Lecture on Black Holes

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1 Basics of Black Holes

§1.1 Concept of Infinity

1.1.1 Conformal Embedding: example

The Minkowski Spacetime with metric

$$g(E^{n,1}) = -dt^2 + dr^2 + r^2 g(S^{n-1}) \tag{1.1.1}$$

can be embedded into the static Einstein universe

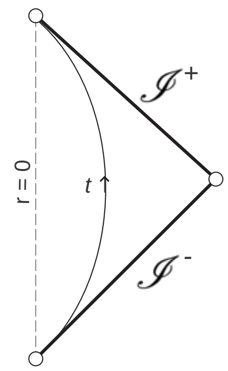
$$\Omega^2 g(E^{n,1}) = -d\eta^2 + d\chi^2 + \sin^2\chi\, g(S^{n-1}) = -d\eta^2 + g(S^n), \tag{1.1.2}$$

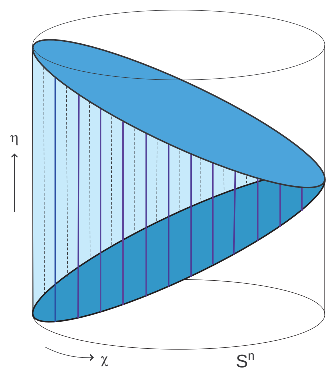
$$\Omega = \cos\frac{\eta+\chi}{2}\cos\frac{\eta-\chi}{2} \tag{1.1.3}$$

by the transformation

$$t - r = 2\tan\frac{\eta-\chi}{2}, \tag{1.1.4a}$$

$$t + r = 2\tan\frac{\eta+\chi}{2}. \tag{1.1.4b}$$





The image is

$$|\eta| < \pi - \chi, \quad 0 \le \chi < \pi. \tag{1.1.5}$$

The boundary of this image in SEU is given by

$$\partial M = \mathscr{I}^+ \cup \mathscr{I}^-, \quad \mathscr{I}^\pm \simeq \mathbb{R} \times S^{n-1} \tag{1.1.6}$$

1.1.2 Conformal Infinity

By generalising the previous example, Penrose proposed the following definition of the infinity boundary $\mathscr{I}$ of a spacetime $\mathscr{M}$ in terms of a conformal mapping $f : \mathscr{M} \to \hat{\mathscr{M}}$ [Penrose R 1963[Pen63]]:

1. $\mathscr{I} = \partial\overline{f(\mathscr{M})} \subset \hat{\mathscr{M}}$: smooth
2. $\hat{g} = \Omega^2 f_* g$: $\exists \Omega : \hat{\mathscr{M}} \to \mathbb{R}$
3. $\Omega|_{\mathscr{I}} = 0, d\Omega|_{\mathscr{I}} \neq 0$

For the Weyl transformation

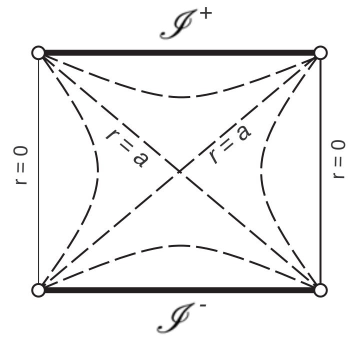
$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}; \quad \Omega = e^{\Phi}, \tag{1.1.7}$$

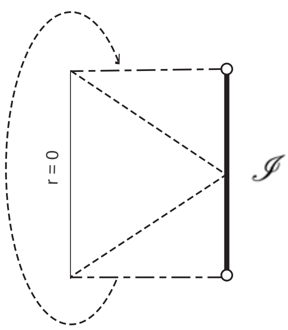
the Christofell symbol and the curvature tensor of an $(n+1)$-dimensional Riemannian space(time) transform as

$$\begin{aligned}
\Gamma^\mu_{\nu\lambda} &= \hat{\Gamma}^\mu_{\nu\lambda} - \hat{\nabla}_\nu \Phi \delta^\mu_\lambda - \hat{\nabla}_\lambda \Phi \delta^\mu_\nu + \hat{\nabla}^\mu \Phi \hat{g}_{\nu\lambda}, && (1.1.8)\\
\Omega^2 R^\mu{}_{\nu\lambda\sigma} &= \Omega^2 \hat{R}^\mu{}_{\nu\lambda\sigma} - 2\Omega \delta^\mu_{[\sigma} \hat{\nabla}_{\lambda]} \hat{\nabla}_\nu \Omega + 2\Omega \hat{g}_{\nu[\sigma} \hat{\nabla}_{\lambda]} \hat{\nabla}^\mu \Omega \\
&\quad -2(\hat{\nabla}\Omega)^2 \delta^\mu_{[\lambda} \hat{g}_{\sigma]\nu}, && (1.1.9)\\
\Omega^2 R_{\mu\nu} &= \Omega^2 \hat{R}_{\mu\nu} + (n-1)\Omega \hat{\nabla}_\mu \hat{\nabla}_\nu \Omega + \hat{g}_{\mu\nu} \left\{ \Omega \hat{\nabla}^2 \Omega - n(\hat{\nabla}\Omega)^2 \right\}, && (1.1.10)\\
R &= \Omega^2 \hat{R} + 2n\Omega \hat{\nabla}^2 \Omega - n(n+1)(\hat{\nabla}\Omega)^2. && (1.1.11)
\end{aligned}$$

From this, it follows that for vacuum spacetime satisfying

$$R_{\mu\nu} = \frac{2\Lambda}{n-1} g_{\mu\nu} + \kappa^2 \left(T_{\mu\nu} - \frac{T}{n-1} g_{\mu\nu} \right), \tag{1.1.12}$$





or with the energy-momentun tensor decreasing as $O(\Omega)$ at infinity, $\hat{\nabla}\Omega$ has to satisfy the condition

$$(\hat{\nabla}\Omega)^2 = -\frac{2\Lambda}{n(n-1)}. \tag{1.1.13}$$

This implies that

$$\begin{aligned} \Lambda = 0 \Rightarrow &\quad \mathscr{I}\text{: Null} \\ \Lambda > 0 \Rightarrow &\quad \mathscr{I}\text{: Spacelike} \\ \Lambda < 0 \Rightarrow &\quad \mathscr{I}\text{: Timelike} \end{aligned}$$

【**Definition 1.1.1** (WAS spacetime)】 In general, when a spacetime $\mathscr{M}$ has a neighborhood of infinity that is isomorphic to a neighborhood of infinity of either $E^{n,1}$, dS^{n+1} or adS^{n+1}, $\mathscr{M}$ is called to be weakly asymptotically simple. □

§1.2 Definition

1.2.1 Causal sets

【**Definition 1.2.1** (Chronological past/future)】 For a set $\mathscr{S}$ in a spacetime $\mathscr{M}$, its chronological past $I^-(\mathscr{S},\mathscr{M})$ (future $I^+(\mathscr{S},\mathscr{M})$) is defined as the set of all points that can be connected to a point in $\mathscr{S}$ by a future (past)-directed timelike curve of non-zero length in $\mathscr{M}$. □

【**Definition 1.2.2** (Causal past/future)】 For a set $\mathscr{S}$ in a spacetime $\mathscr{M}$, its causal past $J^-(\mathscr{S},\mathscr{M})$ (future $J^+(\mathscr{S},\mathscr{M})$) is defined as the set of all points that can be connected to a point in $\mathscr{S}$ by a future (past)-directed causal curve in $\mathscr{M}$. □

【**Proposition 1.2.3**】 The boundary of $J^\pm(\mathscr{S},\mathscr{M})$ is a union of a null hypersurface $\mathscr{N}^\pm$ and an acausal subset $\mathscr{S}_0 = J^+(\mathscr{S},\mathscr{M}) \cap J^-(\mathscr{S},\mathscr{M})$. Each null geodesic generator of $\mathscr{N}^+$ ($\mathscr{N}^-$) has no past (future) end point except in $\mathscr{S}$. □

Proof. Let us consider the case of $J^-(\mathscr{S},\mathscr{M})$. Let p be a point in $\mathscr{B} = \partial J^-(\mathscr{S},\mathscr{M})$. Then, if p is not a point in S_0, there exists a sequence of points q_j converging to p and future-directed causal curves γ_j connecting q_j to a point in S_0, A subset of γ_j converges to a null geodesic passing through p in $\mathscr{B}$. □

【**Definition 1.2.4** (Cauchy development)】 For an acausal hypersurface $\mathscr{S}$ in $\mathscr{M}$,

- Future (past) Cauchy development (or domain of dependence) $D^+(\mathscr{S})$ ($D^-(\mathscr{S})$)= the set of all points p such that any past-directed (future-directed) non-extendible curve passing through p intersects §.
- Cauchy development (or domain of dependence) $D^(\mathscr{S}) = D^+(\mathscr{S}) \cup D^-(\mathscr{S})$.
- Cauchy horizon $H^\pm(\mathscr{S}) = \overline{D^\pm(\mathscr{S})} - I^\mp(D^\pm(\mathscr{S}))$.

□

1.2.2 Causality conditions

【**Definition 1.2.5** (Various causality conditions)】

Chornonology condition ⇔ There exists no closed timelike curve.

Causality condition ⇔ There exists no closed causal curve.

Strong causality condition at p ⇔ Every neighborhood of p contain a neighborhood of p which no causal curve intersects more than once.

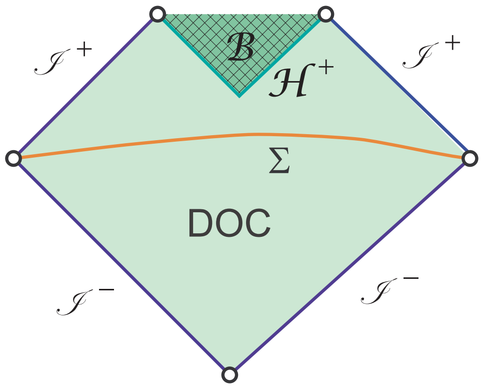
Stable causality condition ⇔ Chronogy condition holds for any metric in an open neighborhood of the metric in the space of metric ⇔ There is a global time function t whose gradient is everywhere timelike.

□

【**Definition 1.2.6** (global hyperbolicity (Leray 1952))】 A set $\mathscr{N}$ in $\mathscr{M}$ is said to be *globally hyperbolic* if the strong causality condition holds on $\mathscr{N}$ and if for any two points $p, q \in \mathscr{N}$, $J^+(p) \cap J^-(q)$ is compact and contained in $\mathscr{N}$. □

【**Theorem 1.2.7** (HE1973)】 If $\mathscr{S}$ is a closed achronal set, then $\mathrm{int}(D(\mathscr{S})) := D(\mathscr{S}) - \dot{D}(\mathscr{S})$ is globally hyperbolic. □

【**Theorem 1.2.8** (Geroch 1970)】 If an open set $\mathscr{N}$ is globally hyperbolic, then $\mathscr{N} \approx \mathbb{R} \times \mathscr{S}$ with $\mathscr{S}$ a spacelike manifold, and for each $t \in \mathbb{R}$, $\{t\} \times \mathscr{S}$ is a Cauchy surface for $\mathscr{N}$. In particular, $\mathscr{N}$ is stably causal. □



1.2.3 Horizon

Let $\mathscr{M}$ be an AS spacetime and $\mathscr{I}$ be its conformal infinity.

- **Asymoptotically predictable**

$$\mathscr{I} \subset \overline{D(\Sigma)} \quad \text{in } \hat{\mathscr{M}} \tag{1.2.1}$$

- **Horizon**

$$H^+ = \partial(J^-(\mathscr{I})) \cap J^+(\mathscr{I}) \tag{1.2.2}$$

- **Black hole region**

$$\mathscr{B} = \overline{\mathscr{M} - J^-(\mathscr{I})} \tag{1.2.3}$$

- **DOC** (Domain of outer communiation)

$$\mathrm{DOC} = J^-(\mathscr{I}, \mathscr{M}) \cap J^+(\mathscr{I}, \mathscr{M}) \tag{1.2.4}$$

§1.3
Raychaudhuri equation

1.3.1 Jacobi equation

For a congruence of curves $\Gamma : x^\mu = x^\mu(\tau, z)$, let $u^\mu = \dot{x}^\mu$ be its tangent vector field and $Z^\mu = \delta z^i \partial x^\mu / \partial z^i$ be the relative dispacement of neighboring curves of a reference curve. Then, from

$$\nabla_u Z = \nabla_Z u \tag{1.3.1}$$

it follows that

$$\nabla_u^2 Z = \nabla_u \nabla_Z u = R(u, Z)u + \nabla_Z \nabla_u u. \tag{1.3.2}$$

Hence, we have

$$\nabla_u^2 Z = R(u, Z)u + \nabla_Z A \tag{1.3.3}$$

where

$$\nabla_u u = A. \tag{1.3.4}$$

In particular, when Γ is a geodesic congruence, i.e., $A = 0$, this equation is called the Jacobi equation.

1.3.2 Representation in terms of the Fermi basis

When u^μ is a timelike vector field (a velocity field of particles),

$$u = \partial_\tau; \; u \cdot u = -1. \tag{1.3.5}$$

let us take an orthonormal basis E_a satisfying the condition

$$E_0 = u, \; u \cdot E_I = 0, \; \dot{E}_I \equiv \nabla_u E_I = A_I u. \tag{1.3.6}$$

Then, because A_I can be expressed as

$$A_I = -u \cdot \dot{E}_I = \dot{u} \cdot E_I = A \cdot E_I, \tag{1.3.7}$$

E_a is uniquely determined if one specifies its value at a point on the flow line.

Let us express the relative position vector Z between two fluid line in terms of a Fermi basis as

$$Z = Z^0 u + Z^I E_I. \tag{1.3.8}$$

Then, we have

$$\begin{aligned}\dot{Z}^I &= \dot{E}_I \cdot Z + E_I \cdot \nabla_u Z = -A_I Z^0 + E_I \cdot \nabla_Z u \\ &= -A_I Z^0 + Z^0 E_I \cdot \dot{u} + E_I \cdot \nabla_{E_J} u Z^J, \end{aligned} \tag{1.3.9}$$

hence,

$$\dot{Z}^I = M_{IJ} Z^J; \quad M_{IJ} = E_I \cdot \nabla_{E_J} u. \tag{1.3.10}$$

Further,

$$\begin{aligned}\dot{Z}^0 &= -\dot{u} \cdot Z - u \cdot \nabla_u Z = -\dot{u} \cdot E_I Z^I - u \cdot \nabla_Z u \\ &= -A_I Z^I. \end{aligned} \tag{1.3.11}$$

These can be combined into the single expression

$$\dot{Z} = Z^0 \dot{u} + M_{IJ} Z^J E^I. \tag{1.3.12}$$

1.3.3 Expansion, shear, rotation

Let us define a symmetric tensor θ_{IJ} and an anti-symmetric tensor ω_{IJ} by

$$\theta_{IJ} = \sigma_{IJ} + \frac{1}{d}\delta_{IJ}\theta := M_{(IJ)}; \ \sigma_{II} = 0, \tag{1.3.13}$$

$$\omega_{IJ} := M_{[IJ]}. \tag{1.3.14}$$

Then, from

$$\begin{aligned}M_{IJ} E^I_\mu E^J_\nu &= (E^I_\mu E^\lambda_I)(E^J_\nu E^\sigma_J)\nabla_\sigma u_\lambda = (\delta^\lambda_\mu + u_\mu u^\lambda)(\delta^\sigma_\nu + u_\nu u^\sigma)\nabla_\sigma u_\lambda \\ &= \nabla_\nu u_\mu + u_\nu \dot{u}_\mu \end{aligned} \tag{1.3.15}$$

it follows that

$$\nabla_\nu u_\mu = \theta_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_\nu; \ \theta_{\mu\nu} := E^I_\mu E^J_\nu \theta_{IJ}, \ \omega_{\mu\nu} := E^I_\mu E^J_\nu \omega_{IJ}. \tag{1.3.16}$$

1.3.4 Geodesic deviation equations

From $[u, Z] = 0$, it follows that

$$\ddot{Z} = \nabla_u \nabla_Z u = R(u, Z)u + \nabla_Z \dot{u}. \tag{1.3.17}$$

Here, by differentiating the expression for $\dot{Z}$ by τ, we obtain

$$\begin{aligned}\ddot{Z} - \nabla_Z \dot{u} = \ & -A_I Z^I \dot{u} - Z^I \nabla_{E_I} \dot{u} + A_I M_{IJ} Z^J u \\ & +(\dot{M} + M^2)_{IJ} Z^J E_I. \end{aligned} \tag{1.3.18}$$

Hence, we have

$$(\dot{M} + M^2)_{IJ} = -R_{I\mu J\nu} u^\mu u^\nu + A_I A_J + E_I \cdot \nabla_{E_J} \dot{u}. \tag{1.3.19}$$

This matrix equation is equivalent to

$$\dot{\theta} + \frac{1}{d}\theta^2 = -2\sigma^2 + 2\omega^2 - R_{\mu\nu}u^\mu u^\nu + \nabla_\mu \dot{u}^\mu, \tag{1.3.20a}$$

$$\begin{aligned}\dot{\sigma}_{IJ} + \frac{2}{d}\theta\sigma_{IJ} &= -\sigma_{IK}\sigma_J^K + \frac{2}{d}\sigma^2\delta_{IJ} + \omega_{IK}\omega_J{}^K - \frac{2}{d}\omega^2\delta_{IJ} \\ &\quad -R_{IuJu} + \frac{1}{d}R_{uu}\delta_{IJ} + A_I A_J + E_{(I}\cdot\nabla_{E_{J)}}\dot{u} - \frac{1}{d}\nabla\cdot\dot{u}\delta_{IJ},\end{aligned} \tag{1.3.20b}$$

$$\dot{\omega}_{IJ} + \frac{2}{d}\theta\omega_{IJ} + \sigma_I^K\omega_{KJ} - \sigma_J^K\omega_{KI} = E_{[I}\cdot\nabla_{E_{J]}}\dot{u}, \tag{1.3.20c}$$

where

$$2\sigma^2 := \sigma_{IJ}\sigma_{IJ}, \quad 2\omega^2 := \omega_{IJ}\omega_{IJ}. \tag{1.3.21}$$

In particular, when $\dot{u} \equiv 0$, the equation for $\dot{\theta}$ is called the *Raychaudhuri equation.*

Similarly, in the case of a null geodesic congruence, the same set of equations with d replaced by $d-1$ and the range of I restricted to $1, \cdots, d-1$ hold by considering a parallel null basis instead of the orthonormal Fermi basis.

1.3.5 Conjugate point

【Definition 1.3.1】 Two points on a geodesic is called conjugate if there exists a Jacobi field that vanishes at these points. Similarly, a point p is said conjugate to a surface $\mathscr{S}$ along a geodesic γ that passes through p and crosses $\mathscr{S}$ orthogonally at q, if there is a Jacobi field that vanishes at p and is tangential to $\mathscr{S}$ at q. □

【Proposition 1.3.2】 If p and q ($\mathscr{S}$) are conjugate along a null geodesic γ, any point on the extension of γ past p can be connected to q ($\mathscr{S}$) bu a timelike curve. □

§1.4
Area Theorem

【Definition 1.4.1 (Strong energy condition)**】** The Ricci curvature satisfy the condition $R_{ab}V^aV^b$ for any timeline (null) vector V, it is said that the spacetime satisfies the timelike (null) strong energy condition or convergence condition. □

【Proposition 1.4.2 ($\dot{\theta} \geq 0$)**】** If the horizon $\mathscr{H}^+$ is future complete and the null strong energy condition is satisfied, the expansion rate θ of the null geodesic generators of $\mathscr{H}^+$ is non-negative. □

Proof. Let u be a non-degenerate function that is defined around $\mathscr{H}^+$ and constant on $\mathscr{H}^+$. Then, the tangent vector k of each null geodesic generator of $\mathscr{H}^+$ is orthogonal to the normal vector of $\mathscr{H}^+$ that is propotional to ∇u: $k = f\nabla u$. From this we obtain $dk_* = df \wedge du \propto df \wedge k_*$, hence $\omega = 0$. Therefore, the Raychaudhuri equation can be written

$$\dot{\theta} + \frac{1}{d-1}\theta^2 = -2\sigma^2 - R_{\mu\nu}k^\mu k^\nu \leq 0. \tag{1.4.1}$$

If θ is negative at some $v = v_0$, it behaves as

$$\theta(v) \leq -\frac{|\theta_0|}{1 - |\theta_0|\frac{v-v_0}{d-1}}. \tag{1.4.2}$$

Hence, θ diverges at $v = v_1$, a finite affine distance from $v = v_0$. The point $v = v_1$ is conjugate to a spacelike 2-surface on the horizon. Hence, the null geodesic generator passing through this point has to go outside $J^-(\mathscr{I})$ beyond this point. This contradict the fact that each null geodesic generator of $\mathscr{H}^+$ has no future end point. □

【**Theorem 1.4.3** (Area Increase Theorem)】 If the horizon is null geodesically future complete and if the null strong energy condition is satisfied, the horizon area does not decrease. □

§1.5
Symmetry

1.5.1 Killing vector

Let $\Phi_t : \mathscr{M} \to \mathscr{M}$ be a one-parameter family of isometries:

$$\Phi_t^* g = g, \tag{1.5.1a}$$

$$\Phi_t \circ \Phi_s = \Phi_{t+s}, \quad \Phi_0 = \mathrm{id}_{\mathscr{M}}. \tag{1.5.1b}$$

Then, its infinitesimal transformation X is defined as the vector field that is tangent to the curve $\Phi_t(p)$ at each point p:

$$X_p = \left.\frac{d\Phi_t(p)}{dt}\right|_{t=0}. \tag{1.5.2}$$

This infinitesimal transformation X is called a Killing vector and satisfies

$$\pounds_X g = \lim_{t\to 0}\frac{\Phi_t^* g - g}{t} = 0. \tag{1.5.3}$$

From

$$\begin{aligned}(\pounds_X g)(Y,Z) &= \pounds_X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= \nabla_X(g(Y,z)) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \end{aligned} \tag{1.5.4}$$

this equation can be written

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0. \tag{1.5.5}$$

Conversely, any vector field satisfying this Killing equation generate a local one parameter family of isometries uniquely.

【Formula 1.5.1】

$$\nabla \cdot X = 0, \tag{1.5.6a}$$

$$\triangle X_a + R_a^b X_b = 0, \tag{1.5.6b}$$

$$\nabla_a \nabla_b X_c = -R_{bca}{}^d X_d. \tag{1.5.6c}$$

□

Proof. Summing

$$\begin{aligned}&\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c = R_{abc}{}^d X_d, \\ &-\nabla_b \nabla_c X_a + \nabla_c \nabla_b X_a = -R_{bca}{}^d X_d, \\ &\nabla_c \nabla_a X_b - \nabla_a \nabla_c X_b = R_{cab}{}^d X_d,\end{aligned}$$

we obtain

$$2\nabla_a \nabla_b X_c = -2R_{bca}{}^d X_d.$$

□

1.5.2 Stationary spacetime

【Definition 1.5.2 (Stationary spacetime)**】** A spacetime $\mathscr{M}$ is said to be stationary if there is a Killing vector ξ that is timelike in some region. □

The metric of a stationary spacetime can be written

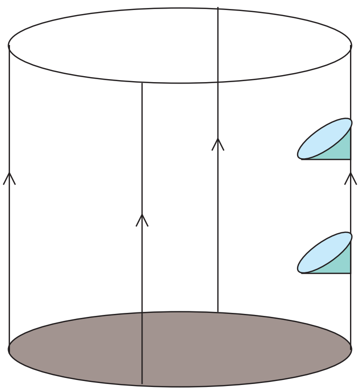
$$ds^2 = -e^{2U(x)}(dt + A(x))^2 + g_{ij}(x)dx^i dx^j, \tag{1.5.7}$$

where $x = (x^i)$ is the spatial coordinates. The Killing vector ξ can be written $\xi = \partial_t$ in this coordinate system, hence

$$\xi_* = -e^{2U}(dt + A(x)). \tag{1.5.8}$$

The rotation of the Killing vector is defined as

$$*(\xi_* \wedge d\xi_*) = -e^{3U} *_n dA. \tag{1.5.9}$$



【Definition 1.5.3 (Static spacetime)**】** A stationary spacetime $\mathscr{M}$ with the time translation Killing vector ξ is called static when the rotation of ξ vanishes. □

When a spacetime $(\mathscr{M}, g)$ is static, from the rotation free condition, we can find a coordinate system locally in which the metric can be written

$$ds^2 = -e^{2U(x)}dt^2 + g_{ij}(x)dx^i dx^j. \tag{1.5.10}$$

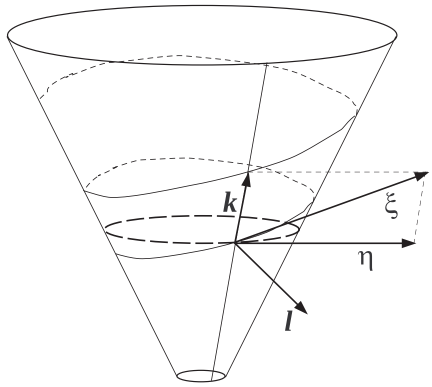
【Definition 1.5.4 (Axisymmetric spacetime)**】** A spacetime $\mathscr{M}$ is said to be axisymmetric if there is a Killing vector field η whose orbits are all closed. □

1.5.3 Killing horizon

【Definition 1.5.5 (Killing horizon)**】** A null hypersurface $\mathscr{H}$ in a stationary spacetime is called a Killing horizon when there is a Killing vector that is parallel to the null geodesic generators on $\mathscr{H}$. □

【Proposition 1.5.6 (Killing horizon of static BHs)**】** A horizon of an asymptotically simple and static spacetime with respect to infinity $\mathscr{I}$ is a Killing horizon if the spacetime is asymptotically predictable and the time translation Killing vector ξ is timelike in a neighborhood of $\mathscr{I}$. □

Proof. The Killing vector ξ is always tangent to the horizon $\mathscr{H}$. If ξ is spacelike, there must exist a hypersurface $\mathscr{S}$ outside $\mathscr{H}$ on which ξ is null and outside of which ξ is timelike. In this region where ξ is timelike, ξ can be written $\xi = -e^{2U}\nabla t$ with $g(\xi, \xi) = -e^{2U}$, where $t = $ const hypersurfaces Σ_t are spacelike. Hence, $\mathscr{S} = \lim_{t\to\infty} \Sigma_t$. Because ξ is orthgonal to Σ_t, it is also orthogonal to $\mathscr{S}$. However, because ξ is tangent to $\mathscr{S}$, this means that $\mathscr{S}$ is a null hypersurface, and ξ is parallel to its null geodesic generators. If $\mathscr{S}$ intersects with $\mathscr{I}$, the assumption on the timelikeness of ξ near $\mathscr{I}$ is violated. This implies that $\mathscr{S} = \mathscr{H}$. □



【**Definition 1.5.7** (rotating BH)】 The black hole of a stationary spacetime is said rotating if the time-translation Killing vector is spacelike on the horizon. □

【**Proposition 1.5.8** (Killing horizon of rotation BH)】 The rotating black hole horizon is a Killing horizon. □

Proof. From the rigidity theorem. □

【**Definition 1.5.9** (Surface gravity)】 For a stationary and axisymmetric spacetime with a Killing horizon $\mathscr{H}$, let ξ and η be the corresponding Killing vectors. Then, a tengent vector of the null generator of $\mathscr{H}$ can be uniquely written as

$$k = \xi + \Omega_h \eta. \tag{1.5.11}$$

Ω_h is called the angular velocity of the horizon. Further, on $\mathscr{H}$, we have

$$\nabla_k k = \kappa k \quad \Leftrightarrow \quad \nabla k^2 = -2\kappa k \tag{1.5.12}$$

The coefficiet κ is called the surface gravity of the black hole. □

【**Proposition 1.5.10** (shear and rotation of the Killing horizon)】 If the null energy condition is satisfied, the expansion, shear and rotation of the null geodesic generators k of the Killing horizon all vanish, and $R_{ab}k^a k^b = 0$. □

【**Theorem 1.5.11** (Zero-th law of BH thermodynamics (Bardeen, Carter, Hawking 1973)[BCH73])】 If the Einstion equations and the dominant energy condition hold, κ and Ω_h is constant on each connected component of the Killing horizon. □

Proof. Let us take a null basis (k, l, e_I) on the Killing horizon. Then, from the definition of κ, we have

$$\begin{aligned}\nabla_V \kappa &= -\nabla_V(g(l, \nabla_V k)) = -g(\nabla_V l,^n abla_k k) - g(l, \nabla_V \nabla_k k) \\ &= -\kappa g(k, \nabla_V l) - g(l, \nabla_{\nabla_V k} k) - g(l, (\nabla^2 k)(V, k)).\end{aligned} \tag{1.5.13}$$

Here, because the expansion, shear and rotation of the null tangent k of the Killing horizon $\mathscr{H}$ vanishes,

$$\nabla_V k \propto k, \quad \forall V_{//} \mathscr{H}. \tag{1.5.14}$$

From this, it follows

$$g(l, \nabla_{\nabla_V k} k) = g(l, \kappa \nabla_V k) = \kappa g(l, \nabla_V k). \tag{1.5.15}$$

Hence, the first two terms in the above expression cancel. Hence, from the identity $\nabla_a \nabla_b k_c = -R_{bca}{}^d k_d$, we have

$$\nabla_V \kappa = R(k, l, V, k). \tag{1.5.16}$$

Here, from

$$R(V, e_J, e_I, k) = g(e_I, \nabla_V \nabla_{e_J} k - \nabla_{e_J} \nabla_V k - \nabla_{[V, e_J]} k) = 0, \tag{1.5.17}$$

we obtain

$$R(k, l, V, k) = -R(k, V) + R(k, e_I, V, e_I) = -R(k, V) \tag{1.5.18}$$

If the dominant energy condition hold, $T(k, *)$ is a non-spacelike vector. However, because $T(k, k) = 0$, this vector must be proportional to k. Hence, $R(k, V) \propto g(k, V) = 0$. This proves the constancy of κ. □

§1.6 Examples

1.6.1 Static spherically symmetric black holes

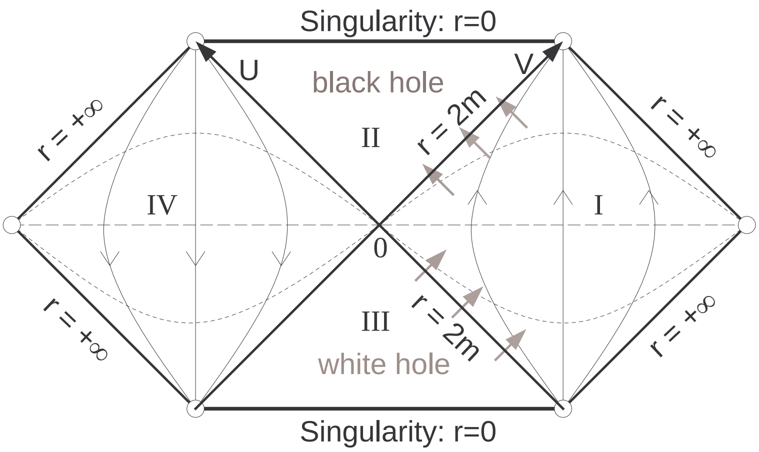
Metric Consider the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma_n^2, \tag{1.6.1}$$

$$f(r) = K - \left(\frac{r_0}{r}\right)^{n-1} - \lambda r^2 \tag{1.6.2}$$

where $d\sigma_n^2$ is a metric of a constant curvatur space with sectional curvature K, and

$$\lambda = \frac{2\Lambda}{n(n+1)}. \tag{1.6.3}$$



This is a static solution to the vacuum Einstein equation

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0 \quad \Leftrightarrow \quad R_{ab} = \frac{2\Lambda}{n}g_{ab}. \tag{1.6.4}$$

Horizon The Killing horizons of this spacetime are expressed as $r = r_h$ in terms of solutions to

$$f(r_h) = 0. \tag{1.6.5}$$

The Kretchmann invariant is given by

$$R^{abcd}R_{abcd} = 2(n+1)(n+2)\lambda^2 + n^2(n^2-1)\frac{r_0^{2(n-1)}}{r^{2(n+1)}}. \tag{1.6.6}$$

Thus, the spacetime is regular on Killing horizons.

The future Finkelstein coordinates (v, r, z) defined by

$$v = t + r_*; \quad dr_* = dr/f(r). \tag{1.6.7}$$

provide a regular chart around the future horizon:

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\sigma_n^2. \tag{1.6.8}$$

In this coordinate system, the killing vector $\xi = \partial_t$ can be written

$$\xi = \tilde{\partial}_v, \quad \partial_r = \frac{1}{f(r)}\tilde{\partial}_v + \tilde{\partial}_r, \tag{1.6.9}$$

From this, we have

$$\nabla(\xi^2) = \nabla f = f'\nabla r = ff'\partial_r = f'(f\tilde{\partial}_r + \tilde{\partial}_v). \tag{1.6.10}$$

Hence, the surface gravity is given by

$$\kappa = \frac{1}{2}f'(r_h) = \frac{n-1}{2r_0}. \tag{1.6.11}$$

Smarr formula From the correspndence with the Newtonian theory,

$$\triangle\phi = 4\pi G\rho, \quad g_{tt} \simeq -(1+2\phi), \tag{1.6.12}$$

the mass of the BH and r_0 is related by

$$8\pi GM = (n-1)r_0^{n-1}V(S^n). \tag{1.6.13}$$

Hence, we obtain the Smarr formula

$$GM = \frac{\kappa}{4\pi}A, \tag{1.6.14}$$

whose variation gives the zeroth law of thermodynamics

$$\delta M = \frac{\kappa}{8\pi G}\delta A. \tag{1.6.15}$$

1.6.2 Ernst formalism

Metric

$$ds^2 = -f(dt + Ad\phi)^2 + f^{-1}[\rho^2 d\phi^2 + e^{2k}(d\rho^2 + dz^2)] \tag{1.6.16}$$

Killing vectors are

$$\xi = \partial_t, \quad \eta = \partial_\phi. \tag{1.6.17}$$

The corresponding 1-forms are

$$\xi_* = -f(dt + Ad\phi), \quad \eta_* = f^{-1}\rho^2 d\phi + A\xi_*. \tag{1.6.18}$$

The orientation:

$$\epsilon_{t\phi\rho z} = \sqrt{-g} > 0. \tag{1.6.19}$$

EM field

$$\mathscr{F} = \sqrt{G}(F + i *F), \tag{1.6.20}$$

$$d\Phi = I_\xi \mathscr{F}, \tag{1.6.21}$$

$$\mathscr{F} = f^{-1}\left[d\Phi \wedge \xi_* + i *(d\Phi \wedge \xi_*)\right] \tag{1.6.22}$$

In the coordinates (ρ, z),

$$G^{-1/2}d\Phi = \left(F_{t\rho} - i\frac{f}{\rho}F_{\phi z}\right)d\rho + \left(F_{tz} + i\frac{f}{\rho}F_{\phi\rho}\right)dz \tag{1.6.23}$$

Ernst Potential : In terms of the rotation 1-form for the Killing ξ,

$$\omega = -I_\xi *d\xi_* = *(\xi_* \wedge d\xi_*), \tag{1.6.24}$$

the Ernst potential is defined as

$$d\mathscr{E} = df + i\omega - 2\bar{\Phi}d\Phi. \tag{1.6.25}$$

In the (ρ, z) coordinates, ω reads

$$\omega = \frac{f^2}{\rho}(-A_\rho dz + A_z d\rho). \tag{1.6.26}$$

In terms of the Ernst potential, the metric components are expressed as

$$f = \mathscr{E}_1 + |\Phi|^2, \tag{1.6.27a}$$
$$\partial_\zeta A = f^{-2}\rho\left[i\partial_\zeta\mathscr{E}_2 + \bar{\Phi}\partial_\zeta\Phi - \Phi\partial_\zeta\bar{\Phi}\right], \tag{1.6.27b}$$
$$\begin{aligned}\partial_\zeta k &= \frac{\rho}{2f^2}\left(\partial_\zeta\mathscr{E} + 2\bar{\Phi}\partial_\zeta\Phi\right)\left(\partial_\zeta\bar{\mathscr{E}} + 2\Phi\partial_\zeta\bar{\Phi}\right) - 2\frac{\rho}{f}\partial_\zeta\Phi\partial_\zeta\bar{\Phi}\\ &= \frac{\rho}{2f^2}(\partial_\zeta f)^2 - \frac{f^2}{2\rho}(\partial_\zeta A)^2 - 2\frac{\rho}{f}\partial_\zeta\Phi\partial_\zeta\bar{\Phi},\end{aligned} \tag{1.6.27c}$$

where

$$\mathscr{E} = \mathscr{E}_1 + i\mathscr{E}_2, \tag{1.6.28}$$
$$\partial_\zeta = \tfrac{1}{2}(\partial_\rho - i\partial_z). \tag{1.6.29}$$

Ernst equation : with $\partial = (\partial_\rho, \partial_z)$

$$f\rho^{-1}\partial\cdot(\rho\partial\mathscr{E}) = \partial\mathscr{E}\cdot(\partial\mathscr{E} + 2\bar{\Phi}\partial\Phi), \tag{1.6.30a}$$
$$f\rho^{-1}\partial\cdot(\rho\partial\Phi) = \partial\Phi\cdot(\partial\mathscr{E} + 2\bar{\Phi}\partial\Phi). \tag{1.6.30b}$$

1.6.3 Kerr-Newman Solution

Ernst potential

$$\mathscr{E} = \frac{(1-|\mu|^2)(px - iqy) - (1+|\mu|^2)}{(1-|\mu|^2)(px - iqy) + (1+|\mu|^2)}, \tag{1.6.31a}$$
$$\Phi = -\frac{2\mu}{(1-|\mu|^2)(px - iqy) + (1+|\mu|^2)}, \tag{1.6.31b}$$

where the (x, y) coordinates are related to (ρ, z) as

$$\rho^2 = \sigma^2(x^2-1)(1-y^2), \quad z = \sigma xy. \tag{1.6.32}$$

From this, we have

$$d\rho^2 + dz^2 = \sigma^2(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right). \tag{1.6.33}$$

Metric in the (x,y) coordinates

$$f = \frac{(1-|\mu|^2)^2(p^2x^2+q^2y^2-1)}{(1-|\mu|^2)^2(p^2x^2+q^2y^2)+2(1-|\mu|^4)px+(1+|\mu|^2)^2}, \tag{1.6.34a}$$

$$A = \frac{2\sigma q}{p}\frac{(1-y^2)\left[(1-|\mu|^4)px+1\right]}{(1-|\mu|^2)^2(p^2x^2+q^2y^2-1)}, \tag{1.6.34b}$$

$$e^{2k} = \frac{p^2x^2+q^2y^2-1}{p^2(x^2-y^2)}. \tag{1.6.34c}$$

Solution parameters

$$p = \frac{\sigma}{\sqrt{M^2-Q^2}}, \tag{1.6.35a}$$

$$q = \frac{a}{\sqrt{M^2-Q^2}}, \tag{1.6.35b}$$

$$\mu = -\frac{Q}{M+\sqrt{M^2-Q^2}}, \tag{1.6.35c}$$

$$\sigma = \sqrt{M^2-a^2-Q^2}. \tag{1.6.35d}$$

Representation in the (r,θ) coordinates

$$\sigma x = r - M, \quad y = \cos\theta. \tag{1.6.36}$$

$$\mathscr{E} = 1 - \frac{2M}{r-ia\cos\theta}, \quad \Phi = \frac{Q}{r-ia\cos\theta}. \tag{1.6.37}$$

$$ds^2 = -\frac{\Delta\rho^2}{\Gamma}dt^2 + \frac{\Gamma\sin^2\theta}{\rho^2}(d\phi-\Omega dt)^2 + \rho^2\left(\frac{dr^2}{\Delta}+d\theta^2\right), \tag{1.6.38}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \tag{1.6.39a}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{1.6.39b}$$

$$\Gamma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta, \tag{1.6.39c}$$

$$\Omega = \frac{a(2Mr-Q^2)}{\Gamma} \tag{1.6.39d}$$

Future Finkelstein coordinates

$$dv = dt + \frac{r^2+a^2}{\Delta}dr, \tag{1.6.40a}$$

$$d\phi_+ = d\phi + \frac{a}{\Delta}dr. \tag{1.6.40b}$$

If we introduce the coordinate basis in the Future Finkelstein as $\partial_v^+, \partial_r^+, \partial_p^+ hi, \partial_\theta^+$, we obtain

$$\partial_t = \partial_v^+,\ \partial_r = \partial_r^+ + \frac{r^2+a^2}{\Delta}\partial_v^+ + \frac{a}{\Delta}\partial_\phi^+,\ \partial_\phi = \partial_\phi^+,\ \partial_\theta = \partial_\theta^+. \tag{1.6.41}$$

Surface gravity From

$$\xi = \partial_t + \Omega_h \partial_\phi, \tag{1.6.42}$$
$$\Omega_h = \frac{a}{r_+^2 + a^2}. \tag{1.6.43}$$

we obtain

$$\xi \cdot \xi = -\frac{\Delta \rho^2}{\Gamma}. \tag{1.6.44}$$

This quantity is regular at horizon. Hence, because

$$\begin{aligned} \nabla X &= \frac{\partial_r X}{\rho^2}\left[\Delta \partial_r^+ + (r^2+a^2)\partial_v^+ + a\partial_\phi^+\right] + \frac{\partial_\theta X}{\rho^2}\partial_\theta^+ \\ &\to \frac{r_+^2 + a^2}{\rho^2}\partial_r X \xi + \frac{\partial_\theta X}{\rho^2}\partial_\theta^+ \end{aligned} \tag{1.6.45}$$

for a quantity X which is regular at horizon, we otain

$$\nabla(\xi \cdot \xi)|_H = \frac{(r_+^2 + a^2)\partial_r(\xi \cdot \xi)}{\rho^2}\xi = -\frac{r_+^2 + a^2}{\Gamma}\partial_r \Delta \xi. \tag{1.6.46}$$

Therefore,

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2}. \tag{1.6.47}$$

Bekenstein formula From the expression for the horizon area

$$A = 4\pi(r_+^2 + a^2) = 4\pi(2Mr_+ - Q^2), \tag{1.6.48}$$

we obtain the Smarr formula

$$M = \frac{\kappa}{4\pi G}A + 2\Omega_h J + Q\Phi_h, \tag{1.6.49}$$

and the Bekenstein formula

$$\frac{\kappa}{8\pi}dA = dM - \Omega_h dJ - \Phi_h dQ, \tag{1.6.50}$$

where

$$\Phi_H = \Phi(r = r_+, \theta = 0) = \frac{Qr_+}{r_+^2 + a^2}. \tag{1.6.51}$$

1.6.4 Degenerate solutions

【**Definition 1.6.1** (degenerate horizon)】 When the null geodesic generators of the horizon is incomplete, we say that the horizon is degenerate. □

【**Proposition 1.6.2**】 The surface gravity of a degenerate horizon vanishes. □

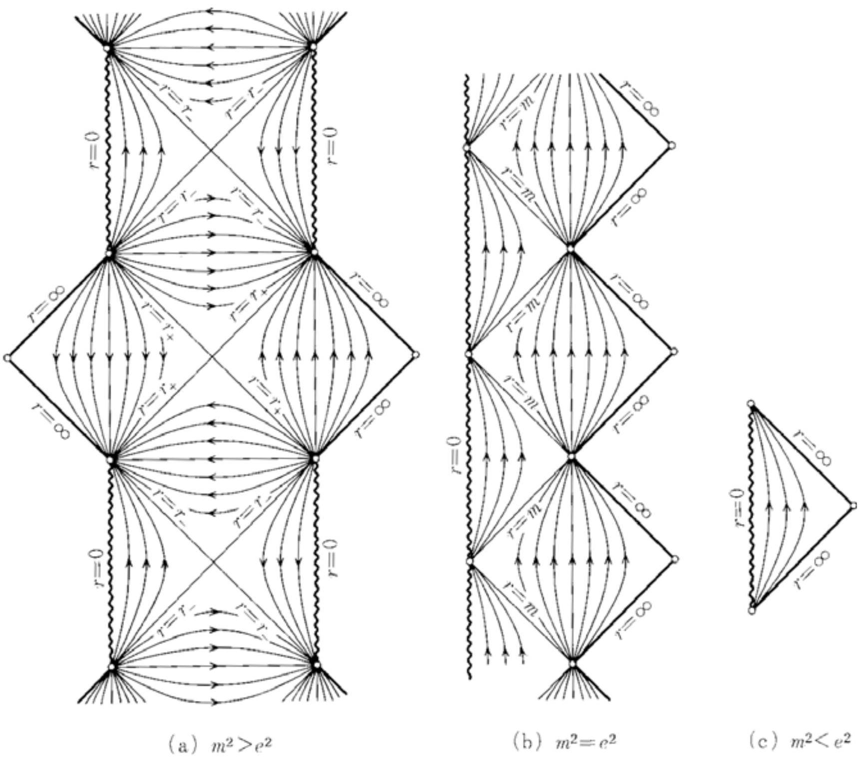


Figure 1.1: Penrose diagrams for RN black holes

Proof. The null generator of the time translation on the horizon, $k = \partial_u$, satisfies

$$\nabla_k k = \kappa k.$$

If we introduce a new coordinate λ by $\lambda = \lambda(u)$, from

$$k = \lambda' \tilde{k}, \quad \tilde{k} = \partial_\lambda,$$

this equation can be written

$$(\lambda')^2 \nabla_{\tilde{k}} \tilde{k} = (\kappa\lambda' - \lambda'')\tilde{k}.$$

Hence, for

$$\lambda = \frac{1}{\kappa}(e^{\kappa u} - 1),$$

λ becomes an affine parameter:

$$\nabla_{\tilde{k}} \tilde{k} = 0. \tag{1.6.52}$$

Thus, the future horizon ($\kappa \geq 0$) is complete iff $\kappa = 0$. □

Extremal Reissner-Nordstrom solution

$$g = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 g(S^n), \tag{1.6.53}$$

$$f(r) = \left(1 - \frac{M}{r^{n-1}}\right)^2 \tag{1.6.54}$$

This can be put into the form

$$ds^2 = -A(\rho)^2 dt^2 + B(\rho)^2 d\boldsymbol{x}^2, \tag{1.6.55}$$

$$\rho^2 = \boldsymbol{x}^2 \tag{1.6.56}$$

in terms of the isotropic coordinates:

$$\frac{dr}{r(f(r))^{1/2}} = \frac{d\rho}{\rho}. \tag{1.6.57}$$

To be explicit, we have

$$ds^2 = -\left(1 + \frac{M}{\rho^{n-1}}\right)^{-2} dt^2 + \left(1 + \frac{M}{\rho^{n-1}}\right)^{2/(n-1)} d\boldsymbol{x}^2, \tag{1.6.58a}$$

$$\rho^{n-1} = r^{n-1} - M. \tag{1.6.58b}$$

5

Majumdar-Papapetrou solution

$$ds^2 = -(1+\Phi)^{-2} dt^2 + (1+\Phi)^{2/(n-1)} d\boldsymbol{x}^2, \tag{1.6.59a}$$

$$\triangle_{n+1}\Phi = 0. \tag{1.6.59b}$$

§1.7 Smarr Formula

Reference

- Bardeen J, Carter B, Hawking SW: CMP31:161 (1973).

1.7.1 Komar integral

Let us define a vector-valued 1-form $d\Sigma_a$ by

$$d\Sigma_a := \frac{1}{(D-1)!}\epsilon_{ab_1\cdots b_{D-1}} dx^{b_1} \wedge \cdots dx^{b_{D-1}}. \tag{1.7.1}$$

Then, we obtain

$$d(\,*dX_*) \equiv \nabla_b(\nabla^a X^b - \nabla^b X^a) d\Sigma_a = -2\triangle X^a d\Sigma_a. \tag{1.7.2}$$

Hence, from the Stokes law, we obtain

$$\frac{1}{2}\int_{\partial\mathscr{S}^{D1}} *dX_* = \int_{\mathscr{S}^{D-1}} R^a_b X^b d\Sigma_a. \tag{1.7.3}$$

1.7.2 Integral at infinity

Let us assume that the spacetime approaches a spherically symmetric solution at spatial infinity:

$$ds^2 \approx -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 ds^2(S^n). \tag{1.7.4}$$

Then, for the time-translation Killing vector $\xi = \partial_t$, we have near infinity

$$\xi_* = -f(r)dt \Rightarrow \quad d\xi_* = f' dt \wedge dr. \tag{1.7.5}$$

Hence, from

$$*(dt \wedge dr) \approx -r^n dV(S^n), \tag{1.7.6}$$

we have

$$*d\xi_* \approx -f' r^n dV(S^n), \tag{1.7.7}$$

and

$$\frac{1}{2}\int_{S^n_\infty} *d\xi_* = -\frac{1}{2}V(S^n)\lim_{r\to\infty} r^n f' = -\frac{n-1}{2} r_0^{n-1} V(S^n). \tag{1.7.8}$$

Similarly, for a rotating AF black hole spacetime with the metric

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + \cdots, \tag{1.7.9}$$

the angular Killing vector is

$$\eta = \partial_\phi \Rightarrow \quad \eta_* = g_{t\phi dt} + g_{\phi\phi}d\phi. \tag{1.7.10}$$

Hence, at infinity,

$$*d\eta_*|_{S^n_\infty} \approx \partial_r g_{t\phi} *(dr \wedge dt) \approx \partial_r g_{t\phi} dV(S^n), \tag{1.7.11}$$

where $g_{t\phi}$ is asymptotically

$$g_{t\phi} \approx \frac{2ar_0^{n-1}}{r^{n-1}}\sin^2\theta. \tag{1.7.12}$$

From

$$dV(S^n) = d\phi d\theta \cos^{n-2} dV(S^{n-2}), \tag{1.7.13}$$

we obtain

$$\int_{S^n} \sin^2\theta dV(S^n) = \frac{1}{n}V(S^n). \tag{1.7.14}$$

Hence,

$$\frac{1}{2}\int_{S^n_\infty} *d\eta_* = \frac{n-1}{n} a r_0^{n-1} V(S^n) = \frac{8\pi}{n} GaM = \frac{8\pi}{n} GJ. \tag{1.7.15}$$

1.7.3 Integral at horizon

Next, let (k, l, e_I) be a null tetrad on the Killing horizon $\mathscr{H}^+$ such that k is parallel to the null geodesic generator of $\mathscr{H}^+$, and $k \cdot l = -1$. Then, dX_* can be expressed as

$$\begin{aligned} dX_* &= -(dX_*)(k,l)k_* \wedge l_* - (dX_*)(l,e_I)k_* \wedge \theta^I \\ &\quad -(dX_*)(k,e_I)l_* \wedge \theta^I + (dX_*)(e_I,e_J)\theta^I \wedge \theta^J. \end{aligned} \quad (1.7.16)$$

The dual of the 2-form basis on the right-hand side is given by

$$*(k \wedge l) = -r^n dV(S^n), \quad (1.7.17a)$$
$$*(k \wedge \theta^I) = -k \wedge r^n I_{e_I} dV(S^n), \quad (1.7.17b)$$
$$*(l \wedge \theta^I) = l \wedge r^n I_{e_I} dV(S^n), \quad (1.7.17c)$$
$$*(\theta^I \wedge \theta^J) = k \wedge l \wedge r^n I_{e_J} I_{e_I} d(V^n). \quad (1.7.17d)$$

Hence,

$$\int_{\mathscr{H}\cap\mathscr{S}^{D-1}} *dX_* = (dX_*)(k,l) r_h^n V(S^n). \quad (1.7.18)$$

For $X = \xi$, we have

$$\begin{aligned} (dxi_*)(k,l) &= 2k^a l^b \nabla_a \xi_b = 2g(l, \nabla_k \xi) \\ &= 2g(l, \nabla_k k) - 2\Omega_h g(l, \nabla_k \eta) = -2\kappa - \Omega_h (d\eta_*)(k,l). \end{aligned} \quad (1.7.19)$$

Inserting these relations into the Komar formula (1.7.3) with $X = \xi$, we obtain the Smarr formula[BCH73]

$$M_h = \frac{\kappa}{4\pi G} A_h + 2\Omega_h J_h, \quad (1.7.20)$$
$$M = M_h - \frac{1}{4\pi G} \int_{\mathscr{S}^{D-1}} R^a_b \xi^b d\Sigma_a, \quad (1.7.21)$$
$$J = J_h + \frac{n}{8\pi G} \int_{\mathscr{S}^{D-1}} R^a_b \eta^b d\Sigma_a. \quad (1.7.22)$$

§1.8
Wald Formulation for BH Thermodynamcis

Reference

- Wald RM: PRD48:R3427 (1993).
- Iyer V, Wald R: PRD50:846 (1994).

1.8.1 Noether charge

Let $\mathscr{L}$ be a Lagrangian n-form for a set of fundamental field ϕ in an n-dimensional spacetime. Then, its variation can be generally written in the form

$$\delta\mathscr{L} = \mathscr{E}\cdot\delta\phi + d(\Theta(\phi;\delta\phi)). \tag{1.8.1}$$

In particular, if the theory is diffeomorphism invariant, for an infinitesimal diffeomorphism corresponding to a vector field X, we have

$$\hat{\delta}\mathscr{L} := \pounds_X\mathscr{L} = d(X\cdot\mathscr{L}) = \mathscr{E}\cdot\pounds_X\phi + d(\Theta(\phi,\pounds_X\phi)), \tag{1.8.2}$$

from $\pounds_X = d\circ i_X + i_X\circ d$ for differential forms. Hence, the current $(n-1)$-form defined by

$$j[X] := \Theta(\phi;\pounds_X\phi) - i_X\mathscr{L} \tag{1.8.3}$$

satisfies

$$dj[X] = -\mathscr{E}\cdot\pounds_X\phi \approx 0, \tag{1.8.4}$$

where $\approx$ implies the equality modulo the EOM. Hence, we can define the Noether charge $(n-2)$-form $Q[X]$ by

$$j[X] \approx dQ[X]. \tag{1.8.5}$$

Next, the variation of j can be calculated as

$$\begin{aligned} \delta j[X] &= \delta[\Theta(\phi;\pounds_X\phi)] - i_X\delta\mathscr{L} \\ &= \Omega(\phi;\delta\phi,\pounds_X\phi) + d(i_X\Theta(\phi;\delta\phi)) - i_X(\mathscr{E}\cdot\delta\phi), \end{aligned} \tag{1.8.6}$$

where Ω is the exterior derivative of Θ in the functional sense:

$$\Omega(\phi;\delta_1\phi,\delta_2\phi) = \delta_1(\Theta(\phi;\delta_2\phi)) - \delta_2(\Theta(\phi;\delta_1\phi)). \tag{1.8.7}$$

Hence, for a variation in the space of solutions to EOM,

$$d(\delta Q[X] - i_X\Theta(\phi;\delta\phi)) \approx \Omega(\phi;\delta\phi,\pounds_X\phi). \tag{1.8.8}$$

Here, note that the integration of Ω on a spatial hypersuface Σ provides the canonical symplectic form

$$\omega(\phi;\delta_1\phi,\delta_2\phi) := \int_\Sigma \Omega(\phi;\delta_1\phi,\delta_2\phi), \tag{1.8.9}$$

for the Hamiltonian formulation of the theory. Therefore, for a vector field X, the function $F[X]$ generating the canonical transformation on the phase space for the dynamical fields induced from X can be expressed as

$$\delta F[X] = \omega(\phi;\delta\phi,\pounds_X\phi) = \int_{\partial\Sigma}(\delta Q[X] - i_X\Theta(\phi;\delta\phi)). \tag{1.8.10}$$

In particular, the existence of $F[X]$ requires the existence of $(n-2)$-form $\mathscr{B}[X](\phi)$ such that

$$i_X\Theta(\phi;\delta\phi) = \delta\mathscr{B}[X](\phi). \tag{1.8.11}$$

1.8.2 Bekenstein formula

Let us consider a stationary and axisymmetric BH solution ϕ with a bifurcating horizon to the theory and a solution $\delta\phi$ for its linear perturbation equation that needs not be stationary or axisymmetric. With the helps of the Killing vector ξ for time translation and the Killing vector η for rotation, the canonical energy and angular momentum at infinity are defined by

$$\delta E = \int_\infty (\delta Q[\xi] - i_\xi \Theta(\phi, \delta\phi)), \tag{1.8.12a}$$

$$\delta J = -\int_\infty \delta Q[\eta]. \tag{1.8.12b}$$

Then, because $\pounds_\xi \phi = \pounds_\eta \phi = 0$ and the null geodesic generator k for the background solution is expressed in terms of ξ and η as

$$k = \xi + \Omega_h \eta, \tag{1.8.13}$$

we have

$$\delta \int_{\mathscr{H}\cap\Sigma} Q[k] = \delta \int_\infty (Q[k] - i_k \Theta(\phi, \delta\phi)) = \delta E - \Omega_h \delta J. \tag{1.8.14}$$

Here, Ω_h is the angular velocity of the background black hole solution and is treated as a constant for perturbations.

In general, $Q[X]$ can be written in the form

$$Q[X] = W_a(\phi) X^a + U^{ab}(\phi) \nabla_{[a} X_{b]} + Y(\phi, \pounds_X \phi) + d(Z(\phi, X)). \tag{1.8.15}$$

For $X = k$, let us define S by

$$S := 2\pi \int_{\mathscr{H}\cap\Sigma} U^{ab} \epsilon_{ab}, \tag{1.8.16}$$

where

$$\epsilon_{ab} = k_a l_b - k_b l_a. \tag{1.8.17}$$

Then, we can show that[IW94]

$$\delta \int_{\mathscr{H}\cap\Sigma} Q[k] = \frac{\kappa}{2\pi} \delta S, \tag{1.8.18}$$

where κ is the surface gravity of the background black hole. This leads to the Bekenstein formula

$$\frac{\kappa}{2\pi} \delta S = \delta E - \Omega_h \delta J. \tag{1.8.19}$$

Note that if the variation of the Lagrangian density with respect to the curvature tensor reads

$$\delta_R \mathscr{L} = \mathscr{E}^{abcd} \delta R_{abcd}, \tag{1.8.20}$$

S can be expressed as[IW94]

$$S = -2\pi \int_{\mathscr{H}\cap\Sigma} i_l i_k \mathscr{E}^{abcd} \epsilon_{ab} \epsilon_{cd}. \tag{1.8.21}$$

2 Solutions with High Symmetries

§2.1 Black-Brane Type Spacetime

2.1.1 Assumptions

1. The spacetime is locally the product of a m-dimensional spacetime $\mathscr{N}$ and n-dimensional Einstein space $\mathscr{K}$:

$$M^{n+m} \approx \mathscr{N} \times \mathscr{K} \ni (z^M) = (y^a, x^i) \tag{2.1.1}$$

2. The metric can be written

$$ds^2 = g_{MN} dz^M dz^N = g_{ab}(y) dy^a dy^b + r(y)^2 d\sigma_n^2, \tag{2.1.2}$$

where $d\sigma_n^2 = \gamma_{ij}(z) dx^i dx^j$ is an n-dimensional metric on $\mathscr{K}$,

For this type of spacetime, the following three types of covariant derivatives appear:

$$M : \quad \nabla_M, \Gamma^M_{NL}(z), R_{MNLS}(z) \tag{2.1.3a}$$

$$\mathscr{N} : \quad D_a, {}^m\Gamma^a_{bc}(y), {}^mR_{abcd}(y). \tag{2.1.3b}$$

$$\mathscr{K} : \quad \hat{D}_i, \hat{\Gamma}^i_{jk}(x), \hat{R}_{ijkl}(x). \tag{2.1.3c}$$

The curvature tensor of M can be decomposed as

$$R^a{}_{bcd} = {}^mR^a{}_{bcd}, \quad R^i{}_{ajb} = -\frac{D_a D_b r}{r}\delta^i_j, \quad R^i{}_{jkl} = {}^nR^i_{jkl} - (Dr)^2(\delta^i_k \gamma_{jl} - \delta^i_l \gamma_{jk}). \tag{2.1.4}$$

From this, we have

$$R_{ab} = {}^mR_{ab} - n\frac{D_aD_br}{r}, \tag{2.1.5a}$$
$$R_{ai} = 0, \tag{2.1.5b}$$
$$R_{ij} = -\left\{r\Box r + (n-1)(Dr)^2\right\}\delta^i_j + {}^nR_{ij}, \tag{2.1.5c}$$
$$R = {}^mR + {}^nR - 2n\frac{\Box r}{r} - n(n-1)\frac{(Dr)^2}{r^2}. \tag{2.1.5d}$$

This leads to the following decomposition formula for the Einstein tensor:

$$G_{ab} = {}^mG_{ab} - \frac{n}{r}D_aD_br - \left[\frac{n(n-1)}{2}\frac{K-(Dr)^2}{r^2} - \frac{n}{r}\Box r\right]g_{ab}, \tag{2.1.6a}$$
$$G^i_a = 0, \tag{2.1.6b}$$
$$G^i_j = \left[-\frac{1}{2}{}^mR - \frac{(n-1)(n-2)}{2}\frac{K-(Dr)^2}{r^2} + \frac{n-1}{r}\Box r\right]\delta^i_j. \tag{2.1.6c}$$

This together with the Einstein equations require for the energy-momentum tensor to have the following structure:

$$T_{ab} = T_{ab}(y),\ T_{ai} = 0,\ T^i_j = P(y)\delta^i_j. \tag{2.1.7}$$

2.1.2 General Robertson-Walker spacetime

$m = 1$ and $\mathscr{K}$ is an Einstein space:

$$ds^2 = -dt^2 + a(t)^2 d\sigma_n^2.$$

For this metric, the Einstein tensors can be written

$$G_{00} = \frac{n(n-1)}{2}\left(\frac{K}{a^2} + H^2\right), \tag{2.1.8a}$$
$$G^i_j = -(n-1)\left[\frac{\ddot{a}}{a} + \frac{n-2}{2}\left(\frac{K}{a^2} + H^2\right)\right]\delta^i_j, \tag{2.1.8b}$$

and the Einstein equations reduce to

$$H^2 = \frac{2\kappa^2}{n(n-1)}\rho - \frac{K}{a^2}, \tag{2.1.9a}$$
$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{n-1}\left(\frac{n-2}{n}\rho + P\right). \tag{2.1.9b}$$

These lead to the energy conservation equation

$$\dot{\rho} = -n(\rho + P)H. \tag{2.1.10}$$

For $n = 3$, this spacetime describes a spatially homogeneous isotripic universe (FLRW universe), which is conformally flat:

$$C^{ab}{}_{cd} := R^{ab}{}_{cd} - \frac{4}{n-2}\delta^{[a}_{[c}R^{b]}_{d]} + \frac{2R}{(n-1)(n-2)}\delta^{[a}_{[c}\delta^{b]}_{d]} \equiv 0. \tag{2.1.11}$$

In contrast, for $n \geq 4$, because the Einstein space need not be a constant curvature space, this class of spacetimes contain anisotropic and inhomogeneous universes. For example, Euclidian Taub-NUT space

$$\begin{aligned} ds^2(\mathrm{ETN}^4) &= f(x)^{-1}dx^2 + (2l)^2 f(x)(d\psi + \cos\theta d\phi)^2 \\ &\quad +(l^2 - x^2)(d\theta^2 + \sin^2\theta d\phi^2); \\ f(x) &= \frac{2mx + l^2 + x^2 + K(x^4 - 6l^2x^2 - 3l^4)}{l^2 - x^2} \end{aligned} \tag{2.1.12}$$

is an inhomogeneous 4D Einstein space satsifying

$$R_{ij} = 3Kg_{ij}. \tag{2.1.13}$$

2.1.3 Braneworld model

$m = 2$. For example, the metric of the anti-de Sitter spacetime AdS^{n+2} reads

$$ds^2 = \frac{dr^2}{1 - \lambda r^2} - (1 - \lambda r^2)dt^2 + r^2 d\Omega_n^2. \tag{2.1.14}$$

This is a special case of general Einstein black hole solutions.

2.1.4 Higher-dimensional static Einstein black holes

$m = 2$, $\mathscr{K}$ is a compact Einstein space.

Electromagnetic fiels Let us consider the case in which matter consists only of electromagnetic fields whose electromagnetic tensor fields take the form

$$\mathscr{F} = \frac{1}{2}E_0\epsilon_{ab}dy^a \wedge dy^b + \frac{1}{2}\mathscr{F}_{ij}dz^i \wedge dz^j. \tag{2.1.15}$$

Then, the Maxwell equations read

$$d\mathscr{F} = 0 \Rightarrow \quad E_0 = E_0(y),\ \mathscr{F}_{ij} = \mathscr{F}_{ij}(z),\ \mathscr{F}_{[ij,k]} = 0. \tag{2.1.16}$$

and

$$0 = \nabla_\nu \mathscr{F}^{a\nu} = \frac{1}{r^n}\epsilon^{ab}D_b(r^n E_0), \tag{2.1.17a}$$

$$0 = \nabla_\nu \mathscr{F}^{i\nu} = \hat{D}_j \mathscr{F}^{ij}. \tag{2.1.17b}$$

The general solution to these equations is

$$E_0 = \frac{q}{r^n}, \tag{2.1.18}$$

$$\hat{\mathscr{F}} = \frac{1}{2}\mathscr{F}_{ij}(z)dz^i \wedge dz^j : \text{harmonic} \tag{2.1.19}$$

The corresponding value of the energy-momentum tensor is

$$T_{ab} = -\frac{1}{4}(2E_0^2 + \mathscr{F}_{ij}\mathscr{F}^{ij})g_{ab}, \tag{2.1.20a}$$

$$T_{ai} = 0, \tag{2.1.20b}$$

$$T^i_j = \mathscr{F}^{ik}\mathscr{F}_{jk} - \frac{1}{4}(2E_0^2 + \mathscr{F}_{ij}\mathscr{F}^{ij})\delta^i_j. \tag{2.1.20c}$$

Therefore, when the Einstein equations are satisfied, $\mathscr{F}^i_j$ has to satisfy the condition

$$\mathscr{F}^{ik}\mathscr{F}_{jk} \propto \delta^i_j. \tag{2.1.21}$$

From this, when $\mathscr{F}_{ij} \neq 0$, $\mathscr{F}_{ij}$ must be a regular matrix. In particular, n has to be a even number. When $\mathscr{K}$ is a sphere, a harmonic form exists only when $n = 2$.

In general, when there exists a nonvanishing anti-symmetric tensor in the background, vector-type and scala-type perturbations couple with each other. From this point, we assume $\mathscr{F}_{ij} = 0$. Under this assumption, the energy-momentum tensor reads

$$T^a_b = -P\delta^a_b,\ T^i_j = P\delta^i_j; \quad P = \frac{1}{2}E_0^2 = \frac{q^2}{2r^{2n}}. \tag{2.1.22}$$

Remark If $\mathscr{F}_{ai} \neq 0$, we have

$$T_{ai} = E_0\epsilon_{ab}\mathscr{F}_i{}^b + \mathscr{F}_{aj}\mathscr{F}_i{}^j. \tag{2.1.23}$$

Hence, $T_{ai} = 0$ requirs

$$\mathscr{F}_{aj}\mathscr{F}_i{}^j = -E_0\epsilon_{ab}\mathscr{F}_i{}^b \Rightarrow \quad \mathscr{F}_{aj}\mathscr{F}_i{}^j\mathscr{F}^i{}_k = -E_0^2\mathscr{F}_{ak}. \tag{2.1.24}$$

However, because $\mathscr{F}_i{}^j\mathscr{F}^i{}_k$ is a non-negative symmetric matrix, this leads to $E_0 = 0$.

Generalised Birkhoff's theorem The Einstein equations

$$R_{\mu\nu} = \frac{2}{n}\Lambda g_{\mu\nu} + \kappa^2\left(T_{\mu\nu} - \frac{1}{n}Tg_{\mu\nu}\right) \tag{2.1.25}$$

are equivalent to

$$n\Box r = r^2R - 2(n+1)\lambda r + \frac{2(n-1)^2Q^2}{r^{2n-1}}, \tag{2.1.26a}$$

$$(n-1)\frac{K-(Dr)^2}{r^2} = \frac{\Box r}{r} + (n+1)\lambda + \frac{(n-1)Q^2}{r^{2n}}, \tag{2.1.26b}$$

$$2D_aD_br = \Box r g_{ab} \tag{2.1.26c}$$

with

$$\lambda := \frac{2\Lambda}{n(n+1)}, \quad Q^2 := \frac{\kappa^2q^2}{n(n-1)}. \tag{2.1.27}$$

From (2.1.26c), it follows that

$$D_a\Box r = \Box D_a r - \frac{{}^2R}{2}D_a r = \frac{1}{2}D_a\Box r - \frac{{}^2R}{2}D_a \Rightarrow \quad D_a\Box r = -{}^2R D_a r \quad (2.1.28)$$

Hence, by differentiating (2.1.26a) w.r.t. r, we obtain

$$D_a\left(r^{n+1}({}^2R - 2\lambda) + \frac{2(n-1)(2n-1)Q^2}{r^{n-1}}\right) = 0. \quad (2.1.29)$$

BH 型： When $\nabla r \neq 0$,

$$\frac{\Box r}{r} = -2\lambda + \frac{2(n-1)M}{r^{n+1}} - \frac{2(n-1)Q^2}{r^{2n}}, \quad (2.1.30a)$$

$$\frac{K-(Dr)^2}{r^2} = \lambda + \frac{2M}{r^{n+1}} - \frac{Q^2}{r^{2n}}, \quad (2.1.30b)$$

$${}^2R = 2\lambda + \frac{2n(n-1)M}{r^{n+1}} - \frac{2(n-1)(2n-1)Q^2}{r^{2n}}. \quad (2.1.30c)$$

Therefore, the spacetime metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma_n^2; \quad (2.1.31)$$

$$f(r) = K - \lambda r^2 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}}. \quad (2.1.32)$$

Nariai 型： When $r = a$(constant), the value of a is determined as a solution to the equation

$$(n-1)\frac{K}{a^2} = (n+1)\lambda + \frac{(n-1)Q^2}{a^{2n}}. \quad (2.1.33)$$

The spacetime metric is

$$ds^2 = -f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + a^2 d\sigma_n^2, \quad (2.1.34)$$

where

$$f(\rho) = 1 - \sigma\rho^2; \quad \sigma = (n+1)\lambda - \frac{(n-1)^2Q^2}{a^{2n}}. \quad (2.1.35)$$

2.1.5 Black branes

When the spacetime is a simple direct product of two spacetime

$$\mathscr{N} = \mathscr{N} \times \mathscr{K} : \quad ds^2(\mathscr{M}) = ds^2(\mathscr{N}) + ds^2(\mathscr{K}), \quad (2.1.36)$$

the vacuum Einstein equation for $\mathscr{M}$,

$$R_{MN} = \frac{2\Lambda}{D-2}g_{MN} = (D-1)\lambda g_{MN}, \quad (2.1.37)$$

is satisifed if

$$R_{ab}(\mathscr{N}) = (D-1)\lambda g_{ab}(\mathscr{N}), \quad R_{ij}(\mathscr{K}) = (D-1)\lambda g_{ij}(\mathscr{K}). \tag{2.1.38}$$

This class contains only products of constant curvature spaces if $n, m \leq 3$, but if $n \geq 4$ or $m \geq 4$, it contains infinitely many non-trivial solutions. For example, SchBH$^m \times \mathbb{R}^n$, KerrBH$^m \times RF^n$, dS-KerrBH$^m \times H^n$, adS-KerrBH$^m \times S^n$ are contained if $m \geq 4$.

$m = 2 + k$ and $\mathscr{K}$ is an Einstein space. The spacetime factor $\mathscr{N}$ is the direct product of a 2D BH and a k-dimensional brane:

$$ds^2 = \frac{dr^2}{f(r)} - f(r)dt^2 + d\boldsymbol{z}\cdot d\boldsymbol{z} + r^2 d\sigma_n^2, \tag{2.1.39}$$

$$R_{\mu\nu} = 0. \tag{2.1.40}$$

For $k = 1$, this can be generalised to a warped solution,

$$ds^2 = e^{2y/\ell}\left(\frac{dr^2}{f(r)} - f(r)dt^2 + r^2 d\sigma_n^2\right) + dy^2, \tag{2.1.41}$$

$$R_{\mu\nu} = -\frac{n}{\ell^2}. \tag{2.1.42}$$

§2.2 Higher-dimensional rotating black hole

2.2.1 GLPP solution

In general, the Myers-Perry solution representing higher-dimensional AF rotating BH solution with spherical horizon can be generalised to the following solution for $\Lambda \neq 0$[Gibbons GW, Lü H, Page DN, Pope CN(2004)[GLPP04]],

$$\begin{aligned} ds^2 &= -\frac{f(r)F}{Z}dt^2 + \frac{F}{f(r)}dr^2 \\ &\quad + \frac{\lambda}{(1-\lambda r^2)W}\left(\sum \frac{r^2+a_i^2}{1+\lambda a_i^2}\mu_i d\mu_i\right)^2 + \sum_i \frac{r^2+a_i^2}{1+\lambda a_i^2}d\mu_j^2 \\ &\quad + {\sum}'_i \frac{r^2+a_i^2}{1+\lambda a_i^2}\mu_i^2(d\phi_i - \Omega^i dt)^2 \\ &\quad + \frac{2M}{U}\left\{{\sum}'_i \frac{a_i\mu_i^2}{1+\lambda a_i^2}(d\phi_i - \Omega^i dt)\right\}^2, \end{aligned} \tag{2.2.1}$$

where

$$U = r^{\epsilon-2}F\prod_{j=1}^{N}(r^2+a_j^2), \quad W = \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{1+\lambda a_i^2}, \tag{2.2.2a}$$

$$F = r^2\sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}, \tag{2.2.2b}$$

$$X = 1+\frac{2M}{U}{\sum_i}'\frac{a_i^2\mu_i^2}{(r^2+a_i^2)(1+\lambda a_i^2)} = \frac{f+\frac{2MW}{U}}{1-\lambda r^2}, \tag{2.2.2c}$$

$$\Omega^i = \frac{2MW}{UX}\frac{a_i}{r^2+a_i^2}, \tag{2.2.2d}$$

$$f(r) = 1-\lambda r^2 - \frac{2Mr^{2-\epsilon}}{\Pi_i'(r^2+a_i^2)}, \tag{2.2.2e}$$

$$Z := \frac{UX}{W}\frac{r^{4-\epsilon}}{\Pi_i'(r^2+a_i^2)}. \tag{2.2.2f}$$

The number N is related to the spacetime dimension D by

$$D = 2N+1+\epsilon; \quad \epsilon = 0,1, \tag{2.2.3}$$

and when $\epsilon = 1$, $a_{N+1} = 0$. Further, the angular coordinates μ_i are constrained by

$$\sum_{i=1}^{N+\epsilon}\mu_i^2 = 1. \tag{2.2.4}$$

Basic Properties

Parameters M is the mass parameter, and $[a_1,\cdots,a_{[(D-1)/2]}]$ are the angular momentum parameters.

Symmetry The isometry group of this solution is $\mathbb{R}\times\mathrm{U}(1)^{(D-2)/2}$ for even D and $\mathbb{R}\times\mathrm{U}(1)^{(D-1)/2}$ for odd D for generic values of the angular momenta a_i. Hence, the cohomogeneity of the spacetime is $[D/2]$. The Killing vectors are given by

$$\xi = \partial_t, \quad \eta_i = \partial_{\phi_i}\ (i=1,\cdots,N). \tag{2.2.5}$$

However, the symmetry is enhanced for special values of a_i.

Killing horizons The determinant of the metric of the Killing orbit is given by

$$\tilde{\Delta} = \det(g(\xi_I,\xi_J)_{I,J=1,\cdots,N+1}) = -f(r)W\prod_{i=1}^{N}\frac{\mu_i^2(r^2+a_i^2)}{1+\lambda a_i^2}. \tag{2.2.6}$$

Hence, Killing horizons are expressed as $r = r_h$ in terms of solutions to

$$f(r_h) = 0. \tag{2.2.7}$$

However, some solutions may not give a Killing horizon. Further, for odd D, the solution can be extended to $r^2 < 0$ for some cases.

Angular velocity of BH ???

Area of BH

Surface gravity of BH

2.2.2 Simply rotating solution

If we choose the angular momentum parameters of the GLPP solution as

$$a_1 = a,\ a_2 = \cdots = a_{[(D-1)/2]} = 0, \tag{2.2.8}$$

the metric can be written

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2 X \sin^2\theta}{\rho^2} dt^2 - \frac{2a\sin^2\theta}{C\rho^2}\left\{\lambda\rho^2(r^2+a^2) + \frac{2M}{r^{n-1}}\right\} dt d\phi \\ &\quad + \frac{\sin^2\theta}{C^2\rho^2}\left[C(r^2+a^2)\rho^2 + \frac{2a^2 M}{r^{n-1}}\sin^2\theta\right] d\phi^2 \\ &\quad + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{X} d\theta^2 + r^2\cos^2\theta d\Omega_n^2. \end{aligned} \tag{2.2.9}$$

where $D = n+4$, and

$$\Delta = (1-\lambda r^2)(r^2+a^2) - \frac{2M}{r^{D-5}}, \tag{2.2.10}$$

$$C = 1 + \lambda a^2, \quad X = 1 + \lambda a^2 \cos^2\theta. \tag{2.2.11}$$

This metric has the structure

$$ds^2 = g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + r^2\cos^2\theta d\Omega_n^2, \tag{2.2.12}$$

where all the metric coefficient functions depend only on r and θ. Thus, this belong to the BB-type solutions with $m = 4$ and $\mathscr{K}^n = S^n$.

Symmetry The symmetry of this spacetime is $\mathbb{R} \times \mathrm{U}(1) \times \mathrm{SO}(n+1)$, and the cohomogeneity is $D - 2 - n = 2$.

Killing horizon The Killing horizon radius is a solution to

$$\Delta(r_h) = 0. \tag{2.2.13}$$

Hence, for $D > 5$, the horizon exists for arbitrarily large values of a in contract to $D \leq 5$.

Angular velocity The equation

$$g(\xi + \Omega_h\eta, \xi + \Omega_h\eta) = g_{tt} + 2g_{t\phi}\Omega_h + g_{\phi\phi}\Omega_h^2 = 0 \tag{2.2.14}$$

has a double root at the horizon where $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = 0$. Hence,

$$\Omega_h = -\frac{g_{t\phi}}{g_{tt}} = \frac{Ca}{r^2+a^2}. \tag{2.2.15}$$

Horizon area

$$A = \frac{2\pi r_h^n(r_h^2+a^2)}{C}V(S^n)\int_0^{\pi/2} d\theta \sin\theta\cos^n\theta$$
$$= \frac{2\pi r_h^n(r_h^2+a^2)}{(n+1)C}V(S^n). \tag{2.2.16}$$

Surface gravity

$$\kappa = \frac{\partial_r \Delta}{2(r^2+a^2)} \tag{2.2.17}$$

§2.3
Special MP Solution

2.3.1 Invariant bases of S^3

Let us define a map σ from $Z = (z^1 = x^4 + ix^3, z^2 = x^1 + ix^2) \in \mathbb{R}^4 \cong \mathbb{C}^2$ to $M(2,\mathbb{C})$ by

$$\sigma(Z) = x^4 + ix^3\sigma_3 + i(x^2\sigma_1 + x^1\sigma_2) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \tag{2.3.1}$$

Then, from

$$\sigma(Z)\sigma(Z)^\dagger = |Z|^2 1_2, \tag{2.3.2}$$

where $|Z|^2 = |z_1|^2 + |z_2|^2$, it induces the diffeomorphism from $S^3 \subset \mathbb{C}^2$ to SU(2).

Utilising this diffeormorphism, we can easily construct left invariant basis on S^3. Let $X_{I0}(I = 1,2,3)$ be a basis of vectors at $Z = (1,0)$ of S^3, and define vector fields X_I on S^3 by

$$\sigma(X_I) = \sigma(Z)\sigma(X_{I0}). \tag{2.3.3}$$

Then, X_I becomes a basis of vector fields on S^3 that are invariant under the left action of SU(2) on SU(2) $\cong S^3$. Let δy_I^i be components of X_I in a local coordinate system (y^i):

$$\sum_{j=1}^{3} \delta y_I^j \partial_j Z = X_I. \tag{2.3.4}$$

Then, the dual basis $\chi^I = \chi_j^I dy^j$ is determined by δy_I^j as

$$\sum_{j=1}^{3} \chi_j^J \delta y_I^j = \delta_I^J. \tag{2.3.5}$$

Hence,

$$dZ = \sum_{j=1}^{3} \partial_j Z dy^j = \sum_{I,j=1,3} X_I \chi^I_j dy^j = \sum_{I=1}^{3} X_I \chi^I. \tag{2.3.6}$$

Taking the σ value of this expression, we obtain

$$\sigma(dZ) = \sum_{I=1}^{3} \sigma(X_I)\chi^I = \sigma(Z) \sum_{I=1}^{3} \sigma(X_{I0})\chi^I, \tag{2.3.7}$$

or equivalently

$$\sigma(Z)^{-1}\sigma(dZ) = \sum_{I=1}^{3} \sigma(X_{I0})\chi^I. \tag{2.3.8}$$

For example, for

$$dZ^J(X_{I0}) = \delta^J_I, \tag{2.3.9}$$

we obtain the invariant dual basis

$$\chi^1 + i\chi^2 = \bar{z}_1 dz_2 - z_2 d\bar{z}_1, \tag{2.3.10a}$$
$$\chi^3 = i(z_1 d\bar{z}_1 + \bar{z}_2 dz_2). \tag{2.3.10b}$$

If we introduce the polar coordinates (θ, ϕ_1, ϕ_2) by

$$z_1 = \sin\theta e^{i\phi_1}, \quad z_2 = \cos\theta e^{-i\phi_2}, \tag{2.3.11}$$

there invariant basis can be written

$$\chi^1 + i\chi^2 = e^{-i(\phi_1+\phi_2)} \left[-d\theta + i \sin\theta\cos\theta(d\phi_1 - d\phi_2)\right], \tag{2.3.12a}$$
$$\chi^3 = d\phi_1 \sin^2\theta + d\phi_2 \cos^2\theta. \tag{2.3.12b}$$

The standard metric on the unit S^3 can be expressed in terms of this basis as

$$ds^2(S^3) = (\chi^1)^2 + (\chi^2)^2 + (\chi^3)^2 \tag{2.3.13}$$

From the relations

$$d\theta = \chi^2 \sin\psi - \chi^1 \cos\psi, \tag{2.3.14a}$$
$$d\phi_1 = \chi^3 + (\chi^1 \sin\psi + \chi^2 \cos\psi)\cot\theta, \tag{2.3.14b}$$
$$d\phi_2 = \chi^3 - (\chi^1 \sin\psi + \chi^2 \cos\psi)\tan\theta, \tag{2.3.14c}$$

where $\psi = \phi_1 + \phi_2$, we obtain

$$a\sin^2\theta d\phi_1 + b\cos^2\theta d\phi_2 = (a+b)\chi^3 + (a-b)(\chi^1 \sin\psi + \chi^2 \cos\psi)\sin\theta\cos\theta, \tag{2.3.15a}$$

$$\cos^2\theta d\theta^2 + \sin^2\theta d\phi_1^2 = (\chi^3)^2 \sin^2\theta + 2\chi^3(\chi^1 \sin\psi + \chi^2\cos\psi)\sin\theta\cos\theta + \left((\chi^1)^2 + (\chi^2)^2\right)\cos^2\theta, \tag{2.3.15b}$$

$$\sin^2\theta d\theta^2 + \cos^2\theta d\phi_2^2 = (\chi^3)^2 \cos^2\theta - 2\chi^3(\chi^1 \sin\psi + \chi^2\cos\psi)\sin\theta\cos\theta + \left((\chi^1)^2 + (\chi^2)^2\right)\sin^2\theta. \tag{2.3.15c}$$

2.3.2 S^1 fibring of S^{2N-1}

The unit sphere S^{2N-1} can be embedded into $\mathbb{C}^N$ by

$$\boldsymbol{z} \cdot \bar{\boldsymbol{z}} \equiv \sum_{j=1}^{N} z_j \bar{z}_j = 1, \tag{2.3.16}$$

where $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ are the column vectors with N entries, $\boldsymbol{z} = (z_j)$ and $\bar{\boldsymbol{z}} = (\bar{z}_j)$, respectively. If we parametrise z_j as

$$z_j = \mu_j e^{i\phi_j}, \tag{2.3.17}$$

this equation is expressed as

$$\sum_{j=1}^{N} \mu_j^2 = 1. \tag{2.3.18}$$

In terms of these coordinates the metric of S^{2N-1} can be written

$$ds^2(S^{2N-1}) = d\boldsymbol{z} \cdot d\bar{\boldsymbol{z}} = \sum_{j=1}^{N} \left(d\mu_j^2 + \mu_j^2 d\phi_j^2 \right). \tag{2.3.19}$$

We can define a natural free U(1) isometric action on S^{2N-1} by

$$z_j \mapsto e^{i\lambda} z_j. \tag{2.3.20}$$

As is well-known, the quotient space of S^{2N-1} by this action is $\mathbb{C}P^{N-1}$, and the original S^{2N-1} can be regarded as a S^1 bundle over $\mathbb{C}P^{N-1}$ with Fubini-Study metric.

The explicit form of the metric in this fibring can be obtained in the following way. First, the infinitesimal transformation X with unit norm corresponding this U(1) action is given by

$$X = i \left(z_j \partial_j - \bar{z}_j \bar{\partial}_j \right), \tag{2.3.21}$$

Then, this vector field X with a set of U(1)-invariant unit vector fields that are orthogonal to X and project onto an orthonormal frame on $\mathbb{C}P^{N-1}$ form an orthonormal basis of vector fields on S^{2N-1}. Let χ^j be the 1-form basis dual to it such that $\chi^{2N-1} = \chi$ is dual to X. Then, χ is expressed as

$$\chi = \operatorname{Im} \bar{\boldsymbol{z}} \cdot d\boldsymbol{z} = \sum_{j=1}^{N} \mu_j^2 d\phi_j, \tag{2.3.22}$$

and the metric of S^{2N-1} can be written

$$ds^2(S^{2N-1}) = \chi^2 + ds^2(\mathbb{C}P^{N-1}), \tag{2.3.23}$$

where $ds^2(\mathbb{C}P^{N-1})$ is the Fubini-Study metric of $\mathbb{C}P^{N-1}$ that is expressed in terms of the homogeneous coordinates z_j as

$$ds^2(\mathbb{C}P^{N-1}) = \frac{d\boldsymbol{z}\cdot d\bar{\boldsymbol{z}}}{\boldsymbol{z}\cdot\bar{\boldsymbol{z}}} - \frac{|\boldsymbol{z}\cdot d\bar{\boldsymbol{z}}|^2}{(\boldsymbol{z}\cdot\bar{\boldsymbol{z}})^2}. \tag{2.3.24}$$

Here, note that the Kähler form of this metric

$$\varphi = \frac{i}{2\boldsymbol{z}\cdot\bar{\boldsymbol{z}}}\sum_j dz_j\wedge d\bar{z}_j - \frac{i}{2(\boldsymbol{z}\cdot\bar{\boldsymbol{z}})^2}\bar{\boldsymbol{z}}\cdot d\boldsymbol{z}\wedge\boldsymbol{z}\cdot d\bar{\boldsymbol{z}} \tag{2.3.25}$$

is related to χ by

$$d\chi = 2\varphi. \tag{2.3.26}$$

Further, it is easy to see that the set of isometries of S^{2N-1} that preserves χ is isomorphic to $\mathrm{U}(N)$, which projects onto the isometry group of $\mathbb{C}P^{N-1}$, $\mathrm{SU}(N)$[KN69].

2.3.3 U(N) MP solution

For an odd spacetime dimension $D = 2N+1$, the Myers-Perry solution[MP86] can be written

$$ds^2 = \frac{r^2}{\Delta}dr^2 + (r^2+a^2)ds^2(S^{2N-1}) + \frac{\mu}{(r^2+a^2)^{N-1}}(dt - a\chi)^2 - dt^2 \tag{2.3.27}$$

where

$$\Delta = r^2 + a^2 - \frac{\mu r^2}{(r^2+a^2)^{N-1}}, \tag{2.3.28}$$

and χ is the $\mathrm{U}(N)$-invariant 1-form (2.3.22) on S^{2N-1} introduced in the previous section. Note that the metric (2.3.27) has $\mathrm{U}(N)$ invariance in addition to the time translation invariance.

In terms of the coordinate y defined by

$$y = r^2 + a^2 \tag{2.3.29}$$

this metric can be regularly extended to $r^2 < 0$ as[MP86]

$$ds^2 = \frac{dy^2}{4\Delta} + ds_K^2, \tag{2.3.30}$$

$$ds_K^2 = y\,ds^2(S^{2N-1}) + \frac{\mu}{y^{N-1}}(dt - a\chi)^2 - dt^2. \tag{2.3.31}$$

where Δ now reads

$$\Delta = y - \frac{\mu(y-a^2)}{y^{N-1}}. \tag{2.3.32}$$

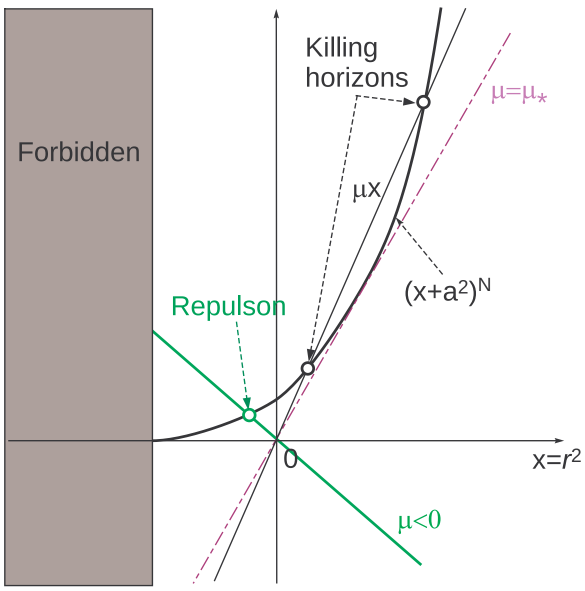


Figure 2.1: Horizon position of the U(N) MP solution

Utilising the decomposition (2.3.23), the metric of the Killing orbits ds^2_K can be written

$$ds_K^2 = -Cdt^2 + B(\chi - \Omega dt)^2 + yds^2(\mathbb{C}P^{N-1}), \tag{2.3.33}$$

where

$$B = \frac{y^N + \mu a^2}{y^{N-1}}, \tag{2.3.34a}$$

$$C = \frac{y^{N-1}\Delta}{y^N + \mu a^2}, \tag{2.3.34b}$$

$$\Omega = \frac{\mu a}{y^N + \mu a^2}. \tag{2.3.34c}$$

From this, it follows that

$$\det g_K = -y^{2N-2}\Delta \prod_j \mu_j^2, \tag{2.3.35a}$$

$$\det g = -\frac{1}{4} y^{2N-2} \prod_j \mu_j^2. \tag{2.3.35b}$$

Hence the possible loci of singularity are $y = 0$ and $\Delta = 0$. Among these, $y = 0$ is a curvature singularity because the Kretchman invariant of this metric is given by

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = \frac{4N(N-1)\mu^2}{y^{2N}} \left\{ (2N-1)^2 - 8N(N+1)\frac{a^2}{y} + 4(N+1)(N+2)\frac{a^4}{y^2} \right\}. \tag{2.3.36}$$

2.3.4 Internal Structure

As in the five-dimensional case, the location of the horizon is determined by the zero of Δ,

$$(x+a^2)^{N-1}\Delta = (x+a^2)^N - \mu x = 0 \tag{2.3.37}$$

where $x = r^2$. As is shown in Fig. 2.1, this equation has two positive roots for $\mu > \mu_* = N^N a^{2(N-1)}/(N-1)^{N-1}$, while it has a single negative root x_h for $\mu < 0$. In the the former case, there exists no CTC in the regular region $y = x + a^2 > 0$, and the two roots correspond to Killing horizons.

In contrast, in the negative mass case, CTC appears in the region $\Delta > 0$ because $B < 0$ from (2.3.32) and (2.3.33), and $-C > 0$ around $x = x_h$. Hence, $x = x_h$ cannot be a horizon as in the $N = 2$ case. Now, we show that it is a quasi-regular singularity in general that can be made regular by some periodic identification of the time coordinate for a discrete set of values for the angular momentum a.

For that purpose, we introduce the new coordinate ξ by

$$\xi = \int_{y_h} \frac{dy}{\sqrt{2\Delta}}. \tag{2.3.38}$$

At around $\xi = 0$, y is expressed in terms of ξ as

$$y - y_h = \Delta'(y_h)\xi^2 + \mathrm{O}\left(\xi^4\right). \tag{2.3.39}$$

Hence, the metric (2.3.30) can be expanded around ξ as

$$\begin{aligned} ds^2 &= d\xi^2 + c^2\xi^2\tilde{\chi}^2 + y ds^2(\mathbb{C}P^{N-1}) - \frac{y_h^2}{a^2-y_h}\left(1+\mathrm{O}\left(\xi^2\right)\right)\tilde{\tau}^2 + \mathrm{O}\left(\xi^4\right) \\ &\approx d\xi^2 + c^2\xi^2\tilde{\chi}^2 + y_h ds^2(\mathbb{C}P^{N-1}) - \frac{y_h^2}{a^2-y_h}\tilde{\tau}^2, \end{aligned} \tag{2.3.40}$$

where c is the constant

$$c = \frac{aN}{\sqrt{a^2-y_h}}\left(1 - \frac{N-1}{N}\frac{y_h}{a^2}\right), \tag{2.3.41}$$

and $\tilde{\tau}$ and $\tilde{\chi}$ are the 1-forms

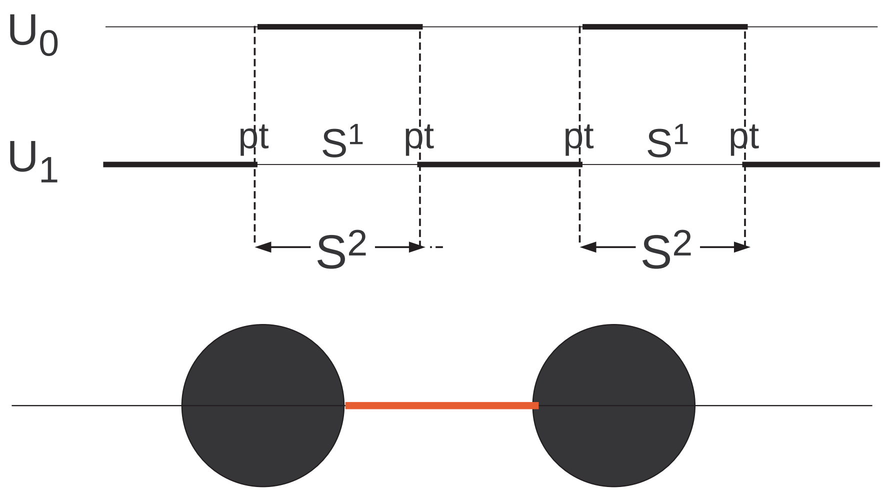
$$\tilde{\tau} = \chi - \frac{a}{y_h}dt, \tag{2.3.42a}$$

$$\tilde{\chi} = \chi + \frac{N-1}{N}\frac{y_h}{a^2}\tilde{\tau}. \tag{2.3.42b}$$

For further details, see G.W. Gibbons and H. Kodama (2009) Prog. Theor. Phys. 121: 1361.

§2.4

Black Ring



2.4.1 Generalised Weyl Formulation

For $\mathscr{R}^{D-2}$ symmetric spacetime of dimension D,

$$ds^2 = -e^{2U_0}dt^2 + \sum_{i=1}^{D-3} e^{2U_i} d\phi_i^2 + e^{2\nu}(d\rho^2 + dz^2) \tag{2.4.1}$$

the Einstein equations reduce to a linear PDE system:

$$\rho^{-1}\partial_\rho\left(\rho\partial_\rho U_i\right) + \partial_z^2 U_i = 0, \quad \sum_{i=0}^{D-3} U_i = \ln\rho \tag{2.4.2}$$

Utilising this formulation in four dimensions, we can construct the Israel-Kahn solutions that represent chains of black holes supported by struts or strings, as superpositions of Schwarzschild black holes.

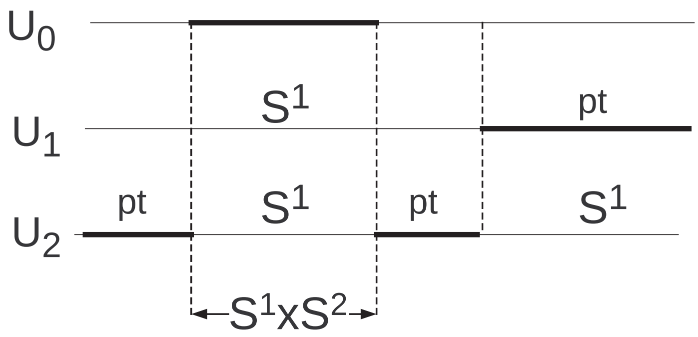
2.4.2 Static black ring solution

In five dimensions, utilising the generalised Weyl formulation, we can construct a static asymptotically flat black hole solution whose horizon has non-trivial topology S1 £ S2 : [Emparan, Reall 2002]

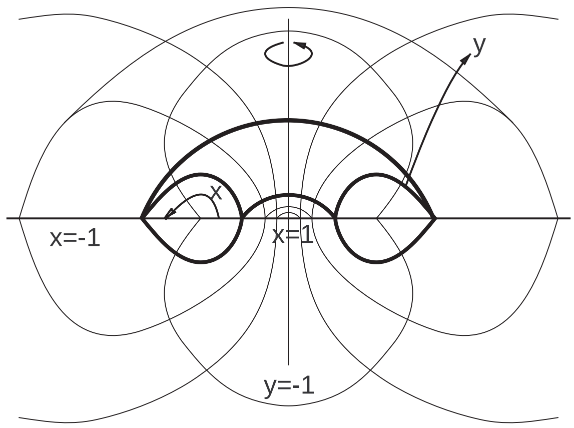
2.4.3 Rotating black ring

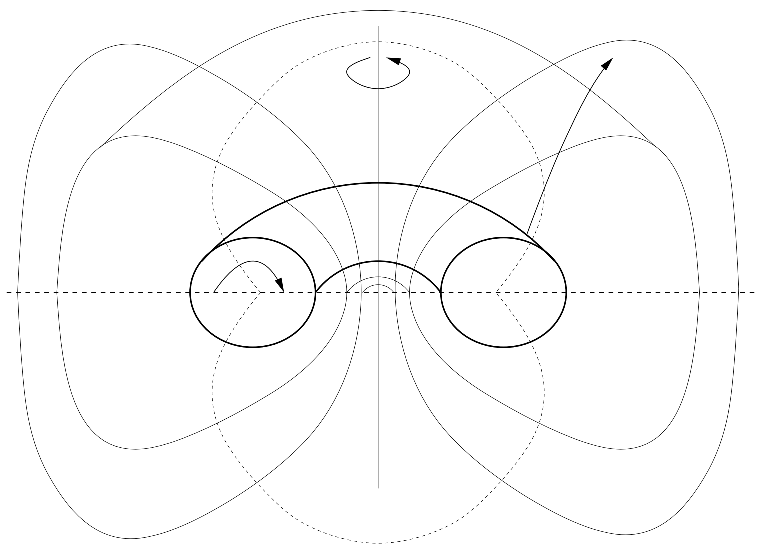
The membrane singularity of a black ring can be removed by rotation. [Emparen, Reall 2002]

- Asymptotically flat regular solution with two parameters: R, ν
- Non-trivial horizon topology: $S^1 \times S^2$



Static Black Ring





- Rotating in a special 2-plane (in the S^1 direction).

$$M = R^2 \hat{M}(\nu), \quad J_\psi = R^3 \hat{J}(\nu), \quad J_\phi = 0, \tag{2.4.3}$$

where $0 < \nu < 1$.

- Non-unique: the parameter ν can not be uniquely determined only by the asymptotic conserved ' charges ' M and J.

$$j^2 := \frac{27\pi}{32G} \frac{J^2}{M^3} = \frac{(1+\nu)^3}{8\nu} \tag{2.4.4}$$

3 Rigidity and Uniqueness

§3.1 Initial value problem

3.1.1 Spacetime Decomposition

Let the Riemannian connection of $(\mathscr{M}, g)$ be ∇, and Σ be a hypersurface in a spacetime $(\mathscr{M}, g)$ with the unit normal n.

Riemannian Connection : The Riemannian connection of a Riemannian manifold $(\mathscr{M}, g)\nabla$ is the unique connection satisfying the following two conditions:

1. (**metric conditoin**) $\nabla g = 0$.
2. (**torsion-free condition**) $\nabla_X Y - \nabla_Y X = [X, Y]$.

Gauss's formula : Let n be the unit normal field on Σ. For any two vector fields X and Y tangential to Σ, we can define $D_X Y$ and $K(X, Y)$ by the orthogonal decomposition

$$\nabla_X Y = D_X Y - K(X, Y)n; \quad \nabla_X Y_{/\!/}\Sigma. \tag{3.1.1}$$

Then, D coincides with the Riemannian connection on Σ corresponding to the Riemannian metric q on Σ induced from g, and $K(X, Y)$ defines a symmetric tensor on Σ called the second fundamental form or the extrinsic curvature:

$$K(X, Y) = K(Y, X). \tag{3.1.2}$$

Weingarten's formula : For $X_{/\!/}\Sigma$, let us define a mixed-type $(1, 1)$-tensor field $K(X)$ on Σ by

$$q(K(X), Y) = K(X, Y). \tag{3.1.3}$$

Then, we have

$$\nabla_X n = \pm K(X)\ {}_{/\!/}\Sigma; \quad g(n,n) = \pm 1. \tag{3.1.4}$$

Riemann curvature tensor The curvature tensor of a linear connection ∇ is defined by

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z. \tag{3.1.5}$$

Then, for a Riemanian connection, the covariant rank 4 tensor defined by

$$R(X,Y,Z,W) = g(Z, R(X,Y)W) \tag{3.1.6}$$

satisfies the following identities:

$$R(X,Y,Z,W) = -R(X,Y,W,Z), \tag{3.1.7a}$$
$$R(X,Y,Z,W) = R(Z,W,X,Y), \tag{3.1.7b}$$
$$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0, \tag{3.1.7c}$$
$$(\nabla_W R)(X,Y,U,V) + (\nabla_U R)(X,Y,V,W) + (\nabla_V R)(X,Y,W,U) = 0. \tag{3.1.7d}$$

The third and fourth of these are called the 1st Bianchi identity and the 2nd Bianchi identity, respectively.

Gauss-Codazzi equation : From the decomposition of $\nabla_X Y$, for vector fields X, Y, Z tangent to Σ, we have

$$\begin{aligned} \tilde{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= R(X,Y)Z \pm (K(X,Z)K(Y) - K(Y,Z)K(X)) \\ &\quad + [-(D_X K)(Y,Z) + (D_Y K)(X,Z)]n. \end{aligned} \tag{3.1.8}$$

These are equivalent to the following two equations:

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) \pm (K(X,W)K(Y,Z) \\ & -K(X,Z)K(Y,W)), \end{aligned} \tag{3.1.9}$$
$$\tilde{R}(X,Y,Z,n) = \pm\left((\nabla_X K)(Y,Z)) - (\nabla_Y K)(X,Z)\right). \tag{3.1.10}$$

In terms of the orthonormal basis of $\mathscr{M}$ consisiting of the orthonormal basis e_I of Σ and $e_0 = n$, these are expressed as

$$\tilde{R}_{IJKL} = R_{IJKL} \pm (K_{IL}K_{JK} - K_{IK}K_{JL}), \tag{3.1.11a}$$
$$\tilde{R}_{0IJK} = n_\mu \tilde{R}^\mu{}_{IJK} = \pm(D_K K_{IJ} - D_J K_{IK}). \tag{3.1.11b}$$

Note : For $\dim\Sigma = 2$, the curvature tensor can be alway written as

$$ {}^2R_{IJKL} = k(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK}), \tag{3.1.12} $$

and has one independent component ${}^2R_{1212} = k$.

Inparticular, when Σ is a 2-surface in E^3, from $R_{abcd} = 0$ and Gauss equation, it follows that

$$ k = K_{12}^2 - K_{11}K_{22} = \det K_{IJ}. \tag{3.1.13} $$

Hence, because the eigenvalue of K_{IJ} can be written $1/R_1, 1/R_2$ in terms of the principal curvature radii R_1 and R_2 of Σ, we obtain the famous Gauss's formula

$$ k = \frac{1}{R_1 R_2}. \tag{3.1.14} $$

3.1.2 $(n+1)$-decomposition

Metric

$$ ds^2 = -N^2dt^2 + q_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{3.1.15} $$

Extrinsic curvature The unit normal n to the $t=$ const hypersurface Σ_t is given by

$$ n = \frac{1}{N}(\partial_t - \beta^i\partial_i). \tag{3.1.16} $$

Then, by putting $T = Nn$,

$$ \begin{aligned} K(X,Y) &= \pm\frac{1}{N}g(\nabla_X T, Y) = \pm\frac{1}{N}g([X,T] + \nabla_T X, Y) \\ &= \pm\frac{1}{2N}(\pounds_T g)(X,Y). \end{aligned} \tag{3.1.17} $$

Hence, we have

$$ K_{ij} = -\frac{1}{2}\pounds_n q_{ij} = \frac{1}{2N}\left(-\partial_t q_{ij} + D_j\beta_k + D_k\beta_j\right) \tag{3.1.18} $$

Constraint equations From the Gauss equation and the trace of the Codazzi equation, the $G_{n\mu}$ components of the Einstein tensor are decomposed as

$$ 2\tilde{G}_{nn} = 2\tilde{R}_{nn} \mp \tilde{R} = \mp R + K^2 - K^i_j K^j_i, \tag{3.1.19a} $$

$$ \tilde{G}_{ni} = \tilde{R}_{ni} = \pm(D_j K^j_i - D_i K). \tag{3.1.19b} $$

Hence, the corresponding components of the Einstein equations give the constraint on the initial data

$$ \begin{aligned} &R + K^2 - K^i_j K^j_i = 2\kappa^2\rho, \\ &-D_j K^j_i + D_i K = \kappa^2 J_i, \end{aligned} $$

where

$$ \rho := T_{nn}, \quad J_i := T_{nj}. \tag{3.1.20} $$

Evolution equations Similarly, the Gauss-Codazzi equation yields

$$\begin{aligned}\tilde{G}_{ij} &= \tilde{R}_{ij} + (\tilde{G}^0_0 - \tilde{R}^0_0) g_{ij}, &(3.1.21)\\ \tilde{R}_{ij} &= g^{kl}\tilde{R}_{kilj} + \tilde{R}^0{}_{i0j}\\ &= R_{ij} \pm (K_{ik}K^k_j - KK_{ij}) + \tilde{R}^0{}_{i0j}. &(3.1.22)\end{aligned}$$

Here, the component $\tilde{R}^0{}_{i0j}$ can be written in terms of $T = \partial_t - \beta^i\partial_i$ as

$$\tilde{R}^0{}_{i0j} = -\frac{1}{N}\text{\pounds}_T K_{ij} \pm K_{ik}K^k_j - \frac{1}{N}D_iD_jN, \quad (3.1.23)$$

$$g^{ij}\text{\pounds}_T K_{ij} = \text{\pounds}_T K \pm 2NK^i_jK^j_i. \quad (3.1.24)$$

Therefore,

$$\begin{aligned}\tilde{R}_{ij} &= R_{ij} \pm (2K_{ik}K^k_j - KK_{ij}) - \frac{1}{N}\text{\pounds}_T K_{ij} - \frac{1}{N}D_iD_jN, &(3.1.25a)\\ \tilde{R}^0_0 &= -\frac{1}{N}\text{\pounds}_T K \mp K^i_jK^j_i - \frac{1}{N}\triangle N, &(3.1.25b)\\ \tilde{R} &= R - \frac{2}{N}\text{\pounds}_T K \mp (K^2 + K^i_jK^j_i) - \frac{2}{N}\triangle N. &(3.1.25c)\end{aligned}$$

From this, we obtain

$$\begin{aligned}\tilde{G}_{ij} &= G_{ij} \pm \left[2K^k_iK_{kj} - KK_{ij} + \frac{1}{2}(K^2 + K^k_lK^l_k)g_{ij}\right]\\ &\quad -\frac{1}{N}(\text{\pounds}_T K_{ij} - g_{ij}\text{\pounds}_T K) - \frac{1}{N}D_iD_jN + \frac{1}{N}\triangle N g_{ij}, &(3.1.26)\end{aligned}$$

where, in the expression $(\text{\pounds}_T K)_{ij} = \partial_t K_{ij} - (\text{\pounds}_\beta K)_{ij}$,

$$(\text{\pounds}_\beta K)_{ij} = (D_\beta K)_{ij} + K_{ik}D_j\beta^k + K_{jk}D_i\beta^k. \quad (3.1.27)$$

Thus, the spatial components of the Einstein equations provide the 1-st order evolution equation system for $(q_{ij}.K^i_j)$:

$$\frac{1}{N}\text{\pounds}_T q_{ij} = -2q_{ik}\left(\hat{K}^k_j + \frac{K}{d}\delta^k_j\right), \quad (3.1.28a)$$

$$\frac{1}{N}\partial_T K = \frac{1}{2}K^2 + \frac{d}{2(d-1)}\hat{K}^2 + \frac{d-2}{2(d-1)}{}^dR - \frac{\triangle N}{N} + \frac{\kappa^2}{d-1}q^{lm}T_{lm}, \quad (3.1.28b)$$

$$\begin{aligned}\frac{1}{N}\text{\pounds}_T \hat{K}^i_j &= K\hat{K}^i_j + {}^dR^i_j - \frac{{}^dR^l_l}{d}\delta^i_j - \frac{1}{N}\left(D^iD_jN - \frac{\triangle N}{d}\delta^i_j\right)\\ &\quad -\kappa^2\left(q^{ik}T_{kj} - \frac{q^{lm}T_{lm}}{d}\delta^i_j\right). &(3.1.28c)\end{aligned}$$

Here,

$$\text{\pounds}_T q_{ij} = \partial_t q_{ij} - D_i\beta_j - D_j\beta_i, \quad (3.1.29a)$$

$$\text{\pounds}_T K^i_j = \partial_t K^i_j - D_\beta K^i_j - K^i_lD_j\beta^l + K^l_jD_l\beta^i. \quad (3.1.29b)$$

and

$$K^i_j = \hat{K}^i_j + \frac{K}{d}\delta^i_j. \quad (3.1.30)$$

§3.2 Positive Energy Theorem

【**Definition 3.2.1** (Spatially asympotitically flat)】 A spatial metric q_{ij} on a 3-manifold $\approx \mathbb{R}^3 - C$ is spatially asymptotically flat if it has the asymptotic behavior at infinity

$$q_{ij} = \left(1 + \frac{M}{r}\right)^4 \delta_{ij} + p_{ij}, \tag{3.2.1a}$$

$$p_{ij} = \mathrm{O}\left(\frac{1}{r^2}\right), \; Dp_{ij} = \mathrm{O}\left(\frac{1}{r^3}\right), \; DDp_{ij} = \mathrm{O}\left(\frac{1}{r^4}\right). \tag{3.2.1b}$$

The constant M in this expression is called the ADM mass. □

【**Definition 3.2.2** (dominant energy condition)】 The energy-monentum tensor T_{ab} satisfies the dominant energy condition if $T_{ab}V^b$ is past-directed time-like vector for any future-directed time-like vector V. This condition is equivalent to

$$\rho \geq 0, \quad \rho^2 \geq J^2. \tag{3.2.2}$$

□

【**Theorem 3.2.3** (Schoen-Yau 1979; Witten 1981)】 Let us consider the initial value problem in GR for the initial surface Σ that satisfies the condition

$$\Sigma - C \approx \cup_j N_j, \quad N_j \approx \mathbb{R}^3 - C,$$

with some compact sets C and C_j. Then, the ADM mass is non-negative for any solution to this initial value problem that is spatially asymptotically flat at each end N_j, if the dominant energy condition is satisfied. Further, if the ADM mass vanishes under the same conditions, the spacetime is flat. □

§3.3 Rigidity Theorem for Static Black Holes

3.3.1 Birkhoff's Theorem

【**Theorem 3.3.1** (Original 4D version)】 A weakly AS and spherically symmetric vacuum solution to the 4D Einstein equations is always static and coincides with the (dS/adS-)Schwarzscild solution. □

【**Theorem 3.3.2** (Generalised version)】 A vacuum or electrovacuum SO$(n+1)$/SO$(n,1)$/ISO(n)-symmetric solution to the $(n+2)$-dimensional Einstein equation is always static and locally isomorphic to one of the following solutions:

- Nariai solution: $\mathscr{M} = \mathrm{dS}^2 \times S^n, \mathrm{adS}^2 \times H^n, E^{n+1,1}$.
- Black hole solution with the metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\sigma_k^2,$$
$$f(r) = k - \frac{m}{r^{n-1}} + \frac{q^2}{r^{2n-2}} - \lambda r^2$$

□

3.3.2 Rigidity theorem: 4D

【**Theorem 3.3.3** (Original version(Israel 1967)[Isr67])】 An asymptotically flat, asymptotically predictable, static solution to the 4D vacuum Einstein equations is spherically symmetric if DOC is homeomorphic to $S^2 \times \mathbb{R}^2$. □

Proof. (Outline) For a static $(n+2)$-dimensional spacetime with the metric

$$ds^2 = -N^2 dt^2 + d\sigma^2(\Sigma), \tag{3.3.1}$$

the Einstein equations read

$$R_{ij}(\Sigma) = \frac{1}{N}\nabla_i \nabla_j N, \quad \triangle N = 0. \tag{3.3.2}$$

If we express the spatial metric $d\sigma^2$ as

$$d\sigma^2 = \rho^2 dN^2 + \gamma_{ab} dz^a dz^b m, \tag{3.3.3}$$

from the Einstein equations, the extrinsic curvature of the surface $S(N)$ with constant N,

$$K_{ab} = \frac{1}{2\rho}\partial_N \gamma_{ab} = \hat{K}_{ab} + \frac{1}{n} K \gamma_{ab} \tag{3.3.4}$$

satisfies

$$\partial_N \int_{S(N)} \frac{K}{N}\rho^{1/n-1} = -\int_{S(N)} \left(\frac{n-1}{n}\frac{(D\rho)^2}{\rho^2} + \hat{K}^2 \right)$$
$$\Rightarrow \quad 2n\Omega_n \left\{(n-1)M\right\}^{1-1/n} \rho_H^{-1/n} \le \int_H R, \tag{3.3.5a}$$
$$\partial_N \int_{S(N)} \left(\frac{KN}{\rho} + \frac{2n}{n-1}\frac{1}{\rho^2} \right) = -\frac{N}{n-1}\int_{S(N)} \left(R + (n-1)\frac{(D\rho)^2}{\rho^2} + n\hat{K}^2 \right)$$
$$\Rightarrow \quad \int_0^1 dN\, N \int_{S(N)} R \le \frac{2nA_H}{\rho_H^2}, \quad M = \frac{A_H}{(n-1)\Omega_n \rho_H} \tag{3.3.5b}$$

In particular for $n = 2$ ($D = 4$) and $S(N) \approx S^2$, from the Gauss-Bonnet theorem, we obtain

$$\int_{S(N)} R = 8\pi \Rightarrow \quad 4\pi \leq \frac{4A_H}{\rho_H^2} = \frac{16\pi M}{\rho_H} \leq 4\pi \Rightarrow \quad D\rho = \hat{K} = 0 \qquad (3.3.6)$$

This implies that $S(N)$ is isometric to a Euclidean 2-sphere, and the spacetime has SO(3) symmetry. □

3.3.3 Rigidity theorem: High D

【**Theorem 3.3.4** (Bunting, Masood-ul-Alam 1987[BMuA87]; Hwang S 1998[Hwa98]; Gibbons, Ida, Shiromizu 2002[GIS03])】 An asymptotically flat, asymptotically predictable, static solution to the electrovacu Einstein equations in four and higher dimensions is spherically symmetric if horizon is non-degenerate. □

Outline of proof

- **Basic equations**

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j, \qquad (3.3.7)$$

$$\triangle V = 0; \quad V = 1 - \frac{2M}{r^{n-1}} + \mathrm{O}\left(\frac{1}{r^n}\right), \ (n = D - 2) \qquad (3.3.8)$$

$$R_{ij} - \frac{1}{V} D_i D_j V = 0; \quad g_{ij} = \left(1 + \frac{4M}{r^{n-1}}\right) \delta_{ij} + \mathrm{O}\left(\frac{1}{r^n}\right). \qquad (3.3.9)$$

- **Conformal trf**

$\Sigma^\pm : \tilde{g}_{ij}^\pm = \Omega_\pm^2 g_{ij}; \quad \Omega_\pm = \left(\frac{1 \pm V}{2}\right)^2$

$\Rightarrow$ $\tilde{\Sigma} = \Sigma^+ \cup \Sigma^- \cup p$: regular

Asymptotically flat, zero mass and $\tilde{R} = 0$

$\Rightarrow$ $(\tilde{\Sigma}, \tilde{g}_{ij})$: flat (PET)

$\Rightarrow$ Horizon is connected.

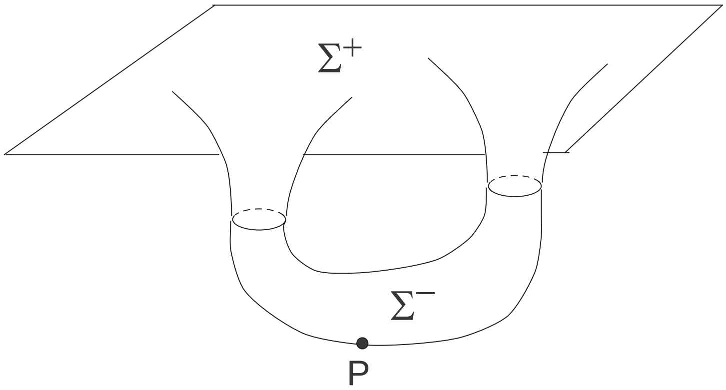
- **Conformally flat $\Rightarrow$ spherically symmetric**

4D case：

Bach tensor $R_{ijk} := D_k R_{ij} - D_j R_{ik} + \frac{1}{4} g_{ik} D_j R - \frac{1}{4} g_{ij} D_k R = 0$

$0 = R_{ijk} R^{ijk} = 4V^{-4} \rho^{-4} (2\rho^2 \hat{K}_{AB} \hat{K}^{AB} + D^A \rho D_A \rho);$

$\rho^{-1} := (DV)^2$



$\Rightarrow$ $V = \text{const}$ is spherically symmetric.

$D \geq 5$ case：

Horizon is a totally umbilic surface in a Euclidean space Σ^+

$\Rightarrow$ The horizon is a Eclidean sphere in Σ^+(rigidity theorem)

Ω_+ is harmonic and constant on the horizon sphereΣ^+

$\Rightarrow$ The spherically symmmetric solution is the unique solution,

§3.4
Uniqueness Theorem: 4D Einstein-Maxwell

3.4.1 Outline

【**Theorem 3.4.1** (4D rigidity (Hawking 1973))】 A stationary rotating black hole solution to the Einstein-Maxwell system is axisymmetric if the horizon is a connected analytic submanifold homeomorphic to $S^2 \times \mathbb{R}$. □

From this theorem, the black hole metric can be written

$$g = -e^{2U}(dt + Ad\phi)^2 + e^{-2U}\left(\rho^2 d\phi^2 + e^{2k}(d\rho^2 + dz^2)\right) \tag{3.4.1}$$

For this metric, the system can be reduced to a non-linear σ model with the help of the Ernst formalism. This leads to

【**Theorem 3.4.2** (4D BH uniqueness (Carter1972; Robinson ; Mazur 1982; Chrusciel 1996))】 An asymptotically flat and predictable stationary rotating black hole solution to the Einstein-Maxwell system is uniquelly determined by its mass, angular momentum and charge and described by the corresponding Kerr-Newman solution if the horizon is a connected analytic submanifold. □

3.4.2 Non-linear σ model

Ernst formalism

- **Geroch decomposition of the metric** When there is a Killing vector ξ, the spacetime metric can be written

$$ds^2 = e^{-2U}\left(\gamma_{_\Sigma} - \epsilon\xi_* \otimes \xi_*\right) \tag{3.4.2}$$

where $N = g(\xi,\xi) = -\epsilon e^{2U}$.

- **Compolex potential** The complex electromagnetic tensor

$$\mathscr{F} := F + i *F, \tag{3.4.3}$$

can te written in terms of a complex potential Φ as

$$\pounds_\xi F = 0 \Rightarrow \quad d\Phi = I_\xi \mathscr{F}. \tag{3.4.4}$$

- **Ernst potential** In terms of the rotation of ξ,

$$\omega := *(\xi_* \wedge d\xi_*) \tag{3.4.5}$$

the Ernst potential is defined as

$$d\mathscr{E} = -dN + i\omega - 2\bar{\Phi}d\Phi. \tag{3.4.6}$$

Embeding into a non-linear σ model

- **Kinnersley vector**

$$v = \frac{1}{2|N|^{1/2}}\left(\mathscr{E} - 1, \mathscr{E} + 1, 2\Phi\right) \tag{3.4.7}$$

In terms of this vector, we define the matrix

$$X \in \mathrm{SU}(1,2) \cap H(3), \tag{3.4.8}$$

as

$$X_{ab} = \eta_{ab} + 2\epsilon\bar{v}_a v_b. \tag{3.4.9}$$

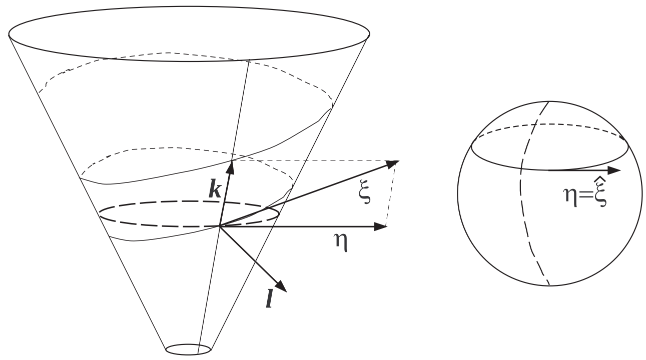
- $\mathrm{SU}(1,2)/S(\mathrm{SU}(2)\times \mathrm{U}(1))$-$\sigma$ **model** The action for X defined by

$$J := X^{-1}dX, \tag{3.4.10}$$

$$S = \frac{1}{2}\int_{(\Sigma,\gamma)} *\mathrm{Tr}(J\cdot J) \tag{3.4.11}$$

leads to the equation equivalent to the Ernst equation:

$$\delta S = 0 \quad \Leftrightarrow \quad d*J = 0. \tag{3.4.12}$$



Mazur identity Let X_1 and X_2 be two solutions to the above equation, and J_1 and J_2 be corresponding currents. Then, we obtain the inequality

$$-d^\dagger d \mathrm{Tr}\Psi = \mathrm{Tr}\left(X_{(1)}^{-1}\Delta J^\dagger, X_{(2)}\Delta J\right) \geq 0 \tag{3.4.13}$$

where

$$\Psi = X_2 X_1^{-1} - 1, \quad \Delta J = J_2 - J_1 \tag{3.4.14}$$

§3.5 Rigidity Theorem in General Dimensions

【**Theorem 3.5.1** (Hollands S, Ishibashi A, Wald R 2007 [HIW07])】 Stationary rotating analytic solution is axisymmetric if the horizon is a connected analytic submanifold. □

Proof. (Outline)

- Let k be the null tangent of $\mathscr{H}^+$ and $\mathscr{S}$ be the orbit space of null geodesic generators. $p : \mathscr{H}^+ \to \mathscr{S}$.
- If the BH is rotating, because the time-translation Killing vector ξ is spacelike on the horizon, $\xi' = p_*\xi$ becomes a non-trivial Killing vector on $\mathscr{S}$.
- When the orbit of ξ' is not ergodic, each orbit is a simple closed curve, and there exists a coordinate ϕ such that $\xi' = \Omega_h\partial_\phi$. Hence, for an appropriate $f(\phi)$, the infinitesimal transformation $\eta = \xi - fk$ preserves the structure of $\mathscr{H}^+$ has closed trajectories on $\mathscr{H}^+$.
- When the spacetime and $\mathscr{H}^+$ is analytic, we can show that η can be analytically extended to a rotational Killing vector on the whole spacetime satisfying $[\xi, \eta] = 0$ with the helps of the Einstein equations.

- Even when ξ' has ergotic orbits, we can show that the same arguments hold by taking another appropriate Killing vector because $\mathscr{S}$ has higher symmetries.

□

§3.6
Spacetime Topology

3.6.1 Topological censorship

【Theorem 3.6.1 (Topological Censorship Theorem (Friedman, Schleich, Witt 1993)[FSW93])**】** If the (average) null strong energy condition holds, each DOC of an asymptotically flat and future predictable spacetime $\mathscr{M}$ is simply connected. □

Proof. (outline) Let C be a causal curve connecting $\mathscr{I}^+$ and $\mathscr{I}^-$ in a DOC of $\mathscr{M}$. Suppose that $\mathscr{M}$ is not simply connected. Then, C can be chosen to be a curve that cannot be continuously deformed into a neighborhood of $\mathscr{I}$.

Let $\hat{\mathscr{M}}$ be the universal covering of $\mathscr{M}$, $\hat{C}$ be the lift of C to $\hat{\mathscr{M}}$. Then, $\hat{C}$ connects two points p and q that belongs to infinities on different covering sheets $p \in \mathscr{I}_1^+ \subset \bar{\mathscr{M}}_1$ and $q \in \mathscr{I}_2^- \subset \bar{\mathscr{M}}_2$.

Let S be a sphere that crosses $\hat{C}$ in a neighborhoof of q. Then, S is a strong outer trapped surface with respect to $\mathscr{I}_1^+$. At the sametime, there should exist a null geodesic generator γ of $\partial J^+(S)$ that terminates at $\mathscr{I}_1^+$.

This contradicts the fact that there exists a pair of conjugate points on any infinitly long null geodesic orthogonal to a strongly outer trapped surface if the (average) null strong energy condition holds. □

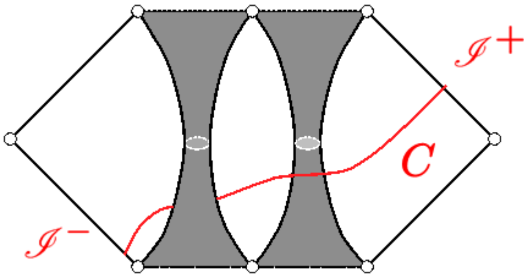
【Theorem 3.6.2 (Galloway GJ, Schleich K, Witt DM, Woolgar E 1999[GSWW99])**】** Let D be the domain of outer communications with respect to $\mathscr{I}$, and assume the following conditions hold.

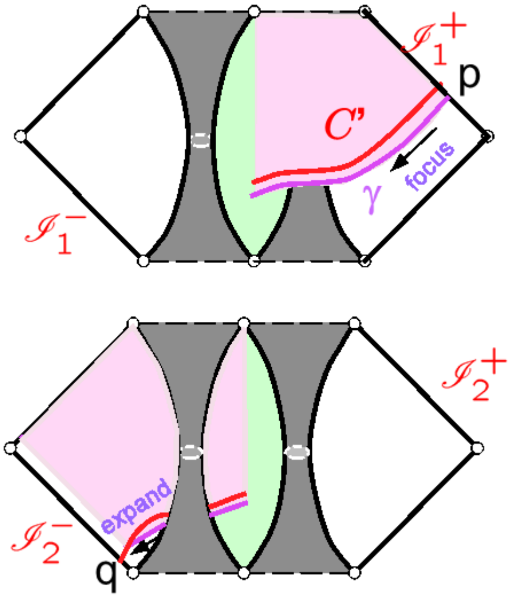
i) $D' = D \cup \mathscr{I}$ is globally hyperbolic.

ii) $\mathscr{I}$ admits a compact spacelike cut.

iii) For each point p in $\mathscr{M}$ near $\mathscr{I}$ and any future complete null geodesic $s \to \eta(s)$ in D starting at p, $\int_0^\infty \mathrm{Ric}(\eta', \eta') ds \geq 0$.

Then the PTC holds on D. □





3.6.2 Horizon topology

【**Definition 3.6.3** (Yamabe constant and type)】 Let $\mathscr{C}(\mathscr{M})$ be the set of all conformal classes of Riemannian metrics on $\mathscr{M}^n$.

Yamabe functional $F : C(\in \mathscr{C}(\mathscr{M})) \to \mathbb{R}$:

$$F(g) = \int_{\mathscr{M}} \frac{R_g d\mu_g}{(\mathscr{M}, g)^{(n-2)/n}}. \tag{3.6.1}$$

Yamabe constant $Y(\mathscr{M}, C) = \inf_{g \in C} F(g)$ (finite).

Yamabe metric g st. $F(g) = Y(\mathscr{M}, C)$.

Yamabe invariant $Y(\mathscr{M}) = \sup_{C \in \mathscr{C}(\mathscr{M})} Y(\mathscr{M}, C)$.

□

【**Theorem 3.6.4** (Yamaba conjecture (Yamabe H 1960; Trudinger NS, Aubin T 1987; Schoen R 1984; Schoen R, Yau T 1988))】 $Y(\mathscr{M}, C) \leq Y(S^n, C_0)$ and the equality holds only when $(\mathscr{M}, C) = (S^n, C_0)$. Further, Each conformal class C contains a Yamabe metric. □

【**Theorem 3.6.5**】 If the dominant energy condition is satisfied, the outmost marginally ourter trapped surface has the positive Yamabe type [Galloway GJ, Schoen R 2005(gr-qc/0509107)] □

【**Proposition 3.6.6**】 From the topological censorship theorem, horizon is cobordant to a sphere by a simply connetect manifold, and as a consequence, has vanishing Pontrjagin number and Stiefel-Whitney number. □

This theorem provides some constraints on possible topology of horizon of 5 or 6 dimensional spacetime.

5D: $\mathscr{H}$ is homeomorphic to S^3 or $S^2 \times S^1$.[Galloway, Schoen 2005; Helfgott C, Oz Y, Yanay Y 2006[HOY06]: hep-th/0509013].

6D If $\mathscr{H}$ is simply connected, it is homeomorphic to S^4 or $S^2 \times S^2$. This was proved using the classification of 4D manifolds in terms of the intersection form by Friedmann and Quinn and the fact that the signature of 4D manifold is cobordism invariant (Helfgott, Oz, Yanay 2005 [HOY06])

4 Gauge-invariant Perturbation Theory

In this section, we explain the basic idea and techniques of the gauge-invariant formulation of perturbations[Bar80, KS84] for a class of background spacetimes that includes static black hole spacetimes as special case. This chapter is based on my lecture note in the 4-th Agean Summer School, Hideo Kodama:Lect. Note Phys. 369: 427 (2009).

§4.1 Background Solution

We assume that a background spacetime can be locally written as the warped product of a m-dimensional spacetime $\mathscr{N}$ and an n-dimensional Einstein space $\mathscr{K}$ as

$$M^{n+m} \approx \mathscr{N} \times \mathscr{K} \ni (z^M) = (y^a, x^i) \tag{4.1.1}$$

and has the metric (2.1.2)

$$ds^2 = g_{MN} dz^M dz^N = g_{ab}(y) dy^a dy^b + r(y)^2 d\sigma_n^2, \tag{4.1.2}$$

where $d\sigma_n^2 = \gamma_{ij} dx^i dx^j$ is an n-dimensional Einstein metric on $\mathscr{K}$ satisfying the condition

$$\hat{R}_{ij} = (n-1) K \gamma_{ij}. \tag{4.1.3}$$

Note that for $n \leq 3$, an Einstein space is automatically a constant curvature space, while for $n > 3$, $\mathscr{K}$ does not have a constant curvature generically.

For this type of spacetimes, we can express the covariant derivative ∇_M, the connection coefficients $\Gamma^M_{NL}(z)$ and the curvature tensor $R_{MNLS}(z)$ in terms of the corresponding quantities for $\mathscr{N}^m$ and $\mathscr{K}^n$. We denote them D_a, ${}^m\Gamma^a_{bc}(y)$, ${}^mR_{abcd}(y)$

and $\hat{D}_i, \hat{\Gamma}^i_{jk}(x), \hat{R}_{ijkl}(x)$ respectively. For example, the curvature tensor can be expressed as

$$R^a{}_{bcd} = {}^m R^a{}_{bcd}, \quad R^i{}_{ajb} = -\frac{D_a D_b r}{r}\delta^i_j, \quad R^i{}_{jkl} = {}^m R_{abcd} - (Dr)^2(\delta^i_k \gamma_{jl} - \delta^i_l \gamma_{jk}), \tag{4.1.4}$$

and the non-vanishing components of the Einstein tensor are given by

$$G_{ab} = {}^m G_{ab} - \frac{n}{r} D_a D_b r - \left[\frac{n(n-1)}{2} \frac{K - (Dr)^2}{r^2} - \frac{n}{r} \Box r \right] g_{ab} \tag{4.1.5a}$$

$$G^i_j = \left[-\frac{1}{2} {}^m R - \frac{(n-1)(n-2)}{2} \frac{K - (Dr)^2}{r^2} + \frac{n-1}{r} \Box r \right] \delta^i_j. \tag{4.1.5b}$$

From this and the Einstein equations $G_{MN} + \Lambda g_{MN} = \kappa^2 T_{MN}$, it follows that the energy-momentum tensor of the background solution should take the form

$$T_{ab} = T_{ab}(y), \ T_{ai} = 0, \ T^i_j = P(y)\delta^i_j. \tag{4.1.6}$$

4.1.1 Examples

This class of background spacetimes include quite a large variety of important solutions to the Einstein equations in four and higher dimensions.

1. **Robertson-Walker universe**: $m = 1$ and $\mathscr{K}$ is a constant curvature space.

   $$ds^2 = -dt^2 + a(t)^2 d\sigma_n^2.$$

   The gauge-invariant formulation was first introduced for perturbations of this background by Bardeen[Bar80] and applied to realistic cosmological models by the author[Kod84, Kod85, KS84].

2. **Braneworld model**: $m = 2$ (and $\mathscr{K}$ is a constant curvature space). For example, the metric of AdS^{n+2} spacetime can be written

   $$ds^2 = \frac{dr^2}{1 - \lambda r^2} - (1 - \lambda r^2)dt^2 + r^2 d\Omega_n^2. \tag{4.1.7}$$

   The gauge-invariant formulation of this background was first discussed by Mukohyama[Muk00] and then applied to the braneworld model taking account of the junction conditions by the author and collaborators[KIS00].

3. **Higher-dimensional static Einstein black holes**: $m = 2$ and $\mathscr{K}$ is a compact Einstein space. For example, for the Schwarzschild-Tangherlini black hole, $\mathscr{K} = S^n$. In general, the generalised Birkhoff theorem says[KI04] that the electrovac solutions of the form (2.1.2) with $m = 2$ to the Einstein equations are exhausted by the Nariai-type solutions such that M is the direct product of a two-dimensional constant curvature spacetime $\mathscr{N}$ and

an Einstein space $\mathscr{K}$ with $r = \text{const}$ and the black hole type solution whose metric is given by

$$ds^2 = \frac{dr^2}{f(r)} - f(r)dt^2 + r^2 d\sigma_n^2; \tag{4.1.8}$$

$$f(r) = K - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}} - \lambda r^2. \tag{4.1.9}$$

The gauge-invariant formulation for perturbations was applied to this background to discuss the stability of static black holes by the author and collaborators[KI03, IK03, KI04]. This application is explained in the next section.

4. **Black branes**: $m = 2 + k$ and $\mathscr{K}$ = Einstein space. In this case, the spacetime factor $\mathscr{N}$ is the product of a two-dimensional black hole sector and a k-dimensional brane sector:

$$ds^2 = \frac{dr^2}{f(r)} - f(r)dt^2 + d\boldsymbol{z} \cdot d\boldsymbol{z} + r^2 d\sigma_n^2. \tag{4.1.10}$$

One can also generalise this background to introducing a warp factor in front of the black hole metric part. The stability of this background for the case in which $\mathscr{K}$ is an Euclidean space is discussed in §5.5.

5. **Higher-dimensional rotating black hole** (a special Myers-Perry solution): $m = 4$ and $\mathscr{K} = S^n$.

$$ds^2 = g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + r^2\cos^2\theta d\Omega_n^2, \tag{4.1.11}$$

where all the metric coefficients are functions only of r and θ. The stability of this background was studied in [Kod07].

6. **Axisymmetric spacetime**: m is general and $n = 1$.

§4.2
Perturbations

4.2.1 Perturbation equations

In order to describe the spacetime structure and matter configuration $(\tilde{M}, \tilde{g}, \tilde{\Phi})$ as a perturbation from a fixed background (M, g, Φ), we introduce a mapping F : background $M \to \tilde{M}$, and define perturbation variables on the fixed background spacetime as follows:

$$h := \delta g = F^*\tilde{g} - g, \quad \phi := \delta\Phi = F^*\tilde{\Phi} - \Phi. \tag{4.2.1}$$

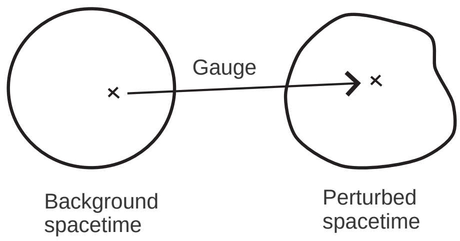


Figure 4.1: Gauge transformation

Then, if these perturbation variables have small amplitudes, the Einstein equations and the other equations for matter can be described by linearised equations well. For example, in terms of the variable $\psi_{MN} = h_{MN} - hg_{MN}/2$, the linearised Einstein equations can be written as

$$\triangle_L \psi_{MN} + \nabla_M \nabla_A \psi^A_N + \nabla_N \nabla_A \psi^A_M - \nabla^A \nabla^B \psi_{AB} g_{MN} + R^{AB} \psi_{AB} g_{MN} - R\psi_{MN} \\ = 2\kappa^2 \delta T_{MN}. \tag{4.2.2}$$

where $\triangle_L$ is the Lichnerowicz operator defined by

$$\triangle_L \psi_{MN} := -\Box \psi_{MN} + R_{MA} \psi^A_N + R_{NA} \psi^A_M - 2R_{MANB} \psi^{AB}. \tag{4.2.3}$$

4.2.2 Gauge problem

For a different mapping F', the perturbation variables defined above change their values, which has no physical meaning and can be regarded as a kind of gauge freedom. Because F and F' are related by a diffeomorphism, the corresponding changes of the variables are identical to the transformation of the variables with respect to the transformation $f = F'^{-1}F$. In the framework of linear perturbation theory, we can restrict considerations to infinitesimal changes of F. Hence, f is expressed in terms of an infinitesimal transformation ξ^M as

$$\bar{\delta} x^M = x^M(f(p)) - x^M(p) = \xi^M, \tag{4.2.4}$$

and the gauge transformations are expressed as

$$\bar{\delta} h_{MN} = -\pounds_\xi g_{MN} \equiv -\nabla_M \xi_N - \nabla_N \xi_M, \quad \bar{\delta}\phi = -\pounds_\xi \Phi. \tag{4.2.5}$$

From its origin, the perturbation equations including the linearised Einstein equations given above are invariant under this gauge transformation.

To be specific, for our background spacetime, the metric perturbation transforms as

$$\bar{\delta} h_{ab} = -D_a \xi_b - D_b \xi_a, \tag{4.2.6a}$$

$$\bar{\delta} h_{ai} = -r^2 D_a \left(\frac{\xi_i}{r^2} \right) - \hat{D}_i \xi_a, \tag{4.2.6b}$$

$$\bar{\delta} h_{ij} = -\hat{D}_i \xi_j - \hat{D}_j \xi_i - 2r D^a r \xi_a \gamma_{ij} \tag{4.2.6c}$$

and the perturbation of the energy-momentum tensor $\tau_{MN} = \delta T_{MN}$ transforms as

$$\bar{\delta}\tau_{ab} = -\xi^c D_c T_{ab} - T_{ac} D_b \xi^c - T_{bc} D_a \xi^c, \tag{4.2.7a}$$
$$\bar{\delta}\tau_{ai} = -T_{ab}\hat{D}_i \xi^b - r^2 P D_a(r^{-2}\xi_i), \tag{4.2.7b}$$
$$\bar{\delta}\tau_{ij} = -\xi^a D_a(r^2 P)\gamma_{ij} - P(\hat{D}_i\xi_j + \hat{D}_j\xi_i) \tag{4.2.7c}$$

In order to remove this gauge freedom, one of the following two approaches is adopted in general:

i) **Gauge fixing method**: this method is direct, but it is rather difficult to find relations between perturbation variables in different gauges in general.

ii) **Gauge-invariant method**: this method describe the theory only in terms of gauge-invariant quantities. Such quantities have non-local expressions in terms of the original perturbation variables in general.

These two approaches are mathematically equivalent, and a gauge-invariant variable can be regarded as some perturbation variable in some special gauge in general. Therefore, the non-locality of the gauge-invariant variables implies that the relation of two different gauges are non-local.

4.2.3 Tensorial decomposition of perturbations

In this lecture, we focus on the gauge-invariant approach to perturbations and explain that in the class of background spacetimes described above, we can locally construct fundamental gauge-invariant variables with help of harmonic expansions. This construction becomes more transparent if we decompose the perturbation variables into components of specific tensorial types. This decomposition also helps us to divide the coupled set of perturbation equations into decoupled smaller subsets, and in some cases into single master equations.

First of all, note that the basic perturbation variables h_{MN} and τ_{MN} can be classified into the following three algebraic types according to their transformation property as tensors on the n-dimensional space $\mathscr{K}$:

i) Spatial scalar: h_{ab}, τ_{ab}

ii) Spatial vector: h_{ai}, τ^a_i

iii) Spatial tensor: h_{ij}, τ^i_j

Among these, spatial vectors and tensors can be further decomposed into more basic quantities. First, we decompose a vector field v_i on $\mathscr{K}$ into a scalar field $v^{(\mathrm{s})}$ and a transverse vector $v^{(\mathrm{t})}_i$ as

$$v_i = \hat{D}_i v^{(\mathrm{s})} + v^{(\mathrm{t})}_i; \quad \hat{D}_i v^{(\mathrm{t})\mathrm{i}} = 0. \tag{4.2.8}$$

Then, from the relation

$$\hat{\triangle} v^{(\mathrm{s})} = \hat{D}_i v^i, \tag{4.2.9}$$

the component fields $v^{(\mathrm{s})}$ and $v_i^{(\mathrm{t})}$ can be uniquely determined from v_i up to the ineffective freedom in $v^{(\mathrm{s})}$ to add a constant, provided that this Poisson equation has a unique solution on $\mathscr{K}$ up to the same freedom. For example, when $\mathscr{K}$ is compact and closed, this condition is satisfied.

Next, we decompose a symmetric tensor field of rank 2 on $\mathscr{K}$ as

$$t_{ij} = \frac{1}{n} t g_{ij} + \hat{D}_i \hat{D}_j s - \frac{1}{n} \hat{\triangle} s g_{ij} + \hat{D}_i t_j + \hat{D}_j t_i + t_{ij}^{(\mathrm{tt})}; \tag{4.2.10a}$$

$$\hat{D}_i t^i = 0, \quad t_i^{(\mathrm{tt})\mathrm{i}} = 0, \quad \hat{D}_i t_j^{(\mathrm{tt})\mathrm{i}} = 0. \tag{4.2.10b}$$

Here, t is uniquely determined as $t = t_i^i$. Further, from the relations derived from this definition,

$$\hat{\triangle}(\hat{\triangle} + nK)s = \frac{n}{n-1}\left(\hat{D}_i \hat{D}_j t^{ij} - \frac{1}{n}\hat{\triangle} t\right), \tag{4.2.11a}$$

$$[\hat{\triangle} + (n-1)K] t^i = (\delta_j^i - \hat{D}^i \hat{\triangle}^{-1} \hat{D}_j)(\hat{D}_m t^{jm} - n^{-1} \hat{D}^j t), \tag{4.2.11b}$$

s and t_i, hence $t_{ij}^{(\mathrm{tt})}$, can be uniquely determined from t_{ij} up to the addition of ineffective zero modes, provided that these Poisson equations have solutions unique up to the same ineffective freedom.

After these decompositions of vectors and tensors to basic components, we can classify these components into the following three types:

i) Scalar type: $v^i = \hat{D}^i v^{(\mathrm{s})}$, $t_{ij} = \frac{1}{n} t g_{ij} + \hat{D}_i \hat{D}_j s - \frac{1}{n} \hat{\triangle} s g_{ij}$.

ii) Vector type: $v_i = v_i^{(\mathrm{t})}$, $t_{ij} = \hat{D}_i t_j + \hat{D}_j t_i$.

iii) Tensor type: $v^i = 0$, $t_{ij} = t_{ij}^{(\mathrm{tt})}$.

We call these types *reduced tensorial types*. In the linearised Einstein equations, through the covariant differentiation and tensor-algebraic operations, quantities of different algebraic tensorial types can appear in each equation. However, in the case in which $\mathscr{K}$ is a constant curvature space, perturbation variables belonging to different reduced tensorial types do not couple in the linearised Einstein equations if we decompose these perturbation equations into reduced tensorial types as well, because there exists no quantity of the vector or the tensor type in the background except for the metric tensor. The same result holds even in the case in which $\mathscr{K}$ is an Einstein space with non-constant curvature, because the only non-trivial background tensor other than the metric is the Weyl tensor that can only transform a 2nd rank tensor to a 2nd rank tensor.

Here, note that gauge transformations can be also decomposed into reduced tensorial types, and the gauge transformation of each type affects only the decomposed perturbation variables of the same reduced tensorial type. Hence, gauge-invariant variables can be constructed in each reduced tensorial types independently.

§4.3
Tensor Perturbation

Let us start from the tensor-type perturbation, for which the argument is simplest.

4.3.1 Tensor Harmonics

We utilise tensor harmonics to expand tensor-type perturbations. They are defined as the basis for 2nd-rank symmetric tensor fields satisfying the following eigenvalue problem:

$$(\hat{\triangle}_L - \lambda_L)\mathbb{T}_{ij} = 0; \quad \mathbb{T}^i_i = 0, \quad \hat{D}_j \mathbb{T}^j_i = 0, \tag{4.3.1}$$

where $\hat{\triangle}_L$ is the Lichnerowicz operator on $\mathscr{K}$ defined by

$$\hat{\triangle}_L h_{ij} := -\hat{D}\cdot\hat{D}h_{ij} - 2\hat{R}_{ikjl}h^{kl} + 2(n-1)Kh_{ij}. \tag{4.3.2}$$

When $\mathscr{K}$ is a constant curvature space, this operator is related to the Laplace-Beltrami operator by

$$\hat{\triangle}_L = -\hat{\triangle} + 2nK, \tag{4.3.3}$$

and, $\mathbb{T}_{ij}$ satisfies

$$(\hat{\triangle} + k^2)\mathbb{T}_{ij} = 0; \quad k^2 = \lambda_L - 2nK. \tag{4.3.4}$$

We use k^2 in the meaning of $\lambda_L - 2nK$ from now on when $\mathscr{K}$ is an Einstein space with non-constant sectional curvature.

The harmonic tensor has the following basic properties:

1. **Identities**: Let T_{ij} be a symmetric tensor of rank 2 satisfying

   $$T^i_i = 0, \quad D^j T_{ij} = 0.$$

   Then, the following identities hold:

   $$2D_{[i}T_{j]k}D^{[i}T^{j]k} = 2D^i(T_{jk}D^{[i}T^{j]k}) + T_{jk}\left[-\triangle T^{jk} + R^j_l T^{lk} + R_i{}^{jk}{}_l T^{il}\right],$$
   $$2D_{(i}T_{j)k}D^{(i}T^{j)k} = 2D^i(T_{jk}D^{(i}T^{j)k}) + T_{jk}\left[-\triangle T^{jk} - R^j_l T^{lk} - R_i{}^{jk}{}_l T^{il}\right].$$

   On the constant curvature space with sectional curvature K, these identities read

   $$2D_{[i}T_{j]k}D^{[i}T^{j]k} = 2D^i(T_{jk}D^{[i}T^{j]k}) + T_{jk}(-\triangle + nK)T^{jk},$$
   $$2D_{(i}T_{j)k}D^{(i}T^{j)k} = 2D^i(T_{jk}D^{(i}T^{j)k}) + T_{jk}(-\triangle - nK)T^{jk}.$$

2. **Spectrum**: When $\mathscr{K}$ is a compact and closed space with constant sectional curvature K, these identities lead to the following condition on the spectrum of k^2:

$$k^2 \geq n|K|. \tag{4.3.5}$$

   In contrast, when $\mathscr{K}$ is not a constant curvature space, no general lower bound on the spectrum k^2 is known.

3. When $\mathscr{K}$ is a two-dimensional surface with a constant curvature K, a symmetric 2nd-rank harmonic tensor that is regular everywhere can exist only for $K \leq 0$: for T^2 $(K = 0)$, the corresponding harmonic tensor T_{ij} becomes a constant tensor in the coordinate system such that the metric is written $ds^2 = dx^2 + dy^2 (k^2 = 0)$; for $H^2/\Gamma(K = -1)$, a harmonic tensor corresponds to an infinitesimal deformation of the moduli parameters.

4. For $\mathscr{K} = S^n$, the spectrum of k^2 is given by

$$k^2 = l(l + n - 1) - 2; \quad l = 2, 3, \cdots , \tag{4.3.6}$$

4.3.2 Perturbation equations

The metric and energy-momentum perturbations can be expanded in terms of the tensor harmonics as

$$h_{ab} = 0, \quad h_{ai} = 0, \quad h_{ij} = 2r^2 H_T \mathbb{T}_{ij}, \tag{4.3.7}$$

$$\tau_{ab} = 0, \ \tau^a_i = 0, \ \tau^i_j = \tau_T \mathbb{T}^i_j. \tag{4.3.8}$$

Since the coordinate transformations contain no tensor-type component, H_T and τ_T are gauge invariant by themselves:

$$\xi^M = \bar{\delta} z^M = 0; \quad \bar{\delta} H_T = 0, \ \bar{\delta} \tau_T = 0. \tag{4.3.9}$$

Only the (i, j)-component of the Einstein equations has the tensor-type component:

$$-\Box H_T - \frac{n}{r} Dr \cdot DH_T + \frac{k^2 + 2K}{r^2} H_T = \kappa^2 \tau_T. \tag{4.3.10}$$

Here, $\Box = D^a D_a$ is the D'Alembertian in the m-dimensional spacetime $\mathscr{N}$. Thus, the Einstein equations for tensor-type perturbations can be always reduced to the single master equation on our background spacetime.

§4.4

Vector Perturbation

4.4.1 Vector harmonics

We expand transverse vector fields in terms of the complete set of harmonic vectors defined by the eigenvalue problem

$$(\hat{\triangle} + k^2)\mathbb{V}_i = 0; \quad \hat{D}_i \mathbb{V}^i = 0. \tag{4.4.1}$$

Tensor fields of the vector-type can be expanded in terms of the harmonic tensors derived from these vector harmonics as

$$\mathbb{V}_{ij} = -\frac{1}{2k}(\hat{D}_i \mathbb{V}_j + \hat{D}_j \mathbb{V}_i). \tag{4.4.2}$$

They satisfy

$$\left[\hat{\triangle} + k^2 - (n+1)K\right] \mathbb{V}_{ij} = 0, \tag{4.4.3a}$$

$$\mathbb{V}^i_i = 0, \quad \hat{D}_j \mathbb{V}^j_i = \frac{k^2 - (n-1)K}{2k} \mathbb{V}_i. \tag{4.4.3b}$$

Here, there is one subtle point; $\mathbb{V}_{ij}$ vanishes when $\mathbb{V}_i$ is a Killing vector. For this mode, from the above relations, we have $k^2 = (n-1)K$. We will see below that the converse holds when $\mathscr{K}$ is compact and closed. We call these modes *exceptional modes.*

Now, we list up some basic properties of the vector harmonics relevant to the subsequent discussions.

1. **Spectrum**: In an n-dimensional Einstein space $\mathscr{K}$ satisfying $R_{ij} = (n-1)Kg_{ij}$, we have

$$2D_{[i}V_{j]}D^{[i}V^{j]} = 2D_i(V_j D^{[i}V^{j]}) + V_j\left[-\triangle + (n-1)K\right]V^j, \tag{4.4.4a}$$

$$2D_{(i}V_{j)}D^{(i}V^{j)} = 2D_i(V_j D^{(i}V^{j)}) + V_j\left[-\triangle - (n-1)K\right]V^j. \tag{4.4.4b}$$

When $\mathscr{K}$ is compact and closed, from the integration of these over $\mathscr{K}$, we obtain the following general restriction on the spectrum of k^2:

$$k^2 \geq (n-1)|K|. \tag{4.4.5}$$

Here, when the equality holds, the corresponding harmonic vector becomes a Killing vector for $K \geq 0$ and a harmonic 1-form for $K \leq 0$, respectively.

2. For $\mathscr{K}^n = S^n$, we have

$$k^2 = \ell(\ell + n - 1) - 1, \quad (\ell = 1, 2, \cdots). \tag{4.4.6}$$

Here, the harmonic vector field $\mathbb{V}_i$ becomes a Killing vector for $l = 1$ and is exceptional.

3. For $K = 0$, the exceptional mode exists only when $\mathscr{K}$ is isometric to $T^p \times \mathscr{C}^{n-p}$, where $\mathscr{C}^{n-p}$ is a Ricci flat space with no Killing vector.

4.4.2 Perturbation equations

Vector perturbations of the metric and the energy-momentum tensor can be expanded in terms of the vector harmonics as

$$h_{ab} = 0, \quad h_{ai} = r f_a \mathbb{V}_i, \quad h_{ij} = 2r^2 H_T \mathbb{V}_{ij}, \tag{4.4.7a}$$

$$\tau_{ab} = 0,\ \tau^a_i = r\tau_a \mathbb{V}_i,\ \tau^i_j = \tau_T \mathbb{V}^i_j. \tag{4.4.7b}$$

For the vector-type gauge transformation

$$\xi_a = 0,\ \xi_i = rL\mathbb{V}_i \tag{4.4.8}$$

the perturbation variables transform as

$$\bar{\delta} f_a = -rD_a\left(\frac{L}{r}\right),\ \bar{\delta} H_T = \frac{k}{r}L,\ \bar{\delta}\tau_a = 0,\ \bar{\delta}\tau_T = 0. \tag{4.4.9}$$

Hence, we adopt the following combinations as the fundamental gauge-invariant variables for the vector perturbation:

$$\text{generic modes:} \quad \tau_a,\ \tau_T,\ F_a = f_a + \frac{r}{k}D_a H_T \tag{4.4.10}$$

$$\text{exceptional modes:} \quad \tau_a,\ F^{(1)}_{ab} = rD_a\left(\frac{f_b}{r}\right) - rD_b\left(\frac{f_a}{r}\right) \tag{4.4.11}$$

Note that for exceptional modes, $F_a = f_a$ because H_T is not defined.

The reduced vector part of the Einstein equations come from the components corresponding to G^a_i and G^i_j. In terms of the gauge-invariant variables defined above, these equations can be written as follows.

- **Generic modes**:

$$\frac{1}{r^{n+1}}D^b\left(r^{n+1}F^{(1)}_{ab}\right) - \frac{k^2-(n-1)K}{r^2}F_a = -2\kappa^2\tau_a, \tag{4.4.12a}$$

$$\frac{k}{r^n}D_a(r^{n-1}F^a) = -\kappa^2\tau_T. \tag{4.4.12b}$$

- **Exceptional modes**: $k^2 = (n-1)K > 0$. For these modes, the second of the above equations coming from G^i_j does not exist.

$$\frac{1}{r^{n+1}}D^b\left(r^{n+1}F^{(1)}_{ab}\right) = -2\kappa^2\tau_a. \tag{4.4.13}$$

§4.5
Scalar Perturbation

4.5.1 Scalar harmonics

Scalar functions on $\mathscr{K}$ can be expanded in terms of the harmonic functions defined by

$$(\hat{\triangle} + k^2)\mathbb{S} = 0. \tag{4.5.1}$$

Correspondingly, scalar-type vector and tensor fields can be expanded in terms of harmonic vectors $\mathbb{S}_i$ and harmonic tensors $\mathbb{S}_{ij}$ define by

$$\mathbb{S}_i = -\frac{1}{k}\hat{D}_i\mathbb{S}, \tag{4.5.2a}$$
$$\mathbb{S}_{ij} = \frac{1}{k^2}\hat{D}_i\hat{D}_j\mathbb{S} + \frac{1}{n}\gamma_{ij}\mathbb{S}. \tag{4.5.2b}$$

These harmonic tensors satisfy the following relations:

$$\hat{D}_i\mathbb{S}^i = k\mathbb{S}, \tag{4.5.3a}$$
$$[\hat{\triangle} + k^2 - (n-1)K]\mathbb{S}_i = 0, \tag{4.5.3b}$$
$$\mathbb{S}^i_i = 0, \quad \hat{D}_j\mathbb{S}^j_i = \frac{n-1}{n}\frac{k^2 - nK}{k}\mathbb{S}_i, \tag{4.5.3c}$$
$$[\hat{\triangle} + k^2 - 2nK]\mathbb{S}_{ij} = 0. \tag{4.5.3d}$$

Note that as in the case of vector harmonics, there are some exceptional modes:

i) $k = 0$: $\mathbb{S}_i \equiv 0$, $\mathbb{S}_{ij} \equiv 0$.

ii) $k^2 = nK$ $(K > 0)$: $\mathbb{S}_{ij} \equiv 0$.

For scalar harmonics, $k^2 = 0$ is obviously always the allowed lowest eigenvalue. Therefore, the information on the second eigenvalue is important. In general, it is difficult to find such information. However, when $\mathscr{K}^n$ is a compact Einstein space with $K > 0$, we can obtain a useful constraint as follows. Let us define Q_{ij} by

$$Q_{ij} := D_iD_jY - \frac{1}{n}g_{ij}\triangle Y.$$

Then, we have the identity

$$Q_{ij}Q^{ij} = D^i(D^iYD_iD_jY - YD_i\triangle Y - R_{ij}D^jY) + Y\left[\triangle(\triangle + (n-1)K)\right]Y - \frac{1}{n}(\triangle Y)^2.$$

For $Y = \mathbb{S}$, integrating this identity, we obtain the constraint on the second eigenvalue

$$k^2 \geq nK. \tag{4.5.4}$$

For $\mathscr{K}^n = S^n$, the equality holds because the full spectrum is given by

$$k^2 = \ell(\ell + n - 1), \quad (\ell = 0, 1, 2, \cdots). \tag{4.5.5}$$

4.5.2 Perturbation equations

The scalar perturbation of the metric and the energy-momentum tensor can be expanded as

$$h_{ab} = f_{ab}\mathbb{S},\ h_{ai} = r f_a \mathbb{S}_i,\ h_{ij} = 2r^2(H_L \gamma_{ij}\mathbb{S} + H_T \mathbb{S}_{ij}), \tag{4.5.6a}$$

$$\tau_{ab} = \tau_{ab}\mathbb{S},\ \tau^a_i = r\tau_a \mathbb{S}_i,\ \tau^i_j = \delta P \delta^i_j \mathbb{S} + \tau_T \mathbb{S}^i_j. \tag{4.5.6b}$$

For the scalar-type gauge transformation

$$\xi_a = T_a \mathbb{S},\ \xi_i = rL\mathbb{S}_i, \tag{4.5.7}$$

these harmonic expansion coefficients for generic modes $k^2(k^2 - nK) > 0$ of a scalar-type perturbation transform as

$$\bar{\delta} f_{ab} = -D_a T_b - D_b T_a, \quad \bar{\delta} f_a = -rD_a\left(\frac{L}{r}\right) + \frac{k}{r}T_a, \tag{4.5.8a}$$

$$\bar{\delta} H_L = -\frac{k}{nr}L - \frac{D^a r}{r}T_a, \quad \bar{\delta} H_T = \frac{k}{r}L, \tag{4.5.8b}$$

$$\bar{\delta}\tau_{ab} = -T^c D_c T_{ab} - T_{ac} D_b T^c - T_{bc} D_a T^c, \tag{4.5.8c}$$

$$\bar{\delta}\tau_a = \frac{k}{r}(T_{ab}T^b - PT_a), \quad \bar{\delta}(\delta P) = -T^a D_a P, \quad \bar{\delta}\tau_T = 0. \tag{4.5.8d}$$

From these we obtain

$$\bar{\delta} X_a = T_a; \quad X_a = \frac{r}{k}\left(f_a + \frac{r}{k}D_a H_T\right). \tag{4.5.9}$$

Hence, the fundamental gauge invariants can be given by τ_T and the following combinations:

$$F = H_L + \frac{1}{n}H_T + \frac{1}{r}D^a r X_a, \tag{4.5.10a}$$

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a, \tag{4.5.10b}$$

$$\Sigma_{ab} = \tau_{ab} + T^c_b D_a X_c + T^c_a D_b X_c + X^c D_c T_{ab}, \tag{4.5.10c}$$

$$\Sigma_a = \tau_a - \frac{k}{r}(T^b_a X_b - PX_a), \tag{4.5.10d}$$

$$\Sigma_L = \delta P + X^a D_a P. \tag{4.5.10e}$$

The scalar part of the Einstein equations comes from G_{ab}, G_{ai} and G^i_j. First, from δG_{ab}, we obtain

$$\begin{aligned}
&-\Box F_{ab} + D_a D_c F^c_b + D_b D_c F^c_a + n\frac{D^c r}{r}(-D_c F_{ab} + D_a F_{cb} + D_b F_{ca}) \\
&\quad + {}^m R^c_a F_{cb} + {}^m R^c_b F_{ca} - 2\,{}^m R_{acbd}F^{cd} + \left(\frac{k^2}{r^2} - R + 2\Lambda\right)F_{ab} - D_a D_b F^c_c \\
&\quad -2n\left(D_a D_b F + \frac{1}{r}D_a r D_b F + \frac{1}{r}D_b r D_a F\right) \\
&- \Bigg[D_c D_d F^{cd} + \frac{2n}{r}D^c r D^d F_{cd} + \left(\frac{2n}{r}D^c D^d r + \frac{n(n-1)}{r^2}D^c r D^d r\right. \\
&\quad \left. - {}^m R^{cd}\right)F_{cd} - 2n\Box F - \frac{2n(n+1)}{r}Dr\cdot DF + 2(n-1)\frac{k^2 - nK}{r^2}F \\
&\quad -\Box F^c_c - \frac{n}{r}Dr\cdot DF^c_c + \frac{k^2}{r^2}F^c_c\Bigg]g_{ab} = 2\kappa^2\Sigma_{ab}.
\end{aligned} \tag{4.5.11}$$

Second, from δG^a_i, we obtain

$$\frac{k}{r}\left[-\frac{1}{r^{n-2}}D_b(r^{n-2}F^b_a)+rD_a\left(\frac{1}{r}F^b_b\right)+2(n-1)D_aF\right]=2\kappa^2\Sigma_a. \tag{4.5.12}$$

Finally, from the trace-free part of δG^i_j, we obtain

$$-\frac{k^2}{2r^2}\left[2(n-2)F+F^a_a\right]=\kappa^2\tau_T, \tag{4.5.13}$$

and from the trace δG^i_i,

$$\begin{aligned}&-\frac{1}{2}D_aD_bF^{ab}-\frac{n-1}{r}D^arD^bF_{ab}\\&+\left[\frac{1}{2}{}^m\!R^{ab}-\frac{(n-1)(n-2)}{2r^2}D^arD^br-(n-1)\frac{D^aD^br}{r}\right]F_{ab}\\&+\frac{1}{2}\Box F^c_c+\frac{n-1}{2r}Dr\cdot DF^c_c-\frac{n-1}{2n}\frac{k^2}{r^2}F^c_c+(n-1)\Box F\\&+\frac{n(n-1)}{r}Dr\cdot DF-\frac{(n-1)(n-2)}{n}\frac{k^2-nK}{r^2}F=\kappa^2\Sigma_L.\end{aligned} \tag{4.5.14}$$

Note that for the exceptional mode with $k^2=nK>0$, the third equation does not exist, and for the mode with $k^2=0$, the second and the third equations do not exist. For these exceptional modes, the other equations hold without change, but the variables introduced above are not gauge invariant.

Although the energy-momentum conservation equation $\nabla_N T^N_M=0$ can be derived from the Einstein equations, it is often useful to know its explicit form. For scalar-type perturbations, they are given by the following two sets of equations:

$$\begin{aligned}&\frac{1}{r^{n+1}}D_a(r^{n+1}\Sigma^a)-\frac{k}{r}\Sigma_L+\frac{n-1}{n}\frac{k^2-nK}{kr}\tau_T\\&\quad+\frac{k}{2r}(T^{ab}F_{ab}-PF^a_a)=0,\end{aligned} \tag{4.5.15a}$$

$$\begin{aligned}&\frac{1}{r^n}D_b\left[r^n(\Sigma^b_a-T^c_aF^b_c)\right]+\frac{k}{r}\Sigma_a-n\frac{D_ar}{r}\Sigma_L\\&\quad+n\left(T^b_aD_bF-PD_aF\right)+\frac{1}{2}\left(T^b_aD_bF^c_c-T^{bc}D_aF_{bc}\right)=0.\end{aligned} \tag{4.5.15b}$$

5 Instabilities of Black Holes

§5.1 Various Instabilities: summary

5.1.1 4D black holes

Test fields

1976 Superradiance instability of a massive scalar field in 4D Kerr [Damour T, Deruelle N, Ruffini R[DDR76]]

1979 Analytic estimations of the SR instability growth rate: WKB [Zouros TJM, EardleyDM 1979[ZE79]], Matched asymptotic expansion[Detweiler S 1980[Det80]]

2007 Numerical estimation of the SR instability growth rate [Dolan SR 2007[Dol07]]

Stability against gravitational perturbations

- Vacuum Einstein with $\Lambda = 0$
  - AF static: Schwarzschild BH (unique &stable)
  - AF rotating, connected: Kerr BH (unique &stable)
- Vacuum Einstein with $\Lambda \neq 0$
  - AS static: dS/adS Schwarzschild BH (uniqueness ?: stable)
  - AS rotating: dS/adS Lerr BH (uniqueness ?: stability ?)
- Einstein-Maxwell with $\Lambda = 0$
  - AF non-deg. static: RN BH (unique, stable)
  - AF rotating, connected : Kerr-Newmann BH (unique: stability ?)

- Einstein-Maxwell with $\Lambda \neq 0$
  - AS non-deg. static: dS/adS RN BH (uniqueness ?: stable)
  - AS rotating, connected: Plebanski-Demianski BH (uniqueness ?: stability ?)
- Einstein-Maxwell-Dilaton
  - AF non-deg. static: Gibbons-Maeda (unique; stability ?) [Masood-ul-Alam 1993; Mars, Simon 2001; Gibbons, Ida, Shiromizu 2002; Rogatko 1999, 2002]
  - Rotating: ?
- Einstein-Harmonic Scalar
  - AF non-deg. rotating: Kerr (unique) [Breitenlohner, Maison, Gibbons 1988, Heusler 1993, 1995]
- Einstein-Maxwell-Dirac
  - AF non-deg. static: RN BH [Finter, Smoller, Yau 2000]
- Einstein-YM system
  - Static: non-unique, 3 families: Schwarzschild + another regular BH solution (unstable) + spherically symmetric, regular soliton (unstable) [Bartnik, McKInnon 1988; Volkov, Gal'tsov 1990; Künzle, Masood-ul-Alam 1990] [Straumann, Zhou 1990; Bizon 1991; Zhou, Straumann 1991]
- Einstein-Skyrme
  - AF non-deg. static: Schwarzschild + spherically symmetric regular bh solution with skyrme hair (stable) [Droz, Heusler, Straumann 1991] [Heusler, Droz, Strumann 1991, 1992; Heusler, Straumann, Zhou 1993]

5.1.2 Higher-dimensional BHs

Instability milestones

1993 Gregory-Laflamme instability of black brane/string [Gregory R, Laflamme C 1993[GL93]]

2003 Conjecture: GL-type instability of HD Kerr ($D \geq 6$) [Emparan R, Myers R [EM03]]

2006 Special adS-MP: SR unstable if rapidly rotating and odd $D \geq 7$ [Kunduri HK, Lucietti J, Reall HS 2006[KLR06]]

2008 Gravitational instability of dS-RN BH for $D \geq 7$ [Konoplya RA, Zhidenko A 2008[KZ09]]

2009 adS-Kerr: SR unstable if rapidly rotating and $D \geq 7$ [Kodama H, Konoplya R, Zhidenko A 2009[KKZ08]

- GL-type instability of HD Kerr ($D \geq 6$) [Dias OJC et al[DFM$^+$09]]

2010 Bar instability of rapidly rotating HD Kerr ($D \geq 6$) [Shibata M, Yoshino H 2010[SY10b, SY10a]]

Weakly asymmptotically simple solutions

- Vacuum Einstein with $\Lambda = 0$
  - AF static, regular: Tangherlini-Schwarzschild BH ($\forall D$: unique & stable)[Hwang S 1998; Rogatko 2003][Ishibashi, Kodama 2003]
  - AF static with conical singularity: static black ring[ER02a]
  - AF rotating, connected: Myers-Perry BH [MP86](unstable for $D \geq$ 6 if simply and rapidly rotating)[Emparan R, Myers R [EM03];Dias OJC et al[DFM$^+$09]; Shibata M, Yoshino H 2010[SY10b, SY10a], black rings[ER02b, PS06]/black saturn[EF07]/black diring[Izu08] (stability ?).
- Vacuum Einstein with $\Lambda \neq 0$
  - AS static: HD dS/adS Schwarzschild BH (unique ?: stable for $D \leq 6$ and $\Lambda > 0$)
  - AS rotating: GLPP soulution[GLPP05](unique ?) > adS-Kerr (SR unstable if rapidly rotating)[Kodama H, Konoplya R, Zhidenko A 2009[KKZ08]
- Einstein-Maxwell with $\Lambda = 0$
  - AF non-deg. regular static: HD RN BH (unique: stable for $D \leq$ 5)[Gibbons, Ida, Shiromizu 2002[GIS03]; Rogatko 2003][Kodama, Ishibashi 2004[KI04]]
  - AF static with conical singularity: charged static BR[IU03] (stability ?)
  - AF rotating : unknown
- Einstein-Maxwell with $\Lambda \neq 0$
  - AS non-deg. static: HD dS/adS RN BH (uniqueness ?: stable for $D \leq 5$ and $\Lambda > 0$, unstable for $D \geq 7$ and large Q and Λ) [Kodama H, Ishibashi A 2004; Konoplya RA, Zhidenko A 2009[KZ09]]

  - AS rotating: unknown

- Einstein-Maxwell-Dilaton
  - AF non-deg. static: Gibbons-Maeda (unique; stable ?) [Gibbons, Ida, Shiromizu 2002]
  - Rotating: ?

- Einstein-Harmonic Scalar
  - AF static: Tangherlini-Schwarzschild BH ($\forall D$: unique) [Rogatko 2002]

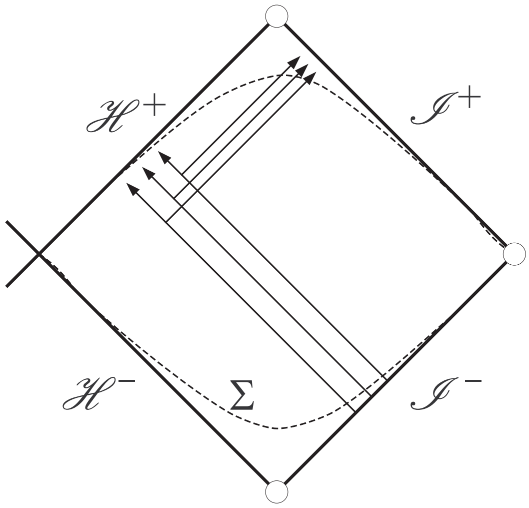
Kaluza-Klein type splutions

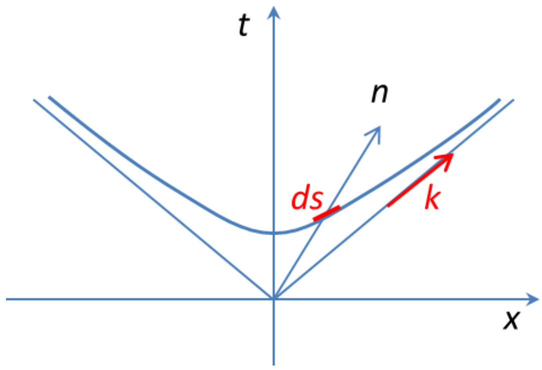
- Vacuum Einstein with $\Lambda = 0$
  - BB static: Uniform solution + Non-Uniform BS solution + Caged BH solution (Gregory-Laflamme instability)[Wiseman T 2003[Wis03]; Kudoh H, Wiseman T 2005[KW05];Kol B2006[Kol06]][Gregory R, Laflamme C 1993[GL93]]
  - BB rotating: Kerr BB (SR-unstable for $\text{Kerr}^4 \times \mathbb{R}^n$, SR-stable for $\text{Kerr}^m \times \mathbb{R}^n$ with $m > 4$)[Cardoso V, Yoshida S2005[CY05]]

- Vacuum Einstein with $\Lambda \neq 0$
  - BS static: 5D warped BS (GR unstable)[Gregorry R 2000[Gre00]]

- Einstein-Maxwell with $\Lambda = 0$
- BB static: BB charged with form field [Gregory R, Laflamme C 1994[GL94]]
- Einstein-Maxwell with $\Lambda \neq 0$
- Einstein-CS with $\Lambda = 0$
- Einstein-CS with $\Lambda \neq 0$
- Einstein-Maxwell-Dilaton
- Einstein-CS-Dilaton
- Einstein-Harmonic Scalar
- Einstein-Dilaton-Axion

§5.2 Superradiance





5.2.1 Klein-Gordon field around a Kerr BH

- **Klein-Gordon product**: From the field equation

$$D^\mu D_\mu \phi = 0; \quad D_\mu = \partial_\mu - iqA_\mu \tag{5.2.1}$$

the KG product defined by

$$K(\phi_1, \phi_2) = -i \int_\Sigma \left(\bar{\phi}_1 D^\mu \phi_2 - (\bar{D}^\mu \bar{\phi}_1)\phi_2\right) d\Sigma_\mu \tag{5.2.2}$$

is independent of the choice of the Cauchy surface Σ in DOC.

- **Scattering problem** When no incoming wave comes from the black hole, the flux conservation yields

$$I_{\mathscr{I}^-} = I_{\mathscr{I}^+} + I_{\mathscr{H}^+} \tag{5.2.3}$$

5.2.2 Integral on a null surface

Let us consider a hyperboloid in Minkowksi spacetime,

$$\Sigma_\epsilon : t^2 - x^2 = \epsilon^2 \tag{5.2.4}$$

Because the tangent vector (dt, dx) satisfies

$$tdt - xdx = 0, \tag{5.2.5}$$

its unit normal is given by

$$n = \frac{1}{\epsilon}(t\partial_t + x\partial_x) \tag{5.2.6}$$

Further, the line element along the hyperboloid can be written

$$ds = \pm\sqrt{dx^2 - dt^2} = \frac{\epsilon}{t}dx. \tag{5.2.7}$$

Hence, we have

$$nds = dx\left(\partial_t + \frac{x}{t}\partial_x\right) \to kdu_\pm \quad \text{as} \quad \epsilon \to 0, \quad u_\pm = t \pm x \tag{5.2.8}$$

5.2.3 Flux integral

Because the massless scalar field behaves at the null infinity $\mathscr{I}^+$ as

$$\phi \approx \frac{1}{r}\left(A^- e^{-i\omega u_-} + A^+ e^{+i\omega u_+}\right) e^{im\varphi}; \quad u_\pm = t \mp \int dr/f, \tag{5.2.9}$$

the flux integral on $\mathscr{I}^+$ is given by

$$I_{\mathscr{I}^\pm} = i\int du_\pm \int_{S^2} d\Omega_2 \lim_{r\to\infty} r^n(\bar{\phi}\overleftrightarrow{\partial}_{u_\pm}\phi) = \sum_m \int d\omega 4\pi\omega\langle |A^\pm_{\omega,m}|^2\rangle_{S^2/U(1)}. \tag{5.2.10}$$

Next, at horizon, the regularity condition on the scalar field is expressed as

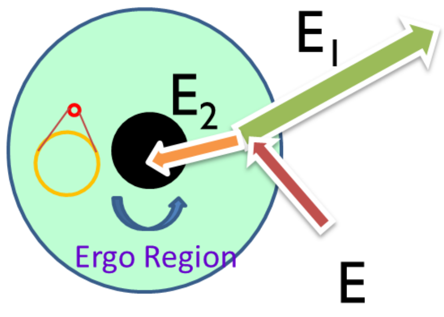
$$\begin{aligned} \phi &= \hat{\phi}(r,z)e^{-i\omega t + im\varphi} = \phi(r,z)e^{-i\omega_* t + im\tilde{\varphi}} \\ &= \phi(r,z)e^{i\omega_* r^*} e^{-i\omega_* v_+ + im\tilde{\varphi}} \approx C(z)e^{-i\omega_* v_+ + im\tilde{\varphi}} \end{aligned} \tag{5.2.11}$$

where

$$\omega_* = \omega - m\Omega_h, \quad \tilde{\phi} = \phi - \Omega_h t. \tag{5.2.12}$$

Hence, the flux integral on the horizon can be written

$$\begin{aligned} I_{\mathscr{H}^+} &= i\int dv_+ \int_{\mathscr{B}} d^{D-2}\sigma \left(\bar{\phi}(\overleftrightarrow{\partial}_{v_+} + 2iq\Phi)\phi\right)_{\mathscr{H}^+} \\ &= \sum_m \int d\omega 4\pi(\omega_* - q\Phi_h)\langle |C_{\omega,m}|^2\rangle_{\mathscr{B}/U(1)} \end{aligned} \tag{5.2.13}$$



5.2.4 Superradiance condition

The flux conservation law can be written

$$\omega\langle|A^{-}_{\omega,m}|^2\rangle = \omega\langle|A^{+}_{\omega,m}|^2\rangle + (\omega - m\Omega_h - q\Phi_h)\,\langle|C_{\omega,m}|^2\rangle \tag{5.2.14}$$

If we devide this expression by the incoming flux integral on the left-hand side, we obtain the transmission and reflection coefficients:

$$1 = R + T; \quad T < 0 \Rightarrow \quad R > 1. \tag{5.2.15}$$

Hence, if the superradiance condition

$$\omega < m\Omega_h + q\Phi_h \tag{5.2.16}$$

is satisfied, the outgoing wave has a larger amplitude than the incoming wave. This condition is equivalent to

$$k \cdot p > 0; \quad p_\mu = -i\partial_\mu - qA_\mu \tag{5.2.17}$$

5.2.5 Penrose process

Energy conservation law Let P be the 4-momentum of a free partilce and ξ be the time-translation Killing vector. Then, the energy defined by $E = -\xi \cdot P$ is conserve:

$$\dot{E} = -\nabla_u(p \cdot \xi) = -p^a u^b \nabla_{(b}\xi_{a)} = 0 \tag{5.2.18}$$

Ergo region of a rotating bh Because ξ is spacelie, a physical partical can have a negative energy $E = -\xi \cdot P < 0$. We can extract energy from a black hole through reactions in the ergo region.

§5.3 Superradiant Instability

5.3.1 Massive scalar around a (adS-)Kerr BH

Kerr metric

$$
\begin{aligned}
ds^2 &= -\frac{\Delta - a^2 X \sin^2\theta}{\rho^2} dt^2 - \frac{2a\sin^2\theta}{C\rho^2}\left\{\lambda\rho^2(r^2+a^2) + \frac{2M}{r^{n-1}}\right\} dt d\phi \\
&\quad + \frac{\sin^2\theta}{C^2\rho^2}\left[C(r^2+a^2)\rho^2 + \frac{2a^2M}{r^{n-1}}\sin^2\theta\right] d\phi^2 \\
&\quad + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{X}d\theta^2 + r^2\cos^2\theta d\Omega_n^2. \qquad (5.3.1)
\end{aligned}
$$

where $D = n+4$, and

$$\Delta = (1-\lambda r^2)(r^2+a^2) - \frac{2M}{r^{D-5}}, \qquad (5.3.2)$$

$$C = 1+\lambda a^2, \quad X = 1 + \lambda a^2\cos^2\theta. \qquad (5.3.3)$$

Klein-Gordon equation Solutions to the Klein-Gordon equation on this background

$$(\Box - \mu^2)\Phi = 0 \qquad (5.3.4)$$

can be written as a superposition of solutions of the form

$$\Phi = e^{-i\omega t + im\phi} Y^l P(r) Q(\theta) \qquad (5.3.5)$$

where Y^l is a harmonic function on S^n satisfying

$$\triangle_n Y^l = -l(l+n-1)Y^l, \qquad (5.3.6)$$

as

$$
\begin{aligned}
& XQ'' - \left(\frac{n\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} + \lambda a^2\left\{n+2-(n+3)\cos^2\theta\right\}\cot\theta\right)Q' \\
& - \left(-\Lambda_m^l + \mu^2 a^2\cos^2\theta + \frac{C}{\sin^2\theta}m^2 + \frac{a^2\sin^2\theta}{X}\omega^2 + \frac{l(l+n-1)}{\cos^2\theta}\right)Q = 0. \qquad (5.3.7)
\end{aligned}
$$

(5.3.8)

and

$$
\begin{aligned}
& \Delta P'' + \left[-\frac{2M}{r^{n+1}} + n + 2 + \frac{na^2}{r^2} - \lambda\left\{(n+4)r^2 + (n+2)a^2\right\}\right] rP' \\
& + \left[-\Lambda_m^l - \mu^2 r^2 + \frac{a^2m^2}{\Delta}\left((1+\lambda a^2)(1-\lambda r^2) - \frac{2\lambda M}{r^{n-1}}\right)\right. \\
& \left. - \frac{4M}{r^{n-1}\Delta}am\omega + \frac{(r^2+a^2)^2\omega^2}{\Delta} - \frac{l(l+n-1)a^2}{r^2}\right] P = 0, \qquad (5.3.9)
\end{aligned}
$$

where

$$\Lambda_m^l = \Lambda_m^l(a^2\omega^2, \lambda a^2, \mu^2 a^2) \qquad (5.3.10)$$

is a separation constant.

The equation for $Q(\theta)$ In terms of the variable defined by

$$x = \cos(2\theta), \ (-1 \le x \le 1), \tag{5.3.11}$$

the equation for Q can be written

$$\begin{aligned}&(1-x^2)(2+\lambda a^2+\lambda a^2 x)\frac{d^2Q}{dx^2}\\&+\left[n-1-(n+3)x+\frac{\lambda a^2}{2}(1+x)\{n+1-(n+5)x\}\right]\frac{dQ}{dx}\\&+\left[\frac{\Lambda^l_m}{2}-\frac{\mu^2a^2}{4}(1+x)+\frac{a^2\omega^2(x-1)}{2(2+\lambda a^2+\lambda a^2 x)}+\frac{1+\lambda a^2}{x-1}m^2\right.\\&\qquad\left.-\frac{l(l+n-1)}{1+x}\right]Q=0.\end{aligned} \tag{5.3.12}$$

The separation constant Λ^l_m is determined by the requirement that this equation has a solution that can be normalised with respect to the norm

$$N(Q) \propto \int_{-1}^{1} dx(1+x)^{(n-1)/2}|Q|^2. \tag{5.3.13}$$

Note that for $a = 0$, we have

$$\Lambda^l_m = (|m|+l+2j)(|m|+l+2j+n+1), \ j = 0,1,2,\cdots, \tag{5.3.14a}$$
$$Q^l_{m,j}(x) = (1-x)^{|m|/2}(1+x)^{l/2}G_j\left(|m|+\gamma,\gamma;\tfrac{1+x}{2}\right), \tag{5.3.14b}$$

where $G_j(m,\gamma,x)$ is the Jacobi polynomial and

$$\gamma = l + \frac{n+1}{2}. \tag{5.3.15}$$

The equation for $P(r)$ In terms of $P_1(r)$ defined by

$$P(r) = r^{-n/2}(r^2+a^2)^{-1/2}P_1(r), \tag{5.3.16}$$

the equation for P can be expressed as

$$D^2P_1 + \left[\left(\omega - \frac{2Mam}{(r^2+a^2)^2 r^{n-1}}\right)^2 - \frac{\Delta}{(r^2+a^2)^2}U\right]P_1 = 0, \tag{5.3.17}$$

where

$$D := \frac{\Delta}{r^2+a^2}\frac{d}{dr} = \left(1-\lambda r^2 - \frac{2M}{r^{n-1}(r^2+a^2)}\right)\frac{d}{dr}, \tag{5.3.18}$$

$$\begin{aligned}U :=\ & \Lambda^l_m + \mu^2r^2 - \frac{a^2m^2}{(r^2+a^2)^2}\left\{(r^2+a^2)(1+\lambda a^2)+\frac{2M}{r^{n-1}}\right\}\\&+\frac{n(n+2)}{4}(1-\lambda a^2)-\lambda a^2-\frac{(n+2)(n+4)}{4}\lambda r^2\\&+\left(l+\frac{n}{2}\right)\left(l+\frac{n}{2}-1\right)\frac{a^2}{r^2}+\frac{a^2(1+\lambda a^2)}{r^2+a^2}\\&+\frac{\{(n+2)r^2+na^2\}^2-8a^2r^2}{2(r^2+a^2)^2}\frac{M}{r^{n+1}}.\end{aligned} \tag{5.3.19}$$

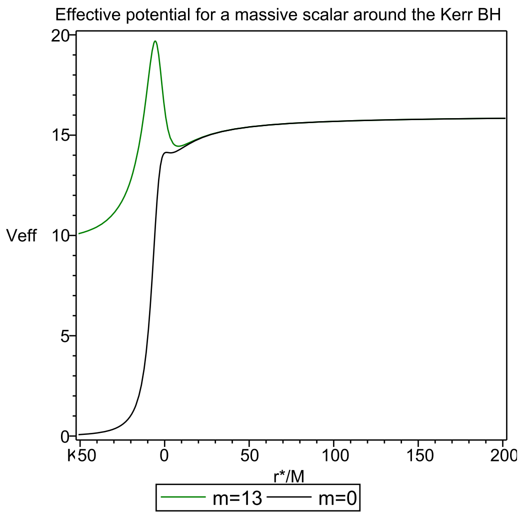


Figure 5.1: Effective potential for a massive scalar in 4D Kerr

Boundary condition at horizon From the regularity condition at the future horizon

$$\begin{aligned}\phi &= \hat{\phi}(r,z)e^{-i\omega t+im\varphi} = \phi(r,z)e^{-i\omega_* t+im\tilde{\varphi}} \\ &= \phi(r,z)e^{i\omega_* r^*}e^{-i\omega_* v_+ +im\tilde{\varphi}} \approx C(z)e^{-i\omega_* v_+ +im\tilde{\varphi}},\end{aligned} \tag{5.3.20}$$

R should behaves near the horizon as

$$R(r) = e^{-i\omega_* r^*}\left[Z_0 + \mathrm{O}((r-r_h))\right] = (r-r_h)^{-i\omega_*/(2\kappa)}\left[Z_0 + \mathrm{O}((r-r_h))\right]. \tag{5.3.21}$$

5.3.2 4D Kerr

Boundary condition In this case in which $n = 0$ and $\lambda = 0$, the boundary conditions are

- At horizon:

$$R_{lm} \approx C_{lm}e^{-i\omega_* r_*}. \tag{5.3.22}$$

- At infinity:

$$R_{lm} \sim \frac{B_{lm}}{r}e^{+ikr^*}: \quad k = (\omega^2 - \mu^2)^{1/2}. \tag{5.3.23}$$

Instability condition From the KG equation, we obtain the energy integral as

$$\begin{aligned} 0 &= \int \frac{d\phi}{2\pi} \int d\theta \sin\theta \int dr \rho^2 (\partial_t \Phi)^* (\Box - \mu^2) \Phi \\ &= \left[\int d\theta \sin\theta (i\omega^*) \Delta \Phi^* \partial_r \Phi \right]_{r=r_h}^{r=r_\infty} \\ &\quad + \int dr \int d\theta \sin\theta \Big[i\rho^2 |\omega|^2 (-\omega g^{tt} + 2m g^{t\phi}) |\Phi|^2 \\ &\qquad -i\omega^* \rho^2 \left(m^2 g^{\phi\phi} + \mu^2 \right) |\Phi|^2 - i\omega^* \left(\Delta |\partial_r \Phi|^2 + |\partial_\theta \Phi|^2 \right) \Big]. \end{aligned} \tag{5.3.24}$$

From this, we obtain the flux condition

$$B\omega_R(m\Omega_h - \omega_R) + \omega_I^2 (C_1 + C_2 \omega_I) = A\omega_I,$$

where

$$B = (r_h^2 + a^2) e^{2\omega_I v_+} |\tilde{R}|^2_{r=r_h}, \tag{5.3.25}$$

$$C_1 = \int dr 2r e^{2\omega_I v_+} |\tilde{R}|^2, \tag{5.3.26}$$

$$C_2 = a^2 \int dr e^{2\omega_I v_+} \int d\theta \sin^3\theta |\tilde{R} S_l^m|^2, \tag{5.3.27}$$

all of which are positive, and

$$\begin{aligned} A &= \int dr e^{2\omega_I v_+} \int d\theta \sin\theta \Big[|\tilde{R}|^2 |\partial_\theta S_l^m|^2 + \rho^2 (-g_{tt}) |\partial_r \tilde{R}|^2 |S_l^m|^2 \\ &\quad + a^2 \left| \sin\theta \partial_r \tilde{R} - i\omega_R^* \frac{(r^2+a^2)}{a^2 \sin\theta} \tilde{R} \right|^2 |S_l^m|^2 \\ &\quad + \left\{ \frac{2(m - Q\omega_R)^2}{\sin^2\theta} + P\omega_R^2 + \mu^2 \rho^2 \right\} |\tilde{R} S_l^m|^2, \end{aligned} \tag{5.3.28}$$

$$\begin{aligned} P &= \rho^2 (r^2+a^2)^2 \left\{ \rho^2 + 4a^2 \sin^2\theta \right\} (g_{tt})^2 \\ &\quad + 8Ma^2 \sin^2\theta \left[r(r^2+a^2)(-g_{tt}) + \frac{a^2 M r^2 \sin^2\theta}{\rho^2} \right] \left(\rho^2 + 2a^2 \sin^2\theta \right). \Big]. \end{aligned} \tag{5.3.29}$$

From these, we obtain the following necessary conditions for instability:

1. The mode is bounded.
2. R is peaked far outside the ergo region $\Rightarrow \quad A > 0$.
3. ω is nearly real: $|\omega_I| \ll \omega_R$.
4. ω satisfies the superradiance condition: $\omega_R < m\Omega_h$. [Zouros TJM, Eardley DM 1979[ZE79]]

Numerical estimates Let us expand $R(r)$ as

$$R(r) = \frac{x^{-i\sigma}}{(r-r_-)^{\chi-1}} e^{qr} \sum_{n=0}^{\infty} a_n x^n, \qquad x = \frac{r-r_+}{r-r_-}, \tag{5.3.30}$$

where

$$\sigma = \frac{2r_+(\omega - m\Omega_h)}{r_+ - r_-} \tag{5.3.31}$$

$$q = -(\mu^2 - \omega^2)^{1/2} \tag{5.3.32}$$

$$\chi = (\mu^2 - 2\omega^2)/q, \tag{5.3.33}$$

Then, the equation for $P(r)$ gives the recurrence relation

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \tag{5.3.34}$$

with

$$\alpha_n = (n+1)(n+c_0), \tag{5.3.35}$$

$$\beta_n = -2n^2 + (c_1+2)n + c_3, \tag{5.3.36}$$

$$\gamma_n = n^2 + (c_2-3)n + c_4, \tag{5.3.37}$$

where $c_1 \sim c_4$ are functions of ω, q, a, m and Λ_{lm}. If we require that the series for $R(r)$ converges, we obtain the infinite continued fraction,

$$\frac{a_{n+1}}{a_n} = -\frac{\gamma_{n+1}}{\beta_{n+1} + \alpha_{n+1}\frac{a_{n+2}}{a_{n+1}}} = -\frac{\gamma_{n+1}}{\beta_{n+1}-} \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-} \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-} \cdots . \tag{5.3.38}$$

Hence, ω can determined as roots to the equation[Dol07]

$$\beta_0 - \frac{\alpha_0\gamma_1}{\beta_1-} \frac{\alpha_1\gamma_2}{\beta_2-} \frac{\alpha_2\gamma_3}{\beta_3-} \cdots = 0. \tag{5.3.39}$$

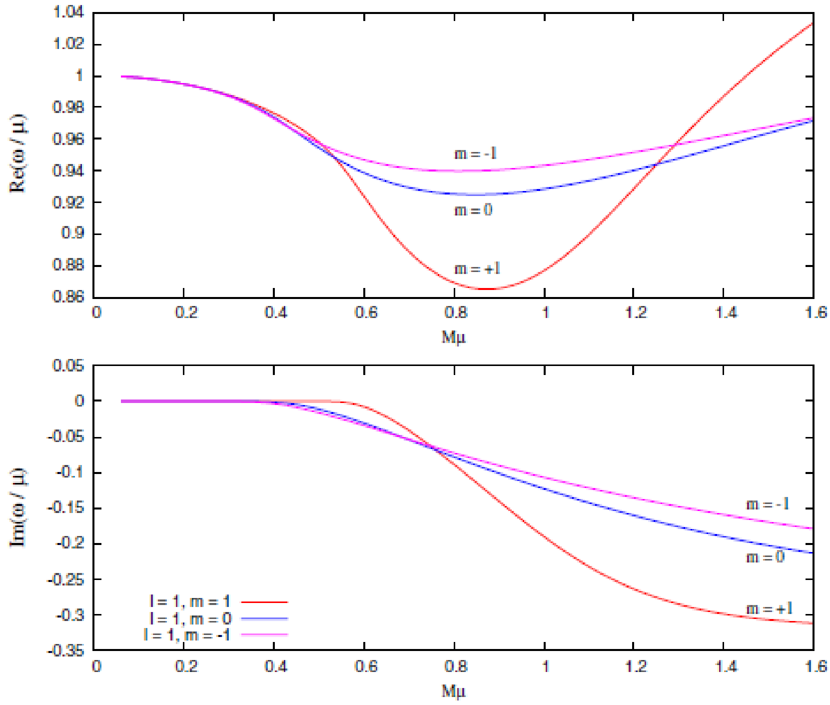
General features

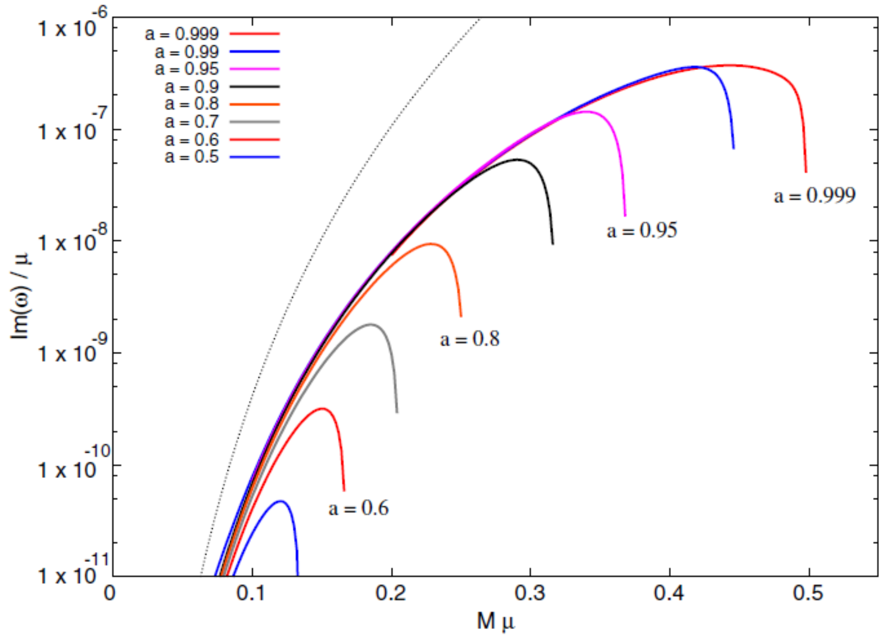
- The growth rate is greatest for $M\mu < 0.5$.
- The mode with $l = m = 1$ is most unstable:
- The maximum growth rate at $a = 0.99$ is

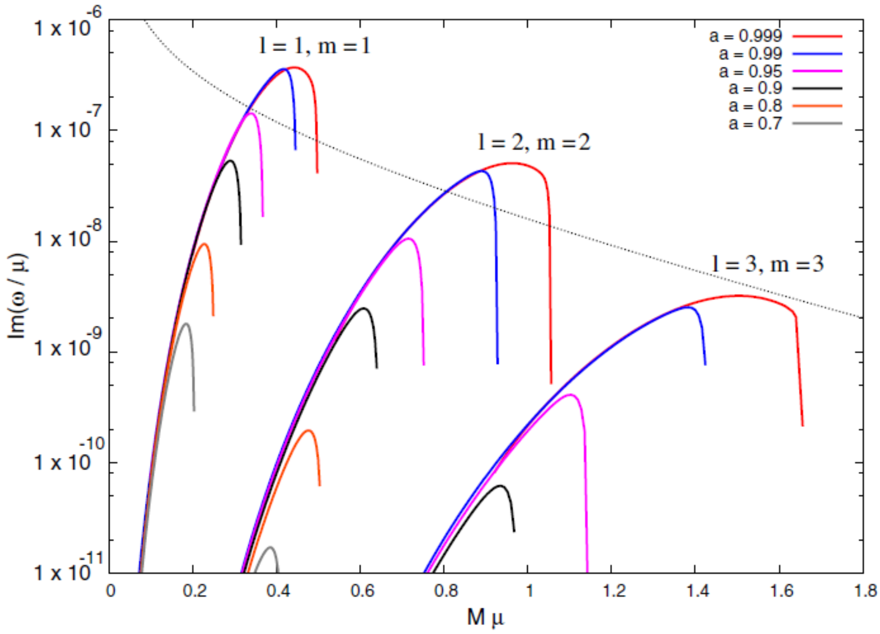
$$\tau^{-1} \sim 1.5 \times 10^{-7} (GM/c^3)^{-1} \tag{5.3.40}$$

- In the asymptotic regions

$$\tau \approx \begin{cases} 10^7 e^{1.84 R_g\mu} R_g & ; R_g\mu \gg 1,\ a = 1 \\ 24 \left(\frac{a}{R_g}\right)^{-1} (R_g\mu)^{-9} R_g & ; R_g\mu \ll 1 \end{cases} \tag{5.3.41}$$


Re(ω / μ)
Mμ
m = -1
m = 0
m = +1
Im(ω / μ)
l = 1, m = 1
l = 1, m = 0
l = 1, m = -1


Im(ω) / μ
M μ
a = 0.999
a = 0.99
a = 0.95
a = 0.9
a = 0.8
a = 0.7
a = 0.6
a = 0.5



§5.4
Stability of Static Black Holes

We study the stability of static black holes utilising the gauge-invariant formulation for perturbations explained in the previous section. We consider the static Einstein black hole which corresponds to the case with $m = 2$ of the general background considered in the previous section and has the metric (4.1.9). The key point is the fact that gauge-invariant perturbation equations can be reduced to decoupled single master equations of the Schrödinger type for any type of perturbations in this background.

5.4.1 Tensor Perturbations

The gauge-invariant equation for tensor perturbations is already given by a single equation for each mode. Assuming that the source term vanishes, it reads

$$-\partial_t H_T^2 + f\partial_r(f\partial_r H_T) - \frac{k^2+2K}{r^2} f H_T = 0. \tag{5.4.1}$$

Here, note that even if there exist electromagnetic fields, τ_T vanishes because the electromagnetic field is vector-like and does not produce a tensor-type quantity in the linear order at least.

With the help of the Fourier transformation with respect to t, i.e., assuming

$H_T \propto e^{-i\omega t}$, this equation can be put into the Schrödinger-type eigenvalue problem;

$$\omega^2\Phi = -f\partial_r(f\partial_r\Phi) + V_t\Phi; \quad H_T = r^{-n/2}\Phi(r)e^{-i\omega t}, \tag{5.4.2}$$

where

$$\begin{aligned} V_t &= \frac{f}{r^2}\left[k^2 + 2K + \frac{nrf'}{2} + \frac{n(n-2)f}{4}\right] \\ &= \frac{f}{r^2}\left[k^2 + \frac{n^2-2n+8}{4}K - \frac{n(n+2)}{4}\lambda r^2 + \frac{n^2M}{2r^{n-1}} - \frac{n(3n-2)Q^2}{4r^{2n-2}}\right]. \end{aligned} \tag{5.4.3}$$

If V_t is non-negative, we can directly conclude the stability. However, it is not so easy to see whether V_t is non-negative or not outside the horizon. This technical difficulty is easily resolved by considering the energy integral

$$E := \int_{r_h}^{r_\infty} dr\left[\frac{1}{f}(\partial_t H_T)^2 + f(\partial_r H_T)^2 + \frac{k^2+2K}{r^2}H_T^2\right]. \tag{5.4.4}$$

From the equation for H_T, we find that

$$\partial_t E = 2\left[f\partial_t H_T \partial_r H_T\right]_{r_h}^{r_\infty} = 0. \tag{5.4.5}$$

Hence, in the case $\mathscr{K}$ is a constant curvature space, the condition on the spectrum $k^2 \geq n|K|$ guarantees the positivity of all terms in E, and as a consequence the stability of the system.

5.4.2 Vector Perturbations

Master equation

For vector perturbations, the energy-momentum conservation law is written

$$D_a(r^{n+1}\tau^a) + \frac{m_v}{2k}r^n\tau_T = 0. \tag{5.4.6}$$

For $m_v \equiv k^2 - (n-1)K \neq 0$, with the help of this equation, the second of the perturbation equations, (4.4.12b), can be written

$$D_a(r^{n-1}F^a) = \frac{2\kappa^2}{m_v}D_a(r^{n+1}\tau^a). \tag{5.4.7}$$

In the case of $m = 2$, from this it follows that F^a can be written in terms of a variable Ω as

$$r^{n-1}F^a = \epsilon^{ab}D_b\Omega + \frac{2\kappa^2}{m_v}r^{n+1}\tau^a. \tag{5.4.8}$$

Further, the first of the perturbation equations, (4.4.12a), is equivalent to

$$D_a\left(r^{n+1}F^{(1)}\right) - m_v r^{n-1}\epsilon_{ab}F^b = -2\kappa^2 r^{n+1}\epsilon_{ab}\tau^b, \tag{5.4.9}$$

where ϵ_{ab} is the two-dimensional Levi-Civita tensor for g_{ab}, and

$$F^{(1)} = \epsilon^{ab} r D_a \left(\frac{F_b}{r} \right) = \epsilon^{ab} r D_a \left(\frac{f_b}{r} \right). \tag{5.4.10}$$

Inserting the expression for F_a in terms of Ω into (5.4.9), we obtain the master equation

$$r^n D_a \left(\frac{1}{r^n} D^a \Omega \right) - \frac{m_v}{r^2} \Omega = -\frac{2\kappa^2}{m_v} r^n \epsilon^{ab} D_a (r \tau_b). \tag{5.4.11}$$

Next, for $m_v = 0$, the perturbation variables H_T and τ_T do not exist. The matter variable τ_a is still gauge-invariant, but concerning the metric variables, only the combination $F^{(1)}$ defined in terms of f_a in (5.4.10) is gauge invariant. In this case, the Einstein equations are reduced to the single equation (5.4.9), and the energy-momentum conservation law is given by (5.4.6) without the τ_T term. Hence, τ_a can be expressed in terms of a function $\tau^{(1)}$ as

$$r^{n+1} \tau_a = \epsilon_{ab} D^b \tau^{(1)}. \tag{5.4.12}$$

Inserting this expression into (5.4.9) with $\epsilon^{cd} D_c (F_d / r)$ replaced by $F^{(1)}/r$, we obtain

$$D_a (r^{n+1} F^{(1)}) = -2\kappa^2 D_a \tau^{(1)}. \tag{5.4.13}$$

Taking account of the freedom of adding a constant in the definition of $\tau^{(1)}$, the general solution can be written

$$F^{(1)} = -\frac{2\kappa^2 \tau^{(1)}}{r^{n+1}}. \tag{5.4.14}$$

Hence, there exists no dynamical freedom in these special modes. In particular, in the source-free case in which $\tau^{(1)}$ is a constant and $K = 1$, this solution corresponds to adding a small rotation to the background static black hole solution.

Neutral black holes

For a neutral static Einstein black hole, the master equation for a generic mode can be put into the canonical form as

$$\omega^2 \Phi = -f \partial_r (f \partial_r \Phi) + V_v \Phi; \quad \Omega = r^{n/2} \Phi(r) e^{-i\omega t}, \tag{5.4.15}$$

where

$$\begin{aligned} V_v &= \frac{f}{r^2} \left[m_v - \frac{nrf'}{2} + \frac{n(n+2)f}{4} \right] \\ &= \frac{f}{r^2} \left[k^2 + \frac{n(n+2)K}{4} - \frac{n(n-2)}{4} \lambda r^2 - \frac{3n^2 M}{2r^{n-1}} \right]. \end{aligned} \tag{5.4.16}$$

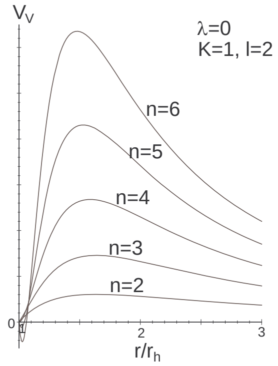


Figure 5.2: V_v for $K = 1, \lambda = 0, l = 2$.

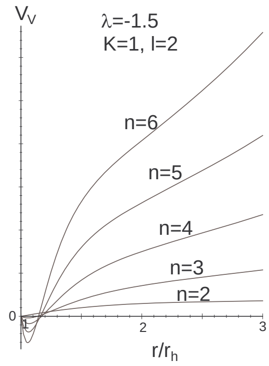


Figure 5.3: V_v for $K = 1, \lambda < 0, l = 2$.

This equation is identical to the Regge-Wheeler equation for $n = 2, K = 1$ and $\lambda = 0$. In this case, we can put V_v into an obviously non-negative form as

$$V_v = \frac{f}{r^2}\left(m_v + 3f\right), \tag{5.4.17}$$

proving the stability of the black hole against vector perturbations (or axial or odd perturbations).

In higher dimensions, the potential V_v is not positive definite anymore and we can not use this type of argument. However, we can still prove the stability with the help of the conserved energy integral as in the case of tensor perturbations. In the present case, if we define E as

$$E := \int_{r_h}^{r_\infty} \frac{dr}{r^n}\left[\frac{1}{r}(\partial_t\Omega)^2 + f(\partial_r\Omega)^2 + \frac{m_v}{r^2}\Omega^2\right], \tag{5.4.18}$$

we have

$$\dot{E} = 2\left[\frac{f}{r^n}\partial_t\Omega\partial_r\Omega\right]_{r_h}^{r_\infty} = 0. \tag{5.4.19}$$

Further, all terms of E is non-negative because $m_v \geq 0$. Hence, the stability can be concluded.

Charged black hole

The formulation for neutral static black holes can be extended to charged static black holes. The final master equations consist of two equations: the extension of the equation for gravitational perturbations with an electromagnetic source and the equation coming from the Maxwell equations[KI04]:

$$r^n D_a\left(\frac{1}{r^n}D^a\Omega\right) - \frac{m_v}{r^2}\Omega = \frac{2\kappa^2 q}{r^2}\mathscr{A}, \tag{5.4.20a}$$

$$\frac{1}{r^{n-2}}D_a(r^{n-2}D^a\mathscr{A}) - \frac{m_v + 2(n-1)K}{r^2}\mathscr{A} = \frac{q}{r^{2n}}\left[m_v\Omega + 2\kappa^2 q\mathscr{A}\right], \tag{5.4.20b}$$

where $\mathscr{A}$ is the gauge invariant representing a vector perturbation of the vector potential of the electromagnetic field defined by

$$\delta A_a = 0, \quad \delta A_i = \mathscr{A}\mathbb{V}_i, \tag{5.4.21}$$

and q is the black hole charge related to the charge parameter Q in the background metric by

$$Q^2 := \frac{\kappa^2 q^2}{n(n-1)}, \tag{5.4.22}$$

By taking appropriate combinations, these equations can be transformed to the two decoupled equations

$$-\partial_t^2 \Phi_\pm = (-\partial_{r_*}^2 + V_\pm)\Phi_\pm, \tag{5.4.23}$$

where the effective potentials are given by

$$V_\pm = \frac{f}{r^2}\left[m_v + \frac{n(n+2)K}{4} - \frac{n(n-2)}{4}\lambda r^2 + \frac{n(5n-2)Q^2}{4r^{2n-2}} + \frac{\mu_\pm}{r^{n-1}}\right], \tag{5.4.24}$$

with

$$\mu_\pm = -\frac{n^2+2}{2}M \pm \Delta; \quad \Delta^2 = (n^2-1)^2 M^2 + 2n(n-1)m_v Q^2. \tag{5.4.25}$$

S-deformation

The effective potentials $V_\pm$ are not positive definite as in the neutral case. In the present case, we prove that the system is still stable not by the energy integral method, but rather by a different method, which we call *the S-deformation*[IK03].

We first explain the basic idea by the eigenvalue equation

$$\omega^2 \Phi = \left(-D^2 + V(r)\right)\Phi, \tag{5.4.26}$$

where $D = \partial_{r_*}$. If there exists an unstable mode with $\omega^2 < 0$ and if V is non-negative at horizon and at infinity, we can show Φ falls off sufficiently rapidly at horizon and at infinity. Hence, we obtain the integral identity,

$$\omega^2 \int_{r_h}^{r_\infty} |\Phi|^2 \frac{dr}{f} = \int_{r_h}^{r_\infty} \left[|D\Phi|^2 + V(r)|\Phi|^2\right]\frac{dr}{f}. \tag{5.4.27}$$

If $V(r)$ is non-negative definite, this leads to contradiction and hence proves the stability because the right-hand side is non-negative. In contrast, in the case in which the sign of V is not definite, we cannot say anything about stability from this equation.

In order to treat such a case, let us replace D by $D = \tilde{D} - S(r)$. Then, by partial integrations, we obtain the modified integral identity with D and V replaced by $\tilde{D}$ and $\tilde{V}$ given by

$$\tilde{V} = V + f\frac{dS}{dr} - S^2. \tag{5.4.28}$$

Hence, if we can find S such that the modified effective potential $\tilde{V}$ is non-negative, we can establish the stability of the system even when the original potential is not non-negative definite.

For example, by the S-transformation with $S = nf/(2r)$, the effective potentials $V_\pm$ above can be modified into

$$\tilde{V}_\pm = V_\pm + f\frac{dS}{dr} - S^2 = \frac{f}{r^2}\left[m_v + \frac{1}{r^{n-1}}\left(\frac{3n^2}{2}M + \mu_\pm\right)\right]. \tag{5.4.29}$$

Here, $\tilde{V}_+$ is obviously positive definite. We can also show that $\tilde{V}_-$ is also positive definite. Hence, a charged static Einstein black hole is stable for vector perturbations.

5.4.3 Scalar Perturbations

Master equation

For a static Einstein black hole background, assuming that $F_{ab}, F \propto e^{-i\omega t}$, we can reduce the whole linearised Einstein equations into a single master equation, as in the case of vector perturbations[KI03]:

$$\omega^2\Phi = -f\partial_r(f\partial_r\Phi) + V_s\Phi, \tag{5.4.30}$$

where the master variable Φ is defined as

$$\Phi = \frac{nr^{n/2}}{H}\left(2F + \frac{F^r_t}{i\omega r}\right); \quad H = m + \frac{n(n+1)}{2}x \tag{5.4.31}$$

with $m = k^2 - nK$ and $x = 2M/r^{n-1}$, and the effective potential V_s is given by $V_s(r) = \frac{fU(r)}{16r^2H^2}$ with

$$\begin{aligned} U(r) \;=\; & -\left[n^3(n+2)(n+1)^2x^2 - 12n^2(n+1)(n-2)mx \right. \\ & \left. \quad +4(n-2)(n-4)m^2\right]\lambda r^2 + n^4(n+1)^2x^3 \\ & +n(n+1)\left[4(2n^2-3n+4)m + n(n-2)(n-4)(n+1)K\right]x^2 \\ & -12n\left[(n-4)m + n(n+1)(n-2)K\right]mx \\ & +16m^3 + 4Kn(n+2)m^2. \end{aligned} \tag{5.4.32}$$

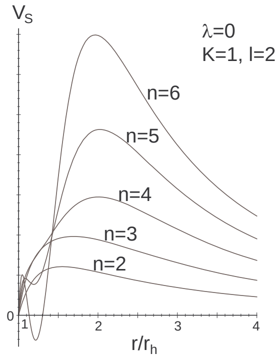


Figure 5.4: V_s for $K = 1, \lambda = 0, l = 2$.

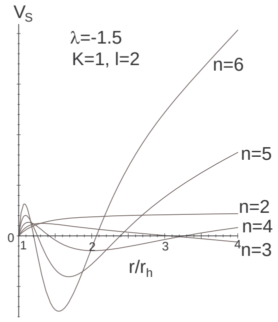


Figure 5.5: V_s for $K = 1, \lambda < 0, l = 2$.

Neutral black holes

The above master equation is identical to the Zerilli equation for the four-dimensional Schwarzschild black hole ($n = 2, K = 1$ and $\lambda = 0$). In this case, from

$$V_s = \frac{f}{r^2 H^2}\left(m^2(m+2) + \frac{6m^2 M}{r} + \frac{36mM^2}{r^2} + \frac{72M^3}{r^3}\right) \geq 0, \tag{5.4.33}$$

where $m = (l-1)(l+2)(l = 2, 3, \cdots)$, we can easily prove the stability of the black hole. In higher dimensions, however, the effective potential V_s is not positive definite. Hence, an instability may arise.

Nevertheless, in the case of $K = 1$ and $\lambda = 0$, i.e. for the Schwarzschild-Tangherlini black hole, we can prove the stability by applying the S-deformation to the energy integral. First, from the above master equation, we obtain

$$E := \int_{r_0}^{r_\infty} \frac{dr}{f}\left[(\partial_t \Phi)^2 + (D\Phi)^2 + V_s \Phi^2\right], \tag{5.4.34}$$

$$\dot{E} = \left[2f\partial_t\Phi\partial_r\Phi\right]_{r_h}^{r_\infty} = 0, \tag{5.4.35}$$

where $D = f\partial_r$. Next, we replace D to $\tilde{D} = f\partial_r + S$. Then, by partial integration we obtain

$$E = \int_{r_0}^{r_\infty} \frac{dr}{f}\left[(\partial_t \Phi)^2 + (\tilde{D}\Phi)^2 + \tilde{V}_s \Phi^2\right], \tag{5.4.36}$$

where

$$\tilde{V}_s = V_s + f\frac{dS}{dr} - S^2. \tag{5.4.37}$$

For example, for

$$S = \frac{f}{h}\frac{dh}{dr}, \quad h \equiv r^{n/2+l-1}\left\{(l-1)(l+n) + n(n+1)x/2\right\}. \tag{5.4.38}$$

we obtain

$$\tilde{V}_s = \frac{f(r)\tilde{Q}(r)}{4r^2\left\{(l-1)(l+n) + n(n+1)x/2\right\}}, \tag{5.4.39}$$

where

$$\tilde{Q}(r) \equiv lx[ln(n+1)x + 2(l-1)\{n^2 + n(3l-2) + (l-1)^2\}]\,. \tag{5.4.40}$$

Clearly $\tilde{V}_s > 0$.

Charged black holes

For charged black holes, we can also reduce the perturbation equations to decoupled single master equations. First, we generalise the master variable Φ for the metric perturbation given in (5.4.31) by replacing H by

$$H = m + \frac{n(n+1)M}{r^{n-1}} - \frac{n^2Q^2}{r^{2n-2}}. \tag{5.4.41}$$

Next, we introduce the gauge-invariant variable $\mathscr{A}$ in terms of which the scalar perturbation of the electromagnetic field is expressed as

$$\delta\mathscr{F}_{ab} + D_c(E_0X^c)\epsilon_{ab}\mathbb{S} = \mathscr{E}\epsilon_{ab}\mathbb{S}, \tag{5.4.42a}$$
$$\delta\mathscr{F}_{ai} - kE_0\epsilon_{ab}X^b\mathbb{S}_i = r\epsilon_{ab}\mathscr{E}^b\mathbb{S}_i, \tag{5.4.42b}$$
$$\delta\mathscr{F}_{ij} = 0, \tag{5.4.42c}$$

with

$$\mathscr{E}_a = \frac{k}{r^{n-1}}D_a\mathscr{A}, \quad r^n\mathscr{E} = -k^2\mathscr{A} + \frac{q}{2}(F^c_c - 2nF). \tag{5.4.43}$$

Then, the Einstein and Maxwell equations for scalar perturbations of a charge Einstein black hole can be reduced to the following two coupled equations[KI04]:

$$\omega^2\Phi = -\frac{d^2\Phi}{dr_*^2} + V_s\Phi + \frac{\kappa^2 qfP_{S1}}{r^{3n/2}H^2}\mathscr{A}, \tag{5.4.44a}$$

$$\begin{aligned}\omega^2\mathscr{A} = & -r^{n-2}\frac{d}{dr_*}\left(\frac{1}{r^{n-2}}\frac{d\mathscr{A}}{dr_*}\right) + f\left(\frac{k^2}{r^2}\mathscr{A} + \frac{2n^2(n-1)^2Q^2f}{r^{2n}H}\right)\mathscr{A} \\ & + f\frac{(n-1)q}{r^{n/2}}\left(\frac{4H^2 - nP_Z}{4nH}\Phi + fr\partial_r\Phi\right).\end{aligned} \tag{5.4.44b}$$

where V_s. P_{S1} and P_Z are the functions of r (see [KI04] for their exiplict expressions).

As in the case of vector perturbations, we can find linear combinations of $\mathscr{A}$ and Φ, in terms of which these equations are transformed to the decoupled equations

$$\frac{\omega^2}{f}\Phi_\pm = -(f\Phi'_\pm)' + \frac{V_\pm}{f}\Phi_\pm; \quad V_\pm = \frac{fU_\pm}{64r^2H_\pm^2}, \tag{5.4.45}$$

Here, $H_+ = 1 - n(n+1)\delta x/2, H_- = m + n(n+1)(1+m\delta)x/2$, and δ is a non-negative constant determined from Q by

$$Q^2 = (n+1)^2M^2\delta(1+m\delta). \tag{5.4.46}$$

The effective potentials $U_\pm$ can be expressed in terms of $x, \lambda r^2, m$ and δ as follows:

$$\begin{aligned} U_+ = & \left[-4n^3(n+2)(n+1)^2\delta^2x^2 - 48n^2(n+1)(n-2)\delta x \right.\\ & \left. -16(n-2)(n-4)\right]\lambda r^2 - \delta^3 n^3(3n-2)(n+1)^4(1+m\delta)x^4 \\ & +4\delta^2 n^2(n+1)^2\left\{(n+1)(3n-2)m\delta + 4n^2 + n - 2\right\}x^3 \\ & +4\delta(n+1)\left\{(n-2)(n-4)(n+1)(m+n^2K)\delta - 7n^3 + 7n^2 - 14n + 8\right\}x^2 \\ & +\left\{16(n+1)\left(-4m + 3n^2(n-2)K\right)\delta - 16(3n-2)(n-2)\right\}x \\ & +64m + 16n(n+2)K, \end{aligned} \tag{5.4.47}$$

$$\begin{aligned} U_- = & \left[-4n^3(n+2)(n+1)^2(1+m\delta)^2x^2 + 48n^2(n+1)(n-2)m(1+m\delta)x \right.\\ & \left. -16(n-2)(n-4)m^2\right]y - n^3(3n-2)(n+1)^4\delta(1+m\delta)^3x^4 \\ & -4n^2(n+1)^2(1+m\delta)^2\left\{(n+1)(3n-2)m\delta - n^2\right\}x^3 \\ & +4(n+1)(1+m\delta)\left\{m(n-2)(n-4)(n+1)(m+n^2K)\delta \right.\\ & \quad \left. +4n(2n^2-3n+4)m + n^2(n-2)(n-4)(n+1)K\right\}x^2 \\ & -16m\left\{(n+1)m\left(-4m + 3n^2(n-2)K\right)\delta \right.\\ & \left. +3n(n-4)m + 3n^2(n+1)(n-2)K\right\}x \\ & +64m^3 + 16n(n+2)m^2K. \end{aligned} \tag{5.4.48}$$

By applying the S-deformation to V_+ with

$$S = \frac{f}{h_+}\frac{dh_+}{dr}; \; h_+ = r^{n/2-1}H_+, \tag{5.4.49}$$

we obtain

$$\tilde{V}_{S+} = \frac{k^2 f}{2r^2 H_+}\left[(n-2)(n+1)\delta x + 2\right]. \tag{5.4.50}$$

Since this is positive definite, the electromagnetic mode Φ_+ is always stable for any values of K, M, Q and λ, provided that the spacetime contains a regular black hole, although V_+ has a negative region near the horizon when $\lambda < 0$ and Q^2/M^2 is small.

Using a similar transformation, we can also prove the stability of the gravitational mode Φ_- for some special cases. For example, the S-deformation of V_- with

$$S = \frac{f}{h_-}\frac{dh_-}{dr}; \; h_- = r^{n/2-1}H_- \tag{5.4.51}$$

leads to

$$\tilde{V}_- = \frac{k^2 f}{2r^2 H_-}\left[2m - (n+1)(n-2)(1+m\delta)x\right]. \tag{5.4.52}$$

Table 5.1: **stability of generalised static black holes.**

		Tensor $\forall Q$	Vector $\forall Q$	Scalar $Q = 0$	Scalar $Q \neq 0$
$K = 1$	$\lambda = 0$	OK	OK	OK	$D = 4, 5$ OK $D \geq 6$?
	$\lambda > 0$	OK	OK	$D \leq 6$ OK $D \geq 7$?	$D = 4, 5$ OK $D \geq 6$?
	$\lambda < 0$	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?
$K = 0$	$\lambda < 0$	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?
$K = -1$	$\lambda < 0$	OK	OK	$D = 4$ OK $D \geq 5$?	$D = 4$ OK $D \geq 5$?

For $n = 2$, this is positive definite for $m > 0$. When $K = 1$, $\lambda \geq 0$ and $n = 3$ or when $\lambda \geq 0, Q = 0$ and the horizon is S^4, from $m \geq n + 2$ $(l \geq 2)$ and the behaviour of the horizon value of x (see Ref.[KI04] for details), we can show that $\tilde{V}_{S-} > 0$. Hence, in these special cases, the black hole is stable with respect to any type of perturbation.

However, for the other cases, $\tilde{V}_{S-}$ is not positive definite for generic values of the parameters. The S-deformation used to prove the stability of neutral black holes is not effective either. Recently, Konoplya and Zhidenko studied the stability of this system for $n > 2$ numerically. They found that if $\lambda \geq 0$, the system is stable for $n \leq 9$, i.e., $D \leq 11$[KZ07].

5.4.4 Summary of the Stability Analysis

The results of the stability analysis in this section can be summarised in Table 5.1. In this table, D represents the spacetime dimension, $n + 2$. The results for tensor perturbations apply only for maximally symmetric black holes, while those for vector and scalar perturbations are valid for black holes with generic Einstein horizons, except in the case with $K = 1, Q = 0, \lambda > 0$ and $D = 6$.

Note that this is a summary of the analytic study. As we mentioned above, the stability of AF/dS black hole is shown for $D < 12$ numerically.

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§5.5

Flat black brane

Static flat black brane solutions are perturbatively unstable in contrast to asymptotically simple static black holes discussed in the previous section. This was first shown by Gregory and Laflamme for the s-mode perturbation, i.e. perturbations that is spherically symmetric in the directions perpendicular to the brane[GL93, GL95]. Later on, it was shown that the system has no other unstable modes numerically[SCM05, Kud06]. These analyses however assumed that the frequency of an unstable mode, if it exists, is pure imaginary. In the static system this assumption may appear to be natural, but it is not the case in reality. In this section, we explain this point explicitly by applying the the gauge-invariant formulation in the previous section to this system.

5.5.1 Strategy

Let us rewrite the $(m+n+2)$-dimensional flat black brane solution

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\sigma_n^2 + d\boldsymbol{x}^2, \tag{5.5.1}$$

which is the product of $(n+2)$-dimensional static black hole solution and the m-dimensional Euclidean space, as

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2 d\sigma_n^2 \tag{5.5.2}$$

with the $(m+2)$-dimensional metric

$$ds_{m+2}^2 = g_{ab}(y)dy^a dy^b = -f(r)dt^2 + f(r)^{-1}dr^2 + d\boldsymbol{x}^2. \tag{5.5.3}$$

Then, we can classify metric perturbations into tensor, vector and scalar types with respect to the n-dimensional constant curvature space $\mathscr{K}^n$ with the metric $d\sigma_n^2 = \gamma_{ij}(z)dz^i dz^j$, and apply the gauge-invariant formulation developed in the previous chapter to them. Further, since the background spacetime is homogeneous in the brane direction $\boldsymbol{x}$, for each type of perturbations, we can apply the Fourier transformation with respect to $\boldsymbol{x} = (x^p)$ to the perturbation variable as

$$\delta g_{\mu\nu} = h_{\mu\nu}(t, r, z^i)e^{ik\cdot x}. \tag{5.5.4}$$

Since the background metric is static, we can further apply the Fourier transformation with respect to t to $h_{\mu\nu}$ if necessary and assume that

$$h_{\mu\nu} \propto e^{-i\omega t}. \tag{5.5.5}$$

Hence, we can reduce the Einstein equations for perturbations to a set of ODEs with respect to r. In this section, we assume that $\mathscr{K}^n$ is compact.

5.5.2 Tensor perturbations

The equation for tensor perturbations (4.3.10) with $\tau_T = 0$ reads for the present system

$$-\partial_t^2 H_T + \frac{f}{r^n}\partial_r(r^n f \partial_r H_T) - f\left(\frac{k_T^2 + 2K}{r^2} + k^2\right) H_T = 0. \tag{5.5.6}$$

Let us define the energy integral for a tensor perturbation by

$$E := \int_{r_h}^{\infty} dr\, r^n \left[\frac{1}{f}\dot{H}_T^2 + f(H_T')^2 + \left(\frac{k_T^2 + 2K}{r^2} + k^2\right) H_T^2\right]. \tag{5.5.7}$$

Then, from the perturbation equation, we have $\dot{E} = 2\left[r^n f \dot{H}_T H_T'\right]_{r_h}^{\infty}$. If there exists an unstable solution $H_T \propto e^{-i\omega t}$ with $\operatorname{Im}\omega < 0$, it must fall off exponentially at $r \to \infty$ and vanish at the horizon from the above equation, provided that the solution is uniformly bounded. For such a solution, E becomes constant and contradicts the assumed exponential growth because all terms in the energy integral is non-negative definite. Hence, the black brane solution is stable for tensor perturbations.

5.5.3 Vector perturbations

Basic perturbation equations

Basic gauge-invariant variables for vector perturbations are given by $F^a(t,r)$ with $a = t, r, p$ ($p = 1, \cdots, m$). Among these components, we decompose the part parallel to the brane, F_p, into the longitudinal component F_k proportional to the wave vector k^p and the transversal components $F_p^\perp$ as

$$F_k = ik^p F_p = \partial_p F^p, \quad F_p^\perp = F_p + \frac{ik_p}{k^2} F_k. \tag{5.5.8}$$

With this decomposition, the perturbation equations (4.4.12a) and (4.4.12b) can be written as the four wave equations

$$\frac{1}{f}\partial_t^2 F_t - \frac{f}{r^n}\partial_r(r^n \partial_r F_t) + \frac{nf + m_v + r^2 k^2}{r^2} F_t = \left(f' - \frac{2f}{r}\right)\partial_t F_r, \tag{5.5.9a}$$

$$\frac{1}{f}\partial_t^2 F^r - \frac{f}{r^{n-2}}\partial_r(r^{n-2}\partial_r F^r) + \frac{2(n-1)f + m_v + k^2 r^2}{r^2} F^r = \frac{f'}{f}\partial_t F_t, \tag{5.5.9b}$$

$$\frac{1}{f}\partial_t^2 F_k - \frac{1}{r^n}\partial_r(r^n f \partial_r F_k) + \frac{rf' + nf + m_v + k^2 r^2}{r^2} F_k = \frac{2k^2}{r} F^r, \tag{5.5.9c}$$

$$\frac{1}{f}\partial_t^2 (F^\perp/r) - \frac{1}{r^{n+2}}\partial_r\left[r^{n+2} f \partial_r (F^\perp/r)\right] + \left(k^2 + \frac{m_v}{r^2}\right)(F^\perp/r) = 0, \tag{5.5.9d}$$

and the constraint

$$-\frac{1}{f}\partial_t F_t + \frac{1}{r^{n-1}}\partial_r(r^{n-1} f F_r) + F_k = 0. \tag{5.5.10}$$

With the help of this constraint, the second of the above can be also written as

$$\frac{1}{f}\partial_t^2 F^r - \frac{1}{r^{n-2}}\partial_r\left(r^{n-2} f\partial_r F^r\right) + \frac{(n-1)(2f - rf') + m_v + k^2r^2}{r^2}F^r = f'F_k. \tag{5.5.11}$$

Clearly, the transversal part $F_p^\perp$ decouple from the other modes and each component obeys the same single wave equation. Further, each of (F_t, F^r) and (F^r, F_k) obeys a closed set of equations, and the remaining components F_k and F_t, respectively, are directly determined from them with the help of the above constraint equation.

Master equation

Let us take F^r and F_k as fundamental variables and set

$$\Psi := \begin{pmatrix} r^{n/2}F_k \\ (n+1)r^{n/2-1}F^r + r^{n/2}F_k \end{pmatrix}. \tag{5.5.12}$$

Then, the perturbation equations can be put into the form

$$\omega^2\Psi = \left(-D^2 + V + fA\right)\Psi, \tag{5.5.13}$$

where V is the scalar potential

$$V = f\left[\frac{m_v}{r^2} + k^2 + \frac{n(n+2)}{4r^2}f\right], \tag{5.5.14}$$

and A is the matrix potential

$$A = \begin{pmatrix} \frac{2k^2}{n+1} + \frac{(n+2)f'}{2r} & -\frac{2k^2}{n+1} \\ \frac{2k^2}{n+1} & -\frac{2k^2}{n+1} - \frac{n}{2r}f' \end{pmatrix}. \tag{5.5.15}$$

In order to see whether this set of equations can be reduced into decoupled single equations, we introduce a new vector variable Φ by

$$\Phi = Q\Psi + P\Psi', \tag{5.5.16}$$

where P and Q are matrix functions of r that are independent of ω. If we require that Φ obeys the equation of the form

$$\Phi'' + (\omega^2 - V - W)\Phi = 0 \tag{5.5.17}$$

with a diagonal matrix W, we obtain constraints on V and B.

For the exceptional mode with $m_v = 0$, these constraints are satisfied, and we find that for the choice $P = 1$ and

$$Q = \begin{pmatrix} -\frac{k^2r}{n+1} - \frac{n+2}{2r}f & \frac{k^2r}{n+1} \\ -\frac{k^2r}{n+1} & \frac{k^2r}{n+1} + \frac{n}{2r}f \end{pmatrix}, \tag{5.5.18}$$

W is given by the diagonal matrix whose entries are

$$W_1 = \frac{n+2}{r^2} f \left(1 - \frac{(n+1)M}{r^{n-1}}\right), \quad W_2 = -\frac{n}{r^2} f \left(1 - \frac{(n+1)M}{r^{n-1}}\right). \tag{5.5.19}$$

The corresponding equations for Φ decouple to

$$\Phi_i'' + (\omega^2 - V_i)\Phi_i = 0, \tag{5.5.20}$$

$$V_1 = f\left[k^2 + \frac{n+2}{4r^2}\left(n + 4 - \frac{2(3n+2)M}{r^{n-1}}\right)\right], \tag{5.5.21}$$

$$V_2 = f\left[k^2 + \frac{n}{4r^2}\left(n - 2 + \frac{2nM}{r^{n-1}}\right)\right]. \tag{5.5.22}$$

V_2 is clearly positive. Further, in terms of the S-deformation with $S = (n+2)f/(2r)$, V_1 is transformed into $\tilde{V}_1 = k^2 f > 0$. Hence, this system is stable for this exceptional mode.

If we apply the same transformation in the case $m_v \neq 0$, we obtain

$$\left[(f\partial_r)^2 - \frac{2m_v fh}{r(r^2\omega^2 - m_v f)} f\partial_r + \omega^2 - V_0\right]\Phi = \frac{fh}{(n+1)(r^2\omega^2 - m_v f)} B\Phi, \tag{5.5.23}$$

where $h = 1 - (n+1)M/r^{n-1}$ and

$$V_0 = f\left[\frac{m_v}{r^2} + k^2 + \frac{n^2+2n+4}{4r^2} - \frac{(n^2+4n+2)M}{2r^{n+1}} + \frac{m_v fh}{r^2(r^2\omega^2 - m_v f)}\right], \tag{5.5.24}$$

$$B = \begin{pmatrix} (n+1)^2\omega^2 + 2m_v k^2 & -2m_v k^2 \\ 2m_v k^2 & -\{(n+1)^2\omega^2 + 2m_v k^2\} \end{pmatrix} \tag{5.5.25}$$

Since B is a constant matrix with eigenvalues

$$\lambda = \pm(n+1)\omega\left[(n+1)^2\omega^2 + 4m_v k^2\right]^{1/2}, \tag{5.5.26}$$

we can reduce the set of equations for Φ to decoupled single second-order ODEs. However, these equations are not useful in the stability analysis because their coefficients depend on ω^2 nonlinearly and have singularities in general[1].

Stability analysis

Since we cannot find a convenient master equation, let us try to analyse the stability by directly looking into the structure of the set of equations (5.5.13). The subtle point of this set of equations is that the operator on the right-hand

[1] In Ref[Kud06] the author derived a well-behaved single master equation of 2nd-order for the black string background. There, the author took the gauge in which $f_z = 0$ and $H_T = 0$. Such a gauge cannot be realised in general because the gauge transformations of f_z and H_T are given by $\bar{\delta} f_z = -S\partial_z(L/S)$ and $\bar{\delta} H_T = k_v L/S$. If we set $f_z = 0$, we cannot change the z-dependence of H_T in general.

side is not self-adjoint because A is not a hermitian matrix. Therefore, we cannot directly conclude that ω^2 is real.

Allowing for the possible existence of the imaginary part of ω^2, we obtain the following two integral relations from the above equation:

$$\mathrm{Re}\,(\omega^2)(\Psi,\Psi) = \int_{r_h}^\infty \frac{dr}{f}\left[(D\Psi_1)^2 + (D\Psi_2)^2 + fU_1|\Psi_1|^2 + fU_2|\Psi_2|^2\right], \tag{5.5.27a}$$

$$\mathrm{Im}\,(\omega^2)(\Psi,\Psi) = -\frac{4k^2}{n+1}\int_{r_h}^\infty dr\mathrm{Im}\,(\bar{\Psi}_1\Psi_2). \tag{5.5.27b}$$

Here, $D = fd/dr$ and

$$U_1 = \frac{m_v}{r^2} + \frac{n+3}{n+1}k^2 + \frac{n(n+2)}{4r^2}f + \frac{(n+2)f'}{2r}, \tag{5.5.28a}$$

$$U_2 = U_1 - \frac{4k^2}{n+1} - \frac{n+1}{r}\frac{f'}{.} \tag{5.5.28b}$$

By applying the S-deformation with $S = \frac{n}{2r}f$ to Ψ_2, the right-hand side of the equation corresponding to $\mathrm{Re}\,(\omega^2)$ is deformed to

$$D\Psi_2 \to (D+S)\Psi_2, \quad U_2 \to \frac{m_v}{r^2} + \frac{n-1}{n+1}k^2. \tag{5.5.29}$$

Therefore, if we assume that ω^2 is real, as is assumed in most work, we can conclude that the system is stable against vector perturbations. However, we cannot exclude the possible existence of an unstable mode with $\mathrm{Im}\,(\omega^2) \neq 0$.

5.5.4 Scalar perturbations

Perturbation variables

The gauge-invariant variable set F_{ab} in the general formulation can be decomposed into the scalar, vector and tensor parts by their transformation behavior with respect to the brane coordinates as

Scalar part: $F_{tt}, F_{tr}, F_{rr}, F_{kt}, F_{kr}, F_{kk}, F_\perp$.

Vector part: $F_{\perp pt}, F_{\perp pr}, F_{\perp pk}$.

Tensor part: $F_{\perp p\perp q}$.

Here,

$$F_{ka} = \partial^p F_{pa}/(ik) = (k^p/k)F_{pa}, \tag{5.5.30a}$$

$$F_{\perp pa} = F_{pa} - (k_p/k^2)k^q F_{qa} = F_{pa} - (k_p/k)F_{ka}, \tag{5.5.30b}$$

$$F_{kk} = (k^p k^q/k^2)F_{pq}, \quad F_\perp = F^p_p - F_{kk}, \tag{5.5.30c}$$

$$F_{\perp p\perp q} = F_{\perp pq} - (k_q/k)F_{\perp pk} - \frac{1}{d-1}F_\perp(\delta_{pq} - k_p k_q/k^2). \tag{5.5.30d}$$

The remaining gauge-invariant variable F in the general formulation also belongs to the scalar part. Note that the vector and tensor parts do not exist for the black string background.

S-mode

First, we consider the exceptional mode with $k_s = 0$, which is often called the S-mode. For this exceptional mode, the general gauge-invariant variables reduce to $F_{ab} = f_{ab}$ and $F = H_L$ due to the non-existence of corresponding harmonic vectors and tensors. These variables are not gauge invariant and subject to the gauge transformation law

$$\bar{\delta} H_L = -\frac{f}{r} T_r, \tag{5.5.31a}$$

$$\bar{\delta} f_{tt} = 2i\omega T_t + f f' T_r, \quad \bar{\delta} f_{tr} = i\omega T_r - f(T_t/f)',$$

$$\bar{\delta} f_{rr} = -2T_r' - (f'/f)T_r, \tag{5.5.31b}$$

$$\bar{\delta} f_{tk} = i\omega T_k - ikT_t, \quad \bar{\delta} f_{rk} = -T_k' - ikT_r, \quad \bar{\delta} f_{kk} = -2ikT_k, \tag{5.5.31c}$$

$$\bar{\delta} f_{\perp pt} = i\omega T_{\perp p}, \quad \bar{\delta} f_{\perp pr} = -T_{\perp p}', \quad \bar{\delta} f_{\perp pk} = -ikT_{\perp p}, \tag{5.5.31d}$$

$$\bar{\delta} f_{\perp} = 0, \quad \bar{\delta} f_{\perp p \perp q} = 0. \tag{5.5.31e}$$

In particular, we have

$$\bar{\delta}\left(f_{tk} + \frac{\omega}{2k} f_{kk}\right) = -ikT_t, \quad \bar{\delta}\left(f_{rk} + \frac{i}{2k} f_{kk}'\right) = -ikT_r. \tag{5.5.32}$$

From these, we can construct the following five gauge invariants for the scalar part:

$$r^{2-n} X = -F_{\perp}, \tag{5.5.33a}$$

$$r^{2-n}(X - Y) = F_r^r - 2rF' - \left(\frac{rf'}{f} - 2\right) F, \tag{5.5.33b}$$

$$r^{2-n} Z = F_t^r + \frac{if^2}{2\omega}(F_t^t)' + i\omega r F - \frac{if^2}{2\omega}\left(\frac{rf'}{f} F\right)', \tag{5.5.33c}$$

$$r^{2-n} V^t = F_k^t + \frac{k}{2\omega}\left(F_t^t - \frac{rf'}{f} F\right) - \frac{\omega}{2kf} F_{kk}, \tag{5.5.33d}$$

$$r^{2-n} V^r = F_k^r - ikrF + \frac{if}{2k} F_{kk}' \tag{5.5.33e}$$

For the vector part, we adopt the following two gauge invariants

$$r^{2-n} W_p^t = F_{\perp p}^t - \frac{\omega}{kf} F_{\perp pk}, \quad r^{2-n} W_p^r = F_{\perp p}^r + \frac{if}{k} F_{\perp pk}'. \tag{5.5.34}$$

(1) Tensor part. First, we study the stability in the tensor part. The perturbation variable of this part, $F_{\perp p \perp q}$, follows the closed equation

$$-f(r^n f F_{\perp p \perp q}')' + (k^2 f - \omega^2) r^n F_{\perp p \perp q} = 0. \tag{5.5.35}$$

From this we obtain the integral relation

$$\omega^2 \int_{r_h}^{\infty} dr_* r^n |F_{\perp p \perp q}|^2 = \int_{r_h}^{\infty} dr r^n \left[f|F_{\perp p \perp q}'|^2 + k^2 |F_{\perp p \perp q}|^2\right] - \left[r^n f \bar{F}^{\perp p \perp q} F_{\perp p \perp q}'\right]_{r_h}^{\infty}.$$

(5.5.36)

If there exists an unstable mode with $\omega = \omega_1 + i\omega_2$ $(\omega_2 > 0)$, a solution that is bounded at the horizon behaves as $F_{\perp p \perp q} \sim e^{-i\omega r_*}$ near the horizon. Next, at infinity, the solution behaves

$$F_{\perp p \perp q} \sim \frac{1}{r^{(n-1)/2}} Z_\nu(\sqrt{\omega^2 - k^2} r) \sim r^{-n/2} \exp(\pm i\sqrt{\omega^2 - k^2} r). \tag{5.5.37}$$

Therefore, for an unstable mode that is uniformly bounded, the boundary term in the above integral relation vanishes and the integral at the left-hand side converges. This implies that $\omega^2 > 0$ and leads to contradiction.

(2) Vector part. Next, for the vector part, we obtain the following two equations for the gauge-invariant variables W^t_p and W^r_p:

$$-i\omega \left[{W^t_{\perp p}}' - \left(\frac{n-2}{r} + \frac{f'}{f} \right) W^t_{\perp p} \right] + \frac{k^2 f - \omega^2}{f^2} W^r_{\perp p} = 0, \tag{5.5.38a}$$

$$-i\omega W^t_{\perp p} + {W^r_{\perp p}}' + \frac{2}{r} W^r_{\perp p} = 0. \tag{5.5.38b}$$

Therefore, we can set

$$W^r_{\perp p} = r^{-2}\Phi, \quad i\omega W^t_{\perp p} = r^{-2}\Phi', \tag{5.5.39}$$

and the perturbation equations can be reduced to the following single master equation for Φ;

$$-f(r^{-n} f \Phi')' + (k^2 f - \omega^2) r^{-n} \Phi = 0. \tag{5.5.40}$$

By the same argument for the tensor part, we can show that this equation does not have a uniformly bounded solution with $\mathrm{Im}\,(\omega) > 0$.

(3) Scalar part. Finally for the scalar part, the perturbation equations gives the closed 1st-order set of equations for X, Y, Z, V^t,

$$\begin{aligned} X' = & \frac{1}{k^2 r H f^2}\left[r^2\omega^4 - \omega^2\left\{k^2r^2f + n - n(n+1)x + \frac{3n^2+2n-1}{4}x^2\right\}\right. \\ & \left. -\left(2+\frac{n-5}{2}x\right)k^2Hf\right]X + \frac{1}{k^2rfH}\left\{n\omega^2\left(1-\frac{n+1}{2}x\right)+k^2H^2\right\}Y \\ & +\frac{2i\omega}{k^2f^2H}(n\omega^2-k^2H)Z + \frac{\omega}{krfH}\left\{2\omega^2r^2+(n-1)xH\right\}V^t, \qquad (5.5.41a) \end{aligned}$$

$$\begin{aligned} Y' = & \frac{1}{k^2rf^2H}\left[r^2\omega^4 - \omega^2\left\{2k^2r^2f + n - n(n+1)x + \frac{3n^2+2n-1}{4}x^2\right\}\right. \\ & \left. +r^2k^4f^2 - \left\{n-(n^2+1)x+\frac{(n+1)^2}{4}x^2\right\}k^2f\right]X \\ & +\frac{1}{k^2rfH}\left[n\omega^2\left(1-\frac{n+1}{2}x\right)+(n-1)k^2\left\{n-\frac{5n}{2}x+\frac{3(n+1)}{4}x^2\right\}\right]Y \\ & +\frac{2in\omega}{k^2f^2H}(\omega^2-k^2f)Z+\frac{\omega}{krfH}\left\{2r^2\omega^2-2k^2r^2f+(n-1)xH\right\}V^t, \end{aligned}$$
(5.5.41b)

$$\begin{aligned} Z' = & -i\frac{(n-1)^2x}{2\omega r^2}X + \frac{i}{2r^2\omega}\left\{r^2\omega^2+(n-1)^2x\right\}Y \\ & -\frac{2}{rf}\left(1-\frac{n+1}{2}x\right)Z + ikfV^t, \qquad (5.5.41c) \end{aligned}$$

$$\begin{aligned} (V^t)' = & \frac{\omega}{2k^3f^3H}\left[-r^2\omega^4+\omega^2\left\{2k^2r^2f+n-n(n+1)x+\frac{3n^2+2n-1}{4}x^2\right\}\right. \\ & \left. -r^2k^4f^2 - k^2f\left\{n-2n^2x+\frac{5n^2+2n-3}{4}x^2\right\}\right]X \\ & +\frac{\omega}{2rk^3f^2H}\left[-n\omega^2\left(1-\frac{n+1}{2}x\right)+k^2\left\{n-n(n+1)x+\frac{3n^2+2n-1}{4}x^2\right\}\right]Y \\ & +\frac{i}{k^3f^3H}\left\{-n\omega^4+k^2\omega^2\left(2n-\frac{3n+1}{2}x\right)-k^4fH\right\}Z \\ & +\frac{1}{rk^2f^2H}\left\{-r^2\omega^4+\omega^2\left(k^2r^2f-\frac{n-1}{2}xH\right)+(n-2)k^2f^2H\right\}V^t. \quad (5.5.41d) \end{aligned}$$

and the expression for V^r in terms of these quantities,

$$V^r = \frac{i(\omega^2+k^2f)}{2nk^3rf^2}\left[\left\{(\omega^2+k^2f)r^2+nf^2\right\}X + nf^2Y + \frac{2in\omega^3r}{\omega^2+k^2f}Z + 2\omega kr^2fV^t\right].$$
(5.5.42)

Here, $x = 2M/r^{n-1}$.

From these equations, we find that X obeys the closed 2nd-order ODE

$$\begin{aligned} & -f(fX')' + (n-4)\frac{f^2}{r}X' \\ & \quad + \left[-\omega^2 + f\left(k^2+\frac{n-2}{r^2}\left\{1+(n-2)x\right\}\right)\right]X = 0, \qquad (5.5.43) \end{aligned}$$

which can be put into the canonical form in terms of Φ defined by

$$X = r^{n/2-2}\Phi \tag{5.5.44}$$

as

$$-f(f\Phi')' + \left[-\omega^2 + f\left\{k^2 + \frac{n}{4r^2}(n-2+nx)\right\}\right]\Phi = 0. \tag{5.5.45}$$

It is clear that this equation does not have an unstable mode.

Next, let us define the new variable Ω by

$$\Omega := PX + nf\left(1 - \frac{n+1}{2}x\right)Y + 2in\omega rZ + 2k\omega r^2 fV^t; \tag{5.5.46}$$

$$\begin{aligned} P := &\left[\frac{n+1}{2n}x - \frac{(n-1)x}{2k^2r^2}\left(n - \frac{n+1}{2}x\right)\right]\omega^2 r^2 + \frac{n-1}{2n}xk^2r^2 \\ &-n + n(n+1)x - (3n^2+2n-1)x^2, \end{aligned} \tag{5.5.47}$$

Then, we find that Ω satisfies a closed 2nd-order ODE mod $X = 0$:

$$-f(f\Omega')' + Af\Omega' + (-\omega^2 + V_\Omega)\Omega = BX, \tag{5.5.48}$$

where

$$\begin{aligned} A = &\frac{f}{4r^3gH}\left[\left\{4n^2 + 2(n+1)(n-2)x\right\}k^2r^2\right. \\ &\left.+n^2(n-1)x\left\{3(n+1)x - 2(n+2)\right\}\right], \end{aligned} \tag{5.5.49a}$$

$$\begin{aligned} V_\Omega = &\frac{f}{8r^4g^2H}\left[2\left\{2n-(n+1)x\right\}^2k^4r^4\right. \\ &+\left\{8n^2(n+2) - 4n(n+2)(3n^2+n+2)x\right. \\ &\left.+2n(n+1)(8n^2+5n+5)x^2 - (n+2)(3n-1)(n+1)^2x^3\right\}k^2r^2 \\ &+n^2(n-1)x\left\{n(n+1)^2x^3 - 3(3n-1)(n+1)x^2\right. \\ &\left.\left.+4(2n^2+2n-1)x - 4n^2\right\}\right], \end{aligned} \tag{5.5.49b}$$

$$B = \frac{fg}{nr^2H}\left\{(n+1)\omega^2 - k^2\right\}\left[-2k^2r^2(1-nx) + n(n-1)x(n-x)\right], \tag{5.5.49c}$$

$$H := k^2 + \frac{n(n-1)}{2r^2}x, \quad g := n - \frac{n+1}{2}x. \tag{5.5.49d}$$

By the transformation

$$\Omega = r^{n/2}gH\Psi, \tag{5.5.50}$$

we can put this equation into the canonical form

$$-f(f\Psi')' + (-\omega^2 + V)\Psi = r^{n/2}gHBX, \tag{5.5.51}$$

where $V = fU/H^2$ with

$$\begin{aligned} U = &k^6 + \frac{(n+4)k^4}{4r^2}(n+2-3nx) - \frac{n(n-1)k^2}{4r^4}\left\{3n(n+2)\right. \\ &\left.-(2n^2+3n+4)x\right\}x + \frac{n^3(n-1)^2}{16r^6}x^2(n-2+nx). \end{aligned} \tag{5.5.52}$$

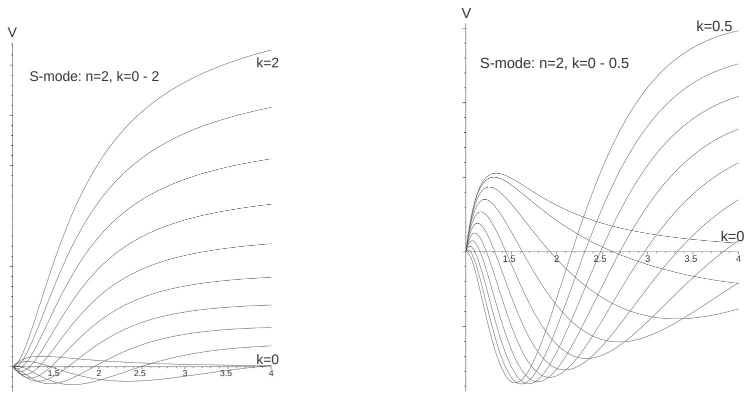

Figure 5.6: The effective potential for S-modes

This potential has a deep negative region for $0 < k < k_n$ with some constant k_n dependent on n. It has been shown by numerical calculations[SCM05, Kud06] that the eigenvalue ω^2 becomes negative for some range $0 < k < k_c$, as first pointed out by Gregory and Laflamme using a different reduction[GL93].

Generic scalar perturbation

(1) Tensor part. The tensor part of generic scalar-type perturbations obeys the decoupled 2nd-order ODE

$$-f(r^n f F'_{\perp p \perp q})' + \left(-\omega^2 + k^2 f + \frac{n+m}{r^2} f\right) r^n F_{\perp p \perp q} = 0. \tag{5.5.53}$$

It is obvious that this equation has no unstable mode.

(2) Vector part. In terms of the gauge-invariant fundamental variables

$$V_t = r^{n-2} F_{t\perp p}, \quad V_r = r^{n-2} F_{r\perp p}, \quad V_k = r^{n-2} F_{k\perp p}, \tag{5.5.54}$$

the perturbation equations for the vector part are expressed as

$$-(fV_r)' - i\omega f^{-1} V_t - ikV_k = 0, \tag{5.5.55a}$$

$$i\frac{\omega}{f}\left(V'_t - \frac{n-2}{r} f V_t\right) + \left(-\frac{\omega^2}{f} + k^2 + \frac{n+m}{r^2}\right) V_r + ik\left(V'_k - \frac{n-2}{r} V_k\right) = 0, \tag{5.5.55b}$$

$$-r^{n-4}(r^{4-n} f V'_k)' + \left(-\frac{\omega^2}{f} + \frac{n+m}{r^2} + \frac{n-2}{r} f' + \frac{n-2}{r^2} f\right) V_k - \frac{k\omega}{f} V_t + ik\left((fV_r)' + \frac{2f}{r} V_r\right) = 0, \tag{5.5.55c}$$

$$-r^{n-4} f (r^{4-n} V'_t)' + \left(\frac{m+n}{r^2} + \frac{n-2}{r^2}\right) V_t - i\omega f\left(V'_r + \frac{2}{r} V_r\right) + k\omega V_k = 0. \tag{5.5.55d}$$

By eliminating V_t and introducing the new variables Y and Z by

$$\Phi = \begin{pmatrix} Z \\ Y \end{pmatrix}; \quad V_k = -ir^{n/2-2}Z, \quad V_r = f^{-1}r^{n/2-1}Y, \tag{5.5.56}$$

this set of equations are reduced to a set of two ODEs,

$$D^2\Phi - \left\{-\omega^2 + k^2 f + \frac{n+m}{r^2}f + \frac{n(n-2)}{4}f^2\right\}\Phi = A\Phi; \tag{5.5.57}$$

$$A = \begin{pmatrix} \frac{nf'}{2r}f & -2kf \\ -\frac{kf'}{r}f & -\frac{(n-2)f'}{2r}f \end{pmatrix}. \tag{5.5.58}$$

This set of equations has the same structure as that for vector perturbations and can be shown to have no unstable mode if ω^2 is real.

(3) Scalar part. Finally, we discuss the scalar part of the generic scalar-type perturbation. Utilising one of the Einstein equations

$$E_T \equiv 2(n-2)F + F^a_a = 0, \tag{5.5.59}$$

the basic perturbation variables can be expressed in terms of X, Y, Z, V^t, V^r, S and Ψ as

$$\tilde{F}^t_t = X + 2\tilde{F} - k^2 f V^t, \quad \tilde{F}^r_r = Y + 2\tilde{F}, \quad \tilde{F}^r_t = i\omega Z, \tag{5.5.60a}$$
$$\tilde{F}^r_k = ikV^r, \quad \tilde{F}^t_k = \omega k V^t, \quad \tilde{F}_{kk} = S + \omega^2 V^t + 2\tilde{F}, \tag{5.5.60b}$$
$$2(n+1)\tilde{F} = -\Psi - X - Y - S - (\omega^2 - k^2 f)V^t, \quad \tilde{F}_\perp = \Psi. \tag{5.5.60c}$$

Here, $\tilde{Q} = r^{n-2}Q$ in general.

In terms of these variables, the Einstein equations can be reduced to the decoupled single equation for Ψ,

$$-r^{-n}f(r^n f\Psi')' + \left[-\omega^2 + \left(k^2 + \frac{n+m}{r^2}\right)f\right]\Psi = 0. \tag{5.5.61}$$

and the regular 1st-order set of ODEs for X, Y, Z, V^t, V^r and S,

$$Z' = X, \tag{5.5.62a}$$
$$X' = \frac{n-2}{r}X + \left(\frac{f'}{f} - \frac{2}{r}\right)Y + \frac{1}{f}\left(-\frac{\omega^2}{f} + k^2 + \frac{m+n}{r^2}\right)Z + k^2 f' V^t, \tag{5.5.62b}$$
$$Y' = \frac{f'}{2f}(X - Y) + \frac{\omega^2}{f^2}Z + \frac{k^2}{f}\left(V^r - \frac{ff'}{2}V^t\right), \tag{5.5.62c}$$
$$(V^r)' = -S, \tag{5.5.62d}$$
$$S' = \frac{n-2}{r}S - \frac{2}{r}Y + \omega^2\frac{f'}{f}V^t + \frac{1}{f}\left(\frac{\omega^2}{f} - k^2 - \frac{n+m}{r^2}\right)V^r, \tag{5.5.62e}$$

$$
\begin{aligned}
k^2r^2f'f^2(V^t)' = &\left[2\omega^2r^2 + (n-1)x\left(n - \frac{n+1}{2}x\right)\right]X \\
&+\left[2\omega^2r^2 - 2(k^2r^2+n+m)f + 2n - 4nx + \frac{(n+1)^2}{2}x^2\right]Y \\
&+\frac{1}{r}\left[-2n\omega^2r^2 + (n-1)x(k^2r^2+n+m)\right]Z \\
&-(n-1)k^2x\left(2+\frac{n-5}{2}x\right)fV^t - 2k^2r\left(n-\frac{n+1}{2}x\right)V^r \\
&-2k^2r^2fS, \qquad (5.5.62f)
\end{aligned}
$$

where $x = 2M/r^{n-1}$.

If we define X_1, X_2, X_3 by

$$
\begin{aligned}
&X_1 = Z, \quad X_2 = r(X+Y) - nZ + k^2rfV^t, \\
&X_3 = -\left(1+\frac{rf'}{2nf}\right)[r(X+Y)-nZ] - \frac{k^2r^2}{2n}f'V^t, \qquad (5.5.63)
\end{aligned}
$$

and introduce Φ by

$$
\Phi := \begin{pmatrix} r^{-n/2}X_1 \\ r^{-n/2+1}f^{-1}X_2 \\ r^{-n/2}X_3 \end{pmatrix}, \qquad (5.5.64)
$$

we can reduce the above set of 1st-order ODEs to the set of 2nd-order ODEs of the normal eigenvalue type as

$$
\omega^2\Phi = (-D^2 + V_0 + W)\Phi. \qquad (5.5.65)
$$

Here, V_0 is the scalar potential

$$
V_0 = \frac{f}{4r^2}\left[4(m+k^2r^2) + n^2 - 2n + n(n+4)x\right], \qquad (5.5.66)
$$

and W is the following matrix of rank 3:

$$
\begin{aligned}
&W_{11} = 0, \quad W_{12} = \frac{(n^2-1)xf^2}{nr^3}, \quad W_{13} = \frac{2f^2}{r^2}, \qquad (5.5.67a) \\
&W_{21} = \frac{\{4-2(n+1)x\}k^2r^2 - 2(n-1)mx - n(n^2-1)x}{rf}, \qquad (5.5.67b) \\
&W_{22} = \frac{nf^2}{r^2}, \quad W_{23} = 0, \qquad (5.5.67c) \\
&W_{31} = \frac{1}{2nr^2f}\left[2(n-1)x(n-2+x)r^2k^2 + \left\{4n + 2n(n-5)x + 2(n+1)x^2\right\}m\right. \\
&\qquad \left. + n(n+1)x\left\{2n^2 - (2n^2+3n-1)x + n(n+1)x^2\right\}\right], \qquad (5.5.67d) \\
&W_{32} = \frac{(n^2-1)x\{2-(n+1)x\}f}{2nr^3}, \quad W_{33} = \frac{(n+1)\{2-(n+1)x\}f}{r^2}. \qquad (5.5.67e)
\end{aligned}
$$

Unfortunately, it is not possible to analyse the stability of this 2nd-order system by an analytic method, partly because it is not a self-adjoint system. However, all numerical calculations done by various authors have found no evidence of instability for this system[SCM05, Kud06].

6 Lovelock Black Holes

When Einstein derived the field equations for gravity, he adopted the three requirements as the guiding principle in addition to the principle of general relativity. The first is the metric ansatz that gravitational field is completely determined by the spacetime metric. The second is that the energy-momentum tensor sources gravity, hence it is balanced by a second-rank symmetric tensor constructed from the metric. The last is the requirement that the field equations contain the second derivatives of the metric at most and are quasi-linear, i.e., the coefficients of the second derivatives contain only the metric and its first derivatives. These requirements determine the gravitational field equation uniquely.

In 4-dimensions, we can obtain a similar result even if we loosen the third condition and require only that the field equations contain second derivatives of the metric at most. To be precise, gravity theories satisfying these weaker requirements are the Einstein gravity and the $f(R)$ gravity, the latter of which is mathematically equivalent to the Einstein gravity coupled with a scalar field with a non-trivial potential.

When we extend general relativity to higher dimensions, however, the difference between the two versions of the third requirement becomes important. In fact, we require the stronger version, we obtain the same field equations for gravity in higher dimensions. In contrast, we require the weaker version, we obtain a larger class of theories that contains the higher-dimensional general relativity as a special case. Such extensions to higher dimensions were first studied systematically by D. Lovelock in 1971[Lov71]. What he found was that the second-rank gravitational tensor $E_{\mu\nu}$ balancing the energy-momentum tensor $T_{\mu\nu}$ is a sum of polynomials in the curvature tensor and that the polynomial of each degree is unique up to a proportionality constant. The physical importance of such an extension was later recognised when B. Zwiebach pointed out that the special quadratic combinations of the curvature tensor named the Gauss-Bonnet term naturally arises when we add quadratic terms of the Ricci curvature to the quadratic term in the Riemann curvature tensor obtained as the α' correction to the field equations in the heterotic string theory to obtain a ghost-free theory[Zwi85]. In this section, we briefly overview the Lovelock theory, its static black hole solution and perturbation theory recently developed by Takahashi and Soda[TS09a, TS10b].

§6.1
Lovelock theory

In 1986, B. Zumino pointed out that the class of theories obtained by Lovelock has a natural mathematical meaning. That is, the Lovelock equations for gravity can be obtained from the action that is a linear combination of the terms each of which corresponds to the Euler form in some even dimensions:

$$\begin{aligned} S &= \int \sum_{k=0}^{[(D-1)/2]} \alpha_k \mathscr{L}_k; && (6.1.1)\\ \mathscr{L}_k &:= \frac{1}{(D-2k)!}\epsilon_{L_1\cdots L_{2k}M_1\cdots M_{D-2k}}\mathscr{R}^{L_1L_2}\wedge\cdots\wedge\mathscr{R}^{L_{2k-1}L_{2k}}\wedge\theta^{M_1\cdots M_{D-2k}} \\ &= I_k\Omega_D, && (6.1.2)\end{aligned}$$

where $\mathscr{R}^L{}_M$ is the curvature form with respect to the orthonormal 1-form basis θ^L $(L, M = 0, \cdots, D-1)$, $\theta^{L\cdots M} = \theta^L\wedge\cdots\theta^M$, Ω_D is the volume form, and

$$I_k = \frac{1}{2^k}\delta^{L_1\cdots L_{2k}}_{M_1\cdots M_{2k}} R_{L_1L_2}{}^{M_1M_2}\cdots R_{L_{2k-1}L_{2k}}{}^{M_{2k-1}M_{2k}}. \tag{6.1.3}$$

The explicit expressions for small values of k are

$$\begin{aligned} I_0 &= 1, && (6.1.4a)\\ I_1 &= R, && (6.1.4b)\\ I_2 &= R^2 - 4R^L_M R^M_L + R_{LM}{}^{NP}R_{NP}{}^{LM}, && (6.1.4c)\\ I_3 &= R^3 - 3R(-4R^L_M R^M_L + R_{LM}{}^{NP}R_{NP}{}^{LM}) + 24R^L_M R^M_N R^N_L + 3R_{LM}{}^{NP}R_{NP}{}^{QS}R_{QS}{}^{LM}. && (6.1.4d)\end{aligned}$$

Hence, if we require that the theory has the Einstein theory in the low energy limit, α_0 and α_1 are related to the cosmological constant Λ and the Newton constant $\kappa^2 = 8\pi G$ as

$$\alpha_0 = -\frac{\Lambda}{\kappa^2}, \quad \alpha_1 = \frac{1}{2\kappa^2}. \tag{6.1.5}$$

Further, if the Gauss-Bonnet term $\mathscr{L}_2$ comes from the $O(\alpha')$ correction in the heterotic string theory, $\alpha_2 > 0$.

From the Bianchi identity $\mathscr{D}\mathscr{R}_{LM} \equiv 0$, the variation of the Lagrangian density can be written

$$\begin{aligned} \delta\mathscr{L}_k &= d(*) + \frac{k}{(D-2k-1)!}\epsilon_{L_1\cdots L_{2k}M_1\cdots M_{D-2k}}\delta\omega^{L_1L_2}\wedge\mathscr{R}^{L_3L_4}\wedge\cdots\Theta^{M_1}\wedge\theta^{M_2\cdots M_{D-2k}} \\ &\quad + \frac{(-1)^{D-1}}{(D-2k-1)!}\epsilon_{L_1\cdots L_{2k}M_1\cdots M_{D-2k}}\mathscr{R}^{L_1L_2}\wedge\cdots\mathscr{R}^{L_{2k-1}L_{2k}}\wedge\theta^{M_1\cdots M_{D-2k-1}}\wedge\delta\theta^{M_{D-2k}} \\ &= d(*) + \left\{kT^{(k)L}{}_{M_1M_2}(\delta\omega^{M_1M_2})_L + (-1)^{D-1}E^{(k)L}{}_M\delta\theta^M_\mu e^\mu_L\right\}\Omega_D, && (6.1.6)\end{aligned}$$

where $\omega^L{}_M$ is the connection form with respect to θ^L, e_L is the vector basis dual to θ^L, $\mathscr{D}$ is the corresponding covariant exterior derivative, $\Theta^L = \mathscr{D}\theta^L$ is the torsion 2-form, and

$$T^{(k)L}{}_{M_1M_2} = \delta^{LN_1\cdots N_{2k}}_{M_1M_2P_1\cdots P_{2k-1}} R^{P_1P_2}{}_{N_1N_2}\cdots R^{P_{2k-3}P_{2k-2}}{}_{N_{2k-3}N_{2k-2}} T^{P_{2k-1}}_{N_{2k-1}N_{2k}}, \tag{6.1.7a}$$
$$E^{(k)L}{}_M = -\frac{1}{2^k}\delta^{LL_1\cdots L_{2k}}_{MM_1\cdots M_{2k}} R_{L_1L_2}{}^{M_1M_2}\cdots R_{L_{2k-1}L_{2k}}{}^{M_{2k-1}M_{2k}}. \tag{6.1.7b}$$

Hence, if we treat θ^L and $\omega^L{}_M$ as independent dynamical variables, the field equations are given by

$$\sum_k \alpha_k E^{(k)L}{}_M = 0, \tag{6.1.8a}$$
$$\sum_k k\alpha_k T^{(k)L}{}_{MN} = 0. \tag{6.1.8b}$$

If we require the connection to be Riemannian, the second equation becomes trivial due to the torsion free condition $T^L_{MN} = 0$. Examples of the explicit expressions for $E^{(k)L}_M$ and $T^{(k)L}_{MN}$ are

$$E^{(0)L}_M = -\delta^L_M, \tag{6.1.9a}$$
$$E^{(1)L}_M = 2R^L_M - R\delta^L_M, \tag{6.1.9b}$$
$$E^{(2)L}_M = -\delta^L_M I_2 + 4RR^L_M - 8R^{LN}{}_{MP}R^P_N + 8R^L_N R^N_M + 4R^{LN_1N_2N_3}R_{MN_1N_2N_3}, \tag{6.1.9c}$$
$$T^{(1)L}_{MN} = T^L_{MN} + 2\delta^L_{[M}T^P_{N]P}, \tag{6.1.9d}$$
$$T^{(2)L}_{MN} = 8\delta^L_{[M}(-2R^P_{N]}T_P + R^{P_1P_2}{}_{N]P_3}T^{P_3}_{P_1P_2} + RT_{N]} - 2R^{P_1}_{P_2}T^{P_2}_{N]P_1}) + 12(R^{LP}{}_{MN}T_P - 2R^L_{[M}T_{N]} - 2R^{LP_1}{}_{P_2[M}T^{P_2}_{N]P_1} - 12R^L_P T^P_{MN}). \tag{6.1.9e}$$

§6.2
Static black hole solution

6.2.1 Constant curvature spacetimes

In general relativity, a constant curvature spacetime is always a vacuum solution and the curvature is uniquely determined by the value of the cosmological constant. This feature is not shared by the Lovelock theory. In fact, the Lovelock theory does not allow a vacuum constant curvature solution for some range of the coupling constants $\{\alpha_k\}$, and have multiple constant curvature solutions with different curvatures for other range of the coupling constants.

To see this, let us insert the constant Riemann curvature

$$R_{\mu\nu\lambda\sigma} = \lambda(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) \tag{6.2.1}$$

into the field equation (6.1.7b). Then, we obtain

$$P(\lambda) = 0, \tag{6.2.2}$$

where

$$P(X) := \sum_{k=0}^{[(D-1)/2]} \alpha_k \frac{X^k}{(D-2k-1)!}. \tag{6.2.3}$$

For general relativity for which $\alpha_k = 0$ for $k \geq 2$, this equation has a unique solution. In contrast, when $D > 4$ and $\alpha_k \neq (k \geq 2)$, the equation can have no solution or multiple solutions depending on the functional shape of $P(X)$.

6.2.2 Black hole solution

Now, let us look for spherically symmetric black hole solutions. Because the Birkhoff-type theorem holds for the Lovelock theory except for the case in which $P(\lambda) = 0$ has a root with multiplicity higher than one[Zeg05, Wil86], we only consider static spacetimes whose metric can be put into the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\sigma_n^2, \tag{6.2.4}$$

where $d\sigma_n^2$ represents the metric of a constant curvature space with sectional curvature K. For a spherically symmetric solution, $K = 1$. However, because the argument in this section holds for any value of K, we consider this slightly general spacetime.

Then, the non-vanishing components of the curvature tensor are given up to symmetry by

$$R^{01}{}_{01} = \frac{h}{2}\left(-\frac{f''}{f} + \frac{(f')^2}{2f^2}\right) - \frac{h'f'}{4f}\,, \tag{6.2.5a}$$

$$R_{0i0j} = \frac{hf'}{2rf} g_{ij}\,, \quad R_{1i1j} = -\frac{h'}{2r} g_{ij}\,, \tag{6.2.5b}$$

$$R_{ijkl} = X(g_{ik}g_{jl} - g_{il}g_{jk})\,, \tag{6.2.5c}$$

where $X(r) := (K - h(r))/r^2$. Inserting these into (6.1.7b), we find that the field equations reduce to

$$(r^{n+1}P(X(r)))' = 0\,, \quad P^{(1)}(X(r))(f(r)/h(r))' = 0\,. \tag{6.2.6}$$

The first of these is integrated to yield

$$P(X(r)) = \frac{C}{r^{n+1}}\,, \tag{6.2.7}$$

where C is an integration constant. This determines the function $h(r)$ implicitly. In particular, for $C = 0$, we have $h(r) = K - \lambda r^2$ for each solution to $P(\lambda) = 0$. If

$P^{(1)}(\lambda) \neq 0$, after an appropriate scaling of t, $f(r) = h(r)$ follows from the second of the above field equations:

$$ds^2 = -(K - \lambda r^2)dt^2 + \frac{dr^2}{K - \lambda r^2} + r^2 d\sigma_n^2 . \tag{6.2.8}$$

This represents a constant curvature spacetime with sectional curvature λ irrespective of the value of K, as is well known.

This implies that for $C \neq 0$, we have in general multiple solutions corresponding to multiple solutions to $P(\lambda) = 0$. Each solution approaches a constant curvature spacetime with sectional curvature λ at large r asymptotically. For these solutions, we can always put $f(r) = h(r)$ and the metric can be written[Whe86]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma_n^2, \quad f(r) = K - X(r)r^2. \tag{6.2.9}$$

In general, the constant C is proportional to the total mass M of the system and positive if $M > 0$. We can easily show that $X(r)$ changes monotonically with r from infinity to some value of r where the metric becomes singular. This singularity may or may not be hidden by a horizon depending on the functional shape of $P(X)$. In the former case, we obtain a regular black hole solution.

§6.3 Perturbation equations for the static solution

The linear perturbation of the Lovelock tensor (6.1.7b) in general reads

$$\delta E^\mu_\nu = -\sum_{k=1}^{[(D-1)/2]} \frac{k\alpha_k}{2^k} \delta^{\mu\mu_1\cdots\mu_{2k}}_{\nu\nu_1\cdots\nu_{2k}} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} \cdots R_{\mu_{2k-3}\mu_{2k-2}}{}^{\nu_{2k-3}\nu_{2k-2}} \delta R_{\mu_{2k-1}\mu_{2k}}{}^{\nu_{2k-1}\nu_{2k}} . \tag{6.3.1}$$

Inserting (6.2.5) with $f(r) = h(r) = K - X(r)r^2$ into this yields[TS10b]

$$r^{n-1}\delta E^t_t = -\frac{rT'}{n-1}\delta R_{ij}{}^{ij} - 2T\delta R_{ir}{}^{ir}, \tag{6.3.2a}$$

$$r^{n-1}\delta E^r_t = -2T\delta R_{it}{}^{ir}, \tag{6.3.2b}$$

$$r^{n-1}\delta E^r_r = -\frac{rT'}{n-1}\delta R_{ij}{}^{ij} - 2T\delta R_{it}{}^{it}, \tag{6.3.2c}$$

$$r^{n-1}\delta E^i_a = \frac{2rT'}{n-1}\delta R_{aj}{}^{ij} + 2T\delta R_{ab}{}^{ib}, \tag{6.3.2d}$$

$$\begin{aligned} r^{n-1}\delta E^i_j = {} & \frac{2rT'}{n-1}\delta R_{aj}{}^{ai} + \frac{2r^2T''}{(n-1)(n-2)}\delta R_{jk}{}^{ik} \\ & -\delta^i_j\left[2T\delta R_{tr}{}^{tr} + \frac{2rT'}{n-1}\delta R_{ak}{}^{ak} + \frac{r^2T''}{(n-1)(n-2)}\delta R_{kl}{}^{kl}\right], \end{aligned} \tag{6.3.2e}$$

where

$$T(r) := r^{n-1}P^{(1)}(X(r)). \tag{6.3.3}$$

Thus, we can obtain perturbation equations for the metric in the Lovelock theory simply by calculating the perturbation of the curvature tensor.

6.3.1 Tensor perturbations

For tensor perturbations, the metric perturbation can be expanded in terms of the harmonic tensor $\mathbb{T}_{ij}$ as (4.3.7). The non-vanishing components of the curvature tensor for this type of perturbations read

$$\delta R_{ai}{}^{aj} = -\left(\Box H_T + \frac{2}{r}Dr\cdot DH_T\right)\mathbb{T}_i^j, \tag{6.3.4a}$$

$$\delta R_{ik}{}^{jk} = \left[-(n-2)\frac{f'}{r}H_T' + \frac{2K+k_t^2}{r^2}H_T\right]\mathbb{T}_i^j. \tag{6.3.4b}$$

Hence, from the above expression for δE_i^j, we obtain the following wave equation for H_T:

$$\frac{1}{f}\ddot{H}_T - fH_T'' - \left(f\frac{T''}{T'} + \frac{2f}{r} + f'\right)H_T' + \frac{2K+k_t^2}{(n-2)r}\frac{T''}{T'}H_T = 0. \tag{6.3.5}$$

If we introduce the mode function $\Psi(r)$ by

$$H_T(t,r) = \frac{\Psi(r)}{r\sqrt{T'(r)}}e^{-i\omega t}, \tag{6.3.6}$$

this wave equation can be put into the standard form

$$-\frac{d^2\Psi}{dr_*^2} + V_t\Psi = \omega^2\Psi, \tag{6.3.7}$$

with the effective potential

$$V_t(r) = \frac{(2K+k_t^2)f}{(n-2)r}\frac{T''}{T'} + \frac{1}{r\sqrt{T'}}\frac{d^2(r\sqrt{T'})}{dr_*^2}, \tag{6.3.8}$$

where $dr_* = dr/f(r)$ as in the Einstein black hole case.

6.3.2 Vector perturbations

For vector perturbations, the perturbation of components of the curvature tensor that are relevant to the field equations can be expressed in terms of the basic gauge-invariant quantities F_a as

$$\delta R_{aj}{}^{ai} = -\frac{k}{r^2} D^a(rF_a)\mathbb{V}^i_j, \tag{6.3.9a}$$

$$\delta R_{jk}{}^{ik} = -\frac{(n-2)k}{r}\frac{D^a r}{r} F_a \mathbb{V}^i_j, \tag{6.3.9b}$$

$$\delta R_{aj}{}^{ij} = \frac{n-1}{2r^2}\left[-D^b F^{(1)}_{ba} - \frac{k^2-(n-1)K}{(n-1)r}F_a + \frac{2(K-f)+rf'}{r} f_a\right]\mathbb{V}^i, \tag{6.3.9c}$$

$$\delta R_{ab}{}^{ib} = \left[-\frac{1}{2r^3}D^b(r^2 F^{(1)}_{ba}) + \frac{rf''-f'}{2r^2} f_a\right]\mathbb{V}^i. \tag{6.3.9d}$$

Inserting these into (6.3.2d), we obtain the following equations for the gauge-invariant variables:

$$\frac{1}{r^2}D^b\left(r^2 T F^{(1)}_{ba}\right) + T'\frac{k^2-(n-1)K}{(n-1)r}F_a = 0, \tag{6.3.10a}$$

$$\frac{1}{r}D^a(rT'F_a) = 0. \tag{6.3.10b}$$

The gauge-dependent residuals in the expressions for the curvature tensor cancel exactly owing to the identity

$$P^{(1)}X'' + P^{(2)}(X')^2 + \frac{n+2}{r}P^{(1)}X' = 0 \tag{6.3.11}$$

obtained from the background equation $(r^{n+1}P(X))' = 0$. We can easily confirm that for general relativity for which $T = r^{n-1}/(2\kappa^2)$, these reduce to (4.4.12a) and (4.4.12b) with no source terms.

A master equation for vector perturbations in the Lovelock theory can be derived in the same way as that in general relativity. First, the second perturbation equation implies the existence of a potential Ω in which F_a can be expressed as

$$rT'F_a = \epsilon_{ab}D^b\Omega. \tag{6.3.12}$$

Inserting this into the first perturbation equation, we easily find that it is equivalent to

$$rTD_a\left(\frac{1}{r^2T'}D^a\Omega\right) - \frac{k^2-(n-1)K}{(n-1)r^2}\Omega = 0. \tag{6.3.13}$$

6.3.3 Scalar perturbations

For scalar perturbations, we have

$$\begin{aligned}
\delta R^{ai}{}_{bi} &= \frac{k^2}{2r^2}F^a_b + \frac{nf}{r}D_{[b}F^a_{r]} + \frac{n}{2r}D^aF^r_b + \frac{nf'}{2r}F^a_b - nD_bD^aF \\
&\quad -\frac{n}{r}(D^arD_b + D_brD^a)F + \frac{n}{2}X^cD_c\left(\frac{f'}{r}\right)\delta^a_b, && (6.3.14a)\\
\delta R^{ai}{}_{aj} &= -\frac{k^2}{2r^2}F^a_a\mathbb{S}^i_j + \left\{\frac{D_ar}{r}D_bF^{ab} - \frac{1}{2r}Dr\cdot DF^a_a\right. \\
&\quad \left. + \left(\frac{f'}{2r} + \frac{k^2}{2nr^2}\right)F^a_a - \frac{1}{r^2}D^a(r^2D_aF) + X^aD_a\left(\frac{f'}{r}\right)\right\}\delta^i_j\mathbb{S}, && (6.3.14b)\\
\delta R^{ik}{}_{jk} &= (n-1)\left[\frac{D^arD^br}{r^2}F_{ab} - \frac{2}{r}Dr^aD_aF + \frac{2(k^2-nK)}{nr^2}F\right. \\
&\quad \left. -D^b\left(\frac{K-f}{r^2}\right)X_b\right]\delta^i_j\mathbb{S} - (n-2)\frac{k^2}{r^2}F\mathbb{S}^i_j, && (6.3.14c)\\
\delta R^{ib}{}_{ab} &= \left[-\frac{k}{r}D_{[b}\left(\frac{1}{r}F^b_{a]}\right) + \frac{1}{2kr}(f' - rf'')D_aH_T\right]\mathbb{S}^i, && (6.3.14d)\\
\delta R^{ij}{}_{aj} &= (n-1)k\left[-\frac{D_br}{2r^3}F^b_a + \frac{1}{r^2}D_aF - \frac{2(K-f)+rf'}{2r^2k^2}D_aH_T\right]\mathbb{S}^i, && (6.3.14e)\\
\delta R^{ab}{}_{ab} &= \left\{-\Box F^a_a + D^aD^bF_{ab} + \frac{f''}{2}F^a_a + X^aD_af''\right\}\mathbb{S}\,. && (6.3.14f)
\end{aligned}$$

Inserting these into (6.3.2), we obtain

$$\begin{aligned}
r^{n-1}\delta E^t_t &\equiv \left[-\frac{nfT}{r}(F^r_r)' - \left(\frac{nf'}{r}T + \frac{nT'}{r}f + \frac{k^2}{r^2}T\right)F^r_r + 2nfTF''\right. \\
&\quad \left. + \left(\frac{4nf}{r}T + 2nfT' + nf'T\right)F' - 2T'\frac{k^2-nK}{r}F\right]\mathbb{S} = 0, && (6.3.15a)\\
r^{n-1}\delta E^r_t &\equiv \left[-\frac{nfT}{r}\left[\dot{F}^r_r + \frac{k^2}{nrf}F^r_t - 2\sqrt{f}\left(\frac{r}{\sqrt{f}}\dot{F}\right)'\right]\right]\mathbb{S} = 0\,, && (6.3.15b)\\
r^{n-1}\delta E^r_r &\equiv \left[\frac{Tnf}{r}\left\{(F^t_t)' - \frac{k^2}{nfr}F^t_t\right\} - \frac{2nfT}{r}\dot{F}^t_r - \frac{nfT}{r}\left(\frac{f'}{f} + \frac{T'}{T}\right)F^r_r\right. \\
&\quad \left. -\frac{2nT}{f}\ddot{F} + nfT\left(\frac{f'}{f} + 2\frac{T'}{T}\right)F' - 2T'\frac{k^2-nK}{r}F\right]\mathbb{S} = 0, && (6.3.15c)\\
r^{n-1}\delta E^i_a &\equiv \frac{k}{r}\left[-D_b\left(\frac{T}{r}F^b_a\right) + TD_a\left(\frac{1}{r}F^b_b\right) + 2T'D_aF\right]\mathbb{S}^i = 0\,, && (6.3.15d)\\
r^{n-1}\delta E^i_j &\equiv -\frac{k^2}{n-1}\left(\frac{T'}{r}F^a_a + 2T''F\right)\mathbb{S}^i_j \\
&\quad - \left[-D^a(TD_aF^b_b) + \left(\frac{(f'T)'}{2} + \frac{k^2T'}{nr}\right)F^a_a + D^aD^b(TF_{ab})\right. \\
&\quad \left. -\frac{2}{r}D^a\left(r^2T'D_aF\right) + \frac{2(k^2-nK)}{n}T''F\right]\delta^i_j\mathbb{S} = 0\,. && (6.3.15e)
\end{aligned}$$

As was first shown by Takahashi and Soda[TS10b], we can reduce these equations to a single master equation in terms of the master variable Φ defined by

$$F^r_t = r(\dot{\Phi} + 2\dot{F}), \qquad (6.3.16)$$

as

$$\ddot{\Phi} - \frac{fA^2}{r^2T'}\left(\frac{r^2fT'}{A^2}\Phi'\right)' + Q\Phi = 0, \tag{6.3.17}$$

where

$$A = 2k^2 - 2nf + nrf', \tag{6.3.18a}$$

$$Q = \frac{f}{nr^2T}\left[r(k^2T + nrfT')\left(2\frac{(AT)'}{AT} - \frac{T''}{T'}\right) - n(r^2fT')'\right]. \tag{6.3.18b}$$

The other gauge-invariant variables are expressed in terms of Φ as

$$F = -\frac{1}{A}\left\{nrf\Phi' + \left(k^2 + nrf\frac{T'}{T}\right)\Phi\right\}, \tag{6.3.19a}$$

$$F^r_r = -\frac{k^2}{nf}\Phi + 2rF' - \frac{A}{nf}F, \tag{6.3.19b}$$

$$F^t_t = -F^r_r - \frac{2rT''}{T'}F. \tag{6.3.19c}$$

§6.4
Instability of Lovelock BHs

As we have seen in the revious section, there are known exact solutions of static black holes in Lovelock theory, and the master equations for all types of perturbations of static vacuum black holes in general Lovelock theory have recently been derived by Takahashi and Soda[TS10b]. Using the master equations, they have found that an asymptotically flat, static Lovelock black hole with small mass is unstable in arbitrary higher dimensions; it is unstable with respect to tensor type perturbations in even-dimensions[TS09b, TS09a] and with respect to scalar type perturbations in odd-dimensions[TS10a]. The stability under vector type perturbations in all dimensions has also been shown[TS10a] by applying the S-deformation technique.

In fact, such an instability against tensor and scalar type perturbations, as well as the stability under vector type perturbations, have already been indicated by earlier work[DG05a, DG05b, GD05, BDG07] performed within the framework of second-order Lovelock theory, often called the Einstein-Gauss-Bonnet theory. For such a restricted class of Lovelock theory–though most generic in $d = 5, 6$, the master equations for metric perturbations have previously been derived by Dotti and Gleiser[DG05b, GD05]. A numerical analysis of the (in)stability of static black holes in Einstein-Gauss-Bonnet theory in dimensions $d = 5, ..., 11$ has been performed in Ref. [KZ08].

It is interesting to note that the instability found in small Lovelock black holes is typically stronger in short distance scales rather than long distance scale/low multipoles as one may expect.

References

[Bar80] J.M. Bardeen. Gauge invariant cosmological perturbations. *Phys. Rev. D*, 22:1882–905, 1980.

[BCH73] J.B. Bardeen, B. Carter, and S.W. Hawking. The four laws of black hole mechanics. *Comm. Math. Phys.*, 31:161–170, 1973.

[BDG07] M. Beroiz, G. Dotti, and R.J. Gleiser. Gravitational instability of static spherically symmetric einstein-gauss-bonnet black holes in five and six dimensions. *Phys. Rev. D*, 76:024012, 2007.

[BMuA87] G.L. Bunting and A.K.M. Masood-ul Alam. Nonexistence of multiple black holes in asymptotically euclidean static vacuum space-time. *Gen. Rel. Grav.*, 19:147, 1987.

[CY05] V. Cardoso and S. Yoshida. Superradiant instabilities of rotating black branes and strings. *JHEP*, 0507:009, 2005.

[DDR76] T. Damour, N. Deruelle, and R. Ruffini. *Nuovo Cimento Lett.*, 15:257, 1976.

[Det80] S. Detweiler. Klein-gordon equation and rotating black holes. *Phys. Rev. D*, 22:2323–6, 1980.

[DFM+09] O.J.C. Dias, P. Figueras, R. Monteiro, J.E. Santos, and R. Emparan. Instability and new phases of higher-dimensional rotating black holes. *Phys. Rev. D*, 80:111701R, 2009.

[DG05a] G. Dotti and R.J. Gleiser. Gravitational instability of einstein-gauss-bonnet black holes under tensor mode perturbations. *Class. Quantum Grav.*, 22:L1, 2005.

[DG05b] G. Dotti and R.J. Gleiser. Linear stability of einstein-gauss-bonnet static spacetimes. part i. tensor perturbations. *Phys. Rev. D*, 72:044018, 2005.

[Dol07] S.R. Dolan. Instability of the massive klein-gordon field on the kerr spacetime. *Phys. Rev. D*, 76:084001, 2007.

[EF07] H. Elvang and P. Figueras. Black saturn. *JHEP*, 05:050, 2007.

[EM03] R. Emparan and R. C. Myers. Instability of ultra-spinning black holes. *JHEP*, 0309:025, 2003.

[ER02a] R. Emparan and H. S. Reall. Generalized weyl solutions. *Phys. Rev. D*, 65:084025, 2002.

[ER02b] R. Emparan and H. S. Reall. A rotating black ring in five dimensions. *Phys. Rev. Lett.*, 88:101101, 2002.

[FSW93] J.L. Friedman, K. Schleich, and D.M. Witt. Topological censorshiop. *Phys. Rev. Lett.*, 71:1486–1489, 1993.

[GD05] R.J. Gleiser and G Dotti. Linear stability of einstein-gauss-bonnet static spacetimes. part ii: Vector and scalar perturbations. *Phys. Rev. D*, 72:124002, 2005.

[GIS03] G. W. Gibbons, D. Ida, and T. Shiromizu. Uniqueness and non-uniqueness of static vacuum black holes in higher dimensions. *Prog. Theor. Phys. Suppl.*, 148:284–290, 2003.

[GL93] R. Gregory and R. Laflamme. Black strings and p-branes are unstable. *Phys. Rev. Lett.*, 70:2837–2840, 1993.

[GL94] R. Gregory and R. Laflamme. The instability of charged black strings and p-branes. *Nucl. Phys. B*, 428:399–434, 1994.

[GL95] R. Gregory and R. Laflamme. Evidence for the stability of extremal black p-branes. *Phys. Rev. D*, 51:305–309, 1995.

[GLPP04] G.W. Gibbons, H. Lu, D.N. Page, and C.N. Pope. Rotating black holes in higher dimensions with a cosmological constant. *Phys. Rev. Lett.*, 93:171102, 2004.

[GLPP05] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope. The general kerr-de sitter metrics in all dimensions. *J. Geom. Phys.*, 53:49–73, 2005.

[Gre00] R. Gregory. Black string instabilities in anti-de sitter space. *Class. Quantum Grav.*, 17:L125–32, 2000.

[GSWW99] G.J. Galloway, K. Schleich, D.M. Witt, and E. Woolgar. Topological censorship and higher genus black holes. *Phys. Rev. D*, 60:104039, 1999.

[HIW07] S. Hollands, A. Ishibashi, and R. Wald. A higher dimensional stationary rotating black hole must be axisymmetric. *Comm. Math. Phys.*, 271:699–722, 2007.

[HOY06] C. Helfgott, Y. Oz, and Y. Yanay. On the topology of black hole event horizons in higher dimensions. *JHEP*, 0602:025, 2006.

[Hwa98] S. Hwang. *Geometriae Dedicata*, 71:5, 1998.

[IK03] A. Ishibashi and H. Kodama. Stability of higher-dimensional schwarzschild black holes. *Prog. Theor. Phys.*, 110:901–919, 2003.

[Isr67] W. Israel. Event horizon in static vacuum space-times. *Phys. Rev.*, 164:1776–1779, 1967.

[IU03] D. Ida and Y. Uchida. Stationary einstein-maxwell fields in arbitrary dimensions. *Phys. Rev. D*, 68:104014, 2003.

[IW94] V. Iyer and R.M. Wald. Some properties of noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D*, 50:846–64, 1994.

[Izu08] K. Izumi. Orthogonal black di-ring solution. *Prog. Theor. Phys.*, 119:757–774, 2008.

[KI03] H. Kodama and A. Ishibashi. A master equation for gravitational perurbations of maximally symmetric black holes in higher dimensions. *Prog. Theor. Phys.*, 110:701–722, 2003.

[KI04] H. Kodama and A. Ishibashi. Master equations for perturbations of generalised static black holes with charge in higher dimensions. *Prog. Theor. Phys.*, 111:29–73, 2004.

[KIS00] H. Kodama, A. Ishibashi, and O. Seto. Brane world cosmology — gauge-invariant formalism for perturbation —. *Phys. Rev. D*, 62:064022, 2000.

[KKZ08] H. Kodama, R.A. Konoplya, and A. Zhidenko. Gravitational instability of simply rotating myers-perry-ads black holes. *Phys. Rev. D*, 79:044003, 2008.

[KLR06] H. Kunduri, J. Lucietti, and H.S. Reall. Gravitational perturbations of higher dimensional rotating black holes. *Phys. Rev. D*, 74:084021, 2006.

[KN69] S. Kobayashi and K. Nomizu. Interscience Pub., 1969.

[Kod84] H. Kodama. Generation and evolution of the density fluctuation in the evolving universe. i. - the effect of weak transient phenomena -. *Prog. Theor. Phys.*, 71:946–959, 1984.

[Kod85] H. Kodama. Generation and evolution of the density fluctuation in the evolving universe. ii. - the radiation-dust universe -. *Prog. Theor. Phys.*, 73:674–682, 1985.

[Kod07] H. Kodama. Superradiance and instability of black holes. *arXiv:0711.4184 [hep-th]*, 2007.

[Kol06] B. Kol. The phase transition between caged black holes and black strings - a review. *Phys. Rep. C*, 422:119–65, 2006.

[KS84] H. Kodama and M. Sasaki. Cosmological perturbation theory. *Prog. Theor. Phys. Suppl.*, 78:1–166, 1984.

[Kud06] H. Kudoh. Origin of black string instability. *Phys. Rev. D*, 73:104034, 2006.

[KW05] H. Kudoh and T. Wiseman. Connecting black holes and black strings. *Phys. Rev. Lett.*, 94:161102, 2005.

[KZ07] R.A. Konoplya and A. Zhidenko. Stability of multidimensional black holes: Complete numerical analysis. *Nucl. Phys. B*, 777:182–202, 2007.

[KZ08] R.A. Konoplya and A. Zhidenko. (in)stability of d-dimensional black holes in gauss-bonnet theory. *Phys. Rev. D*, 77:104004, 2008.

[KZ09] R.A. Konoplya and a. Zhidenko. Instability of higher dimensional charged black holes in the de-sitter world. *Phys. Rev. Lett.*, 103:161101, 2009.

[Lov71] D. Lovelock. The einstein tensor and its generalizations. *J. Math. Phys.*, 12:498–501, 1971.

[MP86] R.C. Myers and M.J. Perry. Black holes in higher dimensional space-times. *Ann. Phys.*, 172:304–347, 1986.

[Muk00] S. Mukohyama. Brane-world solutions, standard cosmology, and dark radiation. *Phys. Lett. B*, 473:241, 2000.

[Pen63] R. Penrose. Asymptotic properties of fields and space-times. *Phys. Rev. Lett.*, 10:66–68, 1963.

[PS06] A.A. Pomeransky and R.A. Sen'kov. Black ring with two angular momenta. *hep-th/0612005*, 2006.

[SCM05] S.S. Seahra, C. Clarkson, and R. Maartens. Detecting extra dimensions with gravity wave spectroscopy: the black string brane-world. *Phys. Rev. Lett.*, 94:121302, 2005.

[SY10a] M. Shibata and H. Yoshino. Bar-mode instability of rapidly spinning black hole in higher dimensions: Numerical simulation in general relativity. *Phys. Rev. D*, 81:104035, 2010.

[SY10b] M. Shibata and H. Yoshino. Nonaxisymmetric instability of rapidly rotating black hole in five dimensions. *Phys. Rev. D*, 81:021501R, 2010.

[TS09a] T. Takahashi and J. Soda. Instability of small lovelock black holes in even-dimensions. *Phys. Rev. D*, 80:104021, 2009.

[TS09b] T. Takahashi and J. Soda. Stability of lovelock black holes under tensor perturbations. *Phys. Rev. D*, 79:104025, 2009.

[TS10a] T. Takahashi and J. Soda. Catastrophic instability of small lovelock black holes. *Prog. Theor. Phys.*, 124:711, 2010.

[TS10b] T. Takahashi and J. Soda. Master equations for gravitational perturbations of lovelock black holes in higher dimensions. *Prog. Theor. Phys.*, 124:911–24, 2010.

[Whe86] J.T. Wheeler. Symmetric solutions to the gauss-bonnet extended einstein equations. *Nucl. Phys. B*, 268:737, 1986.

[Wil86] D.L. Wiltshire. Spherically symmetric solutions of einstein-maxwell theory with a gauss-bonnet term. *Phys. Lett. B*, 169:36–40, 1986.

[Wis03] T. Wiseman. Static axisymmetric vacuum solutions and non-uniform black strings. *Class. Quantum Grav.*, 20:1137–1176, 2003.

[ZE79] T.J.M. Zouros and D.M. Eardley. Instabilities of massive scalar perturbations of a rotating black hole. *Ann. Phys.*, 118:139–55, 1979.

[Zeg05] R. Zegers. Birkhoff's theorem in lovelock gravity. *J. Math. Phys.*, 46:072502, 2005.

[Zwi85] B. Zwiebach. Curvature squared terms and string theories. *Phys. Lett. B*, 156:315–7, 1985.