lambda,\mu} = 1$. We will find that the ansatz (5.38) applies to the other cases we study.

## 5.3 Scattering amplitudes with conformal gravitons

We now study amplitudes which scatter conformal gravitons. Here, following [2], we study only amplitudes scattering the 2-derivative "Einstein" modes, the computation for the other states, in particular the oscillating helicity $\pm1$ modes remains to be done.

### 5.3.1 $22 \to 22$ scattering

In order to discuss the full amplitude, we focus on the exchange diagrams which have higher spin fields on the internal line - ie. even spins with $s \geq 4$. Following the analysis from above, we expect the t-channel exchange or same-helicity particles to be (cf. (5.25),(5.29) ), for $J > 4$:

$$A^{(J)}_{\mathsf{t}\,\pm2,\pm2;\pm2,\pm2}(\mathsf{t},\cos\theta) = (J+\tfrac{1}{2})F^{(J)}_{\pm2,\pm2;\pm2,\pm2}\,\mathsf{t}^2\,\frac{\mathsf{s}^4}{\mathsf{t}^4}\,P^{(0,8)}_{J-4}(-1-2\,\frac{\mathsf{s}}{\mathsf{t}})\;. \tag{5.39}$$

where the factor of $\mathsf{t}^2$ factor arises due to the fact that the field $h_{ab}$ has mass dimension 0. The total spin $J = s$ exchange should then be as in (5.16) (cf. (5.10)) :

$$\pm2\pm2 \to \pm2\pm2: \qquad A^{(s)} = k_s\,\mathsf{s}^2\,[(\tfrac{\mathsf{s}}{\mathsf{t}})^{s-2}\,P_s(\tfrac{\mathsf{t}}{\mathsf{s}}) + (\tfrac{\mathsf{s}}{\mathsf{u}})^{s-2}\,P_s(\tfrac{\mathsf{u}}{\mathsf{s}})], \tag{5.40}$$

$$P_s(x) = x^{s-2}\,P^{(8,0)}_{s-4}\,(\tfrac{x+2}{x})\;. \tag{5.41}$$

Furthermore, if (5.38) stays valid for all $F^{(J)}_{\{\lambda_i\}}$, we can expect

$$k_s = \mathsf{k}\,\frac{2s+1}{(s-3)(s-2)(s-1)s(s+1)(s+2)(s+3)(s+4)}\;, \tag{5.42}$$

where $\mathsf{k}$ is some numerical factor independent of $s$.

It turns out that explicit computation for several (even) $s$ agrees with this for the value:

$$k = \tfrac{9}{8}\,. \tag{5.43}$$

The derivation of (5.40)–(5.43) can be performed along the lines of the corresponding one for the $11 \to 11$ scattering, described in Appendix C.6. The same analysis can be carried out for the helicity-preserving $\pm 2 \mp 2 \to \pm 2 \mp 2$ amplitude.

Summing all these exchange amplitudes we find:

$$\sum_{s=4,6,\ldots}^{\infty} A^{(s)}(x) = \mathrm{s}^2 \left[\sigma(x) + \sigma(-1-x)\right], \qquad x \equiv \frac{\mathrm{t}}{\mathrm{s}}, \tag{5.44}$$

$$\sigma(x) = \lim_{z\to 1} \sum_{s=4,6,\ldots}^{\infty} k_s\, x^{2-s}\, P_s(x)\, z^{s-4}\,. \tag{5.45}$$

by following the same steps as (5.17)–(5.23). Again, using the generating function representation of Jacobi polynomial, one is led to the form:

$$\sigma(x) = \tfrac{1}{4320}\left[60\,(x+1)^3\,x^3\,\log\tfrac{x+1}{x} - 60\,x^5 - 150\,x^4 - 110\,x^3 - 15\,x^2 + 3\,x - 1\right]. \tag{5.46}$$

Remarkably, this function satisfies:

$$\sigma(x) + \sigma(-1-x) = 0\,, \tag{5.47}$$

indicating that the t and u-channel contributions involving higher spin exchanges cancel against each other.

Next, we must compute the low spin $s < 4$ contributions, as well as the contact term appearing in (5.1). The results following from the exchange of a $h_0$ field are:

$$\begin{aligned}
&\pm 2 \pm 2 \to \pm 2 \pm 2: \quad && A_{\mathrm{s}}^{(0)} = \tfrac{\mathrm{s}^2}{4608}, \quad && A_{\mathrm{t}}^{(0)} = \tfrac{\mathrm{t}^2\,\mathrm{u}^4}{512\,\mathrm{s}^4}, \quad && A_{\mathrm{u}}^{(0)} = \tfrac{\mathrm{t}^4\,\mathrm{u}^2}{512\,\mathrm{s}^4},\\
&\pm 2 \mp 2 \to \pm 2 \mp 2: \quad && A_{\mathrm{s}}^{(0)} = 0, \quad && A_{\mathrm{t}}^{(0)} = \tfrac{\mathrm{t}^2\,\mathrm{u}^4}{512\,\mathrm{s}^4}, \quad && A_{\mathrm{u}}^{(0)} = \tfrac{(\mathrm{s}+3\mathrm{t})^2\,\mathrm{u}^4}{4608\,\mathrm{s}^4}\,.
\end{aligned} \tag{5.48}$$

The spin 2 exchange, which uses the vertex (3.57)) yields:

$$\begin{aligned}
\pm 2 \pm 2 \to \pm 2 \pm 2: \quad A_{\mathrm{s}}^{(2)} &= \tfrac{\mathrm{s}^2+6\,\mathrm{s}\,\mathrm{t}+6\,\mathrm{t}^2}{23040}, \quad A_{\mathrm{t}}^{(2)} = \tfrac{\mathrm{u}^2(2\,\mathrm{s}^4-10\,\mathrm{s}^3\,\mathrm{t}+33\,\mathrm{s}^2\,\mathrm{t}^2-24\,\mathrm{s}\,\mathrm{t}^3+3\,\mathrm{t}^4)}{7680\,\mathrm{s}^4},\\
A_{\mathrm{u}}^{(2)} &= \tfrac{\mathrm{t}^2\,(2\,\mathrm{s}^4-10\,\mathrm{s}^3\,\mathrm{u}+33\,\mathrm{s}^2\,\mathrm{u}^2-24\,\mathrm{s}\,\mathrm{u}^3+3\,\mathrm{u}^4)}{7680\,\mathrm{s}^4},
\end{aligned} \tag{5.49}$$

$$\pm 2 \mp 2 \to \pm 2 \mp 2: \quad A_{\mathrm{s}}^{(2)} = 0, \quad A_{\mathrm{t}}^{(2)} = \frac{\mathrm{u}^4\,(2\,\mathrm{s}^2+2\,\mathrm{s}\,\mathrm{t}+3\,\mathrm{t}^2)}{7680\,\mathrm{s}^4}, \quad A_{\mathrm{u}}^{(2)} = \frac{\mathrm{u}^4\,(10\,\mathrm{s}^2+18\,\mathrm{s}\,\mathrm{u}+9\,\mathrm{u}^2)}{23040\,\mathrm{s}^4}\,.$$

Finally, the contact term diagrams give:

$$\begin{aligned} \pm 2 \pm 2 \to \pm 2 \pm 2: &\qquad A^{(\mathrm{cont})} = -\frac{\mathrm{s}^6-\mathrm{s}^5\,\mathrm{t}+26\,\mathrm{s}^4\,\mathrm{t}^2+63\,\mathrm{s}^3\,\mathrm{t}^3+54\,\mathrm{s}^2\,\mathrm{t}^4+27\,\mathrm{s}\,\mathrm{t}^5+9\,\mathrm{t}^6}{1920\,\mathrm{s}^4}\,, \\ \pm 2 \mp 2 \to \pm 2 \mp 2: &\qquad A^{(\mathrm{cont})} = -\frac{\mathrm{u}^4\,(\mathrm{s}^2+3\,\mathrm{s}\,\mathrm{t}+9\,\mathrm{t}^2)}{1920\,\mathrm{s}^4}\,. \end{aligned} \tag{5.50}$$

As anticipated, for all helicity configurations, the sum of these contributions vanishes:

$$[A_{\mathrm{s}}^{(0)} + A_{\mathrm{t}}^{(0)} + A_{\mathrm{u}}^{(0)}] + [A_{\mathrm{s}}^{(2)} + A_{\mathrm{t}}^{(2)}] + A_{\mathrm{u}}^{(2)} + A^{(\mathrm{cont})} = 0\,. \tag{5.51}$$

This is in fact equivalent to the fact that amplitudes in conformal gravity are trivial, as alluded to in Chapter 2. Indeed, going back to the lower spin truncation of the CHS action of Chapter 3, we saw that it was equivalent to the scalar-Maxwell-Weyl action (3.44) after the field redefinition (3.43). Said field redefinition will not affect the amplitude: it simply repackages the $h_0 h_{ab}^2$ interaction into another 2222 contact term.

Let us note a similarity in the structure of (5.44) and (5.23). This suggests that for higher spin jj$\to$ jj scattering one may be able to guess the expression for $\sigma(x)$ and then check that the coefficients in its expansion in a suitable set of Jacobi polynomials reproduces the $k_s$ prefactor. Similar ideas have been exploited in [151].

### 5.3.2 $11 \to 22$ scattering

We now consider mixed amplitudes involving spin 2 and spin 1 fields. Among these, the amplitudes involving an odd number of vectors vanish identically [56]. As such, we consider the $11 \to 22$ exchange. Starting with the helicity preserving $\pm 1 \mp 1 \to \pm 2 \mp 2$ amplitude, for the s-channel, the higher spin $s \geq 4$ exchanges, one finds:

$$A_{\mathrm{s}\ \pm1,\mp1;\pm2,\mp2}^{(s)} = k_s\,\mathrm{s}\,\frac{\mathrm{t}\,\mathrm{u}^3}{\mathrm{s}^4}\,P_{s-4}^{(6,2)}(-1-2\,\tfrac{\mathrm{t}}{\mathrm{s}})\,, \tag{5.52}$$

$$k_s = (s+\tfrac{1}{2})F_{\pm1,\pm1;\pm2,\mp2}^{(s)} = \frac{3}{2}\,\frac{2s+1}{(s-3)(s-2)(s-1)s(s+1)(s+2)}\,, \tag{5.53}$$

[56] Indeed, if one vertex has external legs of spin 2 and 1, the exchange helicity must be odd, implying the other two external legs must be 2 and 1 as well.

with $\lambda = 2, \mu = 4, M = 4, N = 2$. Once again, the expression of $k_s$ is consistent with (5.38). For the t-channel, the exchange particle must have odd spin, and the expression is therefore of the form:

$$A^{(s)}_{\mathsf{s}\ \pm1,\mp1;\pm2,\mp2} = k'_s\, \mathsf{s}\, \tfrac{\mathsf{u}^3}{\mathsf{st}^2}\, P^{(6,0)}_{s-4}(-1-2\,\tfrac{\mathsf{s}}{\mathsf{t}})\,, \tag{5.54}$$

$$k'_s = \tfrac{2s+1}{(s-2)(s-1)s(s+1)(s+2)(s+3)}\,. \tag{5.55}$$

There is no u-channel contribution. Summing the s-channel and t-channel, one finds:

$$A_{s>2} = \frac{\mathsf{u}^3}{\mathsf{s}^2}\bar{A}(\frac{\mathsf{t}}{\mathsf{s}})\,, \qquad \bar{A}(x) = x\,S(x) + x^{-2}\,T(x^{-1})\,, \tag{5.56}$$

$$S(x) \equiv \sum_{s=4,6,8,\ldots}^{\infty} k_s\, P^{(6,2)}_{s-4}(-1-2\,x)\,, \qquad T(x) \equiv \sum_{s=3,5,7,\ldots}^{\infty} k'_s\, P^{(6,0)}_{s-4}(-1-2\,x)\,. \tag{5.57}$$

In the interval $-1 < x < 0$, the functions $S(x)$ and $T(x)$ are given by:

$$S(x) = -\tfrac{x^3+5x^2+13x-3}{96(x+1)^5} - \tfrac{x\log(-x)}{8(x+1)^6}, \qquad T(x) = -\tfrac{(x-1)(x^2+8x+1)}{96(x+1)^5} - \tfrac{x^2\log(-x)}{8(x+1)^6}\,, \tag{5.58}$$

so that finally, $\bar{A} = -\frac{1}{96\,(x+1)}$. The total amplitude is then:

$$A_{s>2} = -\tfrac{1}{96}\,\tfrac{\mathsf{u}^3}{\mathsf{s}^2}\,\tfrac{1}{\frac{\mathsf{t}}{\mathsf{s}}+1} = \tfrac{1}{96}\,\tfrac{\mathsf{u}^2}{\mathsf{s}}\,. \tag{5.59}$$

Next, one must add the lower spin exchanges, and the contact diagram arising from the 1122 vertex[57]. The spin 0 exchange vanishes, as the 110 vertex (3.61) vanishes for $(\pm,\mp)$ helicities. Similarly, the spin 2 exchange is trivial. The spin 1 exchange has non-trivial contributions for the t and u-channels:

$$\pm1\mp1 \to \pm2\mp2: \qquad A^{(1)}_{\mathsf{t}} = \tfrac{\mathsf{u}^3\,(2\mathsf{s}+\mathsf{t})}{192\,\mathsf{s}^3}, \qquad A^{(1)}_{\mathsf{u}} = \tfrac{\mathsf{t}\,\mathsf{u}^2\,(\mathsf{s}-\mathsf{t})}{192\,\mathsf{s}^3}\,. \tag{5.60}$$

The 1122 vertex is found to give:

$$\pm1\mp1 \to \pm2\mp2: \qquad A^{(\mathrm{cont})} = -\tfrac{1}{96}\,\tfrac{\mathsf{tu}^3}{\mathsf{s}^3}\,. \tag{5.61}$$

[57]This is just given by the graviton-Maxwell coupling up to field redefinitions, see Section 3.3.4

Once again, remarkably the sum of contrbutions (5.59), (5.60) and (5.61) is found to vanish:

$$\pm 1 \mp 1 \to \pm 2 \mp 2: \qquad A_{s>2} + \left[A_{\mathsf{t}}^{(1)} + A_{\mathsf{u}}^{(1)}\right] + A^{(\text{cont})} = 0\,. \tag{5.62}$$

Next, we attack the $\pm 1 \pm 1 \to \pm 2 \pm 2$ amplitude. For the t-channel, it is of the form:

$$\pm 1 \pm 1 \to \pm 2 \pm 2 \;:\quad A_{\mathsf{t}}^{(s)} = k_s \tfrac{\mathsf{s}^3}{\mathsf{t}^2} P_{s-3}^{(0,6)}(-1-2\tfrac{\mathsf{s}}{\mathsf{t}}), \qquad A_{\mathsf{u}}^{(s)} = k_s \tfrac{\mathsf{s}^3}{\mathsf{u}^2} P_{s-3}^{(0,6)}(-1-2\tfrac{\mathsf{s}}{\mathsf{u}})\,,$$
$$k_s = -\tfrac{2s+1}{(s-2)(s-1)s(s+1)(s+2)(s+3)}\,, \qquad s = 3,5,7,\ldots\,. \tag{5.63}$$

Here one can notice that $k_s = -k'_s$ (cf. (5.55)). Combined with the fact that $P_{s-3}^{(0,6)}(-1-2x) = P_{s-3}^{(6,0)}(1+2x)$, we can use the expression for $T(x)$ of (5.57),(5.58) to find that the sum of the t and u-channels vanishes:

$$x^2\, T(-1-x) + (\tfrac{x}{1+x})^2\, T(-\tfrac{1}{1+x}) = 0\,. \tag{5.64}$$

Considering the lower spin exchanges and the contact term, one finds the following non-vanishing contributions:

$$\pm 1 \pm 1 \to \pm 2 \pm 2 \;:\quad A_{\mathsf{s}}^{(0)} = -\tfrac{\mathsf{s}}{128}, \quad A_{\mathsf{t}}^{(1)} = \tfrac{\mathsf{u}^2\,(\mathsf{s}^2 - 6\,\mathsf{s}\,\mathsf{t} + 2\mathsf{t}^2)}{128\,\mathsf{s}^3}, \quad A_{\mathsf{u}}^{(1)} = \tfrac{\mathsf{t}^2\,(\mathsf{s}^2 - 6\,\mathsf{s}\,\mathsf{u} + 2\mathsf{u}^2)}{128\,\mathsf{s}^3},$$
$$A^{(\text{cont})} = -\tfrac{\mathsf{t}\,\mathsf{u}\,(\mathsf{t}^2 + 3\,\mathsf{t}\,\mathsf{u} + \mathsf{u}^2)}{32\,\mathsf{s}^3}, \tag{5.65}$$

which also sum up to zero:

$$A_{\mathsf{s}}^{(0)} + [A_{\mathsf{t}}^{(1)} + A_{\mathsf{t}}^{(1)}] + A^{(\text{cont})} = 0\,, \tag{5.66}$$

### 5.3.3 Conclusion and Global Symmetry

In conclusion, the 11→11, 22→22 and 11→22 amplitudes all vanish. A natural conjecture is then that CHS tree-level amplitudes (of 2-derivative states) all vanish. In Appendix C.5, we show that this is true for four point amplitudes at the special kinematic point of backward scattering, where $\mathsf{u} = 0$, if we use the expressions (5.25),(5.29),(5.38).

These vanishings are related to the action of the global CHS symmetry, as we saw in Chapter 4. Let us sketch how this works on the 11→11 amplitudes. The global CHS symmetry acts on the vector fields as:

$$\delta h^a = \sum_k \left[\epsilon^{ab(k)}\partial_{b(k)}h_0 + \epsilon^{b(k)}\partial_{b(k)}h^a\right] \tag{5.67}$$

When acting on the amplitude, this variation will give a term that corresponds to the $01 \to 11$ amplitude, which we argued vanishes due to parity symmetry considerations. As for the second term in (5.67), it is of the same form as that acting on the scalar $\varphi$, (4.33), so one may repeat the arguments used there, (ie. pick $\epsilon_{a(k)}$ to be proportional to $y_{a_1}\ldots y_{a_k}$. As such, the symmetry constrains the amplitude to vanish. Similar arguments apply to the spin 2 case.

## 5.4 3-point amplitudes of CHS theory

In Section 3.4, we introduced twistor-spinors describing on-shell states of the theory. It is possible to derive formulae for the on-shell action of CHS theory in this formalism (ie. obtain tree-level amplitudes). The advantage of this is that it forgoes the computation of Feynman diagrams, and it conveniently allows us to scatter *any* states of the theory, not just the 2-derivative ones. We will state the formula for 3-point amplitudes here, in generalisation of the discussion in Chapter 2. The amplitudes must be $\overline{\text{MHV}}$ or MHV, and the spin of the external legs have the following constraint:

$$s_1 \geq s_2, s_3\,, \qquad s_1 \leq s_2 + s_3\,. \tag{5.68}$$

The formula for 3-point $\overline{\text{MHV}}$ scattering is:[58]

$$\begin{aligned}\mathcal{M}_3 \sim \mathcal{N}^{(\mathbf{s})} &\left[\frac{(s_2-1)!}{(s_1-s_3)!}\, A_2^{B_K(A_J}\,\tilde{C}_{3\,B_K}\left(B_{1\,A_I}\,A_3^{A_{I-J})}\,\frac{[2\,3]^{s_1+2}}{[1\,2]^{s_1-s_2+1}\,[3\,1]^{s_2}}\right)\right.\\ &\left.+(-1)^{s_2+s_3-s_1}\frac{(s_3-1)!}{(s_1-s_2)!}\,A_3^{B_K(A_{I-J-K}}\,\tilde{C}_{2\,B_K}\left(B_{1\,A_I}\,A_2^{A_{J+K})}\,\frac{[2\,3]^{s_1+2}}{[1\,2]^{s_3}\,[3\,1]^{s_1-s_3+1}}\right)\right],\end{aligned} \tag{5.69}$$

where the spinor objects $A^A$, $B^A$ are defined in (3.75)–(3.79) and (3.81)–(3.84), $C^A$ is the operator defined in (2.96). Here, the spin $s_1$ particle has negative helicity, while the other two carry the positive helicities. Also, $I, J, K$ are multi-twistor

[58]See Appendix C of [3] for a derivation

indices satisfying:

$$|I| = s_1 - 1\,, \qquad |J| = s_1 - s_3\,, \qquad |K| = s_2 + s_3 - s_1 - 1\,, \tag{5.70}$$

and the constants $\mathcal{N}^{(s)}$ are defined by:

$$\mathcal{N}^{(\mathbf{s})} \equiv \frac{1}{(s_2 + s_3 - s_1 - 1)!} \tag{5.71}$$

The first result that this formula yields is that :

$$\mathcal{M}_3(-s_1, s_2, s_3) = 0\,, \tag{5.72}$$

which is to say that scattering amplitudes involving only the maximal helicity, 2-derivative modes vanish. This is in agreement with our earlier conjecture.

Much like the case of Conformal Gravity, one can easily check that this triviality *does not* extend to all modes. Indeed, while scattering the helicity $h_{s_1}$ mode for the spin $s_1$ external leg yields:

$$\mathcal{M}_3(-h_{s_1}, s_2, s_3) = 0\,, \qquad \forall\, h_{s_1} = 2, \ldots, s_1\,, \tag{5.73}$$

scattering a helicity $h_{s_1} = 1$ mode is non-zero:

$$\mathcal{M}_3(-1_{s_1}, s_2, s_3) = \kappa_{s_2}\,\kappa_{s_3}\,\mathcal{K}\,\frac{[2\,3]^{s_2+s_3+1}}{[1\,2]^{s_3-s_2+1}\,[3\,1]^{s_2-s_3+1}}\,\delta^{(4)}\left(\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i\right)\,. \tag{5.74}$$

where we defined the numerical factor:

$$\mathcal{K} \equiv \mathcal{N}^{(\mathbf{s})}\left(\frac{1}{2\,\mathrm{i}}\right)^{s_1-1}\left[(-1)^{s_1-s_3}\,\frac{(s_2-1)!}{(s_1-s_3)!} + (-1)^{s_2}\,\frac{(s_3-1)!}{(s_1-s_2)!}\right]. \tag{5.75}$$

Here once again, $\kappa_s$ are parameters of dimension $s-1$ that arise from picking solutions that are solved by lower derivative equations.

We close this chapter by making a few comments. It is possible to generalise the AdS arguments presented earlier in Section 2.4.4 and obtain a version of the formula (5.69) for AdS. In particular, for the scattering of the maximal helicity modes, one gets, for the $\overline{\text{MHV}}$ amplitude:

$$\mathcal{M}_3^{\Lambda}(-s_1, s_2, s_3) = \Lambda^{s_1-1}\, \kappa_{s_1}\kappa_{s_2}\kappa_{s_3}\, \frac{\mathfrak{n}^{(\mathbf{s})}\ \ [2\,3]^{s_1+s_2+s_3}}{[1\,2]^{s_1-s_2+s_3}\,[3\,1]^{s_1+s_2-s_3}} \\ \times\ (1+\Lambda\Box_K)^{s_2+s_3-s_1-1}\, \delta^{(4)}\Big(\sum_{i=1}^{3}\lambda_i\tilde{\lambda}_i\Big)\,, \quad (5.76)$$

with:

$$\mathfrak{n}^{(\mathbf{s})} \equiv \Big(1+(-1)^{s_2+s_3-s_1}\Big)\ \frac{(-2\,\mathrm{i})^{s_1-1}}{(s_2+s_3-s_1-1)!}\,\frac{(s_1-s_2+s_3-1)!\,(s_1+s_2-s_3-1)!}{(s_1-s_2)!\,(s_1-s_3)!}\,. \quad (5.77)$$

As suggested earlier, this is proportional to the 3-point 2-derivative scattering amplitude. Furthermore, there is an overall factor of $\Lambda^{s_1}$, which leads the vanishing of the expression in the flat-space limit. So this suggests, that like for Conformal Gravity, it is possible to obtain the classical data of massless higher spin theories by taking the correct boundary conditions of CHS theory in AdS (cf. [62, 87]).

# Chapter 6

# Conformal Anomaly in CHS Theory

In this chapter we look more closely at the question of conformal anomalies in CHS theory. We aim to give some exposition into the current state of affairs, as well as present some previously unpublished work on the contribution of mixed terms.

As we said, the phenomenon of cancellation upon performing infinite sums in CHS theory seems to extend to anomaly coefficients. For instance, the $a$ and $c$-coefficient of its conformal anomaly vanish (again, after regularising the infinite sum over spin contributions) [127]. In that paper, several assumptions are made.[59] In particular it requires that there are no contributions coming from possible off-diagonal terms in the CHS action. However, in [130], it was suggested that such non-diagonal quadratic terms proportional to the curvature of the spacetime existed. These terms need to be accounted for on their own, and indeed, in [131], the mixed term of spin 1 and 3 was computed, and it was found to contribute non-trivially to the $c$-coefficient. Since then, [143] has given evidence an independent check for a new value of the $c$-coefficient - which also vanishes after summation.

Here, we will review this and propose an alternative approach that allows us to compute mixed contributions up to spin 5. To do so, we first introduce the curved space version of the scalar-CHS action, and show that it implies a modification to the global CHS symmetries. By fixing this modification, we see that the existence of non-diagonal quadratic terms are implied (away from flat space), and we compute them on the basis of gauge invariance. Finally, this allows us to

[59] A discussion of these assumptions can be found in [152], where the results of [127] are obtained using holographic techniques. We also discuss this in Section 6.2

compute new contributions to the anomaly $c$-coefficient.

## 6.1 Curved Space Scalar-CHS action symmetry

So far, we've only introduced CHS theory on a flat background. Our starting point was the action (3.34) (with N = 1, for convenience):

$$S_{flat} = \int \mathrm{d}^4x\, \phi^* \Box \phi - \sum_s \frac{1}{s!} \int d^dx\, J^{a_1 \cdots a_s} h_{a_1 \cdots a_s} \,. \tag{6.1}$$

where we assumed that there is only one scalar (i.e. $N = 1$). With the linearised symmetries (3.3):

$$\delta h_{a_1 \cdots a_s} = \partial_{(a_1} \varepsilon_{a_2 \cdots a_s)} + \eta_{(a_1 a_2} \alpha_{a_3 \cdots a_s)} \,. \tag{6.2}$$

In a general curved spacetime, we consider the action of a conformally coupled complex scalar with the current coupling

$$S_{curved} = \int \mathrm{d}^4x\, \sqrt{g}\, \phi^* \left(D^2 - R/6\right) \phi - \sum_s \frac{1}{s!} \mathcal{J}_{a(s)} h^{a(s)} \tag{6.3}$$

where $\mathcal{J}_a$ are the curved space (on-shell) traceless currents of the theory. The currents are such that (6.3) has conformal invariance, i.e. is invariant under:

$$\delta g_{ab} = -2w\, g_{ab} \qquad \delta\phi = -w\,\phi \qquad \delta h_{(a(s)} = 2(s-1)w\, h_{a(s)} \,. \tag{6.4}$$

We also require that they have the correct flat space limit, ie. $\mathcal{J}^{a(s)} \to J^{a(s)}$. It was observed in [131], that starting at spin 3, the currents $J_{a(s)}$ are no longer necessarily conserved on-shell if we impose conformal invariance first.
This has important implications: (6.3) can no longer be invariant under the curved-space version of (6.2) alone. Instead, we need to supplement these variations to compensate for the obstruction of the current being conserved.
These modifications carry over to the full CHS theory, after integrating out $\phi$ and in particular, they imply that the latter must contain non-diagonal quadratic terms. In principle, these mixed terms can contribute to the $c$-coefficient of CHS anomaly.
Schematically, the currents contain $s$ derivatives, so they look like:

$$\mathcal{J}_s \sim D^s(\phi\phi^*) + D^{s-2}(R\phi\phi^*) + \ldots \,, \tag{6.5}$$

where $R$ stands for a Riemann tensor or its contractions. One then only needs to make an ansatz by writing all possible terms that appear on the RHS with an arbitrary coefficient. These coefficients can then be uniquely fixed by requiring that $\mathcal{J}_{a(s)}$ be Hermitian, on-shell traceless, have the correct Weyl scaling (6.4) and have the correct flat-space limit, (3.35).

Once we know the form of the current, we can compute its divergence on-shell. It turns out that for $s > 2$, this is non vanishing. This is precisely the obstruction to (6.3) being invariant under $\delta h_s = D\epsilon_{s-2}$. Luckily, we find that we can express this divergence in terms of currents of lower spins. Schematically this looks like:

$$D \cdot \mathcal{J}_s \sim D(R\,\mathcal{J}_{s-2}) + D^3(R\,\mathcal{J}_{s-4}) + D(R^2\,\mathcal{J}_{s-4}) + \dots \tag{6.6}$$

Once we work out this form exactly, it is possible to modify the variations of the lower spin CHS fields to compensate. This means that schematically, we need:

$$\delta h_s = \delta^{(s)} h_s + \delta^{(s+2)} h_s + \delta^{(s+4)} h_s + \dots \tag{6.7}$$

$$\delta^{(s)} h_s = D\epsilon_{s-1} + g_2 \alpha_{s-2} \tag{6.8}$$

$$\delta^{(s+2p)} h_s = \left( DR^p + D^3 R^{p-1} + \dots + D^{2p-1} R \right) \epsilon_{s+2p-1} \tag{6.9}$$

Knowing the precise form of these gauge variations is important as it will allow us to fix the CHS action without directly inducing it. Indeed, in general the diagonal spin $s$ quadratic terms will not be invariant on their own, but only when added to a mixed term. Keeping terms proportional to the gauge parameter $\epsilon_{s+2p-1}$:

$$\int \mathrm{d}^d\, x \frac{\delta \mathcal{L}_{s,s}}{\delta h_s} \delta^{(s+2p)} h_s = - \int \mathrm{d}^d\, x \left( \frac{\delta \mathcal{L}_{s,s+2}}{\delta h_{s+2}} \delta^{(s+2p)} h_{s+2} + \dots + \frac{\delta \mathcal{L}_{s,s+2p}}{\delta h_{s+2p}} \delta^{(s+2p)} h_{(s+2p)} \right) \tag{6.10}$$

This equation allow us to fix the non-diagonal terms $\mathcal{L}_{s,s+2p}$ of the quadratic action (while the diagonal terms are invariant on their own under the "normal" part of the symmetry, $\delta^{(s)} h_s$).
These modified gauge variations were found for spin 3. In the following section, we will focus on the spin 4 case as an illustrative example, though the computation has been performed up to spin 5.

### 6.1.1 Curved space spin 4 current

The computations required here are intensive and require the use of a computer. We use the `xTras` add-on to the `xAct` tensor manipulation package [153, 154].

We can simplify matters by assuming a Ricci-flat background, $R_{ab} = 0$. As will be explained later, this is sufficient for our goal of calculating the $c$-coefficient of the anomaly.
Before we start the computation, let us note that we can fix the algebraic part of the gauge variation to take make the CHS fields traceless, which means explicitly:

$$\delta^{(4)} h^{abcd} = D^{(a}\epsilon^{bcd)} - \frac{3}{8} g^{(ab} D_e \epsilon^{cd)e} \tag{6.11}$$

$$\delta^{(2)} h^{ab} = D^{(a}\epsilon^{b)} - \frac{1}{4} g^{ab} D_c \epsilon^c \tag{6.12}$$

$$\epsilon^{ab}{}_{b} = 0 \tag{6.13}$$

The calculation proceeds as outlined above. We simply state the solution for the spin 0, 2 and spin 4 current thus obtained:

$$\mathcal{J}_0 = \phi^* \phi \tag{6.14}$$

$$\mathcal{J}^{ab} = \tfrac{1}{3} D^a \phi^* D^b \phi - \tfrac{1}{12} g^{ab} D_c \phi^* D^c \phi - \tfrac{1}{6} \phi^* D^{ab} \phi + h.c. \tag{6.15}$$

$$\begin{aligned}\mathcal{J}^{abcd} =& \tfrac{9}{35} D^{ab}\phi^* D^{cd}\phi - \tfrac{9}{70} g^{ab} D^c{}_e \phi^* D^{de}\phi + \tfrac{1}{140} g^{ab} C^c{}_e{}^d{}_f \phi^* D^{ef}\phi \\ &+ \tfrac{3}{280} g^{ab} g^{cd} D_{ef}\phi^* D^{ef}\phi - \tfrac{8}{35} D^a \phi^* D^{bcd}\phi + \tfrac{3}{35} g^{ab} D^e \phi^* D^{cd}{}_e \phi \\ &+ \tfrac{1}{70} \phi^* D^{abcd}\phi + h.c.\end{aligned} \tag{6.16}$$

Where above, it is implicit that one must symmetrise over the free indices with the usual weight.
The next step is to look at the divergences of these currents. The spin 2 current is conserved on-shell:

$$D_a \mathcal{J}^{ab} \approx 0 \tag{6.17}$$

where we remind the reader that "≈" indicates equality on equations of motion.
For the spin 4 current, one can show that

$$D_a \mathcal{J}^{abcd} \approx \tfrac{1}{30} C^{bedf} C^c{}_{eaf} D^a \mathcal{J}_0 + \tfrac{1}{30} C^b{}_{fae} C^{cedf} D^a \mathcal{J}_0 - \tfrac{1}{240} g^{cd} C_{aefg} C^{aefg} D^b \mathcal{J}_0$$

$$-\tfrac{1}{2}C^{c}{}_{a}{}^{d}{}_{e}D^{b}\mathcal{J}^{ae}+\tfrac{3}{14}\mathcal{J}^{ae}D^{b}C^{c}{}_{a}{}^{d}{}_{e}+\tfrac{1}{2}C^{c}{}_{a}{}^{d}{}_{e}D^{e}\mathcal{J}^{ba}$$
$$+\tfrac{1}{2}C^{c}{}_{e}{}^{d}{}_{a}D^{e}\mathcal{J}^{ba}-\tfrac{3}{14}g^{cd}C^{b}{}_{aef}D^{f}\mathcal{J}^{ae} \tag{6.18}$$

where again, one needs to symmetrise over the free indices on the RHS, and $C^{abcd}$ is the Weyl curvature tensor[60]. In order to obtain this form, we made use of Dimensionally-Dependent Identities (DDIs) - see Appendix B.3.

A short calculation shows that can compensate for the RHS of (6.18) by supplementing the gauge variations in the following way:

$$\delta^{(4)}h_0 = -\tfrac{1}{360}C_{a}{}^{e}{}_{b}{}^{f}C_{cedf}D^{d}\epsilon^{abc}+\tfrac{1}{360}C_{a}{}^{def}\epsilon^{abc}D_{f}C_{bdce} \tag{6.19}$$

$$\delta^{(4)}h^{ab} = \tfrac{1}{48}C_{adbe}D_{c}\epsilon^{cde}+\tfrac{5}{168}\epsilon^{cde}D_{e}C_{acbd}+\tfrac{1}{42}C_{bcde}D^{e}\epsilon_{a}{}^{cd}$$
$$+\tfrac{1}{42}C_{acde}D^{e}\epsilon_{b}{}^{cd}\,. \tag{6.20}$$

Note that these expressions exhibit the expected behaviour that they vanish in flat-space. We also note that $\delta^{(4)}h^{a}{}_{a}=0$ so that (6.20) does not break our traceless gauge.

These modified gauge variations imply that the $S_{00}$ and $S_{22}$ actions are no longer invariant on their own. This means that the full action needs to be supplemented by the non-diagonal terms $S_{02}$, $S_{04}$, $S_{24}$ which will be able to compensate.

### 6.1.2 Computing new non-diagonal CHS terms.

In order to compute a non-diagonal term $S_{ss'}$, we will simply write down all possible terms which can appear in it. This is possible since the theory is conformal and there are no dimensionful parameters, which allows us to determine that $S_{ss'}$ will contain exactly $s+s'$ derivatives. As we've said, we can then fix these terms by making sure that the variation of this new term cancels the variation of the usual quadratic terms, in this case $S_{22}$ and $S_{00}$.

#### $\mathbf{S_{02}}$

The simplest case is $S_{02}$. In principle this sector is relevant since varying it as in (6.19) - (6.20) might contribute to cancelling variations from $S_{00}$ and $S_{22}$. However, this term is trivial for the following reason: in general, $S_{ss'}$ must vanish

[60] In a Ricci-flat spacetime $C^{abcd}=R^{abcd}$

in the flat-space limit, i.e. in a Ricci-flat space, it must contain at least one factor of the Weyl curvature. Furthermore, dimensionally, the action contains exactly two derivatives. However, there is no non-trivial way to contract the tensors $C^{abcd}$, $h^{ab}$ and $h_0$ together, so this action is vanishes.

**$\mathbf{S_{04}}$**

This term is the first non-trivial correction we obtain. However, it is not very interesting for us as it cannot give new contributions to the Weyl anomaly, since $h_0$ is actually non-dynamical and can be integrated out from the start, providing corrections only to the $S_{44}$ sector and higher order vertices.
We state the result here for completeness, but save the details of the calculation for the next section.

$$S_{04} = \tfrac{1}{180} h_0 \, h_{bdef} \, C_a{}^e{}_c{}^f \, C^{abcd} \tag{6.21}$$

**$\mathbf{S_{24}}$**

Computing this sector of the action is computationally challenging. Indeed, the action contains 6 derivatives, which means that the following schematic terms are allowed:

$$S_{24} \sim h_4 \, D^4(Ch_2) + h_4 \, D^2(C^2 h_2) + h_4 \, C^3 h_2 \tag{6.22}$$

Out of these, only terms with one Weyl curvatures and without derivatives acting on it can contribute to the anomaly, as will be explained in Section 6.2. As such, we can split the action into an "anomalous" part, $S_{24,an} \sim h_4 C D^4 h_2$, and the "rest", $S_{24,re}$. Explicitly, the ansatz for $S_{24,an}$ is

$$\begin{aligned} S_{24,an} = {} & k_1 \, h_a{}^{efg} C^{abcd} D_{bdfg} h_{ce} + k_2 \, h_{ac}{}^{ef} C^{abcd} D_{bd}{}^g{}_g h_{ef} + k_3 \, h_a{}^{efg} C^{abcd} D_{defg} h_{bc} \\ & + k_4 \, h_{ac}{}^{ef} C^{abcd} D_{df}{}^g{}_g h_{be} + k_5 \, h_{ac}{}^{ef} C^{abcd} D_{ef}{}^g{}_g h_{bd} \end{aligned} \tag{6.23}$$

while $S_{24,re}$ is too long to write here. Our goal will then to fix the constants $\{k_1, k_2, k_3, k_4, k_5\}$.
The other relevant part of the full action here will be $S_{22}$, which can be computed by expanding the Weyl action (and using (3.48) as normalisation in the flat-space limit). We then fix the constants in (6.23) by using

$$\frac{\delta(S_{24,an} + S_{24,re})}{\delta^{(0)} h_4} + \frac{\delta S_{22}}{\delta^{(4)} h_2} = 0 \tag{6.24}$$

This equation is solved by systematically symmetrising derivatives, using projections of the Weyl tensor and its derivatives on the relevant Young tableaux as well as including DDIS. The final result for the ”anomalous” part of the action is:

$$S_{24,An} = -\tfrac{1}{1680}h_{ac}{}^{ef}C^{abcd}D_{bd}{}^{g}{}_{g}h_{ef} + \tfrac{1}{1260}h_{ac}{}^{ef}C^{abcd}D_{df}{}^{g}{}_{g}h_{be} - \tfrac{1}{5040}h_{ac}{}^{ef}C^{abcd}D_{ef}{}^{g}{}_{g}h_{bd} \tag{6.25}$$

## 6.2 Contribution to the conformal anomaly

Let us now compute the contribution that the new non diagonal terms may have to the conformal anomaly.
The conformal anomaly of the theory is computed by looking at the logarithmic divergence of the partition function. In $d = 4$ it looks like [155]:

$$\Gamma = -\log Z_{CHS} = -\frac{1}{16\pi^2}\log\Lambda \int \sqrt{g}\, b_4 \tag{6.26}$$

$$b_4 = -a\, R^\star R^\star + c\, C^2 \tag{6.27}$$

where $b_4$ is called a Seeley coefficent and $R^\star R^\star \equiv C^2 - 2R_{ab}R^{ab} + \frac{2}{3}R^2$ is proportional to the Euler number density. Since we are interested in computing the $c$-coefficient, it suffices to choose a Ricci-flat background where $b_4 = (c-a)C^2$. One can further note, as in [131], that in a conformally-flat Einstein background, $b_4 = -aR^\star R^\star$ and furthermore all non-diagonal terms drop out. Therefore, mixed terms can only contribute to the $c$-coefficient.

To compute the coefficient, we follow the steps in [131], namely we compute the log divergence in (6.26). To be more precise, we only compute the terms which can be proportional to $C^2$. Using the newly found $S_{04}$ and $S_{24}$ sector, we see that the only relevant diagram is the one given in Figure 6.1. There, we have two insertions of the $S_{ss'}$ vertex, and the only terms linear in the Weyl tensor are relevant. Note that this justifies the comments made in Section 6.1.2 - we immediately see that only $S_{24,an}$ can contribute.

This observation leads to an important simplification. The flat-space kinetic action, $S_{ss}$ contains $2s$ derivatives. Meanwhile, the diagonal $S_{ss'}$ “vertex” linear in Weyl curvature contains $s + s' - 2$ derivatives. This means that diagram 6.1 is

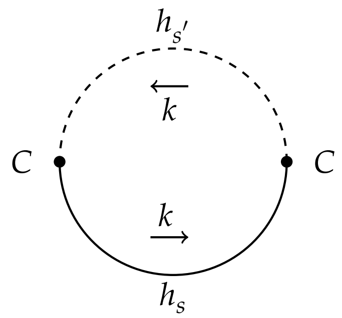



Figure 6.1: Contribution from $S_{s's}$ to the conformal anomaly

superficially only logarithmically divergent, for any spins involved. If we include any part of $S_{ss}$ which is not part of the flat limit, we will have a finite integral. As such, we can simply take the flat-space kinetic terms (3.48). Furthermore, since we only care about that divergence, we can set any external momentum to 0. We can therefore easily write down the relevant terms:

$$-\log Z_{CHS} = \frac{n_2 n_4}{2} \int \frac{\mathrm{d}^d k}{(2\phi)^d} V_{24}(k)_{abcd,a'b'} V_{24}(k)^{efgh,e'f'} \frac{\mathcal{P}(k)^{abcd,}{}_{efgh} \mathcal{P}(k)^{a'b',}{}_{e'f'}}{(k^2)^6} \Bigg|_{\log \Lambda} \tag{6.28}$$

where $\mathcal{P}(k)^{a(s)}_{b(s)}$ correspond to the spin s symmetric TT propagators, while $V_{24}(k)^{a(4),b(2)}$ is the vertex which comes from Fourier transforming $S_{24}$. The factor of $1/2$ is a symmetry factor while $n_2 = 240$ and $n_4 = 2903040$ are normalisation coefficients from the propagators, see (3.49).

This is a straightforward integral which can be done using the tricks in Appendix B.3. Note that if we use dimensional regularisation with $d = 4 - \epsilon$, the logarithmic divergence appears instead as a pole in $-\frac{1}{\epsilon}$, similar to Chapter 3. We now write the results obtained for the first few off-diagonal contributions:

$$c_{02} = 0 \qquad c_{13} = \frac{98}{75} \qquad c_{24} = \frac{972}{35} \qquad c_{35} = \frac{1936}{9} , \tag{6.29}$$

the first two of these values having first been obtained in [131].

Let us put these results in context. In [127], the $a$ and $c$-coefficients of the anomaly were computed. In particular, the spin $s$ contribution to the $c$-coefficient was found to be:

$$c_s = \frac{1}{720} \nu_s \left(4 - 42\nu_s + 29\nu_s^2\right) . \tag{6.30}$$

A few assumptions were made to obtain this result. First of all, it was assumed that we can factorise the spin $s$ kinetic operator into a product of $s$ $2^{\text{nd}}$ order Lichnerowicz-type operators in Ricci-flat spacetimes. In [156], it was shown that there exist obstructions to that factorisation of the form $\sim DR + \ldots$. However, it was noted in [131], these obstructions cannot contribute to the $c$-coefficient.

The second assumption, which we mentioned at the opening of this chapter, was that the kinetic operator is diagonal in spin. Since this is not true, the mixed contribution we have computed must be added to those given in (6.30).

There exists another value for $c_s$ in the literature

$$c'_s \equiv \frac{1}{360}\nu_s \left(-4 - 17\nu_s + 14\nu_s^2\right) , \tag{6.31}$$

which was found by computing it on a conically deformed-sphere, where fewer assumptions need to be made [143]. This value is expected to be the correct one: it matches the $AdS_5$ massless higher spins results [135], which are related to the CHS anomaly via AdS/CFT [54, 134].

A natural expectation is then that the mixed contributions we computed account for the difference between (6.30) and (6.31). This does not appear to be the case - at least naively. Indeed, if we compare our results with the first few values for $c_s$ and $c'_s$:

$$c_1 = \frac{1}{10} \qquad c_2 = \frac{199}{30} \qquad c_3 = \frac{919}{15} \qquad c_4 = 299 \qquad c_5 = \frac{6211}{6} \tag{6.32}$$

$$c'_1 = \frac{1}{10} \qquad c'_2 = \frac{199}{30} \qquad c'_3 = \frac{904}{15} \qquad c'_4 = 291 \qquad c'_5 = \frac{6043}{6}, \tag{6.33}$$

it is not obvious how one should combine our values of $c_{ss'}$ to account for the discrepancy. It is possible that the contribution from a general mixed term $c_{ss'}$ can be non-trivially mixed with $c_s$.[61] One way to highlight this difficulty comes from the fact that in [127], it was found somewhat unexpectedly that $c_s$ is a cubic polynomial in $\nu_s$, whereas on general grounds, it was simply expected to be a polynomial of order 6 in $s$. The same is true of $c_{s'}$. This seems to come from the particular SO(3, 1) representation of the fields. However, from (6.29), this does not appear to be the case for mixed contributions.

[61] For instance one might perform a field redefinition to diagonalise the curved CHS action, thus confirming the assumption of [127], but one would have to compute a non-trivial Jacobian.

Both results values (6.30) and (6.31) have the important property that they vanish under the natural regularisation we looked at in Chapter 4, ie.

$$\sum_{s=0}^{\infty} e^{-(s+1/2)\epsilon}\, c_s \bigg|_{fin} = 0 = \sum_{s=0}^{\infty} e^{(s+1/2)\epsilon}\, c_s' \bigg|_{fin} , \tag{6.34}$$

where we are taking the $\epsilon \to 0$ limit and dropping poles. The fact that both functions vanish under this regularisation implies that the mixed term contribution should as well (especially if it is found to account for the difference). Once again, this raises the question of how one should re-sum mixed contributions to combine them with $c_s$ - specifically, how one should sum there under the regulator $e^{-(s+1/2)\epsilon}$.

In order to say more, it would be very helpful if we could obtain a closed formula for mixed contributions. At present, this seems difficult, although our current approach does highlight a simplification: we can show that anomaly contributions are controlled by only one type of term. Indeed, in the computation of the diagram 6.1, we noticed that the propagators can be taken to be the flat space TT propagators (cf. (3.49), (3.50)). Furthermore, we saw that in general the vertices that are relevant $V_{ss'}$ will come from the part of the action with only one Weyl tensor and no derivatives acting on it, $S_{ss',an}$. But it is even simpler than that: since those vertices are contracted with TT propagators, we can restrict to the TT part of $S_{ss',an}$. This implies that only one type of term can contribute; for $s > s'$, then we have:

$$\mathcal{L}_{s\,s-2,an}\bigg|_{TT} = k_s h^{a(s-2)bc} C_b{}^d{}_c{}^e D_{deb_1\dots}{}^{b_1\dots}{}_{b_{[\frac{s}{2}]}}{}^{b_{[\frac{s}{2}]}} h_{a(s-2)} \tag{6.35}$$

$$\mathcal{L}_{s\,s-p,an}\bigg|_{TT} = 0 \qquad p > 2\,. \tag{6.36}$$

The last equality holds true because there are no non-trivial contractions. As such, there is only one coefficient, $k_s$, that seems to control the conformal anomaly. A streamlined version of the analysis used here might yet provide an expression for generic values of $s$.

# Chapter 7

# Conclusion

Let us summarise this thesis and make some parting comments. Chapter 1 provided some context and motivation for the work presented. We opened by giving a basic introduction to massless higher spin theory and then related it to CHS theory via AdS/CFT.

Chapter 2 focused Conformal Gravity, as a prelude to CHS theory. This gave us the opportunity to talk about higher derivative theories and to give a generalised notion of scattering states.

Notably, out of the 6 degrees of freedom contained in the theory, we found that there were 4 admissible states, the two usual "Einstein modes" but also some helicity $\pm 1$ states that lead to perfectly well-defined amplitudes.

We introduced the fact that conformal field theories find a natural description in the language of twistors. This led to a compact formula for three-point scattering of *any* states. Indeed, this formula matched explicit results computed from usual covariant methods.

This result has remarkable interesting features. First of all, it shows that scattering amplitudes containing only Einstein modes vanish in an interesting way: they are given by Einstein Gravity amplitudes times the cosmological constant, which is related to the results of [62]. This was obtained by extending the formula to AdS spacetime - a generalisation which can readily be done since Minkowski and AdS are conformal to each other and the theory is conformally invariant.

Furthermore, we observed that not *all* amplitudes are zero. Indeed, scattering helicity $\pm 1$ modes gives results which are proportional to Einstein-Maxwell amplitudes.

In Chapter 3 we introduced CHS theories. We described how the infinite

CHS algebra arises, and how the theory can be obtained as the logarithmically divergent part of the 1-loop effective action of free scalars coupled to a general higher spin background. We determined certain parts of the action in preparation for later computations.
Following the logic of the previous chapter, we also characterised the various scattering states which arise as a result of CHS theory being higher in derivatives. Finally, we gave a twistor-spinor description for these states.

Chapter 4 pertained to the computation of scattering amplitudes of scalars coupled to CHS theory. This allowed us to illustrate an interesting aspect of higher spin theories: their observables involve infinite sums over spins which must be regularised. Remarkably, we saw that picking a regularisation scheme which is compatible with the underlying symmetry of the theory led to a vanishing of these amplitudes. We showed how this was due to the action of the large global symmetry group. We also pushed this computation to the 1-loop level but found a non-zero contribution. An argument was made that this contribution might have been expected to be the same as the $c$-coefficient of the conformal anomaly of CHS theory, but the latter is expected to vanish. The conclusion remained unclear though the resolution likely lies in the definition of the summation over spins: the regularisation procedure may not commute with certain simplifying limits that we used.

Chapter 5 contained another set of amplitude computations, this time in pure CHS theory. We first focused on four-point scattering Einstein states and Maxwell modes, and were once again faced with the task of summing an infinite set of diagrams. The same result emerged: all the amplitudes we computed ended up vanishing upon summation, once again due to the global symmetry group.
We also generalised the discussion of Conformal Gravity for three-point functions, ie. we used the twistor-spinor formulation of scattering states in CHS theory to give an expression for *all* scattering three-point amplitudes. The same pattern emerged; the scattering amplitudes involving only two-derivative modes vanish in flat space. In AdS we saw that they were given by massless higher spin amplitudes times powers of the cosmological constant. We also observed that some of the amplitudes containing non-standard modes are not zero.

Chapter 6 looked at CHS theory in curved spacetimes. The action has a more complicated form there, due to the fact that the gauge symmetry is affected by the presence of curvature. This leads to the existence of non-diagonal terms in the quadratic sector of the theory which may contribute to the $c$-coefficient of the CHS conformal anomaly. We computed the modification to the gauge invariance, and used that to deduce the relevant off-diagonal terms up to spin 5. These off-diagonal contributions do not seem to be able to account for a discrepancy which exists in the literature between two different expressions for $c$, though this once again could be due to the exact definition of the sum over spins. One would need to obtain a closed expression for the off-diagonal contribution to the $c$-coefficient from any spin. This remains out of reach as the computation is technically difficult, though we were able to highlight some simplifying structure in the form of the contributing terms.

The main message is that we are starting to understand how to perform classical and quantum computations in CHS theory. This is important as this theory appears to be intimately connected to massless higher spins. Furthermore, evidence points to the fact that its conformal anomaly vanishes meaning that it is well defined at the quantum level. Following the work presented here, there are many directions one could take.

In particular, our 1-loop computations have been somewhat inconclusive. In the context of infinite sums, some of the assumptions we have made should be relaxed as they may not be valid.

Another topic one could pursue is the computation of four-point functions of higher derivative modes. In the covariant formalism, this should be doable at the level of Conformal Gravity, though it may be difficult for general CHS theory. Indeed, the presence of a propagator in the CHS theory complicates things. On the other hand, the twistor approach lends itself nicely to this type of computation.

The theory being conformal, one should also be able to generalise this to AdS, which would allow us to further clarify the relationship between CHS theories and massless higher spin theories.

Finally, our discussions of the action of the global CHS symmetry on scattering amplitudes has been rather ad-hoc. Indeed, we just picked out very particular generators of the algebra to explain some surprising results following amplitude computations. A promising direction would therefore lie in analysing the implications of this algebra in more generality - this could lead to some powerful statements about the theory's triviality.

It would be interesting if one could understand its role in constraining general symmetries in a more general way.

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# Bibliography

[1] E. Joung, S. Nakach, and A. A. Tseytlin, *Scalar scattering via conformal higher spin exchange*, *JHEP* **02** (2016) 125, [arXiv:1512.08896].

[2] M. Beccaria, S. Nakach, and A. A. Tseytlin, *On triviality of S-matrix in conformal higher spin theory*, *JHEP* **09** (2016) 034, [arXiv:1607.06379].

[3] T. Adamo, S. Nakach, and A. A. Tseytlin, *Scattering of conformal higher spin fields*, arXiv:1805.00394.

[4] E. S. Fradkin and A. A. Tseytlin, *Conformal supergravity*, *Phys.Rept.* **119** (1985) 233–362.

[5] A. Y. Segal, *Conformal higher spin theory*, *Nucl.Phys.* **B664** (2003) 59–130, [hep-th/0207212].

[6] S. Weinberg, *The quantum theory of fields. Vol.1, Foundations*. Cambridge University Press, Cambridge, 1995.

[7] X. Bekaert and N. Boulanger, *The Unitary representations of the Poincare group in any spacetime dimension*, hep-th/0611263.

[8] R. Rahman and M. Taronna, *From Higher Spins to Strings: A Primer*, arXiv:1512.07932.

[9] V. Bargmann and E. P. Wigner, *Group Theoretical Discussion of Relativistic Wave Equations*, *Proc. Nat. Acad. Sci.* **34** (1948) 211.

[10] E. P. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group*, *Annals Math.* **40** (1939) 149–204. [Reprint: Nucl. Phys. Proc. Suppl.6,9(1989)].

[11] C. Fronsdal, *Massless Fields with Integer Spin*, *Phys. Rev.* **D18** (1978) 3624.

[12] X. Bekaert, N. Boulanger, and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, *Rev. Mod. Phys.* **84** (2012) 987–1009, [arXiv:1007.0435].

[13] H. van Dam and M. J. G. Veltman, *Massive and massless Yang-Mills and gravitational fields*, *Nucl. Phys.* **B22** (1970) 397–411.

[14] V. I. Zakharov, *Linearized Gravitation Theory and the Graviton Mass*, *Soviet Journal of Experimental and Theoretical Physics Letters* **12** (1970) 312.

[15] Y. Iwasaki, *Consistency condition for propagators*, *Phys. Rev.* **D2** (1970) 2255–2256.

[16] M. Porrati, *Fully covariant van Dam-Veltman-Zakharov discontinuity, and absence thereof*, *Phys. Lett.* **B534** (2002) 209–215, [hep-th/0203014].

[17] S. Weinberg, *Photons and gravitons in s-matrix theory: Derivation of charge conservation and equality of gravitational and inertial mass*, *Phys. Rev.* **135** (Aug, 1964) B1049–B1056.

[18] S. Weinberg and E. Witten, *Limits on Massless Particles*, *Phys. Lett.* **96B** (1980) 59–62.

[19] M. Porrati, *Universal Limits on Massless High-Spin Particles*, *Phys. Rev.* **D78** (2008) 065016, [arXiv:0804.4672].

[20] C. Aragone and S. Deser, *Consistency Problems of Hypergravity*, *Phys. Lett.* **B86** (1979) 161.

[21] C. Aragone and H. La Roche, *Massless Second Order Tetradic Spin 3 Fields and Higher Helicity Bosons*, *Nuovo Cim.* **A72** (1982) 149.

[22] R. R. Metsaev, *Cubic interaction vertices of massive and massless higher spin fields*, *Nucl.Phys.* **B759** (2006) 147–201, [hep-th/0512342].

[23] S. R. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, *Phys. Rev.* **159** (1967) 1251–1256.

[24] R. Haag, J. T. Lopuszanski, and M. Sohnius, *All Possible Generators of Supersymmetries of the s Matrix*, *Nucl. Phys.* **B88** (1975) 257. [,257(1974)].

[25] D. Ponomarev and E. D. Skvortsov, *Light-Front Higher-Spin Theories in Flat Space*, arXiv:1609.04655.

[26] E. S. Fradkin and M. A. Vasiliev, *On the Gravitational Interaction of Massless Higher Spin Fields*, *Phys.Lett.* **B189** (1987) 89–95.

[27] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B243** (1990) 378–382.

[28] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions*, *Phys.Lett.* **B285** (1992) 225–234.

[29] M. A. Vasiliev, *Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions*, *Int.J.Mod.Phys.* **D5** (1996) 763–797, [hep-th/9611024].

[30] M. A. Vasiliev, *Higher spin gauge theories: Star product and AdS space*, hep-th/9910096.

[31] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. Vasiliev, *Nonlinear higher spin theories in various dimensions*, hep-th/0503128.

[32] V. Didenko and E. Skvortsov, *Elements of Vasiliev theory*, arXiv:1401.2975.

[33] S. Giombi, *TASI Lectures on the Higher Spin - CFT duality*, arXiv:1607.02967.

[34] M. A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett.* **B567** (2003) 139–151, [hep-th/0304049].

[35] M. Vasiliev, *Higher spin gauge theories in various dimensions*, *Fortsch.Phys.* **52** (2004) 702–717, [hep-th/0401177].

[36] X. Bekaert, *Higher spin algebras as higher symmetries*, *Ann. U. Craiova Phys.* **16** (2006), no. II 58–65, [arXiv:0704.0898].

[37] C. Iazeolla, *On the Algebraic Structure of Higher-Spin Field Equations and New Exact Solutions*, arXiv:0807.0406.

[38] N. Boulanger and P. Sundell, *An action principle for Vasiliev's four-dimensional higher-spin gravity*, *J. Phys.* **A44** (2011) 495402, [arXiv:1102.2219].

[39] R. Bonezzi, N. Boulanger, E. Sezgin, and P. Sundell, *Frobenius–Chern–Simons gauge theory*, *J. Phys.* **A50** (2017), no. 5 055401, [arXiv:1607.00726].

[40] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[41] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[42] S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys.Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[43] H. Nastase, *Introduction to AdS-CFT*, arXiv:0712.0689.

[44] J. Polchinski, *Introduction to Gauge/Gravity Duality*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010). String Theory and Its Applications: From meV to the Planck Scale: Boulder, Colorado, USA, June 1-25, 2010*, pp. 3–46, 2010. arXiv:1010.6134.

[45] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015.

[46] I. R. Klebanov and A. M. Polyakov, *AdS dual of the critical O(N) vector model*, *Phys. Lett.* **B550** (2002) 213–219, [hep-th/0210114].

[47] N. S. Craigie, V. K. Dobrev, and I. T. Todorov, *Conformally Covariant Composite Operators in Quantum Chromodynamics*, *Annals Phys.* **159** (1985) 411–444.

[48] D. Anselmi, *Theory of higher spin tensor currents and central charges*, *Nucl. Phys.* **B541** (1999) 323–368, [hep-th/9808004].

[49] V. E. Didenko and E. D. Skvortsov, *Exact higher-spin symmetry in CFT: all correlators in unbroken Vasiliev theory*, *JHEP* **04** (2013) 158, [arXiv:1210.7963].

[50] S. Giombi, S. Prakash, and X. Yin, *A Note on CFT Correlators in Three Dimensions*, *JHEP* **07** (2013) 105, [arXiv:1104.4317].

[51] S. Giombi and X. Yin, *Higher Spins in AdS and Twistorial Holography*, *JHEP* **1104** (2011) 086, [arXiv:1004.3736].

[52] S. Giombi and X. Yin, *Higher Spin Gauge Theory and Holography: The Three-Point Functions*, *JHEP* **1009** (2010) 115, [arXiv:0912.3462].

[53] C. Sleight and M. Taronna, *Higher Spin Interactions from Conformal Field Theory: The Complete Cubic Couplings*, *Phys. Rev. Lett.* **116** (2016), no. 18 181602, [arXiv:1603.00022].

[54] S. Giombi and I. R. Klebanov, *One Loop Tests of Higher Spin AdS/CFT*, *JHEP* **1312** (2013) 068, [arXiv:1308.2337].

[55] M. Beccaria and A. Tseytlin, *On higher spin partition functions*, *J.Phys.* **A48** (2015), no. 27 275401, [arXiv:1503.08143].

[56] S. Giombi, I. R. Klebanov, and Z. M. Tan, *The ABC of Higher-Spin AdS/CFT*, *Universe* **4** (2018), no. 1 18, [arXiv:1608.07611].

[57] M. Beccaria and A. A. Tseytlin, *Higher spins in $AdS_5$ at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, *JHEP* **1411** (2014) 114, [arXiv:1410.3273].

[58] K. S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, *Phys. Rev.* **D16** (1977) 953–969.

[59] A. Salvio, *Quadratic Gravity*, arXiv:1804.09944.

[60] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, *JHEP* **08** (2004) 009, [hep-th/0406051].

[61] C. Caprini, P. Hölscher, and D. J. Schwarz, *Astrophysical Gravitational Waves in Conformal Gravity*, arXiv:1804.01876.

[62] J. Maldacena, *Einstein Gravity from Conformal Gravity*, arXiv:1105.5632.

[63] T.-j. Chen, M. Fasiello, E. A. Lim, and A. J. Tolley, *Higher derivative theories with constraints: Exorcising Ostrogradski's Ghost*, *JCAP* **1302** (2013) 042, [arXiv:1209.0583].

[64] M. Kaku, P. K. Townsend, and P. Van Nieuwenhuizen, *Gauge theory of the conformal and superconformal group*, *Physics Letters B* **69** (Aug., 1977) 304–308.

[65] M. Ostrogradsky, *Mém. de l'acad. de st. pétersbourg, Sc. math. et phys.* **4 tome** (1850) 385. cited By 2.

[66] R. P. Woodard, *Avoiding dark energy with 1/r modifications of gravity*, *Lect. Notes Phys.* **720** (2007) 403–433, [astro-ph/0601672].

[67] P. D. Mannheim, *Solution to the ghost problem in fourth order derivative theories*, *Found. Phys.* **37** (2007) 532–571, [hep-th/0608154].

[68] C. M. Bender and P. D. Mannheim, *No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model*, *Phys. Rev. Lett.* **100** (2008) 110402, [arXiv:0706.0207].

[69] C. M. Bender and P. D. Mannheim, *Exactly solvable PT-symmetric Hamiltonian having no Hermitian counterpart*, *Phys. Rev.* **D78** (2008) 025022, [arXiv:0804.4190].

[70] C. M. Bender and P. D. Mannheim, *Giving up the ghost*, *J. Phys.* **A41** (2008) 304018, [arXiv:0807.2607].

[71] C. M. Bender and P. D. Mannheim, *PT symmetry and necessary and sufficient conditions for the reality of energy eigenvalues*, *Phys. Lett.* **A374** (2010) 1616–1620, [arXiv:0902.1365].

[72] M. Raidal and H. Veermäe, *On the Quantisation of Complex Higher Derivative Theories and Avoiding the Ostrogradsky Ghost*, *Nucl. Phys.* **B916** (2017) 607–626, [arXiv:1611.03498].

[73] D. Anselmi and M. Piva, *Perturbative unitarity of Lee-Wick quantum field theory*, *Phys. Rev.* **D96** (2017), no. 4 045009, [arXiv:1703.05563].

[74] D. Anselmi and M. Piva, *A new formulation of Lee-Wick quantum field theory*, *JHEP* **06** (2017) 066, [arXiv:1703.04584].

[75] D. Anselmi, *Fakeons And Lee-Wick Models*, *JHEP* **02** (2018) 141, [arXiv:1801.00915].

[76] H. Johansson, G. Mogull, and F. Teng, *Unraveling conformal gravity amplitudes*, arXiv:1806.05124.

[77] R. R. Metsaev, *Ordinary-derivative formulation of conformal low spin fields*, *JHEP* **01** (2012) 064, [arXiv:0707.4437].

[78] A. Pais and G. E. Uhlenbeck, *On Field theories with nonlocalized action*, *Phys. Rev.* **79** (1950) 145–165.

[79] I. Y. Aref'eva, A. A. Slavnov, and L. D. Faddeev, *Generating functional for the s matrix in gauge-invariant theories*, *Theoretical and Mathematical Physics* **21** (1974), no. 3 1165–1172.

[80] A. Jevicki and C. Lee, *S -matrix generating functional and effective action*, *Phys. Rev. D* **37** (Mar, 1988) 1485–1491.

[81] T. Adamo, P. Hähnel, and T. McLoughlin, *Conformal higher spin scattering amplitudes from twistor space*, *JHEP* **04** (2017) 021, [arXiv:1611.06200].

[82] R. J. Riegert, *The particle content of linearized conformal gravity*, *Physics Letters A* **105** (1984), no. 3 110 – 112.

[83] E. S. Fradkin and A. A. Tseytlin, *One Loop Beta Function in Conformal Supergravities*, *Nucl. Phys.* **B203** (1982) 157–178.

[84] S. C. Lee and P. van Nieuwenhuizen, *Counting of States in Higher Derivative Field Theories*, *Phys. Rev.* **D26** (1982) 934.

[85] M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, *Gauge Theory of the Conformal and Superconformal Group*, *Phys. Lett.* **69B** (1977) 304–308.

[86] R. M. Wald, *General relativity*. Chicago Univ. Press, Chicago, IL, 1984.

[87] T. Adamo and L. Mason, *Conformal and Einstein gravity from twistor actions*, *Class. Quant. Grav.* **31** (2014), no. 4 045014, [arXiv:1307.5043].

[88] L. Dolan and J. N. Ihry, *Conformal Supergravity Tree Amplitudes from Open Twistor String Theory*, *Nucl. Phys.* **B819** (2009) 375–399, [arXiv:0811.1341].

[89] H. Johansson and J. Nohle, *Conformal Gravity from Gauge Theory*, arXiv:1707.02965.

[90] Z. Bern, J. J. M. Carrasco, and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev.* **D78** (2008) 085011, [arXiv:0805.3993].

[91] Z. Bern, J. J. M. Carrasco, and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602, [arXiv:1004.0476].

[92] E. S. Fradkin and A. A. Tseytlin, *Conformal Anomaly in Weyl Theory and Anomaly Free Superconformal Theories*, *Phys.Lett.* **B134** (1984) 187.

[93] D. Butter, F. Ciceri, B. de Wit, and B. Sahoo, *Construction of all N=4 conformal supergravities*, *Phys. Rev. Lett.* **118** (2017), no. 8 081602, [arXiv:1609.09083].

[94] A. A. Tseytlin, *On divergences in non-minimal $N = 4$ conformal supergravity*, *J. Phys.* **A50** (2017), no. 48 48LT01, [arXiv:1708.08727].

[95] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189–258, [hep-th/0312171].

[96] P. Benincasa and F. Cachazo, *Consistency Conditions on the S-Matrix of Massless Particles*, arXiv:0705.4305.

[97] J. Broedel and B. Wurm, *New Twistor String Theories revisited*, *Phys. Lett.* **B675** (2009) 463–468, [arXiv:0902.0550].

[98] E. Conde and A. Marzolla, *Lorentz Constraints on Massive Three-Point Amplitudes*, *JHEP* **09** (2016) 041, [arXiv:1601.08113].

[99] E. Conde, E. Joung, and K. Mkrtchyan, *Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions*, *JHEP* **08** (2016) 040, [arXiv:1605.07402].

[100] N. Arkani-Hamed, T.-C. Huang, and Y.-t. Huang, *Scattering Amplitudes For All Masses and Spins*, arXiv:1709.04891.

[101] E. Cartan, *Les espaces a connexion conforme*, *Ann.Soc.Po.Math.* **2** (1923) 171–221.

[102] K. Dighton, *An introduction to the theory of local twistors*, *Int.J.Theor.Phys.* **11** (1974) 31–43.

[103] H. Friedrich, *Twistor connection and normal conformal Cartan connection*, *Gen.Rel.Grav.* **8** (1977) 303–312.

[104] R. Penrose and M. A. H. MacCallum, *Twistor theory: An Approach to the quantization of fields and space-time*, *Phys. Rept.* **6** (1972) 241–316.

[105] R. Penrose and W. Rindler, *Spinors and Space-Time*, vol. 2. Cambridge University Press, 1988.

[106] R. Penrose and W. Rindler, *Spinors and Space-Time*, vol. 1. Cambridge University Press, 1986.

[107] L. J. Mason, *The relationship between spin-2 fields, linearized gravity and linearized conformal gravity*, in *Further Advances in Twistor Theory* (L. J. Mason and L. P. Hughston, eds.), vol. 1, pp. 67–69. Longman, 1990.

[108] L. J. Mason, *Local twistors and the Penrose transform for homogeneous bundles*, in *Further Advances in Twistor Theory* (L. J. Mason and L. P. Hughston, eds.), vol. 1, pp. 62–66. Longman, 1990.

[109] L. J. Mason, *Twistor actions for non-self-dual fields: A Derivation of twistor-string theory*, *JHEP* **10** (2005) 009, [hep-th/0507269].

[110] T. Adamo, *Twistor actions for gauge theory and gravity*. PhD thesis, University of Oxford, 2013. arXiv:1308.2820.

[111] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference Mumbai, India, January 5-10, 2001*, 2001. hep-th/0106109.

[112] A. Strominger, *The dS / CFT correspondence*, *JHEP* **10** (2001) 034, [hep-th/0106113].

[113] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013, [astro-ph/0210603].

[114] J. M. Maldacena and G. L. Pimentel, *On graviton non-Gaussianities during inflation*, *JHEP* **09** (2011) 045, [arXiv:1104.2846].

[115] S. Raju, *New Recursion Relations and a Flat Space Limit for AdS/CFT Correlators*, *Phys. Rev.* **D85** (2012) 126009, [arXiv:1201.6449].

[116] N. Arkani-Hamed and J. Maldacena, *Cosmological Collider Physics*, arXiv:1503.08043.

[117] S. Deser and R. I. Nepomechie, *Anomalous Propagation of Gauge Fields in Conformally Flat Spaces*, *Phys.Lett.* **B132** (1983) 321.

[118] M. T. Anderson, *$L^2$ curvature and volume renormalization of AHE metrics on 4-manifolds*, *Math. Res. Lett.* **8** (2001) 171–188, [math/0011051].

[119] T. Adamo and L. Mason, *Einstein supergravity amplitudes from twistor-string theory*, *Class. Quant. Grav.* **29** (2012) 145010, [arXiv:1203.1026].

[120] T. Adamo and L. Mason, *Twistor-strings and gravity tree amplitudes*, *Class. Quant. Grav.* **30** (2013) 075020, [arXiv:1207.3602].

[121] T. Adamo, *Gravity with a cosmological constant from rational curves*, *JHEP* **11** (2015) 098, [arXiv:1508.02554].

[122] E. S. Fradkin and V. Y. Linetsky, *Cubic interaction in conformal theory of integer higher spin fields in four-dimensional space-time*, *Phys.Lett.* **B231** (1989) 97.

[123] E. S. Fradkin and V. Y. Linetsky, *Superconformal Higher Spin Theory in the Cubic Approximation*, *Nucl.Phys.* **B350** (1991) 274–324.

[124] A. A. Tseytlin, *On limits of superstring in $AdS_5 \times S^5$*, *Theor.Math.Phys.* **133** (2002) 1376–1389, [hep-th/0201112].

[125] R. R. Metsaev, *Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields*, *JHEP* **06** (2012) 062, [arXiv:0709.4392].

[126] X. Bekaert, E. Joung, and J. Mourad, *Effective action in a higher-spin background*, *JHEP* **1102** (2011) 048, [arXiv:1012.2103].

[127] A. A. Tseytlin, *On partition function and Weyl anomaly of conformal higher spin fields*, *Nucl.Phys.* **B877** (2013) 598–631, [arXiv:1309.0785].

[128] M. Beccaria, X. Bekaert, and A. A. Tseytlin, *Partition function of free conformal higher spin theory*, *JHEP* **08** (2014) 113, [arXiv:1406.3542].

[129] P. Haehnel and T. McLoughlin, *Conformal Higher Spin Theory and Twistor Space Actions*, arXiv:1604.08209.

[130] M. Grigoriev and A. A. Tseytlin, *On conformal higher spins in curved background*, *J. Phys.* **A50** (2017), no. 12 125401, [arXiv:1609.09381].

[131] M. Beccaria and A. A. Tseytlin, *On induced action for conformal higher spins in curved background*, *Nucl. Phys.* **B919** (2017) 359–383, [arXiv:1702.00222].

[132] H. Liu and A. A. Tseytlin, *$D = 4$ superYang-Mills, $D = 5$ gauged supergravity, and $D = 4$ conformal supergravity*, *Nucl.Phys.* **B533** (1998) 88–108, [hep-th/9804083].

[133] R. R. Metsaev, *Gauge invariant two-point vertices of shadow fields, AdS/CFT, and conformal fields*, *Phys.Rev.* **D81** (2010) 106002, [arXiv:0907.4678].

[134] S. Giombi, I. R. Klebanov, S. S. Pufu, B. R. Safdi, and G. Tarnopolsky, *AdS Description of Induced Higher-Spin Gauge Theory*, *JHEP* **1310** (2013) 016, [arXiv:1306.5242].

[135] S. Giombi, I. R. Klebanov, and B. R. Safdi, *Higher Spin $AdS_{d+1}/CFT_d$ at One Loop*, *Phys.Rev.* **D89** (2014) 084004, [arXiv:1401.0825].

[136] M. Beccaria and A. A. Tseytlin, *Iterating free-field AdS/CFT: higher spin partition function relations*, *J. Phys.* **A49** (2016), no. 29 295401, [arXiv:1602.00948].

[137] M. G. Eastwood, *Higher symmetries of the Laplacian*, *Annals Math.* **161** (2005) 1645–1665, [hep-th/0206233].

[138] X. Bekaert, E. Joung, and J. Mourad, *On higher spin interactions with matter*, *JHEP* **0905** (2009) 126, [arXiv:0903.3338].

[139] D. Z. Freedman, K. Johnson, and J. I. Latorre, *Differential regularization and renormalization: A New method of calculation in quantum field theory*, *Nucl. Phys.* **B371** (1992) 353–414.

[140] G. 't Hooft and M. J. G. Veltman, *One loop divergencies in the theory of gravitation*, *Annales Poincare Phys. Theor.* **A20** (1974) 69–94.

[141] S. Giombi, I. R. Klebanov, and A. A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin $AdS_{d+1}/CFT_d$*, *Phys.Rev.* **D90** (2014), no. 2 024048, [arXiv:1402.5396].

[142] D. Ponomarev and A. A. Tseytlin, *On quantum corrections in higher-spin theory in flat space*, *JHEP* **05** (2016) 184, [arXiv:1603.06273].

[143] M. Beccaria and A. A. Tseytlin, *$C_T$ for conformal higher spin fields from partition function on conically deformed sphere*, *JHEP* **09** (2017) 123, [arXiv:1707.02456].

[144] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, arXiv:1308.1697.

[145] T. Gleisberg, F. Krauss, K. T. Matchev, A. Schalicke, S. Schumann, and G. Soff, *Helicity formalism for spin-2 particles*, *JHEP* **09** (2003) 001, [hep-ph/0306182].

[146] A. Datta, E. Gabrielli, and B. Mele, *More on violation of angular-momentum selection rules in quantum gravity*, hep-ph/0406149.

[147] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, *arXiv preprint math/9602214* (1996).

[148] M. Jacob and G. C. Wick, *On the general theory of collisions for particles with spin*, *Annals Phys.* **7** (1959) 404–428. [Annals Phys.281,774(2000)].

[149] R. J. Eden, *High energy collisions of elementary particles*. Cambridge UP, 1967.

[150] P. Collins and E. J. Squires, *Regge poles in particle physics*. Springer, 1968.

[151] J. Gustavsson, *Some sums of Legendre and Jacobi polynomials*, *Mathematica Bohemica* **126** (2001), no. 1 141–149.

[152] S. Acevedo, R. Aros, F. Bugini, and D. E. Díaz, *On the Weyl anomaly of 4D Conformal Higher Spins: a holographic approach*, arXiv:1710.03779.

[153] J. M. Martin-Garcia, *xAct: Efficient tensor computer algebra for Mathematica*, . http://xact.es/.

[154] T. Nutma, *xTras : A field-theory inspired xAct package for mathematica*, *Comput. Phys. Commun.* **185** (2014) 1719–1738, [arXiv:1308.3493].

[155] M. Duff, *Twenty years of the Weyl anomaly*, *Class.Quant.Grav.* **11** (1994) 1387–1404, [hep-th/9308075].

[156] T. Nutma and M. Taronna, *On conformal higher spin wave operators*, *JHEP* **1406** (2014) 066, [arXiv:1404.7452].

[157] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Graduate texts in contemporary physics. Springer, New York, NY, 1997.

[158] T. Adamo, *Lectures on twistor theory*, *PoS* **Modave2017** (2018) 003, [arXiv:1712.02196].

[159] H. Weyl, *Quantum mechanics and group theory*, *Z. Phys.* **46** (1927) 1.

[160] E. P. Wigner, *On the quantum correction for thermodynamic equilibrium*, *Phys. Rev.* **40** (1932) 749–760.

[161] J. E. Moyal, *Quantum mechanics as a statistical theory*, *Proc. Cambridge Phil. Soc.* **45** (1949) 99–124.

[162] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, *Introduction to representation theory*, *ArXiv e-prints* (Jan., 2009) [arXiv:0901.0827].

[163] R. Metsaev, *BRST invariant effective action of shadow fields, conformal fields, and AdS/CFT*, *Theor.Math.Phys.* **181** (2014), no. 3 1548–1565, [arXiv:1407.2601].

# Appendix A

# Definitions and Background

## A.1 Conventions

### Index Conventions

Indices are symmetrised with a weight:

$$A^{(a_1 \dots a_s)} = \frac{1}{s!} \left( A^{a_1 \dots a_s} + (permutations) \right) \tag{A.1}$$

Sometimes, it is convenient to subsume many symmetric indices into one. The notation for this is:

$$A^{a(s)} \equiv A^{(a_1 \dots a_s)} \tag{A.2}$$

We will also sometimes want to condense the notation, by using repeated indices. When that is the case, one is simply to symmetrise over all repeated indices on the same level. For example:

$$\partial^a A^{a(s-1)} \equiv \partial^{(a_1} A^{a_2 \dots a_s)} , \qquad \eta_{b(2)} B_{b(s-2)} \equiv \eta_{(b_1 b_2} B_{b_3 \dots b_s)} . \tag{A.3}$$

Often time, when we simply want to refer to the "spin $s$" tensor, we make use of the short hand notation:

$$f_s \sim f_{a_1 \dots a_s} . \tag{A.4}$$

### Generating functions

We will often make use of generating functions to collect fields of all spins into one object for example. A generating function $f(x,u)$ with the auxiliary variable $u$ is defined as:

$$f(x,u) \equiv \sum_{s=0}^{\infty} f_s(x,u) = \sum_{s=0}^{\infty} \frac{1}{s!} f_{a_1 \ldots a_s} u^{a_1} \ldots u^{a_s} \,. \tag{A.5}$$

In this language, traces and divergences can be easily represented as $\partial_u^2$ and $\partial_u \partial_x$ respectively:

$$\partial_u^2 f(x,u) \quad = \textstyle\sum_{s=2}^{\infty} \frac{1}{(s-2)!} f_{a_1 \ldots a_{s-2} b}{}^{b}\, u^{a_1} \ldots u^{a_{s-2}} \tag{A.6}$$

$$(\partial_u \cdot \partial_x)\, f(x,u) \qquad = \textstyle\sum_{s=1}^{\infty} \partial^b f_{b a_2 \ldots a_s}\, u^{a_1} \ldots u^{a_s} \tag{A.7}$$

Contractions can be written in terms of generating functions:

$$f(x,u) g(x\partial_u)\Big|_{u=0} = \sum_{s=0}^{\infty} \frac{1}{s!} f_{a(s)} g^{a(s)} \,. \tag{A.8}$$

We will sometimes write this using "bra-ket"-like notation, $\langle f | g \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} f_{a(s)} g^{a(s)}$.

**Derivative notation**

The d'Alembertian operator $\Box$ is given by

$$\Box \equiv \partial^a \partial_a \,. \tag{A.9}$$

It is often useful to use the multi-index partial derivative notation:

$$\partial^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} \equiv \partial^{a_1} \ldots \partial^{a_p} \partial_{b_1} \ldots \partial_{b_q} \tag{A.10}$$

In fact, we extend this notation for covariant derivatives as well, where one must explicitly symmetrise over indices:

$$D^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} \equiv g^{a_1 c_1} \ldots g^{a_p c_p} D_{(c_1} \ldots D_{c_p} D_{b_1} \ldots D_{b_q)} \tag{A.11}$$

## A.2 Algebras

### Lorentz Algebra

Algebra of $SO(D-p,p)$ is spanned by $M_{AB}$ following:

$$i[M_{AB}, M_{CD}] = \eta_{BC}M_{AD} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC} - \eta_{BD}M_{AC} \tag{A.12}$$

where $A, B = ((-p+1), \ldots, D)$ and

$$\eta_{AB} = \begin{cases} -1 & \text{for } A = B \leq 0 \\ 1 & \text{for } A = B > 0 \\ 0 & \text{otherwise.} \end{cases}$$

is the metric preserved by $SO(D-p,p)$.

### Poincaré Algebra

Meanwhile, the inhomogeneous group $ISO(D-p,p)$ is spanned by $M_{AB}$ and $P_A$ satisfying:

$$i[M_{AB}, M_{CD}] = \eta_{BC}M_{AD} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC} - \eta_{BD}M_{AC} \tag{A.13}$$
$$i[P_A, M_{BC}] = \eta_{AB}P_B - \eta_{AC}P_B \tag{A.14}$$
$$i[P_A, P_B] = 0 \tag{A.15}$$

### Isometries of $AdS_{d+1}$

The isometry algebra of $d+1$-dimensional $AdS$ space is given by $SO(d,2)$. It is spanned by the generators $M_{AB}$ which satisfy the algebra (A.12) where $A, B$ take on the $d+2$ possible values $(-1, 0, \ldots, d)$. These generators can be restricted to ones spanning only $d+1$ values by isolating the " $-1$" direction and writing write $M_{-1,a} \equiv lP_a$ where $a \in (0, \ldots, d)$ and $l$ is some real constant. The algebra is then:

$$i[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{ac}M_{bd} + \eta_{ad}M_{bc} - \eta_{bd}M_{ac} \tag{A.16}$$
$$i[P_a, M_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b \tag{A.17}$$
$$i[P_a, P_b] = -\frac{1}{l^2}M_{ab} \tag{A.18}$$

where $\eta_{ab}$ is the metric preserved by $SO(d,1)$.

## Conformal Algebra

Conformal transformations can be introduced as diffeomorphisms which rescale the metric. If we are in $d$-dimensional Lorentzian space, so that $a, b \in (0, \ldots, d-1)$, this means:

$$g_{ab}(x+\epsilon) = C g_{ab}(x) \tag{A.19}$$

where the metric has signature $(-1, 1, \ldots, 1)$. Expanding this expression and taking its trace, we obtain the conformal Killing equation:

$$\partial^{(a}\epsilon^{b)} - \frac{2}{d} g_{ab}\partial^c \epsilon_c = 0 \tag{A.20}$$

The action of such a diffeomorphism on a scalar field $\phi$ induces an infinitesimal transformation:

$$\delta\phi = \epsilon_a \partial^a \phi \tag{A.21}$$

By solving (A.20), we can find the different generators which enact the transformations (A.21). These are, explicitly [157]:

$$P^a = i\partial^a \tag{A.22}$$

$$M^{ab} = i(x^{(a}\partial^{b)}) \tag{A.23}$$

$$D = -ix^a \partial_a \tag{A.24}$$

$$K^a = -i(2x^a x_b \partial^b - x^2 \partial^a) \tag{A.25}$$

These generators satisfy the same algebra as (A.13)- (A.15) supplemented with:

$$[D, P^a] = iP^a \qquad [D, K^a] = -iK^a \tag{A.26}$$

$$[K^a, P^b] = 2i(\eta^{ab}D - M^{ab}) \qquad [K^c, M^{ab}] = i(\eta^{ca}K^b - \eta^{cb}K^a) \tag{A.27}$$

It turns out that this algebra is that of $SO(d-1, 2)$. Indeed, if we set:

$$\tilde{M}^{ab} \equiv M^{ab} \qquad \tilde{M}^{-1a} \equiv \frac{1}{2}(P^a - K^a) \tag{A.28}$$

$$\tilde{M}^{-10} \equiv D \qquad \tilde{M}^{0a} \equiv \frac{1}{2}(P^a + K^a) \tag{A.29}$$

then we have defined $\tilde{M}^{AB}$ with $A, B \in (-1, 0, \ldots, d-1)$ which satisfy (A.12) for the case of $SO(d-1, 2)$.

## A.3 Curvatures and identities

The Weyl tensor is defined by:

$$C^{abcd} = R^{abcd} + \frac{1}{(d-2)}\left(R^{ad}g^{bc} - R^{ac}g^{bd} + R^{bc}g^{ad} - R^{bd}g^{ac}\right) \tag{A.30}$$

$$+ \frac{1}{(d-2)(d-1)}R\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right) \tag{A.31}$$

It has the same symmetries as the Riemann tensor and is defined such that it is traceless: ${C^a}_{abc} = {C^a}_{bac} = 0$.

The Bach tensor is defined by by the variation of the Conformal Gravity action. In 4 dimensions it is given by:

$$\begin{aligned} B^{ab} &= -\frac{1}{4}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta^g_{ab}}\int \mathrm{d}^4x\, C^{abcd}C_{abcd} \\ &= R^{cd}{C^a}_c{}^b{}_d - R^{ac}{R^b}_c + \tfrac{1}{4}g^{ab}R_{cd}R^{cd} + \tfrac{1}{3}R^{ab}R - \tfrac{1}{12}g^{ab}R^2 \\ &- \tfrac{1}{6}D^{ab}R + \tfrac{1}{2}{D^c}_cR^{ab} - \tfrac{1}{12}g^{ab}{D^c}_cR \end{aligned} \tag{A.32}$$

One can check that it is traceless ${B^a}_a = 0$, and it can be expressed in in the more compact form:

$$B_{ab} = \left(D_cD_d + \frac{1}{2}R^{cd}\right)C_{acdb} \tag{A.33}$$

**Linearizations**

We write down the first order expansions for the following curvature tensors:

$$R_{ab} = -\tfrac{1}{2}\partial_{ab}{h^c}_c + \partial_{(a}{}^c h_{b)c} - \tfrac{1}{2}{\partial^c}_c h_{ab} + \mathcal{O}(h^2) \tag{A.34}$$

$$R = \partial_{ab}h^{ab} - {\partial^b}_b{h^a}_a + \mathcal{O}(h^2) \tag{A.35}$$

$$G_{ab} = -\tfrac{1}{2}\partial_{ab}{h^c}_c + \partial_{(a}{}^c h_{b)c} - \tfrac{1}{2}{\partial^c}_c h_{ab} - \tfrac{1}{2}\eta_{ab}\partial_{cd}h^{cd} + \tfrac{1}{2}\eta_{ab}{\partial^d}_d{h^c}_c + \mathcal{O}(h^2)$$

## A.4 TT projector

The Traceless transverse projector, $\mathcal{P}^{\alpha(s)}_{\beta(s)}$, is built out of products of $\mathcal{P}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - \frac{p^{\alpha}\, p_{\beta}}{p^2}$. The rank $s$ projector is:

$$\mathcal{P}^{\nu(s)}_{\mu(s)} = \sum_{l=0}^{[\frac{s}{2}]} \mathrm{a}_{s,l}\ M^{\nu(s-2l)}_{\mu(s-2l)}\, N^{\nu(2l)}_{\mu(2l)}\,, \qquad\qquad \mathrm{a}_{s,l} = \frac{(-1)^l s!\, \Gamma(s-l+\frac{1}{2})}{2^{2l}(s-2l)!\, l!\, \Gamma(s+\frac{1}{2})} \tag{A.36}$$

where :

$$M^{\nu(p)}_{\mu(p)} = \mathcal{P}^{(\nu_1}_{\mu_1} \dots \mathcal{P}^{\nu_p)}_{\mu_p} \tag{A.37}$$

$$N^{\nu(2q)}_{\mu(2q)} = \mathcal{P}_{(\mu_1\mu_2} \dots \mathcal{P}_{\mu_{q-1}\mu_q)}\, \mathcal{P}^{(\nu_1\nu_2} \dots \mathcal{P}^{\nu_{q-1}\nu_q)} \tag{A.38}$$

For example, the first few projectors are given by

$$\mathcal{P}^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \mathcal{P}^{\alpha_1}_{(\beta_1}\, \mathcal{P}^{\alpha_2}_{\beta_2)} - \tfrac{1}{3}\, \mathcal{P}^{\alpha_1\alpha_2}\mathcal{P}_{\beta_1\beta_2},$$
$$\mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\beta_1\beta_2\beta_3\beta_4} = \mathcal{P}^{(\alpha_1}_{(\beta_1} P^{\alpha_2}_{\beta_2} P^{\alpha_3}_{\beta_3} \mathcal{P}^{\alpha_4)}_{\beta_4)} - \tfrac{6}{7}\, P^{(\alpha_1\alpha_2}\mathcal{P}_{(\beta_1\beta_2} \mathcal{P}^{\alpha_3}_{\beta_3} \mathcal{P}^{\alpha_4)}_{\beta_4)} + \tfrac{3}{35}\, \mathcal{P}^{(\alpha_1\alpha_2}\mathcal{P}^{\alpha_3\alpha_4)}\mathcal{P}_{(\beta_1\beta_2}\, \mathcal{P}_{\beta_3\beta_4)}\,.$$

One can check explicitly that the trace of the TT projector, ${\mathcal{P}^{a(s)}}_{a(s)} = 2s+1$. In fact this is the dimension of symmetric representation of SO(3) - which are the same as traceless transverse representations of SO(4), see the discussion below (1.6).

# Appendix B

# Tools and formalism

## B.1 Spinor Helicity

Here we give a quick introduction to the formalism used in Section 2.3. We use the conventions of [158].

The formalism uses the fact $SL(2,\mathbb{C})$ is the double cover of $SO(3,1)$. Indeed, we can simply map from a vector valued in $SO(3,1)$ to a bispinor of $SL(2,\mathbb{C})$ using the usual van der Waerden matrices:

$$p_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} p_\mu = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} . \tag{B.1}$$

The determinant of the bispinor is related to the norm of the vector:

$$p^2 = -(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -\det(p_{\alpha\dot{\alpha}}) \tag{B.2}$$

If that vector is null, the rank of the matrix is reduced. This means it can be expressed in terms of two spinors:

$$k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} . \tag{B.3}$$

These bispinors are defined up to a scale:

$$\tilde{\lambda}_{\dot{\alpha}} \to t\tilde{\lambda}_{\dot{\alpha}} , \qquad \lambda_\alpha \to t^{-1}\lambda_\alpha \tag{B.4}$$

The power of this scaling is called the little group weight.
At this stage, it bears mentioning the “helicity” in “spinor helicity”. $SL(2,\mathbb{C})$ is

isomorphic to $SU(2)\times SU(2)$. This is why we use dotted and undotted spinors, they correspond to different $SU(2)$ factors. Furthermore, the representations of $SU(2)$ are labelled by positive half-integers. As such they account for the notion of positive and negative helicity helicities of $SO(3,1)$. Namely, in these conventions, dotted indices carry positive helicity while undotted ones carry negative helicity. More precisely, one can determine the helicity of an object by simply looking at its little group weight.

Raising and lowering is done via:

$$\tilde{\lambda}_{\dot{\beta}} = \tilde{\lambda}^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad \lambda_{\beta} = \lambda^{\alpha}\,\epsilon_{\alpha\beta}$$

Writing the dotted and undotted spinors as square and angle bracket spinors respectively:

$$\tilde{\lambda}^{\dot{a}} \equiv [\lambda|^{\dot{a}}\tilde{\lambda}_{\dot{a}} \equiv |\lambda]_{\dot{a}}\,, \qquad \lambda^{a} \equiv \langle\lambda|^{a}\,, \qquad \lambda_{a} \equiv |\lambda\rangle_{a}\,, \tag{B.5}$$

and we write the contractions:

$$[\tilde{\lambda}|^{\dot{\alpha}}|\tilde{\beta}]_{\dot{\alpha}} \equiv [\tilde{\lambda}\tilde{\beta}]\,, \qquad \langle\lambda|^{a}|\beta\rangle_{\alpha} \equiv \langle\lambda\beta\rangle\,. \tag{B.6}$$

From the antisymmetry of $\epsilon_{ab}$,$\epsilon_{\dot{a}\dot{b}}$ we get:

$$\langle ij\rangle = -\langle ji\rangle\,, \qquad [ij] = -[ji]\,, \qquad \langle\lambda\lambda\rangle = [\tilde{\lambda}\tilde{\lambda}] = 0\,. \tag{B.7}$$

## B.2 Wigner quantization

In this appendix, we give a brief introduction to the methods of Weyl/Wigner quantization. For a slightly more in-depth and general review, see for example Appendix A of [138] of [126] [62].

The main idea is to use maps between symmetric operators and functions of an auxiliary variable. To do so, first introduce the Weyl map, which associates the

[62]For the original works, see [159–161].

operator $\hat{F}$ to the Fourier-transformed function $\mathcal{F}$

$$\hat{F} = \int \frac{\mathrm{d}^d k\, \mathrm{d}^d y}{(2\pi)^d}\, \mathcal{F}(k,y) e^{\mathrm{i}(k\cdot\hat{X} - y\cdot\hat{P})}\,, \tag{B.8}$$

with:

$$\mathcal{F}(k,y) = \int \frac{\mathrm{d}^d x\, \mathrm{d}^d p}{(2\pi)^d}\, f(x,p)\, e^{\mathrm{i}(k\cdot x - y\cdot p)}\,, \tag{B.9}$$

and $f(x,p)$ can interpreted as generating functions defined in Appendix A.1. One can show that there exists an inverse map, called the Wigner map, which goes back from $f(x,p)$ to the operator $\hat{F}$ defined via:

$$f(x,p) = \int \mathrm{d}^d q\, \langle x - q/2|\, \hat{F}\, |x + q/2\rangle\, e^{\mathrm{i}q\cdot p}\,. \tag{B.10}$$

We call the function $f(x,p)$ the "Weyl symbol " of $\hat{F}$, and we can use these maps to switch between both descriptions.

In this formalism, one needs to define a product, $\star$, between Weyl symbols, which is equivalent to operator action. This is done by computing the pull back of this operator action and is known as the "Moyal star product". Explicitly, the symbol of $\hat{F}\hat{G}$ is $f \star g$ with:

$$f(x,p) \star g(x,p) = \exp\left[\frac{\mathrm{i}}{2}(\partial_{x_1}\cdot\partial_{p_2} - \partial_{x_2}\cdot\partial_{p_1})\right] f(x_1,p_1) g(x_2,p_2)\Big|_{x_i = x, p_i = p}\,. \tag{B.11}$$

In the language of Weyl symbols (anti)-commutators of operators are translated to their Moyal counterparts:

$$[f \stackrel{\star}{,} g] \equiv f\star g - g \star p\,, \qquad \{f \stackrel{\star}{,} g\} \equiv f\star g + g\star p\,. \tag{B.12}$$

Finally, the trace of an operator takes a simple form in Weyl symbol formalism. Indeed, one can show: [63]

$$\mathrm{Tr}\left[\hat{F}\right] \equiv \langle x|\hat{F}|x\rangle = \int \frac{\mathrm{d}^d x\, \mathrm{d}^d p}{(2\pi)^d}\, f(x,p)\,. \tag{B.13}$$

[63]Here one needs to make use of the standard relation $\langle x|e^{\mathrm{i}\,y\cdot\hat{P}}|p\rangle = \int \frac{\mathrm{d}^d p}{(2\pi)^d} e^{\mathrm{i}y\cdot p}$ and the Baker-Campbell-Hausdorff formula on (B.8) .

In particular, the trace of two operators simplifies:

$$\mathrm{Tr}\left[\hat{F}\hat{G}\right] = \int \frac{\mathrm{d}^d x\, \mathrm{d}^d p}{(2\pi)^d}\, f(x,p) \star g(x,p) \tag{B.14}$$

$$= \int \frac{\mathrm{d}^d x\, \mathrm{d}^d p}{(2\pi)^d}\, f(x,p)\, g(x,p)\,, \tag{B.15}$$

where we get to the second line by using the fact that all the higher terms in the start product are total derivatives. Finally, consider the "p-Fourier transformed" function $\hat{f}$:

$$\hat{f}(x,u) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, f(x,p)\, e^{\mathrm{i}\, u\cdot p}\,, \tag{B.16}$$

Now consider the contraction:

$$\hat{f}(x,-\mathrm{i}\partial_u)\, g(x,u)\Big|_{u=0} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, f(x,p)\, e^{\partial_u \cdot p} g(x,u)\Big|_{u=0}\,, \tag{B.17}$$

$$= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, f(x,p)\, g(x,p)\,, \tag{B.18}$$

where in the last line used that $g(x,u)$ is just a generating function defined, as defined in Appendix A.1. So in conclusion, we see that a trace can be written as the simple contraction:

$$\mathrm{Tr}\left[\hat{F}\hat{G}\right] = \int \mathrm{d}^d x\, \hat{f}(x,-\mathrm{i}\partial_u)\, g(x,u)\Big|_{u=0} \tag{B.19}$$

In particular, in section 3.1, we have the bra and ket vectors associated with the scalar fields, $\phi_i(x) \equiv \langle x|\phi\rangle$. We consider the operator $|\phi\rangle\langle\phi|$ (summation over the index $i$ is implied), whose associated symbol is found to be:

$$\rho(x,p) = \int \mathrm{d}^d u\, \langle x - u/2|\phi\rangle\langle\phi|x + u/2\rangle\, e^{\mathrm{i}u\cdot p}$$

$$= \int \mathrm{d}^d u\, \phi_i^*(x - u/2)\, \phi_i(x + u/2)\, e^{-\mathrm{i}u\cdot p}\,. \tag{B.20}$$

This defines the Fourier transform $\hat{\rho}(x,u) \equiv \phi_i^*(x - u/2)\, \phi_i(x + u/2)$. For convenience, we define the set of traceful currents $\mathfrak{J}(x,u) \equiv \hat{\rho}(x, -\mathrm{i}\, u)$, which allows us to write, using (B.19):

$$\mathrm{Tr}\left[|\phi\rangle\langle\phi|\hat{H}\right] = \int \mathrm{d}^d x\, \mathfrak{I}(x,\partial_u)\, \mathfrak{h}(x,u)\bigg|_{u=0} . \tag{B.21}$$

# B.3 Identities

## B.3.1 Symmetrisation of Derivatives

When performing the exercise of exhaustively listing all possible terms of a given form, it becomes extremely useful to systematically symmetrize all covariant derivatives. Indeed, since anticommuting two derivatives yields a curvature tensor, symmetrising all derivatives greatly helps in identifying inequivalent tensor contractions. This process is used continuously in the computations of Section 6.1.2.

## B.3.2 Dimensionally-Dependent Identities (DDIS)

When one writes down all possible forms of a tensorial expressions, it turns out that some of them may be related to each other in particular dimensions. Indeed, in $d$ dimensions, it is obvious that $A^{a_1 b_1 \ldots b_k a_k} \equiv g^{[a_1 b_1} \ldots g^{b_k a_k]}$ vanishes for $d < 2k$. Then given any tensorial expression with $k$ free indices, $T^{b_1 \ldots b_k}$, we can simply contract it with $A^{a_1 b_1 \ldots b_k a_k}$ to obtain an identity.
In practice, it is also useful to compute contractions of these identities. This is implemented using the `xTras` package [154]. For example, the identity:

$$C^{acde} C^{b}{}_{cde} = \tfrac{1}{4} g^{ab} C_{cdef} C^{cdef} , \tag{B.22}$$

which is valid only in 4-dimensions, can be found this way.

## B.3.3 Generalised Bianchi identities

When dealing with derivatives acting on curvature tensors, there exist multiterm symmetries coming from Bianchi identities. These can simply be enforced by projecting expressions of Weyl tensors and their derivatives on the corresponding Weyl Tableau.

For instance, the Weyl tensor $C_{abcd}$ can be projected on its young tableau $\begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array}$ by successively symmetrising indices in the same row and anti-symmetrising indices in the same column with a particular normalisation (see e.g. [154] [162]). The result for this case is[64]:

$$P^{\begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array}} C_{abcd} = \frac{2}{3} C_{acbd} + \frac{1}{3} C_{abcd} - \frac{1}{3} C_{adbc} \tag{B.23}$$

Doing this is useful because this makes the Bianchi identity $C_{[abc]d} = 0$ manifest.

In general, if we are given the an expression containing $D_{(a_1 \ldots a_k)} C_{bcde}$, we can obtain the corresponding generalised Bianchi identities by projecting it on the Young Tableau: $\begin{array}{|c|c|c|c|c|}\hline b & d & a_1 & \ldots & a_k \\ \hline c & e \\ \cline{1-2}\end{array}$

### B.3.4 Standard Loop Integrals

Here we introduce some useful identities for dealing with the standard loop integral in the paper. First we have the Feynman parametrisation:

$$\frac{1}{A_1 \ldots A_n} = (n-1)! \int_{[0,1]^n} d^n x \, \frac{\delta(x_1 + \cdots + x_n - 1)}{(x_1 A_1 + \cdots + x_n A_n)^n} \,. \tag{B.24}$$

Next, the following integral can be obtained just from Lorentz invariance, in $d = 4$ dimensions

$$I_s^{a_1 \ldots a_s} = \int \mathrm{d}^d k \, f(k^2) k^{a_1} \ldots k^{a_s} = A_s \, g^{(a_1 a_2} \ldots g^{a_{s-1} a_s)} \int \mathrm{d}^d k \, (k^2)^{s/2} f(k^2) \tag{B.25}$$

$$A_s = \frac{(s-1)!!}{2^{s/2} (\frac{l}{2} + 1)!} \tag{B.26}$$

for even $s$ - the integral vanishes for odd $s$. Above we used the double factorial, defined as $s!! \equiv s(s-2)(s-4)\ldots$.
Finally, we use the following integral for dimensional regularisation:

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^a}{(k^2 + M^2)^b} = \frac{\Gamma(b - a - d/2)\Gamma(a + d/2)}{(4\pi)^{d/2} \Gamma(b) \Gamma(d/2)} \, (M^2)^{d/2 + a - b} \,, \tag{B.27}$$

[64] We use the manifestly antisymmetric convention.

# Appendix C

# Computational Details

## C.1 Helicity structure of conformal graviton modes

In this appendix, we will fill in some of the computational details of 2.2.2. To do so, we will go to the frame where the momentum is in the 3 direction: $k^a = (\omega, 0, 0, \omega)$ and choose $n^a$ to be the unit time-like vector $n^a = (1, 0, 0, 0)$. Since our goal is to discern the helicity of the propagating modes, we first introduce the helicity basis. We then proceed to fix the remnant part of the gauge both in the conformal and TT gauges, and in each case express the result in the helicity basis.

#### Helicity Basis

Helicity is given by the behaviour of under rotations in the plane transverse to momentum. Since we picked $k^a$ to be in the 3 direction, this means that we look at how tensors $M$ behave under $\mathbf{R}\, M\, \mathbf{R}^T$, where $\mathbf{R}$ is given by:

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

To this end, we introduce the helicity basis tensors:

$$T^{\pm\pm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad T^{\pm} = \begin{pmatrix} 0 & 1 & \pm i & 0 \\ 1 & 0 & 0 & 0 \\ \pm i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad \tilde{T}^{\pm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \pm i \\ 0 & 1 & \pm i & 0 \end{pmatrix} .$$

which are chosen to satisfy:

$$\mathbf{R}T^{\pm\pm}\mathbf{R}^T = e^{\pm 2i\theta}T^{\pm\pm}\,, \qquad \mathbf{R}T^{\pm}\mathbf{R}^T = e^{\pm i\theta}T^{\pm}\,, \qquad \mathbf{R}\tilde{T}^{\pm}\mathbf{R}^T = e^{\pm i\theta}\tilde{T}^{\pm}\,,$$

meaning that $T^{\pm}$ and $\tilde{T}^{\pm}$ span helicity $\pm 1$ tensors and $T^{\pm\pm}$ helicity $\pm 2$ tensors.

### Conformal Gauge

In the conformal gauge, one needs to fix the residual transformations (2.41) and (2.42). One possible choice, made in [82] is to fix:

$$\begin{aligned} A^{11}+A^{22}&=0 & B^{11}+B^{22}&=0 & A^{03}&=0 & A^{13}&=0 \\ B^{23}&=0 & B^{03}&=0 & B^{13}&=0 & B^{23}&=0 \end{aligned} \tag{C.1}$$

Given those, the conditions (2.38) impose:

$$A^{00}=0 \qquad B^{00}=0 \qquad B^{01}=0 \qquad B^{02}=0 \tag{C.2}$$

Finally, we have the 6 DoFs left are arranged as:

$$A^{ab} = \begin{pmatrix} 0 & A^{01} & A^{02} & 0 \\ A^{01} & A^{11} & A^{12} & 0 \\ A^{02} & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad B^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B^{11} & B^{12} & 0 \\ 0 & B^{12} & -B^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.3}$$

In the helicity basis, these become

$$\begin{aligned} \mathrm{A}^{ab} &= \left(A^{++}T^{++}+A^{--}T^{--}\right)^{ab} + \left(A^{+}T^{+}+A^{-}T^{-}\right)^{ab}\,, \\ \mathrm{B}^{ab} &= \left(B^{++}T^{++}+B^{--}T^{--}\right)^{ab}\,, \end{aligned} \tag{C.4}$$

So we see that the oscillatory part of the field contains spin 1 and spin 2 modes, while the growing part is purely helicity 2.

### TT gauge

We repeat the analysis for the TT gauge. This time, only 4 conditions can be fixed from (2.46). In this frame, one can show that these conditions can be fixed:

$$B^{00}=0\,, \qquad A^{00}=0 \tag{C.5}$$

$$A^{01}+A^{13}=0\,, \qquad A^{02}+A^{23}=0 \tag{C.6}$$

The gauge conditions 2.43 then give use the following relations:

$$
\begin{array}{llll}
A^{22} = -A^{11} & B^{00} = 0 & B^{01} = -2iwA^{01} & B^{02} = -2iwA^{02} \\
B^{03} = 0 & B^{13} = -2iwA^{01} & B^{22} = -B^{11} & B^{23} = -2iwA^{02}
\end{array}
$$

So that the polarization tensors become

$$
A^{ab} \propto \begin{pmatrix} 0 & A^{01} & A^{02} & 0 \\ A^{01} & A^{11} & A^{12} & -A^{01} \\ A^{02} & A^{12} & -A^{11} & -A^{02} \\ 0 & -A^{01} & -A^{02} & 0 \end{pmatrix}, \quad B^{ab} \propto \begin{pmatrix} 0 & -2iwA^{01} & -2iwA^{02} & 0 \\ -2iwA^{01} & B^{11} & B^{12} & -2iwA^{01} \\ -2iwA^{02} & B^{12} & -B^{11} & -2iwA^{02} \\ 0 & -2iwA^{01} & -2iwA^{02} & 0 \end{pmatrix}
$$

In the helicity basis, this becomes:

$$
\begin{aligned}
\mathrm{A}^{ab} &= \left(A^{++}T^{++} + A^{--}T^{--}\right)^{ab} + \left(A^{+}(T^{+} - \tilde{T}^{+}) + A^{-}(T^{-} - \tilde{T}^{-})\right)^{ab}, \\
\mathrm{B}^{ab} &= \left(B^{++}T^{++} + B^{--}T^{--}\right)^{ab} - 2i\omega \left(A^{+}(T^{+} + \tilde{T}^{+}) + A^{-}(T^{-} + \tilde{T}^{-})\right)^{ab}.
\end{aligned}
\tag{C.7}
$$

So we see that in this gauge, there are oscillatory and growing spin 2 modes, while the spin 1 modes seem to be split between both. The conclusion is that the helicity decomposition of $h_{ab}$ is gauge-dependent.

We note here for completeness, that it is possible to fix the remnant part of the conformal gauge such that $k_a A^{ab} = k_a B^{ab} = 0$, which leads to having the helicity 1 becoming purely growing.

## C.2 Vertices in CHS action from scalar loop integrals

In this Appendix, we fill in the details of the loop computations done in Section 3.3.

### Quadratic Sector

To determine the quadratic sector of the CHS action, we need to compute the UV divergence of the loop diagram (3.46). For spin $s$ insertions, it is given by

$\tilde{h}^{a(s)}(p)\tilde{h}^{b(s)}(-p)I(p)_{a(s),b(s)}$ with:

$$\begin{aligned}
I(p)_{a(s),b(s)} &\equiv \int \frac{d^dk}{(2\pi)^d}\frac{N(k,p)_{a(s)b(s)}}{k^2\,(k+p)^2} \\
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{N(k,p)_{a(s)b(s)}}{[(k+x\,p)^2+x(1-x)\,p^2]^2} \\
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{N(k-x\,p,p)_{a(s)b(s)}}{(k^2+M^2)^2}, \qquad\qquad M^2 = x(1-x)\,p^2\,,
\end{aligned} \tag{C.8}$$

where the numerator can be read off from (3.46) to be
$N(k,p)_{a(s)b(s)} = \frac{1}{(s!)^2}k_{a_1}k_{b_1}\dots k_{a_s}k_{b_s}$. Remembering that this amplitude will be contracted with CHS fields (carrying momentum $p$) which are in the TT gauge, we can write the shifted numerator to be:

$$N_{a(s)b(s)}(k-px,p) \leftrightarrow \tfrac{1}{(s!)^2}k_{a_1}k_{b_1}\dots k_{a_s}k_{b_s}\,, \tag{C.9}$$

where $\leftrightarrow$ means equivalence under contraction with spin $s$ symmetric TT fields. Finally, using Lorentz covariance of the integral $I_{a(s),b(s)}(p)$, the numerator becomes:

$$N_{a(s)b(s)}(k-px,p) \leftrightarrow\rightarrow \tfrac{1}{(s!)^2}\tfrac{1}{2^s(s+1)}(k^2)^s\,\eta_{a_1b_1}\cdots\eta_{a_sb_s}\,. \tag{C.10}$$

Finally, performing the integral, we can get the quadratic action (cf. (3.39)):

$$SS_2[h_s] = \tfrac{(-1)^s}{2^s\,\Gamma(2s+2)}\int\frac{d^4p}{(2\pi)^4}\,h_{a(s)}(p)(p^2)^s\,h^{a(s)}(-p)\,, \tag{C.11}$$

which is just the momentum space version of (3.48).

### Cubic Sector

Next, we must compute diagrams of the form (4.55). The corresponding integral is given by:

$$I_{a,b,c(s)}(p_1,p_2) = \frac{1}{2!}\frac{1}{s!}\int\frac{d^dk}{(2\pi)^d}\frac{k_a(k+p_1)_b(k+p_1+p_2)_{c(s)}}{k^2(k+p_1)^2(k+p_1+p_2)^2}\,. \tag{C.12}$$

Using Feynman parametrisation (B.24) and shifting the loop momentum $k$ we obtain:

$$
\begin{aligned}
I_{a,b,c(s)}(p_1,p_2) &= \frac{1}{2\,s!}\int \frac{d^dk}{(2\pi)^d}\frac{k_a k_{c(s)}(k+p_1)_b}{k^2(k+p_1)^2(k+p_1+p_2)^2} \\
&= \frac{1}{s!}\int_0^1 dx \int_0^{1-x} dy \int \frac{d^dk}{(2\pi)^d}\frac{k_a k_{c(s)}(k+p_1)_b}{[(k+xp_1+y(p_1+p_2))^2+M^2]^3} \\
&\leftrightarrow \frac{1}{s!}\int_0^1 dx \int_0^{1-x} dy \int \frac{d^dk}{(2\pi)^d}\frac{(k-yp_2)_a(k-xp_1)_{c(s)}(k+(1-x-y)p_1)_b}{(k^2+M^2)^3} \,,
\end{aligned}
\tag{C.13}
$$

$$
M^2 = x(1-x)p_1^2 + y(1-y)(p_1+p_2)^2 - 2xyp_1\cdot(p_1+p_2)\,, \tag{C.14}
$$

where once again, we make use of the fact that this is to be contracted with symmetric TT fields. Here, we separate terms of different order in $k$ and use Lorentz covariance and integrate over $x$ and $y$ (cf. (B.25) and (B.27)). The resulting pole part is:

$$
\begin{aligned}
I_{a,b,c(s)}(p_1,p_2)\Big|_{\frac{1}{4\pi^2\varepsilon}} = \tfrac{1}{2(s+2)!}\Big\{&\eta_{ab}(p_1)_{c(s)} - \eta_{ac_1}p_{1b}p_{1c_2}\cdots p_{1c_s} + \eta_{bc_1}p_{2a}p_{1c_2}\cdots p_{1c_s} \\
&- \eta_{ac_1}\eta_{bc_2}\,p_{1c_3}\cdots p_{1c_s}\big[p_1\cdot p_2 + \tfrac{s}{2}\,(p_1^2+p_2^2)\big]\Big\}\,.
\end{aligned}
\tag{C.15}
$$

There is actually another integral contributing at the same level, with the loop running in reverse. As such, the full $1-1-s$ vertex is given by:

$$
\mathrm{V}_{a,b,c(s)}(p_1,p_2) = I_{a,b,c(s)}(p_1,p_2) + I_{b,a,c(s)}(p_2,p_1)\Big|_{\frac{1}{4\pi^2\varepsilon}}\,. \tag{C.16}
$$

This indeed yields the vertex given in (3.56) (when translated to position space).

The $2-2-s$ vertex is computed through the integral

$$
\begin{aligned}
I_{a_1a_2,b_1b_2,c(s)} &= \frac{1}{2!}\frac{1}{(2!)^2}\frac{1}{s!}\int\frac{d^dk}{(2\pi)^d}\frac{k_{a_1}k_{a_2}(k+p_1)_{b_1}(k+p_1)_{b_2}(k+p_1+p_2)_{c(s)}}{k^2(k+p_1)^2(k+p_1+p_2)^2} \\
&\leftrightarrow \frac{1}{4\,s!}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{k_{a_1}k_{a_2}k_{c(s)}(k+p_1)_{b_1}(k+p_1)_{b_2}}{[(k+xp_1+y(p_1+p_2))^2+M^2]^3} \\
&\leftrightarrow \frac{1}{4\,s!}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{N_{a_1a_2,b_1b_2,c(s)}(p_1,p_2,k;x,y)}{(k^2+M^2)^3}\,,
\end{aligned}
\tag{C.17}
$$

where $M^2$ is the same as in (C.14) and

$$
N_{a_1a_2,b_1b_2,c(s)} = (k-yp_2)_{a_1}(k-yp_2)_{a_2}(k-xp_1)_{c(s)}[k+(1-x-y)p_1]_{b_1}[k+(1-x-y)p_1]_{b_2}
$$

Here, the UV pole gets contributions from terms which are of order $k^2, k^4, k^6$ and $k^8$ in the numerator - terms with higher powers of $k$ will lead to traces of the spin $s$ field, and can be discarded. Once again, adding the diagram with the opposite loop orientation, one can get the vertex (3.57)

Finally, we look at vertices involving the CHS scalar $h_0$, which is non propagating (since it has no derivatives in its kinetic term). The relevant loop diagram is given by:

$$
I_{a,c(s)}(p) = \frac{4}{s!}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{(k-yp)_a(k+xp)_{c(s)}}{(k^2+M^2)^3}\,. \tag{C.18}
$$

This yields the vertex given in (3.58). Next, for $2-0-s$, the diagram is:

$$
\begin{aligned}
V_{a_1a_2,c(s)} &= \frac{1}{s!}\int\frac{d^dk}{(2\pi)^d}\frac{k_{a_1}k_{a_2}k_{c(s)}}{k^2(k+p_1)^2(k+p_1+p_2)^2} \\
&\to \frac{2}{s!}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{k_{a_1}k_{a_2}k_{c(s)}}{[(k+xp_1+y(p_1+p_2))^2+M^2]^3} \\
&\to \frac{2}{s!}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{(k-yp)_{a_1}(k-yp)_{a_2}(k+xp)_{c(s)}}{(k^2+M^2)^3}\,,
\end{aligned}
\tag{C.19}
$$

where $M^2$ is as in (C.14). This leads to the vertex (3.59). The final computation, for the $1-2-s$ vertex is follows along the same lines and leads to (3.60).

#### Quartic vertices

For the quartic vertices, one needs to compute diagrams of the form (4.57). For the 1111 diagram, the relevant integral is

$$\begin{aligned}&\tfrac{1}{4!}\times 6\times 3!\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz\int\frac{d^dk}{(2\pi)^d}\frac{k_a k_b k_c k_\sigma}{(k^2+M^2)^4}\\ \leftrightarrow\ &\tfrac{1}{16\pi^2\epsilon}\,\tfrac{1}{48}\left(\eta_{ab}\eta_{c\sigma}+\eta_{ac}\eta_{b\sigma}+\eta_{a\sigma}\eta_{\nu c}\right).\end{aligned} \tag{C.20}$$

which gives the vertex (4.57) when contracted with $h_{\mu_1\mu_2}(p_1)h_{\nu_1\nu_2}(p_2)h_{\rho(s)}$ and translated to position space . As explained in 3.3.4, the others can simply be obtained by expanding Scalar-Maxwell-Weyl action and performing the right field redefinitions.

## C.3 Cubic and quartic vertices in the CHS action involving constant $h_0$ field

Here we provide details as they were originally presented in [1], for computations done in Section 4.3. In [126], the CHS action is computed via heat kernel techniques. This was done for the dressed fields $\mathfrak{h}$ which are related to $h(x,u)$ via $\mathfrak{h}(x,u)=\Pi_d(u,\partial_x)h(x,u)$ where $\Pi_d$ was defined in (3.30) which we rewrite for convenience:

$$\Pi_d(u,\partial_x)=\sum_{n=0}^{\infty}\frac{1}{n!\,(-u\cdot\partial_u-\frac{d-5}{2})_n}\left(\frac{u^2\,\partial_x^2-(u\cdot\partial_x)^2}{16}\right)^n, \tag{C.21}$$

Two observations are in order. First, one can check explicitly that the scalar CHS field is the same in both bases, $\mathfrak{h}_0(x)=h_0(x)$. Above, one can see that if $h(x,u)$ are in the TT gauge, only the $n=0$ term survives, and both dressed and undressed fields are equivalent. As such, while we continue to use $\mathfrak{h}$ in this appendix to make ground with [126], they are actually interchangeable.

Schematically, the partition for a complex operator in a background of CHS fields $\mathfrak{h}$ is expanded as:

$$\mathrm{Tr}\left[e^{-t\left(\hat{p}^2+\hat{\mathfrak{h}}\right)}\right]=\sum_{n=0}^{\infty}t^{n-2}\,a_n[\mathfrak{h}]\,, \tag{C.22}$$

As explained in Section 3.2, in $d=4$ the local action is proportional to the logarithmically divergent term

$$S_{\text{CHS}}[\mathfrak{h}] \propto a_2[\mathfrak{h}]\,. \tag{C.23}$$

Since $h_0$ is constant, let us separate its dressed cousin from the other fields by writing:

$$\mathfrak{h}(x,u) = \mathfrak{h}_0(x) + \mathfrak{h}'(x,u)\,. \tag{C.24}$$

Using the fact that $\mathfrak{h}_0$ is constant, we can re-expand (C.22) as a power series in $\mathfrak{h}_0$ first, and find (by matching powers of $t$ ):

$$a_n[\mathfrak{h}] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\,(\mathfrak{h}_0)^m\, a_{n-m}[\mathfrak{h}']\,. \tag{C.25}$$

In particular,

$$a_2[\mathfrak{h}] = a_2[\mathfrak{h}'] - \mathfrak{h}_0\, a_1[\mathfrak{h}'] + \frac{1}{2}\,(\mathfrak{h}_0)^2\, a_0[\mathfrak{h}'] + \mathcal{O}(\mathfrak{h}_0^3)\,. \tag{C.26}$$

Expressions for these heat kernel coefficients were given, up to quadratic order, in [126]:

$$\begin{aligned} a_{2+m}[\mathfrak{h}] \;&=\; \int \frac{d^4x}{(4\pi)^2}\sqrt{\tfrac{\pi}{8}}\left(\tfrac{1}{2}\partial^2_{x_{12}}\right)^m U_{m+\frac{1}{2}}\left((\partial_{x_{12}}\cdot\partial_{u_{12}})^2 - \partial^2_{x_{12}}\partial^2_{u_{12}}\right) \\ &\qquad\times \mathfrak{h}(x_1,u_1)\,\mathfrak{h}(x_2,u_2)\Big|_{\substack{x_1=x_2=x\\ u_1=u_2=0}} + \mathcal{O}(\mathfrak{h}^3)\,, \end{aligned} \tag{C.27}$$

$$\begin{aligned} a_{1-m}[\mathfrak{h}] \;&=\; \int \frac{d^4x}{(4\pi)^2}\Big[\delta_{m,1} + \left(\tfrac{1}{4}\,\partial_u^2\right)^m \mathfrak{h}(x,u)\Big|_{u=0} \\ &\quad+ \sqrt{\tfrac{\pi}{8}}\,V_m(\partial_{x_{12}},\partial_{u_{12}})\,\mathfrak{h}(x_1,u_1)\,\mathfrak{h}(x_2,u_2)\Big|_{\substack{x_1=x_2=x\\ u_1=u_2=0}} + \mathcal{O}(\mathfrak{h}^3)\Big]\,, \end{aligned} \tag{C.28}$$

where

$$V_m(\partial_x,\partial_u) \;=\; \left(\tfrac{1}{4}\,\partial_u^2\right)^{m+1} \textstyle\sum_{k=0}^{\infty} \frac{\left(\frac{1}{8}\,\partial_x^2\,\partial_u^2\right)^k}{\Gamma(k+m+2)}\, U_{k+\frac{1}{2}}\left((\partial_x\cdot\partial_u)^2\right)\,, \tag{C.29}$$

$$U_\nu(z) \;=\; \left(\tfrac{\sqrt{z}}{2}\right)^{-\nu} J_\nu\left(\tfrac{\sqrt{z}}{2}\right) = \textstyle\sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+m+1)\,2^\nu}\left(-\frac{z}{16}\right)^m\,. \tag{C.30}$$

As a result, in momentum space and in the TT gauge, the CHS action with constant $\mathfrak{h}_0$ is given by:

$$\tilde{\mathcal{L}}_{\text{CHS}}[\mathfrak{h}] \;\propto\; \sum_{s=0}^{\infty} \Big[1 - \frac{4}{p^2}\left(s+\frac{1}{2}\right)\tilde{\mathfrak{h}}_0(0) + \frac{8}{p^4}\left(s+\frac{1}{2}\right)\left(s-\frac{1}{2}\right)\left(\tilde{\mathfrak{h}}_0(0)\right)^2 + \mathcal{O}\left(\tilde{\mathfrak{h}}_0^3\right)\Big] \\ \times \frac{\left(p^2\right)^s \tilde{\mathfrak{h}}_s(p,\partial_u)\,\tilde{\mathfrak{h}}_s(-p,u)}{2^{3s}\,\Gamma(s+\frac{3}{2})} + \mathcal{O}\left(\tilde{\mathfrak{h}}'^3\right), \tag{C.31}$$

In this expression, the non-local terms with negative powers of $p^2$ should be dropped.

## C.4 Gauge fixing and ghost action

Here we fill in some details for Section 4.4.2, namely we compute the action of the ghost which arises after fixing the TT gauge.[65] In this appendix, we will use the dressed field formalism, as the action of the CHS symmetry on it is simpler, cf. (3.26), (3.27). As mentioned in Appendix C.3, in any gauge we have that $\mathfrak{h}_0(x) = h_0(x)$ and in the TT gauge, both bases are equivalent, ie. $\mathfrak{h}(x,u)|_{\text{TT}} = h(x,u)|_{\text{TT}}$.

We are interested in keeping the $h_0$ field as a background field, while the others will be integrated over. The part of the CHS gauge symmetry (3.26) and (3.27) which is independent of the (non-background) fields is then:

$$\delta\,\mathfrak{h}(x,u) = u\cdot\partial_x\,\mathfrak{e}(x,u) + \left(u^2 - \tfrac{1}{4}\,\partial_x^2 + h_0\right)\mathfrak{a}(x,u)\,, \tag{C.32}$$

where we take the fields $\mathfrak{h}$ are doubly traceless while $\mathfrak{a}$ and $\mathfrak{e}$ are traceless. We first use $\mathfrak{a}$ to gauge away the trace of $\mathfrak{h}$ . We thus impose $\partial_u^2(\mathfrak{h}+\delta\,\mathfrak{h}) = 0$ to get the relation:

$$\mathfrak{a}(x,u) = -\frac{1}{2(2+u\cdot\partial_u)}\,\partial_x\cdot\partial_u\,\mathfrak{e}(x,u)\,, \tag{C.33}$$

where we note that the operator $u\cdot\partial_u$ on the denominator simply counts the degree in $u$ of everything to its right. The field now transforms with the remnant symmetry, $\delta\mathfrak{h} \equiv T(h_0,\mathfrak{e})$ where:

$$T(h_0,\mathfrak{e})(x,u) = \mathrm{P}_{\mathrm{T}}\Big(u\cdot\partial_x + \frac{\frac{1}{4}\,\partial_x^2 - h_0}{2(2+u\cdot\partial_u)}\,\partial_u\cdot\partial_x\Big)\mathfrak{e}(x,u)\,, \tag{C.34}$$

[65] For details on fixing a different gauge, and the ghost action which comes with it, see [163].

where $\mathrm{P}_{\mathrm{T}} \equiv 1 - \frac{u^2(\partial_u)^2}{4(s-2)+2d+u^2(\partial_u)^2}$ defines the projector onto symmetric traceless spaces in $d$ dimensions.

We now use this remnant gauge to impose transversality. Following the usual Faddeev-Popov procedure, this is accompanied by the appearance of a ghost action:

$$\begin{aligned} S_{gh} &= \int d^4x \, \langle \bar{c} \,|\, \partial_x \cdot \partial_u \, \frac{\delta T(h_0, \mathfrak{e})}{\delta \, \mathfrak{e}} \,|\, c \rangle \\ &= \int d^4x \sum_{s=0}^{\infty} \langle \bar{c}_s |\, \partial_x \cdot \partial_u \, \mathrm{P}_{\mathrm{T}} \Big( u \cdot \partial_x | c_s \rangle + \frac{\frac{1}{4}\,\partial_x^2 - h_0}{2(s+3)} \, \partial_u \cdot \partial_x \, | c_{s+2} \rangle \Big) . \end{aligned} \tag{C.35}$$

where $c(x,u)$ and $\bar{c}(x, U0$ are the generating functions for the ghosts and anti-ghosts and $\langle a|b\rangle$ are contractions, see Appendix A.1. From the fact that $\mathfrak{e}$ is traceless, it follows that the ghost and anti-ghosts are both traceless as well.

We can proceed by decomposing the spin $s$ ghost into TT components, as:

$$c_s(x,u) = \mathrm{P}_{\mathrm{T}} \sum_{r=0}^{s} (u \cdot \partial_x)^{s-r} \, c_{s,r}(x,u) \,, \qquad \partial_u^2 \, c_{s,r} = 0 = \partial_x \cdot \partial_u \, c_{s,r} \,. \tag{C.36}$$

Plugging this into (C.35), one can see that the first two comopnents $c_{s+2,s+2}$ and $c_{s+2,s+1}$ of $c_{s+2}$ will drop out due to the presence of two divergence factors $(\partial_u \cdot \partial_x)$. One obtains:

$$S_{gh} = \int d^4x \sum_{s=0}^{\infty} \sum_{r=0}^{s} \langle \bar{c}_s |\, \partial_x \cdot \partial_u \, \mathrm{P}_{\mathrm{T}} \, (u \cdot \partial_x)^{s+1-r} \Big( |c_{s,r}\rangle + k_{s,r} (\tfrac{1}{4}\,\partial_x^2 - h_0) \partial_x^2 \, |c_{s+2,r}\rangle \Big) \,, \tag{C.37}$$

with:

$$k_{s,r} = \frac{(s-r+2)(s+r-3)}{4(s+2)(s+3)} \,. \tag{C.38}$$

This form of the ghost action allows us to see explicitly that it is possible to redefine away the $h_0$ dependence, by simply making the field redefinition:

$$c'_{s,r} = c_{s,r} + k_{s,r} (\tfrac{1}{4}\,\partial_x^2 - h_0) \partial_x^2 \, c_{s+2,r} \,. \tag{C.39}$$

For a fixed value of $r$, one can view this as an infinite dimensional matrix acting on a matrix of ghost fields arranged by spin: $(c_{0,r}\,, \, c_{1,r}\,, \ldots)^T$. The form of (C.39) tells us that this "matrix" is the identity matrix plus some upper triangular components. The Jacobian of such a matrix can be seen to be simply 1. As such, this shift of coordinates can be done without re-introducing any $h_0$-dependence, meaning the ghosts don't couple to $h_0$.

## C.5 Triviality of 4-point amplitude for a special kinetmatic point

In this appendix, we focus on the kinetmatic point $\mathsf{u} = 0$, which corresponds to backwards scattering ($\theta = \pi$), to support our conjecture that the tree level amplitudes are trivial. For simplicity, we look at $++ \to ++$ higher spin exchange diagrams, assuming that the lower spin contributions cancel against each other, as we observe in Sections 5.1 - 5.3.

### C.5.1 $11 \to 11$ scattering

Let us first check the results of section 5.1.1. Using the fact that at the special kinematic point, $\mathsf{s} = -\mathsf{t}$, the amplitude sum is given by:

$$\sum_{s=2,4,\ldots}^{\infty} A^{(s)}\Big|_{\mathsf{u}\to 0} = \sum_{s=2,4,\ldots}^{\infty} \tfrac{2\,s+1}{2\,(s-1)\,s\,(s+1)\,(s+2)} + \lim_{\gamma\to\infty} \textstyle\sum_{s=2,4,\ldots}^{\infty} k_s\, \gamma^s\, P_s(0)\,, \quad \text{(C.40)}$$

where the first term comes from the t-channel and, in the second term we used $\gamma \equiv \frac{\mathsf{s}}{\mathsf{u}}$ which diverges at the special kinetmatic point. The first sum can easily be performed to find:

$$\sum_{s=2,4,\ldots}^{\infty} \tfrac{2\,s+1}{2\,(s-1)\,s\,(s+1)\,(s+2)} = \tfrac{1}{8} \quad \text{(C.41)}$$

Using (5.10) and (5.11) we get:

$$c_s\, P_s(0) = \frac{\Gamma(2s+2)}{2\left[\Gamma(s+3)\right]^2}\,. \quad \text{(C.42)}$$

This can be summed to give Hypergeometric functions whose limit yields:

$$\lim_{\gamma\to\infty} \sum_{s=2,4,\ldots}^{\infty} k_s\, \gamma^s\, P_s(0) = -\frac{1}{8} \quad \text{(C.43)}$$

so that the amplitude is indeed zero at this point.

### C.5.2 j j $\to$ j j scattering

Let us first make a gues for the $jj \to jj$scattering amplitude by using the expected structure for the $J \geq 2j$ exchange amplitude (5.25),(5.29),(5.38).

The amplitude will be given by a t and u-channel contribution where the exchange particle has even spin. From (5.17),(5.24) and (5.44),(5.47), we get:

$$A = \mathsf{s}^{2j-2}\left[\sigma(x) + \sigma(-1-x)\right] , \qquad x = \frac{\mathsf{t}}{\mathsf{s}} , \tag{C.44}$$

$$\sigma(x) = \frac{2}{x^2} \sum_{s=0,2,4,\ldots}^{\infty} \left(s + 2j + \tfrac{1}{2}\right) \tfrac{s!}{(s+4j)!} \, P_s^{(4j,0)}\left(\tfrac{x+2}{x}\right) . \tag{C.45}$$

We must now study the $x \to -1$ limit of this expression. For the first term, we simply find:

$$\begin{aligned} \sigma(-1) &= \sum_{s=0,2,4,..}^{\infty} (2s + 4j + 1) \tfrac{s!}{(s+4j)!} \\ &= \tfrac{1}{4\,(2j-1)^2\,\Gamma(4j-2)} . \end{aligned} \tag{C.46}$$

For the second term, one needs to use the expansion for the Jacobi polynomials:

$$P_s^{(4j,0)}\left(\tfrac{x+2}{x}\right)\Big|_{x\to 0} = \frac{1}{x^s} \frac{4^{s+2j}}{\sqrt{\pi}} \frac{(s+2j)!}{s!} \frac{\Gamma(s+2j+\frac{1}{2})}{\Gamma(s+4j+1)} + \cdots . \tag{C.47}$$

Plugging this in (C.45), we find that it exactly cancels the contributions given by (C.46), confirming that the amplitude vanishes for this amplitude at the special kinematic point.

## C.6 Deriving the 11$\to$11 amplitude for a generic spin $s$ exchange

The 1-1-$s$ vertices in (5.4),(C.15) can be written:

$$V^{a,b,c(s)}(p,q) = \tfrac{1}{(s+2)!} q^{c_3} \ldots q^{c_s} \hat{V}^{a,b,c_1c_2}(p,q) \, , bo \tag{C.48}$$

$$\hat{V}^{a,b,c_1c_2}(p,q) = \eta^{ab} q^{c_1} q^{c_2} - \eta^{c_1a} q^b q^{c_2} + \eta^{c_1b} p^a q^{c_2} - \left[p \cdot q + \tfrac{s}{2}(p^2+q^2)\right] \eta^{ac_1} \eta^{bc_2} \, , \tag{C.49}$$

where symmetrisation over $c_1, ..., c_s$ is assumed. Using an auxiliary vector $u^c$, we contract all the $c$ indices to define:

$$V^{a,b}(p,q,u) = \tfrac{1}{2(s+2)!}(q\cdot u)^{s-2}\hat{V}^{a,b}(p,q,u)\,, \tag{C.50}$$
$$\hat{V}^{a,b}(p,q,u) = \eta^{ab}(q\cdot u)^2 - (u^a q^b - u^b p^a)\,q\cdot u + -[p\cdot q + \tfrac{s}{2}(p^2+q^2)]u^a u^b\,.$$

In the amplitude, these vertices are contrcated with the TT projector coming from the propagator (cf. (5.5)). The projector can be written as:

$$\mathcal{P}^{(s)}(\partial_{u_1},\partial_{u_2},k) = \frac{1}{(s!)^2}\sum_{l=0}^{[s/2]} a_{s,l}\, H_1^l H_2^l G^{s-2l}\,, \tag{C.51}$$
$$a_{s,l} = (-1)^l \tfrac{s!\,\Gamma(s-l+\frac{1}{2})}{2^{2l}l!\,(s-2l)!\,\Gamma(s+\frac{1}{2})}\,, \qquad G = \partial_{u_1}\cdot\partial_{u_2} - \tfrac{(\partial_{u_1}\cdot k)(\partial_{u_2}\cdot k)}{k^2}\,, \qquad H_i = \partial_{u_i}^2 - \tfrac{(\partial_{u_i}\cdot k)(\partial_{u_i}\cdot k)}{k^2}\,.$$

So for the s channel $+-\ \to +-$ exchange amplitude, we would have to compute:

$$\mathcal{P}^{(s)}(\partial_{u_1},\partial_{u_2},p_1+p_2)\,V^{a_1,a_2}(p_1,p_2,u_1)\,V^{b_1,b_2}(p_3,p_4,u_2)\Big|_{u_i=0}\,. \tag{C.52}$$

In order to compute (C.52), we will need to see how the operators $G, H_1, H_2$ commute with the $u$-dependence of the vertices. We introduce the operators:

$$W_1 = p_4\cdot\partial_{u_1} + \tfrac{1}{2}(p_1+p_2)\cdot\partial_{u_1}\,, \qquad W_2 = p_2\cdot\partial_{u_2} - \tfrac{1}{2}(p_1+p_2)\cdot\partial_{u_2}\,,$$
$$Z_1 = -\tfrac{1}{2}(p_1+p_2)\cdot\partial_{u_1}\,, \qquad Z_2 = p_4\cdot\partial_{u_2} - \tfrac{1}{2}(p_1+p_2)\cdot\partial_{u_2}\,, \tag{C.53}$$

which allows us to write the commutation relations:

$$[G,(u_1\cdot p_2)(u_2\cdot p_4)] = \tilde{\mathrm{t}} + (p_2\cdot u_1)W_2 + (p_4\cdot u_2)W_1\,,$$
$$[H_1,(u_1\cdot p_2)^2] = \tilde{\mathrm{s}} + (p_2\cdot u_1)Z_1\,, \qquad \left[H_2,(p_4\cdot u_2)^2\right] = \tilde{\mathrm{s}} + (p_4\cdot u_2)Z_2 \tag{C.54}$$
$$[Z_1,(p_2\cdot u_1)] = \tilde{\mathrm{s}}\,, \quad \left[Z_2,(p_4\cdot u_2)\right] = \tilde{\mathrm{s}}\,, \quad [W_1,(p_2\cdot u_1)] = \tilde{\mathrm{t}}\,, \quad [W_2,(p_4\cdot u_2)] = \tilde{\mathrm{t}}$$

where $\tilde{\mathrm{s}} \equiv \frac{\mathrm{s}}{4}$ and $\tilde{\mathrm{t}} \equiv \frac{1}{2}\left(\mathrm{t}+\frac{\mathrm{s}}{2}\right)$. This can be used to compute (C.52), but in general we need to commute many of these operators with powers of $(u_1\cdot p_2)$ and $(u_2\cdot p_4)$. In order to get the resulting combinatorial coefficient, one can use the generating function:

$$\mathcal{T}^{(s)} = \sum_{j=0}^{\infty}(t_1t_2)^j\,\mathcal{T}_j^{(s)} = \sum_{l=0}^{[\frac{s}{2}]} a_{s,l}\,\tilde{\mathrm{t}}^{s-2l}\,\tilde{\mathrm{s}}^{2l}\left[1+\tilde{\mathrm{t}}^{-1}(t_1W_1+t_2W_2+t_1t_2G)\right]^{s-2l}$$

$$\times \left[1+\tilde{\mathsf{s}}^{-1}(t_1 Z_1 + t_1^2 H_1)\right]^l \left[1+\tilde{\mathsf{s}}^{-1}(t_2 Z_2 + t_2^2 H_2)\right]^l . \tag{C.55}$$

Here, the coefficients $\mathcal{T}_j^{(s)}$ will contain $j$ powers of $\partial_{u_1}$ and $\partial_{u_2}$, allowing us to explicitly compute (C.52), with the correct numerical prefactor. Not forgetting the multiplicative factor of $\mathsf{s}^{-s}$ coming from the propagator, we find:

$$A^{(s)}_{\mathsf{s}\;+,-;+,-} = k_s\, x^{-2}\, P^{(4,0)}_{s-2}\left(\tfrac{x+2}{x}\right) , \qquad k_s = 2(2s+1)\tfrac{(s-2)!}{(s+2)!} , \qquad x = \tfrac{\mathsf{s}}{\mathsf{u}} , \tag{C.56}$$

which agrees with (5.10),(5.11). This procedure can be extended for other scattering amplitudes like 22→ 22, where one simply needs to use the correct vertices.