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ON SOLVING EQUATIONS OF THE GELL-MANN-LOW AND CALLAN-SYMANZIK TYPE*

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ABSTRACT

The dimensional analysis usually employed to solve the renormalization group equations for the asymptotic region is examined. It is argued that if this analysis is done systematically, one must in general add new inhomogeneous terms to the asymptotic equations even after invoking Weinberg's theorem to discard the (generalized) mass insertion term. These new inhomogeneities are entirely determined by the physical thresholds of the theory. They are shown to provide a natural explanation of Bjorken scaling in interacting field theories.

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Recently, some of the most exciting work[1] in the context of renormalizable quantum field theory has been done by employing the Gell-Mann-Low[2] and Callan-Symanzik[3] equations in the deep Euclidean region. These equations relate the responses of the one particle irreducible (1PI) Green's functions of a renormalizable field theory to changes in the parameters of the theory. For example, in a theory with one field we have

$$\left(\mu \frac{\partial}{\partial\mu} + \beta(g) \frac{\partial}{\partial g} - n\gamma(g)\right)\Gamma^{(n)}_{asy} = 0 \qquad (1)$$

where $\Gamma^{(n)}_{asy}$ is the ultraviolet asymptotic part of the 1PI renormalized n-particle Green's function, β and γ are finite functions of the renormalized coupling constant g, and μ is the mass parameter of the theory, being either the renormalized mass or, for massless theories, the Euclidean renormalization point. Of course, in writing (1) for theories with masses, we are using Weinberg's theorem.[4]

Equation (1) provides, among other things, a convenient starting point for the discussion of Bjorken scaling in the context of renormalizable quantum field theory. And, indeed, it has recently been shown[1] that in non-Abelian gauge theories, the origin $\bar{g}=0$ of the effective coupling constant $\bar{g}$ defined by

$$\frac{d}{dt}\bar{g}(t,g) = \beta(\bar{g}) \ , \qquad \bar{g}(0,g) = g \qquad (2)$$

is ultravioletly attractive in the sense of Wilson.[5] Thus, when g is in the corresponding region of attraction, these theories are viewed[1] as being asymptotically free and exhibiting Bjorken scaling, within calculable logarithmic corrections,with the following form for the solution of (1)

$$\Gamma^{(n)}_{asy}(\lambda p_1,\ldots,\lambda p_n;g;\mu) \underset{\lambda\to\infty}{\longrightarrow} I_n \Gamma^{(n)}_{asy}(p_1,\ldots,p_n;0;\mu)\,\lambda^{4-n}(\ell n\,\lambda)^{-n\gamma_1/b_0} \qquad (3)$$

where I_n is an unknown constant and $a = \gamma_1/b_0$ with γ_1 and b_0 defined by

$$\begin{aligned} \gamma &= \gamma_1 g^2 + 0(g^4) \\ \bar{g}^2(t,g) &= b_0^{-1}\, t^{-1} + 0\left(\frac{1}{t^2}\, \ln t\right) \qquad t \to \infty. \end{aligned} \tag{4}$$

As is apparent from (1) and (3), this formulation of the solutions of (1) has relied quite crucially on dimensional analysis. In this note we should like to question the way in which this analysis has been effected.

To be specific, we recall the strict Callan-Symanzik equation corresponding to (1):

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\Gamma^{(n)} = i\Gamma_\Delta^{(n)} \tag{5}$$

where $\Gamma^{(n)}$ is the renormalized 1PI n-particle Green's function, and $\Gamma_\Delta^{(n)}$ is the renormalized 1PI n-particle Green's function with one (generalized) mass insertion[3] at zero momentum transfer. In passing to the deep Euclidean region we shall ultimately again use Weinberg's theorem when necessary to discard $\Gamma_\Delta^{(n)}$ in (5). However, in solving the resulting equation we should like to note that $\Gamma^{(n)}$ will in general possess step functions and other singularities associated with the physical thresholds of the theory. As a result, when we write (in a theory with only a massless boson field, for example)

$$\Gamma^{(n)}(\lambda p_j; g, \mu) = \mu^{4-n}\phi \tag{6}$$

as is customarily done in solving (1), the function ϕ, because of these generalized thresholds, may not satisfy

$$\left(\mu \frac{\partial}{\partial\mu} + \lambda\frac{\partial}{\partial\lambda}\right)\phi = 0 \quad . \tag{7}$$

For example, if

$$\phi = \rho(\lambda p_j/\mu)\, \theta(\lambda^2(\Sigma p_j)^2 - m^2) \tag{8}$$

where ρ respects (7), then ϕ clearly does not in general. We therefore let $I\phi$ denote the part of ϕ which violates (7). We may rewrite (5) in the form

$$\left[-\lambda \frac{\partial}{\partial \lambda} + \beta(g) \frac{\partial}{\partial g} + 4 - n(1 + \gamma(g))\right] \Gamma^{(n)} = i\Gamma_{\Delta}^{(n)} - \mu^{4-n}\left(\mu \frac{\partial}{\partial \mu} + \lambda \frac{\partial}{\partial \lambda}\right) I\phi \tag{9}$$

Thus, on using Weinberg's theorem we obtain for the deep Euclidean region

$$\left[-\lambda \frac{\partial}{\partial\lambda} + \beta(g) \frac{\partial}{\partial g} + 4 - n(1 + \gamma(g))\right] \Gamma_{asy}^{(n)} = R\Gamma^{(n)} \tag{10}$$

where we have defined

$$R\Gamma^{(n)} = -\mu^{4-n} \left(\mu \frac{\partial}{\partial\mu} + \lambda \frac{\partial}{\partial\lambda}\right) I\phi \tag{11}$$

For massless non-abelian gauge theories, the solutions of (10) with $R\Gamma^{(n)}$ set equal to zero (the homogeneous solutions) are precisely the results of references 1 as illustrated by (3) above. The complete solution of (10) for any $\Gamma_{asy}^{(n)}$ is of course just a constant multiple of the respective homogeneous solution plus a particular solution of (10). In particular, in the event that the homogeneous solutions are absent due to boundary conditions, $\Gamma_{asy}^{(n)}$ is essentially determined by $R\Gamma^{(n)}$.

The form of $R\Gamma^{(n)}$ is under investigation. In the absence of precise knowledge about its structure, let us construct an example of the form it may take in order to illustrate its possible effect on $\Gamma_{asy}^{(n)}$. We take a theory described by a bare Lagrangian with one massive particle (a fermion, say) and a massless boson and let μ^2 denote the boson renormalization point. We take the relevant physical thresholds to generate only step function discontinuities and delta-functions in ${}^{b}\Gamma^{(n)}$ (the n-boson 1PI Greens function) and, hence, to make the following contribution to ${}^{b}\Gamma^{(n)}(\lambda p_j; \mu; g)$ (the dependences on the fermion mass will be suppressed where possible)

$$I\phi = \mu^{4-n} \left\{ \sum_{\alpha} \rho_{\alpha}^{1} (\lambda^2 (\Sigma p_{j_\beta})^2/\mu^2) \theta(\lambda^2 (\Sigma_{i_\alpha} p_{i_\alpha})^2 - m_{\alpha}^2) \right.$$

$$\left. + \sum_{\nu} \rho_{\nu}^{2} (\lambda^2 (\Sigma p_{j_\delta})^2/\mu^2)\, \mu^2 \; \delta(\lambda^2 (\Sigma p_{i_\nu})^2 - m_{\nu}^2) \right\} \tag{12}$$

where the ρ^i are presumed to satisfy the analogue[7] of (7). In general, the physical thresholds may generate more general terms in $\Gamma^{(n)}$, but, for the purpose of illustration, we shall just consider (12). The form of $R\,\Gamma^{(n)}$ following from (12) is

$$R\,\Gamma^{(n)} = -2\mu^{4-n} \sum_{\alpha'} \widetilde{\rho}_{\alpha'} \left(\frac{m_{\alpha'}^2 (\Sigma p_{j_\beta})^2}{\mu^2 (\Sigma p_{i_{\alpha'}})^2} \right) m_{\alpha'}^2 \delta(\lambda^2 (\Sigma_{i_{\alpha'}} p_{i_{\alpha'}})^2 - m_{\alpha'}^2) \tag{13}$$

where $\widetilde{\rho}$ is clearly determined by $\{\rho_{\alpha}^{1}\}$ and $\{\rho_{\nu}^{2}\}$. Equation (10) now reads

$$\left[-\lambda \frac{\partial}{\partial \lambda} + \beta \frac{\partial}{\partial g} + 4 - n(1+\gamma) \right] {}^{b}\Gamma_{asy}^{(n)} = -2\mu^{4-n} \sum_{\alpha} \widetilde{\rho}_{\alpha}\, m_{\alpha}^2\, \delta(\lambda^2 (\Sigma p_{i\alpha})^2 - m_{\alpha}^2) \tag{14}$$

Let

$$g' = 2 \int_{g_0}^{g} \frac{dx}{\beta(x)} \tag{15}$$

We introduce $\bar{\rho}_\alpha$ and σ_α by

$$-\mu^{4-n}\, m_\alpha^2\, \tilde{\rho}_\alpha \left(\frac{m_\alpha^2 (\Sigma p_{j_\beta})^2}{\mu^2 (\Sigma p_{i_\alpha})^2} \right) = \int d\ell\, \bar{\rho}_\alpha\, e^{i\ell g'}$$

$$-\frac{m_\alpha^2}{(\Sigma p_{i_\alpha})^2} = \int dr\, \sigma_\alpha(r)\, e^{irg'} \tag{16}$$

We are assuming we can invert (15). Defining

$$t = \ell n\, \lambda^2$$

we have from (14) and (16)

$$\sum_\alpha \int \frac{dk d\ell}{2\pi} \sum_{j_1, j_2} \int \frac{dr_1 \cdots dr_{j_2}}{(\Sigma p_{i_\alpha})^2}\, \bar{\rho}_\alpha\, \frac{\sigma(r_1) \cdots \sigma(r_{j_2})(ik)^{j_1+j_2}}{j_1!\, j_2!}\, e^{j_1 t + i(\sum_i r_i + \ell) g'}$$

$$= \left[-\frac{\partial}{\partial t} + \frac{\partial}{\partial g'} + 2 - \frac{n}{2}(1+\gamma) \right] {}^b\Gamma^{(n)}_{asy} \tag{17}$$

This last equation has the general solution (see Symanzik[3])

$${}^b\Gamma^{(n)}_{asy} = \frac{1}{2\pi} \int_t^{\frac{1}{2}(t+g' + h(t+g'))} dt' \Bigg\{ \sum_\alpha \int dk d\ell \sum_{j_1, j_2} \int \frac{dr_1 \cdots dr_{j_2}}{(\Sigma p_{i_\alpha})^2}\, \bar{\rho}_\alpha\, \frac{\sigma_\alpha(r_1) \cdots \sigma_\alpha(r_{j_2})(ik)^{j_1+j_2}}{j_1!\, j_2!}$$

$$\exp\left(j_1 t' + i(\Sigma r_i + \ell)(g' + t - t') + \int_{g'+t-t'}^{g'} dx \left[\frac{n(1+\gamma(g(x)))}{2} - 2 \right] \right) \Bigg\}$$

$$+ B_n\, {}^b\Gamma^{(n)}_{asy,\ homogeneous} \tag{18}$$

where h is arbitrary, B_n is a constant, and ${}^{b}\Gamma^{(n)}_{asy,\,homogeneous}$ is a homogeneous solution of (10). Note that in general h may not be set equal to ∞ in (18). We write

$$h(g' + t) \;=\; g' + t + 2h_0(g' + t) \tag{19}$$

where h_0 is a function which we may constrain by the requirement that the first term on the RHS of (18), which we denote by ${}^{b}\Gamma^{(n)}_{asy,\,p}$, agrees with the following formal particular integral of (14)

$$\begin{aligned}
{}^{b}\Gamma^{(n)}_{\substack{asy,\,p\\ formal}} \;=\;& \int \frac{dk\,d\ell}{2\pi} \sum_{\alpha,\,j_1,\,j_2} \int \frac{dr_1 \cdots dr_{j2}}{(\Sigma p_{i\alpha})^2}\, \bar{\rho}_\alpha\, \frac{\sigma(r_1)\cdots\sigma(r_{j2})(ik)^{j_1+j_2}\, e^{j_1 t + i(\Sigma r_i + \ell)g'}}{j_1!\; j_2!\; (-j_1 + i(\ell + \Sigma r_i) + 2 - n/2)} \\
&+ \frac{n}{2} \int dk\,d\ell \sum_{j} \frac{\overline{\gamma\Gamma}^{(n)}_{asy,\,p}(ik)^j\, e^{jt + i\ell g'}}{j!\,(-j + i\ell + 2 - n/2)}
\end{aligned} \tag{20}$$

where $\overline{\gamma\Gamma}^{(n)}_{asy,\,p}$ is defined by

$$\gamma\,{}^{b}\Gamma^{(n)}_{asy,\,p} \;=\; \int dk\,d\ell\; \overline{\gamma\Gamma}^{(n)}_{asy,\,p}\; e^{i(\ell g' + \lambda^2 k)} \tag{21}$$

Making a change of variable $t' - t = s$ in (18), we see that as $\lambda \to \infty$,

$$ {}^{b}\Gamma^{(n)}_{\text{asy}} \to \frac{1}{2\pi\lambda^2} \left[\int dk d\ell \sum_{\alpha, j_1} \frac{\bar{\rho}_\alpha(ik)^{j_1} e^{i\ell g'}}{(\Sigma p_{i\alpha})^2 j_1!} \lim_{t\to\infty} \int_0^{g'+h_0} ds \right. $$

$$ \left. \exp \left(j_1 + \frac{n}{2} - 2 - i\ell\right) s + \frac{n}{2} \int_{g'-s}^{g'} dx\, \gamma(g(s)) \right] $$

$$ + 0\left(\frac{1}{\lambda^4}\right) + B_n\, {}^{b}\Gamma^{(n)}_{\text{asy, homogeneous}} \tag{22} $$

where we have used (20). Of course, here, we are assuming the interchanges in the orders of limits and integrations which we have made are legitimate. Clearly, if the boundary conditions which determine B_n are such that $B_n = 0$, then it is possible that ${}^{b}\Gamma^{(n)}(\lambda p_j)$ behaves like

$$ 1/\lambda^2 $$

in the deep Euclidean region, for $n \geq 4$.

One of the most pleasing features of this development is the possibility of a natural explanation of Bjorken scaling. Indeed, consider the familiar Wilson[6] expansion for the product of two electromagnetic or weak currents at light-like distances:

$$ J(y/2)\, J(-y/2) = \sum C_n(y^2; g)\, O^{(n)}_{\mu_1 \cdots \mu_n}(0)\, y^{\mu_1} \cdots y^{\mu_n} \tag{23} $$

where we suppress all tensor and quantum number labels. The Fourier transforms of the functions C_n satisfy an equation analogous to (10) in the deep Euclidean region:

$$ \left(-\lambda \frac{\partial}{\partial\lambda} + \beta \frac{\partial}{\partial g} - 2 - \gamma_n\right) C_{n,\text{asy}}(\lambda^2 q^2, \mu; g) = R\, C_n \tag{24} $$

where γ_n is the anomalous dimension of O_n and RC_n represents the physical thresholds. Assuming step function discontinuities and delta functions in C_n due to thresholds $\{m_\alpha^2\}$ we have

$$RC_n = -2\mu^{-2} \sum_\alpha \xi_\alpha^{(n)} (m_\alpha^2/\mu^2)\, m_\alpha^2\, \delta(\lambda^2 q^2 - m_\alpha^2) \tag{25}$$

where we take ξ_α^n to respect the analogue of (7). From (22) it is clear that, for $q^2 < 0$,

$$C_{n,\mathrm{asy}}(\lambda^2 q^2) \underset{\lambda \to \infty}{\rightarrow} \frac{\overset{(n)}{\mathrm{Const}}(g',\mu)}{\lambda^2 q^2} + E_n C_{n,\mathrm{asy,homogeneous}} + 0\left(\frac{1}{\lambda^4}\right) \tag{26}$$

where

$$\overset{(n)}{\mathrm{Const}}(g',\mu) = \frac{1}{2\pi} \int dk d\ell \sum_{\alpha, j} \frac{\bar{\xi}_\alpha^{(n)} (ik)^j e^{i\ell g'}}{j!} \lim_{t \to \infty} \int_0^{g'+h_0} ds$$

$$\exp\left[(j_1 + 1 - i\ell)\, s + \frac{1}{2} \int_{g'-s}^{g'} dx\, \gamma_n(g(x))\right] \tag{27}$$

with $\bar{\xi}_\alpha^{(n)}$ defined by

$$-\mu^{-2} m_\alpha^2\, \xi_\alpha^{(n)} = \int d\ell\, \bar{\xi}_\alpha^{(n)}(\ell)\, e^{i\ell g'} \tag{28}$$

In (26), E_n is a constant and $C_{n,\mathrm{asy,homogeneous}}$ is a homogeneous solution of (24). The homogeneous solutions $C_{n,\mathrm{asy,homogeneous}}$ have been discussed[8] in references 1, for example, and are known to give at least logarithmic deviations from Bjorken scaling in the absence of RC_n. Thus, in view of the deep inelastic data we take such solutions to be absent from (27). Then as $\lambda \to \infty$

$$C_{n,\,asy}(\lambda^2 q^2) \to \frac{\overset{(n)}{\text{Const}}\,(g',\mu)}{\lambda^2 q^2} + 0\left(\frac{1}{\lambda^4}\right) \tag{29}$$

which is clearly the desired naive free field theory scaling result (Bjorken scaling). We emphasize that $\overset{(n)}{\text{Const.}}$ is essentially determined by the physical thresholds. Note that our argument need not depend on the sign of $\beta(g)$!

We should also mention that in the asymptotic time-like q^2 region, the contribution of $RC^{(n)}$ will also be given by (29). However, in this region, for theories with a massive Lagrangian, one may not in general neglect the mass insertion inhomogeneous term in (5) above. This insertion may in general generate deviations from the scale invariant result (29). We conjecture that this is the reason for the fast onset of scaling in the deep inelastic scattering region compared with the annihilation region. Of course, this is well-known.

To conclude, we have argued that, contrary to the approach of reference (1), it is possible that it is the physical thresholds, serving as sources, that generate naive Bjorken scaling in interacting field theories, the boundary conditions being such as to disallow scale violating homogeneous solutions to contribute to the appropriate asymptotic physical solutions of the Callan-Symanzik equations in the deep Euclidean region. Specifically, the step function discontinuities and delta functions which are familiar characteristics of these thresholds can naturally generate Bjorken scaling behavior in the deep inelastic scattering region. This latter statement is independent of the sign of $\beta(g)$ in the renormalization group equation.

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7. By "analogue of (7)" we mean the obvious:

$$\left(\sum_i m_i \frac{\partial}{\partial m_i} + \sum_j \mu_j \frac{\partial}{\partial \mu_j} + \lambda \frac{\partial}{\partial \lambda}\right) \phi = 0$$

where m_i are the fundamental masses of the theory and the μ_j represent the possible (Euclidean) renormalization points.

8. For further references to the discussion of C_n in this connection, see the references in Ref. 1.