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Foundations of Quantum Theory 1: The Square-Root Operator, Proper-Time and a Particle Interpretation for the Klein-Gordon Equation

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ABSTRACT

In this paper we use the theory of semigroups of operators and fractional powers to provide an analytic factorization of the square-root operator. We study the free-particle solutions and construct a conserved current. In passing, we also provide a direct representation for the photon equation and construct its solutions. We apply our results to give a rigorous meaning to the notion of proper-time as an operator, which in turn is used to provide a complete particle interpretation of the Klein-Gordon equation.

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I. INTRODUCTION

History

In the transition from non-relativistic to relativistic quantum mechanics, the Hamiltonian

$$H = \frac{(\vec{P} - e/c\,\vec{A})^2}{2m} + V \qquad (1.0)$$

is replaced by:

$$H = \sqrt{c^2(\vec{P} - e/c\,\vec{A})^2 + m^2c^4} + V \qquad (1.1)$$

It was quite natural to expect that the first choice for a relativistic wave equation would be:

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = (\sqrt{c^2(\vec{P} - e/c\vec{A})^2 + m^2c^4})\psi(\vec{x},t) + V\,\psi(x,t) \qquad (1.2)$$

Where $\vec{P} = -i\hbar\vec{\nabla}$. In the free particle case we get:

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = (\sqrt{-c^2\hbar^2\Delta + m^2c^4})\;\psi(\vec{x},t) \qquad (1.3)$$

In a survey article on relativistic wave equations, Foldy[1] points out that in the absence of interaction, <u>equation (1.3) gives a perfectly good relativistic wave equation</u> for the description of a (spin zero) free particle. When $\vec{A}$ is not zero, the non-commutativity of $\vec{P}$ with $\vec{A}$ appeared to make it impossible to give an unambiguous meaning to the radical operator. Historically, Schrödinger,[2] Gordon,[3] Klein[4] and others[5] attempted to circumvent this problem by starting with the relationship

$$(H - V)^2 = m^2c^4 + C^2(\vec{P} - e/c\;A)^2 \qquad (1.4)$$

which lead to the Klein-Gordon equation. The problems with this equation were so great, that all involved became frustrated and it was dropped from serious consideration for a few years. Dirac[6] argued that the proper equation should be first order in both the space and time variables, in order to be a true relativistic wave equation. This lead to the well-known Dirac equation.

In the same paper that Dirac provided the basic ideas which lead to the Feynman integral,[7] he noted that "the Hamiltonian method is essentially non-relativistic in form, since it marks out a particular time variable as the canonical conjugate of the Hamiltonian function."

Dirac's position, that the equation should be first order in the space and time variables, emphasizes the relativistic invariance point of view in the merge of special relativity with quantum mechanics. From the quantum mechanical point of view, one could argue that a proper relativistic wave equation would elevate the time coordinate to the same level as the space coordinate, in that both become operators. In the relativistic quantum theory of present day, the time coordinate does not have equal status with the space coordinate.

Purpose

This paper represents the first step in showing that the problem of the square-root operator, the question of time as an operator and the inability to give a particle interpretation for the Klein-Gordon equation are all deeply interrelated issues, whose joint resolution provides a new approach to the age old question of how best to bring about the merge of special relativity with quantum mechanics.

In the 1950's, mathematicians,[8] while studying the foundations of stochastic processes, discovered a method to construct fractional powers of generators of semigroups of operators. At that time it became possible to give an unambiguous meaning to the interacting square-root operator in Eq. (1.2). In this paper, we restrict ourselves to the free particle case.

In Sec. II, we provide a brief outline of the main results concerning semigroups of operators and fractional powers. In Sec. III we use the results of Sec. II to obtain an analytic factorization of the free-particle square-root operator. The factorization leads to an operator with many of the properties expected of the Yukawa potential. In Sec. IV, we construct free-particle solutions, discuss Lorentz invariance and construct a conserved current. In Sec. V we look at the special case $m=0$ which produces the photon equation. In Sec. VI we bring together our previous results to give meaning to proper-time as an operator. We use this operator to provide a particle interpretation for the Klein-Gordon equation. In the appendix we outline some properties of Bessel potentials which are used in the paper.

II. SEMIGROUPS AND FRACTIONAL POWERS OF OPERATORS

This section provides a brief survey of the general theory of strongly continuous semigroups of operators, which is in turn used to explain the theory of fractional powers of operators. The definitions and basic results are recorded here for reference so as to make the article self-contained. Yosida[8] is the most general reference, Butzer and Berens[9] has a very nice introduction to semigroups and Tanabe[10] has a good chapter on fractional powers.

<u>Def</u> 2.1 Let $\{T(t) \mid t \geq 0\}$ be a bounded family of linear operators in a Banach space X. This family is called a strongly continuous semigroup of operators (also known as a C_0-semigroup) if the following conditions are satisfied:

1) $T(t+s) = T(t)T(s) = T(s)T(t), \forall\, t, s \geq 0, T(0) = I$

2) $\lim_{t \to s} T(t)u = T(s)u, \forall\, u \in X$

If the family is defined on $(-\infty, \infty)$ then it is called a C_0-group, $T(-t) = T^{-1}(t)$. By further restriction, we obtain the well-known definition of a unitary group.

<u>Theorem 2.1</u> Let $(T(t) \mid t \geq 0)$ be a C_0 - semigroup, let $D = \{u \mid \lim_{h \downarrow 0} h^{-1}[T(h) - I]u \text{ exists}\}$. Define $AU = \lim_{h \downarrow 0} h^{-1}[T(h) - I]u$ for $u \in D$; then:

1) D is dense in X

2) $u \in D \Rightarrow T(t)u \in D$ for each $t \geq 0$

3) $$\frac{d}{dt}[T(t)u] = A(T(t)u] = T(t)AU \quad \text{for} \quad u \in D. \tag{2.1}$$

Proof: (see Tanabe[10] page 53)

<u>Remark</u> In the theory of semigroups, it's customary to suppress all variables except the time variable; for example, if we set $u(0) = u$ and define $u(t) = T(t)'u(0)$, then from (2.1) we have:

$$\frac{du(t)}{dt} = Au(t), \quad u(0) = u \tag{2.2}$$

Eq. (2.2) is called the "Abstract initial value problem." A is called the semigroup generator and we may write

$$T(t) = \exp(tA) \tag{2.3}$$

<u>Theorem 2.2</u> The generator A of the semigroup $\{T(t) \mid t \geq 0\}$ is a closed linear operator. If $T(t) \leq Me^{\beta t}$ for fixed constants M and β, then the half-plane $\{\lambda \mid Re(\lambda) > \beta\}$ is contained in the resolvent set $\rho(A)$ and for each such λ, we have: (Tanabe page 55)

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)dt = R(\lambda, A) \tag{2.4}$$

$R(\lambda, A)$ is called the resolvent operator of A and

$$||R(\lambda, A)|| \leq M[Re(\lambda) - \beta]^{-1}$$

<u>Theorem 2.3</u> If the semigroup $\{T(t) \mid t \geq 0\}$ is C_0, and for each $t>0$, $T(t)X \subset D$; and, there exists a $N>0$ such that:

$$t||AT(t)|| \leq N \quad \text{for} \quad 0 \leq t \leq 1$$

then the semigroup has a holomorphic extension $\{T(z)|z\epsilon\Delta\}$ where

$$\Delta = \{z \mid Re(z) > 0, \ |\arg z| < (eN)^{-1}\}$$

In this case the family $\{T(t)|t\geq 0\}$ is called a holomorphic C_0-semigroup of operators. (See Butzer and Berens[9] page 16).

Introduce the function $f_{t,\alpha}(s)$ defined by: (Yosida[8] page 259)

$$f_{t,\alpha}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \exp(z\lambda - tz^\alpha)dz, \quad \lambda \geq 0 \tag{2.5}$$

$$= 0 \quad \text{when} \quad \lambda < 0$$

where $t>0$, $0<\alpha<1$ and $\sigma>0$, and the branch of z^α is taken so that $Re(z^\alpha)>0$ when $Re(z)>0$. The branch is a single valued function in the complex z-plane cut along the negative real axis. The convergence of the integral (2.5) is insured by the factor $\exp\{-tz^\alpha\}$, define $\hat{T}_\alpha(t)$ by:

$$\hat{T}_\alpha(t)u = \int_0^\infty f_{t,\alpha}(s)T(s)u\,ds, \quad t > 0 \tag{2.6}$$

$$= u \qquad t = 0$$

Where $\{T(t)|t \geq 0\}$ is a C_0-semigroup of operators on X.

<u>Theorem 2.4</u> Let A be the generator of the family $\{T(t)|t\geq 0\}$ then:

1) $\{\hat{T}_\alpha(t)|t\geq 0\}$ is a holomorphic C_0-semigroup on X for each $\alpha\epsilon(0,1)$.

2) $\hat{A}_\alpha$, the generator of $\{\hat{T}_\alpha(t)|t\geq 0\}$ is defined by:

$$\hat{A}_\alpha u = -(-A)^\alpha u, \quad u \in D \text{ and} \tag{2.7a}$$

$$\hat{A}_\alpha u = \frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{\alpha-1} R(\lambda, A)[-Au]d\lambda \tag{2.7b}$$

3) $$R(\mu, \hat{A}_\alpha)u = \frac{\sin\alpha\pi}{\pi}\int_0^\infty \frac{R(\lambda, A)u\,\lambda^\alpha d\lambda}{\mu^2 - 2\mu\lambda^\alpha \cos\alpha\pi + \lambda^{2\alpha}} \tag{2.8}$$

4) If A is invertible then:

$$(-A)^{-\alpha}u = (-\hat{A}_\alpha)^{-1}u = \frac{\sin\alpha\pi}{\pi}\int_0^\infty R(\lambda,A)u\,\lambda^{-\alpha}d\lambda \tag{2.9}$$

Proof: (see Yosida[8] page 260) 4) follows from 3) if $\mu=0$.

<u>Theorem 2.5</u> Suppose $u \in D(A^2)$ (A^2u exists) then:

1) $\hat{A}_\alpha \hat{A}_\beta u = \hat{A}_{\alpha+\beta} u$, $\alpha, \beta > 0$, $\alpha + \beta < 1$

2) $\lim_{\alpha \uparrow 1} \hat{A}_\alpha u = Au$

3) $\lim_{\alpha \downarrow 0} \hat{A}_\alpha u = -u$ (if $\lim_{\lambda \downarrow 0} \lambda R(\lambda, A)u = 0$)

Of particular interest to our work is the case $\alpha=1/2$. Let us deform the path of integration in Eq. (2.5) into a union of two paths, $re^{-i\theta}$ when $-r \in (-\infty, 0)$ and $re^{i\theta}$, $r \in (0, \infty)$, where $\pi/2 \leq \theta \leq \pi$, we get:

$$f_{t,1/2}(s) = \frac{1}{\pi}\int_0^\infty \exp\{sr \cos\theta - tr^{1/2} \cos\theta/2\}.$$

$$\times \sin\{sr \sin\theta - tr^{1/2} \sin(\theta/2) + \theta\}dr \tag{2.10}$$

set $\theta = \pi$, (2.10) becomes:

$$f_{t,1/2}(s) = \frac{1}{\pi}\int_0^\infty \exp(-sr) \sin(+ tr^{1/2})dr \tag{2.11}$$

Using a standard table of LaPlace transforms, we have:

$$f_{t,1/2}(s) = \frac{t}{2\sqrt{\pi}\, s^{3/2}} \exp(-\frac{t^2}{4s}) \tag{2.12}$$

III. SQUARE-ROOT OPERATOR EQUATION

In this section, we apply the theory of fractional powers to provide a direct analytical meaning to Eq. (1.3). For now we restrict ourselves to the free particle case, the study of interaction, brings in new effects and additional issues, hence will be taken up at a later date. In order to compute:

$$\hbar c L[\psi(\vec{x})] = (\sqrt{-c^2\hbar^2\Delta + m^2c^4})\psi(\vec{x}), \tag{3.1}$$

set $\omega = mc/\hbar$, we may then write (3.1) as:

$$\hbar c L[\psi(\vec{x})] = \hbar c(\sqrt{-\Delta + \omega^2})\psi(\vec{x})$$

or

$$L[\psi(\vec{x})] = (\sqrt{-\Delta + \omega^2})\psi(\vec{x}). \tag{3.2}$$

Let $A = \Delta-\omega^2$, so that (3.2) becomes:

$$L[\psi(\vec{x})] = [-A]^{1/2}\psi(\vec{x}) \tag{3.3}$$

We may now use equation (2.7) since $\Delta-\omega^2$ generates a (semigroup) solution to the (abstract) diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u - \omega^2 u \tag{3.4}$$

We have:

$$[-A]^{1/2}\psi = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}}\, R(\lambda, A)[-A\psi] \tag{3.5}$$

In order to compute $R(\lambda,A)$ we need the LaPlace transform of the fundamental solution to (3.4) (see Eq. (2.3)). This is done in many textbooks on partial differential equations (for example Trèves,[11] page 41) the result is:

$$P[\omega^2, t, \vec{x} - \vec{y}] = \frac{1}{[4\pi t]^{3/2}} \exp\left(-\omega^2 t - \frac{||\vec{x} - \vec{y}||^2}{4t}\right) \tag{3.6}$$

and

$$R(\lambda, A)u(\vec{x}) = \int_0^\infty e^{-\lambda t}\, \tilde{u}(\vec{x}, t)dt \tag{3.7}$$

where:

$$\tilde{u}(\vec{x}, t) = \int_{\mathbb{R}^3} P[\omega^2, t, \vec{x} - \vec{y}]u(\vec{y})d\vec{y} \tag{3.8}$$

Using (3.8) in (3.7) and inverting the order of integration (e.g. Fubini's theorem) we have:

$$R(\lambda, A)u(\vec{x}) = \int_{\mathbb{R}^3} u(\vec{y})d\vec{y}\int_0^\infty \exp(-\lambda t)\, P[\omega^2, t, \vec{x} - \vec{y}]dt \tag{3.9}$$

The inner integral can be compute using a Laplace transform table, we get:

$$R(\lambda, A)u(\vec{x}) = \int_{\mathbb{R}^3} u(\vec{y}) \exp\left(-\sqrt{\lambda + \omega^2}\, ||\vec{x} - \vec{y}||\right) \frac{d\vec{y}}{4\pi||\vec{x} - \vec{y}||} \tag{3.10}$$

We now use (3.10) in (3.5) and identify $u(y) = -A\psi(y)$ to get:

$$[-A]^{1/2}\psi(\vec{x}) = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}}\int_{\mathbb{R}^3} \exp\left(-\sqrt{\lambda + \omega^2}\, ||\vec{x} - \vec{y}||\right) \frac{d\vec{y}}{4\pi||\vec{x} - \vec{y}||}[-A\psi(\vec{y})] \tag{3.11}$$

Invert the order of integration in (3.11), we get:

$$[-A]^{1/2}\psi(\vec{x}) = \frac{-1}{4\pi^2}\int_{\mathbb{R}^3} \frac{d\vec{y}}{||\vec{x} - \vec{y}||}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \exp\left(-\sqrt{\lambda + \omega^2}\, ||\vec{x} - \vec{y}||\right) [A\psi(y)] \tag{3.12}$$

To compute the inner integral, set $P = ||\vec{x} - \vec{y}||$, $\theta^2 = \lambda + \omega^2$, $d\lambda = 2\theta d\theta$, $\sqrt{\lambda} = \sqrt{\theta^2 - \omega^2}$,

the inner integral becomes:

$$\int_{\omega^2}^\infty \exp(-P\theta)\, \frac{2\theta d\theta}{\sqrt{\theta^2 - \omega^2}} \tag{3.13}$$

if we integrate by parts we have:

$$P\int_{\omega^2}^{\infty} \exp(-P\theta)\ \sqrt{\theta^2 - \omega^2}\,d\theta \tag{3.14}$$

Compute (3.14) using a table of Laplace transforms, to get:

$$\int_0^{\infty} \exp(-\sqrt{\lambda + \omega^2}\ ||\vec{x} - \vec{y}||)\ \frac{d\lambda}{\sqrt{\lambda}} = \frac{2\omega\Gamma(3/2)}{\sqrt{\pi}}\, K_1[\omega||\vec{x} - \vec{y}||] \tag{3.15}$$

Where K_1 [z] is the modified Bessel function of the third kind of order one. Putting (3.15) into (3.12) we obtain:

$$[-A]^{1/2}\psi(\vec{x}) = \left(\frac{-1}{4\pi^2}\right) \frac{2\omega\Gamma(3/2)}{\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{K_1[\omega||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||} [(\Delta_y - \omega^2)\psi(\vec{y})]d\vec{y} \tag{3.16}$$

Since $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$, we have:

$$L[\psi] = \frac{-\omega}{4\pi^2} \int_{\mathbb{R}^3} \frac{K_1[\omega||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||} [(\Delta_y - \omega^2)\psi(\vec{y})]d\vec{y} \tag{3.17}$$

We note that A is invertible and:

$$\sqrt{-A} = \frac{-A}{\sqrt{-A}}, \tag{3.18}$$

or we may use Eq. (2.9) directly to get that:

$$[-A]^{-1/2}\psi(\vec{x}) = \frac{\omega}{4\pi^2} \int_{\mathbb{R}^3} \frac{K_1[\omega||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||} \psi(\vec{y})d\vec{y} \tag{3.19}$$

We shall have use for this result in Sec. VI.

Returning to Eq. (3.17) we may now further refine our result so that physical implications may be more obvious, define $G_1[\omega|\vec{x}-\vec{y}|]$ by:

$$G_1[\omega\ |\vec{x} - \vec{y}|] = \frac{1}{2\pi^2} \frac{K_1[\omega||\vec{x} - \vec{y}||]}{\omega||\vec{x} - \vec{y}||} \tag{3.20}$$

G_1 is known as the Bessel potential of order one (c.f. Aronszajn and Smith[12] or Donoghue[13]). Let us note that $G_\beta[\omega|\vec{x}-\vec{y}|]$ is defined in general by:

$$G_\beta[\omega||\vec{x} - \vec{y}||] = \frac{1}{2^{\frac{3+\beta-2}{2}}\ \pi^{3/2}\ \Gamma(\beta/2)}\ \frac{K_{\frac{3-\beta}{2}}[\omega||\vec{x} - \vec{y}||]}{[\omega||x - y||]^{\frac{3-\beta}{2}}} \tag{3.21}$$

So that, in particular, $G_2[\omega|\vec{x}-\vec{y}|]$ is the well-known Yukawa potential (except for a factor). A more detailed discussion will be given later, see also the appendix.

Using (3.20) in (3.18) we get:

$$L[\psi] = -\frac{\omega^2}{2}\int_{\mathbb{R}^3} G_1[\omega|\vec{x}-\vec{y}|][(\Delta_y - \omega^2)\psi(\vec{y})]d\vec{y} \tag{3.22}$$

$$= -\frac{\omega^2}{2}\int_{\mathbb{R}^3} G_1[\omega|\vec{x}-\vec{y}|]\,\Delta_y\,\psi(\vec{y})d\vec{y} \qquad \text{(I)}$$

$$+\frac{\omega^4}{2}\int_{\mathbb{R}^3} G_1[\omega|\vec{x}-\vec{y}|]\psi(\vec{y})d\vec{y} \qquad \text{(II)} \tag{3.23}$$

In order to transform (I) above, let us construct a small ball of radius ρ about $\vec{x}$, $B_\rho(\vec{x}) = \{\vec{y}\in R^3 \mid |\vec{x}-\vec{y}|\leq\rho\}$ on this ball, $\vec{y} = \vec{x}+\vec{\mu}\rho$, where $\vec{\mu} = -\vec{\nu}$ and $\vec{\nu}$ is the inward normal (so that $\vec{\mu}$ is the outward normal). We now have:

$$\int_{\mathbb{R}^3} G_1[\omega|\vec{x}-\vec{y}|]\Delta_y\,\psi(\vec{y})d\vec{y} = \int_{\mathbb{R}^3\setminus B_\rho(\vec{x})} G_1[\omega|x-y|]\Delta_y\,\psi(\vec{y})d\vec{y}$$

$$+\int_{B_\rho(\vec{x})} G_1[\omega|\vec{x}-\vec{y}|]\Delta_y\,\psi(\vec{y})d\vec{y} \tag{3.24}$$

Let us now apply Green's theoreum to get:

$$\int_{\mathbb{R}^3\setminus B_\rho(\vec{x})} G_1[\omega|\vec{x}-\vec{y}|]\Delta_y\,\psi(\vec{y})d\vec{y} = \int_{\mathbb{R}^3\setminus B_\rho(\vec{x})} \Delta_y G_1[\omega|\vec{x}-\vec{y}|]\psi(\vec{y})d\vec{y}$$

$$+\int_{\partial B_\rho(\vec{x})} [G_1[\omega|\vec{x}-\vec{y}|]\psi_\nu(\vec{y})dy - (G_1[\omega|\vec{x}-\vec{y}|])_\nu\,\psi(\vec{y})]ds \tag{3.25}$$

where $f_\nu = \frac{\partial f}{\partial\vec{\nu}} = \vec{\nabla}_{\vec{\nu}} f$, ds is the surface area element of the ball, $ds = \rho^2 \sin\theta\, d\theta\, d\phi$, $0\leq\theta\leq\pi$, $0\leq\phi\leq 2\pi$, $ds = \rho^2 d\Omega$

$\nu_1 = \cos\phi\sin\theta$

$\nu_2 = \sin\phi\sin\theta$

$\nu_3 = \cos\theta$

We want to integrate:

$$\text{(i)}\quad \int_{|\vec{\mu}|=1} G_1[\omega\rho]\,\psi_\mu[\vec{x}+\vec{\mu}\rho]ds$$

and

$$\text{(ii)}\quad \int_{|\vec{\mu}|=1} G_1[\omega\rho]_\mu\,\psi(\vec{x}+\vec{\mu}\rho)ds$$

For (i) we have: (in the limit as $\rho\to 0$)

$$\int_{|\vec{\mu}|=1} G_1[\omega\rho]\,\psi_\mu[\vec{x}+\vec{\mu}\rho]ds = G_1[\omega\rho]\rho^2\sum_{j=1}^{3}\int_0^{2\pi}\int_0^{\pi}\frac{\partial\psi}{\partial y_i}\mu_j\,d\Omega \tag{3.26}$$

$$= G_1[\omega\rho]\rho^2\,\Big[\frac{\partial\psi}{\partial x_j}\int_0^{2\pi}\int_0^{\pi} -\cos\phi\sin^2\theta\, d\theta d\theta$$

$$+\frac{\partial\psi}{\partial x_2}\int_0^{2\pi}\int_0^{\pi} -\sin\phi\sin^2\theta d\theta d\theta$$

$$+\frac{\partial\psi}{\partial x_3}\int_0^{2\pi}\int_0^{\pi} -\cos\theta\sin^2\theta d\theta d\theta\Big] \tag{3.27}$$

Each integral in (3.27) is zero, since

$$\lim_{\rho \to 0} \rho^2 G_1[\omega\delta] = \frac{1}{\omega^2}$$

(see the appendix) we see that (i) gives a zero contribution. In order to compute (ii), note that on $|\mu|=1$, $(G_1[\omega\rho])_\mu = +(G_1[\omega\rho])_\rho = +\omega\, d/d(\omega\rho)\, G_1[\omega\rho]$ set $r = \omega\rho$ so that $(G_1[\omega\rho])_\rho$ becomes

$$\omega \frac{d}{dr} G_1[r],$$

(ii) reduces to:

$$\int_{|\vec{\mu}|=1} (G_1[\omega\rho])_\mu\, \psi(\vec{x} + \vec{\mu}\rho)ds = + \omega \frac{dG_1[r]}{dr} \left(\frac{r^2}{\omega^2}\right) \int_{|\vec{\mu}|=1} \psi(\vec{x} + \vec{\mu}\rho)d\Omega \qquad (3.28a)$$

By Taylor's theorem,

$$\psi(\vec{x} + \vec{\mu}\rho) = \psi(\vec{x}) + \sum_{j=1}^{3} \rho\mu_j \left.\frac{\partial\psi}{\partial y_j}\right|_x + R_1[\vec{\mu}\rho, \vec{x}]$$

where $R_1[\vec{\mu}\rho, \vec{x}]/||\vec{\mu}\rho|| \to 0, \rho \to 0$

Using this in (3.28a), we have:

$$\int_{|\vec{\mu}|=1} (G_1[\omega\rho])_\mu\, \psi(\vec{x} + \vec{\mu}\rho)ds = + \frac{r^2}{\omega} (G_1[r])[\psi(\vec{x}) \int d\Omega$$

$$+ \rho \sum_{j=1}^{3} \int \mu_j \left.\frac{\partial\psi}{\partial y_j}\right|_{\vec{x}} d\Omega + \int R_1[\vec{\mu}\rho, \vec{x}]d\Omega]$$

In the limit we have:

$$\lim_{\rho \to 0} \int_{|\vec{\mu}|=1} (G_1[\omega\rho])_\mu\, \psi(\vec{x} + \vec{\mu}\rho)ds = \lim_{\rho \to 0} + [4\pi\psi(x) \frac{r^2}{\omega} (G_1[r])'] \qquad (3.28)$$

We show in the appendix that

$$(G_1[r])' = - \frac{G_3[r]}{r} - 2\, \frac{G_1[r]}{r} \qquad (3.29)$$

and that $r\,G_3[r]\to 0$, $r\to 0$ while $r\,G_1[r]$ becomes divergent (this also follows from the result after Eq.(3.27)). This means that Eq. (3.28) is singular at $r=0$. In order to see why this term is needed, let us compute

$$\int_{\mathbb{R}^3} \Delta_y\, G_1[\omega\,||\vec{x}-\vec{y}||]\,\psi(\vec{y})d\vec{y}$$

Since (appendix) $(I-\Delta)\,G_\beta = G_{\beta-2}$ we have:

$$\Delta_y\, G_1[\omega\,||\vec{x}-\vec{y}||] = \omega^2\{G_1[\omega\,||\vec{x}-\vec{y}||] - G_{-1}[\omega\,||\vec{x}-\vec{y}||]\} \tag{3.30}$$

$$G_{-1}[\omega\,||\vec{x}-\vec{y}||] = -\left\{\frac{G_3\,[\omega\,||\vec{x}-\vec{y}||] + 2G_1[\omega\,||\vec{x}-\vec{y}||]}{\omega^2\,||\vec{x}-\vec{y}||^2}\right\} \tag{3.31}$$

Putting (3.31) and (3.30) into the integral, we get:

$$\int_{\mathbb{R}^3} \Delta_y G_1[\omega\,||\vec{x}-\vec{y}||]\,\psi(\vec{y})d\vec{y} = \omega^2\int_{\mathbb{R}^3} G_1[\omega\,|\vec{x}-y^*|]\,\psi(\vec{y})d\vec{y}$$
$$+\int_{\mathbb{R}^3}\frac{G_3[\omega\,||\vec{x}-\vec{y}||] + 2G_1[\omega\,||\vec{x}-\vec{y}||]}{||\vec{x}-\vec{y}||^2}\,\psi(y)dy \tag{3.32}$$

It is not hard to see that the last term in Eq. (3.22) is divergent at $\vec{x}=\vec{y}$. Returning to Eq. (3.28) we may write it as:

$$-4\pi\int_{\mathbb{R}^3}\frac{G_3[\omega||\vec{x}-\vec{y}||] + 2G_1[\omega\,||\vec{x}-\vec{y}||]}{||\vec{x}-\vec{y}||}\,\delta(\vec{x}-\vec{y})\,\psi(\vec{y})d\vec{y} \tag{3.33}$$

Using (3.5) to acquire a (-1) factor which changes the sign of (3.33), combining with (3.32) we have:

$$\int_{\mathbb{R}^3} G_1[\omega\,||\vec{x}-\vec{y}||]\Delta_y\,\psi(\vec{y})d\vec{y} = \omega^2\int_{\mathbb{R}^3} G_1[\omega\,||\vec{x}-\vec{y}||]\,\psi(\vec{y})d\vec{y}$$
$$+\int_{\mathbb{R}^3}\frac{\{G_3[\omega||\vec{x}-\vec{y}||] + 2G_1[\omega||\vec{x}-\vec{y}||]\}}{||\vec{x}-\vec{y}||}\,\left(\frac{1}{||\vec{x}-\vec{y}||}\right.$$
$$\left.-4\pi\delta(\vec{x}-\vec{y})\right)\psi(\vec{y})d\vec{y} \tag{3.4}$$

Returning to (3.23), we see that:

$$L[\psi(\vec{x})] = -\frac{\omega^2}{2}\int_{\mathbb{R}^3}\frac{G_3[\omega\,||\vec{x}-\vec{y}||] + 2G_1[\omega\,||\vec{x}-\vec{y}||]}{||\vec{x}-\vec{y}||}\,\left[\frac{1}{||\vec{x}-\vec{y}||} - 4\pi\delta(\vec{x}-\vec{y})\right]\psi(y)d\vec{y} \tag{3.35}$$

Eq. (1.3) may now be written as:

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) = -\int_{\mathbb{R}^3} G[\omega\,||\vec{x}-\vec{y}||]\,\psi(\vec{y},t)d\vec{y} \tag{3.36}$$

Where the kernal G is given by:

$$G[\omega\ ||\vec{x} - \vec{y}||] = \frac{\hbar c\omega^2}{2}\left\{\frac{G_3[\omega\ ||\vec{x} - \vec{y}||] + 2G_1[\omega\ ||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||}\right\} \times \left\{\frac{1}{||\vec{x} - \vec{y}||} - 4\pi\delta(\vec{x} - \vec{y})\right\} \tag{3.37}$$

Discussion

From Eq. (3.21) and the results of the appendix, it is easy to see that:

$$-g^2 4\pi \sqrt{\omega}\, G_2[\omega\ ||\vec{x} - \vec{y}||] = g^2 \frac{\exp(-\omega\ ||\vec{x} - \vec{y}||)}{||\vec{x} - \vec{y}||} \tag{3.38}$$

This of course is the well-known Yukawa[14] potential, proposed in 1935 in order to account for the short range of the nuclear interaction which was then conjectured to be mediated by massive particles (mesons). Where g represents the "charge" of the exchange field. Yukawa assumed that the range of the interaction was $1/\omega$ = 1.4 fermi which lead to a mass value of

about 270 times that of the electron. In 1936, Anderson and Neddermeyer[15] discovered what was believed to be Yukawa's meson with a mass of 207 times that of the electron. This particle interacted so weakly with nuclei and had such a long lifetime, it was rejected as a participant in the nuclear interaction. In 1947,[16] the π-meson (pion) was identified and it had all the expected properties.

It is helpful at this point to directly review (3.35) in terms of the Bessel kernals $K_0[Z]$ and $K_1[Z]/Z$. The spherical symmetry means that the effective kernal in (3.35) is: (assuming $\vec{X} \neq \vec{Y}$)

$$G_3[\omega\ ||\vec{x} - \vec{y}||] + 2G_1[\omega\ ||\vec{x} - \vec{y}||],$$

using (3.21) we get that the effective kernal is:

$$\frac{1}{2\pi^2} K_0[\omega\ ||\vec{x} - \vec{y}||] + \frac{1}{\pi^2\omega} \frac{K_1[\omega\ ||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||} \tag{3.39}$$

We follow Donoghue;[13] for $0<Z<<1$ (appendix)

$$\left.\begin{aligned} \frac{K_1[z]}{z} &= c_1[1 + \theta_1(z)]z^{-2} \\ K_0[z] &= c_1[1 + \theta_1(z)]\log z^{-1} \end{aligned}\right\} \tag{3.40}$$

where $\theta_1(Z)\downarrow 0$, $Z\downarrow 0$. For $Z>>1$,

$$\left.\begin{aligned} \frac{K_1[z]}{z} &= c_2[1 + \theta_2(z)]\,\frac{\exp(-z)}{z^{3/2}} \\ K_0[z] &= c_2[1 + \theta_2(z)]\,\frac{\exp(-z)}{z^{1/2}} \end{aligned}\right\} \tag{3.41}$$

In particular, we note that for z close to zero;

$$\frac{K_1[z]}{z} > \frac{\exp(-z)}{z} > K_0[z] \tag{3.42}$$

in the sense of strength of singularity. The $K_0[Z]$ singularity is the weakest possible at $Z=0$, in that:

$$\lim_{z \to 0} z^{\epsilon} K_0[z] = 0, \quad \epsilon > 0 \tag{3.43}$$

On the other hand, for Z large, the inequality (3.42) is reversed so that: (see (3.41))

$$K_0[z] > \frac{\exp(-z)}{z} > \frac{K_1[z]}{z} \tag{3.44}$$

Using the constant $-\omega^2/2$ from (3.35) in (3.39) we get:

$$-\frac{\omega^2}{4\pi^2} \cdot K_0[\omega\, ||\vec{x} - \vec{y}||] - \frac{\omega}{2\pi^2} \frac{K_1[\omega\, ||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||} \tag{3.45}$$

The coefficient of the K_0 term has an additional factor of $\omega/2$; assuming $m=m_e$, this gives a value of $\approx 5\times10^{13}\text{cm}^{-1}$. We may thus

draw the following conclusions concerning the operator $L[\psi]$ in Eq. (3.35):

1) L [ ] acts like a <u>short range potential</u> (nonlocal) with effective range $\hbar/mc$ where m is the given mass of the spin zero particle; <u>two parts are attractive and one part is hard core repulsive</u>.
2) The $G_3[z]$ term has a very weak ($\ln 1/z$) singularity at $z=0$, while the $G_1[z]$ term has a very strong (z^{-2}) singularity at $z=0$. The Yukawa singularity is (z^{-1}) between (halfway) them.
3) Although both terms are short-range, the $G_3[z]$ term has a slightly longer range compared to both the $G_1[z]$ and the Yukawa term. In addition, the coefficient of the $G_3[z]$ term is larger than that of the $G_1[z]$ term by at least a factor of 10^9.
4) L [ ] looks like the <u>zero operator</u> outside a few (≈ 3) Compton wavelengths. This follows from a graph of the $G_3[z]$ term.

In 1) we have noted that L [ ] acts "like" a short range attractive potential. This is certainly a new aspect which occurs because of the confluence of special relativity and quantum mechanics. It is very interesting that there are three parts. We are accustomed to think that free relativistic particles have kinetic and rest energies but never having an "intrinsic" potential energy. Related ideas on this subject were discussed by Brillouin[17] in the classical case, and lead him to conclude that it was incorrect to assume that the total energy (relativistic) could always be written as the sum of the kinetic plus potential. In a simple analysis of two interacting charged spheres, he showed how additional mass occurred because of the potential energy.

IV. FREE-PARTICLE SOLUTIONS

We may use Eq. (2.12) to construct explicit solutions to (3.36). In order to keep the units correct, we must replace t by ct so that (2.12) becomes:

$$f_{t,1/2}(s) = \frac{ct}{\sqrt{4\pi}} s^{3/2} \exp\left(-\frac{c^2t^2}{4s}\right) \tag{4.1}$$

Let us note that had we kept $L[\psi]$ in the form, $L[\psi] = \sqrt{-c^2\hbar^2\Delta+m^2c^4}$, this change would not be required. Using (4.1) in (2.6), we get:

$$\hat{T}_{1/2}(t)\, v(\vec{x}) = \int_0^\infty \left(\frac{ct}{4\pi}\right) \exp\left(-\frac{c^2t^2}{4s}\right) \cdot T(s)\, v(\vec{x})\, \frac{ds}{s^{3/2}} \tag{4.2}$$

Where, as in (4.2), $T(s)v(\vec{x})$ is defined by:

$$T(s)\, v(\vec{x}) = \int_{\mathbb{R}^3} \exp\left(-\omega^2 s - \frac{||\vec{x}-\vec{y}||}{4s}\right) v(\vec{y})\, \frac{d\vec{y}}{(4\pi s)^{3/2}} \tag{4.3}$$

Combining (4.3) with (4.2), we obtain:

From any table of Laplace transforms, we get that:

$$\int_0^\infty e^{-ps}\, e^{-a/s}\, \frac{ds}{s^3} = 2(P/a)\, K_2[2(aP)^{1/2}] \tag{4.5}$$

With:

$$a = \frac{||\vec{x}-\vec{y}||^2 + c^2t^2}{4}, \quad P = \omega^2,$$

we can interchange the order of integration to get:

$$\hat{T}_{1/2}(t)\, v(\vec{x}) = \int_{\mathbb{R}^3} \frac{ct}{(4\pi)^2} \frac{8\omega^2 K_2(\omega\sqrt{||\vec{x}-\vec{y}||^2 + c^2t^2})}{||\vec{x}-\vec{y}||^2 + c^2t^2}\, v(\vec{y})d\vec{y} \tag{4.6}$$

To put (4.6) in a more appropriate form, we note that

$$K_2[z] = K_0[z] + \frac{2K_1[z]}{z}$$

$$G_3[z] = \frac{1}{2\pi^2} K_0[z], \quad G_1[z] = \frac{1}{2\pi^2} \frac{K_1[z]}{z}$$

so that

$$\hat{T}_{1/2}(t)\, v(\vec{x}) = 4ct\omega^2 \int_{\mathbb{R}^3} \frac{G_3\left[\omega\sqrt{||\vec{x}-\vec{y}|^2 + c^2t^2}\right] + 2G_1\left[\omega\sqrt{||x-y||^2 + c^2t^2}\right)}{||\vec{x}-\vec{y}||^2 + c^2t^2} \times v(\vec{y})d\vec{y} \tag{4.7}$$

We now use the fact that $\hat{T}_{1/2}(t)$ has a holomorphic extension into the complex plane so that we may replace t by it in (4.7). Setting $U(t) = \hat{T}_{1/2}(it)$, identifying $\psi(\vec{y}, 0)$ with $v(\vec{y})$ leads to:

$$\psi(\vec{x}, t) = U(t)\psi(\vec{x}, 0)$$

$$\cdot\,(4ict\omega^2) \int_{\mathbb{R}^3} \frac{G_3[\omega\sqrt{||\vec{x} - \vec{y}||^2 - c^2t^2}] + 2G_1[\omega\sqrt{||\vec{x} - \vec{y}||^2 - c^2t^2}]}{||\vec{x} - \vec{y}||^2 - c^2t^2}\,\psi(\vec{y}, 0)\,d\vec{y} \tag{4.8}$$

To compute the integral in (4.8), we must replace $|\vec{x} - \vec{y}|^2 - c^2t^2$ by $||\vec{x} - \vec{y}||^2 - c^2t^2 + i\varepsilon$, afterward, let $\varepsilon \downarrow 0$; with this understanding, ε is omitted.

The representation (4.8) of $\psi(\vec{x}, t)$, makes it clear that whatever the initial state $\psi(\vec{x}, 0)$, the wave packet spreads instantaneously over all space at any later time. This result implies a conflict with (Einstein) causality since the speed of propagation is instantaneous; of course the measurable effects are in a region of a few fermi due to the mass cutoff.

It has been shown by Hegerfeldt[18] and Ruijsenaars, that this problem is quite widespread, however, their methods do not apply in general to equations which are not bounded below in energy, as in the case of the Klein-Gordon and the Dirac equations. Equation (4.8) is a solution to the Klein-Gordon equation which does violate causality. This is quite interesting since signal propagation still occurs with velocity bounded above by the speed of light. This apparent contradiction clearly indicates the need for additional study.

Let us recall that the Foldy-Wouthuysen[19] transformation is a unitary operator which relates solutions of the Dirac equation to those of the square-root equation. This of course means that the Dirac equation is unitarily equivalent to a causality violating solution.

The question of Lorentz invariance of the square-root operator equation has been discussed by many writers. Sucher[20] has shown that the expected behavior under the action of the Lorentz group, namely:

$$S_\Lambda^{-1} K S_\Lambda = K \tag{4.9}$$

which holds if K is the Dirac or Klein-Gordon operator, does not hold if

$$K = i\hbar\frac{\partial}{\partial t} \pm \sqrt{-c^2\hbar^2\Delta + m^2c^4} \tag{4.10}$$

In this case, K is shown to be invariant but under the more general condition:

$$S_\Lambda^{-1} K S_\Lambda = JK \tag{4.11}$$

Where J is not the identity operator.

We close this section with the construction of a conserved current density. We remark that $\hat{A}_\alpha = -(-A)^\alpha$, $\alpha \epsilon (0,1)$ is essentially self-adjoint if $-A$ is self-adjoint (see Kato[21]). This means that $\hat{A}_\alpha$ has a unique self-adjoint closure. Set $H = \sqrt{c^2\vec{p}^2 + m^2c^4}$, define a linear form on $\{L^2(R^3), (\ ,\ )\}$ (denoted $<,>$) by:

$$<\psi, \phi> = (H^{1/2}\psi, H^{1/2}\phi) \tag{4.12}$$

The self-adjoint property implies that:

$$<\psi, \phi> = (H\psi, \phi) \tag{4.13}$$

It is now easy to show that the linear manifold generated by solutions (4.8), with the inner $<,>$ is a pre-Hilbert space. Define $\rho(\vec{x})$ by:

$$\begin{aligned} \rho(\vec{x}) &= \frac{1}{2mc^2} |H^{1/2}\psi|^2 \\ &= \frac{1}{2mc^2} (H^{1/2}\psi)(H^{1/2}\bar{\psi}) \end{aligned} \tag{4.14}$$

We then have that:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \rho(\vec{x}) d\vec{x} &= \frac{1}{2mc^2} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (H^{1/2}\psi)(H^{1/2}\bar{\psi}) d\vec{x} \\ &= \frac{1}{2mc^2} \int_{\mathbb{R}^3} (H^{1/2} \frac{\partial\psi}{\partial t})(H^{1/2}\bar{\psi}) + (H^{1/2}\psi)(H^{1/2} \frac{\partial\bar{\psi}}{\partial t})] dx) \end{aligned}$$

$$\begin{aligned} &= \frac{-i}{2mc^2\hbar} \left\{ \int_{\mathbb{R}^3} [(H^{1/2}H\psi)(H^{1/2}\bar{\psi}) - (H^{1/2}\psi)(H^{1/2}H\bar{\psi})] d\vec{x} \right\} \\ &= \frac{-i}{2mc^2\hbar} \left\{ \int_{\mathbb{R}^3} [(H^2\psi)\bar{\psi} - \psi(H^2\bar{\psi})] d\vec{x} \right\} \end{aligned} \tag{4.15}$$

In (4.15) we have used the self-adjoint property; simplifying we get:

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \rho d\vec{x} = \frac{i\hbar}{2m} \int_{\mathbb{R}^3} (\bar{\psi}\Delta\psi - \psi\Delta\bar{\psi}) d\vec{x} = \frac{i\hbar}{2m} \int_{\mathbb{R}^3} \vec{\nabla} \cdot [\bar{\psi}\vec{\nabla}\psi - \psi\vec{\nabla}\bar{\psi}] d\vec{x} \tag{4.16}$$

It is clear that (4.16) will be zero by use of the divergence theorem and the fact that the functions ψ vanish at infinity. We thus have a conserved current four-vector defined by:

$$\rho = |H^{1/2}\psi|^2, \ \vec{J} = \frac{i\hbar}{2m}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) \tag{4.17}$$

It should be noted that our definition does not differ much from the one in Schweber[22] (page 57).

V. THE PHOTON EQUATION

In this section, we consider the special case, $\omega=0$, this corresponds to the photon equation in case $\psi(\vec{x},t)$ is a three component vector valued function. It is normal to also impose the subsidiary constraint (gauge) condition:

$$\vec{\nabla} \cdot \psi = 0 \tag{5.1}$$

The method of limits is quite complicated and tedious; it is easier to compute both the square-root operator and the solution generator directly. First, we use Eq. (3.6) with $\omega=0$ in Eq. (3.10) to get:

$$R(\lambda, \Delta)u(\vec{x}) = \int_{\mathbb{R}^3} u(\vec{y})d\vec{y} \int_0^\infty \exp(-\lambda t) \exp\left(-\frac{||\vec{x}-\vec{y}||^2}{4t}\right) \frac{dt}{(4\pi t)^{3/2}} \tag{5.2}$$

$$= \int_{\mathbb{R}^3} \exp(-\sqrt{\lambda}\,||\vec{x}-\vec{y}||)\, u(\vec{y}) \frac{d\vec{y}}{4\pi\,||\vec{x}-\vec{y}||} \tag{5.3}$$

We now use (5.3) in (3.5) to get:

$$[-\Delta]^{1/2}\psi = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} R(\lambda, \Delta)(-\Delta\psi) \tag{5.4}$$

Using the fact that: $\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \exp(-\sqrt{\lambda}\,||\vec{x}-\vec{y}||) = \frac{2}{||\vec{x}-\vec{y}||}$

(5.4) becomes:

$$[-\Delta]^{1/2}\psi = \frac{-1}{2\pi^2}\int_{\mathbb{R}^3} \frac{dy}{||\vec{x}-\vec{y}||^2} \Delta_y\, \psi(\vec{y}) \tag{5.5}$$

Set $Q = \frac{1}{2\pi^2}\frac{1}{||\vec{x}-\vec{y}||} = Q[||\vec{x}-\vec{y}||]$ (5.6)

Construct a sphere of radius ρ about $\vec{x}$, so that: $\vec{y} = \vec{x}-\vec{\nu}\rho = \vec{x}+\vec{\mu}\rho$ where $\vec{\mu} = -\vec{\nu}$ is the outward normal on the sphere. We may now use Green's theorem to get:

$$-\int_{\mathbb{R}^3} Q[\vec{x}-\vec{y}]\Delta_y\,\psi(\vec{y})d\vec{y} = \lim_{\rho\to 0}\Big\{\int_{\mathbb{R}^3\setminus B_\rho(\vec{x})} -\Delta_y\, Q[\vec{x}-\vec{y}]\psi(\vec{y})d\vec{y}$$
$$-\int_{\partial B_\rho(\vec{x})} [(Q\,\psi_\nu) - (Q_\nu\,\psi)]ds\Big\} \tag{5.7}$$

Now

$$-\Delta_y Q[\vec{x}-\vec{y}] = \frac{-1}{\pi^2}\frac{1}{||\vec{x}-\vec{y}||^4} \tag{5.8}$$

and

$$Q_\nu[\vec{x}-\vec{y}] = -Q_\mu[\vec{x}-\vec{y}] = -\frac{dQ}{d\rho}[\vec{x}-\vec{y}] \tag{5.9}$$

since $Q[\vec{x}-\vec{y}] = Q(\rho)$, we have:

$$Q_\nu[\vec{x}-\vec{y}] = \frac{1}{\pi^2}\frac{1}{||\vec{x}-\vec{y}||^3} \tag{5.10}$$

It is easy to see that the other surface integral in (5.7) is zero as in Sec. IV, combining we have:

$$-\int_{\mathbb{R}^3} Q(\vec{x}-\vec{y})\Delta_y\,\psi(\vec{y})d\vec{y} = \lim_{\rho\to 0}\left\{\int_{\mathbb{R}^3\setminus B_\rho(\vec{x})} \frac{-\psi(\vec{y})d\vec{y}}{\pi^2\,||\vec{x}-\vec{y}||^4} + \frac{4\pi\,\psi(\vec{x})}{\pi^2\,||\vec{x}-\vec{y}||}\bigg|_{\vec{x}=\vec{y}}\right\} \quad (5.11)$$

We may rewrite (5.11) as: [using (5.5)]

$$[-\Delta]^{1/2}\,\psi(\vec{x}) = -\frac{1}{\pi^2}\int_{\mathbb{R}^3}\frac{1}{||\vec{x}-\vec{y}||^3}\left[\frac{1}{||\vec{x}-\vec{y}||} - 4\pi\,\delta(\vec{x}-\vec{y})\right]\psi(\vec{y})d\vec{y} \quad (5.12)$$

We note for later reference that:

$$[-\Delta]^{-1/2}\,\psi(\vec{x}) = \frac{+1}{2\pi^2}\int_{\mathbb{R}^3}\frac{\psi(\vec{y})d\vec{y}}{||\vec{x}-\vec{y}||} \quad (5.13)$$

The photon equation becomes:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar c}{\pi^2}\int_{\mathbb{R}^3}\frac{1}{||\vec{x}-\vec{y}||^3}\left[\frac{1}{||\vec{x}-\vec{y}||} - 4\pi\,\delta(\vec{x}-\vec{y})\right]\psi(\vec{y})d\vec{y} \quad (5.14)$$

$$\vec{\nabla}\cdot\psi = 0$$

To compute the general solution to (5.14), we use Eq. (4.1) and (3.6) with $\omega=0$, to obtain:

$$\hat{T}_{1/2}(t)v(\vec{x}) = \frac{ct}{(4\pi)^2}\int_0^\infty\frac{ds}{s^3}\int_{\mathbb{R}^3}\exp\left(-\frac{(||\vec{x}-\vec{y}||^2+c^2t^2)}{4s}\right)v(\vec{y})d\vec{y} \quad (5.15)$$

Integrate by parts twice with $u=\frac{1}{s}$, $dv=\frac{ds}{s^2}e^{-a/s}$, to get:

$$\hat{T}_{1/2}(t)v(\vec{x}) = \frac{(ct)(16)}{(4\pi)^2}\int_{\mathbb{R}^3}\frac{v(\vec{y})d\vec{y}}{(||\vec{x}-\vec{y}||^2+c^2t^2)^2} \quad (5.16)$$

Using the fact that $\hat{T}_{1/2}(t)$ has a holomorphic extension we have:

$$\psi(\vec{x},t) = U(t)\,\psi(\vec{x},0) = \lim_{\varepsilon\to 0}\frac{ict}{\pi^2}\int_{\mathbb{R}^3}\frac{\psi(\vec{y},0)d\vec{y}}{[||\vec{x}-\vec{y}||^2-c^2t^2]^2+i\varepsilon} \quad (5.17)$$

A simple computation shows that the gauge condition $\vec{\nabla}\cdot\psi(\vec{x},t)=0$, becomes:

$$\lim_{\varepsilon\to 0}\frac{-4ict}{\pi^2}\int_{\mathbb{R}^3}\frac{(\vec{x}-\vec{y})\cdot\psi(\vec{y},0)d\vec{y}}{||\vec{x}-\vec{y}||^2-c^2t^2+i\varepsilon} = 0 \quad (5.18)$$

Equations (5.17) and (5.18) provide an alternate (and direct) approach to the physical interpretation of the Photon equation. (Cf Schweber[22] page 116).

VI. PROPER-TIME AND THE KLEIN-GORDON PROBLEM

If one attempts to implement the successful procedures and methods of non-relativistic quantum mechanics with the special theory of relativity, it is well-known that problems of physical interpretation appear. In the case of the Klein-Gordon equation, the inability to give mathematical meaning to the square-root operator forced the use of the square of the physical energy as a logical substitute. This meant that a fundamental deviation had been made from the known methodology since a priori, there was no way of knowing if the standard rules of quantization applied in this case. In order to maintain the standard rules, the concept of a quantum mechanical particle with a meaningful probability density had to be abandoned. The problems are well-known, and discussed by many writers.[23]

In a different direction, in non-relativistic quantum mechanics, the space coordinate becomes an operator, while the time maintains its status as a parameter. A fundamental assumption of relativistic quantum theory is that space and time should have an equal status. It is clear that this does not occur, in that the time variable does not become an operator when relativity and quantum mechanics are merged. This may lead to some problems of an internal consistency nature, but at present things are not clear. In order to clearly see one apparent problem, let us note that the three fundamental relationships of classical special relativity:

$$\frac{d\tau}{dt} = \sqrt{1 - v^2/c^2}$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{6.1}$$

$$E = \sqrt{c^2\vec{p}^2 + m^2c^4}\ ,$$

may be combined uniquely to give:

$$\frac{d\tau}{dt} = \frac{mc^2}{\sqrt{c^2\vec{p}^2 + m^2c^4}} \tag{6.2}$$

If we now make the transition to quantum mechanics, $(\vec{p} \to -i\hbar\vec{\nabla})$ we obtain:

$$\frac{d\tau}{dt} = \frac{mc^2}{\sqrt{-c^2\hbar^2\Delta + m^2c^4}} \tag{6.3}$$

<u>This result is consistent with quantum mechanics but inconsistent with the many attempts[23] to treat proper-time as a parameter</u> (moving along the world line of the particle). In this particular case we may integrate (6.3) to obtain:

$$\tau(\vec{x}, t) = \int_0^t \left(\frac{d\tau}{dt}\right)dt = \frac{mc^2 t}{\sqrt{-c^2\hbar^2\Delta + m^2c^4}} \tag{6.4}$$

From Eq. (3.19) we see that $\tau(\vec{x},t)$ may be explicitly represented by:

$$\tau(\vec{x}, t)\ \psi(\vec{x}) = \frac{mc^2 t\omega}{(\hbar c)4\pi^2} \int_{\mathbb{R}^3} \frac{K_1[\omega\ ||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||}\ \psi(\vec{y})d\vec{y} \tag{6.5}$$

$$= \frac{\omega^2 t}{4\pi^2} \int_{\mathbb{R}^3} \frac{K_1[\omega\ ||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||}\ \psi(\vec{y})d\vec{y} \tag{6.6}$$

As an operator, we see that $\tau(\vec{x},t)$ is non-local, however, we know that $K_1[\omega|\vec{x}-\vec{y}|]$ is zero outside a few Compton wavelengths. From Eq. (6.6) we see that the appropriate definition of $d\tau(\vec{x},t)$ for the free particle case is:

$$d\tau(\vec{x}, t)\ \psi(\vec{x}, t) = \frac{dt\omega^2}{4\pi^2} \int_{\mathbb{R}^3} \frac{K_1[\omega\ ||\vec{x} - \vec{y}||]}{||\vec{x} - \vec{y}||}\ \psi(\vec{y}, t)d\vec{y} \tag{6.7}$$

Using Eq. (3.20) we may write (6.7) as

$$d\tau(\vec{x}, t)\ \psi(\vec{x}, t) = \frac{dt\omega^2}{2} \int_{\mathbb{R}^3} G_1[\omega\ ||\vec{x} - \vec{y}||]\ \psi(\vec{y}, t)d\vec{y} \tag{6.8}$$

It is clear that $d\tau/dt$ is a self-adjoint, bounded and invertible operator, its inverse $dt/d\tau$ is easily seen to be:

$$\left(\frac{dt}{d\tau}\right) \psi(\vec{x}, t) = \left(\frac{\omega}{\sqrt{-\Delta + \omega^2}}\right)^{-1} \psi(\vec{x}, t) = \frac{1}{\omega} L[\psi(\vec{x}, t)] \tag{6.9}$$

Using (3.35) and (3.36) we have:

$$\left(\frac{dt}{d\tau}\right) \psi(\vec{x}, t) = \frac{-\omega}{2} \int_{\mathbb{R}^3} G[\omega\ ||\vec{x} - \vec{y}||]\ \psi(\vec{y}, t)d\vec{y} \tag{6.10}$$

We are now in a position to show that it is possible to give a unambiguous particle interpretation for the Klein-Gordon equation, but first it is necessary to show that it is possible to give meaning to the symbol $\partial/\partial\tau$. Let us note that as an operator, the following expression is well defined:

$$\frac{dt}{d\tau}\frac{\partial\psi}{\partial t} = -\frac{\omega}{2} \int_{\mathbb{R}^3} G[\omega\ ||\vec{x} - \vec{y}||]\ \frac{\partial\psi(\vec{y}, t)}{\partial t}d\vec{y} \tag{6.11}$$

We therefore define $\partial/\partial\tau$ by:

$$\frac{\partial\psi(\vec{x}, t)}{\partial\tau} = \frac{dt}{d\tau}\frac{\partial\psi(\vec{x}, t)}{\partial t} \tag{6.12}$$

Let us now suppose that $\psi(\vec{x},t)$ is a solution to Eq. (1.3) so that:

$$i\hbar\frac{\partial\psi(\vec{x}, t)}{\partial t} = \left(\sqrt{-c^2\hbar^2\Delta + m^2c^4}\right) \psi(\vec{x}, t)$$

Operating on both sides with $dt/d\tau$, we have:

$$i\hbar\frac{\partial\psi(\vec{x}, t)}{\partial\tau} = \frac{(-c^2\hbar^2\Delta + m^2c^4)}{mc^2}\ \psi(\vec{x}, t) \tag{6.13}$$

If we now note that in this case $dt/d\tau$ and $\partial/\partial t$ commute so that from (6.12) we have:

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial}{\partial t}\left[\frac{dt}{d\tau}\psi\right] = \frac{\partial}{\partial t}\left[\frac{H}{mc^2}\psi\right] = \frac{i\hbar}{mc^2}\frac{\partial}{\partial t}\left[\frac{\partial\psi}{\partial t}\right] = \frac{i\hbar}{mc^2}\frac{\partial^2\psi}{\partial t^2} \tag{6.14}$$

combining with Eq. (6.13) we have:

$$\frac{(i\hbar)^2}{mc^2}\frac{\partial^2\psi}{\partial t^2} = \frac{(-c^2\hbar^2\Delta + m^2c^4)\psi}{mc^2} \tag{6.15}$$

This is clearly the Klein-Gordon equation; we thus see that in the free-particle case, we may view the Klein-Gordon equation as an alternate representation for (6.13). Let us rewrite (6.13) as:

$$i\hbar\frac{\partial\psi}{\partial\tau} = \frac{H^2}{mc^2}\psi \tag{6.16}$$

In this form, we see that if we define the proper energy $K = H^2/mc^2$, then K is never negative. If we replace H by $-H$, it is clear from (6.12) that the operator $dt/d\tau$ changes sign. We may

now correctly view antiparticles as particles moving backward in proper time. <u>Thus, in addition to a physical justification for the Klein-Gordon equation, we may now interpret it as a true, particle equation</u>. In order to see this, note that $\int_{R^3}\rho(\vec{x})d\vec{x}$ is a constant function in the space variable so that:

$$\frac{\partial}{\partial\tau}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} = \left(\frac{dt}{d\tau}\right)\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} \tag{6.17}$$

can be replaced by

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x}.$$

This is so because by (3.32) and (3.23) we have: ([a] is any constant) note $\Delta_y[a] = 0$)

$$\frac{dt}{d\tau}[a] = \frac{\omega^4}{2}\int_{\mathbb{R}^3}G_1[\omega\,||x - y||][a]d\vec{y} \tag{6.18}$$

We now use the fact that Lebesgue measure is translation invariant which allows us to replace $d\vec{y}$ by $d(\vec{x}-\vec{y})$ in (6.18). It is shown in Aronszajn and Smith that:

$$\int_{\mathbb{R}^3}G_1[z]dz = 1 \tag{6.19}$$

so that (6.18) becomes:

$$\frac{dt}{d\tau}[a] = \frac{\omega^3}{2}[a] \tag{6.20}$$

This means that Eq. (6.17) becomes:

$$\frac{\partial}{\partial \tau}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} = \frac{\omega^3}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} \tag{6.21}$$

Thus, we may interpret (6.16) as a true particle equation with $\rho(\vec{x})$ and $\vec{J}$ as defined in (4.17). If we want the anti-particle evolution, we replace $\partial/\partial\tau$ by $-\partial/\partial\tau$ in (6.16).

The question of interaction requires additional study and will be the subject of future work. The question of generality, at least in the free particle case, can be dispensed with in the affirmative. In order to see this, let us note that we may rewrite Eq. (6.3) as:

$$\frac{d\tau}{dt} = \frac{mc^2}{H[\vec{P}, mc^2]} \tag{6.22}$$

Where $H[\vec{P},mc^2]$ is now used to denote the relativistic Hamiltonian for a free particle of arbitrary spin. The explicit representation for any integer spin is easy to compute using (6.5). The explicit representation for half-integer spin can be computed and is much easier.

We pause to compute $dt/d\tau$ when H is the free particle Dirac Hamiltonian. We do not need it for the purposes of this paper; we present it for completeness. In this case (6.22) becomes:

$$\frac{d\tau}{dt} = \frac{mc^2}{c\vec{\alpha}\cdot\vec{P} + mc^2\beta} \tag{6.23}$$

If we multiply by

$$\frac{c\vec{\alpha}\cdot\vec{P} + mc^2\beta}{c\vec{\alpha}\cdot\vec{P} + mc^2\beta}$$

we get:

$$\frac{d\tau}{dt} = \frac{mc^2[c\vec{\alpha}\cdot\vec{P} + mc^2\beta]}{c^2\vec{P}^2 + m^2c^4} \tag{6.24}$$

Let us now note that:

$$\frac{1}{c^2\vec{P}^2 + m^2c^4} = \frac{1}{\hbar^2c^2[-\Delta + \omega^2]}$$

From Eqs. (2.3) and (3.11), we see that:

$$\left(\frac{1}{-\Delta + \omega^2}\right)u(\vec{x}) = R(\omega^2, \Delta)u(\vec{x}) = \int_{\mathbb{R}^3}\exp\{-\omega\,||\vec{x}-\vec{y}||\}\,\frac{u(\vec{y})d\vec{y}}{4\pi\,||\vec{x}-\vec{y}||} \tag{6.26}$$

Using this in (6.24) and simplifying, we have:

$$\left(\frac{d\tau}{dt}\right)\psi(\vec{x}) = \frac{\omega}{4\pi}\int_{\mathbb{R}^3}(-i\vec{\alpha}\cdot\vec{\nabla}_y + \omega\vec{\beta})\left[\frac{\exp(-\omega\,||\vec{x}-\vec{y}||)}{||\vec{x}-\vec{y}||}\right]\psi(\vec{y})d\vec{y} \tag{6.27}$$

It should be noted that the operator in Eq. (6.27) only acts on the enclosed expression and not on $\psi(\vec{y})$. If we integrate by parts, it is easy to check that (6.27) can also be written as:

$$\left(\frac{d\tau}{dt}\right)\psi(\vec{x}) = \frac{\omega}{4\pi}\int_{\mathbb{R}^3}\frac{\exp(-\omega\,||\vec{x}-\vec{y}||)}{||\vec{x}-\vec{y}||}(-i\alpha\cdot\vec{\nabla}_y + \omega\vec{\beta})\psi(\vec{y})d\vec{y} \tag{6.28}$$

This also follows from the commutativity of the two operators that make up (6.24).

It is clear that $(d\tau/dt)$ is also an invertible operator with an unbounded inverse so that the class of functions on which the inverse acts must be restricted:

$$\left(\frac{dt}{d\tau}\right) = \frac{(c\vec{\alpha}\cdot\vec{p} + mc^2\vec{\beta})}{mc^2} \tag{6.29}$$

Returning to (6.22) in the general case, we see that:

$$\left(\frac{dt}{d\tau}\right) = \frac{H(P, mc^2)}{mc^2} \tag{6.30}$$

As in (6.12) we define

$$\frac{\partial\psi}{\partial\tau} = \frac{dt}{d\tau}\frac{\partial\psi}{\partial t} \tag{6.31}$$

and once again we get that:

$$i\hbar\frac{\partial\psi}{\partial\tau} = \frac{(-c^2\hbar^2\Delta + m^2c^4)\psi}{mc^2} \tag{6.32}$$

We now see that there is a distinct difference between the proper time operators in the integer and half-integer spin cases, and yet the form of the free particle equation is the same.

In the half-integer spin case, we once again may use the standard probability density function, in particular, for a Dirac particle, we have:

$$\frac{\partial}{\partial\tau}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} = \frac{c\vec{\alpha}\cdot\vec{p}}{mc^2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} + \vec{\beta}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} \tag{6.33}$$

As the above integral is constant in the space variable, the first term on the right of (6.33) is zero so that:

$$\frac{\partial}{\partial\tau}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} = \vec{\beta}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\rho(\vec{x})d\vec{x} \tag{6.34}$$

We thus see that we need not change the probability density function or the current density for a half-integer spin paricle. As in the integer spin case, we have a true particle equation, with no need to deal with the concept of negative energy and its interpretive difficulties.

We close in showing that our approach is no more than a strict adherence to standard canonical transformation theory. First let us consider the classical case; returning to Eqs. (6.1) and (6.2), we see that we may write:

$$d\tau = \frac{mc^2}{H}dt \tag{6.35}$$

Where $H = \sqrt{c^2\vec{p}^2+m^2c^4}$, starting with the standard Lagrangian:

$$Ldt = \vec{p}\cdot d\vec{x} - Hdt \tag{6.36}$$

We rewrite the left hand side using (6.35) as:

$$Ldt \equiv \vec{P}\cdot d\vec{x} - \frac{H^2}{mc^2}\left(\frac{mc^2}{H}dt\right) \equiv \vec{P}\cdot d\vec{x} - \frac{H^2}{mc^2}d\tau \tag{6.37}$$

That is: (we use $\equiv$ to indicate identity)

$$\left(\vec{P}\frac{d\vec{x}}{dt} - H\right)dt \equiv \left(\vec{P}\frac{d\vec{x}}{d\tau} - \frac{H^2}{mc^2}\right)d\tau$$

$$Ldt \equiv Ld\tau \tag{6.38}$$

We also have that:

$$Hdt \equiv \frac{H^2}{mc^2}d\tau \tag{6.39}$$

For a clear presentation of canonical transformation theory, we refer to Arnold[24] (page 241). See also Abraham and Marsden.[25] In particular we have that for a general change from observables $(H, \vec{p}, \vec{x}, t)$ to observables $(\hat{H}, \hat{\vec{P}}, \hat{\vec{X}}, T)$. Arnold shows that the new Lagrangian is related to the old one by:

$$\vec{p}\cdot d\vec{x} - Hdt = \hat{\vec{P}}\cdot d\hat{\vec{X}} - \hat{H}dT + dS \tag{6.40}$$

Where $S = S(\hat{\vec{P}}, \hat{\vec{X}}, T)$ is a function to be determined, and is sometimes called the generator of the canonical transformation of variables. In our case, we see that $S \equiv 0$, $\hat{\vec{p}} = \vec{P}$, $\hat{\vec{x}} = \vec{X}$, $\hat{H} = H^2/mc^2$ and $T = \tau$. We see from the above, that our theory belongs to a subgroup of the canonical transformation group, the so called contact group; that is, all canonical transformations such that:

$$\hat{\vec{P}}\cdot d\hat{\vec{X}} = \vec{P}\cdot d\vec{x} \tag{6.41}$$

(See Sudarshan and Mukunda[26], page 47). In our case this is an identity. It follows that H^2/mc^2 and τ are canonical variables, so that we may invoke quantization via the standard rules to obtain:

$$i\hbar\frac{\partial\psi}{\partial\tau} = \frac{H^2}{mc^2}\psi \tag{6.42}$$

This approach is quite general, however it gives no insight into the mathematical meaning that one may attribute to the symbol: $\partial/\partial\tau$; and that explains the more concrete approach we used to derive Eqs. (6.13) and (6.32). The approach via canonical quantization also does not tell us that the symbol $\partial/\partial\tau$ has a different meaning for integer versus half-integer spin particles.

We have given meaning to $\partial/\partial\tau$, however no meaning has been given for the symbol $d/d\tau$, our approach to a rigorous definition of $d/d\tau$ is related to attempts to account directly for extended particles and/or particles acting under forces that are not necessarily derivable from a potential.[27] Let A be any quantum mechanical operator, define $dA/d\tau$ by:

$$\frac{dA}{d\tau} = \frac{1}{i\hbar}(ATH - HTA) + \frac{\partial A}{\partial\tau} \tag{6.43}$$

Where $T = \frac{H}{mc^2}$ and $\frac{\partial A}{\partial\tau} = \frac{dt}{d\tau}\frac{\partial A}{\partial t}$, we may also write (6.43) as:

$$\frac{dA}{d\tau} = \frac{1}{i\hbar}[A, \frac{H^2}{mc^2}] + \frac{\partial A}{\partial\tau}$$

We choose to use the operator T in order to point out that our theory is a particular Lie-isotopic generalization of the Heisenberg equations which automatically preserves known results. (Cf Santilli[28]). To see the connection, write

$$[A, H]^* = ATH - HTA$$

That is, we may view our approach as inducing a new Heisenberg bracket via a change in the definition of multiplication in the underlying Lie Algebra; where:

$$A \cdot B = AB + ATB = A^*B$$

A,B in the algebra of observables. This implies a generalization of the Heisenberg uncertainty relations and other well-known results and principals of quantum mechanics. (i.e. $\Delta A\Delta B = \langle i\hbar[A,B]^*\rangle$ etc.) In closing, note that if A commutes with H, then $dA/d\tau = \partial A/\partial\tau$.

CONCLUSION

In this paper, we have attempted to connect a number of apparently unrelated issues in the foundations of relativistic quantum theory. Many of the topics and approaches touched upon could themselves be the subject of in-depth study, and the study of each will a part of the overall long-range program; however, there are a number of important short-range issues that must be addressed.

From a physical point of view, the first question is interaction:

1) How do we define the square root operator in case we introduce minimal coupling.
2) How do we compute the square root operator with interaction.
3) Does the probability density remain positive definite for the square-root operator Eq. (1.3) during interaction.
4) Does the generalized Eq. (6.32) retain a positive definite probability density during interaction.
5) What is the correct definition of the proper time during interaction.
6) How does one approach the theory when there is more than one particle.

Questions 1) and 2) are mathematical questions of fundamental importance from a physical point of view; questions 3) through

6) are completely physical in nature and must be answered before any claims for a true generalization of relativistic quantum theory can be made.

Let us note that the question of invariance under group transformations has not been included because a little reflection reveals that except for translation, gauge group and O(3) invariance, apparently no other groups appear. In particular, the question of Lorentz invariance does not seem to enter into our approach. Since the proper frame is unique, Lorentz invariance becomes an improper question. This leads to a question of uniqueness which requires a detailed classical analysis. If (for whatever reason) one insists on Lorentz invariance, then it becomes necessary to define transformations on an operator valued metric.

There is one question of a mathematical nature that should be discussed, namely, how does one analyze equations with unbounded operator valued coefficients. That is, equations (abstract) of the form

$$A(t)\frac{du(t)}{dt} + B(t)U(t) = f(t) \tag{a}$$

Where $A(t)$ and $B(t)$ are operators (unbounded in general), $f(t)$ is some given function (abstract) and $U(t)$ is to be found subject to the appropriate initial data and boundary conditions. If we let $A(t) = i\hbar\, dt/d\tau$, $B(t) = -H^2/mc^2$ and $f(t)$

= 0, we obtain (6.32). We are in luck since at least as early as 1957, the more general problem:

$$A(t)\frac{d^2U(t)}{dt^2} + B(t)\frac{dU(t)}{dt} + C(t)U(t) = f(t) \qquad \text{(b)}$$

has been the subject of mathematical inquiry. Questions of existence of solutions, uniqueness, continuous dependence on initial data, and approximation procedures have all been studied within the Hilbert space setting. Those with interest in this question are referred to the report by Lions.[29]

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APPENDIX

BESSEL FUNCTIONS AND POTENTIALS

In this appendix we present some well-known as well as some not so well-known results about Bessel functions. We use these results to prove some statements and derive some formulas used in Sections III and IV.

Let $K_\beta[z]$ be the modified Bessel function of the third kind of order β. The following results are derived in Donoghue:[13]

<u>Theorem 1</u>

1) If $0<\beta<3$ then

a) $$K_\beta[z] = \frac{[1 + \theta(z)]}{z^{3-\beta}} \qquad z \to 0 \qquad \text{(A1)}$$

b) $$K_0[z] = [1 - \theta(z)] \log \frac{1}{z}, \quad z \to 0 \qquad \text{(A2)}$$

$$\theta(z) \to 0 \quad \text{as} \quad z \to 0$$

2) If $z \to \infty$ then

$$K_\beta[z] = \frac{e^{-z}}{z^{\frac{\beta-4}{2}}}(1 + \theta(z))$$

$$\theta(z) \to 0, \qquad z \to \infty$$

Theorem 2

1) $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [K_\beta[z]z^{-\beta}] = (-1)^m K_{\beta+m}[z]z^{-\beta-m} \qquad (A4)$$

2) $K_{-\beta}[z] = K_\beta[z]$ and $K_\beta[z]$ is an entire analytic function everywhere except at $z=0$.

Theorem 3

1) $$K_{\beta-1}[z] - K_{\beta+1}[z] = \frac{-2\beta}{z}(K_\beta[z]) \qquad (A5)$$

2) $$K_{\beta-1}[z] + K_{\beta+1}[z] = \frac{-2d}{dz} K_\beta[z] \qquad (A6)$$

3) $$K_\beta[z] = \frac{\sqrt{\pi}}{\Gamma(\beta + 1/2)}\left(\frac{z}{2}\right)^\beta \int_0^\infty \exp(-z \cosh t)\ (\sinh t)^{2\beta} dt, \quad Re(z) > 0 \qquad (A7)$$

The proof of theorem 2 is in Aronazajn and Smith,[12] and the proof of theorem 3 may be found in Magnus and Oberhettinger.[29]

Using (A7) it is easy to see that

$$K_{1/2}[z] = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}}{\sqrt{z}} \qquad (A8)$$

so that:

$$\sqrt{\frac{2}{\pi}}\,\frac{K_{1/2}[z]}{z^{1/2}} = \frac{e^{-z}}{z}$$

From (A1), (A2) we see that:

$$\lim_{z\to 0} z^2 K_1[z] = \lim_{z\to 0}[1 - \theta(z)] = 1 \qquad (A9)$$

$$\lim_{z\to 0} z\, K_0[z] = 0 \qquad (A10)$$

We now follow Aronszajn and Smith[12] in defining a Bessel potential $G_\beta[z]$ of order β by:

$$G_\beta[z] = \frac{1}{2^{\frac{1+\beta}{2}}\, \Pi^{3/2}\ \Gamma(\beta/2)}\ \frac{K_{\frac{3-\beta}{2}}[z]}{z^{\frac{3-\beta}{2}}} \qquad (A11)$$

Theorem 4

1) $$\int_{\mathbb{R}^3} G_\beta(z)dz = (2\pi)^{3/2}\ \hat{G}\ (0) = 1 \qquad (A12)$$

where $\hat{G}_\beta$ means Fourier transform.

2) $$\hat{G}\ (z) = (2\pi)^{-3/2}\ (1 + |z|^2)^{-\beta/2} \qquad (A13)$$

Theorem 5

1) $$G_\alpha[z] * G_\beta[z] = G_{\alpha+\beta}[z] \qquad (A14)$$

Where * denotes convolution.

2) $(I - \Delta)\, G_\beta[z] = G_{\beta-2}[z]$ (A15)

Proofs of theorems 4 and 5 may be found in Aronszajn and Smith,[12] (A15) shows that $G_2[z]$ satisfies: $(I-\Delta)\, G_2[z] = \delta(z)$, it is easy to see that $G_2[z]$ is the scaled Yukawa potential (A8) (m=1). It is easy to show using (A9) and (A10) that:

$$\left.\begin{aligned} &\lim_{\rho\to 0} \rho^2 G_1[\omega\rho] = \frac{1}{\omega^2} \\ &\lim_{\rho\to 0} \rho G_3[\omega\rho] = 0 \end{aligned}\right\} \tag{A16}$$

From (A6) we have:

$$\frac{d}{dr}(G_1(r)) = -\left[\frac{G_3(r)}{r} + \frac{2G_1(r)}{r}\right] \tag{A17}$$

and from (A5) we see that:

$$G_{-1}[z] = \frac{1}{\pi^{3/2}\,\Gamma(-1/2)}\,\frac{K_2[z]}{z^2} = -\left(\frac{G_3[z] + 2G_1(z)}{z^2}\right) \tag{A18}$$

Where we have used (A5) to expand $K_2[z]$, the negative sign appears because of $\Gamma(-1/2)$.

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