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A Construction of Einstein-Weyl Spaces via LeBrun-Mason Type Twistor Correspondence

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Abstract We construct infinitely many Einstein-Weyl structures on $S^2 \times \mathbb{R}$ of signature $(-++)$ which is sufficiently close to the model case of constant curvature, and on which the space-like geodesics are all closed. Such a structure is obtained as a parameter space of a family of holomorphic disks which is associated to a small perturbation of the diagonal of $\mathbb{CP}^1 \times \overline{\mathbb{CP}^1}$. The geometry of constructed Einstein-Weyl spaces is well understood from the configuration of holomorphic disks. We also review Einstein-Weyl structures and their properties in the former half of this article.

1 Introduction

Twistor type correspondences for the following structures are known (see (6)):

(T1) projective structures on complex 2-manifolds,
(T2) self-dual conformal structures on complex 4-manifolds, and
(T3) Einstein-Weyl structures on complex 3-manifolds.

(T2) is the original twistor theory introduced by R. Penrose (15). (T3) is called Hitchin correspondence or minitwistor correspondence.

There has been much progress on these twistor theories; more detailed or concrete investigation (13; 14), real objects and reduction theory (1; 4; 5; 7; 16), relation with the theory of integrable systems (2; 3), and so on. The geometric structures considered in these papers are either complex or real slices of complex objects, hence they are all analytic.

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On the other hand, the real indefinite case, for example, admits non-analytic solutions. Recently, C. LeBrun and L. J. Mason developed another type of twistor theory by which we can also analyse such non-analytic solutions (9; 10) (see also (11; 12)). The structures investigated by LeBrun and Mason are

(LM1) Zoll projective structures on S^2 or $S^2/\mathbb{Z}_2$, and
(LM2) self-dual conformal structures of signature $(++--)$ on $S^2\times S^2$ or $(S^2\times S^2)/\mathbb{Z}_2$.

Here, a projective structure is called Zoll if and only if all the maximal geodesics are closed. Notice that (LM1) and (LM2) are the real objects corresponding to (T1) and (T2) respectively.

There are several remarkable points for LeBrun-Mason theory. First, the twistor space is given as a pair (Z,N) of a complex manifold Z and a totally real submanifold N in Z. The "twistor lines", also known as the "nonlinear gravitons", are given by holomorphic disks on Z with boundaries lying on N while in Penrose's case or Hitchin's case the twistor lines are embedded $\mathbb{CP}^1$. Second, the structures (LM1) and (LM2) are obtained from a small perturbation of N in Z. By this reason, we have only been able to deal with the objects which are sufficiently close to the model case up to now. Lastly, the corresponding geometry satisfies a global condition, for example, Zoll condition in (LM1) case.

In light of this research, in this article, we investigate another possibility, the LeBrun-Mason type correspondence for Einstein-Weyl structures. We now review the definitions and then we state the conjecture and the main theorem. Let X be a real (or complex) manifold.

Definition 1.1 *Let $[g]$ be the conformal class of a definite or an indefinite metric g (or holomorphic bilinear metric for the complex case) on X, and ∇ be a (holomorphic) connection on TX. The pair $([g],\nabla)$ is called a* **Weyl structure** *on X if there exists a (holomorphic) 1-form a on X such that*

$$\nabla g = a\otimes g. \tag{1.1}$$

Definition 1.2 *A Weyl structure $([g],\nabla)$ is called* **Einstein-Weyl** *if the symmetrized Ricci tensor $R_{(ij)}=\frac{1}{2}(R_{ij}+R_{ji})$ is proportional to the metric tensor g_{ij}, that is, if we can write*

$$R_{(ij)} = \Lambda\, g_{ij} \tag{1.2}$$

using a function Λ which depends on the choice of $g\in[g]$.

Let $[g]$ be an indefinite conformal structure on a real manifold X. A tangent vector v on X is called *time-like* if $g(v,v)<0$, *space-like* if $g(v,v)>0$ and *light-like* or *null* if $g(v,v)=0$. We introduce the following global condition.

Definition 1.3 *An indefinite Weyl structure $([g],\nabla)$ is called* **space-like Zoll** *if and only if every maximal space-like geodesic is closed.*

Now we state the conjecture for the LeBrun-Mason type correspondence for Einstein-Weyl structures.

Conjecture 1.4 *There is a natural one-to-one correspondence between*

- *equivalence classes of space-like Zoll Einstein-Weyl structures on $S^2 \times \mathbb{R}$; and*
- *equivalence classes of totally real embeddings $\iota : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$,*

at least in a neighborhood of the standard objects.

Here the standard embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ is given by $\zeta \mapsto (\zeta, \bar{\zeta}^{-1})$ using the inhomogeneous coordinate ζ of $\mathbb{CP}^1$. The standard Einstein-Weyl structure is explained in Sect. 5. Before we state the main theorem, we define the following notion.

Definition 1.5 *Let Z be a complex manifold and $\mathfrak{D} \subset Z$ be a holomorphic disk with boundary embedded in Z. Let $v \in T_pZ$ be a non-zero tangent vector at $p \in \partial\mathfrak{D}$. Then v is said to be adapted to $\mathfrak{D}$ (denoted by $v \parallel \mathfrak{D}$) if and only if $v \in T_p\partial\mathfrak{D}$ and v has the same orientation as the orientation of $\partial\mathfrak{D}$ which is induced from the complex orientation of $\mathfrak{D}$.*

The main theorem (Theorem 1.6) gives half of the correspondence in the above con-
jecture; from the embedding ι to the Einstein-Weyl space. We also claim that the geometry of the constructed Einstein-Weyl space is characterized by the holomorphic disks in the following way.

Theorem 1.6 *Let N be the image of any embedding of $\mathbb{CP}^1$ into $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ which is C^{2k+5} close to the standard one. Then there is a unique family of holomorphic disks $\{\mathfrak{D}_x\}_{x \in S^2 \times \mathbb{R}}$ such that each boundary $\partial\mathfrak{D}_x$ lies on N, and that the parameter space $M = S^2 \times \mathbb{R}$ has a unique C^k indefinite Einstein-Weyl structure $([g], \nabla)$ satisfying the following properties.*

1. *For each $p \in N$, $\mathfrak{S}_p = \{x \in M \mid p \in \partial\mathfrak{D}_x\}$ is a maximal connected null surface on M and every null surface can be written in this form.*
2. *For each $p \in Z \setminus N$, $\mathfrak{C}_p = \{x \in M \mid p \in \mathfrak{D}_x\}$ is a maximal connected time-like geodesic and every time-like geodesic on M can be written in this form.*
3. *For each $p \in N$ and non zero $v \in T_pN$, $\mathfrak{C}_{p,v} = \{x \in M \mid p \in \partial\mathfrak{D}_x, v \parallel \mathfrak{D}_x\}$ is a maximal connected null geodesic on M and every null geodesic on M can be written in this form.*
4. *For each distinguished $p, q \in N$, $\mathfrak{C}_{p,q} = \{x \in M \mid p, q \in \partial\mathfrak{D}_x\}$ is a connected closed space-like geodesic on M and every space-like geodesic on M can be written in this form.*

In particular, this Einstein-Weyl structure is space-like Zoll.

The organization of this paper is as follows. We first review projective structures in Sect. 2. Next, we study complex, definite or indefinite Einstein-Weyl spaces in Sect. 3. We prove that, in each case, the Einstein-Weyl condition can be translated to an integrability condition for certain distributions. Applying this method, we review the proof of the Hitchin correspondence in Sect. 4. In Sect. 5, the model case of the LeBrun-Mason type correspondence is explained. The standard Einstein-Weyl space is obtained as a double cover of a real slice of Hitchin's example. We also study some properties of this model case.

From Sect. 6, we deal with the perturbation of the model case. In Sect. 6, we prove that, for a small perturbation of the real submanifold N, there is a unique family of holomorphic disks with boundaries lying on N. This family has similar properties to the model case, especially for the double fibration, which is studied in Sect. 7. Finally in Sect. 8, we prove that there is a unique Einstein-Weyl structure on the parameter space of the constructed family of holomorphic disks. We also prove that the geometry of the Einstein-Weyl space is characterized by the holomorphic disks as in Theorem 1.6.

2 Projective Structures

In this section, we review projective structures. Let X be a real smooth n-manifold and $x^i\,(i=1,\dots,n)$ be a local coordinate on X. The following argument also works well in the complex case by considering x^i as a complex coordinate, and using holomorphic functions instead of smooth functions.

Definition 2.1 *Two connections ∇ and ∇' on the tangent bundle TX are called* **projectively equivalent** *if their geodesics coincide without considering parameterizations. A projectively equivalent class $[\nabla]$ is called a* **projective structure** *on X.*

Let ∇ and ∇' be connections on TX, and let Γ^i_{jk} and ${\Gamma'}^i_{jk}$ be their Christoffel symbols respectively, that is, $\nabla_{\partial_k}\partial_j = \sum \Gamma^i_{jk}\partial_i$ and so on, where we denote $\partial_i = \frac{\partial}{\partial x^i}$. Notice that ∇ is torsion-free if and only if $\Gamma^i_{jk} = \Gamma^i_{kj}$. The following proposition is readily checked (see (6)).

Proposition 2.2 *Suppose that both ∇ and ∇' are torsion-free, then they are projectively equivalent if and only if there exist functions $f_i\,(i=1,\dots,n)$ on X such that the following condition holds:*

$$\Gamma^i_{jk} = {\Gamma'}^i_{jk} + \frac{1}{2}(\delta^i_j f_k + \delta^i_k f_j). \tag{2.1}$$

In the complex case, we can prove the following.

Proposition 2.3 *Let X be a complex n-manifold, and $\mathscr{F}$ be a holomorphic family of holomorphic curves on X. Suppose that, for each non-zero tangent vector $v \in TX$, there is a unique member of $\mathscr{F}$ which tangents to v. Then there is a unique projective structure $[\nabla]$ on X so that $\mathscr{F}$ coincides to the family of geodesics.*

Proof Let $p : TX \setminus 0_X \to X$ and $\pi : TX \setminus 0_X \to \mathbb{P}(TX)$ be the projections, where 0_X is the zero section and $\mathbb{P}(TX)$ is the projectivization of TX. We use a local coordinate (x^i) on X, and let (y^i) be the fiber coordinate on TX with respect to the frame $\left(\frac{\partial}{\partial x^i}\right)$. First we consider a curve $c : (-\varepsilon,\varepsilon) \to X$ given by $c(t) = (x^i(t))$. We also write c for the image of c. Then the natural lift $\tilde{c} : (-\varepsilon,\varepsilon) \to TX$ is given by $\tilde{c}(t) = \left(x^i(t); \frac{dx^i}{dt}(t)\right)$. We obtain the velocity vector field of $\tilde{c}$, and this vector field uniquely extends to the vector field along $p^{-1}(c)$ of the form

$$v = y^i \frac{\partial}{\partial x^i} + G^i(x,y)\frac{\partial}{\partial y^i} \tag{2.2}$$

so that G^i satisfies $G^i(x,ay) = a^2 G^i(x,y)$ for each non-zero $a \in \mathbb{C}$. Notice that v descends to a line distribution on $\pi(\tilde{c}) \subset \mathbb{P}(TX)$ by π_*. This is the tangent distribution of the lift of c on $\mathbb{P}(TX)$, hence it does not depend on the parametrization of c.

Now we go back to the statement. Since the statement is local, we can assume $\mathbb{P}(TX) = X \times \mathbb{CP}^{n-1}$. Let $\mathbb{CP}^{n-1} = \cup W_\alpha$ be an affine open cover. By the assumption, a foliation $\tilde{\mathscr{F}}$ on $\mathbb{P}(TX)$ is defined so that each leaf of $\tilde{\mathscr{F}}$ is the natural lift of a curve in $\mathscr{F}$. We notice the curves in $\mathscr{F}$ of which the lift intersects with $X \times W_\alpha$. Taking a parametrization of them, we obtain a holomorphic vector field

$$v_\alpha = y^i \frac{\partial}{\partial x^i} + G^i_\alpha(x,y) \frac{\partial}{\partial y^i}$$

on $\pi^{-1}(X \times W_\alpha)$ by the above construction.

In this way, we have obtained the vector fields $\{v_\alpha\}$. Since v_α and v_β descend to the same line distribution on $X \times W_\alpha \cap X \times W_\beta$, we can write $v_\alpha - v_\beta = f_{\alpha\beta}(x,y) y^i \frac{\partial}{\partial y^i}$ on $\pi^{-1}(X \times W_\alpha) \cap \pi^{-1}(X \times W_\beta)$, where $f_{\alpha\beta}$ is a holomorphic function satisfying $f_{\alpha\beta}(x,ay) = a f_{\alpha\beta}(x,y)$ for each non-zero $a \in \mathbb{C}$. Since $\mathrm{H}^1(\mathbb{P}^{n-1}, \mathscr{O}(1)) = 0$, we can take $\{v_\alpha\}$ so that $f_{\alpha\beta} = 0$ by changing the parametrizations. Hence we obtain a vector field on the whole of $TX \setminus 0_X$ of the form (2.2). Then G^i must be a degree-two polynomial for y, so we obtain a torsion-free connection ∇ so that $G^i(y) = \Gamma^i_{jk} y^j y^k$. For this ∇, each curve of $\mathscr{F}$ is a geodesic by construction. Here ∇ is determined up to projective equivalence since the ambiguity of taking v remains. □

3 Einstein-Weyl Structures

In this section, we study the basic properties of 3-dimensional Einstein-Weyl structures. We will prove that the Einstein-Weyl condition is equivalent to the integrability condition of certain distributions. We consider the complex, definite, and indefinite cases separately.

Complex case. Let X be a complex 3-manifold and $([g], \nabla)$ be a Weyl structure on X. We pick a $g \in [g]$, however, the statements do not depend on the choice of g. We denote

$$T_{\mathbb{C}}X = TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$
$$\text{and} \quad T^*_{\mathbb{C}}X = T^*X \otimes \mathbb{C} = T^{*1,0}X \oplus T^{*0,1}X.$$

Notice that g induces complex bilinear metrics on $T^{1,0}X, T^{0,1}X, T^{*1,0}X$ and $T^{*0,1}X$ which we also denote g.

Definition 3.1 *For each $x \in X$, a complex two-dimensional subspace $V \subset T^{1,0}_x X$ is called a* **null plane** *if the restriction of g on V degenerates.*

The following property is easily checked.

Lemma 3.2 *If $v \in T^{1,0}_x X$ is null, then $v^\perp$ is a null plane. Conversely, every null plane is written as $v^\perp$ for some null vector v.*

Notice that $v^{\perp} = \ker v^*$ for every $v \in T_x^{1,0}X$, where $v^* = g(v,\cdot) \in T_x^{*\,1,0}X$, and that v is null if and only if v^* is null. Let $N(T^{*\,1,0}X)$ be the null cotangent vectors, and $\mathscr{Z} = \mathbb{P}(N(T^{*\,1,0}X))$ be its complex projectivization. Notice that each point $u \in \mathscr{Z}$ corresponds to the null plane $V_u = \ker\lambda$, where $\lambda \in N(T^{*\,1,0}X)$ is the cotangent vector satisfying $u = [\lambda]$. We can define a complex 2-plane distribution $\subset T^{1,0}\mathscr{Z}$ so that ${}_u \subset T_u^{1,0}\mathscr{Z}$ is the horizontal lift of the null plane V_u with respect to ∇. Notice that the horizontal lift is well-defined since $N(T^{*\,1,0}X)$ is parallel to ∇ because of the compatibility condition (1.1).

Proposition 3.3 *Let X be a complex 3-manifold. A Weyl structure $([g],\nabla)$ with torsion-free ∇ on X is Einstein-Weyl if and only if the induced distribution on $\mathscr{Z}$ is integrable, in other words, involutive.*

Proof Let $\{e_1,e_2,e_3\}$ be an orthonormal complex local frame on $T^{1,0}X$ with respect to $g \in [g]$, and $\{e^1,e^2,e^3\}$ be the dual frame on $T^{*\,1,0}X$. Let $\omega = (\omega^i_j)$ be the connection form of ∇ with respect to $\{e_i\}$, and let $K^i_j = K^i_{jkl}e^k \wedge e^l$ be its curvature form. Then from the compatibility condition (1.1), we obtain the following symmetry for K:

$$
\begin{aligned}
&K^i_{jkl} = A^i_{jkl} + \delta^i_j B_{kl},\\
&A^i_{jkl} = -A^i_{jlk} = -A^j_{ikl} \quad \text{and} \quad B_{kl} = -B_{lk}.
\end{aligned}
\tag{3.1}
$$

Since the frame is orthonormal, the Einstein-Weyl equations are

$$
R_{(12)} = R_{(23)} = R_{(31)} = 0 \quad \text{and} \quad R_{(11)} = R_{(22)} = R_{(33)},
$$

and this is equivalent to

$$
A^1_{213} + A^1_{312} = A^2_{321} + A^2_{123} = A^3_{132} + A^3_{231} = 0 \quad \text{and} \quad A^1_{212} = A^2_{323} = A^3_{131}. \tag{3.2}
$$

Now let $\mathscr{N} = N(T^{*\,1,0}X)\backslash 0_X$, and $\pi : \mathscr{N} \to \mathscr{Z}$ be the projection where 0_X is the zero section. Then is integrable if and only if the pull-back π^* is integrable. Here $\pi^* \subset T^{1,0}\mathscr{N}$ is the complex 3-plane distribution defined by $\pi^* = \{v \in T\mathscr{N} \mid \pi_*(v) \in \}$. On the other hand, there is a 2-plane distribution $\tilde{} \subset T^{1,0}\mathscr{N}$ which is defined in a similar way to , that is, ${}_u$ is the horizontal lift of the null plane V_u. These distributions are related by $\pi^* = \tilde{} \oplus \langle \Upsilon \rangle$, where

$$
\Upsilon = \sum \lambda_i \frac{\partial}{\partial \lambda_i} \tag{3.3}
$$

is the Euler differential. Now we define several 1-forms on $\mathscr{N}$ by

$$
\theta = \sum \lambda_i e^i, \quad \theta_i = d\lambda_i - \sum \lambda_j \omega^j_i \quad \text{and} \quad \tau_{ij} = \lambda_i\theta_j - \lambda_j\theta_i. \tag{3.4}
$$

Then $\tilde{} = \{v \in T\mathscr{N} \mid \theta(v) = \theta_i(v) = 0\,(\forall i)\}$ and $\pi^* = \{v \in T\mathscr{N} \mid \theta(v) = \tau_{ij}(v) = 0\,(\forall i,j)\}$. Hence is integrable if and only if the 1-forms $\{\theta, \tau_{ij}\}$ on $\mathscr{N}$ are involutive. Notice that $\tau_{23}/\lambda_1 = \tau_{31}/\lambda_2 = \tau_{12}/\lambda_3$, hence τ_{ij} are proportional to each other.

Let us prove that is integrable if and only if (3.2) holds. First, we claim that $d\theta \equiv 0 \mod \langle \theta, \tau_{ij} \rangle$ always holds. Indeed, since $\theta_1/\lambda_1 \equiv \theta_2/\lambda_2 \equiv \theta_3/\lambda_3$, we have

$$\sum \theta_i \wedge e^i \equiv \frac{\theta_1}{\lambda_1} \wedge \theta \equiv 0 \mod \langle \theta, \tau_{ij} \rangle .$$

On the other hand, we have the torsion-free condition: $de^i + \sum \omega^i_j \wedge e^j = 0$. Then

$$d\theta = \sum d\lambda_i \wedge e^i + \sum \lambda_i de^i = \sum \theta_i \wedge e^i + \sum \lambda_i (de^i + \omega^i_j \wedge e^j) \equiv 0 \mod \langle \theta, \tau_{ij} \rangle .$$

Next, a direct calculation shows that

$$d\tau_{12} \equiv -\sum \lambda_1 \lambda_j K^j_2 + \sum \lambda_2 \lambda_j K^j_1 \mod \tau_{12}, \tag{3.5}$$

and we can check that $d\tau_{12} \equiv 0$ holds if and only if

$$\begin{aligned} 0 = \lambda_3 \big[& -A^2_{323}\lambda_1^2 - A^3_{131}\lambda_2^2 - A^1_{212}\lambda_3^2 \\ & + (A^3_{132} + A^3_{231})\lambda_1\lambda_2 + (A^2_{321} + A^2_{123})\lambda_3\lambda_1 + (A^1_{213} + A^1_{312})\lambda_2\lambda_3 \big] \end{aligned}$$

for every (λ_i) satisfying $\sum \lambda_i^2 = 0$. Hence is integrable if and only if the Einstein-Weyl equation (3.2) holds. □

The distribution can be explicitly described in the following way. As in the above proof, let us take a local orthonormal frame $\{e_1, e_2, e_3\}$ on an open set $U \subset X$. From the compatibility condition (1.1), the connection form ω of ∇ is written

$$\omega = \begin{pmatrix} \phi & \eta^1_2 & \eta^1_3 \\ \eta^2_1 & \phi & \eta^2_3 \\ \eta^3_1 & \eta^3_2 & \phi \end{pmatrix}, \quad \text{with } \eta^j_i = -\eta^i_j. \tag{3.6}$$

We can write

$$\begin{aligned} N(T^{*1,0}X)|_U &= \big\{ \sum \lambda_i e^i \,\big|\, \sum \lambda_i^2 = 0 \big\} \\ \text{and} \quad \mathscr{Z}|_U &= \big\{ [\lambda_1 : \lambda_2 : \lambda_3] \,\big|\, \sum \lambda_i^2 = 0 \big\}. \end{aligned}$$

Then we obtain

$$\tau_{23} = \lambda_2 d\lambda_3 - \lambda_3 d\lambda_2 + \lambda_1 \big(\lambda_1 \eta^2_3 + \lambda_2 \eta^3_1 + \lambda_3 \eta^1_2 \big). \tag{3.7}$$

Let $U \times \mathbb{CP}^1 \xrightarrow{\sim} \mathscr{Z}|_U$ be a trivialization given by

$$(x, \zeta) \longmapsto [i(1+\zeta^2) : 1 - \zeta^2 : 2\zeta], \tag{3.8}$$

where $\zeta \in \mathbb{C} \cup \{\infty\}$ is a inhomogeneous coordinate. The horizontal lift $\tilde{v}$ of $v \in T_x U$ at $(x, \zeta) \in \mathscr{Z}|_U$ is

$$\tilde{v} = v + \left\{ \frac{\eta^2_3 + i\eta^1_3}{2} - i\zeta\eta^1_2 + \zeta^2 \frac{\eta^2_3 - i\eta^1_3}{2} \right\} (v) \frac{\partial}{\partial \zeta}. \tag{3.9}$$

For $(x,\zeta) \in \mathscr{Z}|_U$, the corresponding null plane on $T_x^{1,0}X$ is spanned by

$$\mathfrak{m}_1(\zeta) = ie_1 + e_2 + \zeta e_3 \quad \text{and} \quad \mathfrak{m}_2(z) = \zeta(-ie_1 + e_2) - e_3. \tag{3.10}$$

Hence $_{(x,\zeta)}$ is spanned by $\tilde{\mathfrak{m}}_1(\zeta)_x$ and $\tilde{\mathfrak{m}}_2(\zeta)_x$. Therefore the Einstein-Weyl condition is equivalent to the involutive condition $[\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2] \in$. Proposition 3.3 could also be proved in this way. However, it is rather easier to check the integrability condition for π^* as we did.

Definite case. Let X be a real 3-manifold and $([g],\nabla)$ be a definite Weyl structure, that is, a Weyl structure on X with positive definite $[g]$. In this case, we can define complex null planes on $T_{\mathbb{C}}X$. If we put $\mathscr{Z} = \mathbb{P}(N(T^*_{\mathbb{C}}X))$, then we can define the complex 2-plane distribution $\subset T_{\mathbb{C}}\mathscr{Z}$ in the same manner as the complex case by using the horizontal lift defined by (3.9). The complex conjugation $T^*_{\mathbb{C}}X \to T^*_{\mathbb{C}}X$ induces a fixed-point-free involution $\sigma : \mathscr{Z} \to \mathscr{Z}$ which is fiber-wise antiholomorphic. Notice that satisfies $\sigma^* =$. We also define a complex 3-plane distribution $\subset T_{\mathbb{C}}\mathscr{Z}$ by $= \oplus V^{0,1}$, where $V^{0,1} \subset T_{\mathbb{C}}\mathscr{Z}$ is the $(0,1)$-tangent vectors on the fiber of $\varpi : \mathscr{Z} \to X$. Here, we also obtain $\sigma^* =$.

Proposition 3.4 *Let $([g],\nabla)$ be a definite Weyl structure on a 3-manifold X. Let $\varpi : \mathscr{Z} \to X$ be the $\mathbb{CP}^1$-bundle and be the distribution on $\mathscr{Z}$ constructed above. Then there is a unique continuous distribution L of real lines on $\mathscr{Z}$ which satisfies $L \otimes \mathbb{C} = \cap$ on $\mathscr{Z}$. Moreover the projection $\varpi(C)$ of each integral curve C of L is a geodesic.*

Proof If we take a real local frame $\{e^i\}$, then we can describe the situations in a similar way to (3.6) to (3.10). Then $= \mathrm{Span}\langle \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2 \rangle$ and $= \mathrm{Span}\left\langle \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2, \frac{\partial}{\partial \bar{\zeta}} \right\rangle$. Since $+ \ = T_{\mathbb{C}}\mathscr{Z}$, L exists uniquely by a dimension counting argument.

Now let us define

$$l = \bar{\zeta}\mathfrak{m}_1 + \mathfrak{m}_2 = 2(\mathrm{Im}\,\zeta)e_1 + 2(\mathrm{Re}\,\zeta)e_2 + (|\zeta|^2 - 1)e_3.$$

Notice that l is real. We can take a unique function γ on $\mathscr{Z}$ so that

$$l^\dagger := \bar{\zeta}\tilde{\mathfrak{m}}_1 + \tilde{\mathfrak{m}}_2 + \gamma \frac{\partial}{\partial \bar{\zeta}}$$

is real. Then we obtain $L = \mathrm{Span}\langle l^\dagger \rangle$. Let $p : \ \to \ $ be the natural projection, then $p(L) = \mathrm{Span}\langle \tilde{l} \rangle$, where $\tilde{l} = \bar{\zeta}\tilde{\mathfrak{m}}_1 + \tilde{\mathfrak{m}}_2$. By construction, the image of an integral curve of $p(L)$ by ϖ is a geodesic. Pulling back to by p, we obtain the statement. □

Proposition 3.5 *Let X be a real 3-manifold, and $([g],\nabla)$ be a definite Weyl structure on X with torsion-free ∇. Then $([g],\nabla)$ is Einstein-Weyl if and only if is integrable, in other words, involutive.*

Proof The distribution is integrable, if and only if π^* is integrable, where $\pi : \mathscr{N} = N(T^*_{\mathbb{C}}X)\backslash 0_X \to \mathscr{Z}$. If we take an orthonormal frame field $\{e_1, e_2, e_3\}$ of $T_{\mathbb{C}}X$, and if we use the complex fiber coordinate $\{\lambda_i\}$ for $T^*_{\mathbb{C}}X$, then we can define 1-forms θ, θ_i, τ_{ij} on $\mathscr{N}$ by (3.4). In this case, we obtain $\pi^* = \pi^* + \pi^* V^{0,1}$, and $\pi^* = \{v \in T^*\mathscr{N} \mid \theta(v) = \tau_{ij}(v) = 0\,(\forall i,j)\}$. Hence is integrable if and only if $\langle \theta, \tau_{ij} \rangle$ is involutive. By similar arguments, this occurs if and only if $([g],\nabla)$ is Einstein-Weyl. □

Remark 3.6 Locally speaking, $/L$ defines an almost complex structure on the space of geodesics on X. Proposition 3.5 means that this almost complex structure is integrable if and only if $([g],\nabla)$ is Einstein-Weyl (see also (14)).

Indefinite case. Let X be a real 3-manifold and $([g],\nabla)$ be a Weyl structure on X for which the conformal structure $[g]$ has signature $(-++)$. Let $\{e_1,e_2,e_3\}$ be a local frame field on TX such that

$$(g_{ij}) = (g(e_i,e_j)) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{3.11}$$

A non-zero tangent vector $v \in TX$ is called time-like, space-like or null when $g(v,v)$ is negative, positive, or zero respectively. The following properties are easily checked.

Lemma 3.7 1. *For each space-like vector v, there are just two real null planes which contain v.*
2. *Each time-like vector is transverse to every real null plane.*

Similar to the definite case, we define $N(T^*_{\mathbb{C}}X)$, the space of complex null cotangent vectors, and $\mathscr{Z} = \mathbb{P}(N(T^*_{\mathbb{C}}X))$, the space of complex null planes. In the indefinite case, we can also define $N(T^*X)$, the space of *real* null cotangent vectors, and $\mathscr{Z}_{\mathbb{R}} = \mathbb{P}(N(T^*X))$, the space of *real* null planes. There is a natural embedding $\mathscr{Z}_{\mathbb{R}} \hookrightarrow \mathscr{Z}$. The complex conjugation $T^*_{\mathbb{C}}X \to T^*_{\mathbb{C}}X$ induces an involution $\sigma : \mathscr{Z} \to \mathscr{Z}$ which is fiber-wise antiholomorphic and for which the fixed point set coincides with $\mathscr{Z}_{\mathbb{R}}$.

Let us describe the situation explicitly using the above frame $\{e_i\}$ and its dual $\{e^i\}$. From the compatibility condition (1.1), the connection form ω of ∇ is written:

$$\omega = \begin{pmatrix} \phi & \eta^1_2 & \eta^1_3 \\ \eta^1_2 & \phi & \eta^2_3 \\ \eta^1_3 & -\eta^2_3 & \phi \end{pmatrix}. \tag{3.12}$$

We can write

$$\begin{aligned} N(T^*_{\mathbb{C}}X)|_U &= \left\{ \textstyle\sum \lambda_i e^i \,\middle|\, -\lambda_1^2+\lambda_2^2+\lambda_3^2 = 0 \right\} \\ \text{and} \quad \mathscr{Z}|_U &= \left\{ [\lambda_1:\lambda_2:\lambda_3] \,\middle|\, -\lambda_1^2+\lambda_2^2+\lambda_3^2 = 0 \right\}. \end{aligned} \tag{3.13}$$

Let $U \times \mathbb{CP}^1 \xrightarrow{\sim} \mathscr{Z}_{\mathbb{R}}|_U$ be a trivialization over an open set $U \subset X$ such that

$$(x,\zeta) \longmapsto \left[(1+\zeta^2)e^1 + (1-\zeta^2)e^2 + 2\zeta e^3\right]. \tag{3.14}$$

Here $\mathscr{Z}_{\mathbb{R}}$ corresponds to $\{(x,\zeta) \in U \times \mathbb{CP}^1 \mid \zeta \in \mathbb{R} \cup \{\infty\}\}$. The horizontal lift $\tilde{v}$ of $v \in T_xU$ at $(x,\zeta) \in \mathscr{Z}_{\mathbb{R}}|_U$ is

$$\tilde{v} = v + \left\{ \frac{\eta^2_3 + \eta^1_3}{2} - \zeta\eta^1_2 + \zeta^2 \frac{\eta^2_3 - \eta^1_3}{2} \right\}(v) \frac{\partial}{\partial\zeta}. \tag{3.15}$$

If we define

$$\mathfrak{m}_1(\zeta) = -e_1 + e_2 + \zeta e_3 \quad \text{and} \quad \mathfrak{m}_2(\zeta) = \zeta(e_1 + e_2) - e_3, \tag{3.16}$$

then $\mathfrak{m}_1(\zeta)$ and $\mathfrak{m}_2(\zeta)$ span the null plane corresponding to $(x,\zeta) \in \mathscr{Z}_{\mathbb{R}}$. Define the real 2-plane distribution $_{\mathbb{R}} \subset T\mathscr{Z}_{\mathbb{R}}$ so that $_{\mathbb{R}} = \mathrm{Span}\,\langle \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2 \rangle$, where $\tilde{\mathfrak{m}}_i$ are the vector fields on $\mathscr{Z}_{\mathbb{R}}$ such that $\tilde{\mathfrak{m}}_{i(x,\zeta)}$ is the horizontal lift of $\mathfrak{m}(\zeta)_x$.

We can extend $\tilde{\mathfrak{m}}_i$ meromorphically on $\mathscr{Z}$, and define the complex 2-plane distribution $\subset T_{\mathbb{C}}\mathscr{Z}$ by $= \mathrm{Span}\,\langle \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2 \rangle$. We also define a complex 3-plane distribution by $= \oplus V^{0,1}$, where $V^{0,1} \subset T_{\mathbb{C}}\mathscr{Z}$ is $(0,1)$-tangent vectors. Then we obtain

$$\sigma^* = , \quad \sigma^* = \dot{} ,$$
$$_{\mathbb{R}} \otimes \mathbb{C} = |_{\mathscr{Z}_{\mathbb{R}}} \quad \text{and} \quad _{\mathbb{R}} = \cap T\mathscr{Z}_{\mathbb{R}} = \cap T\mathscr{Z}_{\mathbb{R}}.$$

Proposition 3.8 *Let $([g],\nabla)$ be an indefinite Weyl structure on a 3-manifold X. Let $\varpi : \mathscr{Z} \to X$ be the $\mathbb{CP}^1$-bundle and be the distribution on $\mathscr{Z}$ constructed above. Then there is a unique continuous distribution L of real lines on $\mathscr{Z}$ which satisfies $L \otimes \mathbb{C} = \cap \dot{}$ on $\mathscr{Z} \backslash \mathscr{Z}_{\mathbb{R}}$ and $L \subset {}_{\mathbb{R}}$ on $\mathscr{Z}_{\mathbb{R}}$. Moreover each integral curve C of L is contained in either $\mathscr{Z} \backslash \mathscr{Z}_{\mathbb{R}}$ or $\mathscr{Z}_{\mathbb{R}}$, and the projection $\varpi(C)$ is time-like geodesic if $C \subset \mathscr{Z} \backslash \mathscr{Z}_{\mathbb{R}}$, and null-geodesic if $C \subset \mathscr{Z}_{\mathbb{R}}$.*

Proof Let us define a real vector field l on X by

$$l = \mathfrak{m}_1 - \bar{\zeta}\mathfrak{m}_2 = -(1+|\zeta|^2)e_1 + (1-|\zeta|^2)e_2 + (\zeta + \bar{\zeta})e_3. \tag{3.17}$$

Notice that l is time-like if $\mathrm{Im}\,\zeta \neq 0$, and null if $\mathrm{Im}\,\zeta = 0$. We can take a unique function γ on $\mathscr{Z}$ so that

$$l^\dagger = \tilde{\mathfrak{m}}_1 - \bar{\zeta}\tilde{\mathfrak{m}}_2 + \gamma \frac{\partial}{\partial \bar{\zeta}}$$

is real. Since $\tilde{l} = \tilde{\mathfrak{m}}_1 - \bar{\zeta}\tilde{\mathfrak{m}}_2$ is real on $\mathscr{Z}_{\mathbb{R}}$, $\gamma = 0$ and $l^\dagger = \tilde{l}$ on $\mathscr{Z}_{\mathbb{R}}$. If we put $L = \langle l^\dagger \rangle$, then we obtain $L \otimes \mathbb{C} = \cap \dot{}$ on $\mathscr{Z} \backslash \mathscr{Z}_{\mathbb{R}}$ and $L \subset {}_{\mathbb{R}}$ on $\mathscr{Z}_{\mathbb{R}}$. L is unique since $E + \bar{E} = T_{\mathbb{C}}\mathscr{Z}$ on $\mathscr{Z} \backslash \mathscr{Z}_{\mathbb{R}}$. The remaining statements are proved in a similar way to the definite case (Proposition 3.4). □

Proposition 3.9 *Let X be a real 3-manifold, and $([g],\nabla)$ be an indefinite Weyl structure on X with torsion-free ∇. Then the following conditions are equivalent:*

- *$([g],\nabla)$ is Einstein-Weyl,*
- *the real distribution $_{\mathbb{R}}$ is integrable,*
- *the complex distribution is integrable.*

Proof If we put

$$\Upsilon = -\lambda_1 \frac{\partial}{\partial \lambda_1} + \lambda_2 \frac{\partial}{\partial \lambda_2} + \lambda_3 \frac{\partial}{\partial \lambda_3},$$
$$\tau_{12} = \lambda_1\theta_2 + \lambda_2\theta_1, \quad \tau_{13} = \lambda_1\theta_3 + \lambda_3\theta_1 \quad \text{and} \quad \tau_{23} = \lambda_2\theta_3 - \lambda_3\theta_2$$

instead of (3.3) and (3.4), then the situation is parallel to the complex or definite case. □

A direct calculation shows

$$\tau_{23} = \lambda_2 d\lambda_3 - \lambda_3 d\lambda_2 - \lambda_1\left(\lambda_1\eta_3^2 + \lambda_2\eta_3^1 - \lambda_3\eta_2^1\right). \tag{3.18}$$

Equation (3.18) will be used in Sect. 8.

Remark 3.10 We can write $= \langle \tilde{\mathfrak{m}}_1\rangle \oplus \langle \tilde{\mathfrak{m}}_2\rangle$ locally, hence $c_1() = c_1(\langle \tilde{\mathfrak{m}}_1\rangle) + c_1(\langle \tilde{\mathfrak{m}}_2\rangle) = -2$ along each $\mathbb{CP}^1$-fiber of $\varpi : \mathscr{Z} \to X$. Since $c_1(V^{0,1}) = -2$, we also obtain $c_1() = -4$ along each fiber.

4 Hitchin Correspondence

In this section, we recall the twistor correspondence for complex Einstein-Weyl structures introduced by Hitchin (6).

Let Z be a complex 2-manifold and Y be a non-singular rational curve on Z with the normal bundle $N_{Y/Z} \cong \mathscr{O}(2)$. Let X be the space of twistor lines, that is, the rational curves which are obtained by small deformation of Y in Z. By Kodaira's theorem, X has a natural structure of a 3-dimensional complex manifold, and its tangent space at $x \in X$ is identified with the space of sections of the normal bundle $N_{Y_x/Z}$, where Y_x is the twistor line corresponding to x.

Proposition 4.1 *There is a unique Einstein-Weyl structure on X such that*

- *each non-null geodesic on X corresponds to a one-parameter family of twistor lines on Z passing through two fixed points, and*
- *each null geodesic on X corresponds to a one-parameter family of twistor lines each of which passes through a fixed point and is tangent to a fixed non-zero vector there.*

Proof We have $N_{Y_x/Z} \cong \mathscr{O}(2)$ for each $x \in X$ since Y_x is a small deformation of Y. We have $T_xX \cong \Gamma(Y_x, N_{Y_x/Z})$ by definition. Each holomorphic section of $N_{Y_x/Z} \simeq \mathscr{O}(2)$ corresponds to a degree-two polynomial $s(\zeta) = a\zeta^2 + b\zeta + c$, where ζ is the inhomogeneous coordinate on Y_x. We can define the conformal structure $[g]$ so that a tangent vector in T_xX is null if and only if the corresponding polynomial $s(\zeta)$ has double roots, that is, when $b^2 - 4ac = 0$.

If we fix two, possibly infinitely near, points in Z, then the twistor lines passing through these points make a one-parameter family. This family corresponds to a holomorphic curve on X. Let $\mathscr{F}$ be the family of such holomorphic curves. Then, by Proposition 2.3, we obtain a unique projective structure $[\nabla]$ on X such that $\mathscr{F}$ coincides with the geodesics.

Now, we prove that there is a unique torsion-free $\nabla \in [\nabla]$ such that $([g], \nabla)$ defines a Weyl structure. For this purpose, we first fix an arbitrary torsion-free $\nabla \in [\nabla]$, and check that the second fundamental form on each null surface with respect to ∇ vanishes.

For each point $p \in Z$, the two-parameter family of twistor lines passing through p corresponds to a null surface S on X. Notice that S is totally geodesic and naturally foliated by null geodesics each of which corresponds to a tangent line at p. Let $N = TX|_S/TS$ be the normal bundle of S. The second fundamental form $I\!I : TS \otimes TS \to N$ is defined by $v \otimes w \to [\nabla_v w]^N$, where the value does not depend on how we extend w. Take a frame field $\{e_1,e_2,e_3\}$ on $TX|_S$ so that e_1 is null and $TS=\langle e_1,e_2\rangle$. Then the metric tensor is

$$g=(g_{ij})=\begin{pmatrix}0&0&*\\0&*&*\\ *&*&*\end{pmatrix}.$$

Since ∇ is torsion-free, $\nabla_{e_1}e_2-\nabla_{e_2}e_1=[e_1,e_2]\in TS$, so $g(\nabla_{e_1}e_2,e_1)=g(\nabla_{e_2}e_1,e_1)$. Since $g_{13}\neq 0$, we obtain

$$\Gamma^3_{12}=\Gamma^3_{21}. \tag{4.1}$$

On the other hand, since S is totally geodesic, we obtain

$$0=g(\nabla_\xi\xi,e_1)=\xi^1\xi^2g_{13}\big(\Gamma^3_{12}+\Gamma^3_{21}\big)$$

for every tangent vector $\xi=\xi^1e_1+\xi^2e_2$ on S. So we obtain $\Gamma^3_{12}+\Gamma^3_{21}=0$, and combining with (4.1), we obtain $\Gamma^3_{12}=\Gamma^3_{21}=0$. Hence $g(\nabla_\xi\eta,e_1)=0$ for every vector field ξ and η on S, and this means $II=0$ on S.

Next we claim that there are functions a_i,b_i $(i=1,2,3)$ on X such that

$$(\nabla g)_{ijk}=a_ig_{jk}+\frac{1}{2}b_jg_{ik}+\frac{1}{2}b_kg_{ij}. \tag{4.2}$$

Since $II=0$ for every null surface, we obtain

$$\nabla_\eta g(\xi,\xi)=0 \tag{4.3}$$

for every null vector ξ and every vector η satisfying $g(\eta,\xi)=0$. Let us fix a local frame $\{e_i\}$ on X. If we put $\xi=\xi^ie_i,\eta=\eta^ie_i$ $(i=1,2,3)$ and $\varphi_{ijk}=\nabla_{e_i}(e_j,e_k)$, then (4.3) is written

$$(\varphi_{ijk}\xi^j\xi^k)\eta^i=0. \tag{4.4}$$

Since ξ runs over all null vectors, (ξ^i) moves the conic

$$C=\left\{\,[\xi^1:\xi^2:\xi^3]\in\mathbb{CP}^2\;\middle|\;\xi^i\xi^jg_{ij}=0\right\}.$$

For fixed ξ, (η^i) moves the line

$$L(\xi)=\{\,[\eta^1:\eta^2:\eta^3]\in\mathbb{CP}^2\;\big|\;\eta^i(\xi^jg_{ij})=0\}.$$

Since (4.4) holds for every $[\eta^i]\in L(\xi)$, we can take a function $b(\xi)$ satisfying

$$\varphi_{ijk}\xi^j\xi^k=b(\xi)\xi^jg_{ij}$$

for every $\xi\in C$ and $i=1,2,3$. Then we can take $b(\xi)$ to be a degree-one polynomial. Actually, since ξ^jg_{ij} $(i=1,2,3)$ do not all vanish at the same time, $b(\xi)$ $=(\varphi_{ijk}\xi^j\xi^k)/(\xi^jg_{ij})$ defines a holomorphic section of $\mathscr{O}(1)$ over $\mathbb{CP}^2$. If we put $b(\xi)=b_k\xi^k$, then we obtain

$$(\varphi_{ijk}-b_kg_{ij})\xi^j\xi^k=0$$

for $i = 1,2,3$. Here the b_k $(k = 1,2,3)$ are functions on X. Since these equations hold for every $\xi \in C$, there are functions a_i on X such that

$$(\varphi_{ijk} - b_k g_{ij})X^j X^k = a_i(g_{jk}X^j X^k)$$

for every $(X^j) \in \mathbb{C}^3$ and $i = 1,2,3$. Noticing the symmetry, we obtain (4.2).

Finally, if we define a new connection $\tilde{\nabla}$ by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2}b_j + \frac{1}{2}b_k, \tag{4.5}$$

then $\tilde{\nabla} \in [\nabla]$ and $\tilde{\nabla}$ satisfies

$$(\tilde{\nabla} g)_{ijk} = (a_i - b_i)g_{jk},$$

which means $\tilde{\nabla}$ is compatible with $[g]$. Moreover, $([g], \tilde{\nabla})$ is Einstein-Weyl, since the integrable condition in Proposition 3.3 is automatically satisfied by construction. Notice that such a connection is unique since the compatibility condition is not satisfied for any other torsion-free connection in $[\nabla]$. □

Remark 4.2 Let $\mathscr{X} = \{(x,p) \in X \times Z \,|\, p \in Y_x\}$, then we obtain the double fibration $X \xleftarrow{\varpi} \mathscr{X} \xrightarrow{\mathfrak{f}} Z$, where ϖ and $\mathfrak{f}$ are the projections. Each $u \in \mathscr{X}$ defines a null plane at $\varpi(u) \in X$ as a tangent plane of the null surface corresponding to $\mathfrak{f}(u) \in Z$. Hence we obtain a natural map $\mathscr{X} \to \mathscr{Z} = \mathbb{P}(N(T^{*\,1,0}_{\mathbb{C}}X))$ which is in fact biholomorphic. Identifying $\mathscr{X}$ with $\mathscr{Z}$, we obtain $= \ker\{\mathfrak{f}_* : T^{1,0}_{\mathbb{C}}\mathscr{X} \to T^{1,0}_{\mathbb{C}}Z\}$.

Hitchin introduced two examples of Einstein-Weyl spaces, each of which is obtained from a complex twistor space (6). The twistor space of one of them is

$$Z = \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \,|\, z_1^2 + z_2^2 + z_3^3 = 0\right\}.$$

In this case, the twistor lines are the plane sections, and the corresponding Einstein-Weyl space is flat. In the other case, the twistor space is

$$Z = \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \,|\, z_0^2 + z_1^2 + z_2^2 + z_3^3 = 0\right\}. \tag{4.6}$$

In this case, the twistor lines are also the plane sections, and the corresponding Einstein-Weyl space is constant curvature space. We study the latter example in more detail in the next section.

5 The Standard Case

In this section, the standard model of LeBrun-Mason type correspondence is explained. We start from Hitchin's example (4.6), and construct the model case as a real slice of it (see also (14)).

If we change the coordinate, (4.6) can be written $\{[z_i] \in \mathbb{CP}^3 \,|\, z_0 z_3 = z_1 z_2\}$ which coincides with the image of the Segre embedding $\mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$,

$$([u_0 : u_1],[v_0 : v_1]) \longmapsto [u_0 v_0 : u_0 v_1 : u_1 v_0 : u_1 v_1].$$

So we usually identify Z with $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since the twistor lines are the plane sections, the twistor lines are parametrized by $X = \mathbb{CP}^{*3}$. We introduce a homogeneous coordinate $[\xi^i] \in \mathbb{CP}^{*3}$ so that $[\xi^i]$ corresponds to the plane $\{[z_i] \in \mathbb{CP}^3 \,|\, \xi^i z_i = 0\}$. Let

$$X_{\text{sing}} = \left\{ \, [\xi^i] \in \mathbb{CP}^{*3} \,\middle|\, \xi^0\xi^3 = \xi^1\xi^2 \right\}$$

be the set of planes tangent to Z. If $[\xi^i] \in X_{\text{sing}}$, then the plane section degenerates to two lines

$$\left(\mathbb{CP}^1 \times [-\xi^1 : \xi^0]\right) \cup \left([-\xi^2 : \xi^0] \times \mathbb{CP}^1\right)$$

intersecting at the tangent point. We call such a plane section a *singular twistor line* on Z. Since Proposition 4.1 does not work on X_{sing}, the Einstein-Weyl structure is defined only on $X \backslash X_{\text{sing}}$.

Next we introduce real structures, that is, antiholomorphic involution on Z. There are several ways to introduce such a structure. For example, if we take the fixed-point-free involution

$$\sigma' : ([u_0 : u_1], [v_0 : v_1]) \longmapsto ([\bar{u}_1 : \bar{u}_0], [\bar{v}_1 : -\bar{v}_0]),$$

then σ' extends to an involution on $\mathbb{CP}^3$ by

$$[z_0 : z_1 : z_2 : z_3] \longmapsto [\bar{z}_3 : -\bar{z}_2 : -\bar{z}_1 : \bar{z}_0].$$

Then we also obtain an antiholomorphic involution on X. Let $X_{\mathbb{R}}$ be its fixed point set. Since $X_{\mathbb{R}} \cap X_{\text{sing}}$ is empty, we obtain a real Einstein-Weyl structure on the whole of $X_{\mathbb{R}} \cong \mathbb{RP}^3$ as a real slice of the complex Einstein-Weyl structure on

$X\backslash X_{\rm sing}$. This is nothing but the definite Einstein-Weyl structure induced from the standard constant curvature metric on $\mathbb{RP}^3$.

Our main interest is, however, in the indefinite case. Let

$$\sigma : ([u_0 : u_1],[v_0 : v_1]) \longmapsto ([\bar{v}_1 : \bar{v}_0],[\bar{u}_1 : \bar{u}_0]),$$

be another involution on Z for which the fixed point set is denoted by $Z_{\mathbb{R}}$. The involution σ extends to an involution on $\mathbb{CP}^3$ by

$$[z_0 : z_1 : z_2 : z_3] \longmapsto [\bar{z}_3 : \bar{z}_1 : \bar{z}_2 : \bar{z}_0].$$

Then we also obtain an involution on X. Let $X_{\mathbb{R}}$ be its fixed point set. In this case, $X_{\mathbb{R},\rm sing} = X_{\mathbb{R}} \cap X_{\rm sing}$ is nonempty.

Let $(\eta_1,\eta_2) = (u_0/u_1, v_0/v_1)$ be a coordinate on $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$, and let us write $\tau(\eta)$ for $\bar{\eta}^{-1}$. Then $\sigma(\eta_1,\eta_2) = (\tau(\eta_2),\tau(\eta_1))$ and $Z_{\mathbb{R}} = \{(\eta,\tau(\eta))\,|\,\eta \in \mathbb{CP}^1\}$. In this coordinate, each non-singular twistor line l is written as a graph of some Möbius transform $f : \mathbb{CP}^1 \to \mathbb{CP}^1$, that is, $l = \{(\eta, f(\eta))\,|\,\eta \in \mathbb{CP}^1\}$. The twistor line l is σ-invariant if and only if $\tau(f(\eta)) = f^{-1}(\tau(\eta))$, and in this case we can write

$$f(\eta) = \frac{A\eta - B}{\bar{B}\eta - C}$$

for some $(A,B,C) \in \mathbb{R}\times\mathbb{C}\times\mathbb{R}$ satisfying $|B|^2 - AC \neq 0$. The intersection $l \cap Z_{\mathbb{R}}$ is nonempty if $|B|^2 - AC > 0$, and is empty if $|B|^2 - AC < 0$.

In the non-singular case, the parameters (A,B,C) can be normalized so that $|B|^2 - AC = \pm 1$. Since (A,B,C) and $(-A,-B,-C)$ defines the same Möbius transform, we obtain $X_{\mathbb{R}}\backslash X_{\mathbb{R},\rm sing} \cong H \sqcup H'$, where

$$H = \left\{(A,B,C) \in \mathbb{R}\times\mathbb{C}\times\mathbb{R} \,\middle|\, |B|^2 - AC = 1 \right\}/\pm$$
$$\text{and}\quad H' = \left\{(A,B,C) \in \mathbb{R}\times\mathbb{C}\times\mathbb{R} \,\middle|\, |B|^2 - AC = -1 \right\}/\pm.$$

We obtain an indefinite Einstein-Weyl structure on H and a definite Einstein-Weyl structure on H' as a real slice of $X\backslash X_{\rm sing}$. The conformal structures are the class of

$$g = |dB|^2 - dAdC,$$

which is indefinite on H and definite on H'.

If we identify $\mathbb{CP}^1 \xrightarrow{\sim} Z_{\mathbb{R}}$ by $\omega \mapsto (\omega, \bar{\omega}^{-1})$, then the intersection of $Z_{\mathbb{R}}$ with the twistor line corresponding to $[A,B,C] \in H$ is the circle

$$\left\{\omega \in \mathbb{CP}^1 \,\middle|\, A|\omega|^2 - B\bar{\omega} - \bar{B}\omega + C = 0 \right\}. \tag{5.1}$$

Hence H is naturally identified with the set of circles on $\mathbb{CP}^1$, and its double cover

$$\tilde{H} = \left\{(A,B,C) \in \mathbb{R}\times\mathbb{C}\times\mathbb{R} \,\middle|\, |B|^2 - AC = 1 \right\} \cong S^2 \times \mathbb{R}$$

is identified with the set of oriented circles on $\mathbb{CP}^1$. Since each circle divides the twistor line into two holomorphic disks, $\tilde{H}$ is identified with the set of holomorphic disks in Z with boundaries lying on $Z_{\mathbb{R}}$.

There is a natural action of $\mathrm{PSL}(2,\mathbb{C})$ on H, H' and $\tilde{H}$ defined in the following way. Each $\phi \in \mathrm{PSL}(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)$ induces an automorphism on Z by

$$\phi_* : (\eta_1, \eta_2) \mapsto (\phi(\eta_1), \tau\phi\tau(\eta_2)). \tag{5.2}$$

The automorphism ϕ_* maps each σ-invariant twistor line to another σ-invariant twistor line. Since ϕ_* preserves $Z_{\mathbb{R}}$, ϕ_* preserves H and H'. Obviously this action lifts to an automorphism on $\tilde{H}$, and we will see later that this action on $\tilde{H}$ is transitive.

Now we introduce an explicit description of the holomorphic disks corresponding to $\tilde{H}$. Let $M \cong \mathbb{CP}^1 \times \mathbb{R} = U_1 \cup U_2$, where the $U_i = \{(\lambda_i, t) \in \mathbb{C} \times \mathbb{R}\}$ are patched by $\lambda_2 = \lambda_1^{-1}$. Let $\varpi : \mathscr{X}_+ \to M$ be the disk bundle

$$\begin{aligned} &\mathscr{X}_+ = (U_1 \times \mathbb{D}) \cup (U_2 \times \mathbb{D}), \\ &(\lambda_1, t; z_1) \sim (\lambda_2, t; z_2) \iff \lambda_2 = \lambda_1^{-1}, z_2 = \frac{\bar{\lambda}_1}{\lambda_1} z_1, \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| \le 1\}$. We denote $\mathscr{X}_{\mathbb{R}} = (U_1 \times \partial\mathbb{D}) \cup (U_2 \times \partial\mathbb{D})$, and notice that $\mathscr{X}_{\mathbb{R}}$ is a circle bundle with $c_1(\mathscr{X}_{\mathbb{R}}) = 2$ along each fiber of ϖ. Let us define a smooth map $\mathfrak{f} : \mathscr{X}_+ \to Z$ by

$$\begin{aligned} & U_1 \times \mathbb{D} \ni (\lambda_1, t; z_1) \longmapsto \Big(\frac{z_1 + r\lambda_1}{-\bar{\lambda}_1 z_1 + r}, \frac{r z_1 - \lambda_1}{r\bar{\lambda}_1 z_1 + 1} \Big) \\ \text{and} \quad & U_2 \times \mathbb{D} \ni (\lambda_2, t; z_2) \longmapsto \Big(\frac{\bar{\lambda}_2 z_2 + r}{-z_2 + r\lambda_2}, \frac{r\bar{\lambda}_2 z_2 - \lambda}{r z_2 + \lambda_2} \Big), \end{aligned}$$

where $r = e^t$. In this way, we have obtained the following double fibration:

$$\begin{array}{ccc} & \mathscr{X}_+ & \\ {\scriptstyle\varpi}\swarrow & & \searrow{\scriptstyle\mathfrak{f}} \\ M & & Z \end{array} \tag{5.3}$$

We use the coordinate $\lambda \in \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ satisfying $\lambda = \lambda_1$ on U_1, and we define $D_{(\lambda,t)} = \mathfrak{f} \circ \varpi^{-1}(\lambda, t)$. Then $\{D_{(\lambda,t)}\}_{(\lambda,t)\in M}$ gives the family of holomorphic disks which coincides with the family corresponding to $\tilde{H}$ above. Hence naturally $M \cong \tilde{H}$. Notice that we made our construction in such a way that the center of $D_{(\lambda,t)}$, that is, the point given by $z = 0$, lies on

$$Q = \big\{ (\lambda, -\lambda) \in Z \,\big|\, \lambda \in \mathbb{CP}^1 \big\}$$

which is a twistor line on Z corresponding to $[1,0,1] \in H'$.

We have already defined a $\mathrm{PSL}(2,\mathbb{C})$-action on $M = \tilde{H}$ by (5.2). For each element $\phi \in \mathrm{PSU}(2) \subset \mathrm{PSL}(2,\mathbb{C})$, we can check that $\phi_*(D_{(\lambda,t)}) = D_{(\phi(\lambda),t)}$. Since $\mathrm{PSU}(2)$ acts transitively on $\mathbb{CP}^1$, $\mathrm{PSU}(2)$ acts transitively on $\mathbb{CP}^1 \times \{t\} \subset M$ for each $t \in \mathbb{R}$. On the other hand,

$$\phi = \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \tag{5.4}$$

gives the automorphism ϕ_* which maps the disk $D_{(0,1)}$ to $D_{(0,2t)}$. Hence the action of $\mathrm{PSL}(2,\mathbb{C})$ on $M = \tilde{H}$ is transitive.

Let $S(TZ_{\mathbb{R}}) = (TZ_{\mathbb{R}} \backslash 0_{Z_{\mathbb{R}}})/\mathbb{R}_+$ be the circle bundle on $Z_{\mathbb{R}}$, where $0_{Z_{\mathbb{R}}}$ is the zero section and $\mathbb{R}_+$ is positive real numbers acting on $TZ_{\mathbb{R}}$ by scalar multiplication. On $\mathscr{X}_{\mathbb{R}}$, we can take a nowhere vanishing vertical vector field $\mathbf{v}$, that is, $\varpi_*(\mathbf{v}) = 0$, so that the orientation matches the complex orientation of the fiber of $\varpi : \mathscr{X}_+ \to M$. Since $\mathfrak{f}_*(\mathbf{v})$ does not vanish anywhere, we can define a smooth map $\tilde{\mathfrak{f}} : \mathscr{X}_{\mathbb{R}} \to S(TZ_{\mathbb{R}})$ by $u \mapsto [\mathfrak{f}_*(\mathbf{v}_u)]$. Then we obtain the following diagram:

$$\begin{array}{ccc} \mathscr{X}_{\mathbb{R}} & \xrightarrow{\tilde{\mathfrak{f}}} & S(TZ_{\mathbb{R}}) \\ & {}_{\mathfrak{f}}\searrow \quad \swarrow & \\ & Z_{\mathbb{R}} & \end{array}$$

Proposition 5.1 *Let $S_t = \mathbb{CP}^1 \times \{t\} \subset M$, and let $\mathfrak{f}_t$ and $\tilde{\mathfrak{f}}_t$ be the restriction of $\mathfrak{f}$ and $\tilde{\mathfrak{f}}$ on $\varpi^{-1}(S_t)$ respectively. Then, for each $t \in \mathbb{R}$,*

1. *$\mathfrak{f}_t : (\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}})|_{S_t} \to Z \backslash Z_{\mathbb{R}}$ is diffeomorphic,*
2. *$\tilde{\mathfrak{f}}_t : \mathscr{X}_{\mathbb{R}}|_{S_t} \to S(TZ_{\mathbb{R}})$ is diffeomorphic, and*
3. *$\mathfrak{f}_t : \mathscr{X}_{\mathbb{R}}|_{S_t} \to Z_{\mathbb{R}}$ is an S^1-fibration such that each fiber is transverse to the vertical distribution of $\varpi : \mathscr{X}_{\mathbb{R}} \to M$.*

In particular, $\{D_{(\lambda,t)}\}_{\lambda \in \mathbb{CP}^1}$ gives a foliation on $Z \backslash Z_{\mathbb{R}}$ for each $t \in \mathbb{R}$.

Remark 5.2 Notice that, from 2 above, the following holds: for each $t \in \mathbb{R}$, $p \in Z_{\mathbb{R}}$ and non-zero $v \in T_p Z_{\mathbb{R}}$, there is a unique $x \in S_t$ such that $p \in \partial D_x$ and $v \parallel D_x$ (see Definition 1.5).

Proof of Proposition 5.1 We can assume $t = 0$ by changing the parameter $t \in \mathbb{R}$ by the automorphism of type (5.4).

When $t = 0$, we can interpret the situation as a geometry on S^2 in the following way. Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = 1\}$ and $\mathfrak{p} : \mathbb{CP}^1 \xrightarrow{\sim} S^2$ be the stereographic projection,

$$\mathfrak{p} : \lambda \longmapsto \Big(\frac{2\operatorname{Re}\lambda}{1+|\lambda|^2}, \frac{2\operatorname{Im}\lambda}{1+|\lambda|^2}, \frac{1-|\lambda|^2}{1+|\lambda|^2}, \Big).$$

We identify Z with $S^2 \times S^2$ by the diffeomorphism $Z \xrightarrow{\sim} S^2 \times S^2 : (\eta_1, \eta_2) \mapsto (\mathfrak{p}(\eta_1), \mathfrak{p} \circ \tau(\eta_2))$, where $\tau(\eta) = \bar{\eta}^{-1}$. Notice that $Z_{\mathbb{R}}$ corresponds to the diagonal in this identification.

Recall that $D_{(\lambda,0)}$ is the image of $\mathbb{D} \to Z$:

$$z \longmapsto (\eta_1, \eta_2) = \Big(\frac{z+\lambda}{-\bar{\lambda}z+1}, \frac{z-\lambda}{\bar{\lambda}z+1}\Big).$$

Then $\partial D_{(\lambda,0)} \subset Z_{\mathbb{R}}$ corresponds to the big circle on the diagonal $S^2 \subset S^2 \times S^2$ cut out by the plane

$$2(\operatorname{Re}\lambda)x_1 + 2(\operatorname{Im}\lambda)x_2 + (1-|\lambda|^2)x_3 = 0. \tag{5.5}$$

Hence we obtain a one-to-one correspondence between $\lambda \in \mathbb{CP}^1$ and the oriented big circle $\mathfrak{p}(\partial D_{(\lambda,0)})$, where the orientation is induced from the natural orientation of $\mathfrak{p}(D_{(\lambda,0)})$. Moreover, we claim that the following conditions are equivalent:

(A1) $(\eta_1, \eta_2) \in Z$ lies on $D_{(\lambda,0)}$,

(A2) the oriented big circle $\mathfrak{p}(\partial D_{(\lambda,0)})$ winds anti-clockwise around $\mathfrak{p}(\eta_1)$, and this big circle coincides with the set of points on S^2 which have the same distance from $\mathfrak{p}(\eta_1)$ and $\mathfrak{p}\circ\tau(\eta_2)$ with respect to the standard Riemannian metric on S^2.

Indeed, if $(\eta_1, \eta_2) \in D_{(\lambda,0)}$, then the point

$$\mathfrak{p}(\eta_1) + \mathfrak{p}\circ\tau(\eta_2) \in \mathbb{R}^3$$

lies on the plane (5.5), hence the big circle $\mathfrak{p}(\partial D_{(\lambda,0)})$ satisfies (A2). The converse is easy. In particular, the following conditions are equivalent:

(B1) $(\eta_1, \eta_2) \in Z_{\mathbb{R}}$ lies on $\partial D_{(\lambda,0)}$,

(B2) the big circle $\mathfrak{p}(\partial D_{(\lambda,0)})$ passes through $\mathfrak{p}(\eta_1) = \mathfrak{p}\circ\tau(\eta_2)$.

The statement follows directly from this interpretation. Actually, for each $p = (\eta_1, \eta_2) \in Z\backslash Z_{\mathbb{R}}$, the big circle satisfying (A2) exists uniquely, hence *1* holds. For each $p = (\eta_1, \eta_2) \in Z_{\mathbb{R}}$, $S(T_p Z_{\mathbb{R}})$ corresponds to the oriented big circles satisfying (B2), hence *2* and *3* follow. □

The geometry on M is characterized by the double fibration (5.3) in the following way:

Proposition 5.3 1. *For each $p \in Z_{\mathbb{R}}$, $\mathfrak{S}_p = \{x \in M \,|\, p \in \partial D_x\} = \varpi\circ\mathfrak{f}^{-1}(p)$ is a maximal connected null surface on M and every null surface can be written in this form.*

2. *For each $p \in Z\backslash Z_{\mathbb{R}}$, $\mathfrak{C}_p = \{x \in M \,|\, p \in D_x\} = \varpi\circ\mathfrak{f}^{-1}(p)$ is a maximal connected time-like geodesic on M and every time-like geodesic can be written in this form.*

3. *For each $p \in Z_{\mathbb{R}}$ and each non-zero $v \in T_p Z_{\mathbb{R}}$, $\mathfrak{C}_{p,v} = \{x \in M \,|\, p \in \partial D_x, v \parallel D_x\} = \varpi\circ\tilde{\mathfrak{f}}^{-1}([v])$ is a maximal connected null geodesic on M and every null geodesic can be written in this form.*

4. *For each distinguished $p, q \in Z_{\mathbb{R}}$, $\mathfrak{C}_{p,q} = \{x \in M \,|\, p, q \in \partial D_x\} = \mathfrak{S}_p \cap \mathfrak{S}_q$ is a closed connected space-like geodesic on M and every space-like geodesic can be written in this form.*

Proof Since $\{\partial D_{(\lambda,t)}\}$ is the set of oriented circles of the form (5.1), we obtain

- $\mathfrak{S}_p \simeq S^1 \times \mathbb{R}$ for each $p \in Z_{\mathbb{R}}$,
- $\mathfrak{C}_{p,v} \simeq \mathbb{R}$ for each $p \in Z_{\mathbb{R}}$ and non zero vector $v \in T_p Z_{\mathbb{R}}$,
- $\mathfrak{C}_{p,q} \simeq S^1$ for each distinguished $p, q \in Z_{\mathbb{R}}$.

Since $\mathfrak{S}_p$ is a real slice of a complex null surface, it is a real null surface. Moreover, it is a maximal connected null surface since $\mathfrak{S}_p$ is closed in M. Hence *1* holds. In a similar way, we can see that $\mathfrak{C}_{p,v}$ is a maximal connected real null geodesic, so *3* holds. $\mathfrak{C}_{p,q}$ is also a maximal connected real non-null geodesic. Notice that $\mathfrak{C}_{p,q}$ is contained in the null surface $\mathfrak{S}_p$. Since a null plane never contains time-like vectors, $\mathfrak{C}_{p,q}$ is a space-like geodesic (see Lemma 3.7). Hence *4* holds.

Now we check 2. Let $p \in Z \backslash Z_{\mathbb{R}}$ and notice that every σ-invariant twistor line passing through p also passes through $\sigma(p)$. So $\mathfrak{C}_p$ is a real slice of the complex geodesic corresponding to the two points p and $\sigma(p)$. Hence $\mathfrak{C}_p$ is a geodesic. From Proposition 5.1, we obtain $\mathfrak{C}_p \simeq \mathbb{R}$ which is closed in M. Hence $\mathfrak{C}_p$ is a maximal connected geodesic. To see that $\mathfrak{C}_p$ is a time-like geodesic, it is enough to check that $\mathfrak{C}_p$ is transversal to every null plane at each point (see Lemma 3.7). Notice that, if we fix three points on Z, there is at most one twistor line containing them. Hence $\mathfrak{C}_p \cap \mathfrak{S}_q = \{x \in M \,|\, p, \sigma(p), q \in D_x\}$ is at most one point for each $q \in Z_{\mathbb{R}}$. Thus $\mathfrak{C}_p$ is time-like. $\square$

In particular, we obtain the following.

Corollary 5.4 *The indefinite Einstein-Weyl structure on M constructed above is space-like Zoll.*

Let $\mathscr{X} = \mathscr{X}_+ \cup_{\mathscr{X}_{\mathbb{R}}} \mathscr{X}_-$ be a $\mathbb{CP}^1$ bundle over M, where $\mathscr{X}_- = \overline{\mathscr{X}_+}$ is the copy of $\mathscr{X}_+$ with fiber-wise opposite complex structure. On the other hand, we have a $\mathbb{CP}^1$-bundle $\mathscr{Z}$ on M equipped with the distributions $_{\mathbb{R}}, , L$ and so on. Then, similar to Remark 4.2, there is a natural identification $\mathscr{X} \xrightarrow{\sim} \mathscr{Z}$ such that

- for each $p \in Z_{\mathbb{R}}$, $\mathfrak{f}^{-1}(p)$ corresponds to an integral surface of $_{\mathbb{R}}$,
- for each $p \in Z \backslash Z_{\mathbb{R}}$, $\mathfrak{f}^{-1}(p)$ corresponds to an integral curve of L in $\mathscr{X} \backslash \mathscr{X}_{\mathbb{R}}$, and
- for each $p \in Z_{\mathbb{R}}$ and $[v] \in S(T_p Z_{\mathbb{R}})$, $\tilde{\mathfrak{f}}^{-1}([v])$ corresponds to an integral curve of L in $\mathscr{X}_{\mathbb{R}}$.

Hence the following holds:

- $_{\mathbb{R}} = \cap T\mathscr{X}_{\mathbb{R}} = \ker\{\mathfrak{f}_* : T\mathscr{X}_{\mathbb{R}} \to TZ_{\mathbb{R}}\}$ on $\mathscr{X}_{\mathbb{R}}$,
- $L = \ker\{\mathfrak{f}_* : T\mathscr{X} \to TZ\}$ on $\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}}$, and
- $L = \ker\{\tilde{\mathfrak{f}}_* : T\mathscr{X}_{\mathbb{R}} \to S(TZ_{\mathbb{R}})\}$ on $\mathscr{X}_{\mathbb{R}}$.

Recall that we denote $S_t = \mathbb{CP}^1 \times \{t\}$ and let us denote $\mathscr{X}_t = \varpi^{-1}(S_t)$, where $\varpi : \mathscr{X} \to M$ is the projection. Let $_t = \cap T_{\mathbb{C}}\mathscr{X}_t$ for each t. Then, since $L \cap T\mathscr{X}_t = 0$, we obtain $= (L \otimes \mathbb{C}) \oplus _t$. From $\cap \bar{} = L \otimes \mathbb{C}$ and $\oplus \bar{} = T\mathscr{X}$, we obtain $_t \oplus \bar{}_t = T\mathscr{X}_t$. Moreover, since is integrable, $_t$ is also integrable. Hence $_t$ defines a complex structure on $\mathscr{X}_t$.

Now we claim that $\mathfrak{f}_t : (\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}})|_{S_t} \to Z \backslash \mathbb{Z}_{\mathbb{R}}$ is holomorphic with respect to the above complex structure. Consider the complex Einstein-Weyl space $M_{\mathbb{C}} = X \backslash X_{\text{sing}}$ defined at the beginning of this section, and let $\mathscr{Z}_{\mathbb{C}} = \mathbb{P}(N(T^{*1,0}M_{\mathbb{C}}))$. Then we obtain the double fibration $M_{\mathbb{C}} \leftarrow \mathscr{Z}_{\mathbb{C}} \xrightarrow{\mathfrak{f}_{\mathbb{C}}} Z$, where $\mathfrak{f}_{\mathbb{C}}$ is holomorphic. On the other hand, there is natural embedding $i_t : (\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}})|_{S_t} \to \mathscr{Z}_{\mathbb{C}}$ which is holomorphic since it preserves the distributions. Since $\mathfrak{f}_t = \mathfrak{f}_{\mathbb{C}} \circ i_t$, $\mathfrak{f}_t$ is holomorphic on $(\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}})|_{S_t}$.

From the above argument, we obtain $_t = (\mathfrak{f}_t)_*^{-1}(T^{0,1}Z) \subset \mathfrak{f}_*^{-1}(T^{0,1}Z)$ on $\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}}$. Since $L \otimes \mathbb{C} = \ker \mathfrak{f}_*$ there, we obtain $= (L \otimes \mathbb{C}) \oplus _t \subset \mathfrak{f}_*^{-1}(T^{0,1}Z)$ on $\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}}$. Then we also have $\bar{} \subset \mathfrak{f}_*^{-1}(T^{1,0}Z)$. Since $+ \bar{} = T_{\mathbb{C}}\mathscr{X}_+$ and $\cap \bar{} = L \otimes \mathbb{C}$, we obtain $= \mathfrak{f}_*^{-1}(T^{0,1}Z)$ on $\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}}$.

In this way, we have proved the following:

Proposition 5.5 *Identifying $\mathscr{X} = \mathscr{X}_+ \cup \mathscr{X}_-$ with $\mathscr{Z}$,*

1. $\;= \mathfrak{f}_*^{-1}(T^{0,1}Z)$ *on* $\mathscr{X}_+$ *where* $\mathfrak{f}_* : T_{\mathbb{C}}\mathscr{X}_+ \to T_{\mathbb{C}}Z$,
2. $\;_{\mathbb{R}} = \cap T\mathscr{X}_{\mathbb{R}} = \ker\{\mathfrak{f}_* : T\mathscr{X}_{\mathbb{R}} \to TZ_{\mathbb{R}}\}$ *on* $\mathscr{X}_{\mathbb{R}}$,
3. $L = \ker\{\mathfrak{f}_* : T\mathscr{X}_+ \to TZ\}$ *on* $\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}}$, *and*
4. $L = \ker\{\tilde{\mathfrak{f}}_* : T\mathscr{X}_{\mathbb{R}} \to S(TZ_{\mathbb{R}})\}$ *on* $\mathscr{X}_{\mathbb{R}}$.

It is convenient to consider the compactification of M and $\mathscr{X}_+$. Let $I = [-\infty, \infty]$ be the natural compactification of $\mathbb{R}$. If we put $\hat{M} = \mathbb{CP}^1 \times I$, then we obtain a natural embedding $\iota : M \hookrightarrow \hat{M}$. Next, let $\Psi : \mathscr{X}_+ \to \hat{M} \times Z$ be the embedding defined by $\Psi(u) = \big(\iota \circ \varpi(u), \mathfrak{f}(u)\big)$. Let us define $\hat{\mathscr{X}_+}$ and $\hat{\mathscr{X}_{\mathbb{R}}}$ as the closure of $\Psi(\mathscr{X}_+)$ and $\Psi(\mathscr{X}_{\mathbb{R}})$ in $\hat{M} \times Z$. Then we obtain the double fibration

$$\begin{array}{ccc} & (\hat{\mathscr{X}_+}, \hat{\mathscr{X}_{\mathbb{R}}}) & \\ {\scriptstyle\hat{\varpi}}\swarrow & & \searrow{\scriptstyle\hat{\mathfrak{f}}} \\ \hat{M} & & (Z, Z_{\mathbb{R}}) \end{array} \tag{5.6}$$

where $\hat{\varpi}$ and $\hat{\mathfrak{f}}$ are the projections.

Notice that $\hat{\varpi}^{-1}(x)$ is no longer a disk for $x = (\lambda, \pm\infty) \in \partial\hat{M}$, but a *marked* $\mathbb{CP}^1$ for which the marking point is $\hat{\varpi}^{-1}(x) \cap \hat{\mathscr{X}_{\mathbb{R}}}$. We denote these marked $\mathbb{CP}^1$ by

$$\begin{aligned} D_{(\lambda,\infty)} &= \hat{\varpi}^{-1}(\lambda, -\infty) = \{\lambda\} \times \mathbb{CP}^1 \\ \text{and} \quad D_{(\lambda,-\infty)} &= \hat{\varpi}^{-1}(\lambda, \infty) \ = \mathbb{CP}^1 \times \{-\lambda\}, \end{aligned} \tag{5.7}$$

where $D_{(\lambda,\infty)}$ is marked at $(\lambda, \bar{\lambda}^{-1})$ and $D_{(\lambda,-\infty)}$ is marked at $(-\bar{\lambda}^{-1}, -\lambda)$.

Recall the definitions of $\mathfrak{C}_p$ and $\mathfrak{C}_{p,v}$ introduced in Proposition 5.3. We define $\hat{\mathfrak{C}}_p$ and $\hat{\mathfrak{C}}_{p,v}$ as the compactification of $\mathfrak{C}_p$ and $\mathfrak{C}_{p,v}$ in $\hat{M}$ respectively. Then the following properties are easily checked.

Proposition 5.6 1. *For each* $p \in Z \backslash Z_{\mathbb{R}}$, $\hat{\mathscr{X}_{\mathbb{R}}}|_{\hat{\mathfrak{C}}_p}$ *is homeomorphic to* S^2 *and the restriction* $\hat{\mathfrak{f}} : \hat{\mathscr{X}_{\mathbb{R}}}|_{\hat{\mathfrak{C}}_p} \to Z_{\mathbb{R}}$ *is a homeomorphism. In particular,* $\{\partial D_x\}_{x \in \mathfrak{C}_p}$ *gives a foliation on* $Z_{\mathbb{R}} \backslash \{2 \text{ points}\}$.

2. *For each* $p \in Z_{\mathbb{R}}$ *and non-zero* $v \in T_p Z_{\mathbb{R}}$, $\hat{\mathscr{X}_{\mathbb{R}}}|_{\hat{\mathfrak{C}}_{p,v}}$ *is homeomorphic to* S^2 *and the restriction* $\hat{\mathfrak{f}} : \hat{\mathscr{X}_{\mathbb{R}}}|_{\hat{\mathfrak{C}}_{p,v}} \to Z_{\mathbb{R}}$ *is surjective. Moreover, this is one-to-one on the complement of the curve* $\hat{\mathfrak{f}}^{-1}(p)$, *hence* $\{(\partial D_x \backslash \{p\})\}_{x \in \mathfrak{C}_{p,v}}$ *gives a foliation on* $Z_{\mathbb{R}} \backslash \{p\}$.

Remark 5.7 For distinguished points $p, q \in Z_{\mathbb{R}} \simeq \mathbb{CP}^1$, there are two families of circles called "Apollonian circles". One of them is the family of the circles passing through p, q, which corresponds to the space-like geodesic $\mathfrak{C}_{p,q}$. The other family gives a foliation on $\mathbb{CP}^1 \backslash \{p, q\}$, which corresponds to a time-like geodesic and the foliation coincides with the one given in *1* of Proposition 5.6. The Case *2* of Proposition 5.6 corresponds to the degenerate case.

6 Perturbation of Holomorphic Disks

We now investigate the deformation of holomorphic disks. For a complex manifold A and its submanifold B, we use the term *holomorphic disk on* (A,B) for a continuous map $(\mathbb{D},\partial\mathbb{D}) \to (A,B)$ which is holomorphic on the interior of $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$.

As in the previous section, we put $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $Z_{\mathbb{R}} = \{(\eta, \bar{\eta}^{-1}) \,|\, \eta \in \mathbb{CP}^1\}$. We have the family of holomorphic disks $\{D_{(\lambda,t)}\}$ defined from the double fibration (5.3), and we call each $D_{(\lambda,t)}$ a *standard disk*. In this section, we treat a small perturbation N of $Z_{\mathbb{R}}$, and prove that there is a natural $(S^2 \times \mathbb{R})$-family of holomorphic disks on (Z,N) each of which is close to some standard disk. From the general theory by LeBrun (8), one can show that there exists a real three-parameter family of holomorphic disks on (Z,N) near each standard disk. We, however, use the method given in (9) so that we can consider the holomorphic disks in more detail.

First of all, we recall the C^k-topology of the space of deformations of $Z_{\mathbb{R}}$ in Z. A small perturbation N of $Z_{\mathbb{R}}$ can be written

$$N = \left\{ \left(\eta, \overline{\phi(\eta)}^{-1}\right) \,\middle|\, \eta \in \mathbb{CP}^1 \right\}$$

using an automorphism $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$ which is sufficiently close to the identity map. Let $\{A_i\}$ be finitely many compact subsets and $\{B_i\}$ be open subsets on $\mathbb{CP}^1$ with complex coordinates η_i, which satisfy $A_i \subset B_i$, $\phi(A_i) \subset B_i$ and $\cup_i A_i = \mathbb{CP}^1$. Then ϕ is identified with a combination of functions $(h_i)_i$, where $h_i \in C^k(A_i,\mathbb{C})$ is defined by $\phi(\eta_i) = \eta_i + h_i(\eta_i)$ for each i. The C^k-topology of the set of deformations of $Z_{\mathbb{R}}$ in Z is defined by the norm

$$\|\phi\|_{C^k} = \sup_i \|h_i\|_{C^k(A_i)},$$

where $\|h_i\|_{C^k(A_i)}$ is the supremum on A_i of absolute values of all partial derivatives of h_i for which the order is less than or equal to k. In particular, let $A \in \mathbb{CP}^1$ be a compact subset contained in a coordinated open subset of $\mathbb{CP}^1$, which we denote B, and suppose $\phi(A) \subset B$, then $\|h\|_{C^k(A)}$ is sufficiently small if ϕ is sufficiently close to the identity where $\phi(\eta) = \eta + h(\eta)$.

Lemma 6.1 *Fix a standard holomorphic disk $D = D_{(\lambda,t)}$. If $N \subset Z$ is the image of any embedding $\mathbb{CP}^1 \hookrightarrow Z$ which is sufficiently close to the standard one in the C^{k+l}-topology with $k,l \geq 1$, then there is a real three-parameter C^l-family of holomorphic disks on (Z,N) each of which is L^2_k close to D.*

Proof Since there is a transitive action of $\mathrm{PSL}(2,\mathbb{C})$ on the standard disks, we can assume $(\lambda,t) = (0,0)$, that is,

$$D = \{(z,z) \in Z \,|\, z \in \mathbb{D}\},$$

where $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$. If we put $A = \{\eta \in \mathbb{C} \,|\, \frac{1}{2} \leq |\eta| \leq 2\}$, then N can be written

$$\left\{ \left(\eta\,, \overline{(\eta + h(\eta))}^{-1}\right) \in Z \,\middle|\, \eta \in A \right\}$$

near ∂D using a function $h \in C^{k+l}(A)$ for which the C^{k+l}-norm is sufficiently small.

Then a small perturbation of ∂D is given as the image of

$$S^1 \to N:\ \theta \mapsto \left(e^{i(\theta+u(\theta))}, \left[e^{-i(\theta+\bar{u}(\theta))} + \bar{h}\left(e^{i(\theta+u(\theta))}\right)\right]^{-1}\right),$$

where u is a $\mathbb{C}$-valued function on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Here we write $\bar{u}(\theta)$ for $\overline{u(\theta)}$ and $\bar{h}(\eta)$ for $\overline{h(\eta)}$. Then we define the maps $\mathfrak{F}_i : L^2_k(S^1,\mathbb{C}) \times C^{k+l}(A,\mathbb{C}) \to L^2_k(S^1,\mathbb{C})$ by

$$\begin{aligned} &[\mathfrak{F}_1(u,h)](\theta) = e^{i(\theta+u(\theta))} \\ \text{and}\quad &[\mathfrak{F}_2(u,h)](\theta) = \left[e^{-i(\theta+\bar{u}(\theta))} + \bar{h}\left(e^{i(\theta+u(\theta))}\right)\right]^{-1}. \end{aligned} \tag{6.1}$$

For a given h, we want to choose $u \in L^2_k(S^1,\mathbb{C})$ so that $[\mathfrak{F}_i(u,h)](\theta)$ extends holomorphically to $\{|z| < 0\}$ for $z = e^{i\theta}$. Taking the derivation $\mathfrak{F}_i$, we obtain

$$\begin{aligned} &[\mathfrak{F}_{1*(0,0)}(\dot{u},\dot{h})](\theta) = ie^{i\theta}\dot{u}(\theta) \\ \text{and}\quad &[\mathfrak{F}_{2*(0,0)}(\dot{u},\dot{h})](\theta) = ie^{i\theta}\bar{\dot{u}}(\theta) - e^{2i\theta}\bar{\dot{h}}(e^{i\theta}). \end{aligned} \tag{6.2}$$

Now, we introduce some bounded operators (see (9)). Set

$$\begin{aligned} L^2{\downarrow} &= \left\{ \sum_{l<0} a_l e^{il\theta} \ \middle|\ a_l \in \mathbb{C},\ \sum_{l<0} |a_l|^2 < \infty \right\} \\ \text{and}\quad L^2_k{\downarrow} &= \left\{ \sum_{l<0} a_l e^{il\theta} \ \middle|\ a_l \in \mathbb{C},\ \sum_{l<0} l^{2k}|a_l|^2 < \infty \right\} = L^2_k(S^1,\mathbb{C}) \cap L^2{\downarrow}, \end{aligned}$$

and define $\Pi : L^2_k(S^1,\mathbb{C}) \to L^2_k{\downarrow}$ by

$$\Pi\Big(\sum_{l=-\infty}^{\infty} a_l e^{il\theta}\Big) = \sum_{l<0} a_l e^{il\theta}.$$

Similarly let us define п $: L^2_k(S^1,\mathbb{C}) \to \mathbb{C}$ by

$$\text{п}\Big(\sum_{l=-\infty}^{\infty} a_l e^{il\theta}\Big) = a_0.$$

Then, for $k,l \geq 1$, we define a C^l-map

$$\mathfrak{F} : L^2_k(S^1,\mathbb{C}) \times C^{k+l}(A,\mathbb{C}) \longrightarrow L^2_k{\downarrow} \times L^2_k{\downarrow} \times C^{k+l}(A,\mathbb{C}) \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$$

given by

$$\mathfrak{F} = (\Pi \circ \mathfrak{F}_1) \times (\Pi \circ \mathfrak{F}_2) \times \text{Л} \times (\text{п} \circ \mathfrak{F}_1) \times (\text{п} \circ \mathfrak{F}_2) \times \text{ш},$$

where

$$\text{Л} : L^2_k(S^1,\mathbb{C}) \times C^{k+l}(A,\mathbb{C}) \longrightarrow C^{k+l}(A,\mathbb{C})$$

is the factor projection, and

$$\text{ш} : L^2_k(S^1,\mathbb{C}) \times C^{k+l}(A,\mathbb{C}) \longrightarrow \mathbb{C}$$

is given by

$$\text{ш}(u,h) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)d\theta,$$

in other words, $\text{ш}(u,h) = \text{п}(u)$. The map $\mathfrak{F}$ is C^l since Π, Л, п and ш are all bounded linear operators, and its derivative is given by

$$\mathfrak{F}_* = (\Pi\circ\mathfrak{F}_{1*})\times(\Pi\circ\mathfrak{F}_{2*})\times \text{Л} \times (\text{п}\circ\mathfrak{F}_{1*})\times(\text{п}\circ\mathfrak{F}_{2*})\times \text{ш}.$$

In particular, if we write $\dot{u}(\theta) = \sum_n u_n e^{in\theta}$, then we obtain

$$\mathfrak{F}_{*(0,0)}\begin{bmatrix}\dot{u}\\ \dot{h}\end{bmatrix} = \begin{bmatrix}\sum_{n<0} iu_{n-1}e^{in\theta}\\ i\sum_{n<0}\bar{u}_{1-n}e^{in\theta} - \Pi\big(e^{2i\theta}\bar{\dot{h}}(e^{i\theta})\big)\\ \dot{h}\\ iu_{-1}\\ i\bar{u}_1 - \text{п}\big(e^{2i\theta}\bar{\dot{h}}(e^{i\theta})\big)\\ u_0\end{bmatrix}.$$

Since $\mathfrak{F}_{*(0,0)}$ has a bounded inverse, the Banach-space inverse function theorem tells us that there is an open neighborhood $\mathfrak{U}$ of $(0,0)\in L^2_k(S^1,\mathbb{C})\times C^{k+l}(A)$ and open neighborhood $\mathfrak{V}$ of $\mathbf{0}\in L^2_k{\downarrow}\times L^2_k{\downarrow}\times C^{k+l}(A,\mathbb{C})\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ such that $\mathfrak{F}|_{\mathfrak{U}} : \mathfrak{U}\to\mathfrak{V}$ is a C^l-diffeomorphism.

Hence, for a given h, we obtain a complex three-parameter C^l-family of holomorphic disks defined from $(u,h) = \mathfrak{F}^{-1}(0,0,h,\alpha_1,\alpha_2,\beta)$, where α_1,α_2,β are small complex numbers. It contains, however, real three-dimensional ambiguity which comes from the disk automorphism. To kill this ambiguity, it is enough to use the inverse of

$$(0,0,h,\alpha,-\alpha,i\beta)\in L^2_k{\downarrow}\times L^2_k{\downarrow}\times C^{k+l}(A,\mathbb{C})\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}, \qquad (6.3)$$

in $\mathfrak{F}$ for $(\alpha,\beta)\in\mathbb{C}\times\mathbb{R}$ which is sufficiently close to $(0,0)$. Now the statement follows since $\|h\|_{C^{k+l}(A)}$ is sufficiently small if N is sufficiently close to $Z_{\mathbb{R}}$. □

Remark 6.2 1. Let $\mathfrak{D}$ be any holomorphic disk on (Z,N) constructed as in the above lemma. Then $\mathfrak{D}$ intersects with N only on the boundary $\partial\mathfrak{D}$. Actually, let $\mathbb{D}\to Z : z\mapsto(\varphi_1(z),\varphi_2(z))$ be the map corresponding to $\mathfrak{D}$ and denote $N = \{(\eta,\overline{\phi(\eta)}^{-1}) \mid \eta\in\mathbb{CP}^1\}$. Notice that $\eta\mapsto\overline{\phi(\eta)}^{-1}$ maps $\varphi_1(\partial\mathbb{D})$ to $\varphi_2(\partial\mathbb{D})$ and maps the interior of $\varphi_1(\mathbb{D})$ to the outside of $\varphi_2(\mathbb{D})$. Suppose that there is an interior point $z\in\mathbb{D}$ such that $\varphi_2(z) = \overline{\phi(\varphi_1(z))}^{-1}$. Then $\varphi_1(z)$ is contained in the interior of $\varphi_1(\mathbb{D})$, and $\overline{\phi(\varphi_1(z))}^{-1}$ is outside of $\varphi_2(\mathbb{D})$. This is a contradiction.

2. We can take $\mathfrak{V}$ so that

$$\mathfrak{V} = \mathfrak{V}_1 \times \mathfrak{V}_2 \times \mathfrak{W} \times V_1 \times V_2 \times V_3$$
$$\text{with} \quad \mathfrak{W} = \left\{ h \in C^{k+l}(A, \mathbb{C}) \,\middle|\, \|h\|_{C^{k+l}(A)} < \varepsilon_0 \right\}, \tag{6.4}$$

where $\mathfrak{V}_i \subset L^2_k\!\downarrow$ and $V_i \subset \mathbb{C}$ are small open sets and $\varepsilon_0 > 0$ is a constant. This notation is used in the following arguments.

Next we want to prove that, if N is sufficiently close to $Z_{\mathbb{R}}$, then the method of Lemma 6.1 works for *all* standard disks at once. Then we need a uniform estimate of the deformation N of $Z_{\mathbb{R}}$ among all standard disks. In LeBrun-Mason's case (9; 10), the parameter spaces of holomorphic disks are compact and homogeneous, so the uniform estimate is automatically deduced from the local estimate. In our case, however, the parameter space is a non-compact space $S^2 \times \mathbb{R}$, so we need more detailed arguments. For this purpose, it is enough to show that the deformations of the disks are "tame", as in the following lemma, on the neighborhood of the boundary of the parameter space.

Lemma 6.3 *Let $\{D_{(\lambda,t)}\}$ be the standard disks. Suppose $N \subset Z$ is sufficiently close to $Z_{\mathbb{R}}$ in the C^{k+l}-topology. Then a three-parameter family of holomorphic disks on (Z,N) near $D_{(\lambda,t)}$ always exists for each $(\lambda,t) \in \mathbb{CP}^1 \times \mathbb{R}$ with $t \gg 0$.*

Proof It is enough to consider the case $\lambda = 0$. We fix a small constant $c > 0$ and let $B_c = \{z \in \mathbb{C} \mid |z| < c\}$. Notice that the compact subset $B_c \times \mathbb{CP}^1 \subset Z$ contains all holomorphic disks of the form $D_{(0,t)}$ if $e^t > 2c^{-1}$. We can write

$$N \cap (B_c \times \mathbb{CP}^1) = \left\{ \left(\eta, \overline{(\eta + h(\eta))}^{-1}\right) \,\middle|\, \eta \in B_c \right\} \tag{6.5}$$

using $h \in C^{k+l}(B_c, \mathbb{C})$. We claim that if $\|h\|_{C^{k+l}(B_c)} < \frac{\varepsilon_0}{4\sqrt{2}}$, then a three-parameter family of holomorphic disks on (Z,N) near $D_{(0,t)}$ exists for all $e^t > 2c^{-1}$. Here ε_0 is the constant defined in (6.4).

Now we show that it is enough to prove the case when $h(0) = 0$ and $\|h\|_{C^{k+l}(B_c)} < \frac{\varepsilon_0}{2\sqrt{2}}$. In the general case, if we change the coordinate $(\eta_1, \bar{\eta}_2^{-1}) \in Z$ to $(\xi_1, \bar{\xi}_2^{-1})$ by the relation

$$\xi_1 = \eta_1, \quad \xi_2 = \eta_2 + h(0),$$

then we can write

$$N \cap (B_c \times \mathbb{CP}^1) = \left\{ \left(\xi, \overline{(\xi + g(\xi))}^{-1}\right) \,\middle|\, \xi \in B_c \right\}$$

using $g(\xi) = h(\xi) - h(0)$. Here we obtain $\|g\|_{C^{k+l}(B_c)} < \frac{\varepsilon_0}{2\sqrt{2}}$, because

$$\sup_{\xi \in B_c} |g(\xi)| < \sup_{\xi \in B_c} |h(\xi)| + |h(0)| < \frac{\varepsilon_0}{2\sqrt{2}}$$
$$\text{and} \quad \sup_{\xi \in B_c} |Dg(\xi)| = \sup_{\xi \in B_c} |Dh(\xi)| < \frac{\varepsilon_0}{4\sqrt{2}},$$

where D is any partial derivative of degree less than or equal to $k+l$. Hence, if we replace g with h, we can assume $h(0)=0$ and $\|h\|_{C^{k+l}(B_c)} < \frac{\varepsilon_0}{2\sqrt{2}}$.

From now on we write r for e^t. A small perturbation of $\partial D_{(0,t)}$ is given as the image of

$$S^1 \to N:\ \theta \mapsto \big(r^{-1}e^{i(\theta+u(\theta))}, \Big[r^{-1}e^{-i(\theta+\bar{u}(\theta))} + \bar{h}\big(r^{-1}e^{i(\theta+u(\theta))}\big)\Big]^{-1}\big),$$

where u is a $\mathbb{C}$-valued function on S^1.

Let $A^r = \{z\in\mathbb{C} \mid \frac{r^{-1}}{2} \le |z| \le 2r^{-1}\}$ and $A = A^1$, then A^r is a compact subset of B_c if $r > 2c^{-1}$. We define the maps $\mathfrak{F}_i^r : L_k^2(S^1,\mathbb{C}) \times C^{k+l}(A^r,\mathbb{C}) \to L_k^2(S^1,\mathbb{C})$ by

$$\begin{aligned} [\mathfrak{F}_1^r(u,h)](\theta) &= r^{-1}e^{i(\theta+u(\theta))} \\ \text{and} \quad [\mathfrak{F}_2^r(u,h)](\theta) &= \Big[r^{-1}e^{-i(\theta+\bar{u}(\theta))} + \bar{h}\big(r^{-1}e^{i(\theta+u(\theta))}\big)\Big]^{-1}. \end{aligned}$$

Putting $h^r(z) = rh(r^{-1}z)$, we obtain

$$[\mathfrak{F}_1^r(u,h)](\theta) = r^{-1}[\mathfrak{F}_1(u,h^r)](\theta) \quad \text{and} \quad [\mathfrak{F}_2^r(u,h)](\theta) = r[\mathfrak{F}_2(u,h^r)](\theta), \tag{6.6}$$

where $\mathfrak{F}_i$ is the map given by (6.1). Notice that the map $\rho^r : h \mapsto h^r$ gives an isomorphism of Banach spaces $C^{k+l}(A^r,\mathbb{C}) \to C^{k+l}(A,\mathbb{C})$.

In a similar way to how we defined $\mathfrak{F}$ in Lemma 6.1, we define

$$\mathfrak{F}^r : L_k^2(S^1,\mathbb{C}) \times C^{k+l}(A^r,\mathbb{C}) \longrightarrow L_k^2{\downarrow} \times L_k^2{\downarrow} \times C^{k+l}(A^r,\mathbb{C}) \times \mathbb{C}\times\mathbb{C}\times\mathbb{C}$$

given by

$$\mathfrak{F}^r = (\Pi\circ\mathfrak{F}_1^r) \times (\Pi\circ\mathfrak{F}_2^r) \times \text{Л} \times (\text{п}\circ\mathfrak{F}_1^r) \times (\text{п}\circ\mathfrak{F}_2^r) \times \text{ш},$$

where Л is the projection. Then we can relate $\mathfrak{F}^r$ to $\mathfrak{F}$ in the following way. Let $m(r)$ be multiplication of r on $L_k^2{\downarrow}$ or $\mathbb{C}$, then we obtain the following commutative diagram

$$\begin{array}{ccc} L_k^2(S^1,\mathbb{C})\times C^{k+l}(A^r,\mathbb{C}) & \xrightarrow{\ \mathfrak{F}^r\ } & L_k^2{\downarrow} \times L_k^2{\downarrow} \times C^{k+l}(A^r,\mathbb{C}) \times \mathbb{C}\times\mathbb{C}\times\mathbb{C} \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\rho^r} & & \Big\downarrow{\scriptstyle \Phi^r} \\ L_k^2(S^1,\mathbb{C})\times C^{k+l}(A,\mathbb{C}) & \xrightarrow{\ \mathfrak{F}\ } & L_k^2{\downarrow} \times L_k^2{\downarrow} \times C^{k+l}(A,\mathbb{C}) \times \mathbb{C}\times\mathbb{C}\times\mathbb{C} \end{array} \tag{6.7}$$

where $\Phi^r = m(r)\times m(r^{-1})\times\rho^r\times m(r)\times m(r^{-1})\times \mathrm{id}$. Notice that the vertical arrows in the above diagram are isomorphisms, and that the restriction $\mathfrak{F}|_{\mathfrak{U}} : \mathfrak{U} \to \mathfrak{V}$ gives a C^l-diffeomorphism from the proof of Lemma 6.1. Hence the restriction

$$\mathfrak{F}^r : (\mathrm{id}\times\rho^r)^{-1}(\mathfrak{U}) \longrightarrow (\Phi^r)^{-1}(\mathfrak{V})$$

is a C^l-diffeomorphism. If we take $\mathfrak{V}$ to be the product as in (6.4), then

$$(\Phi^r)^{-1}(\mathfrak{V}) = r^{-1}\mathfrak{V}_1 \times r\mathfrak{V}_2 \times (\rho^r)^{-1}(\mathfrak{W}) \times r^{-1}V_1 \times rV_2 \times V_3.$$

We want to show that $h|_{A^r} \in (\rho^r)^{-1}(\mathfrak{W})$, or equivalently $\|h^r\|_{C^{k+l}(A)} < \varepsilon_0$, for all $r > 2c^{-1}$. Let x, y be the real coordinate such that $\eta = x + iy$, and let $D = \partial^m/\partial x^j \partial y^{m-j}$ be a derivation of degree $m \le l + k$, then we obtain

$$Dh^r(\eta) = r^{1-m} Dh(r^{-1}\eta).$$

Hence

$$\sup_{\eta \in A} |Dh^r(\eta)| \le r^{1-m} \sup_{\eta \in A} |Dh(r^{-1}\eta)| \le r^{1-m} \sup_{\zeta \in A^r} |Dh(\zeta)| < \frac{\varepsilon_0}{2\sqrt{2}} r^{1-m} < \varepsilon_0,$$

if $m \ge 1$. For $m = 0$, notice that

$$\begin{aligned} |h(\eta)| &\le \int_0^1 \left| \frac{dh}{dt}(t\eta) \right| dt \le \int_0^1 \left| \frac{\partial h}{\partial x}(t\eta) \right| |x| dt + \int_0^1 \left| \frac{\partial h}{\partial y}(t\eta) \right| |y| dt \\ &< \frac{\varepsilon_0}{2\sqrt{2}} (|x| + |y|) < \frac{\varepsilon_0}{2} |\eta|, \end{aligned}$$

hence we obtain

$$\sup_{\eta \in A} |h^r(\eta)| = r \sup_{\eta \in A} |h(r^{-1}\eta)| = r \sup_{\zeta \in A^r} |Dh(\zeta)| < \frac{r\varepsilon_0}{2} \sup_{\zeta \in A^r} |\zeta| = \varepsilon_0.$$

In this way, we have obtained $\|h^r(\eta)\|_{C^{k+l}(A)} < \varepsilon_0$ for all $r > 2c^{-1}$. □

Remark 6.4 Lemma 6.3 also holds for $t \ll 0$. Exchange the role of factors of $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ and replace t with $-t$ to prove this case.

From Lemmas 6.1 and 6.3, we obtain the following statement.

Proposition 6.5 *If $N \subset Z$ is the image of any embedding $\mathbb{CP}^1 \hookrightarrow Z$ which is sufficiently close to the standard one in the C^{k+l}-topology with $k, l \ge 1$, then there is a C^l family of holomorphic disks on (Z, N), each of which is L^2_k close to some standard disk on $(Z, Z_{\mathbb{R}})$.*

We will strengthen this statement in Proposition 7.3.

7 The Double Fibration

In this section, we investigate some properties for the family of holomorphic disks constructed in Sect. 6. We continue to use the notation $\mathfrak{F}, \mathfrak{F}_i, \mathfrak{U}, \mathfrak{V}$ and so on.

For each $h \in C^{k+l}(A, \mathbb{C})$, we define C^l-maps $\Xi^h, F^h_i : U \to L^2_k(S^1, \mathbb{C})$ so that

$$\begin{aligned} \big(\Xi^h(\alpha,\beta), h\big) &= \mathfrak{F}^{-1}(0,0,h,\alpha,-\alpha,i\beta), \\ F^h_1(\alpha,\beta)(e^{i\theta}) &= \mathfrak{F}_1\big(\Xi^h(\alpha,\beta),h\big)(\theta) = \exp i\big\{\theta + \Xi^h(\alpha,\beta)(\theta)\big\}, \\ \text{and} \quad F^h_2(\alpha,\beta)(e^{i\theta}) &= \mathfrak{F}_2\big(\Xi^h(\alpha,\beta),h\big)(\theta), \end{aligned}$$

where $U \subset \mathbb{C} \times \mathbb{R}$ is a small open neighborhood of $(0,0)$ depending on h. By definition, the functions $F_i^h(\alpha,\beta)(z)$ extend to holomorphic functions on $\mathbb{D} = \{|z| \le 1\}$, and satisfy $F_1^h(\alpha,\beta)(0) = \alpha$ and $F_2^h(\alpha,\beta)(0) = -\alpha$. If we expand

$$\Xi^h(\alpha,\beta)(\theta) = \sum_k \Xi^h(\alpha,\beta)_k e^{ik\theta}, \tag{7.1}$$

then we obtain $\Xi^h(\alpha,\beta)_0 = i\beta$ by definition. Notice that we can also define the derivatives Ξ^h_* and F^h_{i*} which satisfy

$$\begin{aligned}
\big(\Xi^h_*(\dot\alpha,\dot\beta),0\big) &= \mathfrak{F}_*^{-1}(0,0,0,\dot\alpha,-\dot\alpha,i\dot\beta),\\
F^h_{1*}(\dot\alpha,\dot\beta)(e^{i\theta}) &= \mathfrak{F}_{1*}\big(\Xi^h(\dot\alpha,\dot\beta),0\big)(\theta) = iF_1^h(e^{i\theta})\,\Xi^h_*(\dot\alpha,\dot\beta)(\theta),\\
F^h_{2*}(\dot\alpha,\dot\beta)(e^{i\theta}) &= \mathfrak{F}_{2*}\big(\dot\Xi^h(\dot\alpha,\dot\beta),0\big)(\theta),\\
F^h_{1*}(\dot\alpha,\dot\beta)(0) &= \dot\alpha, \qquad F^h_{2*}(\dot\alpha,\dot\beta)(0) = -\dot\alpha \quad\text{and}\quad \Xi^h_*(\dot\alpha,\dot\beta)_0 = i\dot\beta.
\end{aligned}$$

Let $N \subset Z$ be the image of any embedding $\mathbb{CP}^1 \hookrightarrow Z$ which satisfies Proposition 6.5. Let us denote by $\mathfrak{B}^N_{(\alpha,\beta)}$ the holomorphic disk on (Z,N) which corresponds to the element $(0,0,h,\alpha,-\alpha,i\beta) \in \mathfrak{V}$ in the notation of the proof of Lemma 6.1. Then

$$\mathfrak{B}^N_{(\alpha,\beta)} = \left\{ \big(F_1^h(\alpha,\beta)(z), F_2^h(\alpha,\beta)(z)\big) \in Z \,\middle|\, z \in \mathbb{D} \right\}, \tag{7.2}$$

and $\{\mathfrak{B}^N_{(\alpha,\beta)}\}_{(\alpha,\beta)\in U}$ gives a three-parameter family of holomorphic disks, each of which is L^2_k-close to the standard disk $D_{(0,0)}$. Notice that $\mathfrak{B}^N_{(\alpha,\beta)}$ passes through $(\alpha,-\alpha)$ when $z=0$, hence, for fixed α, $\{\mathfrak{B}^N_{(\alpha,\beta)}\}_\beta$ defines a one-parameter family of holomorphic disks which pass through $(\alpha,-\alpha)$.

In the standard case, the following statement holds.

Proposition 7.1 $\mathfrak{B}^{Z_\mathbb{R}}_{(\alpha,\beta)}$ *coincides with the standard disk* $D_{(\alpha,\beta)}$.

Proof Since the disk

$$\mathfrak{B}^{Z_\mathbb{R}}_{(\alpha,\beta)} = \left\{ \big(F_1^0(\alpha,\beta)(z), F_2^0(\alpha,\beta)(z)\big) \in Z \,|\, z \in \mathbb{D} \right\}$$

coincides with one of the standard disks near $D_{(0,0)}$, there is a unique element $(\lambda,t) \in \mathbb{CP}^1 \times \mathbb{R}$ near $(0,0)$ such that

$$F_1^0(\alpha,\beta)(e^{i\theta}) = \exp i\big(\theta + \Xi^0(\alpha,\beta)(\theta)\big) = \frac{e^{i\theta} + e^t\lambda}{-\bar\lambda e^{i\theta} + e^t}. \tag{7.3}$$

Then we obtain $\alpha = F^0(\alpha,\beta)(0) = \lambda$. On the other hand, taking the derivative of (7.3), we obtain

$$i\,\Xi^0_*(\dot\alpha,\dot\beta)(\theta) = \frac{(\dot\lambda + \lambda\dot t)e^{-(i\theta - t)}}{1 + \lambda e^{-(i\theta - t)}} + \frac{e^{i\theta - t}\bar{\dot\lambda} - \dot t}{1 - \bar\lambda e^{i\theta - t}}.$$

If we expand the right hand side and compare the constant terms, then we find

$$i\dot\beta = \Xi^0_*(\dot\alpha,\dot\beta)_0 = i\dot t.$$

On the other hand, it is easy to see that $t = \beta$ when $\alpha = 0$. Hence $(\lambda,t) = (\alpha,\beta)$ for each $(\alpha,\beta) \in U$. □

Let M be the parameter space of the family of holomorphic disks on (Z,N) constructed in Proposition 6.5. Then M has the natural structure of a real 3-manifold and we can take a coordinate system on M in the following way. For each $(\lambda,t)\in\mathbb{CP}^1\times\mathbb{R}$, choose an element $T=T(\lambda,t)\in\mathrm{PSL}(2,\mathbb{C})$ such that $T_*(D_{(\lambda,t)})=D_{(0,0)}$, where $\{D_{(\lambda,t)}\}$ are the standard disks. Let $U^T\subset\mathbb{C}\times\mathbb{R}$ be an open neighborhood of $(0,0)$ such that $\mathfrak{B}^{T_*(N)}_{(\alpha,\beta)}$ is defined for all $(\alpha,\beta)\in U^T$. Then $\left\{T_*^{-1}\mathfrak{B}^{T_*(N)}_{(\alpha,\beta)}\right\}_{(\alpha,\beta)\in U^T}$ gives the family of holomorphic disks on (Z,N) each of which is close to $D_{(\lambda,t)}$, and $\{U^{T(\lambda,t)}\}$ gives a coordinate system on M.

Using the above coordinates, we prove the following lemma.

Lemma 7.2 *Suppose $N\subset Z$ is sufficiently close to $Z_{\mathbb{R}}$ so that Proposition 6.5 holds, and consider the constructed family of holomorphic disks on (Z,N). Then, for each $q=(\alpha,-\alpha)\in Z$, there is an $\mathbb{R}$-family of holomorphic disks each of which passes through q. Moreover there is a natural compactification of this family and the boundary points $\pm\infty$ correspond to marked $\mathbb{CP}^1$.*

Proof We can assume $\alpha=0$. Take any t so that $|t|$ is sufficiently small, and consider the standard disk $D_{(0,t)}$. If we define $T\in\mathrm{PSL}(2,\mathbb{C})$ by

$$T=\begin{bmatrix} e^{\frac{t}{2}} & \\ & e^{-\frac{t}{2}}\end{bmatrix},$$

then $T_*(\eta_1,\eta_2)=(e^t\eta_1,e^{-t}\eta_2)$ and $T_*(D_{(0,t)})=D_{(0,0)}$. Now $\left\{T_*^{-1}\mathfrak{B}^{T_*(N)}_{(0,\beta')}\right\}_{\beta'\in V}$ gives a one-parameter family of holomorphic disks on (Z,N) each of which is close to $D_{(0,t)}$ and pass through $(0,0)$. Here V is the set $\{\beta'\in\mathbb{R}\,|\,(0,\beta')\in U^T\}$.

Since $|t|$ is small, there is an open set $V'\subset V$ such that $T_*^{-1}\mathfrak{B}^{T_*(N)}_{(0,\beta')}$ is sufficiently close to $D_{(0,0)}$ for all $\beta'\in V'$. Hence, for each $\beta'\in V'$, there is a unique (α,β) such that

$$T_*^{-1}\mathfrak{B}^{T_*(N)}_{(0,\beta')}=\mathfrak{B}^{N}_{(\alpha,\beta)}. \tag{7.4}$$

Now N and $T_*(N)$ can be written locally as

$$N:\left\{\left(\eta,\overline{(\eta+h(\eta))}^{-1}\right)\,\middle|\,\eta\in A\right\}\quad\text{and}\quad T_*(N):\left\{\left(\eta,\overline{(\eta+h^T(\eta))}^{-1}\right)\,\middle|\,\eta\in A\right\}, \tag{7.5}$$

using a C^{k+l}-function h which is defined on a neighborhood of $A=\{z\in\mathbb{C}\,|\,\frac{1}{2}\le|z|\le 2\}$. Here we write h^T to mean ThT^{-1}. Then (7.4) is equivalent to

$$e^{-t}F_1^{h^T}(0,\beta')(z)=F_1^h(\alpha,\beta)(z)\qquad\text{on } z\in\mathbb{D}.$$

Evaluating for $z=0$, we obtain $\alpha=0$. Moreover, this is also equivalent to

$$it+\Xi^{h^T}(0,\beta')(\theta)=\Xi^h(\alpha,\beta)(\theta)\qquad\text{on } \theta\in S^1.$$

Comparing the constant terms for $e^{i\theta}$, we obtain $\beta = \beta' + t$. Hence (7.4) is equivalent to $(\alpha, \beta) = (0, \beta' + t)$. So the one-parameter family $\{\mathfrak{B}^N_{(0,\beta)}\}_{(0,\beta)\in U}$ extends to

$$\left\{\beta \in \mathbb{R} \;\middle|\; (0,\beta) \in U \,\text{or}\, (0, \beta - t) \in U^T \right\}$$

by putting $\mathfrak{B}^N_{(0,\beta)} = T_*^{-1}\mathfrak{B}^{T_*(N)}_{(0,\beta-t)}$. In this way, we can define the one-parameter family $\{\mathfrak{B}^N_{(0,\beta)}\}$ for all $\beta \in \mathbb{R}$.

The statement of the compactification is obtained from Lemma 6.3 and its proof. Indeed, in the notation of (6.5), if we take the limit $t \to \infty$, the holomorphic disk parametrized by t degenerates to $\{0\} \times \mathbb{CP}^1$ marked at $(0, \overline{h(0)}^{-1})$. As we explained in Remark 6.4, we also obtain another marked $\mathbb{CP}^1$ by taking the limit $t \to -\infty$. □

Now the following statement is easily proved.

Proposition 7.3 *If $N \subset Z$ is the image of any embedding $\mathbb{CP}^1 \hookrightarrow Z$ which is sufficiently close to the standard one in the C^{k+l}-topology with $k, l \geq 1$, then there is a C^l family of holomorphic disks on (Z, N) parametrized by $S^2 \times \mathbb{R}$ which satisfies the following properties:*

- *each disk is L^2_k-close to some standard disk,*
- *there is a natural compactification of the family such that the compactified family is parameterized by $S^2 \times I$, and each boundary point on $S^2 \times I$ corresponds to a marked $\mathbb{CP}^1$ embedded in (Z, N),*

where $I = [-\infty, \infty]$ is the compactification of $\mathbb{R}$.

Proof Let $Q = \{(\lambda, -\lambda) \in Z \,|\, \lambda \in \mathbb{CP}^1\}$. For each $q \in Q$, there is an $\mathbb{R}$-family of holomorphic disks constructed in Lemma 7.2. Since this family varies continuously, we obtain the family of holomorphic disks parametrized by $Q \times \mathbb{R} \simeq S^2 \times \mathbb{R}$. The statement for the compactification is obvious from Lemma 7.2. □

For each $(\lambda, t) \in \mathbb{CP}^1 \times \mathbb{R}$, we define

$$\mathfrak{D}_{(\lambda,t)} = T_*^{-1}\mathfrak{B}^{T_*(N)}_{(0,0)},$$

where $T = T(\lambda, t) \in \mathrm{PSL}(2, \mathbb{C})$ is an element which satisfies $T_*(D_{(\lambda,t)}) = D_{(0,0)}$. Then we obtain the continuous map $j : \mathbb{CP}^1 \times \mathbb{R} \to M : (\lambda, t) \mapsto \mathfrak{D}_{(\lambda,t)}$. Moreover, we can prove that j is an isomorphism in the following way. For each constructed holomorphic disk $\mathfrak{D}$ on (Z, N), we can choose (λ, t) and $T = T(\lambda, t)$ so that $\mathfrak{D} = T_*^{-1}\mathfrak{B}^{T_*(N)}_{(0,\beta)}$. Here λ is uniquely defined so that the center of $\mathfrak{D}$ is $(\lambda, -\lambda)$. Then $\mathfrak{D} = \mathfrak{D}_{(0,\beta+t)}$ from Lemma 7.2 and its proof, so j is surjective. The injectivity and the continuity of j^{-1} is also deduced from the above procedure of choosing (λ, t), hence j is isomorphism.

Let us construct the double fibration. Let $U \subset \mathbb{CP}^1 \times \mathbb{R}$ be a sufficiently small neighborhood of $(0, 0)$. For each $(\lambda, t) \in U$, we define $T = T(\lambda, t) \in \mathrm{PSL}(2, \mathbb{C})$ by

$$T = \frac{1}{e^{-\frac{t}{2}}\sqrt{1 + |\lambda|^2}} \begin{bmatrix} 1 & -e^t\lambda \\ \bar{\lambda} & e^t \end{bmatrix},$$

then we obtain $T_*(D_{(\lambda,t)}) = D_{(0,0)}$. Introducing C^{k+l}-functions h and h^T similar to those in (7.5), we define a map $\mathfrak{f} : U \times \mathbb{D} \to Z$ by

$$\mathfrak{f}(\lambda,t;z) = T_*^{-1}\left(F_1^{h^T}(z), F_2^{h^T}(z)\right).$$

Then $\mathfrak{f}$ is C^l for (λ,t) and C^{k-1} for z, and we obtain $\mathfrak{D}_{(\lambda,t)} = \{\mathfrak{f}(\lambda,t;z) \in Z \,|\, z \in \mathbb{D}\}$.

Constructing a similar map for each neighborhood of $\mathbb{CP}^1 \times \mathbb{R}$, and patching them, we obtain the double fibration

$$\begin{array}{ccc} & (\mathscr{X}_+, \mathscr{X}_{\mathbb{R}}) & \\ {}^{\varpi}\swarrow & & \searrow^{\mathfrak{f}} \\ M \simeq \mathbb{CP}^1 \times \mathbb{R} & & (Z,N) \end{array} \tag{7.6}$$

where ϖ is a disk bundle. By construction, $\mathscr{X}_+$ is the same disk bundle as the standard case. In particular, we obtain $c_1(\mathscr{X}_{\mathbb{R}}) = 2$ along each fiber of ϖ and that ϖ is C^∞.

Lemma 7.4 *Let $N \subset Z$ be the image of any embedding $\mathbb{CP}^1 \hookrightarrow Z$ which is sufficiently close to the standard one in the C^{k+l}-topology with $k,l \geq 1$, and consider the double fibration* (7.6)*. Then $f_*(\mathbf{v}) \neq 0$ for each non-zero vector $\mathbf{v} \in T\mathscr{X}_{\mathbb{R}}$ such that $\varpi_*(\mathbf{v}) = 0$.*

Proof For each $(u,h) \in L^2_k(S^1,\mathbb{C}) \times C^{k+l}(A,\mathbb{C})$, we have

$$\frac{d}{d\theta}[\mathfrak{F}_1(u,h)](\theta) = \frac{d}{d\theta} e^{i(\theta+u(\theta))} = e^{i(\theta+u(\theta))}(i + iu'(\theta)),$$

so this does not vanish if $\|u\|_{L^2_1}$ is sufficiently small. Hence, by shrinking $\mathfrak{U}$ and $\mathfrak{V}$ smaller if needed, the statement holds for $\mathbf{v} \in \ker \varpi_*$ over $U \subset M$, where U is the neighborhood introduced above.

Now, recall the diagram (6.7) in the proof of Lemma 6.3. Notice that the $L^2_k(S^1,\mathbb{C})$ component does not change by the vertical arrow, so we can estimate $u \in L^2_k(S^1,\mathbb{C})$ uniformly so that $\frac{d}{d\theta}[\mathfrak{F}_1^r(u,h)](\theta)$ does not vanish for all r. Hence the statement holds for all $\mathbf{v} \in \ker \varpi_*$. $\square$

By Lemma 7.4, we can define the lift $\tilde{\mathfrak{f}}$ of $\mathfrak{f}$ by $\tilde{\mathfrak{f}} : \mathscr{X}_{\mathbb{R}} \to S(TN) : u \mapsto [\mathfrak{f}_*(\mathbf{v}_u)]$. Here $\mathbf{v}$ is a nowhere vanishing vertical vector field, that is, $\varpi_*(\mathbf{v}) = 0$, for which the orientation matches the complex orientation of the fiber of $\varpi : \mathscr{X}_+ \to M$. The next proposition is the perturbed version of Proposition 5.1.

Proposition 7.5 *Let $N \subset Z$ be the image of any embedding $\mathbb{CP}^1 \hookrightarrow Z$ which is sufficiently close to the standard one in the C^{k+l}-topology with $l \geq 1, k \geq 2$. Consider the double fibration* (7.6)*, let $S_t = \mathbb{CP}^1 \times \{t\} \subset M$, and let $\mathfrak{f}_t$ and $\tilde{\mathfrak{f}}_t$ be the restriction of $\mathfrak{f}$ and $\tilde{\mathfrak{f}}$ on $\varpi^{-1}(S_t)$ respectively. Then, for each $t \in \mathbb{R}$,*

1. *$\mathfrak{f}_t : (\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}})|_{S_t} \to Z \backslash N$ is diffeomorphic,*
2. *$\tilde{\mathfrak{f}}_t : \mathscr{X}_{\mathbb{R}}|_{S_t} \to S(TN)$ is diffeomorphic,*
3. *$\mathfrak{f}_t : \mathscr{X}_{\mathbb{R}}|_{S_t} \to N$ is an S^1-fibration such that each fiber is transverse to the vertical distribution of $\varpi : \mathscr{X}_{\mathbb{R}} \to M$.*

In particular, $\{\mathfrak{D}_{(\lambda,t)}\}_{\lambda\in\mathbb{CP}^1}$ *gives a foliation on* $Z\backslash N$ *for each* $t\in\mathbb{R}$.

Remark 7.6 From *2* above, it follows that: for each $t\in\mathbb{R}$, $p\in N$ and non zero $v\in T_pN$, there is a unique $x\in S_t$ such that $p\in\partial\mathfrak{D}_x$ and $v\parallel\mathfrak{D}_x$.

Proof of Proposition 7.5 Since S_t is compact and $\mathfrak{f}$ is C^1-close to the standard case, we can assume the derivation of $\mathfrak{f}_t$ to be non-zero everywhere by shrinking $\mathfrak{W}$ smaller if required. Here $\mathfrak{W}$ is the open set defined in Remark 6.2. Notice that we can define $\mathfrak{W}$ so that this property holds for all $t\in\mathbb{R}$ at once by Lemma 6.3 and its proof. Thus $\mathfrak{f}_t$ gives a proper local diffeomorphism on $(\mathscr{X}_+\backslash\mathscr{X}_\mathbb{R})|_{S_t}$, and this is actually a diffeomorphism since $\mathfrak{f}_t$ is close to the standard case.

By a similar argument for the lift $\tilde{\mathfrak{f}}:\mathscr{X}_\mathbb{R}|_{S_t}\to S(TN)$, we obtain property *2*. If there are $x\in S_t$ and $p\in N$ such that $\varpi^{-1}(x)$ and $\mathfrak{f}_t^{-1}(p)$ are not transversal at $u\in\mathscr{X}_\mathbb{R}$, then $(\mathfrak{f}_t)_*(\mathbf{v}_u)=0$. This contradicts Lemma 7.4, hence *3* holds. □

From Proposition 7.3, we obtain the natural compactification of ϖ and $\mathfrak{f}$ which gives the following double fibration:

$$\begin{array}{ccc} & (\hat{\mathscr{X}}_+,\hat{\mathscr{X}}_\mathbb{R}) & \\ {}^{\hat{\varpi}}\swarrow & & \searrow^{\hat{\mathfrak{f}}} \\ \hat{M} & & (Z,N) \end{array} \tag{7.7}$$

which is studied in Sect. 8.

In the last part of this section, we prove the following technical lemma which enables us to prove the non-degeneracy of the induced conformal structure. Let us denote $\mathfrak{C}_p=\varpi\circ\mathfrak{f}^{-1}(p)=\{x\in M\,|\,p\in\mathfrak{D}_x\}$ for each $p\in Z\backslash N$, then $\mathfrak{C}_p$ is an embedded $\mathbb{R}$ in M from Proposition 7.5. Notice that $\mathfrak{C}_p$ is a closed subset in M since it connects two boundaries of $\hat{M}$.

Lemma 7.7 *Let* $x\in M$*, then there are two points* $p_1,p_2\in\mathfrak{D}_x\backslash\partial\mathfrak{D}_x$ *such that* $\mathfrak{C}_{p_1}$ *and* $\mathfrak{C}_{p_2}$ *intersect transversally at x.*

Proof We can assume $x=(0,0)$, and we use the local coordinate $(\alpha,\beta)\in U$ around x. Each tangent vector on $T_{(0,0)}M$ is given by $(\dot\alpha,\dot\beta)\in\mathbb{C}\times\mathbb{R}\cong T_{(0,0)}(\mathbb{C}\times\mathbb{R})$. Notice that the tangent vector $(\dot\alpha,\dot\beta)\in T_{(0,0)}M$ induces the vector field

$$\big(F_{1*}(\dot\alpha,\dot\beta)(z),F_{2*}(\dot\alpha,\dot\beta)(z)\big)$$

along $\mathfrak{D}_{(0,0)}$. Here we identified $\mathbb{C}\times\mathbb{C}$ with the tangent vectors on each point of $\mathbb{C}\times\mathbb{C}\subset Z$. $F_{1*}(\dot\alpha,\dot\beta)$ and $F_{2*}(\dot\alpha,\dot\beta)$ are holomorphic functions on $\mathbb{D}$ and their zeros coincide since

$$\begin{aligned} F_{1*}(\dot\alpha,\dot\beta)(e^{i\theta}) &= ie^{i\theta}\,\Xi_*(\dot\alpha,\dot\beta)(\theta) \\ \text{and}\quad F_{2*}(\dot\alpha,\dot\beta)(e^{i\theta}) &= ie^{i\theta}\,\overline{\Xi_*(\dot\alpha,\dot\beta)(\theta)} \end{aligned}$$

by (6.2). If $\dot\beta\neq 0$, then $F_{1*}(0,\dot\beta)(z)$ is not a zero function since $\mathfrak{F}_*$ is bijective, and $F_{1*}(0,\dot\beta)(0)=0$ by definition. This means that $(0,\dot\beta)\in T_{(0,0)}M$ is tangent to

$\mathfrak{C}_{(0,0)}$ since the one-parameter family of holomorphic disks fixing $(0,0) \in \mathfrak{D} \subset Z$ is unique and this family corresponds to the vector field $\big(F_{1*}(0,\dot\beta)(z), F_{2*}(0,\dot\beta)(z)\big)$ along $\mathfrak{D}$.

Now consider the vector field

$$\big(F_{1*}(t\dot\alpha,\dot\beta)(z), F_{2*}(t\dot\alpha,\dot\beta)(z)\big)$$

for $t \in [0,1]$ and non-zero $\dot\alpha \in \mathbb{C}$ with sufficiently small $|\dot\alpha|$. Then $F_{1*}(t\dot\alpha,\dot\beta)$ is a non-zero holomorphic function on $\mathbb{D}$ for all t, and its zeros vary continuously depending on t. Hence there exists a $z_1 \in \mathbb{D}$ near 0 such that $F_{1*}(\dot\alpha,\dot\beta)(z_1) = 0$ but z_1 cannot be 0 because $F_{1*}(\dot\alpha,\dot\beta)(0) = \dot\alpha \neq 0$. If we put $p_2 = \big(F_1(0,0)(z_1), F_2(0,0)(z_1)\big) \in \mathfrak{D}_{(0,0)}$, then we find that $(\dot\alpha,\dot\beta) \in T_{(0,0)}M$ is tangent to $\mathfrak{C}_{p_2}$. Hence $p_1 = (0,0)$ and p_2 satisfies the statement. □

8 Construction of Einstein-Weyl spaces

In this section, we construct an Einstein-Weyl structure on the parameter space of the family of holomorphic disks on (Z,N) constructed in the previous sections. The following proposition is critical.

Proposition 8.1 *Let M be a smooth connected 3-manifold and let $\varpi : \mathscr{X} \to M$ be a smooth $\mathbb{CP}^1$-bundle. Let $\rho : \mathscr{X} \to \mathscr{X}$ be an involution which commutes with ϖ, and is fiber-wise anti-holomorphic. Suppose ρ has a fixed-point set $\mathscr{X}_\rho$ which is an S^1-bundle over M, and which disconnects $\mathscr{X}$ into two closed 2-disk bundles $\mathscr{X}_\pm$ with common boundary $\mathscr{X}_\rho$. Let $\text{Д} \subset T_{\mathbb{C}}\mathscr{X}$ be a distribution of complex 3-planes which satisfies the following properties:*

- $\rho_*\text{Д} = \overline{\text{Д}}$,
- *the restriction of Д to $\mathscr{X}_+$ is C^k, $k \geq 1$ and involutive,*
- $\text{Д} + \overline{\text{Д}} = T_{\mathbb{C}}\mathscr{X}$ *on* $\mathscr{X} \setminus \mathscr{X}_\rho$,
- $\text{Д} \cap \ker \varpi_*$ *is the $(0,1)$ tangent space of the $\mathbb{CP}^1$ fibers of ϖ,*
- *the restriction of Д to a fiber of $\mathscr{X}$ has $c_1 = -4$ with respect to the complex orientation, and*
- *the map $\mathscr{X} \to \mathbb{P}(TM) : z \mapsto \varpi_*(\text{Д} \cap \overline{\text{Д}})_z$ is not constant along each fiber of ϖ.*

Then M admits a unique C^{k-1} indefinite Einstein-Weyl structure $([g],\nabla)$ such that the null-surfaces are the projections via ϖ of the integral manifolds of real 2-plane distribution $\text{Д} \cap T\mathscr{X}_\rho$ on $\mathscr{X}_\rho$.

Proof Let $V^{0,1}$ be the $(0,1)$ tangent space of the fibers, then $\mho = \text{Д}/V^{0,1}$ is a rank two vector bundle on $\mathscr{X}$. We can define a continuous map $\psi : \mathscr{X} \to Gr_2(T_{\mathbb{C}}X)$ by $z \mapsto \varpi_*(\text{Д}|_z)$ which makes the following diagrams commute:

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\psi} & Gr_2(T_{\mathbb{C}}X) \\ & \searrow \quad \swarrow & \\ & X & \end{array} \qquad \begin{array}{ccc} \mathscr{X} & \xrightarrow{\psi} & Gr_2(T_{\mathbb{C}}X) \\ \rho \downarrow & & \downarrow c \\ \mathscr{X} & \xrightarrow{\psi} & Gr_2(T_{\mathbb{C}}X) \end{array} \qquad (8.1)$$

Using the involutiveness of $Д$, we can prove that ψ is fiber-wise holomorphic by a similar argument to that in (9; 10).

Let $\mathfrak{P} : Gr_2(T_{\mathbb{C}}X) \longrightarrow \mathbb{P}(\wedge^2 T_{\mathbb{C}}X) \cong \mathbb{P}(T^*_{\mathbb{C}}X)$ be the natural isomorphism. Then we obtain the fiber-wise holomorphic map $\hat{\psi} = \mathfrak{P} \circ \psi : \mathscr{X} \to \mathbb{P}(T^*_{\mathbb{C}}X)$. By definition, we obtain $\hat{\psi}^*\mathscr{O}(-1) = \wedge^2 \mho$. On the other hand, since $c_1(V^{0,1}) = -2$ and $c_1(Д) = -4$ on any fiber of ϖ, we have $c_1(\wedge^2\mho) = c_1(\mho) = -2$. Hence $\hat{\psi}$ is fiber-wise degree 2. For each fiber, there are only two possibilities for $\hat{\psi}$; either a non-degenerate conic or a ramified double cover of a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$. The latter is, however, removable. Indeed, any line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ corresponds to the planes in $\mathbb{C}^3$ containing a fixed line. Notice that, for each $z \in \mathscr{X} \setminus \mathscr{X}_{\mathbb{R}}$,

$$\varpi_*(Д \cap \overline{Д})_z = \varpi_*(Д|_z) \cap \varpi_*(\overline{Д}|_z) = \varpi_*(Д|_z) \cap \varpi_*(Д|_{\rho(z)})$$

is independent on z if the image of $\varpi^{-1}(x)$ under $\hat{\psi}$ is a line. This contradicts the hypothesis.

Now we define a conformal structure $[g]$. Let $U \subset M$ be an open set and let $U \times \mathbb{CP}^1 \xrightarrow{\sim} \mathscr{X}|_U$ be a trivialization on U. Let ζ be an inhomogeneous coordinate on $\mathbb{CP}^1$ such that $\rho(x,\zeta) = (x,\bar{\zeta})$. Then we can choose a C^k frame field $\{e_1, e_2, e_3\}$ on $TM|_U$ so that

$$\hat{\psi}(x,\zeta) = \left[(1+\zeta^2)e^1 + (1-\zeta^2)e^2 + 2\zeta e^3\right], \tag{8.2}$$

where $\{e^i\}$ is the dual frame. Define an indefinite metric g on U so that $g(e_i, e_j)$ is given by (3.11). Here, the frame $\{e_i\}$ is uniquely defined by (8.2) up to scalar multiplication, and the coordinate change of ζ causes an $SO(1,2)$ action on the frame $\{e_i\}$. Hence the conformal structure $[g]$ is well-defined by $\hat{\psi}$. So we can obtain an indefinite conformal structure $[g]$ on M.

Next we prove that a unique torsion-free connection ∇ on TM is induced, and $([g], \nabla)$ gives an Einstein-Weyl structure on M. We also prove that $Д$ agrees with the distribution defined in Sect. 3.

We fix an indefinite metric $g \in [g]$, and take a local frame field $\{e_1, e_2, e_3\}$ of TM on an open set $U \subset M$ as above. It is enough to construct ∇ on U. Notice that (8.2) gives a natural identification $\mathscr{X} \xrightarrow{\sim} \mathscr{Z} = \mathbb{P}(N(T^*_{\mathbb{C}}M))$ on U. If we define the maps $\mathfrak{m}_i : U \times \mathbb{C} \to TM$ for $i = 1,2$ by

$$\mathfrak{m}_1 = -e_1 + e_2 + \zeta e_3 \quad \text{and} \quad \mathfrak{m}_2 = \zeta(e_1 + e_2) - e_3, \tag{8.3}$$

then we obtain $\varpi_*(Д|_{(x,\zeta)}) = \mathrm{Span}\,\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle$ (see (3.16)).

Let $\tilde{\mathfrak{m}}_i$ be the vector fields on $U \times \mathbb{C} \subset U \times \mathbb{CP}^1 \simeq \mathscr{X}|_U$ such that $\tilde{\mathfrak{m}}_i \in Д$ and the $\tilde{\mathfrak{m}}_i$ are written in the following form:

$$\tilde{\mathfrak{m}}_1 = \mathfrak{m}_1 + \alpha \frac{\partial}{\partial \zeta} \quad \text{and} \quad \tilde{\mathfrak{m}}_2 = \mathfrak{m}_2 + \beta \frac{\partial}{\partial \zeta}, \tag{8.4}$$

where α and β are functions on $\mathscr{X}$. Then α and β are uniquely defined and C^k. Moreover, α and β are holomorphic for ζ, since

$$\left[\frac{\partial}{\partial \bar{\zeta}}, \tilde{\mathfrak{m}}_1\right] = \frac{\partial \alpha}{\partial \bar{\zeta}} \frac{\partial}{\partial \zeta} \equiv 0 \qquad \mathrm{mod}\ Д,$$

and so on.

By a similar argument for $\zeta^{-1}\mathfrak{m}_i$ on $\{(x,\zeta)\in U\times\mathbb{CP}^1 \mid \zeta\neq 0\}$, we find that $\zeta^{-1}\alpha\frac{\partial}{\partial\zeta}$ and $\zeta^{-1}\beta\frac{\partial}{\partial\zeta}$ extends to holomorphic vector fields on $\{\zeta\neq 0\}$, hence we can write

$$\begin{aligned} \tilde{\mathfrak{m}}_1 &= \mathfrak{m}_1 + (\alpha_0+\alpha_1\zeta+\alpha_2\zeta^2+\alpha_3\zeta^3)\frac{\partial}{\partial\zeta}, \\ \text{and}\quad \tilde{\mathfrak{m}}_2 &= \mathfrak{m}_2 + (\beta_0+\beta_1\zeta+\beta_2\zeta^2+\beta_3\zeta^3)\frac{\partial}{\partial\zeta}, \end{aligned} \tag{8.5}$$

where α_i and β_i are C^k functions on U.

Recall that the compatibility condition $\nabla g = a\otimes g$ holds if and only if the connection form ω of ∇ is written

$$\omega = (\omega^i_j) = \begin{pmatrix} \phi & \eta^1_2 & \eta^1_3 \\ \eta^1_2 & \phi & \eta^2_3 \\ \eta^1_3 & -\eta^2_3 & \phi \end{pmatrix} \tag{8.6}$$

with respect to the frame $\{e_i\}$ (see (3.12)). For each vector $v\in TU$, the horizontal lift $\tilde{v}$ with respect to the connection defined from (8.6) is given by (3.15). If $\tilde{\mathfrak{m}}_{i(x,\zeta)}$ is the horizontal lift of $\mathfrak{m}_i(\zeta)_x$, then η^i_j must be

$$\eta^2_3 = \eta^2_{3,0}+fe^1, \quad \eta^1_3 = \eta^1_{3,0}+fe^2, \quad \eta^1_2 = \eta^1_{2,0}-fe^3, \tag{8.7}$$

where f is an unknown function on U and

$$\begin{aligned} \eta^2_{3,0} &= \frac{\alpha_0+\alpha_2+\beta_1+\beta_3}{2}e^1 + \frac{-\alpha_0-\alpha_2+\beta_1+\beta_3}{2}e^2 + (-\alpha_3-\beta_0)e^3, \\ \eta^1_{3,0} &= \frac{\alpha_0-\alpha_2+\beta_1-\beta_3}{2}e^1 + \frac{-\alpha_0+\alpha_2+\beta_1-\beta_3}{2}e^2 + (\alpha_3-\beta_0)e^3, \\ \text{and}\quad \eta^1_{2,0} &= \frac{-\alpha_1+\alpha_3+\beta_0-\beta_2}{2}e^1 + \frac{\alpha_1+\alpha_3-\beta_0-\beta_2}{2}e^2. \end{aligned} \tag{8.8}$$

We claim that there is a unique pair (f,ϕ) such that the connection (8.6) is torsion-free, that is, ω satisfies

$$de^i + \sum \omega^i_j e^j = 0. \tag{8.9}$$

First, we fix a connection for which the connection form is

$$\omega_0 = (\omega^i_{j,0}) = \begin{pmatrix} 0 & \eta^1_{2,0} & \eta^1_{3,0} \\ \eta^1_{2,0} & 0 & \eta^2_{3,0} \\ \eta^1_{3,0} & -\eta^2_{3,0} & 0 \end{pmatrix}.$$

Let λ_i be the fiber coordinate on $T^*_{\mathbb{C}}X$ with respect to $\{e^i\}$. We consider the distribution $\pi^*\Pi$ on $\mathscr{N} = N(T^*_{\mathbb{C}}M)\backslash 0_M$, where $\pi:\mathscr{N}\to\mathscr{Z}\simeq\mathscr{X}$ is the projection. We define 1-forms $\theta,\theta_{i,0},\tau_{ij,0}$ on $\mathscr{N}$ (see (3.4)) by

$$\theta = \sum \lambda_i e^i, \quad \theta_{i,0} = d\lambda_i - \sum \lambda_j \omega^j_{i,0}, \quad \tau_{ij,0} = \lambda_i\theta_{j,0} - \lambda_j\theta_{i,0}.$$

If we simply write $\tau = \tau_{23,0}$, then we have (see (3.18))

$$\tau = \lambda_2 d\lambda_3 - \lambda_3 d\lambda_2 - \lambda_1\big(\lambda_1\eta^2_{3,0} + \lambda_2\eta^1_{3,0} - \lambda_3\eta^1_{2,0}\big).$$

Similar to the proofs of Propositions 3.3 or 3.5, we obtain $\pi^*\varPi = \{v \in T\mathscr{N} \mid \theta(v) = \tau_{ij,0}(\theta) = 0\}$. Hence the 1-forms $\{\theta, \tau_{ij}\}$ are involutive.

Since $\sum \theta_{i,0}\wedge e^i \equiv 0 \mod \langle\theta,\tau_{ij}\rangle$, we obtain $d\theta \equiv \mu \mod \langle\theta,\tau_{ij}\rangle$, where

$$\begin{aligned}\mu =& (\lambda_2\eta^1_{2,0} + \lambda_3\eta^1_{3,0})\wedge e^1 + (\lambda_1\eta^1_{2,0} - \lambda_3\eta^2_{3,0})\wedge e^2\\ &+(\lambda_1\eta^1_{3,0} + \lambda_2\eta^2_{3,0})\wedge e^3 + \sum\lambda_i de^i.\end{aligned} \tag{8.10}$$

Then we can write

$$\mu = \mu_{23}e^2\wedge e^3 + \mu_{31}e^3\wedge e^1 + \mu_{12}e^1\wedge e^2, \tag{8.11}$$

where the $\mu_{ij} = {\mu_{ij}}^l\lambda_l$ are linear in λ. Notice that the ${\mu_{ij}}^l$ are C^{k-1} functions because θ is C^k. Since $d\theta \equiv 0 \mod \langle\theta,\tau_{ij}\rangle$, there are 1-forms Θ_1 and Θ_2 such that

$$\mu = \Theta_1\wedge\tau + \Theta_2\wedge\theta. \tag{8.12}$$

The 1-form Θ_1 is, however, zero since μ does not contain $d\lambda_i$. Hence we obtain $\mu\wedge\theta = 0$, and this is equivalent to

$$\begin{gathered}-{\mu_{23}}^1 = {\mu_{31}}^2 = {\mu_{12}}^3,\\ {\mu_{12}}^2 + {\mu_{31}}^3 = 0,\quad {\mu_{23}}^3 + {\mu_{12}}^1 = 0 \quad\text{and}\quad {\mu_{31}}^1 + {\mu_{23}}^2 = 0.\end{gathered} \tag{8.13}$$

Thus, if we put $f = \frac{1}{2}{\mu_{12}}^3$ and $\phi = {\mu_{31}}^3 e^1 + {\mu_{12}}^1 e^2 + {\mu_{23}}^2 e^3$, then

$$\mu = -\phi\wedge\theta + f\big(-\lambda_1 e^2\wedge e^3 + \lambda_2 e^3\wedge e^1 + \lambda_3 e^1\wedge e^2\big).$$

Here f and ϕ are C^{k-1}. Comparing the coefficients of λ_i with (8.10), we obtain

$$\begin{aligned}&de^1 + \phi\wedge e^1 + (\eta^1_{2,0} - fe^3)\wedge e^2 + (\eta^1_{3,0} + fe^1)\wedge e^3 = 0,\\ &de^2 + (\eta^1_{2,0} - fe^3)\wedge e^1 + \phi\wedge e^2 + (\eta^2_{3,0} + fe^1)\wedge e^3 = 0,\\ \text{and}\quad &de^3 + (\eta^1_{3,0} + fe^1)\wedge e^1 - (\eta^2_{3,0} + fe^1)\wedge e^2 + \phi\wedge e^3 = 0.\end{aligned}$$

This is nothing but the torsion-free condition for the connection defined from f and ϕ above.

Since (f,ϕ) is uniquely defined, we have obtained the unique torsion-free C^{k-1} connection ∇. For this ∇, the distribution on $\mathscr{Z} \simeq \mathscr{X}$ agrees with $\varPi$ by construction. Hence $([g],\nabla)$ is Einstein-Weyl from Proposition 3.9. The remaining condition is deduced from the fact that $\varPi\cap T\mathscr{X}_\rho$ corresponds to $\mathbb{R}$. □

Remark 8.2 In the statement of Proposition 8.1, the last hypothesis

- $\varpi_*(\text{Д} \cap \overline{\text{Д}})_z$ is not constant along the fiber

is not removable. Actually, $\varpi_*(\text{Д} \cap \overline{\text{Д}})_z$ can be constant when the metric degenerates so that the light cone degenerates to a line, which occurs as a limit of an indefinite metric.

Proposition 8.3 *Let N be any embedding of $\mathbb{CP}^1$ into $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ which is C^{2k+5} close to the standard one. Let $\{\mathfrak{D}_x\}_{x \in S^2 \times \mathbb{R}}$ be the constructed family of closed holomorphic disks on (Z, N). Then a C^k indefinite Einstein-Weyl structure $([g], \nabla)$ is naturally induced on $M = S^2 \times \mathbb{R}$.*

Proof We apply Proposition 7.3 by putting $k+3$ instead of k and $l = k+2$. Let $M \overset{\varpi}{\leftarrow} \mathscr{X}_+ \overset{\mathfrak{f}}{\rightarrow} Z$ be the constructed double fibration (the diagram (7.6)), then $\mathfrak{f}$ is C^{k+2} in this case. Let $\mathscr{X}_-$ be a copy of $\mathscr{X}_+$ and let $\mathscr{X} = \mathscr{X}_+ \cup \overline{\mathscr{X}}_-$ be the $\mathbb{CP}^1$ bundle over X which is obtained by identifying the boundaries $\partial \mathscr{X}_+$ and $\partial \mathscr{X}_-$ where $\overline{\mathscr{X}}_-$ is a copy of $\mathscr{X}_-$ with fiber-wise opposite complex structure. Let $\rho : \mathscr{X} \to \mathscr{X}$ be the involution which interchanges $\mathscr{X}_+$ and $\mathscr{X}_-$.

Let $\mathfrak{f}_* : T_{\mathbb{C}} X \to T_{\mathbb{C}} Z$ be the differential of $\mathfrak{f}$. We define $\text{Д} = \mathfrak{f}_*^{-1}(T^{0,1} Z)$ on $\mathscr{X}_+$. Then, along $\mathscr{X}_{\mathbb{R}} = \partial \mathscr{X}_+$, Д is spanned by $\frac{\partial}{\partial \bar{\zeta}}$ and the distribution of real planes tangent to the fibers of $\mathfrak{f} : \mathscr{X}_{\mathbb{R}} \to N$. So we can extend Д to the whole of $\mathscr{X}$ so that $\text{Д} = \rho^* \overline{\text{Д}}$ on $\mathscr{X}_{\mathbb{R}}$. Let us check the hypotheses in Proposition 8.1:

- $\rho_* \text{Д} = \overline{\text{Д}}$ follows from the construction.
- Д is C^{k+1} on $\mathscr{X}_+ \setminus \mathscr{X}_{\mathbb{R}}$ since $\mathfrak{f}_*$ is C^{k+1}, and Д is involutive since $T^{0,1} Z$ is involutive.
- $\text{Д} + \overline{\text{Д}} = \mathfrak{f}_*^{-1}(T^{0,1} Z) + \mathfrak{f}_*^{-1}(T^{1,0} Z) = \mathfrak{f}_*^{-1}(T_{\mathbb{C}} Z) = T_{\mathbb{C}} \mathscr{X}_+$ on $\mathscr{X}_+ \setminus \mathscr{X}_{\mathbb{R}}$ since $\mathfrak{f}$ is surjective.
- For each fiber $\varpi^{-1}(x) = \mathscr{X}_+|_x$, the restriction $\mathfrak{f}_x : \mathscr{X}_+|_x \to Z$ of $\mathfrak{f}$ is a holomorphic embedding. Hence $\text{Д} \cap \ker \varpi_* = (\mathfrak{f}_x)_*^{-1}(T^{0,1} Z) = V^{0,1}$.
- Д is C^0-close to the Д of the standard case, so $c_1(\text{Д}) = -4$ on each fiber of ϖ.
- For each $x \in M$, there are $p, q \in \mathfrak{D}_x$ such that $\mathfrak{C}_p$ and $\mathfrak{C}_q$ intersects transversally at x (Lemma 7.7). If we put $z = \mathfrak{f}_x^{-1}(p) = \mathfrak{f}^{-1}(p) \cap \varpi^{-1}(x)$, then we obtain

  $$(T_x \mathfrak{C}_p) \otimes \mathbb{C} = \varpi_*(T_{\mathbb{C} z} \mathfrak{f}^{-1}(p)) = \varpi_*(\ker \mathfrak{f}_*)_z = \varpi_*(\text{Д} \cap \overline{\text{Д}})_z.$$

  Similarly $(T_x \mathfrak{C}_q) \otimes \mathbb{C} = \varpi_*(\text{Д} \cap \overline{\text{Д}})_{z'}$ for $z' = \mathfrak{f}_x^{-1}(q)$. Hence $\varpi_*(\text{Д} \cap \overline{\text{Д}})$ is not constant.

Thus all the hypotheses in Proposition 8.1 are fulfilled, so we obtain the unique C^k indefinite Einstein-Weyl structure on M. □

Recall that we obtained a lift $\tilde{\mathfrak{f}} : \mathscr{X}_{\mathbb{R}} \to S(TN)$ of $\mathfrak{f} : \mathscr{X}_{\mathbb{R}} \to N$ in Sect. 7.

Proposition 8.4 *Identifying $\mathscr{X}$ with $\mathscr{Z}$,*

1. $\ = \mathfrak{f}_*^{-1}(T^{0,1} Z)$ *on* $\mathscr{X}_+$ *where* $\mathfrak{f}_* : T_{\mathbb{C}} \mathscr{X}_+ \to T_{\mathbb{C}} Z$,
2. $_{\mathbb{R}} = \cap T \mathscr{X}_{\mathbb{R}} = \ker\{\mathfrak{f}_* : T \mathscr{X}_{\mathbb{R}} \to TN\}$ *on* $\mathscr{X}_{\mathbb{R}}$,
3. $L = \ker\{\mathfrak{f}_* : T \mathscr{X}_+ \to TZ\}$ *on* $\mathscr{X}_+ \setminus \mathscr{X}_{\mathbb{R}}$, *and*

4. $L = \ker\{\tilde{\mathfrak{f}}_* : T\mathscr{X}_{\mathbb{R}} \to S(TN)\}$ *on* $\mathscr{X}_{\mathbb{R}}$.

Proof 1 and *2* follow from Propositions 8.1 and 8.3 and their proofs. We also have $= \mathfrak{f}_*^{-1}(T^{1,0}Z)$, so $L \otimes \mathbb{C} = \cap = \ker \mathfrak{f}_* : T_{\mathbb{C}}\mathscr{X}_+ \to T_{\mathbb{C}}Z$. Hence *3* follows.

Let us prove *4*. Let $U \times \mathbb{CP}^1 \xrightarrow{\sim} \mathscr{X}|_U$ be a trivialization on U such that $\rho(x,\zeta) = (x,\bar{\zeta})$. Notice that $\mathscr{X}_{\pm}|_U = \{(x,\zeta) \in U \times \mathbb{CP}^1 \mid \pm\operatorname{Im}\zeta \geq 0\}$.

Let us denote $\zeta = \xi + \sqrt{-1}\eta$ using a real coordinate (ξ,η). We fix a point $(x_0,\xi_0) \in \mathscr{X}_{\mathbb{R}}|_U$ and let $c(s)$ be a curve defined by $I_\varepsilon \to \varpi^{-1}(x) : s \mapsto (x_0, \xi_0 + \sqrt{-1}s)$, where $I_\varepsilon = (-\varepsilon,\varepsilon)$ is a small interval. Now, we define a map $\Phi : I_\varepsilon \times I_\varepsilon \to \mathscr{X} : (s,t) \mapsto \Phi(s,t)$ so that $\Phi(s,0) = c(s)$ and $\Phi_*(\frac{\partial}{\partial t}) = l^\dagger$, where $l^\dagger$ is a ρ-invariant real vector field such that $L = \operatorname{Span}\langle l^\dagger\rangle$.

Let Σ be the image of Φ, and let $\nu = \Phi(\frac{\partial}{\partial s})$ which is a tangent vector field along Σ such that $T\Sigma = \operatorname{Span}\langle l^\dagger, \nu\rangle$. Moreover, ν is proportional to $\frac{\partial}{\partial\eta}$ on $\Sigma \cap \mathscr{X}_{\mathbb{R}}$. Indeed, we have $\rho \circ \Phi(s,t) = \Phi(-s,t)$ by definition, so $\rho_*\nu = -\nu$. Hence ν is "pure imaginary" on $\mathscr{X}_{\mathbb{R}}$, that is, we can write $\nu = a\frac{\partial}{\partial\eta}$ using a real-valued function a on $\mathscr{X}_{\mathbb{R}}$. Taking ε small, we can assume a is a positive function since $\nu_{(x_0,\xi_0)} = c_*(\frac{\partial}{\partial s}) = \frac{\partial}{\partial\eta}$.

Since $\{l^\dagger, \nu\}$ is involutive, there are functions A, B on Σ such that $[l^\dagger, \nu] = Al^\dagger + B\nu$. Let φ be a positive function on Σ such that $l^\dagger\varphi = -B$, then $[l^\dagger, \varphi\nu] = \varphi A l^\dagger$. We define a positive function ψ on $\Sigma \cap \mathscr{X}_{\mathbb{R}}$ by $\varphi\nu = \psi\frac{\partial}{\partial\eta}$.

Now, $\mathfrak{f} : \mathscr{X}_+ \to Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ is described as $\mathfrak{f}(x,\zeta) = (F_1(x,\zeta), F_2(x,\zeta))$ in the neighborhood of (x_0,ξ_0) using functions F_i which are holomorphic on ζ. Let $p_1 : Z \to \mathbb{CP}^1$ be the first projection. Then its restriction $p_1 : N \to \mathbb{CP}^1$ is diffeomorphism. Hence, identifying N with $\mathbb{CP}^1$ by p_1, $\mathfrak{f} : \mathscr{X}_{\mathbb{R}} \to N$ is described by F_1. Since $L = \operatorname{Span}\langle l^\dagger\rangle = \ker \mathfrak{f}_*$ on $\mathscr{X}_+ \backslash \mathscr{X}_{\mathbb{R}}$, we have $l^\dagger F_i$=0 on $\mathscr{X}_+$. Then

$$l^\dagger(\varphi\nu F_i) = [l^\dagger, \varphi\nu]F_i + \varphi\nu(l^\dagger F_i) = 0,$$

and so $l^\dagger\big(\psi\frac{\partial F_i}{\partial\eta}\big) = 0$ on $\Sigma \cap \mathscr{X}_{\mathbb{R}}$.

Since the F_i are holomorphic for ζ, we have $\frac{\partial F_i}{\partial\xi} = -\sqrt{-1}\frac{\partial F_i}{\partial\eta}$ for $i = 1,2$. Thus we have obtained

$$l^\dagger\big(\psi\frac{\partial F_i}{\partial\xi}\big) = 0 \tag{8.14}$$

on $\Sigma \cap \mathscr{X}_{\mathbb{R}}$ for $i = 1,2$. Since $\tilde{\mathfrak{f}}(x,\xi) = \left[\frac{\partial F_1}{\partial\xi}(x,\xi)\right]$ by definition, and since ψ is a positive function, (8.14) means $\tilde{\mathfrak{f}}_*(l^\dagger) = 0$. From *2* of Proposition 7.5, the fiber of $\tilde{\mathfrak{f}}$ is at most one-dimensional, hence $L = \ker\{\tilde{\mathfrak{f}}_* : T\mathscr{X}_{\mathbb{R}} \to S(TN)\}$ on $\mathscr{X}_{\mathbb{R}}$. □

Proposition 8.5 *The Einstein-Weyl structure* $([g],\nabla)$ *constructed in Proposition 8.3 satisfies the following properties:*

1. *For each* $p \in N$, $\mathfrak{S}_p = \{x \in M \mid p \in \partial\mathfrak{D}_x\}$ *is a connected maximal null surface on M and every null surface can be written in this form.*
2. *For each* $p \in Z \backslash N$, $\mathfrak{C}_p = \{x \in M \mid p \in \mathfrak{D}_x\}$ *is a connected maximal time-like geodesic and every time-like geodesic on M can be written in this form.*

3. *For each* $p \in N$ *and non-zero* $v \in T_pN$, $\mathfrak{C}_{p,v} = \{x \in M \,|\, p \in \partial\mathfrak{D}_x, v \parallel \mathfrak{D}_x\}$ *is a connected maximal null geodesic on* M *and every null geodesic on* M *can be written in this form.*

Proof From Proposition 8.4 and the properties of $_{\mathbb{R}}$ and L, we obtain

- $\mathfrak{S}_p = \varpi \circ \mathfrak{f}^{-1}(p)$ is a null surface for each $p \in N$,
- $\mathfrak{C}_p = \varpi \circ \mathfrak{f}^{-1}(p)$ is a time-like geodesic for each $p \in Z \backslash N$,
- $\mathfrak{C}_{p,v} = \varpi \circ \tilde{\mathfrak{f}}^{-1}([v])$ is a null geodesic for each $p \in N$ and non-zero $v \in T_pN$.

Moreover from Proposition 7.5,

- $\mathfrak{S}_p \simeq S^1 \times \mathbb{R}$ for each $p \in N$,
- $\mathfrak{C}_p \simeq \mathbb{R}$ for each $p \in Z \backslash N$,
- $\mathfrak{C}_{p,v} \simeq \mathbb{R}$ for each $p \in N$ and non-zero $v \in T_pN$,

and they are all closed in M. Hence the statement follows. □

Recall the compactification of the double fibration given by (7.7). Let $\hat{\mathfrak{C}}_p$ and $\hat{\mathfrak{C}}_{p,v}$ be the compactification of $\mathfrak{C}_p$ and $\mathfrak{C}_{p,v}$ in $\hat{\mathscr{X}}_+$ respectively.

Proposition 8.6 1. *For each $p \in Z \backslash N$, $\hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_p}$ is homeomorphic to S^2 and the restriction $\hat{\mathfrak{f}} : \hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_p} \to N$ is a homeomorphism. In particular, $\{\partial \mathfrak{D}_x\}_{x \in \mathfrak{C}_p}$ gives a foliation on $N \backslash \{2 \text{ points}\}$.*
2. *For each $p \in N$ and non-zero $v \in T_pN$, $\hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_{p,v}}$ is homeomorphic to S^2 and the restriction $\hat{\mathfrak{f}} : \hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_{p,v}} \to N$ is surjective. Moreover, this is one-to-one on the complement of the curve $\hat{\mathfrak{f}}^{-1}(p)$, hence $\{(\partial \mathfrak{D}_x \backslash \{p\})\}_{x \in \mathfrak{C}_{p,v}}$ gives a foliation on $N \backslash \{p\}$.*

Proof Let $p \in Z \backslash N$, then $\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_p}$ is an S^1-bundle over $\mathfrak{C}_p \simeq \mathbb{R}$. Since $\hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_p}$ is the compactification of $\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_p}$ with extra two points, it is isomorphic to S^2. Since $\mathfrak{f}$ is C^0-close to the $\mathfrak{f}$ of the standard case, $\hat{\mathfrak{f}} : \hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_p} \to N$ is a degree one map.

Let $\mathfrak{f}_* : T(\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_p}) \to TZ_{\mathbb{R}}$ be the differential. We claim that $\ker \mathfrak{f}_* = 0$ everywhere. Indeed, if there exists a non-zero $w \in T_z(\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_p})$ such that $\mathfrak{f}_*(w) = 0$, then $w \in \mathit{\Pi}_z$ and $\varpi_*(w) \neq 0$. Then $\varpi_*(w)$ must be null with respect to the constructed conformal structure. On the other hand $\varpi_*(w)$ tangents to $\mathfrak{C}_p$, so this is time-like. This is a contradiction.

Hence $\hat{\mathfrak{f}} : \hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_p} \to N$ is locally homeomorphic degree one map, that is, it is a homeomorphism.

Next, let $p \in N$. By a similar argument, $\hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_{p,v}} \simeq S^2$ and $\hat{\mathfrak{f}} : \hat{\mathscr{X}}_{\mathbb{R}}|_{\hat{\mathfrak{C}}_{p,v}} \to N$ is degree one, hence surjective.

We claim that $\ker\{\mathfrak{f}_* : T(\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_{p,v}}) \to TN\} = 0$ on $z \in \left(\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_{p,v}} \backslash \mathfrak{f}^{-1}(p)\right)$. Indeed, if there exists non-zero $w \in T_z(\mathscr{X}_{\mathbb{R}}|_{\mathfrak{C}_{p,v}})$ such that $\mathfrak{f}_*(w) = 0$, then $\varpi_*(w)$ is non-zero and null. Notice that $\varpi_*(w)$ is tangent to the null surface $\mathfrak{S}_{\mathfrak{f}(z)}$.

On the other hand, $\varpi_*(w)$ is tangent to $\mathfrak{C}_{p,v} \subset \mathfrak{S}_p$. Since $\mathfrak{f}(z) \neq p$, $\mathfrak{S}_{\mathfrak{f}(z)}$ and $\mathfrak{S}_p$ are different null surfaces, hence $T_{\varpi(z)}\mathfrak{S}_{\mathfrak{f}(z)}$ and $T_{\varpi(z)}\mathfrak{S}_p$ are different null planes at $\varpi(z)$. Then $\varpi_*(w) \in T_{\varpi(z)}\mathfrak{S}_{\mathfrak{f}(z)} \cap T_{\varpi(z)}\mathfrak{S}_p$ must be a space-like vector which is a contradiction. Hence the statement follows. □

Proposition 8.7 *Let $([g], \nabla)$ be the Einstein-Weyl structure constructed in Proposition 8.3. Then, for each distinguished $p, q \in N$, $\mathfrak{C}_{p,q} = \{x \in M \,|\, p, q \in \partial \mathfrak{D}_x\}$ is a* *connected closed space-like geodesic on M and every space-like geodesic on M can be written in this form. In particular, this Einstein-Weyl structure is space-like Zoll.*

Proof Since $\mathfrak{C}_{p,q}$ is the intersection of the null surfaces $\mathfrak{S}_p$ and $\mathfrak{S}_q$, this is either empty or a space-like geodesic. We claim that $\mathfrak{C}_{p,q}$ is not empty and is homeomorphic to S^1. For each non-zero $v \in T_pN$, there is a unique $x \in \mathfrak{C}_{p,v}$ such that $q \in \partial\mathfrak{D}_x$, since $\{(\partial\mathfrak{D}_x \backslash \{p\})\}_{x\in\mathfrak{C}_{p,v}}$ foliates $N\backslash\{p\}$ by *2* of Proposition 8.6. Then $x \in \mathfrak{C}_{p,q}$, so $\mathfrak{C}_{p,q}$ is not empty. Moreover there is a one-to-one continuous map $S(T_pN) \to \mathfrak{C}_{p,q}$, so $\mathfrak{C}_{p,q} \simeq S^1$. □

The main theorem (Theorem 1.6) follows from Propositions 8.3, 8.5 and 8.7.

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