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Background Independence and Gauge Invariance in General Relativity Part 2—Covariant Quantum Gravity

Massimo Tessarotto [1,2] and **Claudio Cremaschini** [2,*]

[1] Department of Mathematics and Geosciences, University of Trieste, Via Valerio 12, 34127 Trieste, Italy
[2] Research Center for Theoretical Physics and Astrophysics, Institute of Physics, Silesian University in Opava, Bezručovo Nám. 13, CZ-74601 Opava, Czech Republic
* Correspondence: claudiocremaschini@gmail.com

Abstract: Background independence is often being claimed as the characteristic property of several current and past models of Quantum Gravity. In actual fact, such a notion has a wider connotation and must be rooted into the validity of the general covariance principle, demanding its logical connection with the notions of manifest covariance and (quantum) gauge invariance. In fact, as we intend to show here, it involves (a) the existence of a well-defined, albeit arbitrary, classical background space-time; and (b) the suitable realization of a dynamical equation for the related background metric field tensor, referred to as quantum-modified Einstein tensor field equation, which actually determines it in a suitable functional setting. Remarkably, it is proved that in the context of the theory of Covariant Quantum Gravity (CQG-theory), recently developed by Cremaschini and Tessarotto (2015–2022), background independence implies that such an equation "emerges" rigorously from the same CQG-theory. This follows in terms of a stochastic quantum expectation value evaluated with respect to the corresponding characteristic quantum PDE. It is shown that an analogous emergence property applies also to the background metric field tensor in terms of stochastic fluctuations of the corresponding underlying quantum tensor of gravitational field. These results warrant the consistent validity of background independence for the prescription of the space-time metric tensor in CQG-theory.

Keywords: Hamiltonian theory of GR; manifest covariance; covariant quantum gravity; background independence; emergent gravity

PACS: 03.50.-z; 04.20.-q; 04.20.Cv; 04.60.-m; 05.30.-d

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1. Introduction

This paper is about two aspects of Quantum Gravity (QG), namely, background independence and gauge independence, which are actually crucial for the possible unique prescription, understanding and physical interpretation of the corresponding theory of quantum gravitational field (QG-theory). This must be realized by the canonical quantization of the continuum Einstein Field Equations (EFE), which are the equations representing, in turn, the so-called standard formulation of GR (SF-GR).

Both QG and QG-theory are related to the nature of the quantum space-time, to be in principle distinguished from that associated to the underlying classical space-time that characterizes GR, and which therefore must be suitably prescribed (see Section 1.1). For the same reason, the notions of background independence and gauge independence in QG-theory must be, in principle, assigned and distinguished from the analogous properties holding in GR (for the related discussions, we refer the reader to Part I [1]), the main difference being represented by the peculiar phenomenology of the quantum space-time. These properties are exemplified by a number of specific quantum phenomena, often referred to as part of quantum phenomenology [2–4], which are/may be associated with conceptual issues that might also represent actual or ideal physical experiments. However, possible proposals of this type in QG/QG-theory are, in principle, numerous. In fact, besides the verification of fundamental physical principles, they may also concern tentative justifications of selected and/or already theoretically predicted properties. These include, for example, the following:

- *Issue #1*: The first one is related with the possible direct observation of gravitons, which would of course represent the definitive proof for QG. While many aspects of GR have been tested, and general principles of quantum dynamics demand its quantization, there is yet no direct experimental evidence for the existence of gravitons, i.e., the quanta of the gravitational field. On the basis of elementary physical grounds and according to the theoretical estimate pointed out in Ref. [5], the theoretical graviton rest mass m_o should scale with respect to the electron rest mass m_e as $\frac{m_o}{m_e} \cong 1.38 \times 10^{-39}$. This suggests that in order to increase the probability of detection of gravitons and enhance measurements of their rest mass one should look for environments characterized by the presence of intense fluxes of gravitons themselves, e.g., those expected to occur at the formation/merger of black holes. Conversely, the realization of graviton detectors sensitive enough to detect individual particles (gravitons) requires detectors massive enough to generate a black hole event [6].
- *Issue #2:* The second issue that might also lead to an independent firm establishment of quantized gravity is related to the direct observation of a possible cosmological gravitational wave background (CGWB). According to Ref. [7], it is believed that measurement of polarization of the CGWB might unveil the long-wavelength stochastic background of gravitational waves associated with the inflationary stage of the Early Universe.
- *Issue #3:* A notable issue is certainly the measurement and possible identification of the physical origin of the quantum cosmological constant (CC), as predicted in Ref. [8]. The reason is that, at the classical level, the CC remains undetermined (see also the discussion in the subsequent Section 3). It follows that a meaningful (ideal) target concerns identifying the quantum mechanism responsible for its generation, i.e., the quantum physical interaction responsible for the occurrence of the experimentally observed CC. This, in turn, may lead to the proof of the possible stochastic nature of the same quantum cosmological constant, which might not be deterministic in character as usually considered in the literature.
- *Issue #4:* The proof of the possible stochastic (i.e., quantum) rather than classical (i.e., deterministic) behavior, which, contrary to conventional wisdom, might characterize event horizons (EH) associated to BHs [9]. This issue is intrinsically related to the search of the possible quantum regularization mechanisms of singular classical space-time solutions associated with the classical Einstein Field Equations (EFE).

A prerequisite framework for the establishment of these issues is the proof of existence of the so-called background quantum space-time, namely, the Riemannian differential manifold of the form $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ denoting a "quantum" background metric field tensor, solution of a suitable quantum-modified EFE (*q-modified EFE* [10]), i.e., a real and symmetric metric tensor, everywhere regular in $\mathbf{Q}^4 \subseteq \mathbb{R}^4$, whose covariant and counter-variant components $\{\widehat{g}_{\mu\nu}(r)\}$ and $\{\widehat{g}^{\mu\nu}(r)\}$ necessarily satisfy the orthogonality conditions, i.e., $\widehat{g}_{\mu\nu}(r)\widehat{g}^{\alpha\nu}(r) = \delta^{\alpha}_{\mu}$. In this reference, in fact, a fundamental requirement for the establishment of the QG-theory is the proof of the properties (to be suitably defined) of *background and gauge independence* for a manifestly covariant quantum theory of gravity that should realize the quantization of the classical EFE.

1.1. Two Types of Quantum Space-Times: The Multi- and Uni-Verse Space-Time Representations

It is well known that a mandatory prerequisite for the establishment of a quantum (field) theory is the achievement of an appropriate Hamiltonian representation for the relevant classical (field) theory. This means that it should always be possible to cast the relevant equations in canonical form, via the adoption of Lagrangian and Hamiltonian representations. In the present case, this concerns the choice of the Lagrangian continuum coordinates for EFE, the evaluation of the conjugate canonical momenta and the identification of corresponding canonical variables.

This might seem, in principle, an easy task. However, as pointed out elsewhere [5], such a Hamiltonian structure is non-unique. There are in fact two possible representations of EFE, which correspond to constrained and unconstrained abstract Hamiltonian systems, respectively. In turn, these correspond to two possible choices of the quantum functional setting; however, this also implies two different quantum space-times and, therefore, two alternative formulations of QG-theory.

For definiteness and ease of understanding, we briefly recall the basic differences between them. In fact, the question here concerns the precise meaning of the concept of quantum space-time. Conventional wisdom is that this should be somehow associated with a functional "configuration" space $\{g(r)\}$, namely, an appropriate continuous ensemble of symmetric, real and quantum (and possibly stochastic) field tensors of the type

$$g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}. \tag{1}$$

Here, the notation is standard. Thus, $r \equiv \{r^{\mu}\}$ (with $\mu = 0,3$) denotes an arbitrary $4-$position (not a $4-$vector), which spans an appropriately defined space-time, and $g_{\mu\nu}(r)$ and $g^{\mu\nu}(r)$ are the suitably defined counter-variant and covariant components of the tensor $g(r)$. However, in order to properly prescribe/identify the $4-$position $r \equiv \{r^{\mu}\}$ and the quantum tensor $g(r)$ (and hence, the configuration space $\{g(r)\}$ too), there are actually two possible distinct choices available for the prescription of the same quantum space-time. These are referred to as follows:

- *Multi-verse quantum space-time: case of the constrained functional setting* $\{g(r)\}_C$. In this case, the generic quantum tensor $g(r) \in \{g(r)\}$ is itself identified with a metric tensor, which therefore generates its own space-time. For this reason, this representation (of quantum space-time) is denoted intuitively as multi-verse, since each choice of the tensor determines, at least in principle, a new and different space-time. Therefore, the structure of the corresponding quantum space-time is identified with the differential manifold
$$\left\{\mathbf{Q}^4, g(r)\right\}. \tag{2}$$
This means that the counter- and covariant components of the symmetric tensor $g(r)$, namely, $\{g_{\mu\nu}(r)\}$ and $\{g^{\mu\nu}(r)\}$ are necessarily constrained, the constraint conditions being represented by the orthogonality relations
$$g_{\mu\nu}(r)g^{\alpha\nu}(r) = \delta^{\alpha}_{\mu}. \tag{3}$$
These constraints must hold identically at arbitrary $4-$positions $r \equiv \{r^{\mu}\}$ spanning $\{\mathbf{Q}^4, g(r)\}$ and arbitrary indices $\mu, \alpha = 0,3$. As a consequence, the functional setting $\{g(r)\}$ is represented by constrained tensor functions subject to the orthogonality constraints (3), i.e., a suitably defined constrained-function space $\{g(r)\}_C$.
- *Uni-verse or Background quantum space-time: case of the unconstrained functional setting* $\{g(r)\}_U$. In this case, the same generic quantum tensor $g(r)$ defined above is not required to be a metric tensor, which means that it remains unconstrained. However, the same $g(r)$ is assumed to be a $4-$tensor with respect to a suitably prescribed space-time. For this purpose, the assumption is introduced that there exists a suitable and possibly non-unique quantum background metric field tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$, with $\widehat{g}(r)$ itself belonging to an appropriate functional set $\{\widehat{g}(r)\}$, and an associated *"background" quantum space-time*
$$\left\{\mathbf{Q}^4, \widehat{g}(r)\right\}, \tag{4}$$
with $\widehat{g}(r)$ to be prescribed in such a way to coincide with an appropriate, and yet to be defined, quantum expectation value. This means that (a) by construction, $\widehat{g}(r)$

is a metric tensor, and it follows that its covariant and counter-covariant components, namely, $\{\widehat{g}_{\mu\nu}(r)\}$ and $\{\widehat{g}^{\mu\nu}(r)\}$, necessarily satisfy the orthogonality conditions, i.e., $\widehat{g}_{\mu\nu}(r)\widehat{g}^{\alpha\nu}(r) = \delta^{\alpha}_{\mu}$; (b) the transformation properties of the generic quantum tensor $g(r)$ (1) are now defined with respect to the same space-time $\{\mathbf{Q}^4, \widehat{g}(r)\}$. Thus, its covariant and counter-variant components are determined via the equations $g_{\mu\nu}(r) = \widehat{g}_{\mu\alpha}(r)\widehat{g}_{\nu\beta}(r)g^{\alpha\beta}(r)$ and $g^{\mu\nu}(r) = \widehat{g}^{\mu\alpha}(r)\widehat{g}^{\nu\beta}(r)g_{\alpha\beta}(r)$. Therefore, this implies and confirms that in this case the quantum tensor $g(r)$ (1) is indeed unconstrained, while the functional setting $\{g(r)\}$ is now represented by the unconstrained 4−tensor functions $g(r)$ and identified with the unconstrained-function space $\{g(r)\}_U$.

In the same reference, the implications of the two choices indicated above that concern the classical Hamiltonian representations of GR (i.e., of EFE), in which $g(r)$ and $\widehat{g}(r)$ are both intended as classical—i.e., deterministic—fields have been investigated at length, showing that they correspond to the two alternative, possible formulations, respectively: (A) the well-known ADM Hamiltonian representations of GR holding in the constrained functional setting $\{g(r)\}_C$; (B) the manifestly covariant Hamiltonian representation applied in the unconstrained functional setting $\{g(r)\}_U$.

An analogous non-uniqueness feature arises, of course, in QG. In fact, it is immediate to show that, also in QG, the choices indicated above, in which the quantum tensors $g(r)$ (1) are, respectively, constrained or unconstrained, lead to different (and actually incompatible) realizations of the quantum theory of gravity.

1.2. Goals of the Investigation

This is the second part of a two-paper investigation, the present one being devoted to QG, or more precisely, the theory of QG that emerges as the unique choice dictated by the Hamiltonian structure of GR determined in Part I. More precisely, here, the goal of the paper is twofold. The first one is to formulate QG in the extended classical framework developed in Part I, i.e., based on the metric-Ricci extended Hamiltonian representation. This means that the task appears very challenging because it involves developing, at least in principle, a corresponding coupled metric-Ricci quantization of SF-GR. In fact, the metric-Ricci extended Hamiltonian representation of classical GR determined in Part I allows, in principle, the freedom of infinite possible Hamiltonian couplings of the form

$$H_{g+R} \equiv H_g + \alpha_1 H_R, \tag{5}$$

referred to here as metric-Ricci extended Hamiltonian representation, to arise among the two independent Hamiltonian functions $H_g \equiv H_g(g(r), \Pi(r), \widehat{g}(r))$ and $H_R \equiv H_R(R(r), Q(r), \widehat{g}(r))$. Here, α_1 represents, in principle, an arbitrary dimensionless coupling constant; however, this is to be set necessarily equal to zero when the classical cosmological constant $\Lambda^{(c)}$ is set $= 0$. Thus, the question of the possible non-uniqueness of the metric-Ricci quantization and, hence, of QG theory arises only when $\Lambda^{(c)}$ is considered $\neq 0$. This means that, in principle, infinite different (but equivalent) possible routes to quantization of classical gravity may arise only if $\Lambda^{(c)} \neq 0$. Here, we intend to investigate under what conditions, if any, the customary manifestly-covariant $g-$quantization originally developed in [5,11] remains still valid.

The second goal involves understanding the role of *quantum background independence* and *quantum gauge invariance* in the context of QG. In this regard, a general feature that we intend to show is that, like their classical counterparts characterizing the Hamiltonian representation of classical GR (see Part I), quantum background independence as well quantum gauge invariance are, in a specific sense, both consequences of general covariance with respect to the group of local point transformations (i.e., the so-called LPT-group, also known as diffeomorphism group). However, more specifically, we intend to present a new interpretation of the notion of background independence in the context of QG, which holds thanks to the property of quantum gauge invariance, i.e., to the intrinsic gauge indeterminacy of the quantum Hamiltonian operator. Accordingly, we intend to show that background independence in QG should mean that not only must a suitably defined background metric field tensor exist but also that the same tensor field and the related quantum-modified Einstein field equation (EFE) must have a unique "emerging" character, i.e., it must be possible to represent both quantities in terms of suitably defined stochastic quantum expectation values of appropriate underlying quantum tensor fields.

The reason why this happens in the present context will appear clear (only) once the proper functional setting is adopted. Here, we intend to show that such a goal can be realized by an appropriate realization of QG theory. This is achieved by a suitable stochastic modification (referred to as *stochastic quantization*) of the manifestly covariant approach to QG earlier developed and recalled above, in such a way that all physically relevant quantum fields and operators can be expressed in an objective form, namely, in terms of an explicit $4-$tensor representation with respect to the LPT-group. In such a context, the notion of background metric field tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ naturally arises. Indeed, an approach of this type actually presupposes by itself the existence of a background space-time structure, i.e., a Riemannian differential manifold of the type $\{\mathbf{Q}^4 \equiv \mathbb{R}^4, \widehat{g}(r)\}$, with $\widehat{g}(r)$ being a background metric field tensor to be determined. On the other hand, the very existence of such a tensor field in the context of QG, requires that the same tensor field must necessarily exhibit an "emerging" character, i.e., it must be realized by an appropriate quantum expectation value. However, the question is whether and how in the context of QG, besides $\widehat{g}(r)$, the dynamical equation that generates the same tensor field (as its general solution) can be achieved. The main goal of the paper is to provide a definite answer to this important question showing that the same equation must have an emerging character, i.e., it must be possible to determine it in terms of a suitable quantum expectation value.

2. The (Minimal) Requirements of QG-Theory

In this section, the possible requirements to be set for the establishment of a QG-theory are illustrated, based on the issues indicated above. It seems fairly obvious that the choice for a minimal set of such requirements may be crucial for selecting one of the above theories as a preferred one and reaching the appropriate physical interpretation.

First of all, we notice that in all cases indicated above, the Lagrangian tensor fields $g(r)$—and in the special case of the Uni-verse space-time, also $\widehat{g}(r)$—should be regarded or associated with quantum observables. This means that they should be properly related to a suitable quantum state, represented by a wave-function $\psi(r)$. To be more precise, in the two cases discussed above, $\psi(r)$ should be defined on the appropriate functional configuration space U_g spanned by the same Lagrangian tensor fields $g(r)$ and the appropriate space-time; hence, they should take the form $\psi(r) \equiv \psi(g(r), r)$ and, respectively, $\psi(r) \equiv \psi(g(r), \widehat{g}(r), r)$. However, for a quantum theory of classical gravity to make sense, a number of minimal requirements on $\psi(r)$ should be satisfied. In particular, the quantum-gravitational wave function $\psi(r)$ should be such that the following hold:

- *Feature #1:* it satisfies the covariance principle, i.e., it is a $4-$scalar, so its complex value should be frame independent, namely, independent of the choice of the possible coordinate systems that are mutually connected by means of local diffeomorphisms $r \Leftrightarrow r' = r'(r)$ (diffeomorphism invariance).
- *Feature #2:* for the same reason, it should obey a $4-$scalar PDE [12]. This feature automatically preserves for the quantum state $\psi(r)$ its $4-$scalar character everywhere in the corresponding quantum space-time.
- *Feature #3:* it admits, as all quantum theories worthy of this name, a Hilbert-space representation and the prescription of appropriate scalar product [11]. For the same reason, it must be prescribed so that its mod square $|\psi(r)|^2 \equiv \psi(r)\psi^*(r)$ represents the quantum probability density for the occurrence of the quantum metric tensor $g(r) = \{g_{\mu\rho}(r)\} = \{g^{\mu\nu}(r)\}$, with respect to a suitably prescribed functional configuration space, i.e., either $\{g(r)\}_C$ or $\{g(r)\}_U$.

- *Feature #4:* it satisfies, under the assumption of quantum unitarity, the Heisenberg uncertainty principle [11].
- *Feature #5:* it admits a formulation of quantum logic analogous to that of quantum mechanics [13].
- *Feature #6:* it permits to unveil the intrinsic stochastic character of Quantum Gravity and the consequent implications on classical counterpart solutions, in particular, as far as it concerns the definition of the quantum nature of the cosmological constant and the quantum regularization of classical singular solutions predicted by EFE (e.g., black-hole singularities and event horizons).

All of the above features should be mentioned for their absolute censorship character. However, some of them are particularly notable. Features #1 and #2 will be of primary concern here, because they rule out all theories that are not covariant and/or for which the quantum-wave function is not a $4-$scalar. However, Feature #6 may also be considered of primary interest, the reason being the discovery of a characteristic feature of QG-theory, namely, its underlying intrinsic stochastic character. In our view, this feature should actually become of crucial importance for the understanding of quantum-gravity phenomena in relativistic astrophysics and cosmology. For this purpose, the present investigation will be couched in the framework of a stochastic generalization of CQG-theory, i.e., the quantum theory of the gravitational field based on the concept of background quantum space-time and the unconstrained functional setting $\{g(r)\}_U$. At this point, it seems appropriate to briefly summarize the reasons of our choice.

Why CQG-Theory?

Let us illustrate what the precise reasons for adopting the setting provided by CQG-theory are—i.e., the manifestly covariant theory of QG for massive gravitons—among the other quantum models currently available in the literature. The most popularly known (of such quantum models) is Quantum Geometrodynamics (QGD), with its two main variants, namely, the related Wheeler–DeWitt (WDW) equation of QGD [14], and the Ashtekar-variable representation [15] leading to the so-called Loop Quantum Gravity (LQG) [16].

However, the crucial aspect of QGD, which is shared both by the WDW equation and LQG, is that it exhibits characteristic features that are questionable in the light of validity of manifest covariance principle [5]. In particular, first, both theories are based on the notion of multi-verse quantum space-time and the adoption of the constrained functional setting $\{g(r)\}_C$. This means that both theories are *constrained quantum-Hamiltonian theories*, being based on canonical quantization of Hamiltonian systems with constraints. Second, both theories are based on the so-called ADM Hamiltonian representation of GR [17], in which $g \equiv \{g_{\mu\nu}\}$ is actually replaced by a set of $3-$tensors. As a consequence, it follows that both WDW and LQG definitely miss the property of manifest relativistic covariance, since by construction all the relevant equations and variables are not (and cannot be) set in terms of an explicit $4-$tensor representation. Third, the very concept of *constrained canonical quantization*, adopted in both theories, appears questionable too. In fact, serious doubts were raised by Isham [18] (his critic was set originally for WDW; however, a similar argument holds also for LQG), who states that "there is no real justification for extending the Dirac approach to constraint generators that are quadratic functions of the momentum variables". The same author concludes that "although it may be heretical to suggest it, the Wheeler–DeWitt equation—elegant though it be—may be completely the wrong way of formulating a quantum theory of gravity". Fourth, in both theories, the coordinate time variable is missing (i.e., is considered ignorable) [19]. Such a feature, by itself, has raised a long lasting debate on the nature of time in QG. However, the most serious one is that it leads to an intrinsic violation of the relativity principle, an unacceptable feature for all theories having the ambition to provide a frame-independent and background-independent dynamical picture of quantum space-time. Indeed, in greater detail, both WDW and LQG theories are in manifest violation of the relativity principle. The proof is immediate: in fact, the property (of the time-coordinate to be ignorable) is obviously violated by an arbitrary boost that mixes time and space coordinates. In fact, an arbitrary boost of this type destroys the same representation. The absence of the time variable is also contradictory with the requirement of background independence. Indeed, the theory holds only if background metric tensor itself is time-independent. Finally, WDW equation is not an evolution-type equation as required by a quantum-wave equation. Completely analogous conclusions hold for LQG, since the quantum state $\psi(r)$ cannot depend explicitly on the same time coordinate.

It follows that both WDW and LQG cannot determine any (time-)evolution of the quantum state $\psi(r)$. Furthermore, a proper definition of the Hilbert space is missing, so that the quantity $|\psi(r)|^2$ cannot be interpreted as a probability density and, thus, the Heisenberg uncertainty principle does not apply. As a consequence of all previous features, possible logical implications of QG cannot be inferred in a consistent manner, either because they hold only in "special frames", or in "ad hoc" background space-times. Hence, a quantum theory of gravity, based either on WDW or LQG, in a proper sense, does not hold.

In contrast, the features of CQG-theory characterize it as an *unconstrained quantum Hamiltonian theory,* which is associated with a suitably prescribed background metric tensor field $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ and is to be determined self-consistently by CQG-theory itself. Indeed, first, the Hamiltonian representation of GR underlying CQG-theory is intrinsically unconstrained. In fact, the symmetric variational $4-$tensor field $g \equiv \{g_{\mu\nu}\}$, which determines the Lagrangian continuum coordinates, is not required to coincide with a metric tensor, so that its covariant and counter-variant components remain independent. Indeed, the said tensor components are related by means of the "background" metric tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$, so that $g_{\mu\nu}(r) = \widehat{g}_{\mu\alpha}(r)\widehat{g}_{\nu\beta}(r)g^{\alpha\beta}(r)$ and $g^{\mu\nu}(r) = \widehat{g}^{\mu\alpha}(r)\widehat{g}^{\nu\beta}(r)g_{\alpha\beta}(r)$. Second, the notable departure of CQG-theory with respect to former approaches, based on the multi-verse representation, lies in its uni-verse representation: the consequence is that only $4-$scalars or $4-$tensors appear in the theory while the canonical quantization formalism is established in such a way to be consistent with the principle of manifest covariance and the validity of Einstein field equations in the classical domain. Third, it is achieved by means of canonical quantization, which is unconstrained and manifestly covariant, so that no special role or singling-out of the coordinate time variable is required. Fourth, CQG-theory is a trajectory-dependent theory. Already at the classical level, the trajectories are identified with a suitable collection of same and finite-length geodesic curves associated with the background metric field tensor $\widehat{g}(r)$ [5]. The same curves are parametrized with respect to their corresponding proper-time (s) and, thus, take the form

$$\{r(s)\} \equiv C_{(r_0,r_1} \equiv \left\{ r | r = r(s), r_0 = r(s_0), r_1 = r(s_1), s \in [s_0, s_1], r_0 \in \Sigma_0^3, r_1 \in \Sigma_1^3 \right\}, \quad (6)$$

with s being defined through the differential relationship $ds^2 = \widehat{g}_{\mu\nu}(r(s))dr^\mu \, dr^\nu$ and Σ_0^3 and Σ_1^3 denoting two suitable $3D$ subsets of the background space-time $\{\mathbf{Q}^4, \widehat{g}(r)\}$. At the quantum level, however, the same geodesic trajectories represent massive graviton particle trajectories, i.e., necessarily endowed with a subluminal $4-$velocity. As a consequence, the background metric tensor becomes of the form $\widehat{g}(r(s))$, while the Lagrangian coordinates take the general form $g(s) = g(r(s), s)$, thus allowing more generally both explicit and implicit $s-$dependences. Fifth, the evolution of quantum-wave function is determined uniquely by its proper time $s-$dependence (quantum deterministic principle [9]) via a suitable quantum-wave equation. The same quantum-wave equation is realized by a hyperbolic PDE and the quantum state corresponds to a unique prescription of the initial boundary value problem for the same PDE. The corresponding Hilbert-space structure and the quantum observables are identified, warranting that $|\psi(r)|^2$ can be interpreted as a probability density and, consequently, the Principle of Heisenberg indeterminacy holds. This, in turn, preserves the quantum logic structure of CQG-theory.

Needless to say, all such features represent striking differences between the two approaches. However, there may be further issues to be considered. Few researchers today still believe that the ultimate theory of quantum gravity might possibly presuppose some kind of suitably prescribed space-time structure or preferred reference frame of some sort. Thus, the notion of background independence is frequently claimed, together with that of diffeomorphism invariance (i.e., GR-frame independence), to be the distinctive physical property and compelling requirement for most current models of QG-theory, to be realized either by exact or approximate mathematical methods, including perturbative renormalization schemes [20]. This means that background independence and gauge symmetry should necessarily be associated not only with the property of covariance but, preferably, with that of manifest covariance too. In other words, all relevant classical and quantum equations and observables should be cast in explicit $4-$tensor form with respect to a background space-time metric field tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$. Nevertheless, the precise prescription of the same tensor field $\widehat{g}(r)$ should remain not only coordinate-independent but also arbitrary within a suitable functional setting to be identified.

3. The Metric-Ricci and the $g-$Quantization of GR

The quantization strategy to be developed here in the framework of a manifestly covariant theory involves, in principle, the simultaneous quantization of the $4-$tensor variables $(g(r), R(r))$, i.e., realizing the so-called metric-Ricci quantization of GR. Therefore, the starting point is the corresponding metric-Ricci Hamiltonian representation of GR established in Part I. This is expressed in terms of the variational Lagrangian $4-$tensor variables $(g(r), R(r))$, i.e., the independent $4-$tensor variational fields that characterize the same representation. The consequence is that quantization should, in principle, be performed simultaneously (and independently) on the same $4-$tensor variables $(g(r), R(r))$. However, according to Refs. [5,11], $g-$quantization alone requires, by itself, setting identically equal to zero the classical cosmological constant, which appears in the classical Hamiltonian density, i.e., letting

$$\Lambda^{(c)} \equiv 0. \tag{7}$$

Such a choice is consistent with the expectation that the cosmological constant should have purely a quantum origin. That this choice is indeed allowed and makes sense follows by noting that, at the classical level, the cosmological constant remains arbitrary. Indeed, in the framework of the synchronous variational approach [12], the occurrence of the classical CC can be shown to arise merely as a consequence of appropriate variational constraints. As a consequence, the classical CC can actually be identified with one of the Lagrangian multipliers. The virtue of the constrained variational principle reported in the same reference is that the same Lagrange multiplier (i.e., the classical CC) is left undetermined, so that the choice (7) can always be satisfied.

The implication of Equation (7) is that the coupling constant α_1 in Equation (5) must also vanish identically in the same limit. This is also a mandatory prerequisite in order for the same metric-Ricci extended Hamiltonian representation to be defined at all. Based on these premises, one concludes that the Hamiltonian coupling H_{g+R} defined by Equation (5) actually reduces identically to the sole metric-Hamiltonian, i.e., $H_g \equiv H_g(x(r), \widehat{g}(r))$, where $H_g(x(r), \widehat{g}(r))$ identifies the $4-$scalar function

$$H_g(x(r), \widehat{g}(r)) = T_g + \sigma \overline{V}_g(g(r), \widehat{g}(r)). \tag{8}$$

As a consequence, the customary manifestly covariant $g-$quantization (see cited references) is recovered at once. Remarkably, this happens under the same conditions that warrant validity of the same $g-$quantization.

This means that no quantization of the Ricci tensor is actually required; so, in principle, the standard $g-$quantization procedure is recovered. However, here, such a goal is conveniently realized by the theory of stochastic quantization of CQG-theory. In the following, we summarize and extend its formulation, specializing the discussion to the subject of the paper, namely, the implications of background independence and gauge symmetry in the framework of a manifestly covariant quantum theory.

3.1. Stochastic Quantization in CQG-Theory

Why do we need stochastic quantization and what does this mean in practice? An established theoretical result in the context of non-relativistic quantum mechanics is that quantum particle trajectories, which span the 3D Euclidean space, should be non-deterministic in character. Therefore, an analogous feature should be retained also in CQG-theory, where the geodesic trajectories with respect to which the manifestly-covariant Hamiltonian representation of GR is achieved may be interpreted as graviton trajectories [5]. Therefore, one expects that such (geodesic) trajectories should actually be replaced by suitable stochastic quantum trajectories. Although physically intuitive, a possible explanation is required. Stochasticity of space-time quantum trajectories may occur for two independent reasons:

- The first one is the intrinsic stochasticity of space-time quantum trajectories. Their identification with deterministic trajectories, as performed in the original formulation of Bohmian quantum mechanics [21], may be viewed as physically incorrect because this is in potential violation of the Heisenberg indeterminacy principle. The proof of their stochastic character has actually been reached rigorously in the context of QM. Therefore, in close analogy, it appears reasonable to expect that graviton particle trajectories that span the background space-time could similarly acquire a stochastic character and therefore depart from purely deterministic geodetics as typically assumed in QG-theory.
- The second reason that motivates the allowance of stochastic space-time trajectories is the extended feature of quantum particles. Accordingly, all quantum particles (including graviton particles), in a strict sense, should not be treated (any more) as point-like particles. The corresponding physical motivations being either (a) the proper treatment of particle self-interactions, which cannot simply be defined in the case of point-particles (see for example the case of electromagnetic self-interactions), or (b) the issue of possible quantum regularization of space-time singularities.
- Finally, we add here a third and actually crucial motivation: this is realized by the fact that it becomes possible to represent the quantum-modified EFE (which determines the same background field tensor) by performing the quantum-average (or quantum expectation value) of a suitably determined quantum Hamilton equation. However, by construction, the quantum expectation value now also includes suitable stochastic averaging on the stochastic trajectories. This stochastic-averaging effect is believed to be crucial for determining the qualitative behavior—i.e., regularization—of the background metric tensor at microscopic scales, i.e., at the Planck scale length.

However, there is yet another type of stochasticity that may occur. This concerns the possible appearance of explicit stochastic gauge contributions in the quantum Hamiltonian operator. Such a type of contribution—as discussed below in Section 4—may actually be of crucial importance for constructing the q-modified EFE in general form, i.e., in the presence of stochastic trajectories, which is actually the problem of interest here. Indeed, these stochastic gauge contributions are, in turn, responsible for the prescription of the background metric field tensor $\widehat{g}(r)$.

The resulting new scheme of canonical quantization for GR will be denoted as stochastic quantization. Such a quantization scheme is based on the introduction of two additional, non-trivial transformations, referred to as *stochastic quantization transformations*:

(A) The inclusion of stochastic quantum trajectories. In a trajectory-dependent theory such as CQG-theory, this requires replacing the classical (deterministic, i.e., geodesic) space-time trajectories $\{r(s)\}$ with stochastic quantum trajectories $\left\{r^{(q)}(s)\right\}$. Here, the $4-$scalar parameter s identifies the proper-time determined along a suitable subset of classical geodesic curves by the relation $ds^2 = \widehat{g}_{\mu\nu}dr^\mu dr^\nu$. Thus, stochastic quantization must necessarily involve the additional synchronous transformation (i.e., in which the proper time s is left unchanged):

$$\{r(s)\} \rightarrow \left\{r^{(q)}(s)\right\}. \tag{9}$$

Here, $r^{(q)}(s)$ is a stochastic function of the type

$$r^{(q)}(s) \equiv r^{q}(r(s),\alpha), \tag{10}$$

which depends on $r(s)$, i.e., the deterministic $4-$position, at a given proper-time s, along the (similarly deterministic) geodesic curve $\{r(s)\}$ defined with respect to the classical background metric tensor $\widehat{g}(r)$ [12]. For greater simplicity of notations, in the following, we shall use also the symbolic notation

$$r^{q}(r(s),\alpha) \equiv r(s,\alpha). \tag{11}$$

We shall require that, by assumption, all the quantum trajectories $\{r(s,\alpha)\}$ satisfy the constraint

$$\lim_{\alpha\to 0^{\pm}} r(s,\alpha) = r(s). \tag{12}$$

In all previous Equations (10)–(12), α denotes a dimensionless hidden variable, i.e., a stochastic independent $4-$scalar parameter (i.e., not depending on r or s) and $r(s,\alpha)$ is a space-time stochastic curve, which is assumed to be such that the displacement $r(s,\alpha)-r(s)$ is in some sense suitably small with respect to the characteristic scale length of the geodesics $r(s)$. For definiteness, here, we shall assume in particular that α belongs to the finite set (stochasticity domain)

$$I_\alpha = [-a,a] - I_\varepsilon, \tag{13}$$
$$I_\varepsilon = [-\varepsilon,\varepsilon]. \tag{14}$$

Here, ε is a dimensionless parameter such that $\varepsilon^2 \ll 1$, a is either $a = 1,\infty$, and α is assumed to be endowed with a stochastic Gaussian PDF of the type

$$g_\alpha(\varepsilon) = N\exp\left\{-\alpha^2/\varepsilon^2\right\}\overline{\Theta}(\alpha^2-\varepsilon^2). \tag{15}$$

However, we stress that, in principle, the choice of stochastic PDF remains arbitrary, $\overline{\Theta}(x)$ is the strong Heaviside theta function $\overline{\Theta}(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases}$ and N is a normalization constant defined such that

$$\langle 1\rangle_\alpha \equiv \int_{I_\alpha} d\alpha g_\varepsilon(\alpha) = 1, \tag{16}$$

namely,

$$N^{-1} = 2\int_\varepsilon^1 d\alpha \exp\left\{-\alpha^2/\varepsilon^2\right\} = \varepsilon\sqrt{\pi}\left[erf(\frac{1}{\varepsilon}) - erf(1)\right], \tag{17}$$

while $\langle\bullet\rangle_\alpha$ denotes the stochastic $\alpha-$average

$$\langle\bullet\rangle_\alpha \equiv \int_{I_\alpha} d\alpha \bullet g_\varepsilon(\alpha). \tag{18}$$

Thus, according to this choice, α is assumed as being endowed with a vanishing stochastic expectation value $\langle\alpha\rangle_\alpha = 0$ and standard deviation $\sigma_\alpha = \sqrt{\left\langle \alpha^2 - \langle\alpha\rangle_\alpha^2 \right\rangle_\alpha} \sim O(\varepsilon^2)$. As a consequence, it follows that, by construction and without loss of generality, α can always be assumed to coincide with the stochastic parameter introduced previously in Ref. [9], which enters in the quantum cosmological constant Λ_{CQG} and determines its quantum stochastic property.

(B) The inclusion of stochastic contributions in the Quantum Hamiltonian operator (which vanish identically in the case of classical Hamiltonian function). This is represented by a transformation of the type

$$H_R\left(g, \frac{\partial S}{\partial g}, \widehat{g}(s), r(s)s\right) \rightarrow H_R^{(q)}(g, -i\hbar\frac{\partial}{\partial g}, \widehat{g}(r(s,\alpha)), r(s,\alpha), s) = H_R\left(g, -i\hbar\frac{\partial}{\partial g}, \widehat{g}(r(s,\alpha)), r(s,\alpha), s\right) + \Delta V \tag{19}$$

where, for simplicity of notation, on the rhs of the last equation and in all subsequent equations, H_R, $H_R^{(q)}$ and ΔV represent, respectively, the classical Hamiltonian function of GR (see Equation (A1) in Appendix A), quantum Hamiltonian and a suitable stochastic gauge contribution to be discussed in detail below.

3.2. Stochastic Canonical Map and Stochastic CQG-Quantum-Wave Equation

It is to be noted that the inclusion of Equations (9) and (19) in the canonical quantization involves necessarily a suitable reformulation of the quantization scheme developed originally for CQG-theory. In detail, in terms of the Hamilton–Jacobi $g-$quantization scheme first developed in Ref. [11], the starting point is provided by the proper redefinition of the mapping, which prescribes the $4-$scalar quantum wave-function $\psi(s)$ in terms of the corresponding $4-$scalar classical Hamilton principal function $S(s) \equiv S(g, \widehat{g}, r(s), s)$:

$$S(s) \rightarrow \psi(s). \tag{20}$$

Notice that transformation (9) applies to all fields; so, in particular, the background metric tensor $\widehat{g}(r(s))$ and the background Ricci tensor, defined in terms of the background metric field tensor $\widehat{g}(r(s)$ as $\widehat{R}_{\mu\nu} = R_{\mu\nu}(\widehat{g}(r(s))$, transform as

$$\widehat{g}(s) \equiv \widehat{g}(r(s)) \rightarrow \widehat{g}^{(q)}(s) \equiv \widehat{g}(r(s,\alpha)), \tag{21}$$

$$\widehat{R}_{\mu\nu} = R_{\mu\nu}(\widehat{g}(r(s))) \rightarrow \widehat{R}_{\mu\nu}^{(q)} \equiv R_{\mu\nu}(\widehat{g}(r(s,\alpha))), \tag{22}$$

while the canonical fields $x = \left\{g_{\mu\nu}, \pi_{\mu\nu} \equiv \frac{\partial S(g,\widehat{g},r(s),s)}{\partial g^{\mu\nu}}\right\}$ transform according to the stochastic canonical quantization map (where, for simplicity, we use the same symbols for the quantum variables)

$$\left\{\begin{array}{c} g(s) \equiv g(r(s), s) \\ \pi(s) \equiv \pi(r(s), s) = \frac{\partial S(g,\widehat{g},r(s),s)}{\partial g} \\ H_R\left(g, \frac{\partial S}{\partial g}, \widehat{g}(s), r(s)s\right) \end{array}\right. \rightarrow \left\{\begin{array}{c} g_{\mu\nu}(s) = g_{\mu\nu}(r(s,\alpha), s) \\ \pi_{\mu\nu}(s) \equiv -i\hbar\frac{\partial}{\partial g^{\mu\nu}(r(s,\alpha),s)} \\ H_R^{(q)}\left(g, -i\hbar\frac{\partial}{\partial g}, \widehat{g}(r(s,\alpha), r(s,\alpha), s\right) \end{array}\right. . \tag{23}$$

Regarding Equation (20), on its left-hand side, $S(s) \equiv S(g, \widehat{g}, r(s), s)$ and the fields $(g, \widehat{g}) = (g, \widehat{g}(r(s)))$ are all evaluated at the Lagrangian coordinate g and along the same (generic) classical geodesics $\{r(s)\}$. Instead, on its rhs, $\psi(s)$ is assumed as a complex function of the type

$$\psi(s) \equiv \psi(g, \widehat{g}(r(s,\alpha), r(s,\alpha), s), \tag{24}$$

i.e., evaluated for an arbitrary Lagrangian coordinate g, at the proper time s and along an (arbitrary) stochastic space-time trajectory $r^{(q)}(s)$ (defined according to Equation (10)). Similarly, while in the classical Hamiltonian H_R the canonical momenta π are represented in terms of the Hamilton Principal function by letting $\pi_{\mu\nu} = \frac{\partial S(g,\widehat{g},r(s),s)}{\partial g^{\mu\nu}}$, in the corresponding quantum Hamiltonian operator $H_R^{(q)}(g, \pi, \widehat{g}(r(s,\alpha), r(s,\alpha), s)$, they are prescribed according to Equation (23) and the same operator is evaluated with respect to the space-time trajectory $r^{(q)}(s) \equiv r(s,\alpha)$. Finally, the Hamilton–Jacobi quantization, which maps in each other the Hamilton–Jacobi equation and the corresponding quantum-wave equation, now delivers

$$\boxed{\frac{dS}{ds} + H_R = 0} \rightarrow \boxed{\left[-i\hbar\frac{d}{ds} + H_R^{(q)}\right]\psi(s) = 0,} \tag{25}$$

where the equation on the rhs is referred to as stochastic CQG-quantum-wave equation. The same equation advances in time along the space-time stochastic trajectories (9) the stochastic quantum wave-function $\psi(s)$, which is assumed to be of the type (24). Notice that in both of the above equations, the operator $\frac{d}{ds}$ denotes the covariant $s-$derivative (A3), which acts both on the explicit and implicit dependencies contained through $r(s)$ and $r(s,\alpha)$, respectively, and is performed, instead, by keeping constant the Lagrangian coordinate g (see notations reported in Appendix A).

3.3. Quantum Hydrodynamic Equations

The CQG-wave equation on the rhs of Equation (25) has formal similarities with the Schrödinger equation of non-relativistic quantum mechanics, retaining in particular its linear dependence in terms of the wave-function $\psi(s)$ and its evolution type (i.e., hyperbolic PDE). The analogy includes also its equivalence with a corresponding set of suitable quantum hydrodynamics equations (QHE). The same equations are recovered at once from the quantum-wave equation indicated above upon introducing an exponential representation for the complex field $\psi(s)$, i.e., the so-called Madelung representation

$$\psi(g,\widehat{g}(r(s,\alpha),r(s,\alpha),s) = \sqrt{\rho}\exp\left\{\frac{i}{\hbar}\mathcal{S}^{(q)}\right\}. \tag{26}$$

Here, the quantum fluid fields $\{\rho(g),S^{(q)}(g)\} \equiv \{\rho(g,\widehat{g}(r(s,\alpha),r(s,\alpha),s),\mathcal{S}^{(q)}(g,\widehat{g}(r(s,\alpha),r(s,\alpha),s)\}$ identify, respectively, the $4-$scalar configuration-space quantum probability density function (quantum PDF) and quantum phase function, both considered as functions of the Lagrangian coordinates. As a result, the same quantum fluid fields can be shown to satisfy the set of GR-quantum hydrodynamic equations (CQG-QHE). These are identified with the set of PDE in Eulerian form, i.e., the quantum continuity and quantum Hamilton–Jacobi equations [5]:

$$\frac{d\rho(g)}{ds} + \frac{\partial}{\partial g_{\mu\nu}}(\rho(g)V_{\mu\nu}(g)) = 0, \tag{27}$$

$$\frac{d\mathcal{S}^{(q)}(g)}{ds} + H^{(q)}(g) = 0, \tag{28}$$

where $\frac{d}{ds}$ denotes again the covariant $s-$derivative (A7) acting in the sense indicated above, while $\rho(g)$ and $S^{(q)}(g)$ are both considered functions of the Lagrangian coordinate g, which spans the configuration space U_g. Here, κ is a dimensional constant, which is related to the graviton mass estimate given in the same reference while

$$V_{\mu\nu}(g) \equiv V_{\mu\nu}(g,\widehat{g}(r(s,\alpha),r(s,\alpha),s) \equiv \frac{1}{\kappa}\frac{\partial\mathcal{S}^{(q)}}{\partial g^{\mu\nu}}, \tag{29}$$

denotes the so-called tensor "velocity" field, depending on the Lagrangian coordinate g. Furthermore, $H^{(q)}$ denotes the *effective quantum Hamiltonian*

$$H^{(q)}(g) = \frac{1}{2\kappa}\frac{\partial\mathcal{S}^{(q)}}{\partial g^{\mu\nu}}\frac{\partial\mathcal{S}^{(q)}}{\partial g_{\mu\nu}} + V_{QM}(g) + V_o(g) + \Delta V. \tag{30}$$

In the case of vacuum conditions, namely, the absence of classical sources, V_o and V_{QM} identify, respectively, the vacuum effective potential and quantum Bohm interaction potential [21] given by

$$V_o(g) = \kappa\left(2 - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}\right)g^{\alpha\beta}\widehat{R}_{\alpha\beta}, \tag{31}$$

$$V_{QM}(g) \equiv \frac{\hbar^2}{8\kappa}\frac{\partial\ln\rho(g)}{\partial g^{\mu\nu}}\frac{\partial\ln\rho(g)}{\partial g_{\mu\nu}} - \frac{\hbar^2}{4\kappa}\frac{\partial^2\rho(g)}{\rho\partial g_{\mu\nu}\partial g^{\mu\nu}}, \tag{32}$$

where $\widehat{R}_{\alpha\beta}$ denotes now the Ricci tensor evaluated in terms of the background metric tensor $\widehat{g}_{\alpha\beta}(r(s,\alpha))$.

3.4. GLP Representation

The ensemble of QHE (27) and (28) is set in Eulerian form, with the quantum fluid fields being therefore parametrized in terms of the Lagrangian coordinate g. The corresponding Lagrangian representation can be obtained by first introducing the Lagrangian-path (LP) $\{g(s,\alpha)\}$, where the dependence on the stochastic parameter α now appears both explicitly through $V_{\mu\nu}(g(s),\widehat{g}(r(s,\alpha),r(s,\alpha),s)$ and implicitly through $\widehat{g}(r(s,\alpha))$, while the $s-$dependent tensor field $g(s,\alpha) \equiv \{g_{\mu\nu}(s,\alpha)\}$ is a solution of the initial-value problem

$$\left\{ \begin{array}{c} \frac{d}{ds}g_{\mu\nu}(s,\alpha) = V_{\mu\nu}(g(s),\widehat{g}(r(s,\alpha),r(s,\alpha),s) \\ g(s_o,\alpha) = g_o(\widehat{g}(r(s_o,\alpha),r(s_o,\alpha),s_o) \end{array} \right. . \tag{33}$$

Therefore, introducing the operator of Lagrangian covariant derivative

$$\frac{D}{Ds} = \frac{d}{ds} + V_{\mu\nu}(g)\frac{\partial}{\partial g_{\mu\nu}(s)}, \tag{34}$$

the corresponding Lagrangian-path form (LP-form) of the continuity equation for (27) becomes

$$\frac{D}{Ds}\rho(g) = -\rho(g)\left|\frac{\partial V_{\mu\nu}}{\partial g_{\mu\nu}}\right|_{g=g(s,\alpha)}. \tag{35}$$

As shown elsewhere [8], an explicit solution of the quantum continuity Equation (35) can be obtained by introducing the notion of Generalized Lagrangian Path (GLP), namely, the configuration-space trajectory $\{G(s,\alpha)\}$ defined by the tensor field transformation

$$G_{\mu\nu}(s,\alpha) = g_{\mu\nu}(s,\alpha) - \Delta g_{\mu\nu} \tag{36}$$

with $\Delta g_{\mu\nu} \equiv \Delta g_{\mu\nu}(s)$ denoting a symmetric tensor field. Here, by assumption, the tensor displacement field $\Delta g_{\mu\nu}$ is assumed such that 1) its covariant $s-$derivative (A3) vanishes identically, namely,

$$\frac{d}{ds}\Delta g_{\mu\nu}(s) = 0. \tag{37}$$

Notice that, however, in general, still $\Delta g_{\mu\nu} = \Delta g_{\mu\nu}(s)$ because the background metric field $\widehat{g}(s,\alpha)$ is generally $s-$dependent. 2) It is a symmetric and stochastic tensor field, which is assumed to be endowed with a suitably prescribed probability density. As a consequence of Equations (33) and (37), it follows that the GLP $\{G(s)\}$ is a necessary solution of the GLP equations

$$\left\{ \begin{array}{c} \frac{d}{ds}G_{\mu\nu}(s,\alpha) = V_{\mu\nu}(G(s)+\Delta g,\widehat{g}(s,\alpha),r(s,\alpha),s) \\ G(s_o,\alpha) = g_o(\widehat{g}(r(s_o,\alpha),r(s_o,\alpha),s_o) - \Delta g \end{array} \right. . \tag{38}$$

Therefore, the set of QHE can equivalently be set in GLP-form by introducing the formal replacement

$$g(s,\alpha) \equiv \{g_{\mu\nu}(s,\alpha)\} \equiv \{g^{\mu\nu}(s,\alpha)\} \rightarrow G(s,\alpha) \equiv \{G_{\mu\nu}(s,\alpha)\} \equiv \{G^{\mu\nu}(s,\alpha)\}. \tag{39}$$

Notice that here $g(s,\alpha) \equiv g(r(s,\alpha),s)$ and $G(s,\alpha) \equiv G(r(s,\alpha),s)$ denote, respectively, the LP and GLP solutions defined with respect to the stochastic space-time trajectory $r(s,\alpha)$. Thus, in particular, the stochastic character of CQG-theory in this representation emerges as a consequence of Equation (38). The meaning of Equation (38) is that for each Lagrangian trajectory $\{g_{\mu\nu}(s), s \in I\}$ that occurs in the configuration space U_g, there are infinite stochastic GLP's $\{G_{\mu\nu}(s), s \in I\}$ that belong to the same configuration space, each one corresponding to a different value of the stochastic displacement field $\Delta g_{\mu\nu}(s)$.

The GLP approach, despite its obvious non-uniqueness feature that is intrinsic in the concept of GLP, has an important implication: it allows for the construction of an explicit analytical solution for the quantum PDF $\rho(s)$. For definiteness, let us introduce preliminarily the stochastic displacement (see Equation (28) in Ref. [9])

$$\Delta g_{\mu\nu}(s,\alpha) = \Delta g_{\mu\nu}(s) + \alpha \widehat{g}_{\mu\nu}(s,\alpha). \tag{40}$$

Then, as shown in the same reference (extending the formulation of Ref. [10]) the formal solution of the quantum continuity Equation (27) can be represented in terms of a stochastic PDF of the type

$$\rho(g(s,\alpha)) = \rho_G(\Delta g(s,\alpha) - \widehat{g}(s,\alpha)) \exp\left\{ -\int_{s_o}^{s} ds' \frac{\partial V_\nu^\mu(s')}{\partial g_\nu^\mu(s')} \right\}, \tag{41}$$

where $V_\nu^\mu(s)$ is the tensor velocity field (29) assumed independent of α and $\rho_G(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))$ is the shifted Gaussian PDF expressed by

$$\rho_G(\Delta g(s,\alpha) - \widehat{g}(s,\alpha)) = K \exp\left\{ -\frac{(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2}{r_{th}^2} - \frac{\alpha^2}{\varepsilon^2} \right\}, \tag{42}$$

with $\varepsilon \ll 1$ being a suitably small dimensionless number and the normalization factor K defined so that

$$K = \left[\int_{U_g} d(\Delta g(s)) \int_{-\infty}^{+\infty} d\alpha N \exp\left\{ -\frac{(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2}{r_{th}^2} - \frac{\alpha^2}{\varepsilon^2} \right\} \overline{\Theta}(\alpha^2) \right]^{-1}. \tag{43}$$

Here, r_{th}^2 is the constant dimensionless invariant semi-amplitude width of the Gaussian quantum PDF. Furthermore, the exponent $(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2$ stands for the 4−scalar, which can be represented as $(\Delta g(\alpha) - \widehat{g}(s,\alpha))^2 \equiv (\Delta g(s,\alpha) - \widehat{g}(s,\alpha))_{\mu\nu} (\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^{\mu\nu}$. Now, one notices that by construction

$$\frac{d}{ds} (\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2 = 0, \tag{44}$$

which implies, in turn,

$$(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2 = (\Delta g(s_o,\alpha) - \widehat{g}(s_o,\alpha))^2. \tag{45}$$

Finally, we notice that the 4−scalar $(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2$ can also be equivalently represented in terms of the mixed covariant/counter-variant components of the tensor field $\Delta g(s,\alpha) - \widehat{g}(s,\alpha)$. One obtains that $(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))^2 \equiv (\Delta g(s,\alpha) - \widehat{g}(s,\alpha))_\mu^\nu$ $(\Delta g(s,\alpha) - \widehat{g}(s,\alpha))_\nu^\mu$, where $\widehat{g}_\nu^\mu(s,\alpha) = \delta_\nu^\mu$. Thus, the constraint Equation (45) has the consequence that for arbitrary $s, s_o \in I$, one can always set

$$\Delta g_\nu^\mu(s,\alpha) = \Delta g_\nu^\mu(s_o,\alpha) \equiv \Delta g_\nu^\mu(\alpha), \tag{46}$$

with the tensor components $\Delta g_\nu^\mu(\alpha)$ being, therefore, s−independent. As a consequence, the quantum PDF (41) can be equivalently written as

$$\rho(g(s,\alpha)) = \rho_G(\Delta g(s_o,\alpha) - \widehat{g}(s_o,\alpha)) \exp\left\{ -\int_{s_o}^{s} ds' \frac{\partial V_\nu^\mu(s')}{\partial g_\nu^\mu(s')} \right\}, \tag{47}$$

$$\rho(g(s_o,\alpha)) = \rho_G(\Delta g(s_o,\alpha) - \widehat{g}(s_o,\alpha)), \tag{48}$$

where now $\rho(g(s_o,\alpha)) = K\exp\left\{-\frac{(\Delta g(s_o,\alpha)-\widehat{g}(s_o,\alpha))^2}{r_{th}^2} - \frac{\alpha^2}{\varepsilon^2}\right\}$ and one can readily show that the normalization constant K (defined by Equation (43) above) becomes $s-$independent too.

3.5. Quantum Expectation Value and Stochastic Averaging

A crucial aspect of QG-theory is the proper definition of the related Hilbert space based on a suitable definition of the scalar product and, in particular, the relationship between quantum expectation values and stochastic averaging operation. One can show that the two averaging operations coincide in the case of integrable quantum tensor functions [10]. The result is not new; however, it is useful to restate its validity also in the present setting. To reach the proof of the statement, let us first recall the definition of the *local scalar product*, which in the present case is identified with

$$\langle\psi_a|\psi_b\rangle_{L,\alpha} \equiv \langle\langle\psi_a|\psi_b\rangle_L\rangle_\alpha, \tag{49}$$

with $\langle\bullet\rangle_\alpha$ denoting the stochastic average (18) and $\langle\psi_a|\psi_b\rangle_L$ the *configuration-space average:*

$$\langle\psi_a|\psi_b\rangle_L \equiv \int\limits_{U_g} d(g)\psi_a^*(g,\widehat{g}(r),r(s),s)\psi_b(g,\widehat{g}(r),r(s),s). \tag{50}$$

Furthermore, $\psi_a \equiv \psi$, ψ_b is taken of the form $\psi_b \equiv M(g)\psi$ and $M(g)$ denotes a suitable quantum operator/function such that $\langle\psi_a|\psi_b\rangle_L$ exists. Now, upon introducing the parametrization $g = g(s,\alpha)$ and considering $M(g(s,\alpha))$ as an ordinary real tensor function of $g = g(s,\alpha)$, it follows that the quantum expectation value of $M(g(s,\alpha))$ can be equivalently represented as

$$\langle\psi|M\psi\rangle_{L,\alpha} \equiv \langle\langle\psi_a|\psi_b\rangle_L\rangle_\alpha = \left\langle\int\limits_{U_g} d(g(s,\alpha))\rho(g(s.\alpha))M(g(s,\alpha))\right\rangle_\alpha. \tag{51}$$

We further notice that, thanks to Equation (27), by construction

$$d(g(s,\alpha))\rho(g(s.\alpha)) = d(g(s_o,\alpha))\rho(g(s_o.\alpha)) = d(g(s_o,\alpha))\rho_G(\Delta g(s_o,\alpha) - \widehat{g}(s_o,\alpha)), \tag{52}$$

while $d(g(s_o,\alpha)) = d(\Delta g(s_o))$ and due to Equation (46), which warrants that $d(\Delta g(s_o)) = d(\Delta g)$, the desired result follows, namely, the identity between quantum expectation value and stochastic averaging

$$\langle\psi|M\psi\rangle_{L,\alpha} = \langle\psi|M\psi\rangle, \tag{53}$$

applying for an arbitrary integrable ordinary real tensor function $M(g(s,\alpha))$. On the rhs, the bracketed symbol $\langle\psi|M\psi\rangle$ denotes, now, the operation of stochastic averaging, namely, (upon invoking Equation (40)):

$$\langle\psi|M\psi\rangle = \int_{U_g} d(\Delta g)\int\limits_{-\infty}^{+\infty} d\alpha NK\exp\left\{-\frac{(\Delta g-(1-\alpha)\widehat{g}(s,\alpha))^2}{r_{th}^2} - \frac{\alpha^2}{\varepsilon^2}\right\}\overline{\Theta}(\alpha^2)M(g(s,\alpha)). \tag{54}$$

4. Two Construction Methods for the Quantum-Modified EFE

A notable implication of CQG-theory concerns the establishment of the quantum Hamilton–Jacobi Equation (28). In fact, as discovered in Ref. [8], it implies, in analogy to what happens in quantum mechanics, a quantum Hamiltonian structure analogous to that holding for the classical GR-Hamilton equations, which can therefore be established also for the quantum hydrodynamic state. Such a Hamiltonian structure is represented by the set $\left\{x, H^{(q)}\right\}$, with $x(s) \equiv (g_{\mu\nu}(s), \Pi^{\mu\nu}(s))$ denoting a $4-$tensor canonical state, formed by the Lagrangian coordinate $g_{\mu\nu}(s)$ and $\Pi^{\mu\nu}(s) = \frac{\partial S^{(q)}}{\partial g_{\mu\nu}}$ its canonically conjugate moment, and with $H^{(q)}$ being the effective quantum Hamiltonian defined by Equation (30). This permits to represent equivalently the quantum Hamilton–Jacobi (28) in terms of the quantum Hamilton equations:

$$\frac{d}{ds}g^{\mu\nu}(s) = \frac{\Pi^{\mu\nu}}{\alpha L}, \tag{55}$$

$$\frac{d}{ds}\Pi_{\mu\nu}(s) = -\frac{\partial}{\partial g^{\mu\nu}}(V_o + V_{QM} + \Delta V), \tag{56}$$

with $\frac{d}{ds}$denoting the covariant $s-$derivative (A3), which are subject to generic initial conditions of the type $x(s_o) = x_o \equiv \left(g^{\mu\nu}_{(o)} \equiv g^{\mu\nu}(s_o), \Pi_{(o)\mu\nu} \equiv \Pi_{\mu\nu}(s_o)\right)$. We stress that the contribution to the effective potential ΔV remains undetermined. At this point, however, we shall require that the quantum-modified Einstein equations can still be formally recovered, i.e., in this sense, ΔV must be a gauge contribution.

4.1. First Construction Method: Deterministic Limit

Thus, adopting first the guidelines of Ref. [9], the q–modified EFE is obtained under appropriate initial conditions to be set on the quantum Hamilton Equations (55) and (56), and in a suitable deterministic limit, i.e., in which all stochastic parameters vanish. In detail, these conditions are prescribed as follows:

Requirement #1: the initial state $x(s_o)$ for the initial-value problem associated with the quantum Hamilton Equations (55) and (56) is prescribed by imposing vanishing initial conditions, namely, such that

$$x(s_o) = \left(g^{\mu\nu}_{(o)} \equiv g^{\mu\nu}(s_o), \Pi_{(o)\mu\nu} \equiv 0\right). \tag{57}$$

This means that the initial quantum tensor $g^{\mu\nu}$ coincides with the background one and its corresponding momentum (i.e., its covariant derivative) is identically vanishing. Notice that in the GLP-representation given by Equation (38), this means that in validity of (62) one obtains

$$g_{\mu\nu}(s) = \Delta g_{\mu\nu} + G_{\mu\nu}(s) = G_{\mu\nu}(s). \tag{58}$$

Hence, requiring

$$G_{\mu\nu}(s) = \widehat{g}_{\mu\nu}(s) \tag{59}$$

is equivalent to let

$$g_{\mu\nu}(s) = \widehat{g}_{\mu\nu}(s). \tag{60}$$

Requirement #2: this is obtained performing the limit conditions

$$\Delta g_{\mu\nu}(s,\alpha) \rightarrow 0, \tag{61}$$

$$\alpha \rightarrow 0. \tag{62}$$

The first limit (61) is referred to as *deterministic GLP-limit condition*, while the second one (62) is the analogous *deterministic a-limit condition.* Their physical meanings are intuitive: the first one amounts to imposing that the stochastic GLP quantum trajectories $\{g_{\mu\nu}(s), s \in I\}$, which drive the quantum wave-function, collapse on the single deterministic Lagrangian-path (LP) Bohmian trajectory $\{G_{\mu\nu}(s), s \in I\}$ indicated above. The second one means dropping the stochastic effect due to multiple (stochastic) space-time trajectories and, at the same time, due to the effect of a stochastic quantum cosmological constant.

Requirement #3: one takes the limit condition in which the quantum cosmological constant is $s-$independent, i.e.,

$$\Lambda_{CQG}(s) \rightarrow \Lambda_{CQG} \equiv \frac{\hbar^2}{\kappa^2}\frac{1}{r_{th}^4}, \tag{63}$$

where $\Lambda_{CQG}(s)$ and Λ_{CQG} denote, respectively, the $s-$dependent and constant quantum cosmological constant.

Requirement #4: finally, since by assumption ΔV is a (still undetermined) gauge contribution, we shall assume ΔV to be such that (a) it vanishes identically under Requirement #2, namely,

$$\begin{cases} \lim_{\Delta g_{\mu\nu}(s,\alpha)\to 0} \Delta V = 0, \\ \lim_{\alpha\to 0} \Delta V, \end{cases} \tag{64}$$

(b) it is constant, i.e., it satisfies identically the dynamical constraint

$$\frac{d}{ds}\Delta V = 0. \tag{65}$$

This condition warrants that the effective Hamiltonian $H^{(q)}$ defined by Equation (30) preserves its properties for all proper times s.

While it is obvious that Requirement #4 does not determine uniquely the gauge potential ΔV, the order in which assumptions (1)–(4) are taken is irrelevant. This means, in particular, that condition (2) can be equivalently replaced requiring first Equation (60).

For definiteness, let us impose first Requirement #1 and, furthermore, require validity of the limit (61). In such a setting, and without performing the semiclassical limit $\hbar \to 0$, it is straightforward to show that the quantum Hamilton Equations (55) and (56) yield the following tensor equation, which still depends on the stochastic variable α. This determines the so-called *stochastic quantum-modified EFE:*

$$R_{\mu\nu}(\widehat{g}(r(s,\alpha))) - \frac{1}{2}R(\widehat{g}(r(s,\alpha)))\widehat{g}_{\mu\nu}(r(s,\alpha),s) = T_{\mu\nu}(\widehat{g}(r(s,\alpha))) - B_{\mu\nu}(s,\alpha), \tag{66}$$

where

$$B_{\mu\nu}(s,\alpha) \equiv -\frac{1}{\kappa}\frac{\partial}{\partial g^{\mu\nu}}V_{QM}\bigg|_{g=\widehat{g}} = \Lambda_{CQG}(s)(1-\alpha)\widehat{g}_{\mu\nu}(r(s,\alpha)) \tag{67}$$

and

$$\Lambda_{CQG}(s) = \frac{\hbar^2}{\kappa^2}\frac{f(s)}{r_{th}^4} = \Lambda_{CQG}(s_o)f(s) \tag{68}$$

denote, respectively, the Bohm potential source term and the quantum cosmological constant. Regarding $B_{\mu\nu}(s,\alpha)$, this is to be distinguished from the tensor $B_{\mu\nu}$ obtained in Ref. [8] because of the dependence on the stochastic parameter α appearing now in the quantum PDF. Furthermore, $f(s)$ denotes the suitably prescribed $4-$scalar function reported in Ref. [8]. Its value for the initial condition $s = s_o$ is such that $f(s_o) = 1$. This function carries the proper-time dependence of the quantum-gravity cosmological constant, which is transferred to the metric tensor $\widehat{g}_{\mu\nu}$. Finally, performing also the limit (62) and considering the stationary solution $f(s) = 1$, one obtains

$$\widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{R}\widehat{g}_{\mu\nu} = \widehat{T}_{\mu\nu} - \Lambda_{CQG}\widehat{g}_{\mu\nu}, \tag{69}$$

where

$$\Lambda_{CQG} = \frac{\hbar^2}{\kappa^2}\frac{1}{r_{th}^4} \tag{70}$$

denotes the constant quantum cosmological constant.

4.2. Second Construction Method: Stochastic Averaging

Let us now consider an alternative approach for the derivation of the stochastic-averaged q-modified EFE. Such a method is actually appropriate for the treatment of stochastic trajectories; however, at the same time, it is based on inclusion of the quantization transformation (19), which acts on the quantum Hamiltonian operator. This is obtained by replacing the previous *Requirement #2 and #4* with the following ones:

Alternate requirement #2: This is obtained performing the stochastic averaging of the stationary Hamilton equation, namely, by evaluating explicitly the stochastic average

$$\left\langle \psi | \frac{\partial}{\partial g^{\mu\nu}} (V_o + V_{QM}) \psi \right\rangle = \int_{U_g} d(\Delta g) \int_{-\infty}^{+\infty} d\alpha \frac{K}{\varepsilon \sqrt{\pi}} \exp\left\{ - \frac{(\Delta g - (1-\alpha)\widehat{g}(s,\alpha))^2}{r_{th}^2} - \frac{\alpha^2}{\varepsilon^2} \right\} \overline{\Theta}(\alpha^2) \frac{\partial}{\partial g^{\mu\nu}} (V_o + V_{QM}) = 0. \quad (71)$$

Alternate requirement #4: Setting also a further constraint on the gauge potential ΔV, i.e., requiring that ΔV satisfies the constraint equation

$$\left\langle \psi \frac{1}{\kappa} \frac{\partial}{\partial g^{\mu\nu}} \Delta V \bigg|_{g=\widehat{g}} \psi \right\rangle = \Lambda_{CQG} \langle \widehat{g}(r(s,\alpha)) \rangle_\alpha, \quad (72)$$

which implies the stochastic-gauge term $\Delta V \equiv \Delta V(\Delta g, \alpha)$ to be of the form

$$\Delta V(\Delta g, \alpha) = \frac{\kappa}{2} \Delta g(s,\alpha)^2 (1-\alpha) \Lambda_{CQG}, \quad (73)$$

with $\Delta g(s,\alpha)$ and Λ_{CQG} being defined by Equations (40) and (70). Straightforward calculations deliver, in particular,

$$\left\langle \frac{\partial}{\partial g^{\mu\nu}} \kappa \left(2 - \frac{1}{4} g^{\mu\nu} g_{\mu\nu}\right) g^{\alpha\beta} \widehat{R}_{\alpha\beta} \bigg|_{g=\widehat{g}} \right\rangle_\alpha = \kappa \left\langle \left[R_{\mu\nu}(\widehat{g}(r(s,\alpha))) - \frac{1}{2} R(\widehat{g}(r(s,\alpha))) \widehat{g}_{\mu\nu}(r(s,\alpha),s) \right] \right\rangle_\alpha, \quad (74)$$

$$\left\langle \psi - \frac{1}{\kappa} \frac{\partial}{\partial g^{\mu\nu}} V_{QM} \bigg|_{g=\widehat{g}} \psi \right\rangle_\alpha = -\langle \psi | \Lambda_{CQG}(s) [\Delta g_{\mu\nu} - (1-\alpha) \widehat{g}_{\mu\nu}(r(s,\alpha))] \psi \rangle \equiv 0. \quad (75)$$

That this is indeed a gauge term then follows immediately by noting that identically $\frac{d}{ds} \Delta V(\Delta g, \alpha) = 0$. This implies that the stochastic-averaged q-modified EFE *in the presence of stochastic trajectories* now takes the form

$$\langle R_{\mu\nu}(\widehat{g}(r(s,\alpha))) \rangle_\alpha - \frac{1}{2} \langle R(\widehat{g}(r(s,\alpha))) \widehat{g}_{\mu\nu}(r(s,\alpha),s) \rangle_\alpha = \langle T_{\mu\nu}(\widehat{g}(r(s,\alpha))) \rangle_\alpha - \Lambda_{CQG} \langle \widehat{g}(r(s,\alpha)) \rangle_\alpha. \quad (76)$$

The remarkable aspect of this equation is that all relevant tensor fields appear expressed in terms of stochastic averages and not in terms of their local values. This is a feature that may hopefully "cure" the singular behavior, which instead typically affects classical GR, i.e., the Einstein tensor field equation itself.

On the other hand, the same equation can be shown to be approximated asymptotically by Equation (69). In fact, introducing for $r(s,\alpha)$ (see Equation (9)) a Taylor expansion with respect to α delivers an expansion of the form $r(s,\alpha) = r(s) + \alpha r_1(s) + \alpha^2 r_2(s) \ldots$ As a consequence, assuming that all involved tensor fields are smooth and finite, ignoring corrections of $O(\varepsilon^2)$, one obtains

$$\begin{aligned} \langle \widehat{g}(r(s,\alpha)) \rangle_\alpha &\simeq \widehat{g}(r(s)) \left[1 + O(\varepsilon^2)\right], \\ \langle R_{\mu\nu}(\widehat{g}(r(s,\alpha))) \rangle_\alpha &\simeq R_{\mu\nu}(\widehat{g}(r(s)) \left[1 + O(\varepsilon^2)\right], \\ \langle R(\widehat{g}(r(s,\alpha))) \widehat{g}_{\mu\nu}(r(s,\alpha),s) \rangle_\alpha &\simeq R(\widehat{g}(r(s)) \widehat{g}_{\mu\nu}(r(s),s) \left[1 + O(\varepsilon^2)\right], \\ \langle T_{\mu\nu}(\widehat{g}(r(s,\alpha))) \rangle_\alpha &\simeq T_{\mu\nu}(\widehat{g}(r(s))) \left[1 + O(\varepsilon^2)\right]. \end{aligned} \quad (77)$$

The fact that the two equations do not coincide exactly is not unexpected. In fact, Equation (76) retains the average effect of stochastic perturbations, while Equation (69) actually ignores them altogether.

5. Background Independence

The notion of background independence, intended as the independence of the precise structure of the background space-time $\{\mathbf{Q}^4 \equiv \mathbb{R}^4, \widehat{g}(r)\}$, is frequently claimed to be the unique property shared by most current (and past) models of Quantum Gravity (QG). Accordingly, QG should/could not depend on the particular realization of the said structure, to be associated with a properly defined background space-time. This means, more precisely, that QG should not be formulated in a given space-time having "a priori" an intrinsic independent identity; rather, QG itself should be able to generate, through gravitational quantum dynamics, the said background space-time. Unfortunately, no agreement exists in the literature as to the notions of background field tensor and background independence.

This requires, first of all, suitably setting the very notion of *space-time background*.

As pointed out above (see Section 1.1), this involves distinguishing between multi-verse and uni-verse representations of quantum space-time. In fact, any physical theory of gravity should have the goal of determining the background structure of space-time, namely, identifying a Riemannian differential manifold formed by the couple $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with $\mathbf{Q}^4$ being the abstract set representing the space-time parametrized with respect to a suitable (but arbitrary) coordinate system $r \equiv \{r^\mu\}$ (GR-frame). Here, this is identified with the $4-$dimensional vector space $\mathbf{Q}^4 \equiv \mathbb{R}^4$, while $\widehat{g}(r)$ is a suitably defined *background metric field* parametrized in terms of the same GR-frame, which prescribes the geometric properties of $\mathbf{Q}^4$. Thus, $\widehat{g}(r)$ should be realized by a real and symmetric second-order tensor field of the form $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$, with $\widehat{g}_{\mu\nu}(r)$ and $\widehat{g}^{\mu\nu}(r)$ denoting its covariant and counter-variant components, satisfying the orthogonality conditions $\widehat{g}_{\mu\alpha}(r)\widehat{g}^{\mu\beta}(r) = \delta_\alpha^\beta$. In addition, $\widehat{g}(r)$ should be such that it raises and lowers tensor indices of arbitrary tensor fields, determines the Riemann distance in $\{\mathbf{Q}^4, \widehat{g}(r)\}$ and prescribes the standard connections, the corresponding covariant derivative $\widehat{\nabla}_\rho$ (for arbitrary differentiable tensor fields) and finally the Riemann tensor as well the corresponding Ricci tensor, denoted as $\widehat{R} \equiv R(\widehat{g}(r))$.

In the literature, there are different possible meanings/interpretations of the notion of background metric field (which may depend intrinsically on the type of representation chosen for the quantum space-time). Precisely, we first notice that in classical GR, $\widehat{g}(r)$ can be either identified with a particular solution of the Einstein field equations (EFE) subject to appropriate boundary conditions or the actually physically observed, and thus measurable, metric tensor field represented in terms of a classical tensor field $\widehat{g}(r)$. Instead, in QG $\widehat{g}(r)$ can be either identified with a well-defined quantum tensor field (uni-verse representation), possibly identified with a classical, smooth and deterministic field tensor of *emergent character*, i.e., identified with the quantum expectation value of a suitable ensemble of underlying quantum fields; or alternatively identified with an arbitrary variational and stochastic field tensor $g(r)$ (multi-verse representation).

As already evident from such disparate notions (of background metric field), it is obvious that the concept of background independence can also actually acquire different meanings in each of the contexts indicated above. The first one due to Ashtekar and Lewandowski [22] is that background-independence should simply mean independence on the variational metric tensor $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$. As described in the same reference, this can be in principle achieved in the context of the Einstein–Hilbert variational approach by replacing $g(r)$ with corresponding Palatini variables, i.e., the standard connections. Such a task, however, remains a mere formal trick, since by construction the variational tensor $g(r)$ is also the same metric tensor that raises and lowers tensor indices (and therefore, relates covariant and counter-variant tensor indices). As a consequence, its dependence cannot be truly eliminated. An analogous trick can actually be achieved in the contest of the ADM approach by adopting the Ashtekar variables, where again the dependence of the tensor field $g(r)$ actually remains formally hidden. This appears, again, to be just a mathematical trick. In fact, at the end of the corresponding extremal Euler–Lagrange equations (i.e., the Einstein Field equations), the same tensor field must reappear once again (this time identified with the extremal one).

A second possible viewpoint [23] is that background independence, both in the context of General Relativity (GR) and QG, might/should be identified with the property of the metric tensor of space-time ($g(r)$) to be a solution of a "dynamical equation" of some sort, i.e., actually a PDE. In the literature, the adjective "dynamics" refers to a possible underlying abstract dynamical—i.e., either classical or quantum Hamiltonian—structure. Such a property, however, is insufficient to qualify it uniquely unless a precise definition is given to such a Hamiltonian structure. Therefore, the real issue becomes that of actually identifying such a structure and to prescribe what should be meant by Hamiltonian structure.

Background Independence in CQG-Theory

However, it is in the context of CQG-theory that the notion of background independence acquires a precise physical meaning. In fact, CQG-theory is actually referred to as a single background metric field tensor $\widehat{g}(r)$, which is determined by the q-modified EFE. Nevertheless, background independence is warranted because

- $\widehat{g}(r)$ remains still arbitrary as its prescription depends on still undetermined appropriate boundary conditions;
- CQG-theory holds for arbitrary particular solutions $\widehat{g}(r)$;
- $\widehat{g}(r)$ can be represented in arbitrary coordinate systems (GR-frames);
- $\widehat{g}(r)$ has an emergent character, namely, it can be represented in terms of a quantum expectation value in terms of a suitable quantum (i.e., stochastic) field tensor [8];
- finally, as discussed in the previous section, background independence implies also that the same q-modified EFE can be uniquely determined in terms of quantum expectation value (or equivalent stochastic average) of an underlying quantum equation.

Background independence does not come without a price. In fact it requires a well-defined set of physical assumptions, i.e., the validity of the Principle of General Covariance (PGC), together with the closely related one, i.e., the Principle of Manifest Covariance (PMC). This warrants that all observables, operators and quantum equations are expressed in $4-$tensor form so that they are frame-independent, namely, manifestly covariant. This property includes, in particular, the q-modified EFE. Another assumption is the existence of an underlying classical Hamiltonian structure of GR with an intrinsic $4-$tensor character. Finally, background independence demands the existence of a quantum Hamiltonian structure with an analogous $4-$tensor property [24].

6. Conclusions

In this paper, a new approach to the theory of Covariant Quantum Gravity (CQG-theory) is presented based on what is termed here as stochastic CQG-theory. The novelty of the new approach lies in important modifications introduced into the manifestly covariant quantization approach previously developed in our earlier papers. In detail, these include the following:

- Besides the introduction of a stochastic quantum cosmological constant, i.e., a physical property of crucial importance for the possible tunneling effect arising at the event horizons (a topic already in part discussed in our earlier paper [9]).

Additional, new, important features have been introduced as follows:

- First, the adoption of stochastic quantum trajectories. As a consequence, the quantum trajectories that are taken into account in the quantum-wave equation of CQG-theory are actually considered stochastic. This effect is believed to be relevant to extend the validity of quantum theory to the Planck scale.
- Second, the inclusion of a stochastic-modified quantum potential appearing through stochastic gauge contributions. This feature is crucial for the establishment of the property of background independence in Quantum Gravity, in the sense that the equation determining the background metric field tensor—i.e., the quantum expectation value of a suitable quantum tensor field—should itself be realized in terms of a suitable quantum expectation value.

- Third, the representation of the quantum-modified EFE (which determines the same background field tensor) in terms of stochastic quantum-average (or quantum expectation value) of a suitably determined quantum Hamiltonian equation that depends on the stochastic trajectories.

All these effects are achieved at once by including into the formalism a stochastic parameter α, which, interestingly enough, acquires as a consequence a multiple physical interpretation, i.e., in terms of stochastic modifications of

(a) The graviton quantum trajectories;
(b) The quantum potential;
(c) The cosmological constant.

The results appear notable because of their connection with the inner structure of the theory and the connection with the corresponding classical framework treated in Part I, with special reference to the properties of covariance, manifest covariance, background independence and gauge invariance.

In particular, a new meaning is attributed to the notion of background independence. In fact, we have shown that CQG-theory allows to construct rigorously the quantum-modified Einstein Field Equations (q-modified EFE), which determine the same background metric field. The remarkable result has been achieved by direct evaluation of a suitable stochastic average of the underlying quantum Hamilton equations.

In passing, a detailed analysis of different existing approaches in QG-theory has been carried out. Furthermore, this outcome, in our view, is of potential value for unveiling and weighting the possible physical relevance of various QG-theories with respect to a number of notable features, in particular:

- The relevant dynamical equations for the tensor fields are realized by means of evolution-type ODEs, which identify a Hamiltonian system in a proper sense. Such a feature is warranted by the fulfillment of the constraint condition characteristic of the Einstein field equations, namely, the requirement that the components of the Ricci tensor depend on the background metric tensor $\widehat{g}(r)$ only.
- The corresponding Hamiltonian system $\{x_R, H_R\}$ is constraint-free because the tensor components of the Hamiltonian state $x_R(s) \equiv (g(s), \pi(s))$ are independent.
- Suitable quantum-modified Einstein field equations are obtained that prescribe, in a suitable functional class, the background metric field tensor.
- Quantum Hamilton equations are determined, which are related to the quantum hydrodynamic equations.
- The theory is gauge invariant. As a consequence, both the Hamiltonian and Lagrangian densities are intrinsically non-unique, being determined up to an additive gauge function.

  And finally, the crucial aspect which characterizes CQG-theory and its classical counterpart described in Part I, namely:
- The property of manifest covariance, i.e., the theory is set in $4-$tensor form, since its quantum state and its quantum Hamiltonian operator have a $4-$tensor and frame-independent character with respect to the background space-time $\{\mathbf{Q}^4, \widehat{g}(r)\}$.

However, several questions not treated in this paper still remain open and are left to future investigations. One undoubtedly concerns possible comparisons with ideal and actual QG experiments, an area where theoretical and experimental research hopefully will join forces in the near future.

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Appendix A. Classical Hamiltonian Function of GR and Covariant Derivative

The $4-$scalar classical Hamiltonian function of GR $H_R = H_R(g, \pi, \widehat{g}, r(s), s)$ takes the form

$$\begin{cases} H_R = T_R + V \\ T_R(\pi, \widehat{g}, r(s), s) = \frac{1}{2\varkappa}\pi_{\mu\nu}\pi^{\mu\nu} \\ V(g, \widehat{g}, r(s), s) = \sigma V_o + \sigma V_F \\ V_o = \varkappa h(g, \widehat{g})\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda_{Cl}\right] \end{cases}. \tag{A1}$$

Here, the notation is standard [5,12]. Thus, Λ_{Cl} denotes the classical cosmological constant, $\varkappa$ is a suitable dimensional constant [5] and $(g, \pi) \equiv (g(r(s), s), \pi(r(s), s))$ are the canonical fields evaluated along a field geodesics of the background metric tensor $\widehat{g}(r(s))$. Furthermore, σ is the sign factor $\sigma = \pm 1$; $\widehat{R}_{\mu\nu}$ is the extremal Ricci tensor, i.e., evaluated with respect to the background metric field tensor $\widehat{g}$; and finally, $h(g, \widehat{g})$ denotes the variational factor

$$h(g, \widehat{g}) = 2 - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}. \tag{A2}$$

Next, let us introduce the definition of the *covariant s-derivative* used in Section 4. Analogous to Ref. [8], this is defined as

$$\frac{d}{ds} = \left.\frac{d}{ds}\right|_s + \left.\frac{d}{ds}\right|_r \tag{A3}$$

with standard notation. Thus, first,

$$\left.\frac{d}{ds}\right|_s \equiv t^\alpha \widehat{\nabla}_\alpha \tag{A4}$$

identifies the *directional covariant derivative*. Notice that, here, $\widehat{\nabla}_\alpha$ denotes the covariant derivative evaluated with respect to the background metric tensor $\widehat{g}(s)$. This is performed along a geodesic trajectory $r(s)$. Instead, in stochastic-CQG-theory, ∇_α denotes the covariant derivative performed along a stochastic trajectory $r(s, \alpha)$. Thus, t^α identifies the corresponding tangent $4-$vectors, namely,

$$t^\alpha(s) = \left.\frac{d}{ds}\right|_s r(s) \tag{A5}$$

or

$$t^\alpha(s, \alpha) = \left.\frac{d}{ds}\right|_s r(s, \alpha). \tag{A6}$$

Second, $\left.\frac{d}{ds}\right|_r$ denotes similarly the *covariant* $s-$*partial derivative.* When it operates on a $4-$scalar, this coincides with the ordinary partial derivative, so that in both cases,

$$\left.\frac{d}{ds}\right|_r = \frac{\partial}{\partial s}. \tag{A7}$$

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