

The Fedosov deformation quantization with the induced symplectic connection

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Abstract. A symplectic connection on a cotangent bundle $T^*\mathcal{M}$ induced by a Levi-Civita connection on a configuration space \mathcal{M} is constructed. General properties of an Abelian connection built from the induced symplectic connection are presented. An example of a finite Abelian connection determined by the induced symplectic connection has been found.

1. Introduction

Quantum mechanics is usually formulated in terms of theory of a Hilbert space and linear operators acting in it. Unfortunately, serious problems appear when one tries to describe quantum effects in systems with curved configuration spaces. On the other side, classical physics works perfectly on arbitrary differentiable manifolds. Hence the formulation of quantum theory based on differential geometry may eliminate obstacles.

An alternative construction of quantum mechanics on a phase space \mathbb{R}^{2n} was proposed Moyal [1]. In his paper ideas of Weyl [2], Wigner [3] and Groenewold [4] were developed. Generalization of Moyal's results for systems with nontrivial phase spaces was presented by Bayen *et al* [5, 6]. Their fresh look at the problem gave birth to deformation quantization.

One of practical realisations of the Bayen quantization programme was invented by Fedosov [7, 8] and is known as the Fedosov quantization. Roughly speaking his algorithm is composed of four steps. We start from a phase space of a system, which is a symplectic manifold and equip it with some symplectic connection. Then the symplectic connection is lifted to the bundle of formal Weyl algebras. After that we introduce an Abelian connection in the Weyl algebra bundle, which is determined by the symplectic connection. Next we identify functions representing observables with smooth sections of the subalgebra of the flat sections of the Weyl algebra bundle. By a "quantum" $*$ -product of these functions we understand a projection of a product of smooth sections representing them.

Unlike the Riemannian geometry, a symplectic manifold may be equipped with many symplectic connections. Hence the natural question arises, which of them is the most physical and how for it Fedosov quantization works. In our contribution we propose, following [9], the symplectic connection induced by the Levi- Civita connection on the configuration space \mathcal{M} . For such a symplectic connection we analyse properties of the Abelian connection. Another way leading to the induced symplectic connection was indicated by Bordemann, Neumaier and Waldmann [10, 11].

In all formulas in which summation limits are obvious we use the Einstein summation convention.

2. Induced symplectic connection

In this section we present a construction of some symplectic connection determined by the Riemannian structure of a configuration space \mathcal{M} . Reader interested in details is pleased to see [9].

Assume that the configuration space \mathcal{M} , $\dim \mathcal{M} = n$, is a differentiable manifold endowed with a Riemannian metric g and $T^*\mathcal{M}$ is a cotangent bundle over it. The bundle $T^*\mathcal{M}$ is $2n$ -D symplectic manifold. We cover it with so called proper Darboux atlas.

Definition 1. Let $\{(U_\varrho, \phi_\varrho)\}_{\varrho \in I}$ be an atlas on the symplectic manifold $T^*\mathcal{M}$ such that in every chart the coordinates q^i , $1 \leq i \leq n$ determine points on the basic manifold \mathcal{M} and $q^{i+n} = p_i$, $1 \leq i \leq n$, denote momenta in natural coordinates. Every atlas of this form is called the **proper Darboux atlas** and every chart of this atlas the **proper Darboux chart**. The transitions functions are the point transformations

$$Q^k = Q^k(q^i), \quad P_i = \frac{\partial q^k}{\partial Q^i} p_k.$$

We equip the cotangent bundle $T^*\mathcal{M}$ with a metric structure. Locally in a proper Darboux chart, the metric tensor

$$\tilde{g}_{jk} = \begin{pmatrix} -2p_i\Gamma_{jk}^i & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \quad (1)$$

By **1** and **0** we denote the identity and zero matrices $n \times n$ respectively. Coefficients Γ_{jk}^i are components of the Levi- Civita connection of the metric g on \mathcal{M} i.e.

$$\Gamma_{jk}^i = \frac{1}{2}g^{il} \left(\frac{\partial g_{lk}}{\partial q^j} + \frac{\partial g_{lj}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l} \right), \quad 1 \leq i, j, k, l \leq n. \quad (2)$$

It is interesting, that the metric tensor (1) appears as a consequence of the operator ordering in Fedosov quantization [12].

Theorem 1. Let (\mathcal{M}, g) be a Riemannian manifold. Then $(T^*\mathcal{M}, \tilde{g})$ is also a Riemannian manifold.

Definition 2. The **symplectic connection** on a symplectic manifold (\mathcal{W}, ω) is a torsion free connection satisfying locally a set of conditions

$$\omega_{ij;k} = 0, \quad 1 \leq i, j, k \leq \dim \mathcal{W}.$$

The symbol ω denotes a symplectic form.

In every chart (\mathcal{U}_z, ϕ_z) on \mathcal{W}

$$\omega_{ij;k} = \frac{\partial \omega_{ij}}{\partial q^k} - \gamma_{ik}^l \omega_{lj} - \gamma_{jk}^l \omega_{il} = 0, \quad (3)$$

$$\gamma_{ik}^l = \gamma_{ki}^l, \quad (\text{torsion free}).$$

In Darboux coordinates the set of equations (3) takes a simple form

$$\omega_{ij;k} = -\gamma_{ik}^l \omega_{lj} - \gamma_{jk}^l \omega_{il} = -\gamma_{jik} + \gamma_{ijk} = 0,$$

where

$$\gamma_{ijk} \stackrel{\text{def.}}{=} \gamma_{jk}^l \omega_{li}.$$

Hence the connection is symplectic if and only if in every Darboux chart all of components γ_{jk}^l are totally symmetric with respect to indices $\{i, j, k\}$.

Definition 3. *The symplectic manifold (\mathcal{W}, ω) equipped with the symplectic connection γ is called the Fedosov manifold $(\mathcal{W}, \omega, \gamma)$.*

Every symplectic manifold admits a symplectic connection, but it is not uniquely defined. Locally the difference between two symplectic connections $\gamma_{ijk} - \tilde{\gamma}_{ijk} = r_{ijk}$ is a tensor completely symmetric in indices $\{i, j, k\}$.

From now on the symplectic manifold (\mathcal{W}, ω) will be the cotangent bundle $\mathcal{T}^*\mathcal{M}$ with its symplectic structure. As we mentioned (compare Theorem 1), the space $\mathcal{T}^*\mathcal{M}$ is also the Riemannian manifold and the Levi-Civita connection $\bar{\Gamma}$ on it is determined by the tensor \tilde{g} .

Let us lower the upper index of Christoffel symbols $\bar{\Gamma}_{jk}^i$ so $\bar{\Gamma}_{ijk} \stackrel{\text{def}}{=} \omega_{il}\bar{\Gamma}_{jk}^l$. Then in any Darboux chart

$$\gamma_{ijk} \stackrel{\text{def}}{=} \frac{1}{3}(\bar{\Gamma}_{ijk} + \bar{\Gamma}_{jik} + \bar{\Gamma}_{kij}) \quad (4)$$

is the symplectic connection on $\mathcal{T}^*\mathcal{M}$ induced by the Levi-Civita connection on $\mathcal{T}^*\mathcal{M}$. Moreover, since the metric tensor \tilde{g} (see formula (1)) is a function of the Levi- Civita connection on the configuration space \mathcal{M} then in fact the symplectic connection γ_{ijk} is determined by the connection on the base space \mathcal{M} .

Indeed, in proper Darboux coordinates the coefficients of the induced symplectic connection on $\mathcal{T}^*\mathcal{M}$

$$\begin{aligned} \gamma_{\bar{i}jk} &= \Gamma_{jk}^i, \quad \gamma_{\bar{i}\bar{j}k} = 0, \quad \gamma_{\bar{i}\bar{j}\bar{k}} = 0, \\ \gamma_{ijk} &= \frac{1}{3}p_l\left(\frac{\partial\Gamma_{jk}^l}{\partial q^i} + \frac{\partial\Gamma_{ij}^l}{\partial q^k} + \frac{\Gamma_{ik}^l}{\partial q^j} - 2\Gamma_{mi}^l\Gamma_{jk}^m - 2\Gamma_{mk}^l\Gamma_{ij}^m - 2\Gamma_{mj}^l\Gamma_{ik}^m\right). \end{aligned} \quad (5)$$

We use the convention $\bar{i} = i + n$.

The coefficients $\gamma_{\bar{i}jk}$ of the induced symplectic connection are functions of spatial coordinates only. Terms γ_{ijk} depend linearly on momenta and are also functions of spatial coordinates. The curvature tensor of the induced symplectic connection contains three kinds of elements: R_{ijkl} , $R_{ij\bar{k}l}$ and $R_{\bar{i}\bar{j}kl}$. Exact formulas defining them can be seen in [9]. Relations (5) are invariant under proper Darboux transformations.

3. Abelian connection in the Fedosov formalism

In this section we present a brief review of the Fedosov deformation. Its expanded version can be found for example in original Fedosov works [7, 8].

A Fedosov manifold $(\mathcal{W}, \omega, \gamma)$ is given. We deal with formal series

$$a \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a_{k,i_1 \dots i_l} X_{\mathbf{p}}^{i_1} \dots X_{\mathbf{p}}^{i_l}. \quad (6)$$

In the upper expression \hbar denotes some positive parameter identified by physicists with the Dirac constant. $X_{\mathbf{p}}^1, \dots, X_{\mathbf{p}}^{2n}$ are components of an arbitrary vector $\mathbf{X}_{\mathbf{p}}$ belonging to the tangent space $T_{\mathbf{p}}\mathcal{W}$ at the point \mathbf{p} . The components $X_{\mathbf{p}}^1, \dots, X_{\mathbf{p}}^{2n}$ are written in the natural basis. By $a_{k,i_1 \dots i_l}$ we denote components of a covariant tensor symmetric with respect to indices $\{i_1 \dots, i_l\}$ taken in the basis $dq^{i_1} \odot \dots \odot dq^{i_l}$. The part of the series a standing at \hbar^k and containing l components of the vector $\mathbf{X}_{\mathbf{p}}$ will be denoted by $a[k, l]$ so that $a = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a[k, l]$. The **degree** $\deg(a[k, l])$ of the component $a[k, l]$ is the sum $2k + l$. The degree of the series a is the maximal degree of its nonzero components $a[k, l]$.

Let $P_{\mathbf{p}}^*\mathcal{W}[[\hbar]]$ be the set of all elements a of the kind (6) at the point \mathbf{p} .

Definition 4. The product $\circ : P_p^* \mathcal{W}[[\hbar]] \times P_p^* \mathcal{W}[[\hbar]] \rightarrow P_p^* \mathcal{W}[[\hbar]]$ of two elements $a, b \in P_p^* \mathcal{W}[[\hbar]]$ is the mapping

$$a \circ b \stackrel{\text{def.}}{=} \sum_{t=0}^{\infty} \frac{1}{t!} \left(\frac{i\hbar}{2} \right)^t \omega^{i_1 j_1} \dots \omega^{i_t j_t} \frac{\partial^t a}{\partial X_p^{i_1} \dots \partial X_p^{i_t}} \frac{\partial^t b}{\partial X_p^{j_1} \dots \partial X_p^{j_t}}. \quad (7)$$

The tensor ω^{ij} and the symplectic form ω_{jk} are related by $\omega^{ij} \omega_{jk} = \delta_k^i$. The pair $(P_p^* \mathcal{W}[[\hbar]], \circ)$ is a noncommutative associative algebra called the **Weyl algebra**.

A **Weyl bundle** is a triplet $(\mathcal{P}^* \mathcal{W}[[\hbar]], \pi, \mathcal{W})$, where $\mathcal{P}^* \mathcal{W}[[\hbar]] \stackrel{\text{def.}}{=} \bigcup_{p \in \mathcal{W}} (P_p^* \mathcal{W}[[\hbar]], \circ)$ is a differentiable manifold called the total space, \mathcal{W} is the base space and $\pi : \mathcal{P}^* \mathcal{W}[[\hbar]] \rightarrow \mathcal{W}$ the projection. A Weyl bundle is a vector bundle in which the typical fibre is also an algebra.

Definition 5. An *m-differential form* with value in the Weyl bundle is a form written locally

$$a = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a_{k,i_1 \dots i_l, j_1 \dots j_m}(q^1, \dots, q^{2n}) X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_m} \quad (8)$$

where $0 \leq m \leq 2n$. Now $a_{k,i_1 \dots i_l, j_1 \dots j_m}(q^1, \dots, q^{2n})$ are components of smooth tensor fields on \mathcal{W} and $C^\infty(\mathcal{T}\mathcal{W}) \ni \mathbf{X} \stackrel{\text{locally}}{=} X^i \frac{\partial}{\partial q^i}$ is a smooth vector field.

Let Λ^m be a smooth field of *m*-forms on the symplectic manifold \mathcal{W} . Forms of the kind (8) are smooth sections of $\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda \stackrel{\text{def.}}{=} \bigoplus_{m=0}^{2n} (\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^m)$. For simplicity we will omit the variables (q^1, \dots, q^{2n}) .

Definition 6. The commutator of forms $a \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^{m_1})$ and $b \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^{m_2})$ is the form $[a, b] \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^{m_1+m_2})$

$$[a, b] \stackrel{\text{def.}}{=} a \circ b - (-1)^{m_1 \cdot m_2} b \circ a. \quad (9)$$

A form $a \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda)$ is called **central**, if for every $b \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda)$ the commutator $[a, b]$ vanishes. Only forms not containing X^i 's are central.

Definition 7. The antiderivation operator $\delta : C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^m) \rightarrow C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{m+1})$ is defined by

$$\delta a \stackrel{\text{def.}}{=} dq^k \wedge \frac{\partial a}{\partial X^k}.$$

Definition 8. The operator $\delta^{-1} : C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^m) \rightarrow C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^{m-1})$ is

$$\delta^{-1} a = \begin{cases} \frac{1}{l+m} X^k \frac{\partial}{\partial q^k} a & \text{for } l+m > 0 \\ 0 & \text{for } l+m = 0, \end{cases} \quad (10)$$

where l is the degree of a in X^i 's, i.e. the number of X^i 's.

Definition 9. The exterior covariant derivative ∂_ξ of the form $a \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^m)$ determined by a connection 1-form $\xi \in C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^1)$ is the linear operator

$$\partial_\xi : C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^m) \rightarrow C^\infty(\mathcal{P}^* \mathcal{W}[[\hbar]] \otimes \Lambda^{m+1})$$

defined in a Darboux chart by the formula

$$\partial_\xi a \stackrel{\text{def.}}{=} da + \frac{1}{i\hbar} [\xi, a]. \quad (11)$$

In the case of a symplectic connection, we use γ instead of ξ and put $\gamma = \frac{1}{2}\gamma_{ijk}X^iX^jdq^k$.

The curvature form R_γ of a connection 1-form γ in a Darboux chart can be expressed by the formula

$$R_\gamma = d\gamma + \frac{1}{2i\hbar}[\gamma, \gamma] = d\gamma + \frac{1}{i\hbar}\gamma \circ \gamma. \quad (12)$$

A crucial role in the Fedosov deformation quantization is played by an Abelian connection $\tilde{\Gamma}$. By the **Abelian** connection we mean a connection $\tilde{\Gamma}$ whose curvature form $R_{\tilde{\Gamma}}$ is central so $\partial_{\tilde{\Gamma}}(\partial_{\tilde{\Gamma}}a) = 0$ for every $a \in C^\infty(\mathcal{P}^*\mathcal{W}[[\hbar]] \otimes \Lambda)$.

The Abelian connection proposed by Fedosov is of the form

$$\tilde{\Gamma} = \omega_{ij}X^iX^jdq^j + \gamma + r. \quad (13)$$

Its curvature is

$$R_{\tilde{\Gamma}} = -\frac{1}{2}\omega_{j_1j_2}dq^{j_1} \wedge dq^{j_2} + R_\gamma - \delta r + \partial_\gamma r + \frac{1}{i\hbar}r \circ r. \quad (14)$$

The requirement that the central curvature 2-form $R_{\tilde{\Gamma}} = -\frac{1}{2}\omega_{j_1j_2}dq^{j_1} \wedge dq^{j_2}$ means that r must satisfy the equation

$$\delta r = R_\gamma + \partial_\gamma r + \frac{1}{i\hbar}r \circ r. \quad (15)$$

Theorem 2. *The equation (15) has a unique solution*

$$r = \delta^{-1}R_\gamma + \delta^{-1}\left(\partial_\gamma r + \frac{1}{i\hbar}r \circ r\right) \quad (16)$$

fulfilling the following conditions

$$\delta^{-1}r = 0, \quad 3 \leq \deg(r). \quad (17)$$

We work only with the Abelian connection of the form (13) with the correction r defined by (16) and fulfilling (17).

Here some of properties of the Abelian connection are pointed out. More information can be found in [13].

(i) The correction

$$r = \delta^{-1}R_\gamma + \sum_{z=4}^{\infty} \sum_{k=0}^{\lfloor \frac{z-1}{4} \rfloor} \hbar^{2k} r_m[2k, z-4k]dq^m, \quad (18)$$

so r contains only terms with at least one vector component X^i and only even powers of \hbar . By $[z]$ we mean the floor of an integer z .

(ii) If $R_\gamma \neq R_{\bar{\gamma}}$ then corrections r determined by connections γ and $\bar{\gamma}$ respectively are different.

On the contrary, two different symplectic connections γ and $\bar{\gamma}$ with the same curvature may generate the same Abelian correction r .

(iii) Explicit formulas on the part $r[z]$ of the degree z are [14, 15]

$$r[3] = \delta^{-1}R_\gamma,$$

$$r[z] = \delta^{-1}\left(\partial_\gamma r[z-1] + \frac{1}{i\hbar} \sum_{j=3}^{z-2} r[j] \circ r[z+1-j]\right), \quad 4 \leq z. \quad (19)$$

Definition 10. $\mathcal{P}^*\mathcal{W}[[\hbar]]_{\tilde{\gamma}} \subset C^\infty(\mathcal{P}^*\mathcal{W}[[\hbar]] \otimes \Lambda^0)$ is the subalgebra consisting of flat sections, i.e. sections such that $\partial_{\tilde{\gamma}}a = 0$.

Theorem 3. For any $a_0 \in C^\infty(\mathcal{W})$ there exists a unique section $a \in \mathcal{P}^*\mathcal{W}[[\hbar]]_{\tilde{\gamma}}$ such that $\sigma(a) = a_0$.

It can be proved [7, 8] that

$$a = a_0 + \delta^{-1} \left(\partial_\gamma a + \frac{1}{i\hbar} [r, a] \right). \quad (20)$$

By a star product ‘*’ of functions $a_0, b_0 \in C^\infty(\mathcal{M})$ we mean

$$a_0 * b_0 \stackrel{\text{def.}}{=} \sigma(\sigma^{-1}(a_0) \circ \sigma^{-1}(b_0)). \quad (21)$$

The *- product (21) is interpreted as the quantum multiplication of observables.

4. Abelian connection of the induced symplectic connection

In this paragraph we consider the situation when the Abelian connection is determined by the induced symplectic connection. We present some general remarks about a shape of a formal series representing the Abelian connection in that case and look for examples of finite Abelian connections.

After easy but rather long computations it can be checked that the correction r to the Abelian connection does not contain the deformation parameter \hbar . Moreover, each term $r[z]$, $3 \leq z$ is a sum of three kinds of elements:

- (i) $r_{i_1 \dots i_z, j}(q^1, \dots, q^n) X^{i_1} \dots X^{i_z} dp_j$, $1 \leq i_1, \dots, i_z \leq n$,
- (ii) $r_{i_1 \dots i_{z-1}, j, i_z}(q^1, \dots, q^n) X^{i_1} \dots X^{i_{z-1}} X^j dq^{i_z}$, $1 \leq i_1, \dots, i_z \leq n$, $n < j \leq 2n$,
- (iii) $\sum_{k=1}^n p_k \cdot r_{k, i_1 \dots i_z, j}(q^1, \dots, q^n) X^{i_1} \dots X^{i_z} dq^j$, $1 \leq i_1, \dots, i_z, j \leq n$.

Coefficients of the 1st and 2nd kind depend only on spatial coordinates. They are related by a condition $\delta^{-1}r = 0$. Terms of the 3rd type are linear functions in momenta. They also fulfill the requirement $\delta^{-1}r = 0$.

It seems to be extremely difficult to find the general expression of the Abelian connection in the Fedosov formalism. Even in case when the Abelian connection is determined by the induced symplectic connection, this problem has not been completely solved yet.

Hence we look for examples of finite Abelian connections. By definition, we say that r is a **finite formal series** when $\deg(r) = d$, $d \in \mathbb{N}$. The series representing a finite Abelian connection can be easily found even by a computer programme without any additional considerations.

Now we present the only one known example of the finite Abelian connection.

Let $(\mathcal{W}, \omega, \gamma)$ be a Fedosov manifold. Locally the nonvanishing components of a symplectic connection are

$$\gamma_{l_1 l_2 l_3}(q^{l_4}, \dots, q^{l_r}), \quad 1 \leq l_1, \dots, l_r \leq \dim \mathcal{W}$$

such that all of Poisson brackets

$$\{q^{l_i}, q^{l_j}\}_P = 0, \quad 1 \leq i, j \leq r.$$

This connection can be curved only if $4 \leq \dim \mathcal{W}$. The Ricci tensor $K_{ij} \stackrel{\text{def.}}{=} \omega^{kl} R_{likj}$ of γ disappears.

Then the correction

$$r[z] = (\delta^{-1} \partial_\gamma)^{z-3} \delta^{-1} R_\gamma. \quad (22)$$

As it was shown [13], the sufficient and necessary condition for r to be a finite series is

$$\exists_{z \geq 4} \ (\partial_\gamma \delta^{-1})^{z-3} R_\gamma = 0. \quad (23)$$

It would be interesting to apply this example for the induced symplectic connection. Let us consider first the case when the configuration space is 2-dimensional. As it was proved by Plebański *et al* [9], the Fedosov manifold $(T^*\mathcal{M}, \omega, \gamma)$ with the induced symplectic connection γ is Ricci flat (i.e. $K_{ij} = 0$) if and only if the Riemannian manifold (\mathcal{M}, g) is Ricci flat ($R_{ij} = 0$).

The only example of the Ricci flat 2-D Riemannian manifold is just a flat space so this is not the case we look for.

Let \mathcal{M} be the 3-D Riemannian manifold covered by an atlas $\{(U_\rho, \phi_\rho)\}_{\rho \in I}$. Assume that in some chart (U_ρ, ϕ_ρ) the nonvanishing Levi-Civita connection components are only $\Gamma_{11}^3(q^1, q^2)$, $\Gamma_{22}^3(q^1, q^2)$ and $\Gamma_{12}^3(q^1, q^2)$. Then from (5)

$$\gamma_{116} = \Gamma_{11}^3(q^1, q^2), \quad \gamma_{126} = \Gamma_{12}^3(q^1, q^2), \quad \gamma_{226} = \Gamma_{22}^3(q^1, q^2),$$

$$\gamma_{111} = q^6 \frac{\partial \Gamma_{11}^3}{\partial q^1}, \quad \gamma_{112} = \frac{1}{3} q^6 \left(2 \frac{\partial \Gamma_{12}^3}{\partial q^1} + \frac{\partial \Gamma_{11}^3}{\partial q^2} \right), \quad \gamma_{122} = \frac{1}{3} q^6 \left(2 \frac{\partial \Gamma_{12}^3}{\partial q^2} + \frac{\partial \Gamma_{22}^3}{\partial q^1} \right), \quad \gamma_{222} = q^6 \frac{\partial \Gamma_{22}^3}{\partial q^2}.$$

In the chart (U_ρ, ϕ_ρ) the symplectic curvature 2-form $R_\gamma = d\Gamma$ and the correction

$$r[z] = (\delta^{-1}d)^{z-3} \delta^{-1} R_\gamma, \quad 3 \leq z.$$

The term $r[z]$ consists of four kinds of elements:

- (i) $q^6 \sum_{i=0}^z f_{i,z-i,1}(q^1, q^2) (X^1)^i (X^2)^{z-i} dq^1, \quad q^6 \sum_{i=0}^z f_{i,z-i,2}(q^1, q^2) (X^1)^i (X^2)^{z-i} dq^2$
- (ii) $\sum_{i=0}^{z-1} g_{i,z-i-1,1}(q^1, q^2) (X^1)^i (X^2)^{z-i-1} X^6 dq^1, \quad \sum_{i=0}^{z-1} g_{i,z-i-1,2}(q^1, q^2) (X^1)^i (X^2)^{z-i-1} X^6 dq^2$.

Straightforward calculations lead to the conclusion that functions $f_{i,z-i,1}(q^1, q^2)$, $f_{i,z-i,2}$ are linear combinations of partial derivatives of $\Gamma_{11}^3(q^1, q^2)$, $\Gamma_{22}^3(q^1, q^2)$ and $\Gamma_{12}^3(q^1, q^2)$ of the total degree $z+1$. Moreover, coefficients $g_{i,z-i-1,1}(q^1, q^2)$, $g_{i,z-i-1,2}(q^1, q^2)$ are linear combinations of partial derivatives of $\Gamma_{11}^3(q^1, q^2)$, $\Gamma_{22}^3(q^1, q^2)$ and $\Gamma_{12}^3(q^1, q^2)$ of the total degree z .

Hence we deduce that the Abelian connection $\tilde{\gamma}$ on U_ρ is represented by a finite formal series for example if $\Gamma_{11}^3(q^1, q^2)$, $\Gamma_{22}^3(q^1, q^2)$ and $\Gamma_{12}^3(q^1, q^2)$ are polynomials in q^1, q^2 .

5. Results and Perspectives

The fact, that every symplectic manifold may be equipped with many symplectic connections, offers some freedom in frames of the Fedosov deformation quantization. In our opinion the most physically justifiable symplectic connection is the induced symplectic connection built according to the idea presented in [9].

In the current contribution we presented basic properties and the general shape of the Abelian connections generated by the induced symplectic connections. But even in this narrow class of Abelian connections we may search for especially convenient ones namely finite formal series. We found the example of the finite Abelian connection. The correction r determined by the formula (22), for which (23) holds, is the finite series. Unfortunately this example works only on a Ricci flat space and it cannot be considered as an universal solution. Therefore we will search for finite Abelian connections built over non Ricci flat configuration spaces.

The Abelian connection is a step on the way to the $*$ -product. It would be valuable to find the explicit form of the $*$ -product for observables on a symplectic manifold with the induced symplectic connection.

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