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Special Issue
Symmetry in Gravity Research

Edited by
Prof. Dr. Xin Wu and Prof. Dr. Wenbiao Han

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https://doi.org/10.3390/sym14122661

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The Quasi-Keplerian Motion of the Charged Test Particle in Reissner-Nordström Spacetime under the Wagoner-Will-Epstein-Haugan Representation

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Abstract: We present the post-Newtonian solution for the quasi-Keplerian motion of a charged test particle in the field of Reissner-Nordström black hole under the Wagoner-Will-Epstein-Haugan representation. The explicit formulations for the charge effects on perihelion precession and the orbital period are achieved, which may be useful not only in the comparisons with astronomical observations but also in calculating the waveform of the gravitational wave from this kind of system.

Keywords: quasi-Keplerian; Reissner-Nordström spacetime; post-Newtonian approximation

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Citation: Yang, B.; Jiang, C.; He, G.; Lin, W. The Quasi-Keplerian Motion of the Charged Test Particle in Reissner-Nordström Spacetime under the Wagoner-Will-Epstein-Haugan Representation. *Symmetry* **2022**, *14*, 2661. https://doi.org/10.3390/sym14122661

Academic Editors: Xin Wu, Wenbiao Han, Kazuharu Bamba and Sergei D. Odintsov

Received: 17 October 2022
Accepted: 10 December 2022
Published: 16 December 2022

1. Introduction

The motion of charged or neutral, or massive or massless test particles in the gravitational fields is one of the most important problems of relativistic astrophysics. For example, the prediction by general relativity accounts for the perihelion precession of Mercury perfectly [1]. The analytical solutions of the motion are not only important in the theoretical significance, but are also useful in exploring the properties of spacetime in which the bodies move as well as calculating the gravitational wave radiated by the bodies. For the motion of celestial bodies in the strong gravitational fields, a variety of analytical solutions have been obtained using the post-Newtonian (PN) approximation. These solutions include not only the motion of the photon with the small-deflection angle [2–4], but also the quasi-Keplerian motion of the test particles as well as the binary systems [5–22]. The latter is mainly represented in the following two ways: one is the Brumberg–Damour–Deruelle (BDD) representation [5,8,14–17,20] and the other is the Wagoner-Will-Epstein-Haugan (WWEH) representation [6,7,9–11,21,22]. Under the WWEH representation, the solution is expressed by the eccentricity and the semilatus rectum in the Newtonian theory being different from the solution under the BDD representation, which is expressed with the test particle's orbital energy and angular momentum.

The Reissner-Nordström (RN) spacetime is a static, asymptotically flat solution of the Einstein–Maxwell equations in general relativity (GR) [23–25]. The exact harmonic metric for a moving RN black hole with an arbitrary constant speed has been obtained [26]. In fact, there are two paths to solve the orbits of charged particles in the RN spacetime. They are numerical methods and analytical methods. The numerical methods are a main path to solve the orbits of charged particles in the RN spacetime at present. The numerical methods include manifold correction schemes [27], energy-conserving integrators [28], extended phase space explicit symplectic-like methods [29], and explicit and implicit mixed symplectic algorithms [30]. Recently, explicit symplectic methods have been developed for black hole spacetimes [31–34]. Solving the geodesic equations numerically may be straightforward, but it would be time-consuming for the numerical simulations to achieve a good precision. On the other hand, the analytical solutions can exhibit the effects of the source's parameters (e.g., mass, magnetic charge, and angular momentum) on the particle's motion explicitly. The analytical solutions to the equations of motion for the test particles moving in the RN spacetime have been extensively investigated in the literature in a variety of contexts and ways [35–43]. In the GR framework, several others have successfully used PN formalism on binary mergers to evaluate the source parameters from gravitational waves [44–47]. In particular, Zhang et.al discussed equivalence between two charged black holes in dynamics of orbits outside the event horizons [48].

On the other hand, the spherical symmetry of the spacetime in physics or mathematics has an invariant angular momentum and a fourth motion constant such as the Carter constant. Thus, the spacetime is integrable and nonchaotic. In addition, the spherical symmetry of the spacetime is convenient to study black hole shadows because only one impact parameter is used and lots of circular photon orbits are considered [49,50]. Therefore, it is interesting and timely to discuss the charge's effects on the quasi-Keplerian motion for a particle in spherical symmetric RN spacetime.

In the previous work [42,43], we derived the quasi-Keplerian motion for the neutral test particle in the RN spacetime under both the BDD representation and the WWEH one, and that for the charged test particle under the BDD representation. Here, we derive the quasi-Keplerian motion for the charged test particle in the same background under the WWEH representation. The charge effects of the black hole and the test particle on the quasi-Keplerian motion, including the perihelion precession and the orbital period, are shown clearly.

The structure of the paper is as follows: In Section 2, we briefly introduce the quasi-Keplerian dynamics for the charged test particle in the RN spacetime. In Section 3, we give a detailed derivation of the 1PN solution for the quasi-Keplerian motion of the charged test particle. In Section 4, we investigate the relations between the Keplerian parameters and the conserved quantity (orbital energy and angular momentum). The validity of the analytical solution is discussed in Section 5. A summary is given in Section 6.

2. The Quasi-Keplerian Dynamics for the Charged Test Particle

In the harmonic coordinates, the metric of RN spacetime with mass M and electric charge Q in the 1PN approximation can be written as ($G = 1$ and $c = 1$) [26,51]

$$g_{00} = -1 + \frac{2M}{r} - \frac{2M^2}{r^2}\left(1 + \frac{1}{2}\epsilon_0^2\right), \tag{1}$$

$$g_{0i} = 0\,, \tag{2}$$

$$g_{ij} = \left(1+\frac{2M}{r}\right)\delta_{ij}, \tag{3}$$

where $\epsilon_0 \equiv Q/M$ denotes the charge-to-mass of the RN black hole. The condition for the nonsingularity of the RN spacetime is $|\epsilon_0| \leq 1$. $r \equiv |\boldsymbol{x}|$ denotes the distance from the field position $\boldsymbol{x} \equiv (x, y, z)$ to the black hole located at the coordinate origin. The metric has signature of $(- + ++)$. Latin indices i and j range from 1 to 3.

We consider the motion of a charged test particle with mass m and electric charge q. $\epsilon_1 \equiv q/m$ is the charge-to-mass ratio of the test particle. The covariantgeodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = \epsilon_1 F^\mu{}_\nu \frac{dx^\nu}{d\tau}\,, \tag{4}$$

where $\Gamma^\mu_{\nu\lambda}$ denotes the Christoffel's symbols that are given by the derivatives of the chosen metric $g_{\mu\nu}$, and τ is the proper time of the particle along its world line. The electromagnetic Faraday tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu\,, \tag{5}$$

where A_α is the associated electromagnetic potential vector [51]

$$A_0 = -\frac{\epsilon_0 M}{r}\left(1+\frac{M}{r}\right)^{-1} , \tag{6}$$
$$A_i = 0 . \tag{7}$$

Substituting Equations (1)–(3) into Equation (4), we can obtain the 1PN geodesic equations for the charged test particle as follows [43]:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{M\boldsymbol{x}}{r^3}\left[(1-\epsilon_0\epsilon_1)-\frac{M}{r}(4+\epsilon_0^2-5\epsilon_0\epsilon_1)+v^2\left(1+\frac{1}{2}\epsilon_0\epsilon_1\right)\right] , \tag{8}$$

where $\boldsymbol{v}$ denotes the velocity of the test particle. When $\epsilon_1 = 0$, it is the equation of motion for a neutral particle in RN spacetime.

Since the problem has a spherical symmetry, for convenience, we take the plane in which the test particle moves as the equatorial plane, then the particle's trajectory x can be expressed as

$$\boldsymbol{x} = r(\cos\phi\, \boldsymbol{e}_x + \sin\phi\, \boldsymbol{e}_y) , \tag{9}$$

where ϕ is the azimuthal angle, and $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors of the x and y axes.

3. The Quasi-Keplerian Motion for the Charged Test Particle

3.1. Keplerian Motion in the Newtonian Theory

In order to derive the analytical solution for the quasi-Keplerian motion, we first present the Keplerian solution as follows:

$$r = \frac{p}{1+e\cos\phi} = a(1-e\cos E) , \tag{10}$$
$$\phi = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) , \tag{11}$$
$$t\left(\frac{2\pi}{\mathrm{T_K}}\right) = E - e\sin E , \tag{12}$$
$$r^2\dot{\phi} = [M(1-\epsilon_0\epsilon_1)p]^{\frac{1}{2}} , \tag{13}$$
$$\boldsymbol{v}_{\mathrm{K}} = \left[\frac{M(1-\epsilon_0\epsilon_1)}{p}\right]^{\frac{1}{2}}[-\sin\phi\, \boldsymbol{e}_x + (e+\cos\phi)\, \boldsymbol{e}_y] , \tag{14}$$

with

$$p = a(1-e^2) , \tag{15}$$
$$\mathrm{T_K} = 2\pi\left[\frac{a^3}{M(1-\epsilon_0\epsilon_1)}\right]^{\frac{1}{2}} , \tag{16}$$

where p, e denote the semilatus rectum and the eccentricity of Keplerian orbit. $a \equiv p/(1-e^2)$ is the semimajor axis. $\boldsymbol{v}_{\mathrm{K}}$ is the Keplerian solution for the velocity of the charged test particle. E is the eccentric anomaly. $\mathrm{T_K}$ represents the Keplerian period.

3.2. The Quasi-Keplerian Motion under the Wagoner-Will-Epstein-Haugan Representation

We start with the equation of motion for the charged test particle under the harmonic coordinates. Following the derivation in Ref. [6,7,9,11], we can write

$$r^2\dot{\phi} = [M(1-\epsilon_0\epsilon_1)p]^{\frac{1}{2}}\left[1-(4-\epsilon_0\epsilon_1)\frac{M}{p}e\cos\phi\right] , \tag{17}$$

where the sign "dot" denotes the derivative with respect to the time, and the 1PN solution for the velocity of the charged particle can be written as

$$
\begin{aligned}
\boldsymbol{v} = \left[\frac{M(1-\epsilon_0\epsilon_1)}{p}\right]^{\frac{1}{2}}\Big\{&-\sin\phi\,\boldsymbol{e}_x+\frac{M}{p}\Big[\frac{6-6\epsilon_0\epsilon_1-\epsilon_0^2(1-\epsilon_1^2)}{-2(1-\epsilon_0\epsilon_1)}e\phi+\frac{6-9\epsilon_0\epsilon_1+\epsilon_0^2(2+\epsilon_1^2)}{2(1-\epsilon_0\epsilon_1)}\sin\phi \\
&-\frac{1}{2}e^2(2+\epsilon_0\epsilon_1)\sin\phi+\frac{2-4\epsilon_0\epsilon_1+\epsilon_0^2(1+\epsilon_1^2)}{4(1-\epsilon_0\epsilon_1)}e\sin 2\phi\Big]\boldsymbol{e}_x \\
&+(e+\cos\phi)\,\boldsymbol{e}_y+\frac{M}{p}\Big[\frac{6-9\epsilon_0\epsilon_1+\epsilon_0^2(2+\epsilon_1^2)}{-2(1-\epsilon_0\epsilon_1)}\cos\phi \\
&-\frac{3}{2}e^2(2-\epsilon_0\epsilon_1)\cos\phi-\frac{2-4\epsilon_0\epsilon_1+\epsilon_0^2(1+\epsilon_1^2)}{4(1-\epsilon_0\epsilon_1)}e\cos 2\phi\Big]\boldsymbol{e}_y\Big\}\,, \qquad (18)
\end{aligned}
$$

Substituting Equations (9), (17) and (18) into the identity

$$
\frac{d}{d\phi}\frac{1}{r} = -(r^2\dot{\phi})^{-1}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{r}\right), \qquad (19)
$$

and making integration over ϕ, we can achieve

$$
\begin{aligned}
\frac{p}{r} = 1+e\cos\phi-\frac{M}{4p(1-\epsilon_0\epsilon_1)}\{&2(6-2e^2-7e\cos\phi-6e\phi\sin\phi) \\
&-2(9-e^2-8e\cos\phi-6e\phi\sin\phi)\epsilon_0\epsilon_1+\epsilon_0^2[4+e\cos\phi \\
&+2e\phi\sin\phi+(2+2e^2-3e\cos\phi-2e\phi\sin\phi)\epsilon_1^2]\}\,. \qquad (20)
\end{aligned}
$$

From Equation (20), we can obtain the orbital precession per revolution as

$$
\Delta\phi = \frac{6-6\epsilon_0\epsilon_1-\epsilon_0^2(1-\epsilon_1^2)}{1-\epsilon_0\epsilon_1}\,\frac{\pi M}{p}\,. \qquad (21)
$$

In order to obtain the time dependence of the quasi-Keplerian equation, we introduce the true anomaly η as [7,9]

$$
\eta = \left[1-\frac{6-6\epsilon_0\epsilon_1-\epsilon_0^2(1-\epsilon_1^2)}{2(1-\epsilon_0\epsilon_1)}\,\frac{M}{p}\right]\phi\,, \qquad (22)
$$

or

$$
\phi\left(\frac{2\pi}{\Phi}\right) = \eta\,, \qquad (23)
$$

with

$$
\Phi = 2\pi+\Delta\phi\,. \qquad (24)
$$

Then, we can rewrite Equations (17) and (20) as

$$
r^2\dot{\phi} = [M(1-\epsilon_0\epsilon_1)p]^{\frac{1}{2}}\left[1-(4-\epsilon_0\epsilon_1)\frac{M}{p}e\cos\eta\right], \qquad (25)
$$

$$
\begin{aligned}
\frac{p}{r} = 1+e\cos\eta-\frac{M}{4p(1-\epsilon_0\epsilon_1)}\{&2(6-2e^2-7e\cos\eta)-2(9-e^2-8e\cos\eta)\epsilon_0\epsilon_1 \\
&+\epsilon_0^2[4+e\cos\eta+(2+2e^2-3e\cos\eta)\epsilon_1^2]\}\,. \qquad (26)
\end{aligned}
$$

We further introduce the eccentric anomaly E' in the quasi-Keplerian orbit, which is related to the true anomaly η by

$$
\sin\eta = \frac{(1-e^2)^{\frac{1}{2}}\sin E'}{1-e\cos E'}\,; \qquad \cos\eta = \frac{\cos E'-e}{1-e\cos E'}\,, \qquad (27)
$$

we can rewrite Equation (26) as

$$
\begin{aligned}
r = {} & a(1-e\cos E') \\
& + \frac{M}{(1-e^2)^2}\left\{ \frac{24+46e^2-4e^4-2(18+31e^2-e^4)\epsilon_0\epsilon_1+\epsilon_0^2[8+e^2+(4+15e^2+2e^4)\epsilon_1^2]}{8(1-\epsilon_0\epsilon_1)} \right. \\
& - \frac{38+6e^2-4(13+3e^2)\epsilon_0\epsilon_1+\epsilon_0^2[7-e^2+7(1+e^2)\epsilon_1^2]}{4(1-\epsilon_0\epsilon_1)} e\cos E' \\
& \left. + \frac{26-4e^2-2(17-e^2)\epsilon_0\epsilon_1+\epsilon_0^2[3+(5+2e^2)\epsilon_1^2]}{8(1-\epsilon_0\epsilon_1)} e^2\cos 2E' \right\} .
\end{aligned} \tag{28}
$$

Taking the derivative of Equation (22) with respect to the time, and then using Equation (25), we have

$$
\left[1+\frac{6-6\epsilon_0\epsilon_1-\epsilon_0^2(1-\epsilon_1^2)}{2(1-\epsilon_0\epsilon_1)}\frac{M}{p}\right] r^2\frac{d\eta}{dt} = [M(1-\epsilon_0\epsilon_1)p]^{\frac{1}{2}}\left[1-(4-\epsilon_0\epsilon_1)\frac{M}{p}e\cos\eta\right]. \tag{29}
$$

From Equation (27), we can obtain

$$
\frac{d\eta}{dt} = \frac{(1-e^2)^{\frac{1}{2}}}{1-e\cos E'}\frac{dE'}{dt} . \tag{30}
$$

Substituting Equations (28) and (30) into Equation (29) for eliminating r, then integrating over the time, we finally obtain the quasi-Keplerian equation

$$
t\left(\frac{2\pi}{\mathrm{T}_{E'}}\right) = E - g\sin E' - h\sin 2E' , \tag{31}
$$

where g and h are given by

$$
g = e\left\{1+\frac{M}{p}\frac{2(18-e^2-6e^4)-2(24+e^2-9e^4)\epsilon_0\epsilon_1+\epsilon_0^2[3(2-e^2)+(6+7e^2-6e^4)\epsilon_1^2]}{4(1-e^2)(1-\epsilon_0\epsilon_1)}\right\} , \tag{32}
$$

$$
h = \frac{Me^2}{p(1-e^2)}\frac{26-4e^2-2(17-e^2)\epsilon_0\epsilon_1+\epsilon_0^2[3+(5+2e^2)\epsilon_1^2]}{-8(1-\epsilon_0\epsilon_1)} , \tag{33}
$$

and

$$
\begin{aligned}
\mathrm{T}_{E'} = {} & 2\pi\left[\frac{a^3}{M(1-\epsilon_0\epsilon_1)}\right]^{\frac{1}{2}} \\
& \times\left\{1+\frac{3M}{4p}\frac{12+6e^2+4e^4-2(8+5e^2+3e^4)\epsilon_0\epsilon_1+\epsilon_0^2[2+e^2+(2+3e^2+2e^4)\epsilon_1^2]}{(1-e^2)(1-\epsilon_0\epsilon_1)}\right\} ,
\end{aligned} \tag{34}
$$

which can be used to characterize the orbital period of the quasi-Keplerian motion.

4. The Relations between the Keplerian Parameters and the Orbital Energy and Angular Momentum

From the equation of motion Equation (8), we can calculate the corresponding Lagrangian for the charged particle:

$$
L = \frac{1}{2}v^2+\frac{M}{r}(1-\epsilon_0\epsilon_1)+\frac{1}{8}v^4+\frac{3}{2}\frac{M}{r}v^2-\frac{1}{2}\frac{M^2}{r^2}(1+\epsilon_0^2-2\epsilon_0\epsilon_1) . \tag{35}
$$

We study the motion of charged particles in a spherically symmetric RN spacetime, where the Lagrangian quantities of the system have continuous symmetry, which in turn yields two conserved quantities, orbital energy $\mathcal{E}$ and orbital angular momentum $\mathcal{J}$ of the charged particle. Based on Equation (35), we have

$$\mathcal{E} = \frac{1}{2}\boldsymbol{v}^2 - \frac{M}{r}(1-\epsilon_0\epsilon_1) + \frac{3}{8}\boldsymbol{v}^4 + \frac{3}{2}\frac{M}{r}\boldsymbol{v}^2 + \frac{1}{2}\frac{M^2}{r^2}(1+\epsilon_0^2-2\epsilon_0\epsilon_1)\,, \tag{36}$$

$$\mathcal{J} = |\boldsymbol{x}\times\boldsymbol{v}|\left(1+\frac{1}{2}\boldsymbol{v}^2+\frac{3M}{r}\right), \tag{37}$$

which reduces to those of the neutral particle in RN spacetime if $\epsilon_1 = 0$ [42], and energy conservation comes from translational symmetry in time, where angular momentum conservation is from rotational symmetry in space.

Substituting Equations (9), (18) and (20) into Equations (36) and (37), we can obtain

$$\mathcal{E} = \frac{M}{8p}\Big\{4(-1+e^2)(1-\epsilon_0\epsilon_1) + \frac{M}{p}[19+22e^2+3e^4-2(1+e^2)(13+3e^2)\epsilon_0\epsilon_1 + 2(2+e^2)\epsilon_0^2 + (3+8e^2+3e^4)\epsilon_0^2\epsilon_1^2]\Big\}\,, \tag{38}$$

$$\mathcal{J} = [M(1-\epsilon_0\epsilon_1)p]^{\frac{1}{2}}\Big\{1+\frac{M}{2p}[7+e^2-(1+e^2)]\epsilon_0\epsilon_1\Big\}\,. \tag{39}$$

By solving Equations (38) and (39) inversely, we can obtain p and e in terms of $\mathcal{E}$ and $\mathcal{J}$, as follows:

$$p = \frac{\mathcal{J}^2}{M(1-\epsilon_0\epsilon_1)}\Big\{1 - \frac{M^2(1-\epsilon_0\epsilon_1)}{\mathcal{J}^2}\Big[8+\frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2} - \Big(2+\frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2}\Big)\epsilon_0\epsilon_1\Big\}\,, \tag{40}$$

$$e^2 = 1+\frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2}\Big\{1-\frac{7}{2}\mathcal{E}-15\frac{M^2}{\mathcal{J}^2}-\frac{11}{2}\frac{M^4}{\mathcal{E}\mathcal{J}^4}+\Big(21\frac{M^2}{\mathcal{J}^2}+19\frac{M^4}{\mathcal{E}\mathcal{J}^4}\Big)\epsilon_0\epsilon_1 + \frac{M^4}{4\mathcal{E}\mathcal{J}^4}\epsilon_0^3\epsilon_1[6+46\epsilon_1^2-\epsilon_0\epsilon_1(3+7\epsilon_1^2)] - \frac{M^2}{4\mathcal{J}^2}\epsilon_0^2\Big[2+\frac{3M^2}{\mathcal{E}\mathcal{J}^2}+\Big(22+\frac{93M^2}{\mathcal{E}\mathcal{J}^2}\Big)\epsilon_1^2\Big]\Big\}\,. \tag{41}$$

Substituting Equations (40) and (41) into Equations (21) and (34), and keeping the accuracy to the 1PN order, we can obtain

$$\Delta\phi = \frac{\pi M^2}{\mathcal{J}^2}[6-6\epsilon_0\epsilon_1-\epsilon_0^2(1-\epsilon_1^2)]\,, \tag{42}$$

$$\mathrm{T}_{E'} = \frac{2\pi M(1-\epsilon_0\epsilon_1)}{(-2\mathcal{E})^{\frac{3}{2}}}\Big[1-\mathcal{E}\frac{15-3\epsilon_0\epsilon_1}{4-4\epsilon_0\epsilon_1}\Big]\,, \tag{43}$$

which are same as the results of a charged particle in RN spacetime under the Burmberg–Damour–Deruelle representation [43]. It is worth emphasizing that orbital energy, orbital angular momentum, orbital period, and perihelion precession all remain constant during the different coordinate transformations.

5. The Validity of the Analytical Solution

The achieved analytical solution of the quasi-Keplerian motion is based on the Lagrangian formulation, and is kept to the first-order post-Newtonian approximation. Damour et al. [52] showed the equivalence of the post-Newtonian Lagrangian formulation and the post-Newtonian Hamiltonian formulation at same order; however, Wu et al. [30,53] proved that there are some differences between them. They also showed that the truncated post-Newtonian Lagrangian equations, the coherent post-Newtonian Lagrangian equations, and the post-Newtonian Hamiltonian formulation at same order are not exactly equivalent, and even have completely different dynamical behaviors [54]. When the test particle is far away from the RN black hole, for example, the distance is about 10^7 M, where the gravitational field approximately matches that of the solar system, the differences among these methods reach the computer double precision of 10^{-16}, and in this case, they are basically equivalent.

6. Summary

The aim of this work is to obtain the effects of the black hole and the test particle's charges on the quasi-Keplerian motion in the RN spacetime under the Wagoner-Will-Epstein-Haugan representation. In order to achieve this, we expand the metric of the RN black hole into the powers of Newtonian potential to the 1PN order, and the latter is substituted intothe covariant geodesic equation with the Lorentz force. Then, we employ the iterative method to derive the perturbation to the Keplerian motion. The perturbation is expressed in terms of the semilatus rectum and the eccentricity defined in the Newtonian theory, instead of the orbital energy and angular momentum. We also demonstrated the congruency of the solutions for the orbital period and perihelion precession between these two formulations.

The formulation is

$$\frac{p}{r} = 1+e\cos\eta - \frac{M}{4p(1-\epsilon_0\epsilon_1)}\{2(6-2e^2-7e\cos\eta)-2(9-e^2-8e\cos\eta)\epsilon_0\epsilon_1 + \epsilon_0^2[4+e\cos\eta+(2+2e^2-3e\cos\eta)\epsilon_1^2]\}\,, \tag{44}$$

$$\phi\left(\frac{2\pi}{\Phi}\right) = \eta\,, \tag{45}$$

$$\eta = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E'}{2}\right), \tag{46}$$

$$t\left(\frac{2\pi}{\mathrm{T}_{E'}}\right) = E - g\sin E' - h\sin 2E'\,, \tag{47}$$

with

$$\Phi = 2\pi\left[1+\frac{M}{p}\frac{6-6\epsilon_0\epsilon_1-\epsilon_0^2(1-\epsilon_1^2)}{2(1-\epsilon_0\epsilon_1)}\right], \tag{48}$$

$$g = e\left\{1+\frac{M}{p}\frac{2(18-e^2-6e^4)-2(24+e^2-9e^4)\epsilon_0\epsilon_1+\epsilon_0^2[3(2-e^2)+(6+7e^2-6e^4)\epsilon_1^2]}{4(1-e^2)(1-\epsilon_0\epsilon_1)}\right\}, \tag{49}$$

$$h = \frac{Me^2}{p(1-e^2)}\frac{26-4e^2-2(17-e^2)\epsilon_0\epsilon_1+\epsilon_0^2[3+(5+2e^2)\epsilon_1^2]}{-8(1-\epsilon_0\epsilon_1)}\,, \tag{50}$$

$$\mathrm{T}_{E'} = 2\pi\left[\frac{a^3}{M(1-\epsilon_0\epsilon_1)}\right]^{\frac{1}{2}} \times\left\{1+\frac{3M}{4p}\frac{12+6e^2+4e^4-2(8+5e^2+3e^4)\epsilon_0\epsilon_1+\epsilon_0^2[2+e^2+(2+3e^2+2e^4)\epsilon_1^2]}{(1-e^2)(1-\epsilon_0\epsilon_1)}\right\}, \tag{51}$$

where η and E' denote the true anomaly and the eccentric anomaly for the quasi-Keplerian motion in the Wagoner-Will-Epstein-Haugan representation [7,9]. $\mathrm{T}_{E'}$ represents the orbital period of the quasi-Keplerian motion.

The effects of the black hole and the test particle's charges are characterized by the terms containing ϵ_0 and ϵ_1 in the above formulas, which can not only affect the motion of the test particle, but also further affect the emission of gravitational waves. The analytical solution for the quasi-Keplerian motion is valid for the cases of $|\epsilon_0| \le 1$ and $|\epsilon_0\epsilon_1| \ll 1$.

Author Contributions: Methodology, B.Y.; validation, C.J. and G.H.; writing—original draft preparation, B.Y.; writing—review and editing, W.L.; funding acquisition, W.L. and B.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by by the National Natural Science Foundation of China (Grant Nos. 12247157, 11973025, and 12147208).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data generated or analyzed during this study are included in this published article.

Acknowledgments: Thanks the support of the postdoctoral program of Purple Mountain Observatory, Chinese Academy of Sciences.

Conflicts of Interest: The authors declare no conflict of interest.

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