The first law and Wald entropy formula of heterotic stringy black holes at first order in α'

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The world is indeed full of peril, and in it there are many dark places; but still there is much that is fair, and though in all lands love is now mingled with grief, it grows perhaps the greater.

J. R. R. Tolkien: The Lord of the Rings

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Abstract

Black-hole thermodynamics is probably one of the most active fields of research in Theoretical Physics. It interconnects seemingly disparate areas of Physics such as Gravity, Quantum Field Theory, and Information Theory, providing deep insights in all of them. While initially valid only for General Relativity, Wald and collaborators developed a new approach to demonstrate the first law of black hole mechanics in general diffeomorphism-invariant theories, beyond General Relativity. As a by-product, this approach lead to the identification of an expression that plays the role of entropy (Wald entropy) in the first law in theories beyond General Relativity. However, the first laws and the entropy formulas derived in the literature with this formalism (the Iyer-Wald prescription) present severe shortcomings in certain string theories, such as missing work terms in the first laws and lack of gauge invariance of the entropy formula. This prevents a fair comparison with the microscopic entropy computed using other techniques (AdS/CFT correspondence etc.). The main goal of this thesis is to identify the roots of these problems and fix them. As we will see, the root of these problems is the inadequate treatment of the fields that exhibit some kind of gauge freedom. These are, as a matter of fact, all fields except for scalars and the metric (if one does not use the vielbein formalism).

This thesis is divided into two parts. The first section will involve compactifying the heterotic string action on S^1, allowing us to compute re-derive the Buscher rules and prove T duality. We will then use the Iyer-Wald formula in the dimensionally reduced action to derive an entropy formula that can be applied to black-hole solutions which can be obtained by a single non-trivial compactification on a circle and discuss its invariance under the α'-corrected T duality transformations. Specifically, we shall apply it to the Strominger-Vafa extremal black hole. We will demonstrate that in addition to the lack of gauge invariance, there exists an ambiguity in applying the formula, as applying it to $d = 10$ and $d = 5$ yields two different results that differ by a factor of 2.

As previously mentioned, Iyer-Wald formula cannot be applied unambiguously in the case of the heterotic string case, as one of the main assumptions was that all fields behaved as tensors. However, all fields apart from the metric and scalars possess gauge freedoms, and their transformations under diffeomorphisms are always coupled to gauge transformations. This serves as motivation for the second section of the thesis, where we determine the first law of black hole thermodynamics in a gauge-invariant way, introducing gauge-covariant transformations under diffeomorphisms (gauge covariant Lie derivatives). The construction of these transformations involves the definition of "momentum maps" associated to field strengths and the vectors that generate their symmetries. These objects play the role of generalized thermodynamical potentials in the first law and satisfy the restricted generalized zeroth laws.

After testing our ideas on the d-dimensional Reissner-Nordström-Tangherlini black hole in the context of the Einstein-Maxwell theory, we turn our focus to the heterotic string case. Initially, we examine the case of the heterotic string theory up to zeroth order α' compactified on a torus. This theory is interesting because of the black-hole solutions it admits, and because of the Abelian Chern-Simons terms present in the Kalb-Ramond 3-form field strength. The presence of those terms induces the so-called Nicolai-Townsend gauge transformations of the Kalb-Ramond 2-form. These terms and gauge transformations, appear in the 10-dimensional theory at first order in α' in a much more complicated way (non-Abelian, gravitational) and this model can be used as a toy model to test our ideas. We show how to deal with all these gauge symmetries deriving the

first law in terms of manifestly gauge-invariant quantities. Explicitly, we will demonstrate this in the case of a non-extremal, charged, black ring solution of pure $\mathcal{N} = 1$, $d = 5$ supergravity embedded in the Heterotic Superstring effective field theory.

In the final chapter, we arrive at our main result, based on the work of the previous chapters. We derive the first law of black hole mechanics in the context of the Heterotic Superstring effective action to first order in α' using Wald's formalism, taking into account all the symmetries of the theory. This requires additional care due to the presence of the non-Abelian Lorentz and Yang-Mills Chern-Simon terms found in the Kalb-Ramond field strength. As a result, we obtain a manifestly gauge- and Lorentz-invariant entropy formula in which all the terms can be computed explicitly. An entropy formula with these properties allows unambiguous calculations of macroscopic black-hole entropies to first order in α' that can be reliably used in a comparison with the microscopic ones. Such a formula was still lacking in the literature

Resumen

La termodinámica de los agujeros negros es probablemente uno de los campos de investigación más activos de la Física Teórica. Interconecta áreas de la Física tan aparentemente dispares como la Gravedad, la Teoría Cuántica de Campos y la Teoría de la Información, proporcionando una visión profunda de todas ellas. Si bien inicialmente solo era válida para la Relatividad General, Wald y sus colaboradores desarrollaron un nuevo enfoque para demostrar la primera ley de la mecánica de los agujeros negros en teorías generales invariantes bajo difeomorfismos más generales que la Relatividad General. Como subproducto, este enfoque condujo a la identificación de una expresión que juega el papel de entropía (*entropía de Wald*) en la primera ley en teorías más allá de la Relatividad General.

Sin embargo, las primeras leyes y las fórmulas de entropía derivadas en la literatura con este formalismo (la prescripción de Iyer-Wald, en concreto) presentan graves deficiencias en ciertas teorías de cuerdas, como la falta de términos de trabajo en las primeras leyes y la falta de invariancia de gauge de la fórmula de entropía. Esto impide una comparación justa con la entropía microscópica calculada utilizando otras técnicas (correspondencia AdS/CFT, etc.). El objetivo principal de esta tesis es identificar las raíces de estos problemas y solucionarlos. Como veremos, la raíz de estos problemas es el tratamiento inadecuado de los campos que exhiben algún tipo de libertad de gauge. Estos son, de hecho, todos los campos excepto los escalares y la métrica (si no se usa el formalismo de tétradas).

Esta tésis está dividida en dos partes. En la primera sección se realiza la compactificación de la acción efectiva de la cuerda heterótica en S^1 a primer orden en α', lo que nos permitirá volver a calcular las reglas de Buscher y demostrar que es invariante bajo T dualidad. Luego usaremos la fórmula de Iyer-Wald en la acción del modelo dimensionalmente reducido para derivar una fórmula de entropía que se puede aplicar a soluciones de agujeros negros que pueden ser obtenidos por una sola compactifición no trivial en un círculo y discutiremos su invariancia bajo las transformaciones de T dualidad corregidas por α'. En concreto, lo aplicaremos al agujero negro extremo de Strominger-Vafa. Demostraremos que, además de la falta de invariancia de gauge, existe una ambigüedad en la aplicación de la fórmula, ya que al aplicarla a $d = 10$ y $d = 5$ produce dos resultados diferentes que difieren por un factor de 2.

Como se mencionó anteriormente, la fórmula de Iyer-Wald no se puede aplicar sin ambigüedades en el caso de la cuerda heterótica, ya que una de las suposiciones principales en su derivación era que todos los campos se comportaban como tensores y todos los campos, excepto el métrico y el escalar, poseen libertades de gauge y sus transformaciones bajo difeomorfismos siempre están acoplados a transformaciones de gauge. Esto sirve de motivación para la segunda sección de la tesis en la que probamos la primera ley de la termodinámica de agujeros negros de una manera invariante de gauge, introduciendo transformaciones bajo difeomorfismos covariantes de gauge (derivadas de Lie covariantes de gauge). La construcción de estas transformaciones implica la definición de *momentum maps* asociados a los campos y a los vectores que generan sus simetrías. Estos objetos juegan el papel de potenciales termodinámicos generalizados en la primera ley y satisfacen las "leyes cero generalizadas restringidas".

Después de haber puesto a prueba nuestras ideas sobre el agujero negro Reissner-Nordström-Tangherlini en el contexto de la teoría de Einstein-Maxwell d-dimensional, nos

centramos en el caso de la cuerda heterótica. Inicialmente, examinamos el caso de la teoría efectiva de la cuerda heterótica hasta orden cero en α' compactificada sobre un toro. Esta teoría es interesante debido a las soluciones de agujeros negros que admite, y debido a los términos abelianos de Chern-Simons presentes en la intensidades de campo de la 3 forma de Kalb-Ramond. La presencia de esos términos induce las llamadas *transformaciones de gauge de Nicolai-Townsend* de la 2-forma de Kalb-Ramond. Estos términos y transformaciones de gauge aparecen en la teoría de 10 dimensiones a primer orden en α' de una manera mucho más complicada (no-abeliana, gravitacional) y este modelo puede usarse como un modelo de juguete para poner a prueba nuestras ideas. Así, explicamos cómo hay que tratar todas estas simetrías de gauge y derivamos la primera ley en términos de cantidades manifiestamente invariantes de gauge. Explícitamente, demostraremos esto en el caso de una solución de anillo negro cargada no-extrema de supergravedad pura $\mathcal{N} = 1$, $d = 5$ que se puede ver tambi'en como solución de la teoría efectiva de supercuerda heterótica.

En el capítulo final, llegamos a nuestro resultado principal, basado en el trabajo de los capítulos anteriores. En él demostramos la primera ley de la mecánica de los agujeros negros en el contexto de la acción efectiva de la supercuerda heterótica a primer orden en α' utilizando el formalismo de Wald, teniendo en cuenta correctamente todas las simetrías de la teoría. Esto requiere un cuidado adicional debido a la presencia de los términos no-abelianos de Lorentz y Yang-Mills Chern-Simons que se encuentran en la intensidad de campo de Kalb-Ramond. Como resultado, obtenemos una fórmula de entropía manifiestamente invariante de gauge (incluyendo transformaciones de Lorentz locales) en la que todos los términos puede calcularse explícitamente. Una fórmula de entropía con estas propiedades permite cálculos inambiguos de entropías de agujeros negros macroscópicos de primer orden en α' que pueden usarse de forma fiable en una comparación con los microscópicos. Tal fórmula aún faltaba en la literatura.

Contents

1
Introduction

Black holes remain one of the most enigmatic objects in our universe. Interest from scientists and the public has grown within the last few years, due to the recent discovery of the black hole shadow detected by the Event Horizon Telescope: the first direct observation. Formed from the collapse of massive stars, black holes are regions of space with gravity so strong that light cannot escape. There is evidence that at the center of most galaxies, including our galaxy, there are supermassive black holes, possessing masses of $M \sim 10^6 - 10^{10}M$. There are even suggestions that primordial black holes could have been formed in the early universe. Black holes present many useful research opportunities, allowing us to probe extreme regions of general relativity (GR), as well as observing gravitational waves, which may help us better understand what truly is the nature of gravity. In 2015, the first observation of gravitational waves occurred, which were produced by the merging of two black holes [1], with additional discoveries made in the following years. These detections have been used to perform precision tests of GR [2–6], as well as to constrain the parameter space of its possible extensions [7–11].

One of the most important aspects of black holes, though, is that they present one of the few known regimes where GR and quantum field theory both play a significant role. While the gravitational effects of black holes can be described by GR, a true theory of quantum gravity is necessary in order to deal with various issues, such as the black hole information paradox, or the presence of gravitational singularities.

One particular aspect that has received attention in the past decades is black hole thermodynamics. It has been found that, classically, the geometric properties of the black hole horizon can be interpreted as thermodynamic properties. This allows a test to help determine a theory of quantum gravity, as any prospective theory should provide an explanation for the black hole entropy from a counting of microscopic states. One of the most prominent of these theories is string theory, in which particles are replaced by 1-dimensional strings, which we will examine in detail in 1.3.

Before we examine the quantum aspects of black holes and how they appear in string theory, we will begin discussing black holes in the most classical sense, as described by GR. We will first provide a general description of black holes (focusing on the simplest example of the Schwarzschild case), before examining the laws of black hole thermodynamics in detail.

1.1 Black Holes

Defined more rigorously, a black hole region $\mathcal{B}$ in an asymptotically-flat spacetime $(\mathcal{M}, g_{\mu\nu})$ is defined as the set of events from which outgoing null geodesics cannot reach future null infinity, $\mathscr{I}^+$. The event horizon, defined as the boundary of the black hole region $\mathcal{H} = \partial\mathcal{B}$, is a null hypersurface generated by null geodesics that have no future end points [12].

One further assumption that we shall make is that the black hole spacetime is stationary. This means that the metric $g_{\mu\nu}$ admits a one-parameter family of isometries generated by a Killing vector which is timelike in the asymptotic region. In this case, the rigidity theorems [13, 14] establish that the event horizon is a *Killing horizon*: a null hypersurface whose normal vector k^μ is a Killing vector of $g_{\mu\nu}$[1]. Consequently, the null generators of the horizon are given by the integral curves of k^μ, satisfying

$$k^\nu \nabla_\nu k^\mu = \kappa\, k^\mu\,, \tag{1.1}$$

on the horizon. κ can be defined as the *surface gravity* of the black hole. If $\kappa \neq 0$, then the Killing horizon contains a $(D2)$-dimensional spacelike cross section B on which the Killing field k^μ vanishes. B is then known as a bifurcation surface. The fact that κ is constant on the horizon $\mathcal{H}$ is known as the zeroth law of black hole thermodynamics (explained in section 1.1.1).

The simplest example of a black hole is the Schwarzschild black hole, a black hole possessing no rotation, electric, or magnetic charge. The line element of this case can be presented as (using the (+,-,-,-) signature)

$$ds^2 = (1 - \frac{R_s}{r})dt^2 - \frac{dr^2}{1 - \frac{R_s}{r}} - r^2 d\Omega^2, \tag{1.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and R_s is the Schwarzschild radius $R_s = 2GM$ (assuming natural units of $c = \hbar = 1$, where G is the Newton's constant).

It is noted that two singularities occur at $r = 0$ and $r = R_s$ in the above metric (a component blows up in each case). The former case is known as a *curvature singularity.* These singularities are properties of the spacetime itself, and as such are present in the metric regardless of which coordinate system we choose; they are physical singularities. The simplest way to see this is to check that at least one curvature invariant diverges there. In the case of the Schwarzschild solution, the simplest non-trivial curvature invariant is the Kretschmann invariant,

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2\cos^2\theta}{r^6} + \mathcal{O}(r^{-8}), \tag{1.3}$$

which also possesses a singularity at $r = 0$. The latter singularity at $r = R_s$ is a *coordinate singularity*, which can be removed from the metric by changing our coordinate system. In this case, $r = R_s$ coincides with the event horizon of the black hole. This singularity arises due to a poor choice of coordinates.

One of the best ways of removing this singularity is through the use of the Kruskal-Szekeres' coordinates. We wish for the two coordinates to be a linear combination of the

[1] A Killing vector is defined as a vector k_μ that satisfies the Killing equation, $\nabla_{(\mu}k_{\nu)} = 0$. This means the metric does not change along the integral curves of k^μ, and that the metric possesses an isometry along k^μ. This can also be expressed in terms of the Lie derivative $\mathcal{L}$ as $\mathcal{L}_k g_{\mu\nu} = \nabla_{(\mu}k_{\nu)} = 0$

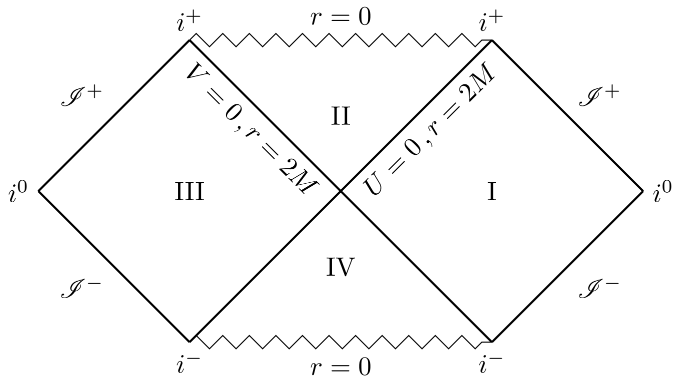

Figure 1.1: Penrose diagram of the extended Schwarzschild metric. Region I corresponds to the area outside the black hole, while region II corresponds to the inside of the black hole. Region IV is the white hole that arises in the new coordinates, and region III corresponds to an alternative universe.

temporal and spatial coordinates. Defining the functions U, V implicitly by

$$UV = (1 - \frac{r}{R_s})e^{r/R_s} \tag{1.4}$$

$$\frac{U}{V} = -e^{t/R_s}, \tag{1.5}$$

our metric now reads as

$$ds^2 = \frac{4R_s^3 e^{-r/R_s}}{r} dUdV - r^2 d\Omega^2. \tag{1.6}$$

Only the physical singularity at $r = 0$ remains in the metric. In addition, this new form of our metric gives us a wider spacetime patch. The original Schwarzschild coordinates only were valid for the region where $r > R_s$, the area outside the event horizon. Our maximally extended spacetime includes the region of $0 < r < R_s$, as well as two other regions, separated by the null hypersurfaces $V = 0$.

The best way to illustrate the black hole in the new coordinate system is through the use of a Penrose diagram, where infinities are brought into a finite distance through conformal transformations, as seen in figure 1.1. These transformations preserve light cones, so light propagates at 45° in the diagram. Region I corresponds to the black hole exterior, $r > R_s$. Region II corresponds to the region beyond the event horizon. Here, we can see that any observer in II will always be doomed to reach the singularity, which is a spacelike hypersurface instead of a timelike one. One notable consequence of the change in coordinates is the appearance of two additional regions, due to the range of values U, V can take. We can visualize this by picturing the constant r curves as hyperbolas with two branches, one occurring in Regions I/II, and the other in Regions III/IV. In the latter case, III can be described as an alternative universe, while IV takes the form of a white hole, where it is impossible for an object to enter, and everything inside is eventually dispersed. The event horizon consists of the case of the two null hypersurfaces: $U = 0$ and $V = 0$. These divide into the future and past event horizons, with the future (past) horizon occurring at $U = 0$ ($V = 0$) in Region I and at $V = 0$ ($U = 0$) in Region III. These null surfaces intersect at the *bifurcation 2-sphere*, where $k^\mu = 0$, which plays an important role in some black hole calculations. Explicitly, in Kruskal-Szekeres' coordinates, this Killing vector takes the form $k = \kappa(V\partial_V - U\partial_U)$.

1.1.1 Black Hole Thermodynamics

Black-hole thermodynamics originates in the analogy between the behaviour of the area of the event horizon A and the second law obeyed by the thermodynamic entropy S noticed by Bekenstein [15, 16] in the results obtained by Christodoulou and Hawking [13, 17, 18]. The non-decreasing area of the black hole as a function of time, known as the Area theorem, was first proposed by Hawking [19]. Shortly afterwards, Bardeen, Carter and Hawking [20] extended this by proving another two laws of black hole mechanics, as well as conjecturing a third, similar to the other three laws of thermodynamics involving the event horizon's surface gravity κ, the angular velocity Ω and angular momentum J, and the black hole's mass M. However, the analogy was only taken seriously after Hawking's discovery that black holes radiate as black bodies with a temperature $T = \kappa/2\pi$ [21], which implied the relation $S = A/4$ (when we work in $c = G_N = \hbar = k = 1$ units). These laws can be expressed in the following form

0. The surface gravity κ is constant across the event horizon. This was initially proven using the Einstein equations. Later on, it would be proven using the geometric properties of the event horizon without the Einstein equations by Racz and Wald [22]. This is analogous to the zeroth law of thermodynamic, which states that the temperature is constant through a body in thermal equilibrium.

1. The variation of the mass is equal to the variation of the area of the black hole horizon multiplied by the surface gravity, plus additional work terms: $\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J$, where M is the total spacetime energy computed from the Hamiltonian, Ω is the angular velocity of the horizon, and J is the angular momentum. This is similar to the standard first law: $\delta E = TdS + VdP +$ work terms. This is true as long as the black hole is stationary, axisymmetric, and asymptotically flat.

2. The area of the black hole horizon is a non-decreasing function of time, which is the area theorem proved by Hawking: $\delta A \geq 0$ assuming the weak energy condition: that for all non-spacelike vector fields k^a, $T_{ab}k^a k^b \geq 0$ for the stress tensor T_{ab}. This is analogous to the second law of thermodynamics, which states that the entropy of an isolated system will be given by $\Delta S \geq 0$. This suggests the association of entropy with the black hole area.

3. No method can reduce κ to zero in a finite time. This conjecture was proved later by Israel [23]. This is analogous to the third law in thermodynamics, which states that $T = 0$ cannot be reached in a finite number of steps.

Shortly after the publication of the black hole thermodynamic laws, Hawking discovered that quantum fluctations of the vacuum in the presence of black holes caused them to behave as black bodies, emitting a steady flux of radiation of temperature.

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi}\,, \tag{1.7}$$

This allowed the fixing of the constants in the Bekenstein Hawking entropy, utilizing the first law:

$$S_{\mathrm{BH}} = \frac{A_H}{4\hbar G_{\mathrm{N}}}\,. \tag{1.8}$$

The physical mechanism behind particle creation by black holes is analogous to the Schwinger pair production in strong electric fields [24]. In the case of black holes, pairs of virtual particles are created just outside the event horizon. One member of the pair has positive energy and escapes to infinity to become part of the Hawking radiation, while the other has negative energy and falls into the black-hole interior, to the region where it can exist as a real particle. The net effect is that the mass and the area of the black hole decrease, hence violating the second law of black-hole mechanics. Still, the evaporation process does not violate *the generalized second law of thermodynamics* [25], which states that the total entropy, i.e. the sum of the black-hole entropy and the entropy of the matter fields in the exterior region, never decreases [26]. One consequence of the particle emission is that black holes evaporate over time. Since the temperature is inversely proportional to the mass in the case of a Schwarzchild black hole, the black hole will get hotter as it evaporates.

Ever since the formulation of these four laws, attempts have been made to extend their original domain of application. Since the surface gravity relation to the Hawking temperature only depends on generic geometric properties of the event horizon, the quantity whose variation it multiplies in the first law is naturally associated to the entropy S. The Bekenstein-Hawking entropy however is derived from GR, and as such, is not necessarily valid in other cases.

In Refs. [22, 27, 28] Wald and collaborators developed a new approach based on the Noether charge, in order to demonstrate the first law of black hole mechanics in general diffeomorphism-invariant theories, beyond and including GR. They introduced the concept of the *Iyer-Wald entropy*, which in GR can be reduced to the Bekenstein-Hawking entropy $\frac{A}{4}$. However, in the presence of α' corrections in Superstring Theories (or any higher order curvature corrections in general), the entropy is no longer solely determined by the area [29–34].

The explicit form of this Iyer-Wald entropy formula is generally written as [22] (taking $\hbar$=1)

$$S = -2\pi \int_\Sigma d^{D-2}x\sqrt{|h|}\mathcal{E}_R^{abcd}\epsilon_{ab}\epsilon_{cd}, \tag{1.9}$$

where we define $|h|$ as the determinant of the metric on the bifurcation surface Σ, ϵ_{ab} is the binormal on the horizon defined such that $\epsilon_{ab}\epsilon^{ab} = -2$, and $\mathcal{E}_R^{abcd}$ represents the equivalent of the equation of motion for the Riemann tensor if it was treated as an independent variable:

$$\mathcal{E}_R^{abcd} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta R_{abcd}}. \tag{1.10}$$

When Iyer and Wald derived their entropy formula, they made a few assumptions, some of which we will address in detail later. Their derivation looked at Lagrangian theories on an n-dimensional oriented manifold M, with dynamic fields consisting of a Lorentz signature g_{ab}, along with other fields ψ. A major assumption that was made, which we shall discuss later, is that for simplicity, all fields ψ were assumed to be tensor fields on M. In addition, they were only interested in diffeomorphic invariant theories, and all fields and spacetimes were assumed to be smooth.

1.2 Conservation laws

In order to understand the Wald entropy, one must first understand the notion of symmetries and conserved charges.

Recall that for some action S written in the form $S = \int_\Sigma d^d x \mathcal{L}(\phi, \partial\phi, \partial^2\phi, \dots)$, the arbitrary infinitesimal variation of the field $\delta\phi$ takes the form

$$\delta S = \int_\Sigma d^d x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \delta \partial_\mu \partial_\nu \phi + \dots \right]. \tag{1.11}$$

By assumption, the variation of the coordinates is zero. This means that the derivatives commute with the variation of the field. We then must integrate by parts, and use Stokes theorem to express the integral of the total derivative as an integral over the boundary.

In theories without higher order derivatives, $\mathcal{L}(\phi, \partial\phi)$, if we impose the condition that the variation of the fields vanishes over the boundary, the boundary terms will vanish. However, if higher order derivatives are present, we require to either impose boundary conditions on the derivative of the variation of the field, or we must introduce boundary terms to the action that keep the equation of motion, but eliminate the $\partial\delta\phi$ terms in the total derivative. Once these are satisfied, and assuming that the action is stationary $\delta S = 0$, we arrive at the Euler-Lagrange formula, which is utilized to find the equations of motion:

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \right) + \partial^\nu \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \partial^\mu \partial^\nu \phi} \right) - \dots \tag{1.12}$$

From here, we will derive an extremely important concept of physics: the notion of a conserved charge. A conserved quantity remains invariant along the classical trajectories of a given dynamical system. For example, in an isolated system, energy is conserved. Mathematically, this can be represented by the continuity equation $\partial_\mu J^\mu = 0$. In this case, J^μ is called the conserved current, because it is used to define a quantity (charge) that is conserved in time: $Q = \int_\Sigma J$ for an arbitrary Cauchy surface Σ.

1.2.1 Noether's first theorem

Conserved charges can be related to the symmetries via Noether's theorems (see [35] for original formulation). Noether's first theorem states that for every global symmetry of an action, there exists an associated conserved charge. Global symmetry transformations apply the same transformation to each point in spacetime. In other words, for some infinitesimal transformation for coordinates and fields x, ϕ respectively, $\delta x^\mu \equiv \sigma^A \delta_A x^\mu$ and $\delta\phi = \sigma^A \delta_A \phi$, where $\delta_A x^\mu$ and $\delta_A \phi$ are given functions of the coordinates and ϕ, and σ^A, $A = 1, \dots, n$ are constant transformation parameters.

Generally, the variation of any action S which is a function of dynamical fields ϕ can be written in the form

$$\delta S = \int d^d x [\mathbf{E} \delta\phi + \partial_\mu \Theta^\mu(\phi, \delta\phi)], \tag{1.13}$$

where summation is understood over all fields. Here, $\mathbf{E}$ are the various equations of motion, and $\partial_\mu \Theta^\mu$ are the additional total derivative terms.

Let $\phi(x)$ be a general set of coordinates; then a transformation of the coordinates $\phi)(x)$ to new coordinates $\phi'(x)$ is a symmetry if the action functional does not change when we evaluate it in these two different sets of coordinates. We want to find the consequences of the invariance, possibly up to a total derivative that depends on the variations, of the action under the above infinitesimal changes of the field $\delta_s\phi$ (which are, then, symmetry transformations). Therefore, we define $\delta_s\phi(x)$ such that for any $\phi(x)$,

$$\begin{aligned}\delta_s S[\phi(x)] &\equiv S[\phi(x)+\delta_s\phi(x)] - S[\phi(x)] \\ &= \int d^d x \partial_\mu K^\mu. \end{aligned} \tag{1.14}$$

Note that we have not imposed the equations of motion.

We can now prove the first Noether theorem by utilizing the variation (1.14) and combining with (1.13). Since equation (1.13) holds for all variation $\delta\phi$, including $\delta_s\phi$, we can use the infinitesimal symmetry transformation $\delta_s\phi$ so that

$$\begin{aligned}\delta_s S &= \int d^d x(\delta_s\phi \mathbf{E} + \partial_\mu \Theta^\mu(\phi, \delta_s\phi)) \\ &= \int d^d x \partial_\mu K^\mu. \end{aligned} \tag{1.15}$$

Using the fact that the domain of integration is arbitrary, we can therefore subtract the equations and arrive at a total derivative term. This yields the conservation law

$$\partial_\mu J^\mu = \mathbf{E}\delta_s\phi \tag{1.16}$$

$$J^\mu \equiv -\partial_\mu \theta^\mu(\phi, \delta_s\phi) + K^\mu(\phi, \delta_s\phi). \tag{1.17}$$

J^μ is what is known as the Noether current, or the conserved current. In the onshell case (when the equations of motion are satisfied), then $\partial_\mu J^\mu = 0$. This is Noether's first theorem: that given a symmetry $\delta_s\phi(x)$, there must exist a conserved current J given by (1.16).

1.2.2 Noether's second theorem

By contrast, Noether's second theorem is applied to local symmetries, which depend on the given point of the manifold (for a more detailed discussion, see for example [36] and references therein). Consider some fields ϕ and a Lagrangian $\mathcal{L}(\phi)$. The generating set of non-trivial gauge symmetries of ϕ are given by $\delta_f \phi^i = R^i_\alpha(f^\alpha) = R^i_f$. Here, $R^i_\alpha = \sum_{k=0} R^{i(\mu_1 \dots \mu_k)}_\alpha \partial_{\mu_1} \dots \partial_{\mu_k}$ are operators, and f_α are arbitrary local functions of coordinates and fields. By definition of gauge symmetry, $R^i_\alpha(f^\alpha)$ must satisfy

$$R^i_\alpha(f^\alpha)\frac{\delta\mathcal{L}}{\delta\phi^i} = \partial_\mu J^\mu_f, \tag{1.18}$$

where $\frac{\delta\mathcal{L}}{\delta\phi^i}$ is the Euler-Lagrange derivative of $\mathcal{L}(\phi)$, and J^μ_f are a set of local functions. J^μ_f are simply the Noether currents associated to the symmetry δ_f. From this, we can arrive at Noether's second theorem, which states that for the Euler-Lagrange equations of motion, there exist associated offshell identities

$$R_\alpha^{i+}\frac{\delta L}{\delta\phi^i}=0 \tag{1.19}$$

$$R_\alpha^{i+}(Q_i)\equiv\sum_{k=0}(-1)^k\partial_{\mu_1}\dots\partial_{\mu_k}[R_\alpha^{i(\mu_1\dots\mu_k)}Q_i], \tag{1.20}$$

where Q_i are local functions. Our operator R_α^{i+} is obtained from the original operator R_α^i by integrating by parts and ignoring the boundary terms. Noether's second theorem also leads to conserved charges, as we shall see below.

We shall show how this theorem applies in our specific case and notation. We shall follow the calculations done in [28] and [22]. It is most useful in our case to convert the coordinate notation to differential forms, using the standard convention:

$$\omega^{(p)}=\frac{1}{p!}\omega_{\mu_1\dots\mu_p}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}. \tag{1.21}$$

In this case, the variation of our Lagrangian takes the d form $\delta\mathbf{L}=\mathbf{E}\delta\phi+d\mathbf{\Theta}(\phi,\delta\phi)$, while the second Noether's theorem then takes the form $\sum_{i=1}^k D^i\mathbf{E}_i=0$, where D^i are the differential operators and $\mathbf{E}_i$ are the equations of motion.

Consider some vector field ξ^a on a manifold M, as well as the field variation $\delta_\xi\phi=-\mathcal{L}_\xi\phi$. Here, we define $\mathcal{L}_\xi$ as the Lie derivative with respect to ξ. Due to the diffeomorphic invariance of $\mathbf{L}$, we can express the variation of our action as

$$\begin{aligned}\delta_\xi S&=\int\delta_\xi\mathbf{L}\\&=-\int d\imath_\xi\mathbf{L},\end{aligned} \tag{1.22}$$

where we utilize the Cartan relation

$$\mathcal{L}_\xi\mathbf{\Lambda}=\imath_\xi d\mathbf{\Lambda}+d(\imath_\xi\mathbf{\Lambda}), \tag{1.23}$$

where $\mathbf{\Lambda}$ is a generic differential form.[2]

Let us then consider the transformations of $\int\delta_\xi\mathbf{L}$:

$$\begin{aligned}\delta_\xi\mathbf{S}&=\int\delta_\xi\mathbf{L}\\&=\int[\mathbf{E}\delta_\xi\phi+d\mathbf{\Theta}].\end{aligned} \tag{1.24}$$

By integrating equation (1.24) by parts, as well as by employing the Noether identities, we arrive at an expression where only the total derivative does not vanish. It can therefore be written as

[2]In our conventions, for a p-form $\omega^{(p)}$ with components $\omega^{(p)}{}_{\mu_1\cdots\mu_p}$, $\imath_\xi\omega^{(p)}$ is the $(p-1)$-form with components $(\imath_\xi\omega^{(p)})_{\mu_1\cdot\mu_{p-1}}=\xi^\nu\omega^{(p)}{}_{\nu\mu_1\cdot\mu_{p-1}}$.

$$\int [\mathbf{E}\delta_\xi \phi + d\Theta[\xi]] = \int d\mathbf{\Theta}'[\xi]$$
$$= \delta_\xi S \tag{1.25}$$

Note that $\mathbf{\Theta}' = \mathbf{\Theta}$ onshell (when the equations of motion are satisfied).

By combining equations (1.22) and (1.25), we arrive at

$$\int d(\mathbf{\Theta}'[\xi] + \imath_\xi \mathbf{L}) = 0$$
$$\mathbf{\Theta}'[\xi] + \imath_\xi \mathbf{L} \equiv \mathbf{J}[\xi]. \tag{1.26}$$

This $n-1$ form $\mathbf{J}[\xi]$ is the Noether current. As this is independent of the equation of motion, $d\mathbf{J} = 0$ identically offshell and in the domain of integration. Since our current is closed for all ξ^a locally, then there must exist some $n-2$ form $\mathbf{Q}$ such that

$$\mathbf{J} = d\mathbf{Q}. \tag{1.27}$$

$\mathbf{Q}$ is known as the Wald-Noether charge relative to our vector field ξ^a.

1.2.3 Issues with the Iyer-Wald formula

In the presence of matter fields, Wald's proof of the first law of black-hole mechanics had to be re-examined because one of the main assumptions of Refs. [22,28] is that all matter fields behave as tensors and, simply put, there are no tensor fields in the Standard Model apart from the metric; all of them have some sort of gauge freedom and their transformations under diffeomorphisms are always coupled to gauge transformations. Indeed, as is well-known, fermionic fields coupled to gravity transform under the local Lorentz group as spinors and bosonic fields must transform under some gauge group if unwanted, typically negative-energy, states are to be eliminated. The only scalar in the Standard Model, the Higgs field, is, in fact, an SU(2) doublet.

The simplest matter field that, coupled to gravity, allows for black-hole solutions is the Maxwell field [37, 38]. The presence of this field introduces an additional term of the form ΦdQ in the first law which takes into account the changes in the mass of the black hole when its electric charge Q changes. In this term Φ is the electric potential on the horizon and a *generalized zeroth law* states that it takes a constant value over the horizon. The value of Φ is customarily taken to be $k^\mu A_\mu$, where k^μ is the Killing vector for which the event horizon is its associated Killing horizon and where it is assumed that the electromagnetic field is in a gauge in which Φ is, indeed, constant.

This definition of Φ is clearly not gauge-invariant. This is a problem of principle,[3] which, as we are going to show, is related to the more fundamental problem we were discussing: the fact that Wald's proof of the first law does not deal properly with fields which have some kind of gauge freedom. In Wald's proof, one considers diffeomorphisms

[3]There are other problems as well: in Wald's approach, the Noether charge, which contains a term in which Φ occurs, is evaluated over the bifurcation surface, but the Maxwell field of the Reissner-Nordström black hole turns out to be singular there in the traditional gauge [39].

which are symmetries of all the dynamical fields, but the naive definition of invariance of fields with gauge freedom under diffeomorphisms through the standard Lie derivative is not gauge invariant. This problem affects the gravitational field itself when it is described in terms of the Vielbein instead of the metric.

This problem was first noticed and solved by Jacobson and Mohd in Ref. [40] for the Einstein-Hilbert action written in terms of the Vielbein. The solution consists of going back to the basic formalism of [27, 28] and dealing carefully with the gauge (local Lorentz) symmetry. In practice, this means taking into account the gauge transformations induced by the diffeomorphisms on the Vielbein. This can be done, for instance, by defining a Lorentz-covariant Lie derivative (*Lie-Lorentz derivative*) which can be decomposed into a standard Lie derivative and a local Lorentz transformation. Apart from being covariant under local Lorentz transformations, this derivative vanishes identically when the diffeomorphism is an isometry of the metric (see Refs. [41, 42][4] which build on earlier work by Lichnerowicz, Kosmann and others [44–50]). Gauge-covariant derivatives arise naturally in the commutator of two local supersymmetry transformations and in the construction of Lie superalgebras of supersymmetric backgrounds [42, 48–50].

A more general mathematically rigorous approach was proposed in [51] using the formalism of principal gauge bundles which encompasses Yang-Mills and Lorentz fields but, unfortunately, not the Kalb-Ramond (KR) field or higher-rank form fields of string theory.[5] Perhaps the most interesting result in that paper is the realization that all the *zeroth-laws* (the constancy of the surface gravity, electric potential etc.) on the horizon fit a common pattern.

In Chapters 3-5, we shall make use of these covariant Lie derivatives, which will be constructed from momentum maps, which we will discuss in detail. We shall also illustrate how the Lie-Lorentz derivatives can be used to extend the proof of the first law of black hole mechanics to supergravity.

1.2.4 Momentum maps

One of the main ingredients in the proofs of the first law of black hole mechanics using Wald's formalism [22, 28] is the use of infinitesimal diffeomorphisms that leave invariant all the dynamical fields.

If we use the metric $g_{\mu\nu}$ as dynamical field, since the metric is just a tensor, its transformation under infinitesimal diffeomorphisms $\delta_\xi x^\mu = \xi^\mu(x)$ is given by (minus) the standard Lie derivative

$$\delta_\xi g_{\mu\nu} = -\pounds_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}\,, \tag{1.28}$$

which vanishes when ξ^μ is a Killing vector of $g_{\mu\nu}$, that we denote by k^μ.

If, as we want to do here, we use as dynamical field the Vielbein $e^a{}_\mu$ instead of $g_{\mu\nu}$, in order to define its symmetries, we face the well-known problem of the gauge freedom of $e^a{}_\mu$, which in this context has been treated in Refs. [40, 51]. The same happens with the electromagnetic potential A_μ, which also has been treated in this context in Refs. [51].

[4] See also Ref. [43] for a more mathematically rigorous point of view.

[5] The first law has been proved for theories including one scalar and one p-form field in [52], although the gauge-invariance problem has not been discussed in it.

One way to deal with this problem is to define a gauge-covariant notion of Lie derivative. The Lie derivative in the corresponding principal bundle, used in Ref. [51] provides the most rigorous definition of such a derivative. Here we will introduce a less sophisticated version that makes use of the so-called *momentum map* and which can be defined for more general fields such as the Kalb-Ramond 2-form of the Heterotic Superstring, which cannot be described in the framework of a principal bundle [53]. Due to its simplicity, we start with the Maxwell field.

The electromagnetic field A_μ is a field with gauge freedom: we must consider physically equivalent two configurations that are related by the gauge transformation

$$\delta_\chi A_\mu = \partial_\mu \chi \,, \tag{1.29}$$

and, furthermore, as a general rule, it is not possible to give a globally regular expression of the electromagnetic field in a single gauge.[6] However, the standard Lie derivative does not commute with these gauge transformations and gives different results in different gauges. This is why a gauge-covariant notion of Lie derivative is needed in this case.

In the subsequent discussion it is convenient to use differential-form language. In terms of the electromagnetic 1-form potential $A \equiv A_\mu dx^\mu$, we define the electromagnetic field strength 2-form by $F = dA$ so that it satisfies the Bianchi identity $dF = 0$. In components we have

$$F \equiv \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \,, \qquad F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \,. \tag{1.30}$$

The field strength is invariant under the gauge transformations $\delta_\chi A = d\chi$ and we can treat it as a standard 2-form whose transformation under infinitesimal diffeomorphisms generated by ξ^μ is given by (minus) the standard Lie derivative which, on p-forms, acts as $\pounds_\xi = \imath_\xi d + d\imath_\xi$. Using the Bianchi identity we find that

$$\delta_\xi F = -d\imath_\xi F \,. \tag{1.31}$$

If k generates a symmetry of all the dynamical fields, we have that $\delta_k F = 0$ and the above equation implies that, locally, there is a gauge-invariant function P_k called *momentum map* such that[7]

$$\imath_k F = -d\mathcal{P}_k \,. \tag{1.32}$$

$\mathcal{P}_k$ is defined by this equation up to an additive constant.

Let us now consider the variation of A under infinitesimal diffeomorphisms, which, according to general arguments (see *e.g.* Refs. [42,51]) has to be given locally by a combination of (minus) the Lie derivative and a "compensating" gauge transformation generated by a ξ-dependent parameter χ_ξ which is to be determined by demanding that $\delta_k A = 0$ when $\delta_k F = 0$:

$$\delta_\xi A = -\pounds_\xi A + d\chi_\xi = -\imath_\xi F + d\left(\chi_\xi - \imath_\xi A\right) \,. \tag{1.33}$$

Then, taking into account Eq. (1.32), we conclude that

[6]The main example of this situation is the magnetic monopole [54].
[7]The sign of $\mathcal{P}_k$ is purely conventional.

$$\chi_\xi = \imath_\xi A - \mathcal{P}_\xi\,, \tag{1.34}$$

where $\mathcal{P}_\xi$ is a function of ξ which satisfies Eq. (1.32) when $\xi = k$ and generates a symmetry of all the dynamical fields.

It is natural to identify the above transformation $\delta_\xi A$ with (minus) a gauge-covariant Lie derivative of A that we can call *Lie-Maxwell derivative*

$$\delta_\xi A = -\mathbb{L}_\xi A\,, \qquad \mathbb{L}_\xi A \equiv \imath_\xi F + d\mathcal{P}_\xi\,. \tag{1.35}$$

While this derivative does not enjoy the most important property of Lie derivatives $[\pounds_\xi, \pounds_\eta] = \pounds_{[\xi,\eta]}$ for generic vector fields ξ, η, it is clear that it does for those that generate symmetries of A, F and $g_{\mu\nu}$ and annihilates them. This is sufficient for our purposes.

For stationary asymptotically-flat black holes, when the Killing vector k is the one normal to the event horizon, the momentum map can be understood as the electric potential Φ which, evaluated on the horizon $\Phi_{\mathcal{H}}$, appears in the first law.[8] In the early literature (see *e.g.* Section 6.3.5 of Ref. [56]) it was assumed from the start that there is a gauge in which

$$\pounds_k A = \imath_k dA + d\imath_k A = 0\,. \tag{1.36}$$

Then, the electric potential Φ was identified with $\imath_k A$ because, according to the above equation, $d\Phi = -\imath_k F$, which can be defined as the electric field for an observer associated to the time direction defined by k.

It is clear that P_k can be identified with Φ (both satisfy the same equation). However, in a general gauge, it will not be given simply by $\imath_k A$ and we will have to compute it. Nevertheless, the main property of Φ, namely the fact that it is constant over the horizon (sometimes called *generalized zeroth law*) still holds because it is, actually, a property of $-\imath_k F$ based on the properties of k, the Einstein equations and the assumption that the energy-momentum tensor of the electromagnetic field satisfies the dominant energy condition.

For a more complex example, we can consider the KR field of the effective string action compactified on the torus, whose field strength is given by

$$H \equiv dB - \tfrac{1}{2}\mathcal{A}_I \wedge d\mathcal{A}^I, \tag{1.37}$$

where $\mathcal{A}_I$ consists of the Kaluza Klein and winding vectors:

$$\mathcal{A}^I \equiv \begin{pmatrix} A^m \\ B_m \end{pmatrix}, \qquad \mathcal{F}^I = d\mathcal{A}^I\,. \tag{1.38}$$

The $O(n,n)$ indices are raised as $\mathcal{A}_I = \Omega_{IJ}\mathcal{A}^J$, where Ω_{IJ} is the off-diagonal form of the $O(n,n)$ metric

$$(\Omega_{IJ}) \equiv \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}, \tag{1.39}$$

[8] See, for instance Ref. [55] for a proof of the first law in the context of 5-dimensional supergravity and the role that Φ plays in it.

It is convenient to start by considering the transformation of the 3-form field strength H defined in Eq.(1.37) under diffeomorphisms. We start by defining the gauge transformations that leave H and F^I invariant.

$$\delta_\chi \mathcal{A}^I = d\chi^I \,, \tag{1.40}$$

$$\delta B = (\delta_\Lambda + \delta_\chi) B = d\Lambda + \tfrac{1}{2}\chi_I d\mathcal{A}^I \,, \tag{1.41}$$

where $\chi^I(x)$ is an $\mathrm{O}(n,n)$ vector if scalar gauge parameters and $\Lambda = \Lambda_\mu(x)dx^\mu$ is a 1-form gauge parameter

Since H is gauge invariant, upon use of its Bianchi identity

$$\delta_\xi H = -\pounds_\xi H = -\imath_\xi dH - d\imath_\xi H = \imath_\xi \mathcal{F}_I \wedge \mathcal{F}^I - d\imath_\xi H \,. \tag{1.42}$$

When $\xi = k$, this expression must vanish by assumption, and we can use Eq. (1.32), which leads to the identity

$$\delta_\xi H = -d\left(\imath_k H + \mathcal{P}_{k\,I}\mathcal{F}^I\right) = 0\,, \tag{1.43}$$

which, in turn, implies the local existence of a gauge-invariant 1-form that we will also call a momentum map, satisfying

$$-\imath_k H - \mathcal{P}_{k\,I}\mathcal{F}^I = dP_k \,. \tag{1.44}$$

The KR momentum map plays a fundamental role in the definition of the variation of the KR 2-form B under diffeomorphisms which should be of the general form

$$\delta_\xi B = -\pounds_\xi B + \left(\delta_{\Lambda_\xi} + \delta_{\chi_\xi}\right) B\,, \tag{1.45}$$

where χ_ξ and Λ_ξ are scalar and 1-form parameters of compensating gauge transformations. They will generically depend on $\mathcal{A}^I$ and B as well as on ξ. ${\chi_\xi}^I$ has to be the same parameter used in the definition of the Lie-Maxwell derivative Eq. (1.34) and we just have to determine Λ_ξ. Now, the Maxwell and Lorentz cases suggest that we try

$$\Lambda_\xi = \imath_\xi B - P_\xi \,, \tag{1.46}$$

which leads to

$$\begin{aligned} \delta_\xi B &= -\pounds_\xi B + d(\imath_\xi B - P_\xi) + \tfrac{1}{2}\chi_{\xi\,I} d\mathcal{A}^I \\ &= -\left(\imath_\xi H + \mathcal{P}_{\xi\,I}\mathcal{F}^I + dP_\xi\right) + \tfrac{1}{2}\mathcal{A}_I \wedge \imath_\xi \mathcal{F}^I + \tfrac{1}{2}\mathcal{P}_{\xi\,I}\mathcal{F}^I \,. \end{aligned} \tag{1.47}$$

When $\xi = k$, though,

$$\delta_k B = d\left(\tfrac{1}{2}\mathcal{P}_{k\,I}\mathcal{A}^I\right) \,. \tag{1.48}$$

This is not zero but it can be absorbed into a redefinition of Λ_ξ:

$$\Lambda_\xi = \imath_\xi B - P_\xi - \tfrac{1}{2}\mathcal{P}_{k\,I}\mathcal{A}^I\,, \tag{1.49}$$

which gives the variation

$$\delta_\xi B = -\left(\imath_\xi H + \mathcal{P}_{\xi\,I}\mathcal{F}^I + dP_\xi\right) - \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I\,. \tag{1.50}$$

This form of the variation makes it evident that $\delta_k B = 0$, because $\delta_k \mathcal{A}^I = 0$ and because of the definition of the KR momentum map 1-form Eq. (1.44).

It remains to check that the vanishing of this variation is a gauge-invariant statement. Indeed, if we perform a gauge transformation in $\delta_\xi B$, taking into account that all the momentum maps and $\delta_\xi \mathcal{A}^I$ are gauge-invariant, we find

$$\delta_{\text{gauge}}\delta_\xi B = -\tfrac{1}{2}\delta_{\text{gauge}}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I\,, \tag{1.51}$$

which vanishes identically for $\xi = k$.

Much like the electrostatic potential, we find that since the field strengths are regular on the horizon,

$$\imath_k \mathcal{F}^I \stackrel{\mathcal{BH}}{=} 0\,, \tag{1.52a}$$

$$\imath_k H \stackrel{\mathcal{BH}}{=} 0\,. \tag{1.52b}$$

It is possible to prove the first law using Wald's formalism working on the bifurcation sphere $\mathcal{BH}$, where the Killing vector k associated to the horizon vanishes. This restricts the necessity of the proof to bifurcate horizon but, on the other hand, it makes it possible to carry out the proof of the first law using a more restricted form of the (generalized) zeroth laws which states the closedness of the electrostatic potential and its higher-rank generalizations on $\mathcal{BH}$. We shall illustrate this proof for our specific example in Chapter 4. In general, our calculations take place on $\mathcal{BH}$, so this restriction does not prove to be a problem.

1.3 String theory

String theory[9] is one of the leading candidates in the attempt to unify quantum field theory with GR. In this theory, point-like particles are replaced with one-dimensional "strings". These strings possess a Regge slope parameter α', which sets the fundamental length and mass of the theory, the string length l_s and string mass m_s, as $\alpha' = l_s^2 = m_s^{-2}$. Besides α', there is also a dimensionless string coupling constant g_s, defined as the vacuum expectation value of the dilaton: $g_s =< e^\phi >$. The spectrum of ordinary particles is then believed to emerge as the spectrum of different string vibrational modes, remarkably leading to a massless graviton, which is the particle that mediates gravitational interaction.

The most basic example of an action describing a free string in a d-dimensional curved background with a metric $g_{\mu\nu}$ is given by the Nambu-Goto action:

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_S d^2\xi\sqrt{|g_{ij}|}, \tag{1.53}$$

[9] For a review of string theory, see for example [57–59]

where $\xi^i : i = 0, 1$, are the worldsheet coordinates and g_{ij} is the induced metric on the worldsheet, given by

$$g_{ij} = g_{\mu\nu}(X)\partial_i X^\mu \partial_j X^\nu. \tag{1.54}$$

$X^\mu(\xi) : \mu = 0, \ldots, d-1$ are the spacetime coordinates of the string. It is convenient to introduce the string tension T, which is given as $\frac{1}{2\pi\alpha'}$.

The Nambu-Goto action is highly non-linear and therefore very difficult to quantize even in Minkowski space. Therefore, it is generally convenient to work with a theory that is quadratic in derivatives, by introducing a worldsheet metric γ_{ij}. This is known as the Polyakov action, and takes the form

$$S_{\mathrm{P}} = -\frac{T}{2}\int_W d^2\xi\, \sqrt{|\gamma|}\, \gamma^{ij} g_{\mu\nu}(X)\, \partial_i X^\mu \partial_j X^\nu. \tag{1.55}$$

The equation of motion of the worldsheet metric gives the vanishing of the energy-momentum tensor,

$$g_{\mu\nu}(X)\, \partial_i X^\mu \partial_j X^\nu - \frac{1}{2}\gamma_{ij}\gamma^{kl} g_{\mu\nu}(X)\, \partial_k X^\mu \partial_l X^\nu = 0\,, \tag{1.56}$$

which can be used to obtain the following (onshell) relation between the worldsheet metric and the pullback of the background metric:

$$\gamma_{ij} = \frac{2g_{ij}}{g_k{}^k}\,, \qquad \text{where} \qquad g_k{}^k = \gamma^{kl} g_{kl}\,. \tag{1.57}$$

If we substitute this solution into equation (1.55), we simply arrive at the Nambu-Goto action(1.53), showing that the two are classically equivalent.

In addition to being invariant under the worldsheet reparametrizations, the Polyakov action is also invariant under the following local scale transformations of the worldsheet metric, known as Weyl transformations:

$$\gamma_{ij} \to \Omega^2(\xi)\gamma_{ij}\,. \tag{1.58}$$

This symmetry has very important consequences, specifically in terms of the quantization of the Polyakov action: it allows one to gauge away the worldsheet metric completely.

It is also possible to add another Weyl-invariant term to the Polyakov action without an additional field: the Einstein-Hilbert term:

$$S_{\mathrm{Euler}} = -\frac{\phi_0}{4\pi}\int_W d^2\xi\, \sqrt{|\gamma|}\, R(\gamma) = \phi_0\chi. \tag{1.59}$$

However, this is a total derivative term, and as such does not change the classical equations of motion. This term is actually just ϕ_0 multiplied by a topological invariant χ, where $\chi = 2 - 2g - b - c$ is the Euler characteristic. Here, g is the genus, b the number of boundaries, and c the number of crosscaps (in the case of a non-orientable theory). This ϕ_0 term is used in the exponential for the coupling constant $g_s = e^{\phi_0}$. In particular, ϕ_0 is just the VEV of ϕ. To see this, consider the calculation of the string amplitudes, which are defined as path integrals over all embeddings X^μ and all worldsheet metrics γ_{ij} with given boundaries and boundary data that determine the string states that are scattered.

The boundary data are included as vertex operators in the path integral. Without vertex operators, we have vacuum amplitudes, given by the path integral

$$Z = \int DXD\gamma e^{-S_P - S_{\text{Euler}}}. \tag{1.60}$$

The sum over metrics can be decomposed into a sum of path integrals over worldsheets with given topologies. The topology of two-dimensional surfaces can be characterized completely by g, b, and c, which are combined into the Euler characteristic χ. Therefore, our result takes the form

$$Z = \sum_t (e^{\phi_0})^{-\chi(t)} \int DXD\gamma e^{-S_{P\Sigma_t}}, \tag{1.61}$$

where t stands for the given topologies and $\{\Sigma_t\}$ are the spaces of surfaces with topology t. The above sum can be understood as a perturbative expansion, where e^{ϕ_0} plays the role of the string coupling constant g.

Let us discuss the canonical quantization of a free bosonic string. To do this, we must first examine the boundary conditions of the bosonic string. The variation of the Polyakov action (1.55) yields the following boundary term

$$\int_{\partial W} dW^i \delta X^\mu \partial_i X^\nu g_{\mu\nu}\,, \tag{1.62}$$

which does not vanish for open strings. In order to make it vanish, one can impose Neumann (N) boundary conditions

$$\partial_i X^\mu|_{\partial W} = 0 \tag{1.63}$$

For a free open string, imposing the Neumann boundary conditions is equivalent to the case of no momentum flowing through the endpoints of the strings,

$$\partial_1 X^\mu|_{\xi^1=0,\ell} = 0\,. \tag{1.64}$$

Alternatively, one could impose Dirichlet (D) boundary conditions,

$$\delta X^\mu|_{\partial W} = c^\mu \tag{1.65}$$

where c^μ are constants. This explicitly breaks translation invariance.

These boundary conditions determine the different possible spectra. In a relativistic theory, the polarization states belong to representations of the little group, the subgroup of the Lorentz group preserving the particle momenta.[10] An analysis of the spectra of closed and open bosonic strings reveals that this only occurs in $d = 26$ spacetime dimensions, which is known as the *anomalous* dimension. We are generally interested in the lightest states of the spectra, which govern the low-energy dynamics. In the case of the closed string, the lightest (non-tachyonic) states are massless, and they fit into representations of SO(24), represented by fields that fit into representation SO(1, 25). These are a spin-2 state, the graviton, represented by a symmetric tensor field (metric) $g_{\mu\nu}$, a spin 1 state,

[10]The little group is SO($d-1$) for massive particles and SO($d-2$) for massless particles.

represented by the KR 2-form $B_{\mu\nu}$, and the dilaton, represented by a scalar field ϕ. In the open-string sector with NN boundary conditions (where both sides possess Neumann boundary conditions), the lightest state is also massless, a spin 1 state, represented by the vector field A_μ. In the case of a DD boundary condition (with both sides possessing Dirichlet boundary conditions), the mass of the lightest state is dependent upon the separation between the D-branes which the string endpoints are allowed to move. For example, if the boundary conditions are imposed on a single spacelike direction X with both ends of the string lying on the same hypersurface ($X|_{\xi_1=0} = X|_{\xi_1=\ell}$), the spectrum contains a massless vector field and a massless scalar. The scalar corresponds to the Goldstone boson, and is associated with the breaking of the translational invariance of the vacuum due to the presence of the D-brane. The vacuum expectation value of the boson gives the position of the brane in the x axis, and its profile describes fluctuation of the brane around this position.

However, in addition to the massless modes, there also arises a tachyonic scalar. These tachyons signal that the bosonic string theory is quantum-mechanically unstable. This can be solved by the addition of supersymmetry (though it should be noted that supersymmetry is not necessary do eliminate the tachyons). In addition, we want spacetime fermions to appear in the spectrum.

1.3.1 Superstring theory

The generalization of the Polyakov action, which is also invariant under local worldsheet supersymmetry transformations, is [60, 61]

$$S = -\frac{T}{2}\int_W d^2\xi\, e\left[\gamma^{ij}\partial_i X^\mu \partial_j X_\mu - i\overline{\psi}^\mu \not{D}\psi_\mu + \overline{\chi}_i \rho^j \rho^i (2\,\psi^\mu \partial_j X_\mu + \frac{1}{2}\chi_j \overline{\psi}^\mu \psi_\mu)\right], \quad (1.66)$$

where ψ^μ and χ_i are the worldsheet spinors, $e^\alpha{}_i$ is the vielbein and $\rho^i = \rho^\alpha\, e_\alpha{}^i$ are the two-dimensional gamma matrices. Due to the invariance of this action under super-Weyl transformations, it is possible to eliminate the vielbein and the gravitino χ_i completely. This gives rise to the Ramond-Neveu-Schwarz (RNS) action [62, 63],

$$S_{\text{RNS}} = -\frac{T}{2}\int_W d^2\xi \left(\eta^{ij}\partial_i X^\mu \partial_j X_\mu - i\overline{\psi}^\mu \not{\partial}\psi_\mu\right). \quad (1.67)$$

Varying with respect to the spinor, a non-trivial boundary term arises. In order to cancel this, we need to impose appropriate boundary conditions. The possibilities depend on whether we are considering open or closed strings, with each having either Ramond(R) or Neveu-Schwarz(NS) conditions, which correspond to .

String type	R	NS
Open	$\psi^\mu_L\vert_{\xi^1=0} = \psi^\mu_R\vert_{\xi^1=0}, \quad \psi^\mu_L\vert_{\xi^1=\ell} = \psi^\mu_R\vert_{\xi^1=\ell}$	$\psi^\mu_L\vert_{\xi_1=0} = \psi^\mu_R\vert_{\xi_1=0}, \quad \psi^\mu_L\vert_{\xi_1=\ell} = -\psi^\mu_R\vert_{\xi_1=\ell}$
Closed	$\psi^\mu_{L,R}\vert_{\xi_1=0} = \psi^\mu_{L,R}\vert_{\xi_1=\ell}$	$\psi^\mu_{L,R}\vert_{\xi_1=0} = -\psi^\mu_{L,R}\vert_{\xi_1=\ell}$

Table 1.1: Boundary conditions for RNS action

Here, $\psi^\mu_{R,L}$ denote the right- and left-moving components of the fields. Notice that in the case of the closed string, the boundary conditions for the left and right moving fields are determined independently of each other. As a result, there are four possible cases compared to the two cases of the open string: NSNS, NSR, RNS, and RR.

Superstring theories are Poincaré invariant only in the critical dimension $d = 10$. It is necessary to introduce the worldvolume fermion number F, defined modulo 2. The R and NS sectors are separated into $R_\pm$ and $NS_\pm$ subsectors with respect to the operator $e^{i\pi F}$. Then, consistency and the absence of tachyons require the combination of these subsectors (GSO projection) in very precise ways. A total of five different theories arise: Type IIA, Type IIB, Type I, and the two heterotic theories SO(32) and $E_8 \times E_8$. The first two preserve $\mathcal{N} = 2$ spacetime supersymmetry, and correspond to the non-chiral (IIA) and chiral (IIB) theories. By contrast, the other theories only preserve $\mathcal{N} = 1$ supersymmetry. The heterotic string is composed of right moving fields of a type II superstring, with a left-moving fields of a closed bosonic string (propagating in 26 dimensions). The 26 dimensions that normally appear in the bosonic theories are compactified down to ten dimensions, with the 16 compactified spacetime dimensions giving rise to the gaugini χ^A and vector fields A^A. The difference between the two heterotic theories depends on the gauge group, and the two are related to each other via T duality. Finally, the type I string theory also possesses the vector fields, but is constructed differently, consisting of unoriented strings as well as both closed and open strings, and also possesses the SO(32) gauge group. A summary of the massless excitations of the theories can be seen in Table 1.2.

Theory	NSNS	RR	Chiral fermions	Non-chiral fermions	Vector multiplets
Type IIA	$g_{\mu\nu}, B_{\mu\nu}, \phi$	${C^{(1)}}_\mu, {C^{(3)}}_{\mu\nu\rho}$		ψ_μ, λ	
Type IIB	$g_{\mu\nu}, B_{\mu\nu}, \phi$	$C^{(0)}, {C^{(2)}}_{\mu\nu}, {C^{(4)}}_{\mu\nu\rho\sigma}$	ψ^i_μ, λ^i		
Type I	$g_{\mu\nu}, \phi$	${C^{(2)}}_{\mu\nu}$	ψ_μ, λ		A^A, χ^A
Heterotic	$g_{\mu\nu}, B_{\mu\nu}, \phi$		ψ_μ, λ		A^A, χ^A

Table 1.2: Massless excitations of various superstring theories.

1.3.2 Effective string action

It is helpful to examine the low energy limit, which corresponds to $\alpha' \to 0$, the limit where strings become infinitely small, and $T \to \infty$. These effective theories are useful, as this scenario corresponds to point-like particles, so a field theory must be recovered. In addition, the effective theory corresponds only to the massless modes, as the massive modes decouple from the low energy dynamics, due to their masses being given as proportional to $\frac{1}{\sqrt{\alpha'}}$. To determine the effective field theories, one traditionally constructs a field theory that reproduces the string amplitudes as $\alpha' \to 0$. Higher order terms of α' are occasionally used for additional corrections, they are higher order derivative terms, though often these are dropped to the lowest order, as the complexity grows rapidly at higher order.

The orthodox procedure to find these effective actions would be to construct a field theory reproducing the string amplitudes in the $\alpha' \to 0$ limit; however, there are other approaches which ultimately yield the same result. A particularly interesting one, which reflects how crucial conformal invariance is in string theory, consists of coupling a string to general background fields and studying which conditions must be satisfied by the latter in order to preserve conformal invariance at the quantum level.

To visualize this, consider a closed bosonic string, whose coupling to the background fields $g_{\mu\nu}$ $B_{\mu\nu}$, and ϕ (the graviton, the KR form, and the dilaton respectively) is given by the generalized Polyakov action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{|\gamma|}\left\{\left[\gamma^{ij}g_{\mu\nu}(X) - \frac{\epsilon^{ij}B_{\mu\nu}(X)}{\sqrt{|\gamma|}}\right]\partial_i X^\mu \partial_j X^\nu + \alpha'\phi(X)R(\gamma)\right\} . \quad (1.68)$$

The conditions under which conformal invariance is preserved were studied in [64], where it was shown that they boil down to the vanishing of the following β-functionals:[11]

$$\beta^g_{\mu\nu} = \alpha'\left(R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma}\right) + \mathcal{O}(\alpha'^2)\,, \quad (1.69)$$

$$\beta^B_{\mu\nu} = \frac{\alpha'}{2}e^{2\phi}\nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right) + \mathcal{O}(\alpha'^2)\,, \quad (1.70)$$

$$\beta^\phi = -\frac{\alpha'}{2}\left(\nabla^2\phi - (\partial\phi)^2 - \frac{1}{4}R - \frac{1}{48}H^2\right) + \mathcal{O}(\alpha'^2)\,, \quad (1.71)$$

where $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the 3-form field strength of the Kalb-Ramond 2-form $B_{\mu\nu}$. At leading order, this is equivalent to the equations of motion that can be derived from

$$S = \frac{g^2}{16\pi G_N^{(d)}}\int d^dx\,\sqrt{|g|}\,e^{-2\phi}\left[R - 4\,(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2\right]\,, \quad (1.72)$$

with the addition of a term

$$\frac{g^2}{16\pi G_N^{(d)}}\int d^dx\,\sqrt{|g|}\;e^{-2\phi}[-2(d-2)\Lambda] \qquad \Lambda = \frac{2(d-26)}{3\alpha'(d-2)}, \quad (1.73)$$

which vanishes at $d = 26$ for bosonic string theories. Thus, we see that quantum conformal invariance leads (in the critical dimension) to the same effective action for the string common sector.

This can also be expanded to higher orders of α'. If $\omega^{ab} = \omega_\mu{}^{ab}dx^\mu$ is the Levi-Civita spin connection,[12] we define the zeroth-order torsionful spin connections[13]

$$\Omega^{(0)}_{(\pm)\,ab} = \omega_{ab} \pm \tfrac{1}{2}\imath_b\imath_a H^{(0)}\,, \quad (1.74)$$

and their corresponding zeroth-order curvature 2-forms and Chern-Simons 3-forms

$$R^{(0)}_{(\pm)}{}^{ab} \equiv d\Omega^{(0)}_{(\pm)}{}^{ab} - \Omega^{(0)}_{(\pm)}{}^{a}{}_{c}\wedge\Omega^{(0)}_{(\pm)}{}^{cb}\,, \quad (1.75a)$$

$$\omega^{(L)(0)}_{(\pm)} = R^{(0)}_{(\pm)}{}^{a}{}_{b}\wedge\Omega^{(0)}_{(\pm)}{}^{b}{}_{a} + \tfrac{1}{3}\Omega^{(0)}_{(\pm)}{}^{a}{}_{b}\wedge\Omega^{(0)}_{(\pm)}{}^{b}{}_{c}\wedge\Omega^{(0)}_{(\pm)}{}^{c}{}_{a}\,. \quad (1.75b)$$

[11] As recently showed in [65], these are a set of sufficient but not necessary conditions.

[12] If $e^a = e^a{}_\mu dx^\mu$ are the Vielbein, the spin connection is defined to satisfy the Cartan structure equation $\mathcal{D}e^a \equiv de^a - \omega^a{}_b\wedge e^b = 0$.

[13] We denote by $\imath_a A$ the inner product of $e_a \equiv e_a{}^\mu\partial_\mu$ ($e_a{}^\mu e^b{}_\mu = \delta^a{}_b$) with the differential form A. If A is a p-form with components $A_{\mu_1\cdots\mu_p}$, $\imath_a A$ is the $(p-1)$ form with components $e_a{}^\nu A_{\nu\mu_1\cdots\mu_{p-1}}$.

$\omega^{(L)(0)}_{(\pm)}$ is known as the Lorentz Chern-Simon 3-form at zeroth order.

When we expand the $d = 26$ bosonic string theory to this higher order, the action will take the form (ignoring the prefactor for convenience) [14]

$$\begin{aligned} S = \int d^dx\,\sqrt{|g|}\,e^{-2\phi}\Bigg[&R - 4\,(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 \\ &-\frac{\alpha'}{8}\left(R_{(-)\mu\nu ab}R_{(-)}{}^{\mu\nu ab} + R_{(+)\mu\nu ab}R_{(+)}{}^{\mu\nu ab}\right) + \mathcal{O}(\alpha'^2)\Bigg]\,, \end{aligned} \tag{1.76}$$

where now the 3-form field strength is defined as

$$H = dB + \frac{\alpha'}{4}\left(\omega^{\rm L}_{(-)} - \omega^{\rm L}_{(+)}\right)\,. \tag{1.77}$$

When we examine superstring theories, the effective actions are fixed by spacetime-supersymmetry up to redefinitions, which strongly limits the possibilities. In the case of $d = 10$, the possibilities vary depending on if $\mathcal{N} = 1, 2$. For theories with $N = 2$ supersymmetry, there are two possibilities: the so-called $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$ [68–71], which describe the low-energy dynamics of type IIA and type IIB theories respectively at lowest order. Then there is the $\mathcal{N} = 1$ supergravity theory, which describes both the low-energy effective actions of heterotic and type I theories. In this theory, the supergravity multiplet consists of the vielbein $e^a{}_\mu$, the dilaton ϕ, the gravitino ψ_μ, the dilatino λ and a 2-form, which can be either the KR 2-form $B_{\mu\nu}$ or the RR 2-form $C^{(2)}{}_{\mu\nu}$, whose main difference at the level of the low-energy action lies on the coupling to the dilaton. This supergravity multiplet can be consistently coupled to a Yang-Mills vector multiplet, which contains a vector field $A^A{}_\mu$ and a gaugino χ^A needed for the heterotic and type I superstring effective actions.

Normally, $\mathcal{N} = 1$ supergravity possesses gauge and gravitational anomalies. However, it has been shown [72] that in the case of the SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8$ gauge groups, these can be cancelled by adjusting our definitions. For this work, we will focus solely on the bosonic portion of the heterotic string.

Making the necessary field redefinitions, the Heterotic Superstring effective action can be described at first order in α' as follows [73]:[15] we start by defining the zeroth-order Kalb-Ramond (KR) field strength $H^{(0)}$ and its components $H^{(0)}{}_{\mu\nu\rho}$ as

$$H^{(0)} \equiv dB = \tfrac{1}{3!}H^{(0)}_{\mu\nu\rho}dx^\mu \wedge dx^\mu \wedge dx^\rho\,, \tag{1.78}$$

where $B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\mu$ is the KR 2-form potential. Next, we define the gauge field strength 2-form and the Chern-Simons 3-form for the YM field $A^A = A^A{}_\mu dx^\mu$ by

$$F^A \;=\; dA^A + \tfrac{1}{2}f_{BC}{}^A A^B \wedge A^C\,, \tag{1.79}$$

$$\omega^{\rm YM} \;=\; F_A \wedge A^A - \tfrac{1}{6}f_{ABC}A^A \wedge A^B \wedge A^C\,, \tag{1.80}$$

[14]See also [66, 67].

[15]We use the conventions of Ref. [42], reviewed for the zeroth-order case in Ref. [74]. In particular, the relation with the fields in Ref. [73] can be found in Ref. [75].

where we have lowered the adjoint group indices $A, B, C, \ldots$ in the structure constants $f_{AB}{}^C$ and gauge fields using the Killing metric.

Then, we can define the first-order KR field strength 3-form as

$$H^{(1)} \equiv H^{(0)} + \frac{\alpha'}{4}\left(\omega^{\mathrm{YM}} + \omega^{(L)(0)}_{(-)}\right) , \tag{1.81}$$

Where $\omega^{(L)(0)}_{-}$ is the Lorentz Chern Simon 3-form defined in (5.3b). Its Bianchi identity takes the well-known form

$$dH^{(1)} = \frac{\alpha'}{4}\left(F_A \wedge F^A + R_{(-)}^{(0)}{}^{a}{}_{b} \wedge R_{(-)}^{(0)}{}^{b}{}_{a}\right) . \tag{1.82}$$

Having made these definitions and adding the dilaton field ϕ, we can write the Heterotic Superstring effective action to first-order in α' as

$$\begin{aligned} S^{(1)}[e^a, B, A^A, \phi] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int e^{-2\phi} \Big[& (-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} - 4 d\phi \wedge \star d\phi \\ & + \tfrac{1}{2} H^{(1)} \wedge \star H^{(1)} + (-1)^d \frac{\alpha'}{4}\left(F_A \wedge \star F^A + R_{(-)}^{(0)}{}^{a}{}_{b} \wedge \star R_{(-)}^{(0)}{}^{b}{}_{a}\right)\Big] \\ \equiv & \int \mathbf{L}^{(1)} . \end{aligned} \tag{1.83}$$

1.3.3 Dualities

While initially thought to be distinct from each other, it was discovered that the five superstring theories were in fact related to each other via various dualities, suggesting they could be just different limits of the same theory.

The first duality, S-duality, is a strong-weak coupling duality, which relates a theory with a coupling constant g_s to a theory with a coupling $\frac{1}{g_s}$. By definition, these are non-perturbative, and as a result, their existence was inferred by the properties of the effective action and the non-perturbative states. One example is the IIB theory: it possesses a global symmetry $\mathrm{SL}(2, \mathbb{R})$ that is broken to $\mathrm{SL}(2, \mathbb{Z})$ by quantum effects [76]. In IIB, the complex field $\tau = C^{(0)} + ie^{-\phi}$ (for a RR 0-form $C^{(0)}$ and a dilaton ϕ) transforms nonlinearly under $\mathrm{SL}(2, \mathbb{R})$. Taking $C^{(0)} = 0$, we see that the transformation $\tau \to -\frac{1}{\tau}$ changes $\phi \to -\phi$. This corresponds to the coupling constant g_s being inverted. Another example relates the Type I theory to the SO(32) heterotic theory.

The second duality, and the one which we shall focus on, is the T duality (see for example [77] for a review). It corresponds to the symmetry of the perturbative spectrum, exchanging the winding and momentum modes. The simplest example of this involves the closed bosonic string where one spacetime coordinate $X^{d-1} \equiv Z$ is compactified on the circle: $Z \sim Z + 2\pi R_z$. This results in two different modes: the momentum modes, and the winding modes.

The momentum modes, also known as the Kaluza-Klein modes, are present in field theory, and are inversely proportional to the size of the internal dimension. The spatial momentum of the string in the circle direction is constrained to these modes as $p = \frac{n}{R_z}$ for an integer n. The winding modes by contrast are purely stringy effects, which corresponds to the ability of closed strings to be wrapped in the compactified dimension. When we go around a string once, $\xi^1 \to \xi^1 + 2\pi l$, we wind w times around the compact dimension. These two new modes modify the mass operator and the level matching constraint as

$$M^2 = \frac{n^2}{R_z^2} + \frac{R_z^2 w^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right), \qquad \text{with} \qquad N = \tilde{N} + nw\,, \tag{1.84}$$

where N and $\tilde{N}$ are the *level operators*. This shows, for example, that for a string with $n > 0$ units of momentum gain a mass contribution of $\frac{n}{R_z}$. It is straightforward to check that (1.84) is invariant under the following transformations

$$n \to n' = w\,, \qquad w \to w' = n\,, \qquad R_z \to R_z' = \frac{\alpha'}{R_z}\,. \tag{1.85}$$

This is actually a symmetry of the full spectrum which, furthermore, has been proven to hold at all orders in perturbation theory [78]. It turns out that it is related to the invariance of the Polyakov action under Poincaré dualization of the embedding coordinate Z, see e.g. [42] and references therein.

We are mostly interested in the manifestation of T-duality at the level of the effective action. In order to study it, we first need to introduce the basics of the Kaluza-Klein (KK) dimensional reduction [79, 80]. The original idea was to unify gravity and electromagnetism by assuming that the spacetime has an extra dimension so that both four-dimensional spacetime and gauge symmetries arise from spacetime symmetries in five dimensions. Although abandoned for its original purpose, this theory remains an extraordinarily powerful tool in the general context of theoretical physics and particularly in string theory, where it is crucial in order to make contact with the four-dimensional world that we experience.

We will follow the modern Scherk-Schwarz formalism [81], which makes use of the vielbein and which is therefore well adapted to describe the dimensional reduction of theories with fermionic degrees of freedom, such as supergravity theories. We will always assume that none of the fields depend on the coordinate $z \sim z + 2\pi R_z$ that parametrizes the compact dimension $\mathbb{S}^1_z$.

As an example, we shall now carry out the dimensional reduction of the Einstein-Hilbert action. We start by decomposing the (d)-dimensional vielbein, $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, and its inverse, $\hat{e}_{\hat{a}}{}^{\hat{\mu}}$, in terms of the lower-dimensional fields as follows, noting that by using local Lorentz rotations, we can always choose a vielbein expressed in an upper triangular form[16]

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & kA_\mu \\ 0 & k \end{pmatrix}, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_a \\ 0 & k^{-1} \end{pmatrix}, \tag{1.86}$$

where $A_a = e_a{}^\mu A_\mu$ and k is the KK scalar. The latter measures the radius of the circle $\mathbb{S}^1_z$ as a function of the non-compact coordinates x^μ:

[16] The d-dimensional fields and indices will be denoted with hats. Then, we have $\hat{a} = (a, z)$ for flat indices and $\hat{\mu} = (\mu, \underline{z})$ for world indices.

$$R(x) = \frac{1}{2\pi}\int_0^{2\pi R_z} dz\sqrt{|g_{\underline{zz}}|} = R_z k(x)\,. \tag{1.87}$$

Our choice of vielbein breaks the $\hat{d}$-dimensional Lorentz invariance into a $d = \hat{d} - 1$ dimensional one.

We can express this using the Palantini identity

$$\int d^{\hat{d}}x\,\sqrt{|\hat{g}|}K\hat{R} = \int d^{\hat{d}}x\,\sqrt{|\hat{g}|}K\left[-\hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}}\hat{\omega}_{\hat{c}}{}^{\hat{c}}{}_{\hat{a}} - \hat{\omega}_{\hat{a}}{}^{\hat{b}\hat{c}}\hat{\omega}_{\hat{b}\hat{c}}{}^{\hat{a}} + 2\hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}}\partial_{\hat{a}}\log K\right]\,, \tag{1.88}$$

In order to apply it, we must first compute the spin connection. In the vielbein basis we have chosen, one obtains

$$\hat{\omega}_{abc} = \omega_{abc}, \qquad \hat{\omega}_{abz} = \frac{1}{2}kF_{ab}, \qquad \hat{\omega}_{zbc} = -\frac{1}{2}kF_{bc}, \qquad \hat{\omega}_{zbz} = -\partial_b \ln k \tag{1.89}$$

where $F_{ab} = 2\nabla_{[a}A_{b]}$ is the field strength of the KK vector field $A_a = e_a{}^{\mu}A_{\mu}$.

Then, making use of (1.88), one obtains

$$\begin{aligned} S_{\text{EH}} =& \frac{1}{16\pi G_{\text{N}}^{\hat{d}}}\int z\int d^{\hat{d}-1}x\,\sqrt{|g|}\,k\left[-\omega_b{}^{ba}\omega_c{}^c{}_a - \omega_a{}^{bc}\omega_{bc}{}^a + 2\omega_b{}^{ba}\partial_a\log k - \frac{1}{4}k^2F^2\right] \\ =& \frac{2\pi l}{16\pi G_{\text{N}}^{(\hat{d})}}\int d^{\hat{d}-1}x\sqrt{|g|}\,k\left[R - \frac{1}{4}k^2F^2\right]\,. \end{aligned} \tag{1.90}$$

In the general case in which a $(d+n)$-dimensional manifold $\mathcal{M}^{(d+n)}$ contains a n-dimensional compact space $\mathcal{C}^{(n)}$, the relation between the Newton's constants is

$$G_{\text{N}}^{(d)} = \frac{G_{\text{N}}^{(d+n)}}{V_n}\,, \tag{1.91}$$

where V_n is the volume of $\mathcal{C}^{(n)}$.

Rewriting the action (1.90) in terms of the metric in the Einstein frame, $g_{\text{E}\,\mu\nu} = k^{\frac{2}{d-2}}g_{\mu\nu}$,[17]

$$S_{\text{EH}} = \frac{1}{16\pi G_{\text{N}}^{(d)}}\int d^dx\sqrt{|g_{\text{E}}|}\,\left[R_{\text{E}} + \frac{d-1}{d-2}(\partial\log k)^2 - \frac{1}{4}k^{\frac{2(d-1)}{d-2}}F^2\right]\,, \tag{1.92}$$

we clearly see that the KK scalar is dynamical and cannot be truncated to a fixed value without imposing the corresponding constraint derived from its equation of motion ($F^2 = 0$ in this case).

[17] The Einstein frame is the one in which there is no conformal factor multiplying the Ricci scalar.

Once we know how to reduce the Einstein-Hilbert term, the last piece of information needed is to learn how to reduce p-forms. The dimensional reduction of a p-form $\hat{C}^{(p)}{}_{\hat{\mu}_1\dots\hat{\mu}_p}$ on a circle gives raise to a p-form $C^{(p)}{}_{\mu_1\dots\mu_p}$ and to a $(p-1)$-form $C^{(p-1)}{}_{\mu_1\dots\mu_{p-1}}$ in d dimensions:

$$
\begin{aligned}
\hat{C}^{(p)}{}_{\mu_1\dots\mu_p} &= C^{(p)}{}_{\mu_1\dots\mu_p} + p\, A_{[\mu_1} C^{(p-1)}{}_{\mu_2\dots\mu_p]}\,, \\
\hat{C}^{(p)}{}_{\mu_1\dots\mu_{p-1}\underline{z}} &= C^{(p-1)}{}_{\mu_1\dots\mu_{p-1}}\,.
\end{aligned}
\tag{1.93}
$$

This is, however, subject to field redefinitions. In the case of the Kalb-Ramond 2-form, we find convenient to define

$$
\hat{B}_{\mu\nu} = B_{\mu\nu} - A_{[\mu}B_{\nu]}\,, \qquad \text{with} \qquad B_\mu = \hat{B}_{\mu\underline{z}}\,, \tag{1.94}
$$

where B_μ is the *winding* vector.

We can begin by defining the lower-dimensional dilaton,

$$
\phi = \hat{\phi} - \frac{1}{2}\log k\,, \tag{1.95}
$$

Using Eq. (1.94) and the dimensional reduction of the Einstein-Hilbert term, one finds that the dimensional reduction on a circle of the effective action of the closed bosonic string (1.72) is

$$
S \sim \int d^dx\,\sqrt{|g|}\,e^{-2\phi}\left[R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 + (\partial \log k)^2 - \frac{1}{4}k^2F^2 - \frac{1}{4}k^{-2}G^2\right]\,, \tag{1.96}
$$

where $G^{(0)}_{\mu\nu} = 2\partial_{[\mu}B^{(0)}_{\nu]}$ is the field strength of the winding vector, and H is the KR field strength written as

$$
H^{(0)}{}_{\mu\nu\rho} = 3\partial_{[\mu}\hat{B}^{(0)}{}_{\nu\rho]} - \tfrac{3}{2}A_{[\mu}G^{(0)}{}_{\nu\rho]} - \tfrac{3}{2}B^{(0)}{}_{[\mu}F_{\nu\rho]}\,. \tag{1.97}
$$

As one can easily check, the action is invariant under the following transformations

$$
A_\mu \to A'_\mu = B_\mu\,, \qquad B_\mu \to B'_\mu = A_\mu\,, \qquad k \to k' = k^{-1}\,, \tag{1.98}
$$

which expressed in terms of the higher-dimensional fields lead to

$$
\begin{aligned}
\hat{g}'_{\underline{zz}} &= 1/\hat{g}_{\underline{zz}}\,, & \hat{B}'_{\mu\underline{z}} &= \hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, \\
\hat{g}'_{\mu\underline{z}} &= \hat{B}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, & \hat{B}'_{\mu\nu} &= \hat{B}_{\mu\nu} + 2\hat{g}_{[\mu|\underline{z}}\hat{B}_{\nu]\underline{z}}/\hat{g}_{\underline{zz}}\,, \\
\hat{g}'_{\mu\nu} &= \hat{g}_{\mu\nu} - (\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}} - \hat{B}_{\mu\underline{z}}\hat{B}_{\nu\underline{z}})/\hat{g}_{\underline{zz}}\,, & e^{-2\hat{\phi}'} &= e^{-2\hat{\phi}}|\hat{g}_{\underline{zz}}|\,.
\end{aligned}
\tag{1.99}
$$

These are known as the Buscher rules [82] [83], and they show that from the perspective of string theory, two backgrounds related by T duality are equivalent, and are both solutions to the classical string effective action, if one of them is only at $\mathcal{O}(1)$ in α'.

While these rules are derived for two backgrounds in the same (bosonic) theory, this does not necessarily need to be the case. For example, these rules can be generalized to relate type IIA and type IIB superstring theories [84].

In our case, we are interested in the heterotic string case. At zeroth order of α', the Buscher rules are identical to that of the bosonic string; we need to modify the rules to take into account the α' corrections. The specific rules for the higher order heterotic string will be discussed in later chapters (see [85] as well).

1.4 Summary of Thesis

We will end this introduction with a brief summary of the chapters and their most important results.

Part I: T-duality and dimensional reduction of the heterotic string on S^1

In the first part, consisting of Chapter 2 (based on paper [33]), we will examine one method of calculating the entropy of the heterotic string effective action up to order α', through dimensional reduction of the action on S^1.

This is achieved through the use of the Scherk-Schwarz formalism, where we split the world indices and field indices into the compactified and remaining dimensions through use of the vielbein. We begin by revisit the dimensional reduction on a circle of the action at zeroth order in α' as a warm-up exercise and also because we will need some of the results when we consider the higher-order terms. We will show that the compactified action takes the form (same as equation (1.96))

$$S = \frac{g_s^2(2\pi\ell_s)}{16\pi G_N^{(10)}} \int d^9x\sqrt{|g|}\, e^{-2\phi} \left\{R - 4(\partial\phi)^2 + (\partial\log k)^2 - \tfrac{1}{4}k^2F^2 - \tfrac{1}{4}k^{-2}G^{(0)\,2} + \tfrac{1}{12}H^{(0)\,2}\right\} . \tag{1.100}$$

We will also determine the T-duality transformations and corresponding Buscher rules (see equation (2.29)).

This will then be followed by the dimensionally-reduced action to first order in α'. This reduction requires additional calculations due to the presence of the Chern-Simons terms. From this reduced action, we will recover the T duality rules found in Janssen et al. [85] and we will prove the invariance of the action under those T duality rules. The main difference between this work and [85] is that in the latter, the complete dimensionally reduced action was not given.

The main result of this chapter illustrates that by dimensionally reducing the T duality-invariant action, it is possible to derive Iyer-Wald entropy for the heterotic version of the α'-corrected Strominger-Vafa black hole of Ref. [29], given by the equations

$$d\hat{s}^2 = \frac{2}{\mathcal{Z}_-}du\left(dv - \tfrac{1}{2}\mathcal{Z}_+du\right) - \mathcal{Z}_0(d\rho^2 + \rho^2 d\Omega^2_{(3)}) - dy^i dy^i\,, \qquad i = 1,\ldots,4\,, \tag{1.101a}$$

$$\hat{H}^{(1)} = d\mathcal{Z}_-^{-1} \wedge du \wedge dv + \star_4 d\mathcal{Z}_0\,, \tag{1.101b}$$

$$e^{-2\hat{\phi}} = e^{-2\hat{\phi}_\infty}\mathcal{Z}_-/\mathcal{Z}_0\,, \tag{1.101c}$$

where $\star_4$ stands for the Hodge dual in the 4-dimensional Euclidean space with metric $d\rho^2 + \rho^2 d\Omega^2_{(3)}$, and where the $\mathcal{Z}$ functions take the values[18]

$$\mathcal{Z}_0 = 1 + \frac{\tilde{q}_0}{\rho^2} - \alpha'\frac{\rho^2 + 2\tilde{q}_0}{(\rho^2 + \tilde{q}_0)^2} + \mathcal{O}(\alpha'^2)\,, \tag{1.102a}$$

$$\mathcal{Z}_- = 1 + \frac{\tilde{q}_-}{\rho^2} + \mathcal{O}(\alpha'^2)\,, \tag{1.102b}$$

$$\mathcal{Z}_+ = 1 + \frac{\tilde{q}_+}{\rho^2} + 2\alpha'\frac{\tilde{q}_+(\rho^2 + \tilde{q}_0 + \tilde{q}_-)}{\tilde{q}_0(\rho^2 + \tilde{q}_0)(\rho^2 + \tilde{q}_-)} + \mathcal{O}(\alpha'^2)\,. \tag{1.102c}$$

Using this, we arrive at the following entropy formula

$$S = \frac{A_{\mathcal{H}}}{4G_N^{(5)}}\left\{1 + \frac{2\alpha'}{\tilde{q}_0}\right\}\,. \tag{1.103}$$

This matches the entropy found by microscopic entropy calculations found in [87] once the relations between integration constants and asymptotic brane charges have been correctly taken into account.

Part II

The second part of this thesis will examine how to modify the Iyer-Wald formalism such that non-tensor fields can be considered. All the fields of the Standard Model, except for the metric, have some kind of gauge freedom and do not transform as tensors under diffeomorphisms. As such, the formalism needs to be adjusted such that it can be applied in these theories.

- In Chapter 3 (which we base on paper [88]), we utilize covariant Lie derivatives, as well as the momentum maps previously discussed in section 1.2.4 in order to determine the Wald entropy of the Reissner-Nordström-Tangherlini black holes. We will consider the Einstein Maxwell theory in d dimensions, which is written in differential

[18] The Regge slope parameter α' in Refs. [29, 86] has been replaced by $\alpha'/8$ here to obtain the correct form of the action and solutions.

form language as

$$S[e^a, A] = \frac{(-1)^{d-1}}{16\pi} \int \left[\frac{1}{(d-2)!} R^{a_1 a_2} \wedge e^{a_3} \wedge \cdots \wedge e^{a_d} \epsilon_{a_1 \cdots a_d} - \tfrac{1}{2} F \wedge \star F \right] \equiv \int \mathbf{L} \,, \tag{1.104}$$

although it is more convenient to rewrite the first (Einstein-Hilbert) term as

$$\frac{1}{(d-2)!} R^{a_1 a_2} \wedge e^{a_3} \wedge \cdots \wedge e^{a_d} \epsilon_{a_1 \cdots a_d} = \star (e^a \wedge e^b) \wedge R_{ab} \,. \tag{1.105}$$

We will then compute the Wald-Noether charge for this theory, using the transformations based on the gauge-covariant Lie derivatives. Specifically, we see that the final Wald-Noether charge can be written in terms of the momentum maps and the field strength as

$$\mathbf{Q}[\xi] = \frac{(-1)^{d-1}}{16\pi} \left[\star F P_{\xi} - \star (e^a \wedge e^b) P_{\xi\, ab} \right] , \tag{1.106}$$

where $P_{\xi\, ab} = \nabla^{[a} \xi^{b]}$ is the Lorentz momentum map and $\imath_k F = -dP_k$ is the Maxwell momentum map.

Finally, we shall verify the first law for this system, identifying the Wald entropy, which we compute for the Reissner-Nordström-Tangherlini black hole solutions. We see that, as expected, the entropy that arises is given by $S = \frac{A}{4}$, where A is the area of the horizon, which arises through proof of the first law.

- Chapter 4 (based on paper [74]) focuses on applying these momentum maps to a non-trivial case: the heterotic string black hole at zeroth-order α'. We will study the heterotic string compactified on a torus, and study the various symmetries that arise. Using the momentum map basics defined in the previous chapter, these symmetries will be used to determine the parameters which leaves all the transformations invariant, which in turn allows us to find the conserved Noether charges. We find that the Noether charge takes the form

$$\begin{aligned} \mathbf{Q}[\xi] = {} & (-1)^d \star (e^a \wedge e^b) \left[e^{-2\phi} P_{\xi\, ab} - 2 \imath_a d e^{-2\phi} \xi_b \right] \\ & + (-1)^{d-1} \mathcal{P}_{\xi}{}^I \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J \right) - P_{\xi} \wedge \left(e^{-2\phi} \star H \right) , \end{aligned} \tag{1.107}$$

where $\mathcal{P}_{\xi}{}^I$ and $P_{\xi\, ab}$ are the Maxwell and Lorentz momentum maps as in the previous chapter, the momentum map P_k is given by $-\imath_k H - \mathcal{P}_{k\, I} \mathcal{F}^I = dP_k$, M_{IJ} is a symmetric $\mathrm{O}(n,n)$ matrix, and $\mathcal{F}^I$ is the $\mathrm{O}(n,n)$ vector of the 2-form field strengths of the KK and winding vectors

$$\mathcal{F}^I \equiv \begin{pmatrix} F^m \\ G_m \end{pmatrix} , \qquad F^m = dA^m \,, \qquad G_m = dB_m \,. \tag{1.108}$$

Using the momentum maps, it is possible to prove the restricted generalized zeroth law. Finally, utilizing the generalized zeroth law as well as our explicit expression of the Noether charge, we are able to prove the first law. We conclude by considering as an example the charged, non-extremal, 5-dimensional black ring solution of pure N = 1, d = 5 supergravity of Ref. [46] and compute its momentum maps.

- Chapter 5 (based on paper [53]) will deal with a more complex example: the heterotic string effective action up to order α'. The additional terms proportional to α' will yield additional complexities, due to the presence of the Chern-Simons terms. We will once more study how the fields of the heterotic string theory change under gauge and general coordinate transformations. We construct variations of the fields that vanish when the parameters of the transformations generate a symmetry of the field configuration and we find the integrals that give the associated conserved charges. The conserved charge associated to the invariance under diffeomorphisms is the Noether-Wald charge. As we have discussed, the correct identification of the conserved charges is essential to obtain for the correct identification of the entropy in the first law. We discuss the restricted generalized zeroth laws of this theory, which also play an essential role in the proof of the first law. Finally, we shall prove the first law using the results obtained in the previous sections, which leads us to identify the Wald entropy formula. We discover that the Wald entropy can be written as

$$S = (-1)^d \frac{g_s^{(d)\,2}}{8G_N^{(d)}} \int_{\mathcal{BH}} e^{-2\phi} \left\{ \left[\star(e^a \wedge e^b) + \frac{\alpha'}{2} e^{-2\phi} \star R_{(-)}^{(0)\,ab} \right] n_{ab} + (-1)^d \frac{\alpha'}{2} \Pi_n \wedge \star H^{(0)} \right\} , \tag{1.109}$$

where we have defined the 1-form Π_n (vertical Lorentz momentum map associated to the binormal) on the bifurcation sphere

$$d\Pi_n \stackrel{\mathcal{BH}}{=} R_{(-)}^{(0)\,ab} n_{ab} \,. \tag{1.110}$$

This is the main result of this thesis, and what the previous chapters have built up towards. We recover the correct form of the Wald entropy, where the last term in (5.102) possesses an additional factor of 2 that is missing in previous derivations.

Notes on conventions

Throughout this thesis, we will make use of the traditional natural units: $c = \hbar = 1$. The gravitational Newton's constant $G_N^{(d)}$ will remain, though we shall occasionally remove it from intermediate calculations in order to simplify computations. Furthermore, all our calculations will be using the convention $g = (+, -, -, \ldots, -)$

Part I

Dimensional reduction of the Heterotic string at α'

2

T duality and Wald entropy formula in the Heterotic Superstring effective action at first-order in α'

Superstring Theory is expected to be a consistent theory of Quantum Gravity. Therefore, one would like to use it to study gravitational systems in which quantum-mechanical effects are believed to play an important role, such as black holes. In particular, one of the results that we expect from Superstring Theory is a microscopical accounting of the entropy attributed to them by macroscopic (thermodynamic) laws and calculations.

Achieving this result demands, first of all, black-hole solutions of Superstring Theory whose macroscopic entropy can be computed. These are classical solutions of the Superstring effective action. Then, if one manages to associate the black-hole solution to a good Superstring Theory background on which the theory can be quantized, the microscopic entropy can be associated to the density of string states in that background.

In a seminal paper, [89] Strominger and Vafa completed the above program for a extremal, static, 3-charge 5-dimensional black-hole solution of the type IIB Superstring Theory at lowest order in the Regge slope parameter α', identifying the associated type IIB string background as one with intersecting D1- and D5-branes with momentum flowing along the intersection. Strominger and Vafa argued that, although the black hole only solved the zeroth-order in α' equations of motion, the higher-order corrections could be made small enough by imposing conditions on the charges carried by the black hole. Under those conditions, the microscopic and macroscopic entropies (the later given simply by the one fourth of the area of the event horizon) matched to lowest order in α'.

Since α' is the square of the string length, the higher-order in α' corrections to the string effective action, its solutions, and the properties of the solutions, describe characteristic "stringy" deviations and this makes their study most interesting. This study requires:

1. The knowledge of the higher-order terms in the string effective field-theory actions.
2. The construction of solutions of those effective actions with higher-order terms. These solutions can often be viewed as α'-corrected zeroth-order solutions (recovered by setting $\alpha' = 0$).
3. The computation of the physical properties of the α'-corrected solutions.

Terms of higher-order in α' are terms of higher order in curvatures and their complexity grows rapidly with the power of α'. This makes them very difficult to compute and, consequently, our knowledge of the α' corrections to the effective field theory actions of different Superstring Theories is very limited. The α' corrections to the Heterotic Superstring effective action are probably the best known, and they have only been computed to cubic order (quartic in curvatures) in Ref. [73], using supersymmetry completion of the Lorentz Chern-Simons terms [90].[1]

We can use, then, the Heterotic Superstring effective action given in Ref. [73] for the next step: computing α' corrections to black-hole solutions. As a matter of fact, the black-hole solution studied by Strominger and Vafa in Ref. [89] can also be considered as a zeroth-order solution of the Heterotic Superstring effective action and it would certainly be interesting to compute its α' corrections, at least to first order. Finding these corrections, though, is a complicated problem. One of the problems is that the complete Heterotic Superstring effective action with higher-order corrections has not been compactified down to the 5 dimensions in which the black hole lives.[2] Effective actions which would capture what are believed to be the most relevant α' corrections in lower dimensions have been proposed and used to compute corrections to black-hole solutions (see, *e.g.* Ref. [98] and references therein). Alternatively, in order to simplify the problem, it has been proposed to work only with the near-horizon solution (see *e.g.* Refs. [99, 100] and references therein and more recent work in the Type IIA compactified on K3 setup [101, 102]). It is fair to say that each of these simplified approaches has problems of its own and that they do not offer a complete picture of what the α'-corrected black-hole solutions are like.

Recently, a different approach for computing α' corrections without making assumptions about the lower-dimensional effective actions or considering only near-horizon limits has been proposed in Ref. [29]: since the 10-dimensional first-order in α' Heterotic Superstring effective action is known without any ambiguities (beyond possible field redefinitions), first-order in α' corrections to solutions should be directly computed in 10 dimensions using the uplift of 4- or 5-dimensional solutions. Then, the α'-corrected solutions can be compactified back to 4- or 5-dimensions. This approach has been successfully used to compute the first-order in α' corrections to 5- and 4-dimensional extremal black holes in Refs. [29] and [30, 86, 103], respectively and, more recently, to 4-dimensional non-extremal Reissner-Nordström black holes in Ref. [32]. The question of the regularity of the so-called *small black holes* has also been reviewed in Ref. [104, 105] in light of those results.

Having the α'-corrected solutions we can compute their physical properties. For black holes, these are their conserved charges and their thermodynamical properties: entropy and temperature. The Hawking temperature is always determined by the value of the surface gravity of the metric. While the metric can receive α' corrections, the relation between Hawking temperature and surface gravity does not change. This is not the case for the Bekenstein-Hawking entropy, which, in presence of α' corrections (higher-order in curvature corrections in general) is no longer determined by the area of the horizon which also receives α' corrections coming from those of the metric. Based on previous

[1] The equivalence of this effective action with previous results obtained in Refs. [64,91–93] was established in Ref. [94].

[2] A toroidal compactification to first order in α' but with no Yang-Mills fields has been recently constructed in Ref. [95]. The toroidal compactification with only Abelian Yang-Mills fields (which occur at first order in α') and no terms involving the torsionful spin connection (so the 10-dimensional action is that of $\mathcal{N} = 1, d = 10$ supergravity coupled to Abelian vector supermultiplets) was carried out in [96]. An earlier compactification of the Heterotic Superstring effective action to just $d = 4$ at zeroth-order in α' (so the 10-dimensional action is that of pure $\mathcal{N} = 1, d = 10$ supergravity) was carried out in [97].

work [27, 28], in Ref. [22] Iyer and Wald gave a prescription to derive an entropy formula in diffeomorphism-invariant theories. The main fact that characterizes this prescription is that the entropy computed using it satisfies the first law of black-hole mechanics [20].

Iyer and Wald's prescription is based on a series of assumptions about the field content, which has to consist of tensor fields only. The only tensor field in our current understanding of Nature is the metric, the rest being connections and sections of different gauge bundles or, in other words, field with some kind of gauge freedom. The validity of Iyer and Wald's prescription has subsequently extended to theories that include fields with gauge freedoms in Refs. [40, 51, 106], but the Heterotic Superstring effective action (and many other string effective actions) include a field which is not a connection or a section of some gauge bundle: the Kalb-Ramond field. This complication has been ignored in most of the string literature[3] and the Iyer-Wald prescription has been naively applied with results that seem to be compatible with the microscopic calculations of the entropy.[4]

For instance, in Ref. [29], the entropy of the (heterotic version of the) α'-corrected Strominger-Vafa black hole was computed using the Iyer-Wald prescription directly in the 10-dimensional action. The result obtained was compatible with that of the microscopic calculation carried out in Ref. [87] to first-order in α', with an appropriate identification between the charges carried by the black hole and associated string background [107]. More precisely, the entropy obtained was interpreted in Ref. [29] as the $\mathcal{O}(\alpha')$ truncation of the expansion in powers of α' of the exact result found in Ref. [87].

This interpretation, however, was a bit puzzling, because in Ref. [29], it was argued that the near-horizon region of the black-hole solution, which determines the entropy, should not receive further α' corrections.[5] Furthermore, an explicit calculation shows that at least the $\mathcal{O}(\alpha'^2)$ corrections to the entropy vanish identically. All this suggests that the result obtained for the entropy in Ref. [29] should be exact to all orders in α' and, therefore, it should be identical to the result of the microscopic calculation of Ref. [87].

This puzzle was solved in Ref. [107], where it was observed that the dependence of the action on the Riemann curvature[6] in the Lorentz Chern-Simons term of the Kalb-Ramond field strength is changed by dimensional reduction. Taking into account this change, which amounts to a factor of 2 with respect to the result of Ref. [29], the macroscopic entropy computed at first order in α', naively using the Iyer-Wald formula, matches the exact microscopic result. This gives further support to the conjecture that the black-hole solution does not receive further α' corrections and may be considered an exact Heterotic Superstring solution.

[3] An independent derivation of an entropy formula using Wald's formalism and dealing with some of the problems that the presence of the Kalb-Ramond field raises has been made in Ref. [31]. The final entropy formula derived there depends on a compensating gauge parameter which was left undetermined. This makes a comparison with the entropy formula we will derive impossible. For instance, it is not possible to compute the entropy of the Strominger-Vafa black hole using this formula, unless one can prove that the unknown term does not contribute to it. Although in that reference it is argued that, at least in certain relevant cases, this is indeed the case. In the same reference it is also shown that the invariance of their entropy formula under local Lorentz transformations depends on it, which seems contradictory.

[4] Wald's formalism's first step consists in the proof of a first law of black-hole mechanics for the theory under consideration. A first law for the Heterotic Superstring effective action to first order in α' has not yet been proven, although it is widely assumed to exist (for instance, in the derivation of the entropy formula of Ref. [31]).

[5] The complete black-hole solution may receive further corrections.

[6] According to the Iyer-Wald prescription, the entropy formula only depends on the occurrences of the Riemann tensor in the action.

The results of Ref. [107] made clear that, in the case of the Heterotic Superstring effective action, the entropy formula has to be derived from the dimensionally-reduced action in order to determine correctly the dependence of the action of the lower-dimensional Riemann tensor. One of our goals in this chapter is to perform the dimensional reduction of the Heterotic Superstring effective action to first order in α' over a circle to apply to it the Iyer-Wald prescription and obtain an entropy formula. This entropy formula can only be applied to d-dimensional black holes that can be obtained by trivial compactification on T^{9-d} and a non-trivial compactification on a circle. For instance, it can be applied to the heterotic version of the Strominger-Vafa black hole because it can be obtained from a 10-dimensional solution by trivial compactification on T^4, to 6 dimensions and a non-trivial compactification on a circle from 6 to 5 dimensions. It can also be applied to the non-supersymmetric 4-dimensional Reissner-Nordström black hole of Ref. [108], which can be obtained from pure 5-dimensional gravity and, therefore, can be obtained from a purely gravitational 10-dimensional solution by trivial compactification on T^5 to 5 dimensions and, then, by a non-trivial compactification on a circle from 5 to 4 dimensions. Actually, the entropy formula Eq. (2.69b) that we are going to derive in Section 2.4 has been applied to a non-extremal version of the 4-dimensional Reissner-Nordström black hole we just discussed, in Ref. [32]. While the microscopic interpretation of the entropy of this black hole is unknown, being a black hole with finite temperature, one can check that the first law of thermodynamics is indeed satisfied because the temperature computed from the α'-corrected metric and the entropy computed from the α'-corrected metric with the α'-corrected entropy formula are related by the thermodynamic relation

$$\frac{\partial S}{\partial M} = \frac{1}{T} \,. \tag{2.1}$$

This paper's second goal has to do with one of the most interesting and characteristic properties of String Theory: T duality.[7] T duality relates two string theories compactified in circles of dual radii. The spectra of the two theories can be put into one-to-one correspondence and, from the lower dimensional point of view, they are essentially identical, up to charge identifications.[8] More generally, Buscher [82, 83] showed that two string backgrounds with one isometry whose background fields are related by the so-called *Buscher T duality rules* are equivalent.

Perhaps not surprisingly, the Buscher rules can be derived from the string effective action: the dual[9] Kaluza-Klein compactifications of two effective actions on a circle give the same $(d-1)$-dimensional action and the same equations of motion. In practice, one can perform identical Kaluza-Klein compactifications, determine the relation between the $(d-1)$-dimensional fields of the two actions (which is usually very simple because it does not involve the $(d-1)$-dimensional string metric or Kalb-Ramond field) and rewrite this relation in terms of the components of the original d-dimensional fields [109]. This relation is just the Buscher T duality rules. This strategy has been successfully used to find the extension of the Buscher T duality rules that relates equivalent type IIA and type IIB superstring backgrounds [110] and higher-rank Ramond-Ramond potentials [84].

In the context of the Heterotic Superstring, this strategy was used in [85] to find

[7]For a review with many early references see Ref. [77].

[8]Charges related to Kaluza-Klein momentum and charges related to the winding number along the compact direction should be interchanged.

[9]That is, with fields related by the Buscher rules.

the first-order in α' corrections to the Buscher rules.[10] Only the Yang-Mills fields were included at order α', but, taking into account that the torsionful spin connection enters the action in exactly the same way as the Yang-Mills fields [90], it was possible to find the α' corrections to the Buscher rules.

The α'-corrected Buscher rules are of no use if there are no α'-corrected solutions at one's disposal to generate new solutions or to check their equivalence. For this reason, the results of Ref. [85] were sleeping the "sleep of the just"[11] until quite recently, when they were first applied to α'-corrected self-T-dual solutions, providing a highly non-trivial test of both the α' corrections of the solutions and of the T duality rules.

Our second goal will be to study the T duality invariance of the complete dimensionally-reduced Heterotic Superstring effective action and of the entropy formula that follows from it. While the α'-corrected Buscher rules will be those of Ref. [85], the complete reduced action will have many more $\mathcal{O}(\alpha')$ terms than the action obtained there. The invariance of the action under T duality suggests that they will contribute to the entropy in a T duality-invariant form, and we will prove that this is the case.[12]

This chapter is organized as follows: we introduce the Heterotic Superstring effective action to first order in α' following Ref. [73] in Section 2.1. In Section 2.2, we revisit the dimensional reduction on a circle of the action at zeroth order in α' as a warm-up exercise and also because we will need some of the results when we consider the higher-order terms in Section 2.3. In that section we will obtain the complete dimensionally-reduced action to first order in α', we will find the T duality rules and we will prove the invariance of the action under those T duality rules. In Section 2.4, we will use the dimensionally-reduced T duality-invariant action to derive an entropy formula using the Iyer-Wald prescription and we will apply it to the heterotic version of the α'-corrected Strominger-Vafa black hole of Ref. [29]. We will end by discussing our results and future work on these topics in Section 5.7.

2.1 The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$

Let us start by reviewing the Heterotic Superstring effective action to $\mathcal{O}(\alpha')$. We will use the formulation given in Ref. [73], but written in the conventions of Ref. [42].[13] In this formulation, the action is constructed recursively order by order in α'.

The zeroth-order 3-form field strength of the Kalb-Ramond 2-form B is defined as

[10] At zeroth-order in α', the Heterotic Superstring effective action only describes the so-called *common sector* of Neveu-Schwarz-Neveu-Schwarz fields, so the Buscher rules are just those found by Buscher.

[11] As a matter of fact, they have partially re-derived several times [111, 112]. Other studies of the effect of α' corrections on T duality and $\mathrm{O}(d,d)$ transformations in toroidal compactifications, sometimes in extended set-ups (such as Double Field Theory) can be found [66, 67, 95, 113–115].

[12] It follows trivially from the invariance of the lower-dimensional string metric and dilaton under T duality that the zeroth-order in α' temperature and entropy (the area) are also T duality invariant. This property was proven by Horowitz and Welch in Ref. [116] before the relation between the Buscher rules and dimensional reduction was established in Ref. [109]. Recently, it has been investigated again from the same point of view in Refs. [31, 117] to first order in α', but, again, the relation between dimensional reduction and T duality and the invariance of the lower-dimensional string metric and dilaton field lead, trivially, to the invariance of the α'-corrected temperature. The invariance of the action under T duality at this order implies that of the entropy formula using the Iyer-Wald prescription because the Riemann curvature is T duality invariant.

[13] The relation with the fields in Ref. [73] can be found in Ref. [75].

$$H^{(0)}{}_{\mu\nu\rho} \equiv 3\partial_{[\mu}B_{\nu\rho]}\,, \tag{2.2}$$

and it contributes as torsion to the zeroth-order torsionful spin connections

$$\Omega^{(0)}_{(\pm)\,\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b} \pm \tfrac{1}{2}H^{(0)}{}_{\mu}{}^{a}{}_{b}\,, \tag{2.3}$$

where $\omega_{\mu}{}^{a}{}_{b}$ is the (torsionless, metric-compatible) Levi-Civita spin connection 1-form.

The corresponding zeroth-order Lorentz curvature 2-forms and Chern-Simons 3-forms are defined as

$$R^{(0)}_{(\pm)\,\mu\nu}{}^{a}{}_{b} = 2\partial_{[\mu|}\Omega^{(0)}_{(\pm)\,|\nu]}{}^{a}{}_{b} - 2\Omega^{(0)}_{(\pm)\,[\mu|}{}^{a}{}_{c}\,\Omega^{(0)}_{(\pm)\,|\nu]}{}^{c}{}_{b}\,, \tag{2.4}$$

$$\omega^{\mathrm{L}\,(0)}_{(\pm)} = 3R^{(0)}_{(\pm)\,[\mu\nu|}{}^{a}{}_{b}\Omega^{(0)}_{(\pm)\,|\rho]}{}^{b}{}_{a} + 2\Omega^{(0)}_{(\pm)\,[\mu|}{}^{a}{}_{b}\,\Omega^{(0)}_{(\pm)\,|\nu|}{}^{b}{}_{c}\,\Omega^{(0)}_{(\pm)\,|\rho]}{}^{c}{}_{a}\,. \tag{2.5}$$

The gauge field 1-form is $A^{A}{}_{\mu}$, where $A, B, C, \ldots$ are the adjoint gauge indices of some group that we will not specify. The gauge field strength and the Chern-Simons 3-forms are defined by

$$F^{A}{}_{\mu\nu} = 2\partial_{[\mu}A^{A}{}_{\nu]} + f_{BC}{}^{A}A^{B}{}_{[\mu}A^{C}{}_{\nu]}\,, \tag{2.6}$$

$$\omega^{\mathrm{YM}} = 3F_{A\,[\mu\nu}A^{A}{}_{\rho]} - f_{ABC}A^{A}{}_{[\mu}A^{B}{}_{\nu}A^{C}{}_{\rho]}\,, \tag{2.7}$$

where we have lowered the adjoint group indices using the Killing metric of K_{AB}: $f_{ABC} \equiv f_{AB}{}^{D}K_{DC}$ and of the gauge fields $F_{A\,\mu\nu} \equiv K_{AB}F^{B}{}_{\mu\nu}$.

Then, at first order

$$H^{(1)}{}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + \frac{\alpha'}{4}\left(\omega^{\mathrm{YM}}{}_{\mu\nu\rho} + \omega^{\mathrm{L}\,(0)}_{(-)\,\mu\nu\rho}\right)\,, \tag{2.8}$$

$$\Omega^{(1)}_{(\pm)\,\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b} \pm \tfrac{1}{2}H^{(1)a}_{\mu}{}_{b}\,, \tag{2.9}$$

$$R^{(1)}_{(\pm)\,\mu\nu}{}^{a}{}_{b} = 2\partial_{[\mu|}\Omega^{(1)}_{(\pm)\,|\nu]}{}^{a}{}_{b} - 2\Omega^{(1)}_{(\pm)\,[\mu|}{}^{a}{}_{c}\,\Omega^{(1)}_{(\pm)\,|\nu]}{}^{c}{}_{b}\,, \tag{2.10}$$

$$\omega^{\mathrm{L}\,(1)}_{(\pm)\,\mu\nu\rho} = 3R^{(1)}_{(\pm)\,[\mu\nu|}{}^{a}{}_{b}\Omega^{(1)}_{(\pm)\,|\rho]}{}^{b}{}_{a} + 2\Omega^{(1)}_{(\pm)\,[\mu|}{}^{a}{}_{b}\,\Omega^{(1)}_{(\pm)\,|\nu|}{}^{b}{}_{c}\,\Omega^{(1)}_{(\pm)\,|\rho]}{}^{c}{}_{a}\,. \tag{2.11}$$

$$H^{(2)}{}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + \frac{\alpha'}{4}\left(\omega^{\mathrm{YM}}{}_{\mu\nu\rho} + \omega^{\mathrm{L}\,(1)}_{(-)\,\mu\nu\rho}\right)\,, \tag{2.12}$$

etc.

Only $\Omega^{(0)}_{(\pm)\,\mu}$, $R^{(0)}_{(\pm)\,\mu\nu}{}^{a}{}_{b}$, $\omega^{\mathrm{L}\,(0)}_{(\pm)\,\mu\nu\rho}$ and $H^{(1)}{}_{\mu\nu\rho}$ (plus the Yang-Mills fields) occur in the action. In practice, though, it is more convenient to work with the higher-order objects, neglecting the terms of higher order in α' when necessary. Thus, from now on we will suppress the (n) upper indices when they do not play a relevant role.

In terms of all these objects, the Heterotic Superstring effective action in the string frame and to first-order in α' can be written as

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \tfrac{1}{12}H^2 - \frac{\alpha'}{8}\left[F_A \cdot F^A + R_{(-)}{}^a{}_b \cdot R_{(-)}{}^b{}_a \right] \right\}, \tag{2.13}$$

where $G_N^{(10)}$ is the 10-dimensional Newton constant, ϕ is the dilaton field, the vacuum expectation value of e^{ϕ} is the Heterotic Superstring coupling constant g_s, R is the Ricci scalar of the string-frame metric $g_{\mu\nu}$ and the dot indicates the contraction of the indices of 2-forms: $F_A \cdot F^A \equiv F_{A\,\mu\nu} F^{A\,\mu\nu}$.

2.2 Dimensional reduction on S^1 at zeroth order in α'

As a warm-up exercise (and also because of the recursive definition of the action that will make necessary the zeroth-order fields in the first-order action), we review the well-known dimensional reduction of the action at zeroth order in α' using the Scherk-Schwarz formalism [81]. We add hats to all the 10-dimensional objects (fields, indices, coordinates) and split the 10-dimensional world indices as $(\hat{\mu}) = (\mu, \underline{z})$ and the 10-dimensional indices as $(\hat{a}) = (a, z)$.

The Zehnbein and inverse-Zehnbein components $\hat{e}_{\hat{\mu}}{}^{\hat{a}}$ and $\hat{e}_{\hat{a}}{}^{\hat{\mu}}$ can be put in an upper-triangular form by a local Lorentz transformation and, then, they can be decomposed in terms of the 9-dimensional Vielbein and inverse Vielbein components $e_\mu{}^a, e_a{}^\mu$, Kaluza-Klein (KK) vector A_μ and KK scalar k as

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & kA_\mu \\ 0 & k \end{pmatrix}, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_a \\ 0 & k^{-1} \end{pmatrix}, \tag{2.14}$$

where $A_a = e_a{}^\mu A_\mu$. We will always assume that all the 9-dimensional fields with Lorentz indices are 9-dimensional world tensors contracted with the 9-dimensional Vielbeins. For instance, the KK fields strength F_{ab} is

$$F_{ab} = e_a{}^\mu e_b{}^\nu F_{\mu\nu}, \qquad F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}, \tag{2.15}$$

The components of the 10-dimensional spin connection $\hat{\omega}_{\hat{a}\hat{b}\hat{c}}$ decompose into those of the 9-dimensional one ω_{abc} and F_{ab} as

$$\begin{aligned} \hat{\omega}_{abc} &= \omega_{abc}, & \hat{\omega}_{abz} &= \tfrac{1}{2}kF_{ab}, \\ \hat{\omega}_{zbc} &= -\tfrac{1}{2}kF_{bc}, & \hat{\omega}_{zbz} &= -\partial_b \ln k. \end{aligned} \tag{2.16}$$

Then, using the Palatini identity, it is not difficult to see that the first two terms in the action Eq. (2.13) take the following 9-dimensional form (up to a total derivative):

$$\int d^{10}\hat{x}\sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left\{\hat{R} - 4(\partial\hat{\phi})^2\right\} =$$
$$\int dz \int d^9x\sqrt{|g|}\, e^{-2\phi} \left\{R - 4(\partial\phi)^2 + (\partial \log k)^2 - \tfrac{1}{4}k^2F^2\right. \quad , \tag{2.17}$$

where the 9-dimensional dilaton field is related to the 10-dimensional one by

$$\phi \equiv \hat{\phi} - \tfrac{1}{2}\log k\,. \tag{2.18}$$

At zeroth order in α', the last term that we have to reduce is the kinetic term of the Kalb-Ramond 2-form $\sim \hat{H}^{(0)\,2}$. Following Scherk and Schwarz, we consider the Lorentz components of the 3-form field strength, because they are automatically gauge-invariant combinations. The $\hat{H}^{(0)}{}_{abz}$ components give

$$\hat{H}^{(0)}{}_{abz} = k^{-1}e_a{}^{\mu}e_b{}^{\nu}\hat{H}^{(0)}{}_{\mu\nu\underline{z}} = k^{-1}e_a{}^{\mu}e_b{}^{\nu}2\partial_{[\mu}\hat{B}_{\nu]\underline{z}}\,. \tag{2.19}$$

It is, then, appropriate to define the zeroth-order “winding”[14] vector field $B^{(0)}{}_{\mu}$ and its field strength $G^{(0)}{}_{\mu\nu}$ by

$$B^{(0)}{}_{\mu} \equiv \hat{B}_{\mu\underline{z}}\,, \qquad G^{(0)}{}_{\mu\nu} \equiv 2\partial_{[\mu}B^{(0)}{}_{\nu]}\,, \tag{2.20}$$

so that

$$\hat{H}^{(0)}{}_{abz} = k^{-1}G^{(0)}{}_{ab}\,. \tag{2.21}$$

The second gauge-invariant combination is

$$\hat{H}^{(0)}{}_{abc} = e_a{}^{\mu}e_b{}^{\nu}e_c{}^{\rho}\left(\hat{H}^{(0)}{}_{\mu\nu\rho} - 3A_{[\mu}\hat{H}^{(0)}{}_{\nu\rho]\underline{z}}\right)\,, \tag{2.22}$$

which suggests the definition

$$H^{(0)}{}_{\mu\nu\rho} \equiv \hat{H}^{(0)}{}_{\mu\nu\rho} - 3A_{[\mu}\hat{H}^{(0)}{}_{\nu\rho]\underline{z}} = 3\partial_{[\mu}\hat{B}_{\nu\rho]} - 6A_{[\mu}\partial_{\nu}B^{(0)}{}_{\rho]}\,. \tag{2.23}$$

We could simply identify $\hat{B}_{\nu\rho}$ with the 9-dimensional Kalb-Ramond field, but it is customary (and convenient) to use a T duality-invariant definition. T duality will interchange KK momentum and winding, and therefore, will interchange A_{μ} with $B^{(0)}{}_{\mu}$, modifying the Chern-Simons term in the above form of $H_{\mu\nu\rho}$. We can, however, rewrite it in the form

$$H^{(0)}{}_{\mu\nu\rho} = 3\partial_{[\mu}\left(\hat{B}_{\nu\rho]} + A_{|\nu}B^{(0)}{}_{\rho]}\right) - \tfrac{3}{2}A_{[\mu}G^{(0)}{}_{\nu\rho]} - \tfrac{3}{2}B^{(0)}{}_{[\mu}F_{\nu\rho]}\,, \tag{2.24}$$

and identify the T duality-invariant 9-dimensional Kalb-Ramond 2-form

[14]This vector couples electrically to the string modes with non-vanishing winding numbers, just as the KK vector field couples to those with non-vanishing momentum in the internal direction.

$$B^{(0)}{}_{\mu\nu} \equiv \hat{B}_{\mu\nu} + A_{[\mu}B^{(0)}{}_{\nu]}\,, \tag{2.25}$$

with the final result

$$H^{(0)}{}_{\mu\nu\rho} = 3\partial_{[\mu}\hat{B}^{(0)}{}_{\nu\rho]} - \tfrac{3}{2}A_{[\mu}G^{(0)}{}_{\nu\rho]} - \tfrac{3}{2}B^{(0)}{}_{[\mu}F_{\nu\rho]}\,. \tag{2.26}$$

Then, after integrating over the length of the compact coordinate z ($2\pi\ell_s$ by convention) the 9-dimensional action to zeroth order in α' takes the form

$$S = \frac{g_s^2(2\pi\ell_s)}{16\pi G_N^{(10)}}\int d^9x\sqrt{|g|}\,e^{-2\phi}\,\left\{R - 4(\partial\phi)^2 + (\partial\log k)^2 - \tfrac{1}{4}k^2F^2 - \tfrac{1}{4}k^{-2}G^{(0)\,2} + \tfrac{1}{12}H^{(0)\,2}\right\}\,. \tag{2.27}$$

This action is invariant under the T duality transformations

$$A'_\mu = B^{(0)}{}_\mu\,, \qquad B^{(0)\prime}{}_\mu = A_\mu\,, \qquad k' = 1/k\,. \tag{2.28}$$

Taking into account the relations between the 10- and 9-dimensional fields, collected in Appendix A.1, it is easy to see that the above T duality transformations correspond to the following transformations of the 10-dimensional fields known as *Buscher rules* [82,83]:

$$\begin{aligned}
\hat{g}'_{\underline{zz}} &= 1/\hat{g}_{\underline{zz}}\,, & \hat{B}'_{\mu\underline{z}} &= \hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,,\\
\hat{g}'_{\mu\underline{z}} &= \hat{B}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, & \hat{B}'_{\mu\nu} &= \hat{B}_{\mu\nu} + 2\hat{g}_{[\mu|\underline{z}|}\hat{B}_{\nu]\underline{z}}/\hat{g}_{\underline{zz}}\,,\\
\hat{g}'_{\mu\nu} &= \hat{g}_{\mu\nu} - (\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}} - \hat{B}_{\mu\underline{z}}\hat{B}_{\nu\underline{z}})/\hat{g}_{\underline{zz}}\,, & \hat{\phi}' &= \hat{\phi} - \tfrac{1}{2}\ln|\hat{g}_{\underline{zz}}|\,.
\end{aligned} \tag{2.29}$$

2.3 Dimensional reduction on S^1 at $\mathcal{O}(\alpha')$

The reduction of the first two terms in the effective action is not modified by the inclusion of α' corrections. The definitions of 9-dimensional metric, dilaton and KK vector and scalar in terms of the 10-dimensional fields are not modified by them either. We expect modifications in the definitions of the 9-dimensional Kalb-Ramond 2-form and of the winding vector, though, because of the presence of the Lorentz and Yang-Mills Chern-Simons 3-forms in $\hat{H}^{(1)}$.

It is convenient to start by studying the dimensional reduction of the Yang-Mills fields. The Lorentz-indices decomposition of the gauge field is

$$\hat{A}^A{}_z = k^{-1}\hat{A}^A{}_{\underline{z}}\,, \tag{2.30a}$$

$$\hat{A}^A{}_a = e_a{}^\mu(\hat{A}^A{}_\mu - \hat{A}^A{}_{\underline{z}}A_\mu)\,, \tag{2.30b}$$

which leads to the definition of the 9-dimensional adjoint scalars ϕ^A and gauge vectors

$$\varphi^{A} \equiv k^{-1}\hat{A}^{A}{}_{\underline{z}}\,, \tag{2.31a}$$

$$A^{A}{}_{\mu} \equiv \hat{A}^{A}{}_{\mu} - \hat{A}^{A}{}_{\underline{z}}A_{\mu}\,. \tag{2.31b}$$

In terms of these variables, it is not difficult to see that the components of 10-dimensional gauge field strength are given by

$$\hat{F}^{A}{}_{az} = \mathfrak{D}_{a}\varphi^{A} + \varphi^{A}\partial_{a}\log k\,, \tag{2.32a}$$

$$\hat{F}^{A}{}_{ab} = F^{A}{}_{ab} + k\varphi^{A}F_{ab}\,, \tag{2.32b}$$

where $F^{A}{}_{\mu\nu}$ is the standard Yang-Mills gauge field strength for the 9-dimensional gauge fields $A^{A}{}_{\mu}$.

The reduction of the first, second and fourth terms in the action Eq. (2.13) gives (up to a total derivative)

$$\begin{aligned}&\int dz \int d^{9}x\sqrt{|g|}e^{-2\phi}\left\{R - 4(\partial\phi)^{2} + \left(1+\frac{\alpha'}{4}\varphi^{2}\right)(\partial\log k)^{2} + \frac{\alpha'}{4}(\mathfrak{D}\varphi)^{2}\right.\\ &\left.-\tfrac{1}{4}\left(1+\frac{\alpha'}{2}\varphi^{2}\right)k^{2}F^{2} + \frac{\alpha'}{4}\partial_{a}\log k\partial^{a}\varphi^{2} - \frac{\alpha'}{8}\left(F_{A}\cdot F^{A} + 2\varphi_{A}F^{A}\cdot kF\right)\right\}\,,\end{aligned} \tag{2.33}$$

where $\varphi^{2} \equiv \varphi_{A}\varphi^{A}$, $\mathfrak{D}_{\mu}\varphi^{A} = \partial_{\mu}\varphi^{A} + f_{BC}{}^{A}A^{B}{}_{\mu}\varphi^{C}$ etc.

Let us now consider the reduction of the Kalb-Ramond 3-form field strength $\hat{H}^{(1)}$, starting with the gauge-invariant combination

$$\hat{H}^{(1)}{}_{abz} = k^{-1}e_{a}{}^{\mu}e_{b}{}^{\nu}\hat{H}^{(1)}{}_{\mu\nu\underline{z}} = k^{-1}e_{a}{}^{\mu}e_{b}{}^{\nu}\left\{2\partial_{[\mu}\hat{B}_{\nu]\underline{z}} + \frac{\alpha'}{4}\left(\hat{\omega}^{\rm YM}{}_{\mu\nu\underline{z}} + \hat{\omega}^{\rm L(0)}_{(-)\,\mu\nu\underline{z}}\right)\right\}\,. \tag{2.34}$$

Using the above results for the Yang-Mills fields we find that

$$\hat{\omega}^{\rm YM}{}_{\mu\nu\underline{z}} = k\varphi_{A}\left(2F^{A}{}_{\mu\nu} + \varphi^{A}kF_{\mu\nu}\right) - 2\partial_{[\mu|}\left(k\varphi_{A}A^{A}_{|\nu]}\right)\,. \tag{2.35}$$

The last term is a total derivative that can be absorbed into the definition of the 9-dimensional vector field $B^{(1)}{}_{\mu}$ and the remaining terms are manifestly gauge-invariant 2-forms.

We can use this result in the reduction of the Lorentz Chern-Simons 3-form; after all, the only difference with the Yang-Mills Chern-Simons 3-form is the gauge group, which now is the 10-dimensional Lorentz group. This is, nevertheless, an important difference because this group is broken down to the 9-dimensional Lorentz group times U(1) and we will have to take this fact into account in a second step.

In order to profit from the previous result, we introduce the following notation

$$\hat{A}^{\hat{a}\hat{b}}{}_{\hat{\mu}} \equiv \hat{\Omega}^{(0)}_{(-)\,\hat{\mu}}{}^{\hat{a}\hat{b}}\,, \tag{2.36a}$$

$$\hat{F}^{\hat{a}\hat{b}}{}_{\hat{\mu}\hat{\nu}} \equiv \hat{R}^{(0)}_{(-)\,\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}}\,. \tag{2.36b}$$

Then, a straightforward application of Eq. (2.35) gives

$$\hat{\omega}^{\rm L(0)}_{(-)\,\mu\nu\underline{z}} = k\varphi^{\hat{a}}{}_{\hat{b}}\left(2F^{\hat{b}}{}_{\hat{a}\,\mu\nu} + \varphi^{\hat{b}}{}_{\hat{a}}kF_{\mu\nu}\right) - 2\partial_{[\mu|}\left(k\varphi^{\hat{a}}{}_{\hat{b}}A^{\hat{b}}{}_{\hat{a}\,|\nu]}\right)\,, \tag{2.37}$$

where

$$\varphi^{\hat{a}\hat{b}} = k^{-1}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{a}\hat{b}}\,, \tag{2.38a}$$

$$A^{\hat{a}\hat{b}}{}_{\mu} = \hat{\Omega}^{(0)}_{(-)\,\mu}{}^{\hat{a}\hat{b}} - A_{\mu}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{a}\hat{b}}\,, \tag{2.38b}$$

and where $F^{\hat{a}\hat{b}}{}_{\mu\nu}$ is the standard field strength of the gauge field $A^{\hat{a}\hat{b}}{}_{\mu}$ defined above. Decomposing now the Lorentz indices, we obtain

$$\begin{aligned}\hat{\omega}^{\rm L(0)}_{(-)\,\mu\nu\underline{z}} &= k\varphi^{a}{}_{b}\left(2F^{b}{}_{a\,\mu\nu} + \varphi^{b}{}_{a}kF_{\mu\nu}\right) + 2k\varphi^{az}\left(2F_{a}{}^{z}{}_{\mu\nu} + \varphi_{a}{}^{z}kF_{\mu\nu}\right)\\ &- 2\partial_{[\mu|}\left[k\left(\varphi^{\hat{a}}{}_{\hat{b}}A^{\hat{b}}{}_{\hat{a}\,|\nu]}\right)\right]\,.\end{aligned} \tag{2.39}$$

The components of these fields are

$$\varphi^{ab} = -\tfrac{1}{2}\left(kF^{ab} + k^{-1}G^{(0)\,ab}\right)\,, \tag{2.40a}$$

$$\varphi^{az} = \partial^{a}\log k\,, \tag{2.40b}$$

$$A^{ab}{}_{\mu} = \omega_{\mu}{}^{ab} - \tfrac{1}{2}H^{(0)}{}_{\mu}{}^{ab} \equiv \Omega^{(0)}_{(-)\,\mu}{}^{ab}\,, \tag{2.40c}$$

$$A^{az}{}_{\mu} = -\tfrac{1}{2}\left(kF_{\mu}{}^{a} - k^{-1}G^{(0)}{}_{\mu}{}^{a}\right)\,, \tag{2.40d}$$

$$F^{ab}{}_{\mu\nu} = R^{(0)}_{(-)\,\mu\nu}{}^{ab} - \tfrac{1}{2}\left(kF_{[\mu}{}^{a} - k^{-1}G^{(0)}{}_{[\mu}{}^{a}\right)\left(kF_{\nu]}{}^{b} - k^{-1}G^{(0)}{}_{\nu]}{}^{b}\right)\,, \tag{2.40e}$$

$$F^{az}{}_{\mu\nu} = -\mathcal{D}^{(0)}_{(-)\,[\mu}\left(kF_{\nu]}{}^{a} - k^{-1}G^{(0)}{}_{\nu]}{}^{a}\right)\,, \tag{2.40f}$$

where $R^{(0)}_{(-)\,\mu\nu}{}^{ab}$ is the standard Lorentz curvature and $\mathcal{D}^{(0)}_{(-)\,\mu}$ is the standard Lorentz-covariant derivative with respect to the 9-dimensional torsionful spin connection $\Omega^{(0)}_{(-)\,\mu}{}^{ab}$.

Replacing the above expressions in Eq. (2.39) we obtain

$$\begin{aligned}\hat{\omega}^{\mathrm{L}(0)}_{(-)\,\mu\nu\underline{z}} = {}& -\tfrac{1}{2}k\left(kF^{a}{}_{b}+k^{-1}G^{(0)\,a}{}_{b}\right)\Big\{2R^{(0)}_{(-)\,\mu\nu}{}^{b}{}_{a}-\left(kF_{[\mu|}{}^{b}-k^{-1}G^{(0)}{}_{[\mu}{}^{b}\right)\left(kF_{\nu]\,a}-k^{-1}G^{(0)}{}_{|\nu]\,a}\right)\\ & -\tfrac{1}{2}\left(kF^{b}{}_{a}+k^{-1}G^{(0)\,b}{}_{a}\right)kF_{\mu\nu}\Big\}-2\partial^{a}k\left[2\mathcal{D}^{(0)}_{(-)\,[\mu}\left(kF_{\nu]\,a}-k^{-1}G^{(0)}{}_{|\nu]\,a}\right)-\partial_{a}kF_{\mu\nu}\right]\\ & -2\partial_{[\mu|}\left[k\left(\varphi^{\hat{a}}{}_{\hat{b}}A^{\hat{b}}{}_{\hat{a}\,|\nu]}\right)\right]\,,\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}\hat{H}^{(1)}{}_{cdz} = {}& k^{-1}e_{c}{}^{\mu}e_{d}{}^{\nu}\Big\{2\partial_{[\mu}\left[\hat{B}_{\nu]\underline{z}}-\frac{\alpha'}{4}k\left(\varphi_{A}A^{A}{}_{|\nu]}+\varphi^{\hat{a}}{}_{\hat{b}}A^{\hat{b}}{}_{\hat{a}\,|\nu]}\right)\right]\\ & +\frac{\alpha'}{4}k\varphi_{A}\left(2F^{A}{}_{\mu\nu}+\varphi^{A}kF_{\mu\nu}\right)\\ & -\tfrac{1}{2}k\left(kF^{a}{}_{b}+k^{-1}G^{(0)\,a}{}_{b}\right)\left[2R^{(0)}_{(-)\,\mu\nu}{}^{b}{}_{a}-\left(kF_{[\mu|}{}^{b}-k^{-1}G^{(0)}{}_{[\mu}{}^{b}\right)\left(kF_{\nu]\,a}-k^{-1}G^{(0)}{}_{|\nu]\,a}\right)\right.\\ & \left.-\tfrac{1}{2}\left(kF^{b}{}_{a}+k^{-1}G^{(0)\,b}{}_{a}\right)kF_{\mu\nu}\right]-2\partial^{a}k\left[2\mathcal{D}^{(0)}_{(-)\,[\mu}\left(kF_{\nu]\,a}-k^{-1}G^{(0)}{}_{|\nu]\,a}\right)-\partial_{a}kF_{\mu\nu}\right]\Big\}\,.\end{aligned} \tag{2.42}$$

Since the right-hand side has to be a gauge-invariant combination, it is natural to define the first-order in α' winding vector and its field strength by

$$\begin{aligned}B^{(1)}{}_{\mu} \equiv {}& \hat{B}_{\mu\,\underline{z}}-\frac{\alpha'}{4}k\left(\varphi_{A}A^{A}{}_{\mu}+\varphi^{\hat{a}}{}_{\hat{b}}A^{\hat{b}}{}_{\hat{a}\,\mu}\right)\\ = {}& \hat{B}_{\mu\,\underline{z}}-\frac{\alpha'}{4}\left[\hat{A}^{A}{}_{\mu}\hat{A}_{A\,\underline{z}}+\hat{\Omega}^{(0)}_{(-)\,\mu}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{b}}{}_{\hat{a}}\right.\\ & \left.-A_{\mu}\left(\hat{A}^{A}{}_{\underline{z}}\hat{A}_{A\,\underline{z}}+\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{b}}{}_{\hat{a}}\right)\right]\,,\end{aligned} \tag{2.43a}$$

$$G^{(1)}{}_{\mu\nu} \equiv 2\partial_{[\mu}B^{(1)}{}_{\nu]}\,. \tag{2.43b}$$

Furthermore, it is also natural to define the combinations

$$K^{(\pm)}{}_{\mu\nu} \equiv kF_{\mu\nu}\pm k^{-1}G^{(0)}{}_{\mu\nu}\,. \tag{2.44}$$

$K^{(+)}{}_{\mu\nu}$ is invariant under the zeroth-order T duality transformations Eq. (2.28) while $K^{(-)}{}_{\mu\nu}$ gets a minus sign under the same transformations. With this notation, we can finally write

$$\hat{H}^{(1)}{}_{abz} = k^{-1}G^{(1)}{}_{ab} + \frac{\alpha'}{4}\Big\{2\varphi_A F^A{}_{ab} + \Big[\varphi^2 - \tfrac{1}{4}K^{(+)\,2} + 2(\partial\log k)^2\Big]kF_{ab}$$
$$+R^{(0)}_{(-)\,ab}{}^{cd}K^{(+)}{}_{cd} - \tfrac{1}{2}K^{(-)}{}_a{}^cK^{(-)}{}_b{}^dK^{(+)}{}_{cd} - 4\mathcal{D}^{(0)}_{(-)\,[a}K^{(-)}{}_{b]\,c}\partial^c\log k\Big\}\,. \tag{2.45}$$

This term contributes as $-\frac{1}{4}\hat{H}^{(1)}{}_{abz}\hat{H}^{(1)\,ab}{}_z$, which, at first order in α' gives

$$-\tfrac{1}{4}\hat{H}^{(1)}{}_{abz}\hat{H}^{(1)\,ab}{}_z = -\tfrac{1}{4}k^{-2}G^{(1)\,2}$$
$$-\frac{\alpha'}{8}\Big\{2\varphi_A F^A\cdot k^{-1}G^{(0)} + \Big[\varphi^2 - \tfrac{1}{4}K^{(+)\,2} + 2(\partial\log k)^2\Big]F\cdot G^{(0)}$$
$$+k^{-1}R^{(0)}_{(-)\,ab}{}^{cd}K^{(+)}{}_{cd}G^{(0)\,ab} - \tfrac{1}{2}k^{-1}G^{(0)\,ab}K^{(-)}{}_a{}^cK^{(-)}{}_b{}^dK^{(+)}{}_{cd}$$
$$-4k^{-1}G^{(0)\,ab}\mathcal{D}^{(0)}_{(-)\,[a}K^{(-)}{}_{b]\,c}\partial^c\log k\Big\}\,. \tag{2.46}$$

Let us now move to the gauge-invariant combination $\hat{H}^{(1)}{}_{abc}$, that we will identify with the 9-dimensional Kalb-Ramond 3-form field strength. Using the zeroth-order result, we get

$$\hat{H}^{(1)}{}_{abc} = H^{(0)}{}_{abc} + \frac{\alpha'}{4}\left(\hat{\omega}^{\rm YM}{}_{abc} + \hat{\omega}^{\rm L\,(0)}{}_{abc}\right)\,. \tag{2.47}$$

Using Eqs. (2.32) it is almost immediately seen that

$$\hat{\omega}^{\rm YM}{}_{abc} = \omega^{\rm YM}{}_{abc} + 3k\varphi_A F_{[ab}A^A{}_{c]}\,. \tag{2.48}$$

Half of the last term should be integrated by parts, and the final result is

$$\hat{\omega}^{\rm YM}{}_{abc} = \omega^{\rm YM}{}_{abc} + 3e_a{}^\mu e_b{}^\nu e_c{}^\rho\left[\partial_{[\mu}\left(kA_{\nu|}\varphi_A A^A{}_{|\rho]}\right) + A_{[\mu}\partial_{\nu|}\left(k\varphi_A A^A{}_{|\rho]}\right) + \frac{3}{2}k\varphi_A A^A_{[\mu}F_{\nu\rho]}\right]\,. \tag{2.49}$$

The second term in the above expression, a total derivative, will combine with $\hat{B}_{\mu\nu}$ (and terms coming from $\hat{\omega}^{\rm L\,(0)}{}_{abc}$) to give $B^{(1)}{}_{\mu\nu}$ and the third term, as we know, combines with $\hat{B}_{\mu\underline{z}}$ (and terms coming from $\hat{\omega}^{\rm L\,(0)}{}_{abc}$) to give $B^{(1)}{}_\mu$.

The above result can be applied to $\hat{\omega}^{\rm L\,(0)}{}_{abc}$, using the definitions Eq. (2.38). We get

$$\hat{\omega}^{\mathrm{L}\,(0)}{}_{abc} = \omega^{\mathrm{L}\,(0)}{}_{abc} + 3e_a{}^{\mu}e_b{}^{\nu}e_c{}^{\rho}\left[\mathcal{D}^{(0)}_{(-)[\mu}K^{(-)}{}_{\nu}{}^{e}K^{(-)}{}_{\rho]\,e} + \partial_{[\mu}\left(kA_{\nu|}\varphi^{\hat{e}}{}_{\hat{f}}A^{\hat{f}}{}_{\hat{e}\,|\rho]}\right)\right.$$
$$\left. +A_{[\mu}\partial_{\nu|}\left(k\varphi^{\hat{e}}{}_{\hat{f}}A^{\hat{f}}{}_{\hat{e}\,|\rho]}\right) + \frac{1}{2}k\varphi^{\hat{e}}A^{\hat{f}}{}_{\hat{e}[\mu}F_{\nu\rho]}\right]\,. \tag{2.50}$$

Defining

$$B^{(1)}{}_{\mu\nu} \equiv \hat{B}_{\mu\nu} + A_{[\mu}\left[\hat{B}_{\nu]\,\underline{z}} + \frac{\alpha'}{4}k\left(\hat{A}^{A}{}_{|\nu]}\hat{A}_{A\,\underline{z}} + \hat{\Omega}^{(0)}_{(-)\,|\nu]}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{b}}{}_{\hat{a}}\right)\right]\,, \tag{2.51}$$

we find

$$\hat{H}^{(1)}{}_{abc} = H^{(1)}{}_{abc}\,, \tag{2.52a}$$

$$H^{(1)}{}_{\mu\nu\rho} \equiv 3\partial_{[\mu}B^{(1)}{}_{\nu\rho]} - \tfrac{3}{2}A_{[\mu}G^{(1)}{}_{\nu\rho]} - \tfrac{3}{2}B^{(1)}{}_{[\mu}F_{\nu\rho]}$$
$$+\frac{\alpha'}{4}\left(\omega^{\mathrm{YM}}{}_{\mu\nu\rho} + \omega^{\mathrm{L}\,(0)}_{(-)\,\mu\nu\rho} + 3\mathcal{D}_{(-)\,[\mu}K^{(-)}{}_{\nu}{}^{e}K^{(-)}{}_{\rho]\,e}\right)\,. \tag{2.52b}$$

Summarizing, the reduction of all the terms in the action but the last one gives, to $\mathcal{O}(\alpha')$,

$$\int dz\int d^9x\sqrt{|g|}e^{-2\phi}\left\{R - 4(\partial\phi)^2 + \left(1+\frac{\alpha'}{4}\varphi^2\right)(\partial\log k)^2 + \frac{\alpha'}{4}(\mathfrak{D}\varphi)^2 + \frac{\alpha'}{4}\partial_a\log k\,\partial^a\varphi^2\right.$$
$$-\tfrac{1}{4}\left(1+\frac{\alpha'}{2}\varphi^2\right)k^2F^2 + \tfrac{1}{12}H^{(1)\,2} - \tfrac{1}{4}k^{-2}G^{(1)\,2} - \frac{\alpha'}{8}\left[F_A\cdot F^A + 2\varphi_A F^A\cdot K^{(+)}\right.$$
$$+\left[\varphi^2 - \tfrac{1}{4}K^{(+)\,2} + 2(\partial\log k)^2\right]F\cdot G^{(0)} + R^{(0)}_{(-)\,ab}{}^{cd}K^{(+)}{}_{cd}k^{-1}G^{(0)\,ab}$$
$$\left.\left.-\tfrac{1}{2}k^{-1}G^{(0)\,ab}K^{(-)}{}_{a}{}^{c}K^{(-)}{}_{b}{}^{d}K^{(+)}{}_{cd} - 4k^{-1}G^{(0)\,ab}\mathcal{D}^{(0)}_{(-)\,[a}K^{(-)}{}_{b]\,c}\partial^c\log k\right]\right\}\,. \tag{2.53}$$

Now, we must deal with the last term. We deal with it in the same way as we dealt with the Yang-Mills kinetic term:

$$\begin{aligned}
\hat{R}^{(0)}_{(-)\,\hat{c}\hat{d}}{}^{\hat{a}}{}_{\hat{b}}\,\hat{R}^{(0)}_{(-)}{}^{\hat{c}\hat{d}\,\hat{b}}{}_{\hat{a}} &= \hat{F}^{\hat{a}}{}_{\hat{b}\,cd}\hat{F}^{\hat{b}}{}_{\hat{a}}{}^{cd} - 2\hat{F}^{\hat{a}}{}_{\hat{b}\,cz}\hat{F}^{\hat{b}}{}_{\hat{a}}{}^{c}{}_{z} \\
&= \left(F^{\hat{a}}{}_{\hat{b}\,cd} + k\varphi^{\hat{a}}{}_{\hat{b}}F_{cd}\right)\left(F^{\hat{b}}{}_{\hat{a}}{}^{cd} + k\varphi^{\hat{b}}{}_{\hat{a}}F^{cd}\right) \\
&\quad - 2\left(\mathcal{D}_c\varphi^{\hat{a}}{}_{\hat{b}} + \varphi^{\hat{a}}{}_{\hat{b}}\partial_c\log k\right)\left(\mathcal{D}^c\varphi^{\hat{b}}{}_{\hat{a}} + \varphi^{\hat{b}}{}_{\hat{a}}\partial^c\log k\right)\,.
\end{aligned} \tag{2.54}$$

The Lorentz-covariant derivatives in the last line must be taken with respect to the connection $A^{\hat{a}\hat{b}}{}_{\mu}$, which means that the ab components contain contributions from $A^{az}{}_{\mu}$ etc. Taking this fact into account, if we split the hatted indices into unhatted indices and z components, we get

$$\begin{aligned}
&\left(F^{a}{}_{b\,\mu\nu} + k\varphi^{a}{}_{b}F_{\mu\nu}\right)\left(F^{b}{}_{a}{}^{\mu\nu} + k\varphi^{b}{}_{a}F^{\mu\nu}\right) + 2\left(F^{az}{}_{\mu\nu} + k\varphi^{az}F_{\mu\nu}\right)\left(F_{a}{}^{z\mu\nu} + k\varphi_{a}{}^{z}F^{\mu\nu}\right) \\
&- 2\left(\mathcal{D}_c\varphi^{a}{}_{b} - A^{az}{}_{c}\varphi_{b}{}^{z} + A_{b}{}^{z}{}_{c}\varphi^{az} + \varphi^{a}{}_{b}\partial_c\log k\right)\left(\mathcal{D}^c\varphi^{b}{}_{a} - A^{bz}{}_{c}\varphi_{a}{}^{z} + A_{a}{}^{z}{}_{c}\varphi^{bz} + \varphi^{b}{}_{a}\partial^c\log k\right) \\
&- 4\left(\mathcal{D}_c\varphi^{az} + A^{bz}{}_{c}\varphi^{a}{}_{b} + \varphi^{az}\partial_c\log k\right)\left(\mathcal{D}^c\varphi_{a}{}^{z} + A^{bz\,c}\varphi_{ab} + \varphi_{a}{}^{z}\partial^c\log k\right)\,,
\end{aligned} \tag{2.55}$$

where, now $\mathcal{D}_c$ is the Lorentz-covariant derivative with respect to the connection $A^{ab}{}_{\mu}$.

Substituting the components $A^{ab}{}_{\mu}, A^{az}{}_{\mu}, \varphi^{ab}, \varphi^{az}$ by their values, we get

$$\begin{aligned}
&\left(R^{(0)}_{(-)\,\mu\nu}{}^{a}{}_{b} - \tfrac{1}{2}K^{(-)}{}_{[\mu}{}^{a}K^{(-)}{}_{\nu]\,b} - \tfrac{1}{2}K^{(+)\,a}{}_{b}kF_{\mu\nu}\right)\left(R^{(0)}_{(-)}{}^{\mu\nu\,b}{}_{a} - \tfrac{1}{2}K^{(-)\,\mu\,b}K^{(-)\,\nu}{}_{a} - \tfrac{1}{2}K^{(+)\,b}{}_{a}kF^{\mu\nu}\right) \\
&+ 2\left(\mathcal{D}^{(0)}_{(-)\,[\mu}K^{(-)}{}_{\nu]\,a} - \partial_a\log k\,kF_{\mu\nu}\right)\left(\mathcal{D}^{(0)}_{(-)}{}^{[\mu|}K^{(-)\,|\nu]\,a} - \partial^a\log k\,kF^{\mu\nu}\right) \\
&+ \tfrac{1}{2}\left(\mathcal{D}^{(0)}_{(-)}{}^{c}K^{(+)\,ab} - 2K^{(-)\,c\,[a}\partial^{b]}\log k + K^{(+)\,ab}\partial^c\log k\right) \\
&\left(\mathcal{D}^{(0)}_{(-)\,c}K^{(+)}{}_{ab} - 2K^{(-)}{}_{c\,[a}\partial_{b]}\log k + K^{(+)}{}_{ab}\partial_c\log k\right) \\
&- 4\left(\mathcal{D}^{(0)}_{(-)}{}^{c}\partial^a\log k - \tfrac{1}{4}K^{(-)\,cb}K^{(+)}{}_{b}{}^{a} + \partial^a\log k\partial^c\log k\right) \\
&\left(\mathcal{D}^{(0)}_{(-)\,c}\partial_a\log k - \tfrac{1}{4}K^{(-)}{}_{c}{}^{b}K^{(+)}{}_{ba} + \partial_a\log k\partial_c\log k\right)\,.
\end{aligned} \tag{2.56}$$

Operating, we finally get

$$
\begin{aligned}
\hat{R}^{(0)}_{(-)\,\hat{c}\hat{d}}{}^{\hat{a}}{}_{\hat{b}}\,\hat{R}^{(0)}_{(-)}{}^{\hat{c}\hat{d}\,\hat{b}}{}_{\hat{a}} = {} & R^{(0)}_{(-)\,\mu\nu}{}^{a}{}_{b}R^{(0)}_{(-)}{}^{\mu\nu\,b}{}_{a} + R^{(0)}_{(-)\,\mu\nu}{}^{ab}K^{(-)\,\mu}{}_{a}K^{(-)\,\nu}{}_{b} + R^{(0)}_{(-)\,\mu\nu}{}^{ab}K^{(+)}{}_{ab}kF^{\mu\nu} \\
& + \tfrac{1}{4}K^{(-)}{}_{[\mu|}{}^{a}K^{(-)}{}_{a}{}^{\nu}K^{(-)}{}_{|\nu]}{}^{b}K^{(-)}{}_{b}{}^{\mu} - \tfrac{1}{2}K^{(-)}{}_{\mu a}K^{(-)}{}_{\nu b}K^{(+)\,ab}kF^{\mu\nu} \\
& - \tfrac{1}{4}(K^{(+)})^2k^2F^2 + 2\mathcal{D}^{(0)}_{(-)}{}^{[\mu|}K^{(-)\,|\nu]\,a}\mathcal{D}^{(0)}_{(-)\,[\mu|}K^{(-)}{}_{|\nu]\,a} \\
& - 4\mathcal{D}^{(0)}_{(-)}{}^{\mu}K^{(-)\,\nu a}\partial_a\log k kF_{\mu\nu} + 2(\partial\log k)^2k^2F^2 \\
& + \tfrac{1}{2}\mathcal{D}^{(0)}_{(-)}{}^{c}K^{(+)\,ab}\mathcal{D}^{(0)}_{(-)\,c}K^{(+)}{}_{ab} - 2\mathcal{D}^{(0)}_{(-)}{}^{c}K^{(+)\,ab}K^{(-)}{}_{c\,a}\partial_b\log k \\
& + \mathcal{D}^{(0)}_{(-)}{}^{c}K^{(+)\,ab}K^{(+)}{}_{ab}\partial_c\log k + 2K^{(-)\,c\,[a}\partial^{b]}\log kK^{(-)}{}_{ca}\partial_b\log k \\
& - 2K^{(-)\,c\,a}\partial^b\log kK^{(+)}{}_{ab}\partial_c\log k + \tfrac{1}{2}(K^{(+)})^2(\partial\log k)^2 \\
& + 4\mathcal{D}^{(0)}_{(-)}{}^{c}\partial^a\log k\mathcal{D}^{(0)}_{(-)\,c}\partial_a\log k + 2\mathcal{D}^{(0)}_{(-)}{}^{c}\partial^a\log kK^{(-)}{}_{c}{}^{b}K^{(+)}{}_{ba} \\
& + 2K^{(-)\,cb}K^{(+)}{}_{b}{}^{a}\partial_c\log k\partial_a\log k - \tfrac{1}{4}K^{(-)\,a}{}_{b}K^{(+)\,b}{}_{c}K^{(-)\,c}{}_{d}K^{(+)\,d}{}_{a} \\
& - 8\mathcal{D}^{(0)}_{(-)}{}^{c}\partial^a\log k\partial_a\log k\partial_c\log k - 4((\partial\log k)^2)^2\,.
\end{aligned}
\tag{2.57}
$$

With all these terms, the action takes the form

$$
\begin{aligned}
S = \frac{g_s^2(2\pi\ell_s)}{16\pi G_N^{(10)}} \int d^9x\sqrt{|g|}e^{-2\phi}\Big\{& R - 4(\partial\phi)^2 + \frac{\alpha'}{4}(\mathfrak{D}\varphi)^2 - \partial_a k^{-1}\partial^a k^{(1)} \\
& - \tfrac{1}{4}k^{(1)\,2}F^2 - \tfrac{1}{4}k^{-2}G^{(1)\,2} + \tfrac{1}{2}(1-k^{(1)}k^{-1})F\cdot G^{(1)} + \tfrac{1}{12}H^{(1)\,2} \\
& - \frac{\alpha'}{8}\Big[F_A\cdot F^A + R_{(-)}^{(0)}{}^{a}{}_{b}\cdot R_{(-)}^{(0)}{}^{b}{}_{a} + R_{(-)}^{(0)}{}_{ab}{}^{cd}\left(K^{(-)\,a}{}_{c}K^{(-)\,b}{}_{d} + K^{(+)\,ab}K^{(+)}{}_{cd}\right) \\
& + 2\varphi_A F^A\cdot K^{(+)} \\
& - \tfrac{3}{4}K^{(+)\,a}{}_{b}K^{(-)\,b}{}_{c}K^{(+)\,c}{}_{d}K^{(-)\,d}{}_{a} + \tfrac{1}{8}K^{(-)\,a}{}_{b}K^{(-)\,b}{}_{c}K^{(-)\,c}{}_{d}K^{(-)\,d}{}_{a} - \tfrac{1}{8}\left(K^{(-)}\cdot K^{(-)}\right)^2 \\
& - 4K^{(+)\,ab}\mathcal{D}_{(-)\,a}^{(0)}K^{(-)}{}_{bc}\partial^c\log k - 2K^{(-)\,ab}\mathcal{D}_{(-)\,a}^{(0)}K^{(+)}{}_{bc}\partial^c\log k \\
& + 2\mathcal{D}_{(-)}^{(0)\,[a|}K^{(-)\,|b]\,c}\mathcal{D}_{(-)\,[a|}^{(0)}K^{(-)}{}_{|b]\,c} + \tfrac{1}{2}\mathcal{D}_{(-)}^{(0)\,c}K^{(+)\,ab}\mathcal{D}_{(-)\,c}^{(0)}K^{(+)}{}_{ab} \\
& - 4\mathcal{D}_{(-)}^{(0)\,c}\partial^a\log k\,\mathcal{D}_{(-)\,c}^{(0)}\partial_a\log k + 2K^{(-)\,ac}K^{(+)}{}_{c}{}^{b}\mathcal{D}_{(-)\,a}^{(0)}\partial_b\log k \\
& + 2K^{(-)\,c\,[a}\partial^{b]}\log k\,K^{(-)}{}_{ca}\partial_b\log k\Big]\Big\}\,,
\end{aligned}
\tag{2.58}
$$

where we have defined

$$
k^{(1)} \equiv k\left[1 + \frac{\alpha'}{4}\left(\varphi^2 - \tfrac{1}{4}K^{(+)\,2} + 2(\partial\log k)^2\right)\right]\,, \tag{2.59}
$$

and we have added some $\mathcal{O}(\alpha'^2)$ terms in order to obtain nicer or simpler expressions.

2.3.1 T duality

All the $\mathcal{O}(\alpha')$ terms of the reduced action Eq. (2.58) are invariant under the zeroth-order T duality transformations Eqs. (2.28), and the whole action is invariant to $\mathcal{O}(\alpha')$ under the transformations

$$
A'_\mu = B^{(1)}{}_\mu\,, \qquad B^{(1)\prime}{}_\mu = A_\mu\,, \qquad k' = 1/k^{(1)}\,, \tag{2.60}
$$

which reduce to the zeroth-order ones in Eqs. (2.28) when we set $\alpha' = 0$. Furthermore, observe that

$$\begin{aligned}
k^{(1)\,\prime} &= k'\left[1+\frac{\alpha'}{4}\left(\varphi^2-\tfrac{1}{4}K^{(+)\,2}+2(\partial\log k)^2\right)\right]\\
&= k^{(1)\,-1}\left[1+\frac{\alpha'}{4}\left(\varphi^2-\tfrac{1}{4}K^{(+)\,2}+2(\partial\log k)^2\right)\right]\\
&= k^{-1}\left[1+\mathcal{O}(\alpha'^{\,2})\right]\,.
\end{aligned}\tag{2.61}$$

Using the relation between the higher- and lower-dimensional fields, these transformations can be expressed in terms of the higher-dimensional ones in the form

$$\begin{aligned}
\hat{g}'_{\mu\nu} &= \hat{g}_{\mu\nu}+\frac{\hat{g}_{\underline{zz}}\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}\mu}\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}\nu}}{\hat{\mathfrak{G}}^{(1)2}_{\underline{zz}}}-\frac{2\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}(\mu}\hat{g}_{\nu)\underline{z}}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}\,,\\
\hat{B}'_{\mu\nu} &= \hat{B}_{\mu\nu}-\frac{\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}[\mu}\hat{\mathfrak{G}}^{(1)}{}_{\nu]\underline{z}}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}\,,\\
\hat{g}'_{\underline{z}\mu} &= -\frac{\hat{g}_{\underline{z}\mu}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}+\frac{\hat{g}_{\underline{zz}}\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}\mu}}{\hat{\mathfrak{G}}^{(1)2}_{\underline{zz}}}\,, &\qquad \hat{B}'_{\underline{z}\mu} &= -\frac{\hat{B}_{\underline{z}\mu}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}-\frac{\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}\mu}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}\,,\\
\hat{g}'_{\underline{zz}} &= \frac{\hat{g}_{\underline{zz}}}{\hat{\mathfrak{G}}^{(1)2}_{\underline{zz}}}\,, &\qquad e^{-2\hat{\phi}'} &= e^{-2\hat{\phi}}|\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}|\,,\\
\hat{A}'^{A}{}_{\underline{z}} &= -\frac{\hat{A}^{A}{}_{\underline{z}}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}\,, &\qquad \hat{A}'^{A}{}_{\mu} &= \hat{A}^{A}{}_{\mu}-\frac{\hat{A}^{A}{}_{\underline{z}}\hat{\mathfrak{G}}^{(1)}{}_{\underline{z}\mu}}{\hat{\mathfrak{G}}^{(1)}{}_{\underline{zz}}}\,,
\end{aligned}\tag{2.62}$$

where the tensor $\hat{\mathfrak{G}}^{(1)}{}_{\hat{\mu}\hat{\nu}}$ is defined by

$$\hat{\mathfrak{G}}^{(1)}{}_{\hat{\mu}\hat{\nu}}\equiv\hat{g}_{\hat{\mu}\hat{\nu}}-\hat{B}_{\hat{\mu}\hat{\nu}}-\frac{\alpha'}{4}\left\{\hat{A}^{A}{}_{\hat{\mu}}\hat{A}_{A\,\hat{\nu}}+\hat{\Omega}^{(0)}_{(-)\,\hat{\mu}}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\hat{\nu}}{}^{\hat{b}}{}_{\hat{a}}\right\}\,.\tag{2.63}$$

These are the α'-corrected Buscher rules first found in Ref. [85] and later rediscovered elsewhere [111, 112].

It is well known that $\mathcal{N}=1, d=10$ supergravity [118, 119] coupled to n_V Abelian vector multiplets [118, 119] and dimensionally reduced on a T^n has a global $\mathrm{O}(n, n+n_V)$ symmetry which was shown in Ref. [96] to be related to string T duality. In the case at hand, the YM vectors are, generically, non-Abelian, which reduces the symmetry to just $\mathrm{O}(n,n)$ [115] or just $\mathrm{O}(1,1)$ here. This group consists of the discrete transformation that give rise to the Buscher rules Eq. (2.60) and rescalings of just certain lower-dimensional fields:

$$A'_{\mu}=\lambda^{-1}A_{\mu}\,,\qquad B^{(1)\,\prime}{}_{\mu}=\lambda B^{(1)\,\prime}{}_{\mu}\,,\qquad k'=\lambda k\,.\tag{2.64}$$

Under these rescalings $K^{\pm}, H^{(1)}$ and the Lorentz curvature terms remain invariant while

$$k^{(1)\,\prime} = \lambda k^{(1)}\,. \tag{2.65}$$

It can be checked that the dimensionally-reduced action Eq. (2.58) is invariant under these transformations and, therefore, under the whole O(1,1) group.

We observe that the kinetic term of the KK and winding vectors is the sum of two separately O(1,1)-invariant terms

$$-\tfrac{1}{4}(F_{\mu\nu}\ ,\ G^{(1)}{}_{\mu\nu}) \begin{pmatrix} k^{(1)\,2} & 0 \\ & \\ 0 & 1/k^2 \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \\ G^{(1)\,\mu\nu} \end{pmatrix} + \tfrac{1}{2}(1 - k^{(1)}/k) F \cdot G^{(1)}\,, \tag{2.66}$$

and that the diagonal kinetic matrix transforms consistently under O(1,1) transformations even though, as different to the zeroth-order case, the kinetic matrix is not an O(1,1) matrix itself. The consistency is related to the fact that it is part of a $\mathrm{O}(1,1+n_V)$ matrix.

2.4 Entropy formula

We can use the dimensionally reduced action we have obtained to calculate the entropy of some d-dimensional heterotic string black holes using the Iyer-Wald prescription [22, 28]. These black holes must be solutions of the theory defined by the action Eq. (2.58) understood as a d-dimensional action. Therefore, they must be solutions of the theory defined by the action Eq. (2.13) understood as a $(d+1)$-dimensional action[15] admitting an isometry. Since this $(d+1)$-dimensional action can be obtained from the 10-dimensional one by a trivial compactification on a $10-(d+1)$-dimensional torus, the metrics of the 10-dimensional solutions corresponding to the d-dimensional black holes are the direct products of non-trivial $(d+1)$-dimensional metrics and the metric of a $10-(d+1)$-dimensional torus. The non-extremal 4-dimensional Reissner-Nordström black hole of Ref. [32] or the heterotic version of the 5-dimensional Strominger-Vafa black hole of Ref. [29] are two interesting examples of this kind of solution.

Applying directly the Iyer-Wald prescription to the d-dimensional action Eq. (2.58) we obtain the following entropy formula expressed in string-frame variables:

[15] The constant in front of the action should now contain the volume of a $(10-d)$-dimensional torus instead of that of circle, that is

$$\frac{g_s^2 (2\pi\ell_s)^{10-d}}{16\pi G_N^{(10)}} = \frac{(g_s^{(d)})^2}{16\pi G_N^{(d)}}\,, \tag{2.67}$$

where $g_s^{(d)}$ is the d-dimensional string coupling constant or the vacuum expected value of the d-dimensional dilaton $< e^{\phi} >= e^{\phi_\infty}$ and $G_N^{(d)}$ the d-dimensional Newton constant. The relations of the 10-dimensional and d-dimensional ones with the volume of the $(10-d)$-dimensional compact space, V_{10-d} is

$$g_s^2 = V_{10-d}/(2\pi\ell_s)^{10-d} g_s^{(d)\,2}\,, \tag{2.68a}$$

$$G_N^{(10)} = G_N^{(d)} V_{10-d}\,. \tag{2.68b}$$

$$S = -2\pi \int_\Sigma d^{d-2}x \sqrt{|h|} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}\,, \tag{2.69a}$$

$$\frac{\partial \mathcal{L}}{\partial R_{abcd}} = \frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_N^{(d)}} \left\{ g^{ab,\,cd} - \frac{\alpha'}{8}\left[H^{(0)\,abg}\left(\omega_g{}^{cd} - H_g^{(0)\,cd}\right) \right.\right.$$

$$\left.\left. -2R_{(-)}^{(0)\,abcd} + K^{(-)\,[a|c}K^{(-)\,|b]d} + K^{(+)\,ab}K^{(+)\,cd}\right]\right\}\,, \tag{2.69b}$$

where $|h|$ is the absolute value of the determinant of the metric induced over the event horizon, $g^{ab,cd} = \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})$, ϵ^{ab} is the event horizon's binormal normalized so that $\epsilon_{ab}\epsilon^{ab} = -2$ and R_{abcd} is the Riemann tensor.

2.4.1 The Wald entropy of the α'-corrected Strominger-Vafa black hole

The entropy formula Eq. (2.69b) has been shown in Ref. [32] to give an entropy which is related to the Hawking temperature by the thermodynamic relation

$$\frac{\partial S}{\partial M} = \frac{1}{T}\,, \tag{2.70}$$

for the particular case of α'-corrected, 4-dimensional, non-extremal Reissner-Nordström black holes. In this section we want to recalculate the Wald entropy of the α'-corrected Strominger-Vafa black hole. Being an extremal black hole, we will not be able to check that the entropy obtained is related to the temperature as above, but, instead, we will be able to compare with other results obtained in the literature and with the microscopic calculations.

The 5-dimensional α'-corrected Strominger-Vafa black hole corresponds to the 10-dimensional solution of the Heterotic Superstring effective action [29, 86]

$$d\hat{s}^2 = \frac{2}{\mathcal{Z}_-} du\left(dv - \tfrac{1}{2}\mathcal{Z}_+ du\right) - \mathcal{Z}_0(d\rho^2 + \rho^2 d\Omega^2_{(3)}) - dy^i dy^i\,, \qquad i = 1,\ldots,4\,, \tag{2.71a}$$

$$\hat{H}^{(1)} = d\mathcal{Z}_-^{-1} \wedge du \wedge dv + \star_4 d\mathcal{Z}_0\,, \tag{2.71b}$$

$$e^{-2\hat{\phi}} = e^{-2\hat{\phi}_\infty}\mathcal{Z}_-/\mathcal{Z}_0\,, \tag{2.71c}$$

where $\star_4$ stands for the Hodge dual in the 4-dimensional Euclidean space with metric $d\rho^2 + \rho^2 d\Omega^2_{(3)}$, and where the $\mathcal{Z}$ functions take the values[16]

[16]The Regge slope parameter α' in Refs. [29, 86] has been replaced by $\alpha'/8$ here to obtain the correct form of the action and solutions.

$$\mathcal{Z}_0 = 1 + \frac{\tilde{q}_0}{\rho^2} - \alpha' \frac{\rho^2 + 2\tilde{q}_0}{(\rho^2 + \tilde{q}_0)^2} + \mathcal{O}(\alpha'^2)\,, \tag{2.72a}$$

$$\mathcal{Z}_- = 1 + \frac{\tilde{q}_-}{\rho^2} + \mathcal{O}(\alpha'^2)\,, \tag{2.72b}$$

$$\mathcal{Z}_+ = 1 + \frac{\tilde{q}_+}{\rho^2} + 2\alpha' \frac{\tilde{q}_+(\rho^2 + \tilde{q}_0 + \tilde{q}_-)}{\tilde{q}_0(\rho^2 + \tilde{q}_0)(\rho^2 + \tilde{q}_-)} + \mathcal{O}(\alpha'^2)\,. \tag{2.72c}$$

Compactifying this solution in a T^4 parameterized by the coordinates y_i is trivial. Then, we just have to compactify the resulting 6-dimensional solution to $d = 5$ using the results obtained here along the coordinate $z \equiv u/k_\infty$, where k_∞ is the asymptotic value of the KK scalar k. It is helpful to rewrite the 6-dimensional solution in the form

$$d\hat{s}^2 = \frac{1}{\mathcal{Z}_+\mathcal{Z}_-}dt^2 - \mathcal{Z}_0(d\rho^2 + \rho^2 d\Omega^2_{(3)}) - \frac{k_\infty^2 \mathcal{Z}_+}{\mathcal{Z}_-}\left(dz - \frac{1}{k_\infty \mathcal{Z}_+}dt\right)^2\,, \tag{2.73a}$$

$$\hat{H}^{(1)} = d\left(-\frac{k_\infty}{\mathcal{Z}_-}dt \wedge dz\right) + \star_4 d\mathcal{Z}_0\,, \tag{2.73b}$$

$$e^{-2\hat{\phi}} = e^{-2\hat{\phi}_\infty}\mathcal{Z}_-/\mathcal{Z}_0\,, \tag{2.73c}$$

where we have set $v = t$, to identify immediately the following 5-dimensional fields:[17]

$$ds^2 = \frac{1}{\mathcal{Z}_+\mathcal{Z}_-}dt^2 - \mathcal{Z}_0(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,, \tag{2.74a}$$

$$H^{(1)} = \star_4 d\mathcal{Z}_0\,, \tag{2.74b}$$

$$F = d\left(-\frac{1}{k_\infty \mathcal{Z}_+}dt\right)\,, \tag{2.74c}$$

$$G^{(0)} = d\left(-\frac{k_\infty}{\mathcal{Z}_-}dt\right)\,, \tag{2.74d}$$

$$e^{-2(\phi - \phi_\infty)} = \sqrt{\mathcal{Z}_+\mathcal{Z}_-}/\mathcal{Z}_0\,, \tag{2.74e}$$

$$k/k_\infty = \sqrt{\mathcal{Z}_+/\mathcal{Z}_-}\,, \tag{2.74f}$$

[17] We have only computed $G^{(0)}$ and not $G^{(1)}$ because of its complication and because it is unnecessary to do it for the calculation of the entropy. On the other hand, the Kalb-Ramond field is customarily dualized into another vector field to which the third charge $\tilde{q}_0$ is associated.

and the T duality even and odd 2-forms

$$K^{\pm} = -\frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}\left(\frac{\mathcal{Z}'_+}{\mathcal{Z}_+} \pm \frac{\mathcal{Z}'_-}{\mathcal{Z}_-}\right) d\rho\wedge dt\,, \tag{2.75}$$

where a prime indicates derivative with respect to ρ.

In the Vielbein basis

$$e^0 = \frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}dt\,, \qquad e^1 = \sqrt{\mathcal{Z}_0}d\rho\,, \qquad e^i = \tfrac{1}{2}\sqrt{\mathcal{Z}_0}\rho\theta^i\,, \tag{2.76}$$

where the θ^i are the left-invariant SU(2) Maurer-Cartan 1-forms that satisfy $d\Omega^2{}_{(3)} = \frac{1}{4}\theta^i\theta^i$, the binormal is given by just $\epsilon^{01} = +1$ and the entropy formula in Eqs. (2.69a) and (2.69b) becomes

$$S = \frac{1}{4G_N^{(5)}}\int_\Sigma d^3x e^{-2(\phi-\phi_\infty)}\sqrt{|h|}\left\{1+\frac{\alpha'}{4}\left[-2R^{0101}+(K^{(-)\,01})^2+(K^{(+)\,01})^2\right]\right\}\,. \tag{2.77}$$

The fields in the integrand are only functions of ρ and we can perform the integral over S^3. Evaluating the zeroth-order term at $\rho = 0$, where the horizon is located, we get

$$\begin{aligned}S = \frac{1}{4G_N^{(5)}}\Bigg\{A_{\mathcal{H}} + \alpha'\pi^2\lim_{\rho\to 0}\rho^3\sqrt{\mathcal{Z}_0\mathcal{Z}_+\mathcal{Z}_-}&\left[-\sqrt{\frac{\mathcal{Z}_+\mathcal{Z}_-}{\mathcal{Z}_0}}\left[\frac{1}{\sqrt{\mathcal{Z}_0}}\left(\frac{1}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-}}\right)'\right]'\right.\\ &\left.+\frac{1}{\mathcal{Z}_0}\left(\frac{\mathcal{Z}'_+}{\mathcal{Z}_+}\right)^2+\frac{1}{\mathcal{Z}_0}\left(\frac{\mathcal{Z}'_-}{\mathcal{Z}_-}\right)^2\right]\Bigg\}\,,\end{aligned} \tag{2.78}$$

where $A_{\mathcal{H}}$, the area of the horizon, is given by

$$A_{\mathcal{H}} = 2\pi^2\lim_{\rho\to 0}\rho^3\sqrt{\mathcal{Z}_0\mathcal{Z}_+\mathcal{Z}_-} = 2\pi^2\sqrt{\tilde{q}_0\tilde{q}_+\tilde{q}_-}\,. \tag{2.79}$$

Finally, we arrive at

$$S = \frac{A_{\mathcal{H}}}{4G_N^{(5)}}\left\{1+\frac{2\alpha'}{\tilde{q}_0}\right\}\,. \tag{2.80}$$

In order to compare this result with the microscopic entropy in Ref. [87], we have to express the charges $\tilde{q}_+, \tilde{q}_-, \tilde{q}_0$ in terms of the asymptotic charges[18]. First, we have to take into account the relation between $\tilde{q}_+, \tilde{q}_-, \tilde{q}_0$ and the numbers of fundamental strings n, momentum w and S5-branes N

$$\tilde{q}_+ = \frac{\alpha'^2 g_s^2 n}{R_z^2}\,, \qquad \tilde{q}_- = \alpha' g_s^2 w\,, \qquad \tilde{q}_0 = \alpha' N\,. \tag{2.81}$$

[18] See Refs. [29, 107], specially Eqs. (2.18),(2.20),(2.21) of the later.

Second, 10-dimensional Newton constant $G_N^{(10)}$ is given in terms of the Regge slope parameter $\alpha' = \ell_s^2$ and the 10-dimensional string coupling constant g_s by

$$G_N^{(10)} = 8\pi^6 g_s^2 \alpha'^4 \,. \tag{2.82}$$

This and Eq. (4.7b) allow us to rewrite the entropy Eq. (2.80) in the form

$$S = 2\pi\sqrt{nwN}\left(1+\frac{2}{N}\right) . \tag{2.83}$$

Finally, in terms of the asymptotic charges Q_+, Q_-, Q_0, which are related to the numbers of branes by

$$Q_+ = n\left(1+\frac{2}{N}\right) \qquad Q_- = w\,, \qquad Q_0 = N-1\,, \tag{2.84}$$

the entropy takes the final form that can be compared with the microscopic formula

$$S = 2\pi\sqrt{Q_+Q_-(Q_0+3)}\,. \tag{2.85}$$

2.5 Discussion

In this chapter we have performed the complete dimensional reduction of the Heterotic Superstring effective action to first order in α' using the formulation based on the supersymmetry completion of the Lorentz Chern-Simons terms that occur in the Kalb-Ramond field strength [73, 90]. We have found a $\mathbb{Z}_2$ transformation of the dimensionally-reduced action that leaves it invariant and that is an $\mathcal{O}(\alpha')$ generalization of the standard transformations that interchange KK and winding vectors and invert the KK scalar. In 10-dimensional variables (the components of the 10-dimensional fields) these transformations are nothing but the α'-corrected Buscher rules of the Heterotic Superstring theory, first found in [85].

Then, we used the dimensionally-reduced action to find, following the Iyer-Wald prescription [22, 28] an entropy formula for stringy black holes that can be obtained from a 10-dimensional solution by a single non-trivial compactification on a circle, supplemented by a trivial compactification on a torus. This formula was successfully applied to a non-extremal 4-dimensional Reissner-Nordström black hole in Ref. [32] and, in this chapter, we have applied it to the α'-corrected heterotic version of the Strominger-Vafa black hole of Ref. [29] obtaining an entropy formula that matches the microscopic result obtained in [87] once the relations between integration constants and asymptotic brane charges have been correctly taken into account. As explained in Ref. [107], the result obtained in Ref. [29] misses a factor of 2 that we recover here.

Part II

Black Hole Thermodynamics through Momentum Maps

3 The first law of black hole thermodynamics in Einstein-Maxwell theory

3.1 Introduction

Black-hole thermodynamics originate in the analogy between the behaviour of the area of the event horizon A and the second law obeyed by the thermodynamic entropy S noticed by Bekenstein [15, 16] in the results obtained by Christodoulou and Hawking [13, 17–19]. Shortly afterwards, in Ref. [20] Bardeen, Carter and Hawking extended this analogy by proving another three laws of black hole mechanics similar to the other three laws of thermodynamics involving the event horizon's surface gravity κ and angular velocity Ω and the black hole's mass M. The analogy, however, was only taken seriously after Hawking's discovery that black holes radiate as black bodies with a temperature $T = \kappa/2\pi$ [21], which implied the relation $S = A/4$, both in $c = G_N = \hbar = k = 1$ units.

Ever since the formulation of these four laws, it has been tried to extend their domain of application and validity with the inclusion of matter fields and terms of higher-order in the curvature, for instance. In Refs. [22, 27, 28] Wald and collaborators developed a new approach to demonstrate the first law of black hole mechanics in general diffeomorphism-invariant theories, beyond General Relativity. Since the surface gravity relation to the Hawking temperature only depends on generic properties of the event horizon, the quantity whose variation it multiplies in the first law is naturally associated to the Bekenstein-Hawking entropy S. This quantity, often called *Wald entropy*, is just $A/4$ in General Relativity but, in more general theories, there can be additional terms which can be understood, for instance, as α' corrections in Superstring Theories [29–34].

In the presence of matter fields, Wald's proof of the first law of black-hole mechanics had to be re-examined because one of the main assumptions Refs. [22, 28] is that all matter fields behave as tensors and, simply put, there are no tensor fields in nature apart form the metric and scalar fields (if any); all of them have some sort of gauge freedom and their transformations under diffeomorphisms are always coupled to gauge transformations. Indeed, as is well-known, fermionic fields coupled to gravity transform under a local Lorentz group as spinors and bosonic fields must transform under some gauge group if unwanted, typically negative-energy, states are to be eliminated. The only scalar in the Standard Model, the Higgs field, is, in fact, SU(2) doublet.

The simplest matter field that, coupled to gravity, allows for black-hole solutions is the Maxwell field [37, 38]. The presence of the field introduces an additional term of the form ΦdQ in the first law which takes into account the changes in the mass of the black hole when its charge Q changes. In this term Φ is the electric potential on the horizon and a *generalized zeroth law* states that it takes a constant value over the horizon. The value of Φ is customarily taken to be $k^\mu A_\mu$, where k^μ is the Killing vector for which the event horizon is its associated Killing horizon and where it is assumed that the electromagnetic field is in a gauge in which Φ is, indeed, constant.

This definition of Φ is clearly not gauge-invariant. This is a problem of principle,[1] which, as we are going to show, is related to the more fundamental problem we were discussing: the fact that Wald's proof of the first law does not deal properly with fields which have some kind of gauge freedom. In Wald's proof, one considers diffeomorphisms which are symmetries of all the dynamical fields, but the naive definition of invariance of fields with gauge freedom under diffeomorphsisms through the standard Lie derivative is not gauge invariant. This problem affects the gravitational field itself when it is described in terms of the Vielbein instead of the metric.

A solution for this particular case was provided in Ref. [40] by defining the variation of the Vielbein under diffeomorphisms through the Lie-Lorentz derivative Refs. [41, 44–47] which can be understood as a generalization of the Lie derivative which transforms covariantly under local Lorentz transformations. If the Vielbein is annihilated by the Lie-Lorentz derivative with respect to some vector field in some gauge it will be annihilated in any gauge and, as a matter of fact, the vector field will be a Killing vector field of the metric. The Lie-Lorentz derivative can be defined on all fields with Lorentz (spinor or vector) indices, a fact that has been used to extend the proof of the first law of black hole mechanics to supergravity in Ref. [106].

A more general mathematically rigorous approach was proposed in [51] using the formalism of principal gauge bundles which encompasses Yang-Mills and Lorentz fields but, unfortunately, not the Kalb-Ramond field or higher-rank form fields of string theory.[2] Perhaps the most interesting result in that paper is the realization that all the *zeroth-laws* (the constancy of the surface gravity, electric potential, etc.) on the horizon fit into a common pattern. In this chapter we are going to recover and reformulate this result in terms of the *momentum map*, using gauge-covariant derivatives in which this object plays a crucial role.[3]

Although gauge-covariant Lie derivatives are, perhaps, not the most mathematically rigorous tool one can use, they can be generalized to frameworks other than principal gauge bundles.[4] Our goal in this chapter is to show they can be consistently used in a simpler context (the Einstein-Maxwell theory described in terms of Vielbeins) and the objects to which the generalized zeroth law applies (here the surface temperature and the electric potential) are the gauge-invariant momentum maps associated to each gauge symmetry (Lorentz and U(1)) evaluated over the horizon.

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[1] There are other problems as well: in Wald's approach, the Noether charge, which contains a term in which Φ occurs, is evaluated over the bifurcation surface, but the Maxwell field of the Reissner-Nordström black hole turns out to be singular there in the traditional gauge [39].

[2] The first law has been proved for theories including one scalar and one p-form field in [52], although the gauge-invariance problem has not been discussed in it.

[3] In Refs. [120, 121], which covers some of the topics studied here this object emerges as an "improved gauge transformation".

[4] In this chapter, we will not consider those more complicated cases involving higher-rank p-form fields with Chern-Simons terms which typically arise in Superstring/Supergravity theories. We will consider the case of the Kalb-Ramond field with Yang-Mills and Lorentz Chern-Simons terms in its field strength in Chapter 5, where we will show how the gauge-covariant derivative approach with momentum maps that we introduce here provides a gauge-covariant, unambiguous results for the Wald-Noether charge.

The emergence of the momentum map in this context may seem a bit strange; for instance, there is no mention of it in Ref. [40] in spite of their use of the (gauge-covariant) Lie-Lorentz derivative. As we will show, however, the momentum map is indeed present in the Lie-Lorentz derivative and plays the same role that the momentum map (change) we will introduce for the Maxwell case. As a matter of fact, gauge-covariant derivatives and the momentum map arise most naturally in the study of superalgebras of symmetry, when all the dynamical fields of a supergravity theory are left invariant by a set of supersymmetry and bosonic transformations that combine diffeomorphisms, gauge, local-Lorentz and local-supersymmetry transformations [42,48–50]. This object also plays a very interesting geometrical role in symmetric Riemannian spaces and in certain spaces of special holonomy when they admit Killing vectors that preserve their geometrical structures. When one wants to gauge the corresponding symmetries in theories with σ-models of that kind (typically supergravity theories) the momentum map plays an essential role in the definition of the gauge-covariant derivative [122].

This chapter is organized as follows: in Section 3.2 we introduce the gauge-covariant derivatives that we are going to use: Lie-Maxwell in Section 3.2.1 and Lie-Lorentz in Section 5.3.3. We also discuss the zeroth laws the respective momentum maps obey. This last section is essentially a review of the literature on the subject where we re-derive the formulae we are going to use in the main part of this chapter using our conventions (those of Ref. [42]). In Section 3.3 we describe the Einstein-Maxwell theory in d dimensions (action and equations of motion) in differential-form language and the d-dimensional Reissner-Nordström-Tangherlini black hole solutions. In Section 3.4 we compute the Wald-Noether charge for this theory using the transformations based on the gauge-covariant Lie derivatives defined in Section 3.2. Then, in Section 3.5 we prove the first law for this system, identifying the Wald entropy, which we compute for the Reissner-Nordström-Tangherlini black hole solutions. In Section 5.7 we briefly discuss our results and future directions of research.

3.2 Covariant Lie derivatives and momentum maps

One of the main ingredients in the proofs of the first law of black hole mechanics using Wald's formalism [22, 28] is the use of infinitesimal diffeomorphisms that leave invariant all the dynamical fields.

If we use the metric $g_{\mu\nu}$ as dynamical field, since the metric is just a tensor, its transformation under infinitesimal diffeomorphisms $\delta_\xi x^\mu = \xi^\mu(x)$ is given by (minus) the standard Lie derivative

$$\delta_\xi g_{\mu\nu} = -\pounds_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}\,, \tag{3.1}$$

which vanishes when ξ^μ is a Killing vector of $g_{\mu\nu}$. We will distinguish Killing vectors from generic vectors ξ^μ denoting them by k^μ.

If, as we want to do here, we use as dynamical field the Vielbein $e^a{}_\mu$ instead of $g_{\mu\nu}$, in order to define its symmetries, we face the well-known problem of the gauge freedom of $e^a{}_\mu$, which in this context has been treated in Refs. [40, 51]. The same happens with the electromagnetic potential A_μ, which also has been treated in this context in Refs. [51].

One way to deal with this problem is to define a gauge-covariant notion of Lie derivative. The Lie derivative in the corresponding principal bundle, used in Ref. [51] provides the most rigorous definition such a derivative. Here we will introduce a less sophisticated version that makes use of the so-called *momentum map* and which can be defined for more general fields such as the Kalb-Ramond 2-form of the Heterotic Superstring, which cannot be described in the framework of a principal bundle [53]. Gauge-covariant derivatives arise naturally in the commutator of two local supersymmetry transformations and in the construction of Lie superalgebras of supersymmetric backgrounds [42, 48–50].

Due to its simplicity, we start with the Maxwell field.

3.2.1 Lie-Maxwell derivatives

The electromagnetic field A_μ is a field with gauge freedom: we must consider physically equivalent two configurations that are related by the gauge transformation

$$\delta_\chi A_\mu = \partial_\mu \chi \,, \tag{3.2}$$

and, furthermore, as a general rule, it is not possible to give a globally regular expression of the electromagnetic field in a single gauge.[5] However, the standard Lie derivative does not commute with these gauge transformations and gives different results in different gauges. This is why a gauge-covariant notion of Lie derivative is needed in this case.

In the subsequent discussion it is convenient to use differential-form language. In terms of the electromagnetic 1-form potential $A \equiv A_\mu dx^\mu$, we define the electromagnetic field strength 2-form by $F = dA$ so that it satisfies the Bianchi identity $dF = 0$. In components we have

$$F \equiv \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \,, \qquad F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \,. \tag{3.3}$$

The field strength is invariant under the gauge transformations $\delta_\chi A = d\chi$ and we can treat it as a standard 2-form whose transformation under infinitesimal diffeomorphisms generated by ξ^μ is given by (minus) the standard Lie derivative which, on p-forms, acts as $\pounds_\xi = \imath_\xi d + d\imath_\xi$.[6]

Using the Bianchi identity we find that

$$\delta_\xi F = -d\imath_\xi F \,. \tag{3.4}$$

If ξ is a symmetry of all the dynamical fields, in which case we will denote it by k, we have that $\delta_k F = 0$ and the above equation implies that, locally, there is a gauge-invariant function P_k called *momentum map* such that[7]

$$\imath_k F = -dP_k \,. \tag{3.5}$$

P_k is defined by this equation up to an additive constant that we will discuss later.

[5]The main example of this situation is the magnetic monopole [54].

[6]In our conventions, for a p-form $\omega^{(p)}$ with components $\omega^{(p)}{}_{\mu_1\cdots\mu_p}$, $\imath_\xi \omega^{(p)}$ is the $(p-1)$-form with components $(\imath_\xi \omega^{(p)})_{\mu_1\cdot\mu_{p-1}} = \xi^\nu \omega^{(p)}{}_{\nu\mu_1\cdot\mu_{p-1}}$.

[7]The sign of P_k is purely conventional.

Let us now consider the variation of A under infinitesimal diffeomorphisms, which, according to general arguments (see *e.g.* Refs. [42, 51]) has to be given locally by a combination of (minus) the Lie derivative and a "compensating" gauge transformation generated by a ξ-dependent parameter χ_ξ which is to be determined by demanding that $\delta_k A = 0$ when $\delta_k F = 0$:

$$\delta_\xi A = -\pounds_\xi A + d\chi_\xi = -\imath_\xi F + d\left(\chi_\xi - \imath_\xi A\right) . \tag{3.6}$$

Then, taking into account Eq. (3.5), we conclude that

$$\chi_\xi = \imath_\xi A - P_\xi \, , \tag{3.7}$$

where P_ξ is a function of ξ which satisfies Eq. (3.5) when $\xi = k$ and generates a symmetry of all the dynamical fields.

It is natural to identify the above transformation $\delta_\xi A$ with (minus) a gauge-covariant Lie derivative of A that we can call *Lie-Maxwell derivative*

$$\delta_\xi A = -\mathbb{L}_\xi A \, , \qquad \mathbb{L}_\xi A \equiv \imath_\xi F + dP_\xi \, . \tag{3.8}$$

While this derivative does not enjoy the most important property of Lie derivatives $[\pounds_\xi, \pounds_\eta] = \pounds_{[\xi,\eta]}$ for generic vector fields ξ, η, it is clear that it does for those that generate symmetries of A and F and annihilates them. This is certainly enough for us.

For stationary asymptotically-flat black holes, when the Killing vector k is the one normal to the event horizon, the momentum map can be understood as the electric potential Φ which, evaluated on the horizon $\Phi_{\mathcal{H}}$, appears in the first law.[8] In the early literature (see *e.g.* Section 6.3.5 of Ref. [56]) it was assumed from the start that there is a gauge in which

$$\pounds_k A = \imath_k dA + d(\imath_k A) = 0 \, . \tag{3.9}$$

Then, the electric potential Φ was identified with $\imath_k A$ because, according to the above equation, $d\Phi = -\imath_k F$, which can be defined as the electric field for an observer associated to the time direction defined by k.

It is clear that P_k can be identified with Φ (both satisfy the same equation). However, in a general gauge, it will not be given by just $\imath_k A$ and we will have to compute it. Nevertheless, the main property of Φ, namely the fact that it is constant over the horizon (sometimes called *generalized zeroth law*) still holds because it is, actually, a property of $-\imath_k F$ based on the properties of k, the Einstein equations and the assumption that the energy-momentum tensor of the electromagnetic field satisfies the dominant energy condition.

3.2.2 Lie-Lorentz derivatives

The original motivation for the definition of a derivative covariant under local Lorentz transformations, often called the Lie-Lorentz derivative, was its need for the proper treat-

[8] See, for instance Ref. [55] for a proof of the first law in the context of 5-dimensional supergravity and the role that Φ plays in it.

ment of spinorial fields in curved spaces in such a way that the flat-space results were correctly recovered.

In Minkowski spacetime, fermionic fields transform in spinorial representations of the Lorentz group, which leaves invariant the spacetime metric $(\eta_{ab}) = \text{diag}(+ - \cdots -)$. Since generic spacetime metrics $g_{\mu\nu}$ do not have any isometries, the Lorentz group will not be realized as a group of general coordinate transformations (g.c.t.s) leaving invariant the spacetime metric. Weyl realized that, if one introduces an orthonormal base in cotangent space at a given point in spacetime

$$\{e^a = e^a{}_\mu dx^\mu\}\,, \qquad e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}\,, \tag{3.10}$$

the Lorentz group arises naturally as the group of linear transformations of the base

$$e^{a\,\prime} = \Lambda^a{}_b e^b \sim (\eta^a{}_b + \sigma^a{}_b)e^b\,, \tag{3.11}$$

($\sigma^a{}_b$ are the infinitesimal transformations) that preserves orthonormality.

$$\Lambda^a{}_c \Lambda^b{}_d \eta^{cd} = \eta^{ab}\,, \quad \Rightarrow \quad \sigma^{(a}{}_c \eta^{b)c} = \sigma^{(ab)} = 0\,. \tag{3.12}$$

In Ref. [123], Weyl proposed to define fermionic fields ψ as fields transforming in the spinorial representation of the Lorentz group that acts in the tangent and cotangent space, that is

$$\delta_\sigma \psi \equiv \tfrac{1}{2}\sigma^{ab}\Gamma_s(M_{ab})\psi\,, \tag{3.13}$$

where $\Gamma_r(M_{ab})$ stands for the matrices that represent the generators of the Lorentz group $\{M_{ab}\}$ in the representation r. As is well-known, the generators in the spinorial representation can be constructed taking antisymmetrized products of the gamma matrices γ^a, $\gamma^{ab} \equiv \gamma^{[a}\gamma^{b]}$

$$\Gamma_s(M_{ab}) = \tfrac{1}{2}\gamma_{ab}\,, \quad \Rightarrow \quad \delta_\sigma\psi \equiv \tfrac{1}{4}\sigma^{ab}\gamma_{ab}\psi\,. \tag{3.14}$$

Since these transformations can be different at each point, the Lorentz parameters σ^{ab} take different values at different points of the spacetime and become functions $\sigma^{ab}(x)$ which will be smooth if the bases of the tangent and cotangent space are assumed to vary smoothly so that they are smooth vector and 1-form fields.

Theories containing fermionic fields in curved spacetimes are required to be invariant under these local Lorentz transformations. Their construction demands the introduction of a gauge field, the so-called spin connection 1-form, conventionally denoted by $\omega^{ab} = \omega_\mu{}^{ab}dx^\mu$. The spin connection enters the Lorentz-covariant derivatives of any field T (indices not shown) transforming in the representation r of the Lorentz group as follows:

$$\mathcal{D}T^{(r)} \equiv \left[d - \tfrac{1}{2}\omega^{ab}\Gamma_r(M_{ab})\right]T^{(r)}\,. \tag{3.15}$$

The transformation properties of $T^{(r)}$ are preserved by the covariant derivative if, under infinitesimal local Lorentz transformations,

$$\delta_\sigma\omega^{ab} = \mathcal{D}\sigma^{ab} = \left[d - \tfrac{1}{2}\omega^{cd}\Gamma_{Adj}(M_{cd})\right]\sigma^{ab} = d\sigma^{ab} - 2\omega^{[a}{}_c\sigma^{|c|b]}\,. \tag{3.16}$$

From now on ∇_μ will denote the full (affine plus Lorentz) covariant derivative satisfying the first Vielbein postulate

$$0 = \nabla_\mu e^a{}_\nu \equiv \partial_\mu e^a{}_\nu - \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma_{\mu\nu}{}^\rho e^a{}_\rho \,. \tag{3.17}$$

On pure Lorentz tensors $\nabla = \mathcal{D}$.

Now, how do spinors and general Lorentz tensors transform under infinitesimal g.c.t.s generated by an vector field ξ?

Customarily, these fields are treated as scalars, so that, if $\pounds_\xi$ stands for the standard Lie derivative,

$$\delta_\xi T = -\pounds_\xi T = -\imath_\xi dT \,. \tag{3.18}$$

There are many reasons why this has to be wrong. For starters, if we consider the particular case of a vector field ξ generating a global Lorentz transformation in Minkowski spacetime $\xi^\mu = \sigma^\mu{}_\nu x^\nu + a^\mu$, the transformation in Eq. (3.18) is completely different from the transformation of a Lorentz tensor

$$\delta_\sigma T = \tfrac{1}{2}\sigma^{ab}\Gamma_r(M_{ab})T \,. \tag{3.19}$$

However, it should reduce to this if the Fermionic fields introduced in curved spacetimes via Weyl's prescription have anything to do with the standard special-relativistic Fermionic fields.

Furthermore, it is clear that the effect of the g.c.t. Eq. (3.18) on T depends on the gauge, or, equivalently, on the choice of tangent space basis. In other words the expression for δ_ξ in Eq. (3.18) is not covariant under local Lorentz transformations.

Indeed, Lorentz tensors are not scalar nor tensor fields under g.c.t.s. They are sections of some bundle or, at a more pedestrian level, they are fields that, under g.c.t.s, transform as world tensors up to a local Lorentz transformation whose parameter depends on the field and on the generator of the g.c.t. σ_ξ^{ab}.

Then, instead of Eq. (3.18) we must write

$$\delta_\xi T = -\pounds_\xi T + \delta_{\sigma_\xi} T \,, \tag{3.20}$$

where $\sigma_\xi{}^{ab}$ makes $\delta_\xi T$ covariant under further local Lorentz transformations.

The parameter of the compensating local Lorentz transformation that renders $\delta_\xi T$ covariant turns out to be given by[9]

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - \nabla^{[a}\xi^{b]} \,, \tag{3.22}$$

and it should be compared with the parameter of the compensating U(1) gauge transformation χ_ξ in Eq. (3.7). By analogy we can define the Lorentz-algebra-valued *momentum map*

[9]After Ref. [40], this parameter is often written in the equivalent, but less transparent, form

$$\sigma_\xi{}^{ab} = -\pounds_\xi e^{[a}{}_\mu e^{b]\,\mu} \,. \tag{3.21}$$

$$P_{\xi}{}^{ab} \equiv \nabla^{[a}\xi^{b]}\,. \tag{3.23}$$

We will see that this object satisfies a generalization of the equation that defines the momentum map in the Maxwell case Eq. (3.5).

It is natural to define the *Lorentz-covariant Lie derivative* (or *Lie-Lorentz derivative*) of any tensor T with Lorentz and world indices with respect to a vector field ξ as (minus) this transformation:[10]

$$\mathbb{L}_{\xi}T \equiv -\delta_{\xi}T = \pounds_{\xi}T - \delta_{\sigma_{\xi}}T\,. \tag{3.24}$$

The properties of the Lie-Lorentz derivative on spinors are reviewed in Refs. [41,42]. Here we are mainly interested in the Lie-Lorentz derivatives of the Vielbein and the spin connection, specially with respect to Killing vectors. According to the general definition, and after trivial manipulations, we find that the Lie-Lorentz derivative of the Vielbein is proportional to the Killing equation

$$\mathbb{L}_{\xi}e^{a}{}_{\mu} = \tfrac{1}{2}\left(\nabla_{\mu}\xi^{a} + \nabla^{a}\xi_{\mu}\right) = \tfrac{1}{2}e^{a\,\nu}\left(\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu}\right)\,, \tag{3.25}$$

and, therefore, it vanishes when ξ is a Killing vector field, independently of the basis chosen, as we should have expected.

We will use this equivalent differential-form expression for the above equation:

$$\mathbb{L}_{\xi}e^{a} = \mathcal{D}\xi^{a} + P_{\xi}{}^{a}{}_{b}e^{b}\,. \tag{3.26}$$

Let us now consider the Lie-Lorentz derivative of the spin connection ω^{ab}. Taking into account the inhomogeneous form of the compensating Lorentz transformation for the spin connection Eq. (3.16) we get[11]

$$\mathbb{L}_{\xi}\omega^{ab} = \pounds_{\xi}\omega^{ab} - \mathcal{D}\sigma_{\xi}{}^{ab}\,, \tag{3.27}$$

where $\sigma_{\xi}{}^{ab}$ is with the same parameter Eq. (3.22). After some massaging, we can rewrite it in a much more suggestive form

$$\mathbb{L}_{\xi}\omega^{ab} = \imath_{\xi}R^{ab} + \mathcal{D}P_{\xi}{}^{ab}\,, \tag{3.28}$$

where the Lorentz curvature 2-form $R^{ab} \equiv \frac{1}{2}R_{\mu\nu}{}^{ab}dx^{\mu}\wedge dx^{\nu}$ is defined as

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{c}\wedge\omega^{cb}\,, \tag{3.29}$$

and where we have replaced $\nabla^{[a}\xi^{b]}$ by $P_{\xi}{}^{ab}$, according to the definition of Eq. (3.23).

The left-hand side of Eq. (3.28) can be shown to vanish identically when ξ is a Killing vector field, because of the identity

[10] The Lie-Lorentz derivative was originally introduced for spinor fields in Refs. [44–47] and its definition was later extended to more general Lorentz tensors T transforming in an arbitrary representation r [41]

[11] The same expression can be found if one considers the variation of the Levi-Civita spin connection as a function of the variation of the Vielbein, given by (minus) the Lie-Lorentz derivative in Eq. (4.46a).

$$\xi^\nu R_{\nu\mu}{}^{ab} + \nabla_\mu(\nabla^{[a}\xi^{b]}) = \nabla^{[a}\left(\nabla^{b]}\xi_\mu + \nabla_\mu \xi^{b]}\right) . \tag{3.30}$$

As desired, for Killing vectors k we have $\mathbb{L}_k e^a = 0$ and $\mathbb{L}_k \omega^{ab} = 0$ and both statements are Lorentz-invariant.[12]

For Killing vectors, Eq. (3.30) can also be written in the form

$$\imath_k R^{ab} = -\mathcal{D} P_k{}^{ab} , \tag{3.31}$$

which is the generalization of Eq. (3.5) and justifies our definition of momentum map Eq. (3.23) for Killing vectors. The main difference with the Lie-Maxwell case is that here we have an explicit expression for $P_\xi{}^{ab}$ for any ξ.

In the context of asymptotically-flat stationary black holes, it is known that, when evaluated on the event (Killing) horizon

$$P_k{}^{ab} = \nabla^{[a}k^{b]} \stackrel{\mathcal{H}}{=} \kappa n^{ab} , \tag{3.32}$$

where κ is the surface gravity and n^{ab} is the binormal, normalized to satisfy $n^{ab}n_{ab} = -2$. The constant[13] κ is related to the Lorentz momentum map just as the electric potential on the horizon was shown to be related to the Maxwell momentum map in Section 3.2.1. This parallelism between zeroth laws was observed in [51].

3.3 The Einstein-Maxwell action and the RNT solutions

In this section we present the d-dimensional Einstein theory and the d-dimensional Reissner-Nordström-Tangherlini (RNT) solutions we are going to study, in order to fix the conventions. We will first give the action and equations of motion in the standard tensorial form, and will then rewrite them in the differential-language form that we will use in the following section.

3.3.1 Action and equations of motion

Setting $G_N^{(d)} = 1$ for simplicity, and choosing as basic dynamical fields the Vielbein $e^a{}_\mu$ and the Maxwell field A_μ, the action of the Einstein-Maxwell theory in d spacetime dimensions

$$S[e^a{}_\mu, A_\mu] = \frac{1}{16\pi}\int d^dx\, e\, \left[R(\omega, e) - \tfrac{1}{4}F^2\right] . \tag{3.33}$$

where $e \equiv \det(e^a{}_\mu)$, $R(\omega, e)$ is the Ricci scalar, defined in terms of the Levi-Civita spin connection $\omega_\mu{}^{ab}$,[14] that is

$$R(\omega, e) = e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(\omega) , \tag{3.34}$$

[12] Observe that $\mathbb{L}_\xi \omega^{ab}$ transforms as a Lorentz tensor even though ω^{ab} is not (it is a connection).

[13] See Ref. [124] for a proof of the constancy of κ over the horizon (the standard zeroth law of black hole mechanics [20]) that makes use of the Einstein equations and the dominant energy condition and Ref. [125] for a proof that does not, relying only on the assumption of geodesic completeness of the null generators of the event horizon.

[14] We are using the second-order formalism.

where $R_{\mu\nu}{}^{ab}(\omega)$ is the curvature 2-form of the Levi-Civita spin connection, defined in Eq. (3.29). The Levi-Civita spin connection (metric compatible and torsion-free, that is $\mathcal{D}e^a = 0$) is given by

$$\omega_{abc} = e_a{}^\mu \omega_{\mu\, ba} = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}\,, \qquad \Omega_{abc} = e_a{}^\mu e_b{}^\nu \partial_{[\mu|} e_{c\,|\nu]}\,. \tag{3.35}$$

Finally, $F^2 = F_{ab}F^{ab}$, $F_{ab} = e_a{}^\mu e_b{}^\nu F_{\mu\nu}$ and $F_{\mu\nu}$ is defined in Eq. (3.3).

The equations of motion are

$$E_a{}^\mu \equiv \frac{\delta S}{\delta e^a{}_\mu} = -\frac{e}{8\pi}\left(G_a{}^\mu - \tfrac{1}{2}T^a{}_\mu\right)\,, \tag{3.36a}$$

$$E^\mu \equiv \frac{\delta S}{\delta A_\mu} = \frac{1}{16\pi}\partial_\nu\left(eF^{\nu\mu}\right)\,, \tag{3.36b}$$

where

$$T_a{}^\mu = F_{ab}F^{\mu b} - \tfrac{1}{4}e_a{}^\mu F^2\,, \tag{3.37}$$

is the electromagnetic field's energy-momentum tensor.

In differential-form language, the action Eq. (3.33) is usually written in this form

$$S[e^a, A] = \frac{(-1)^{d-1}}{16\pi}\int\left[\frac{1}{(d-2)!}R^{a_1a_2}\wedge e^{a_3}\wedge\cdots\wedge e^{a_d}\epsilon_{a_1\cdots a_d} - \tfrac{1}{2}F\wedge\star F\right] \equiv \int \mathbf{L}\,, \tag{3.38}$$

although it is more convenient to rewrite the first (Einstein-Hilbert) term as

$$\frac{1}{(d-2)!}R^{a_1a_2}\wedge e^{a_3}\wedge\cdots\wedge e^{a_d}\epsilon_{a_1\cdots a_d} = \star(e^a\wedge e^b)\wedge R_{ab}\,. \tag{3.39}$$

The $(d-1)$-form equations of motion (which we write in boldface) are given by

$$\mathbf{E}_a = \frac{1}{16\pi}\left\{\imath_a \star(e^c\wedge e^d)\wedge R_{cd} + \tfrac{1}{2}\left(\imath_a F\wedge\star F - F\wedge\imath_a\star F\right)\right\}\,, \tag{3.40a}$$

$$\mathbf{E} = -\frac{1}{16\pi}d\star F\,, \tag{3.40b}$$

where $\imath_c$ stands for i_{e_c}, where $e_c = e_c{}^\mu\partial_\mu$.

3.3.2 The Reissner-Nordström-Tangherlini solutions

The d-dimensional RNT solutions with rationalized mass M and electric charge q are described by the following metric and electromagnetic fields [37, 38, 126]:

$$ds^2 = \lambda dt^2 - \frac{dr^2}{\lambda} - r^2 d\Omega^2_{(d-2)}\,, \qquad F_{tr} = \frac{16\pi}{\omega_{(d-2)}}\frac{q}{r^{d-2}}\,, \tag{3.41}$$

where $d\Omega^2_{(d-2)}$ is the metric of the round $(d-2)$-sphere of unit radius, $\omega_{(d-2)}$ is its volume and

$$\lambda = \frac{(r^{d-3} - r_+^{d-3})(r^{d-3} - r_-^{d-3})}{r^{2(d-3)}}\,, \tag{3.42a}$$

$$r_\pm^{d-3} = \frac{8\pi}{(d-2)\omega_{(d-2)}} M \pm r_0^{d-3}\,, \tag{3.42b}$$

$$r_0^{d-3} = \frac{8\pi}{(d-2)\omega_{(d-2)}}\sqrt{M^2 - \frac{2(d-2)}{(d-3)}q^2}\,. \tag{3.42c}$$

The origin of the annoying normalization factors lies in the standard normalization factor $(16\pi)^{-1}$ of the action, which should be replaced by $[2(d-2)\omega_{(d-2)}]^{-1}$. Instead, we can just define

$$\mathcal{M} \equiv \frac{8\pi}{(d-2)\omega_{(d-2)}} M\,, \qquad \mathcal{Q} \equiv \frac{16\pi}{\omega_{(d-2)}} q\,, \tag{3.43}$$

getting somewhat simpler expressions

$$F_{tr} = \frac{\mathcal{Q}}{r^{d-2}}\,, \tag{3.44a}$$

$$r_\pm^{d-3} = \mathcal{M} \pm r_0^{d-3}\,, \tag{3.44b}$$

$$r_0^{d-3} = \sqrt{\mathcal{M}^2 - \frac{\mathcal{Q}^2}{2(d-2)(d-3)}}\,. \tag{3.44c}$$

The event horizon of these solutions exists when $\mathcal{M} \geq [2(d-2)(d-3)]^{-1/2}|\mathcal{Q}|$ and then it is located at $r = r_+$ and its surface gravity is given by

$$\kappa = (d-3) r_0^{d-3}/r_+^{d-2}\,. \tag{3.45}$$

The surface gravity vanishes in the extremal limit $r_0 = 0$, which is reached when $\mathcal{M} = [2(d-2)(d-3)]^{-1/2}|\mathcal{Q}|$. We will always assume that $\kappa \neq 0$.

The timelike Killing vector that becomes null on the horizon is $k = \partial_t$ in these coordinates, but they do not cover the bifurcate sphere because this expression for k never vanishes. In the region covered by these coordinates we find that

$$P_k{}^{\mu\nu} = \nabla^{[\mu} k^{\nu]} = -\partial_r \lambda g^{\mu\nu}{}_{rt} \stackrel{\mathcal{H}}{=} \kappa n^{\mu\nu}\,, \tag{3.46}$$

where the binormal takes the value

$$n^{\mu\nu} = -2g^{\mu\nu}{}_{rt}\,, \quad \Rightarrow \quad n^{\mu\nu}n_{\mu\nu} = -2\,. \tag{3.47}$$

On the other hand, $\imath_k F = F_{tr} dr$ and

$$P_k = \frac{\mathcal{Q}/(d-3)}{r^{d-3}} \stackrel{\mathcal{H}}{=} \frac{\mathcal{Q}/(d-3)}{r_+^{d-3}} = \Phi\,. \tag{3.48}$$

In order to reach the bifurcation sphere we need to use Kruskal-Szekeres coordinates. For $d = 4$ the change from r, t to Kruskal-Szekeres's U, V is known and given explicitly, for instance, in Ref. [14]. To work in arbitrary d we will just work near the event horizon: expanding the solution in Eq. (3.41) around $r = r_+$ and ignoring terms of second or higher order in $r - r_+$ we get

$$ds^2 = 2\kappa(r-r_+)dt^2 - \frac{dr^2}{2\kappa(r-r_+)} - r_+^2\left[1+2(r-r_+)/r_+\right]d\Omega^2_{(d-2)} + \mathcal{O}(r-r_+)^2\,, \tag{3.49a}$$

$$F_{tr} = \frac{\mathcal{Q}}{r_+^{d-2}}\left[1-(d-2)(r-r_+)/r_+\right] + \mathcal{O}(r-r_+)^2\,. \tag{3.49b}$$

The tortoise coordinate r_* is

$$r_* = \frac{1}{2\kappa}\log\left(\frac{r-r_+}{r_+}\right) + C + \mathcal{O}(r-r_+)^2\,, \tag{3.50}$$

where C is an integration constant that we set to zero for the sake of convenience. Defining

$$v \equiv t + r_*\,, \qquad u \equiv t - r_*\,, \tag{3.51}$$

the solution takes the form

$$ds^2 = 2\kappa r_+ e^{\kappa(v-u)}dudv - r_+^2\left[1+2e^{\kappa(v-u)}\right]d\Omega^2_{(d-2)} + \mathcal{O}(r-r_+)^2\,, \tag{3.52a}$$

$$F_{uv} = \kappa\frac{\mathcal{Q}}{r_+^{d-3}}e^{\kappa(v-u)} + \mathcal{O}(r-r_+)^2\,. \tag{3.52b}$$

Finally, we define the coordinates U, V

$$V \equiv \sqrt{r_+/\kappa}\,e^{\kappa v}\,, \qquad U \equiv -\sqrt{r_+/\kappa}\,e^{-\kappa u}\,, \tag{3.53}$$

in terms of which the solution takes the form

$$ds^2 = -2dUdV - r_+^2\left[1 - 2\kappa UV/r_+\right] d\Omega^2_{(d-2)} + \mathcal{O}(UV)^2\,, \tag{3.54a}$$

$$F_{UV} = -\frac{\mathcal{Q}}{r_+^{d-2}} + \mathcal{O}(UV)^2\,. \tag{3.54b}$$

The Killing vector $k = \partial_t$ becomes, in these coordinates

$$k = \kappa\left(V\partial_V - U\partial_U\right) + \mathcal{O}(UV)^2\,, \qquad \hat{k} \equiv k_\mu dx^\mu = \kappa\left(VdU - UdV\right) + \mathcal{O}(UV)^2\,. \tag{3.55}$$

In these coordinates, the hypersurface $U = 0$ is the past event horizon $\mathcal{H}^-$, generated by $k|_{\mathcal{H}^-} = \kappa V\partial_V = \partial_v$. The hypersurface $V = 0$ is the future event horizon $\mathcal{H}^+$ generated by $k|_{\mathcal{H}^+} = -\kappa U\partial_U = \partial_u$. They cross at the bifurcation sphere, which is defined by $U = V = 0$ and can also be characterized as the spatial cross section of the horizon at which $k = 0$.

On the other hand,

$$\begin{aligned} &P_{k\,\mu\nu}dx^\mu \wedge dx^\nu = d\hat{k} = 2\kappa dV \wedge dU + \mathcal{O}(UV)^2 = 2\kappa g_{VU,\mu\nu}dx^\mu \wedge dx^\nu + \mathcal{O}(UV)^2\,, \\ &\quad\Rightarrow\quad n_{\mu\nu} = -2g_{UV,\,\mu\nu}\,. \end{aligned} \tag{3.56}$$

On the other hand,

$$\begin{aligned} \imath_k F &= \kappa\frac{\mathcal{Q}}{r_+^{d-2}}(VdU + UdV) + \mathcal{O}(UV)^2\,, \\ \Rightarrow\quad P_k &= C + \kappa\frac{\mathcal{Q}}{r_+^{d-2}}UV + \mathcal{O}(UV)^2\,. \end{aligned} \tag{3.57}$$

The constant C clearly has to be identified with the electric potential over the horizon Φ in Eq. (3.48). As observed in Ref. [39], if we use the simplest choice of electromagnetic potential

$$A = \frac{\mathcal{Q}/(d-3)}{r^{d-3}}dt\,, \tag{3.58}$$

we obtain,

$$A = \frac{\mathcal{Q}}{2(d-3)\kappa r_+^{d-3}}\left[1 + (d-3)\kappa UV/r_+ + \mathcal{O}(UV)^2\right]\left(\frac{dV}{V} - \frac{dU}{U}\right)\,, \tag{3.59}$$

which is singular at the horizon.

3.4 Wald-Noether charge for the E-M theory

The general variation of the action of the Einstein-Maxwell theory Eq. (3.38) is

$$\delta S = \int \{\mathbf{E}_a \wedge \delta e^a + \mathbf{E} \wedge \delta A + d\mathbf{\Theta}(e, A, \delta e, \delta A)\} \, , \tag{3.60}$$

where $\mathbf{E}_a$ and $\mathbf{E}$ are, respectively, the $(d-1)$-form Einstein (3.40a) and Maxwell (3.36b) equations multiplied by the volume form $d^d x$ and

$$\mathbf{\Theta}(e, A, \delta e, \delta A) \equiv -\frac{1}{16\pi}\left[\star(e^a \wedge e^b) \wedge \delta\omega_{ab} - \star F \wedge \delta A\right] \, , \tag{3.61}$$

is the *presymplectic* $(d-1)$-*form* defined in Ref. [27] and $\star$ stands for the Hodge dual. For the transformations given by (minus) the covariant Lie derivatives in Eqs. (3.8), (3.26) and (3.28)

$$\delta_\xi S = \int \left\{-\mathbf{E}_a \wedge (\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b) - \mathbf{E} \wedge (\imath_\xi F + dP_\xi) + d\mathbf{\Theta}(e, A, \delta_\xi e, \delta_\xi A)\right\} \, , \tag{3.62}$$

with

$$\mathbf{\Theta}(e, A, \delta_\xi e, \delta_\xi A) = \frac{1}{16\pi}\left[\star(e^a \wedge e^b) \wedge \left(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab}\right) - \star F \wedge (\imath_\xi F + dP_\xi)\right] . \tag{3.63}$$

Let us consider the first term. It is not difficult to see that $\mathbf{E}_a \wedge e^b P_\xi{}^a{}_b = 0$ because the tensor contracted with the Lorentz momentum map give the Einstein equations, which are symmetric in the indices a and b. The rest can be integrated by parts,

$$-\mathbf{E}_a \wedge \mathcal{D}\xi^a = -(-1)^{d-1} d\left(\mathbf{E}_a \xi^a\right) + (-1)^{d-1}\xi^a \mathcal{D}\mathbf{E}_a \, . \tag{3.64}$$

Using the Bianchi identity $\mathcal{D}R^{ab} = 0$,

$$\begin{aligned}
\xi^a \mathcal{D}\mathbf{E}_a &= \frac{1}{32\pi}\xi^a \mathcal{D}\left(\imath_a F \wedge \star F - F \wedge \imath_a \star F\right) \\
&= \frac{1}{32\pi}\xi^a\left[\nabla \imath_a F \wedge \star F - \imath_a F \wedge \nabla \star F - \nabla F \wedge \imath_a \star F - F \wedge \nabla \imath_a \star F\right] \, ,
\end{aligned} \tag{3.65}$$

where we have replaced $\mathcal{D}$ by ∇ is the exterior total covariant derivative operator which satisfies the first Vielbein postulate. Then, using the property

$$\nabla \imath_a \omega = -\imath_a d\omega + \nabla_a \omega \, , \tag{3.66}$$

and replacing ∇ by the exterior derivative when it acts on differential forms with no indices, as well as using the Bianchi identity $dF = 0$, we get

$$\xi^a \mathcal{D}\mathbf{E}_a = \frac{1}{32\pi}\xi^a\left[\nabla_a F \wedge \star F - \imath_a F \wedge d \star F + F \wedge \imath_a d \star F - F \wedge \nabla_a \star F\right] . \tag{3.67}$$

Since ∇_a commutes with the Hodge dual and $F \wedge \star G$ is symmetric in F and G for any 2-forms F, G, the two terms with ∇_a cancel each other. Furthermore,

$$F \wedge \imath_a d \star F \ = \imath_a (F \wedge d \star F) - \imath_a F \wedge d \star F \,, \tag{3.68}$$

and

$$\xi^a \imath_a \omega = \imath_\xi \omega \,, \tag{3.69}$$

for any p-form, we arrive at

$$(-1)^{d-1} \xi^a \mathcal{D} \mathbf{E}_a = -\frac{1}{16\pi} d \star F \wedge \imath_\xi F \,. \tag{3.70}$$

The second term in Eq. (3.62) gives

$$-\mathbf{E} \wedge (\imath_\xi F + dP_\xi) = \frac{1}{16\pi} d \star F \wedge \imath_\xi F - (-1)^{d-1} d\left(\mathbf{E} P_\xi\right) \,, \tag{3.71}$$

and, collecting the partial results, we get

$$\delta S_\xi = \int d\mathbf{\Theta}'(e, A, \delta_\xi e, \delta_\xi A) \,, \tag{3.72}$$

where

$$\begin{aligned} \mathbf{\Theta}'(e, A, \delta_\xi e, \delta_\xi A) &\equiv \mathbf{\Theta}(e, A, \delta_\xi e, \delta_\xi A) + (-1)^d \left(\mathbf{E}_a \xi^a + \mathbf{E} P_\xi\right) \\ &= \frac{1}{16\pi} \Big[\star(e^a \wedge e^b) \wedge (\imath_\xi R_{ab} + \mathcal{D} P_{\xi\, ab}) - \star F \wedge (\imath_\xi F + dP_\xi) \\ &+ (-1)^d \imath_\xi \star (e^a \wedge e^b) \wedge R_{ab} + \frac{(-1)^d}{2} \left(\imath_\xi F \wedge \star F - F \wedge \imath_\xi \star F\right) \\ &+ (-1)^{d-1} d \star F P_\xi \Big] \\ &= -\imath_\xi \mathbf{L} + \frac{(-1)^{d-1}}{16\pi} d\left[\star F P_\xi - \star(e^a \wedge e^b) P_{\xi\, ab}\right] \,. \end{aligned} \tag{3.73}$$

The action of the Einstein-Maxwell theory Eq. (3.38) is exactly invariant under local Lorentz and electromagnetic gauge transformations and it is invariant up to a total derivative under diffeomorphisms. Therefore, under the combined transformations $\delta_\xi \equiv -\mathbb{L}_\xi$ with the covariant Lie derivatives defined in Eqs. (3.8), (3.26) and (3.28),

$$\delta_\xi S = -\int d\imath_\xi \mathbf{L} \,. \tag{3.74}$$

Taking into account the result in Eq. (3.72), the arbitrariness of the domain of integration, of the parameter ξ, and the fact that we have not used the equations of motion, we conclude that, if we define the $(d-1)$-form

$$\mathbf{J} \equiv \mathbf{\Theta}'(e, A, \delta_\xi e, \delta_\xi A) + \imath_\xi \mathbf{L}\,, \tag{3.75}$$

it satisfies

$$d\mathbf{J} = 0\,, \tag{3.76}$$

identically, off-shell. This, in its turn, implies the existence of a $(d-2)$-form $\mathbf{Q}[\xi]$ (the Wald-Noether charge) such that

$$\mathbf{J} = d\mathbf{Q}[\xi]\,. \tag{3.77}$$

The last line of Eq. (3.73) gives the following expression for the Wald-Noether charge:

$$\mathbf{Q}[\xi] = \frac{(-1)^{d-1}}{16\pi}\left[\star F P_\xi - \star(e^a \wedge e^b) P_{\xi\, ab}\right]\,. \tag{3.78}$$

3.5 The first law of black hole mechanics in the E-M theory

Following Ref. [27] we define the pre-symplectic $(d-1)$-form

$$\omega(\phi, \delta_1\phi, \delta_2\phi) \equiv \delta_1\mathbf{\Theta}(\phi, \delta_2\phi) - \delta_2\mathbf{\Theta}(\phi, \delta_1\phi)\,, \tag{3.79}$$

where ϕ stands for the Vielbein and Maxwell fields, and the symplectic form relative to the Cauchy surface Σ

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) \equiv \int_\Sigma \omega(\phi, \delta_1\phi, \delta_2\phi)\,. \tag{3.80}$$

Following now Ref. [22], when ϕ solves the equations of motion $\mathbf{E}_\phi = 0$, for any variation of the fields $\delta_1\phi = \delta\phi$ and the variations under diffeomorphisms $\delta_2\phi = \delta_\xi\phi$

$$\omega(\phi, \delta\phi, \delta_\xi\phi) = \delta\mathbf{J} + d\imath_\xi\mathbf{\Theta}' = \delta d\mathbf{Q}[\xi] + d\imath_\xi\mathbf{\Theta}'\,, \tag{3.81}$$

where, in our case, $\mathbf{J}$ is given by Eq. (3.77), $\mathbf{\Theta}'$ is given in Eq. (3.73) and we observe that, on-shell, $\mathbf{\Theta} = \mathbf{\Theta}'$. Then, if $\delta\phi$ satisfies the linearized equations of motion $\delta d\mathbf{Q} = d\delta\mathbf{Q}$. Furthermore, if the parameter $\xi = k$ generates a transformation that leaves invariant all the fields of the theory, $\delta_k\phi = 0$, $\omega(\phi, \delta\phi, \delta_k\phi) = 0$, and we arrive at

$$d\left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right) = 0\,, \tag{3.82}$$

which, when integrated over a hypersurface Σ with boundary $\delta\Sigma$, gives

$$\int_{\delta\Sigma}\left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right) = 0\,. \tag{3.83}$$

In our case, we are dealing with asymptotically flat, static black holes. k is the timelike Killing vector whose Killing horizon coincides with the event horizon and the hypersurface Σ is the space between infinity and the bifurcation sphere ($\mathcal{BH}$) on which $k = 0$. Infinity and the bifurcate horizon are the two disconnected components of $\delta\Sigma$ and taking into account that $k = 0$ on the bifurcation sphere, we obtain

$$\delta \int_{\mathcal{BH}} \mathbf{Q}[k] = \int_{\infty} \left(\delta\mathbf{Q}[k] + \imath_k \mathbf{\Theta}'\right) . \tag{3.84}$$

As explained in Ref. [22], the right-hand side can be identified with δM, where M is the total mass of the black-hole spacetime. Using Eq. (3.78), we find

$$\delta \int_{\mathcal{BH}} \mathbf{Q}[k] = \frac{(-1)^{d-1}}{16\pi}\delta \int_{\mathcal{BH}} \star F P_k + \frac{(-1)^{d}}{16\pi}\delta \int_{\mathcal{BH}} \star(e^a \wedge e^b) P_{k\,ab} \,. \tag{3.85}$$

According to the discussion at the end of Section 3.2.1, P_k can be identified with the electric potential Φ and it is constant over the horizon. The electric charge contained inside the horizon is given by

$$\mathcal{Q} \equiv \frac{(-1)^{d-1}}{16\pi} \int_{\mathcal{BH}} \star F \,, \tag{3.86}$$

and the first term just gives $+\Phi\delta\mathcal{Q}$, which implies that we get a first-law-like relation if the second term gives $T\delta S$. Let us study that term. Using Eq. (3.32) we get

$$\begin{aligned} \frac{(-1)^{d}}{16\pi}\delta \int_{\mathcal{BH}} \star(e^a \wedge e^b) P_{k\,ab} &= \frac{(-1)^{d}\kappa}{16\pi}\delta \int_{\mathcal{BH}} \star(e^a \wedge e^b) n_{ab} \\ &= -\frac{\kappa}{16\pi}\delta \int_{\mathcal{BH}} d^{d-2}S\, n_{ab}n^{ab} \\ &= T\delta A/4 \,, \end{aligned} \tag{3.87}$$

where we have used the normalization of the binormal $n_{ab}n^{ab} = -2$, A is the area of the horizon and $T = \kappa/2\pi$ is the Hawking temperature.

Thus, we recover the first law of black hole mechanics if we identify the black hole entropy with one quarter of the area of the horizon.

3.6 Discussion

In this chapter, we have showed how to define gauge-covariant Lie derivatives with the momentum map and how to use these derivatives in the proof of the first law of black-hole mechanics in the simple case of the Einstein-Maxwell theory with the Vielbein as the gravitational field. We have also shown that the momentum maps we have introduced in this case satisfy (well known) zeroth laws.

While the formulation of the first law of black-hole mechanics in the Einstein-Maxwell theory is certainly not new, our proposal for dealing with fields with gauge freedoms is a first step towards a generalization of the first law to more complex cases involving p-form fields with Chern-Simons terms such as those occurring in the Heterotic Superstring effective action. The first law in heterotic superstring effective action will be examined in chapters 4 (in the case of zeroth order α') and 5 (at first order).

4

The first law of heterotic stringy black hole mechanics at zeroth order in α'

4.1 Introduction

In Ref. [28], Wald showed that, in a theory of gravity invariant under diffeomorphisms, the black hole entropy is essentially the Noether charge associated to that invariance. The proof consists in showing that this charge plays the role of entropy in the first law of black hole mechanics [20]. As we have previously discussed in Section 1.2.3 though, in presence of matter, some terms in the total Noether charge are identified with other terms in the first law. Therefore, only the "gravitational" part of the Noether charge can be identified with the entropy and, in principle, it is necessary to go through the proof of the first law in order to identify the entropy.

A more general and mathematically rigorous treatment based on the theory of principal bundles was given in Ref. [51] by Prabhu, who was motivated by the problems found by Gao in Ref. [39]. However, String and Supergravity theories have p-form fields with gauge freedom that cannot be described in that framework. Furthermore, the effective action and the field strengths often contain Chern-Simons terms which make the action invariant only up to total derivatives and complicate the gauge transformations of the p-form fields. When the Chern-Simons terms depend on the spin (Lorentz) connection, gauge invariance and diffeomorphism invariance become entangled in a very complex form.

One of the simplest theories with a Chern-Simons term in the action is "minimal" ($\mathcal{N} = 1$) 5-dimensional supergravity [127], which only contains a 1-form coupled to gravity. In order to deal with the lack of exact gauge invariance one has to take into account the total derivative in the definition of the Noether current [128]. However, the entropy obtained by this method in Ref. [129] in the case of the "gravitational" Chern-Simons terms (both in the action or in the Kalb-Ramond field strength) of the Heterotic Superstring effective action turned out to be gauge-dependent.[1] This problem was dealt with in Ref. [130], albeit in a rather complicated form.

In the previous chapter, we studied the use of gauge-covariant Lie derivatives in the context of the Einstein-Maxwell theory using momentum maps to construct the derivatives. Momentum maps arise naturally wherever symmetries of a base manifold have to be related to gauge transformations [42,122] and they are unsurprisingly ubiquitous in gauged supergravity. As a matter of fact, the Lie-Lorentz derivative can be constructed in terms of a Lorentz momentum map and, as previously mentioned in Chapter 3, we also used a

[1] The same happens when one naively uses the Iyer-Wald prescription, as noticed in [33,34].

Maxwell momentum map to construct a *Lie-Maxwell derivative*, covariant under the gauge transformations of the Maxwell field.

This procedure guarantees the gauge-invariance of the results and, as a byproduct, we found a very interesting relation between momentum maps and generalized zeroth laws also observed, in a completely different language by Prabhu in Ref. [51].

In this chapter, we extend this method to a theory with Abelian Chern-Simons terms in a field strength: the effective action of the Heterotic Superstring compactified on a torus to zeroth order in α'. This theory can be seen as a generalization of the theory considered by Compère in Ref. [52] and as a first step towards dealing with the effective action of the Heterotic Superstring to first order in α', which contains non-Abelian and Lorentz ("gravitational") Chern-Simons terms of the kind considered by Tachikawa [73, 90]. The introduction of momentum maps will allow us to obtain invariant results in a rather simple form, basically because they allow us to determine explicitly the gauge parameters that leave invariant all the fields of a given solution [36]. They also allow us to construct forms which are closed on the bifurcation sphere, from which the definitions of the potentials that appear in the first law will follow [52, 131]. The closedness of those forms, therefore, plays the role of the generalized zeroth law, albeit restricted to the bifurcation sphere. Hence, we will refer to these properties as the *restricted generalized zeroth laws*.

As we are going to see in the proof of the first law, there is a very precise, almost clockwork, relation between the closed forms that satisfy the restricted generalized zeroth laws and the definitions of the conserved charges [36, 132–134]. Only when both have been correctly identified is it possible to find the first law and identify the entropy.

In theories with Chern-Simons terms, several different definitions of charges have been proposed and used in the literature (see, for instance, Ref. [135] and references therein). The proof of the first law demands that we use the so-called *Page charge*, which in this context is conserved, localized and on-shell gauge invariant. Only when we use this charge definition for the 1-forms, the closed 1-form associated to the KR potentials Φ^i over the bifurcation sphere appears [52, 131] and the term $\Phi^i \delta \mathcal{Q}_i$ of the first law associated to the "dipole charges" [52, 131, 136–139] can be identified.

In theories with "gravitational" Chern-Simons terms, such as the effective action of the Heterotic Superstring at first order in α', the same mechanism should play a role in the proof of the first law, but the terms that modify the gravitational charges will contribute to the entropy instead [53]. It is in this precise sense that this work is a first step towards the proof of the first law and the determination of a gauge-invariant entropy formula for that theory. The previous discussion should have made clear that such a formula is not yet available, as we have also explained in Refs. [33, 34]. Even though the calculations of some black-hole entropies using the Iyer-Wald prescription seem to give the right value of the entropy in some cases,[2] it is clear that the results obtained using an entropy formula which is not gauge-invariant cannot be trusted in general. It is also clear that the comparison

[2] In Ref. [32] it was shown that the entropy of the α'-corrected non-extremal Reissner-Nordström black hole based in the string embedding of Ref. [108], computed with the entropy formula derived in Ref. [33] using the Iyer-Wald prescription satisfies the thermodynamic relation $\partial S/\partial M = T^{-1}$. That entropy formula is not invariant under Lorentz transformations, though. In a general frame it will give wrong values for the entropy and the reason why it gives the right value in that particular case, in the particular frame in which the calculation was carried out, sill needs to be explained [53]. The same entropy formula has been used to compute the entropy of some α'-corrected extremal black holes and the results, although reasonable, cannot be tested using the same relation.

between entropies computed through macroscopic and microscopic methods [89] only make sense if both computations are reliable, and furthermore, only if the relation between the parameters of the black hole solution and of the microscopic theory is well understood. At first order in α', there is no full-proof entropy formula, as we have explained, and the identification of the parameters of the black-hole solutions (charges) with the numbers of branes and other parameters that appear in the microscopic entropy, has issues that still have not been fully understood [107]. This is one of the main motivations for this work.

This chapter is organized as follows: in Section 4.2 we introduce the effective action of the Heterotic Superstring compactified on a torus at leading order in α'. In Section 4.3 we study the action of the symmetries of the theory on the fields, the parameters of the transformations that leave all of them invariant, and compute the associated conserved charges, including the Wald-Noether charge. In Section 4.4 we study the restricted generalized zeroth laws that we will use in the proof of the first law in Section 4.5. In Section 4.6 we consider as an example the charged, non-extremal, 5-dimensional black ring solution of pure $\mathcal{N}=1, d=5$ supergravity of Ref. [140] and compute its momentum maps. Section 4.7 contains a brief discussion of our results. In the appendix we show how the Heterotic Superstring effective action compactified on $\mathrm{T}^4\times\mathrm{S}^1$ (trivial compactification on T^4) can be understood as a model $\mathcal{N}=1, d=5$ supergravity coupled to two vector supermultiplets, which provides an embedding of this model into the Heterotic Superstring effective action. We also show how this model can be consistently tructated to pure $\mathcal{N}=1, d=5$ supergravity. Again, this provides an embedding of pure $\mathcal{N}=1, d=5$ supergravity and, in particular of the black ring solution of Ref. [140], into the Heterotic Superstring effective action, so we can apply the formulae and results obtained in the main body of the chapter to that solution.

4.2 The Heterotic Superstring effective action on T^n at zeroth order in α'

When the effective action of the Heterotic Superstring at leading order in α' is compactified on a T^n, it describes the dynamics of the $(10-n)$-dimensional (string-frame) metric $g_{\mu\nu}$, Kalb-Ramond 2-form $B_{\mu\nu}$, dilaton field ϕ, Kaluza-Klein (KK) and winding 1-forms $A^m{}_\mu$ and $B_{m\,\mu}$, respectively, and the scalars that parametrize the $\mathrm{O}(n,n)/\mathrm{O}(n)\times\mathrm{O}(n)$ coset space, collected in the symmetric $\mathrm{O}(n,n)$ matrix M that we will write with upper $\mathrm{O}(n,n)$ indices $I, J, \ldots$ as M^{IJ}. This means that M satisfies

$$M^{IJ}\Omega_{JK}M^{KL}\Omega_{LM} = \delta^I{}_M\,, \tag{4.1}$$

where

$$(\Omega_{IJ}) \equiv \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}, \tag{4.2}$$

is the off-diagonal form of the $\mathrm{O}(n,n)$ metric. Eq. (4.1) implies that

$$M_{IJ} \equiv (M^{-1})_{IJ} = \Omega_{IK}M^{KL}\Omega_{LJ}\,. \tag{4.3}$$

Using the notation and conventions of Refs. [34, 42] (in particular, for differential forms, we use those of Ref. [88]), and calling the physical scalars in M_{IJ} ϕ^x, the action of the $d = (10 - n)$-dimensional takes the form

$$
\begin{aligned}
S[e^a, B, \phi, \mathcal{A}^I, \phi^x] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int e^{-2\phi} \Big[& (-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} - 4d\phi \wedge \star d\phi \\
& -\tfrac{1}{8} dM_{IJ} \wedge \star dM^{IJ} + (-1)^d \tfrac{1}{2} M_{IJ} \mathcal{F}^I \wedge \star \mathcal{F}^J + \tfrac{1}{2} H \wedge \star H \Big] \\
\equiv \int \mathbf{L}\, . &
\end{aligned}
\tag{4.4}
$$

In this action $e^a = e^a{}_\mu dx^\mu$ are the string-frame Vielbeins, $\star$ stands for the Hodge dual and, therefore

$$
\star (e^a \wedge e^b) = \frac{1}{(d-2)!} \epsilon_{c_1 \cdots c_{d-2}}{}^{ab} e^{c_1} \wedge \cdots \wedge e^{c_{d-2}}\, . \tag{4.5}
$$

Furthermore, $\omega^{ab} = \omega_\mu{}^{ab} dx^\mu$ is the Levi-Civita spin connection[3] and $R^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu$ is its field strength (the curvature) 2-form, defined as

$$
R^{ab} \equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}\, . \tag{4.6}
$$

$g_s^{(d)}$ and $G_N^{(d)}$ are, respectively, the $d = (10 - n)$-dimensional string coupling and Newton constant. [4]

$\mathcal{F}^I$ is the $\mathrm{O}(n, n)$ vector of the 2-form field strengths of the KK and winding vectors

$$
\mathcal{F}^I \equiv \begin{pmatrix} F^m \\ G_m \end{pmatrix}, \qquad F^m = dA^m\, , \qquad G_m = dB_m\, , \tag{4.8}
$$

which can also be defined in terms of the $\mathrm{O}(n, n)$ vector of 1-forms denoted by $\mathcal{A}^I$

$$
\mathcal{A}^I \equiv \begin{pmatrix} A^m \\ B_m \end{pmatrix}, \qquad \mathcal{F}^I = d\mathcal{A}^I\, . \tag{4.9}
$$

H is the Kalb-Ramond 3-form field strength, defined by

$$
H \equiv dB - \tfrac{1}{2} \mathcal{A}_I \wedge d\mathcal{A}^I\, , \qquad \mathcal{A}_I = \Omega_{IJ} \mathcal{A}^J\, . \tag{4.10}
$$

[3]It is antisymmetric $\omega^{ab} = -\omega^{ba}$ and satisfies $De^a = de^a - \omega^a{}_b \wedge e^b = 0$. We are using the second-order formalism.

[4]They are related to the 10-dimensional constants through the volume of the T^n, V_n, by

$$
g_s^2 = V_n/(2\pi \ell_s)^n g_s^{(d)\,2}\, , \tag{4.7a}
$$

$$
G_N^{(10)} = G_N^{(d)} V_n\, . \tag{4.7b}
$$

The kinetic term of the scalars ϕ^x that parametrize the $\mathrm{O}(n,n)/(\mathrm{O}(n)\times\mathrm{O}(n))$ coset space can also be written in the form

$$-\tfrac{1}{8}dM_{IJ}\wedge\star dM^{IJ}=\tfrac{1}{2}g_{xy}d\phi^x\wedge\star d\phi^y\,, \tag{4.11}$$

where the metric $g_{xy}(\phi)$ is given by

$$g_{xy}\equiv\tfrac{1}{4}\left(\partial_x M_{IK}M^{KJ}\right)\left(\partial_y M_{JK}M^{KI}\right)\,. \tag{4.12}$$

Under a general variation of the fields, the action varies as

$$\delta S=\int\left\{\mathbf{E}_a\wedge\delta e^a+\mathbf{E}_B\wedge\delta B+\mathbf{E}_\phi\delta\phi+\mathbf{E}_I\wedge\delta\mathcal{A}^I+\mathbf{E}_x\delta\phi^x+d\mathbf{\Theta}(\varphi,\delta\varphi)\right. \quad , \tag{4.13}$$

where, suppressing the factors of $g^{(d)\,2}(16\pi G_N^{(d)})^1$ for simplicity, the Einstein equations $\mathbf{E}_a$ are given by

$$\begin{aligned}\mathbf{E}_a=&\,e^{-2\phi}\imath_a\star(e^c\wedge e^d)\wedge R_{cd}-2\mathcal{D}(\imath_b de^{-2\phi})\wedge\star(e^b\wedge e^c)g_{ca}\\&+(-1)^{d-1}4e^{-2\phi}\left(\imath_a d\phi\star d\phi+d\phi\wedge\imath_a\star d\phi\right)\\&+\frac{(-1)^d}{2}e^{-2\phi}g_{xy}\left(\imath_a d\phi^x\star d\phi^y+d\phi^x\wedge\imath_a\star d\phi^y\right)\\&+\frac{1}{2}e^{-2\phi}M_{IJ}\left(\imath_a\mathcal{F}^I\wedge\star\mathcal{F}^J-\mathcal{F}^I\wedge\imath_a\star\mathcal{F}^J\right)\\&+\frac{(-1)^d}{2}e^{-2\phi}\left(\imath_a H\wedge\star H+H\wedge\imath_a\star H\right)\,,\end{aligned} \tag{4.14}$$

the equations of motion of the matter fields are given by

$$\mathbf{E}_B=-d\left(e^{-2\phi}\star H\right)\,, \tag{4.15a}$$

$$\mathbf{E}_\phi=8d\left(e^{-2\phi}\star d\phi\right)-2\mathbf{L}\,, \tag{4.15b}$$

$$\mathbf{E}_I=\tilde{\mathbf{E}}_I+\tfrac{1}{2}\mathbf{E}_B\wedge\mathcal{A}_I\,, \tag{4.15c}$$

$$\tilde{\mathbf{E}}_I\equiv-\left\{d\left(e^{-2\phi}M_{IJ}\star\mathcal{F}^J\right)+(-1)^{d-1}e^{-2\phi}\star H\wedge\mathcal{F}_I\right\}\,, \tag{4.15d}$$

$$\mathbf{E}_x=-g_{xy}\left[d\left(e^{-2\phi}\star d\phi^y\right)+e^{-2\phi}\Gamma_{zw}{}^y d\phi^z\wedge\star d\phi^w\right]+\frac{(-1)^d}{2}e^{-2\phi}\partial_x M_{IJ}\mathcal{F}^I\wedge\star\mathcal{F}^J\,, \tag{4.15e}$$

and

$$\begin{aligned}\mathbf{\Theta}(\varphi,\delta\varphi) = & -e^{-2\phi}\star(e^a\wedge e^b)\wedge\delta\omega_{ab} + 2\imath_a d e^{-2\phi}\star(e^a\wedge e^b)\wedge\delta e_b \\ & -8e^{-2\phi}\star d\phi\delta\phi - \tfrac{1}{4}e^{-2\phi}\star dM^{IJ}\delta M_{IJ} \\ & + e^{-2\phi}M_{IJ}\star\mathcal{F}^J\wedge\delta\mathcal{A}^I + e^{-2\phi}\star H\wedge\left(\delta B + \tfrac{1}{2}\mathcal{A}_I\wedge\delta\mathcal{A}^I\right)\,.\end{aligned} \tag{4.16}$$

The equations of motion of the 1-forms $\mathbf{E}_I$ can be written in the alternative form

$$\mathbf{E}_I = -d\left\{e^{-2\phi}M_{IJ}\star\mathcal{F}^J + \star H\wedge\mathcal{A}_I\right\} - \tfrac{1}{2}\mathbf{E}_B\wedge\mathcal{A}_I\,. \tag{4.17}$$

This form appears naturally in the definition of the electric charges Eq. (4.32).

Here, and in what follows, φ stands for all the fields of the theory. $\mathbf{E}_\varphi$ denotes collectively all their equations of motion.

4.3 Variations of the fields

In this section we are going to study the transformations of the fields under the different symmetries of the action and determine which parameters of the transformations leave a complete field configuration invariant. The conserved charges of those configurations will be associated to those parameters. As a general rule, only if one combines several transformations can one find parameters that simultaneously leave all the fields invariant.

The simplest case in which this happens will involve the gauge transformations of the 1-form fields: the parameters that leave them invariant do not leave the KR field invariant at the same time, unless we perform a KR gauge transformation with a parameter related to that of the other gauge symmetry. As a result, there is an additional term in the formula that gives the electric charges, but it is the presence of this additional term that guarantees the conservation of the charge and the independence of the integration surface (as long as we do not include sources, that is, on-shell).

The transformation of several fields under diffeomorphisms must also be supplemented by "compensating" gauge transformations, including local Lorentz transformations if we want all the fields to be left invariant by those generating isometries (Killing vectors). There are several ways of understanding this need but we believe that the most fundamental is to realize that fields with gauge freedoms (*i.e.* all fields except for the metric and the dilaton field) are not tensors and do not transform as such under diffeomorphisms. The "compensating gauge transformations" can be seen as gauge transformations induced by the diffeomorphisms. Only when they are properly taken into account can one find Killing vector fields that leave all the fields invariant. Furthermore, only then the vanishing of the variations of the fields is invariant under gauge transformations. A more detailed discussion and additional references to this topic can be found in Ref. [88]. The conserved charge associated to diffeomorphisms, the Wald-Noether charge, will therefore include terms related to gauge symmetries and their associated conserved charges, which will ultimately contribute to the first law.

As we will see, only when all these details are properly taken into account can the first law be proven and the entropy identified.

We start by describing the gauge symmetries of the theory (other than diffeomorphisms) and the associated conserved charges.

4.3.1 Gauge transformations

The gauge transformations of the fields are

$$\delta_\sigma e^a = \sigma^a{}_b e^b\,, \tag{4.18a}$$

$$\delta_\chi \mathcal{A}^I = d\chi^I\,, \tag{4.18b}$$

$$\delta B = (\delta_\Lambda + \delta_\chi)B = d\Lambda + \tfrac{1}{2}\chi_I d\mathcal{A}^I\,, \tag{4.18c}$$

where $\sigma^{(ab)}(x) = 0$ are the parameters of local Lorentz transformations, $\chi^I(x)$ is a $\mathrm{O}(n,n)$ vector if scalar gauge parameters and $\Lambda = \Lambda_\mu(x)dx^\mu$ is a 1-form gauge parameter. They leave invariant the field strengths $\mathcal{F}^I$ and H, but they induce the following transformations on the spin connection and curvature

$$\delta_\sigma \omega^{ab} = \mathcal{D}\sigma^{ab} = d\sigma^{ab} - 2\omega^{[a|}{}_c\sigma^{c|b]}\,, \tag{4.19a}$$

$$\delta_\sigma R^{ab} = 2\sigma^{[a|}{}_c R^{c|b]}\,. \tag{4.19b}$$

For the sake of completeness and later use, we quote the Ricci identity in our conventions:

$$\mathcal{D}\mathcal{D}\sigma^{ab} = -2R^{[a|}{}_c\sigma^{c|b]} = \delta_\sigma R^{ab}\,. \tag{4.20}$$

The action is manifestly invariant under these gauge transformations. This leads to the following Noether identities

$$\mathbf{E}^{[a} \wedge e^{b]} = 0\,, \tag{4.21a}$$

$$d\tilde{\mathbf{E}}_I + (-1)^d \mathbf{E}_B \wedge \mathcal{F}_I = 0\,, \tag{4.21b}$$

$$d\mathbf{E}_B = 0\,, \tag{4.21c}$$

4.3.2 Gauge charges

Let us study the conserved charges associated to the gauge transformations δ_χ, δ_Λ and, for the sake of completeness, δ_σ, starting with δ_Λ, which is simpler to deal with.

The variation of the action under δ_Λ transformations follows from Eqs. (4.13) and (4.16)

$$\begin{aligned}\delta_\Lambda S &= \int \left\{ \mathbf{E}_B \wedge \delta_\Lambda B + d\left(e^{-2\phi} \star H \wedge \delta_\Lambda B\right)\right\} \\ &= \int \left\{ \mathbf{E}_B \wedge d\Lambda + d\left(e^{-2\phi} \star H \wedge d\Lambda\right)\right\} .\end{aligned} \tag{4.22}$$

Integrating by parts the first term and using the Noether identity Eq. (4.21c)

$$\delta_\Lambda S = \int d\left(\Lambda \wedge \mathbf{E}_B + e^{-2\phi} \star H \wedge d\Lambda\right) \equiv \int d\mathbf{J}[\Lambda] . \tag{4.23}$$

The invariance of the action under these gauge transformations indicates that the current $\mathbf{J}[\Lambda]$ must be locally exact, so that, locally, there is a $\mathbf{Q}[\Lambda]$ such that $\mathbf{J}[\Lambda] = d\mathbf{Q}[\Lambda]$. It is easy to see that

$$\mathbf{Q}[\Lambda] = \Lambda \wedge \left(e^{-2\phi} \star H\right) . \tag{4.24}$$

The conserved charge is given by the integral of the conserved $(d-2)$-form $\mathcal{Q}[\Lambda]$ over $(d-2)$-dimensional compact surfaces $\mathcal{S}_{d-2}$ for Λs that leave invariant the KR field Bs. These are closed 1-forms. Following [52, 131], using the Hodge decomposition theorem, these closed 1-forms Λ can be written as the sum of an exact and a harmonic form $\Lambda_e = d\lambda$ and Λ_h, respectively. The exact form Λ_e will not contribute to the integral on-shell because

$$Q(\Lambda_e) = \int_{\mathcal{S}_{d-2}} d\lambda \wedge \left(e^{-2\phi} \star H\right) = \int_{\mathcal{S}_{d-2}} d\left[\lambda \wedge \left(e^{-2\phi} \star H\right)\right] - \int_{\mathcal{S}_{d-2}} \lambda \wedge \mathbf{E}_B . \tag{4.25}$$

Therefore,

$$Q(\Lambda) = \int_{\mathcal{S}_{d-2}} \Lambda_h \wedge \left(e^{-2\phi} \star H\right) . \tag{4.26}$$

Then, using the duality between homology and cohomology, if C_{Λ_h} is the $(d-3)$-cycle dual to Λ_h, we arrive at the charges

$$Q(\Lambda) = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{C_{\Lambda_h}} e^{-2\phi} \star H , \tag{4.27}$$

where we have added a conventional sign and recovered the factor of $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$ that we have omitted. From the string theory point of view, these charges are just winding numbers of strings whose transverse space is the cycle C_{Λ_h}. Two homologically equivalent cycles give the same value of the charge on-shell, that is, if there are no sources of the KR field in the $(d-2)$-dimensional volume whose boundary is the union of the two properly oriented $(d-3)$-cycles.

Let us now consider the conserved charges associated to the invariance under δ_χ. This transformation acts on the 1-forms $\mathcal{A}^I$ and on the KR 2-form B. Transformations with constant χ^I (closed 0-forms) leave invariant the 1-forms, but they do not leave invariant B. They only change it by an exact 2-form $d\left(\frac{1}{2}\chi_I\mathcal{A}^I\right)$. Thus, we must add a compensating Λ gauge transformation with parameter $\Lambda_\chi = -\frac{1}{2}\chi_I\mathcal{A}^I$ and consider the transformation of B

$$\delta_\chi B = -\tfrac{1}{2}d\left(\chi_I\mathcal{A}^I\right) + \tfrac{1}{2}\chi_I d\mathcal{A}^I = -\tfrac{1}{2}d\chi_I \wedge \mathcal{A}^I\,. \tag{4.28}$$

Then, from Eqs. (4.13) and (4.16) and the modified transformation rule Eq. (5.35), we get

$$\begin{aligned}\delta_\chi S = & \int \Big\{\mathbf{E}_B \wedge \delta_\chi B + \mathbf{E}_I \wedge \delta_\chi\mathcal{A}^I \\ & +d\left[e^{-2\phi}M_{IJ}\star\mathcal{F}^J\wedge\delta_\chi\mathcal{A}^I + e^{-2\phi}\star H\wedge\left(\delta_\chi B + \tfrac{1}{2}\mathcal{A}_I\wedge\delta_\chi\mathcal{A}^I\right)\right]\Big\}\,, \\ = & \int\left\{\left(\mathbf{E}_I + \tfrac{1}{2}\mathbf{E}_B\wedge\mathcal{A}_I\right)\wedge d\chi^I + d\left[\left(e^{-2\phi}M_{IJ}\star\mathcal{F}^J + e^{-2\phi}\star H\wedge\mathcal{A}_I\right)\wedge d\chi^I\right]\right\}\,.\end{aligned} \tag{4.29}$$

Integrating by parts the first term and using the Noether identities Eqs. (4.21b) and (4.21c) we get

$$\delta_\chi S = \int d\left\{(-1)^{d-1}\chi^I\left(\mathbf{E}_I + \tfrac{1}{2}\mathbf{E}_B\wedge\mathcal{A}_I\right) + \left(e^{-2\phi}M_{IJ}\star\mathcal{F}^J + e^{-2\phi}\star H\wedge\mathcal{A}_I\right)\wedge d\chi^I\right\}\,. \tag{4.30}$$

The usual argument leads to the conserved $(d-2)$-form

$$\mathbf{Q}[\chi] = (-1)^d\chi^I\left(e^{-2\phi}M_{IJ}\star\mathcal{F}^J + e^{-2\phi}\star H\wedge\mathcal{A}_I\right)\,, \tag{4.31}$$

and the definition of electric charges

$$\mathcal{Q}_I = \frac{(-1)^{d-1}g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{\mathcal{S}_{(d-2)}}\left(e^{-2\phi}M_{IJ}\star\mathcal{F}^J + e^{-2\phi}\star H\wedge\mathcal{A}_I\right)\,, \tag{4.32}$$

where we have added a conventional sign. Again, this charge is on-shell invariant under homologically-equivalent deformations of $\mathcal{S}_{(d-2)}$. This follows from the equation of motion written in the alternative form Eq. (4.17). It is also on-shell invariant under the δ_χ transformations, in spite of the explicit occurrence of the vector fields $\mathcal{A}_I$: the second term in the integrand has the same structure as the integrand of the KR charge and, for the same reason, it is invariant on-shell when we add to $\mathcal{A}_I$ exact 1-forms.

This charge is, in the terminology used by Marolf in Ref. [135], a *Page charge* but, as we have explained, apart from localized and conserved, it is also gauge invariant on-shell. The formalism leads us to use precisely this charge, which will be the one occurring in the first law of black hole mechanics.

Finally, let us consider the charge associated to the invariance under local Lorentz transformations δ_σ, which act on the Vielbein and on all the fields derived from it: spin connection and curvature. Let us postpone for the time being the conditions that the parameters that leave all of them invariant have to satisfy and lets study the transformation of the action. From Eqs. (4.13) and (4.16) we find

$$\delta_\sigma S = \int \left\{ \mathbf{E}_a \wedge \delta_\sigma e^a + d\left[-e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta_\sigma \omega_{ab} + 2\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta_\sigma e_b \right] \right\} , \tag{4.33}$$

and using Eqs. (4.18a) and (4.19a) and the Noether identity Eq. (4.21a), we find that the integrand immediately reduces to a total derivative,

$$\begin{aligned} \delta_\sigma S &= \int d\mathbf{J}[\sigma] , \\ \mathbf{J}[\sigma] &= (-1)^{d-1} e^{-2\phi} \mathcal{D}\sigma_{ab} \wedge \star(e^a \wedge e^b) + 2\sigma_{bc}\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge e^c . \end{aligned} \tag{4.34}$$

The standard argument tells us that $\mathbf{J}[\sigma] = d\mathbf{Q}[\sigma]$. Integrating by parts the first term

$$\mathbf{J}[\sigma] = d\left\{ (-1)^{d-1} e^{-2\phi} \sigma_{ab} \star (e^a \wedge e^b) \right\} + 3 \left(\sigma_{[bc}\imath_{a]} d e^{-2\phi} \right) \star (e^a \wedge e^b) \wedge e^c . \tag{4.35}$$

The last term vanishes identically because[5] $\star(e^a \wedge e^b) \wedge e^c = 2\eta^{c[a} \star e^{b]}$ and we arrive at

$$\mathbf{Q}[\sigma] = (-1)^{d-1} e^{-2\phi} \star (e^a \wedge e^b) \wedge \sigma_{ab} . \tag{4.37}$$

Now we have to consider Lorentz parameters that leave all the fields invariant. The spin connection and curvature are left invariant by covariantly constant parameters

$$\mathcal{D}\sigma^a{}_b = 0 , \tag{4.38}$$

but the invariance of the Vielbein $\sigma^a{}_b e^b = 0$ can only be satisfied for $\sigma^a{}_b = 0$, and would automatically imply the vanishing of $\mathbf{Q}[\sigma]$.

The $(d-2)$-form, though, reappears in the proof of the first law for a Lorentz parameter that is covariantly constant over the bifurcation surface. We also notice that terms of higher order in the Lorentz curvature, such as those which arise with α' corrections, lead to a non-vanishing Lorentz charge Ref. [53] .

4.3.3 Diffeomorphisms and covariant Lie derivatives

As we have discussed in the introduction, out of the fundamental fields of our theory, only the dilaton ϕ and the $\mathrm{O}(n,n)/(\mathrm{O}(n)\times\mathrm{O}(n))$ scalars ϕ^x transform as a tensor under

[5]Here we use the property

$$\star\, \omega^{(p)} \wedge \hat{\xi} = \star\imath_\xi \omega^{(p)} , \tag{4.36}$$

which is valid for any p-form $\omega^{(p)}$ and any vector field $\xi = \xi^\mu \partial_\mu$ and its dual 1-form $\hat{\xi} = \xi_\mu dx^\mu$.

diffeomorphisms $\delta_\xi x^\mu = \xi^\mu$, that is[6]

$$\delta_\xi \phi = -\pounds_\xi \phi = -\imath_\xi d\phi\,, \tag{4.40a}$$

$$\delta_\xi \phi^x = -\pounds_\xi \phi^x = -\imath_\xi d\phi^x\,. \tag{4.40b}$$

The Vielbein e^a, the vectors (1-forms), $\mathcal{A}$, and the KR 2-form, B, have gauge freedoms and transform as tensors up to *compensating* gauge transformations. These compensating gauge transformations can be determined by

1. Requiring gauge-covariance of the complete transformation law (which can then be interpreted as a gauge-covariant Lie derivative) and
2. Imposing that, for diffeomorphisms which are symmetries of the field configuration that we are considering (in particular, for isometries), the complete transformation (covariant Lie derivative) vanishes. The first condition ensures that this vanishing is gauge-invariant.

In what follows we will denote by k the vector fields ξ that generate diffeomorphisms that leave invariant the complete field configuration. k is, in particular, a Killing vector of the metric.

In Chapter 3, we reviewed the construction of a Lie derivative of the Vielbein, spin connection and curvature covariant under local Lorentz transformations (*Lie-Lorentz derivative*) of Refs. [41, 42] that build upon earlier work by Lichnerowicz, Kosmann and others [44–47]. We also dealt with Abelian vector fields in similar terms. It is convenient to quickly review these results starting with the Abelian vector case, adapted to the present situation.

The transformation of the Abelian vector fields $\mathcal{A}^I$ under diffeomorphisms can be defined as

$$\delta_\xi \mathcal{A}^I = -\mathbb{L}_\xi \mathcal{A}^I\,, \tag{4.41}$$

where $\mathbb{L}_\xi \mathcal{A}^I$ is the *Lie-Maxwell derivative*, defined by

$$\mathbb{L}_\xi \mathcal{A}^I \equiv \imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\,. \tag{4.42}$$

Here $\mathcal{P}_\xi{}^I$ is a gauge-invariant $\mathrm{O}(n,n)$ vector of functions that depends on $\mathcal{A}^I$ and on the generator of diffeomorphisms ξ and it is assumed to have the property that, when $\xi = k$, it satisfies the equation

[6] The metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\mu$ and the 2- and 3-form field strengths $\mathcal{F}, H$ also transform as tensors:

$$\delta_\xi g_{\mu\nu} = -\pounds_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}\,, \tag{4.39a}$$

$$\delta_\xi \mathcal{F} = -\pounds_\xi \mathcal{F} = -(\imath_\xi d + d\imath_\xi)\mathcal{F}\,, \tag{4.39b}$$

$$\delta_\xi H = -\pounds_\xi H = -(\imath_\xi d + d\imath_\xi)H\,. \tag{4.39c}$$

$$d\mathcal{P}_k{}^I = -\imath_k \mathcal{F}^I \,. \tag{4.43}$$

The invariance of the 2-form $\mathcal{F}^I$ guarantees the local existence of $\mathcal{P}_k{}^I$, which is known as the *momentum map* associated to k. On the other hand, Eq. (4.43) ensures that the two properties of the variations of the fields under diffeomorphisms that we have demanded are satisfied. Finally, observe that the Lie-Maxwell derivative is just a combination of the standard Lie derivative plus a compensating gauge transformation with parameter

$$\chi_\xi{}^I = \imath_\xi \mathcal{A}^I - \mathcal{P}_\xi{}^I \,. \tag{4.44}$$

For fields with Lorentz indices (Vielbein, spin connection and curvature), the variation under diffeomorphisms is also given by (minus) a Lorentz-covariant generalization of the Lie derivative $\delta_\xi = -\mathbb{L}_\xi$ usually called *Lie-Lorentz derivative* Refs. [41,42,44–47]. This derivative can also be constructed by adding to the standard Lie derivative a compensating Lorentz transformation with the parameter

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - \nabla^{[a}\xi^{b]} \,. \tag{4.45}$$

For the Vielbein, the Lie-Lorentz derivative can be expressed in several equivalent and manifestly Lorentz-covariant forms

$$\mathbb{L}_\xi e^a{}_\mu = \tfrac{1}{2} e^{a\,\nu} \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) \tag{4.46a}$$

$$\mathbb{L}_\xi e^a = \mathcal{D}\xi^a + P_\xi{}^a{}_b e^b \,, \tag{4.46b}$$

where

$$P_\xi{}^{ab} \equiv \nabla^{[a}\xi^{b]} \,, \tag{4.47}$$

satisfies, when $\xi = k$, the equation

$$\imath_k R^{ab} = -\mathcal{D} P_k{}^{ab} \,, \tag{4.48}$$

that shows that we can view $P_k{}^{ab}$ as a momentum map as well.[7]

In the form Eq. (4.46a) we immediately see that the Lie-Lorentz derivative of the Vielbein vanishes when $\xi = k$, a Killing vector. The same is true for the connection and curvature.

Observe that $P_\xi{}^{ab}$ transforms covariantly under local Lorentz transformations.

The above transformation of the Vielbein induce the following transformations of the spin connection and curvature that we quote for later use:

$$\delta_\xi \omega^{ab} = -\mathbb{L}_\xi \omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D} P_\xi{}^{ab}\right) \,, \tag{4.49a}$$

$$\delta_\xi R^{ab} = -\mathbb{L}_\xi R^{ab} = -\left(\mathcal{D}\imath_\xi R^{ab} - 2 P_\xi{}^{[a}{}_c R^{b]c}\right) \,. \tag{4.49b}$$

[7] Compare this equation to Eq. (4.43).

Observe that the Lie-Lorentz derivative of the spin connection has the same structure as that of the Abelian connection $\mathcal{A}^I$ in Eq. (4.42), *i.e.* the inner product of ξ with the curvature plus the derivative of the momentum map.

In asymptotically-flat stationary black-hole spacetimes with bifurcate horizon, if k is the Killing vector whose Killing horizon coincides with the event horizon and $\mathcal{BH}$ is the bifurcation sphere,

$$P_k{}^{ab} = \nabla^{[a}k^{b]} \stackrel{\mathcal{BH}}{=} \kappa n^{ab} \,, \tag{4.50}$$

where κ is the surface gravity and n^{ab} is the binormal to the event horizon, with the normalization $n^{ab}n_{ab} = -2$. The zeroth law of black-hole mechanics stating that κ is constant over the horizon [20, 125] is associated to the Lorentz momentum map, just as the generalized zeroth law that states that the electric potential is also constant over the horizon in the Einstein-Maxwell theory is associated to the Maxwell momentum map [88].[8] We are going to see that further "generalized zeroth laws" are also associated to momentum maps when we restrict ourselves to the bifurcation surface. We will call them *restricted generalized zeroth laws.*

Let us now consider the KR field. It is convenient to start by considering the transformation of the 3-form field strength H defined in Eq. (4.10) under diffeomorphisms. Since it is gauge invariant, upon use of its Bianchi identity

$$\delta_\xi H = -\pounds_\xi H = -\imath_\xi dH - d\imath_\xi H = \imath_\xi \mathcal{F}_I \wedge \mathcal{F}^I - d\imath_\xi H \,. \tag{4.51}$$

When $\xi = k$, this expression must vanish and we can use Eq. (4.43), which leads to the identity

$$\delta_\xi H = -d\left(\imath_k H + \mathcal{P}_{k\,I}\mathcal{F}^I\right) = 0 \,, \tag{4.52}$$

which, in turn, implies the local existence of a gauge-invariant 1-form that we will also call a momentum map, satisfying

$$-\imath_k H - \mathcal{P}_{k\,I}\mathcal{F}^I = dP_k \,. \tag{4.53}$$

The KR momentum map plays a fundamental role in the definition of the variation of the KR 2-form B under diffeomorphisms which should be of the general form

$$\delta_\xi B = -\pounds_\xi B + \left(\delta_{\Lambda_\xi} + \delta_{\chi_\xi}\right) B \,, \tag{4.54}$$

where χ_ξ and Λ_ξ are scalar and 1-form parameters of compensating gauge transformations. They will generically depend on $\mathcal{A}^I$ and B as well as on ξ. $\chi_\xi{}^I$ has to be the same parameter used in the definition of the Lie-Maxwell derivative Eq. (4.44) and we just have to determine Λ_ξ. Now, the Maxwell and Lorentz cases suggest that we try

$$\Lambda_\xi = \imath_\xi B - P_\xi \,, \tag{4.55}$$

which leads to

[8] This parallelism between zeroth laws was observed in [51], also in the wider context of Einstein-Yang-Mills theories.

$$\begin{aligned}\delta_\xi B &= -\pounds_\xi B + d(\imath_\xi B - P_\xi) + \tfrac{1}{2}\chi_{\xi\, I} d\mathcal{A}^I \\ &= -\left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) + \tfrac{1}{2}\mathcal{A}_I \wedge \imath_\xi \mathcal{F}^I + \tfrac{1}{2}\mathcal{P}_{\xi\, I}\mathcal{F}^I \,.\end{aligned} \tag{4.56}$$

When $\xi = k$, though,

$$\delta_k B = d\left(\tfrac{1}{2}\mathcal{P}_{k\, I}\mathcal{A}^I\right) \,. \tag{4.57}$$

This is not zero but it can be absorbed into a redefinition of Λ_ξ:

$$\Lambda_\xi = \imath_\xi B - P_\xi - \tfrac{1}{2}\mathcal{P}_{k\, I}\mathcal{A}^I \,, \tag{4.58}$$

which gives the variation

$$\delta_\xi B = -\left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) - \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I \,. \tag{4.59}$$

This form of the variation makes it evident that $\delta_k B = 0$, because $\delta_k \mathcal{A}^I = 0$ and because of the definition of the KR momentum map 1-form Eq. (4.53).

It remains to check that the vanishing of this variation is a gauge-invariant statement. Indeed, if we perform a gauge transformation in $\delta_\xi B$, taking into account that all the momentum maps and $\delta_\xi \mathcal{A}^I$ are gauge-invariant, we find

$$\delta_{\text{gauge}}\delta_\xi B = -\tfrac{1}{2}\delta_{\text{gauge}}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I \,, \tag{4.60}$$

which vanishes identically for $\xi = k$.

4.3.4 The Wald-Noether charge

The Wald-Noether charge is the conserved $(d-2)$-form associated to the invariance of the action under diffeomorphisms [28]. The transformations that we are going to consider (combinations of standard Lie derivative and gauge transformations, as we have explained) are

$$\delta_\xi \phi = -\imath_\xi d\phi \,, \tag{4.61a}$$

$$\delta_\xi \phi^x = -\imath_\xi d\phi^x \,. \tag{4.61b}$$

$$\delta_\xi \mathcal{A}^I = -\left(\imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\right) \,, \tag{4.61c}$$

$$\delta_\xi e^a = -\left(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\right) \,, \tag{4.61d}$$

$$\delta_\xi \omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab}\right) \,, \tag{4.61e}$$

$$\delta_\xi B + \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I = -\left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) \,. \tag{4.61f}$$

From Eq. (4.13), and using the definition of $\tilde{\mathbf{E}}_I$ in Eqs. (4.15c) and (4.15d) to cancel the terms of the form $\mathbf{E}_B \wedge \mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I$, we get

$$\begin{aligned}\delta_\xi S = -\int \Big\{ & \mathbf{E}_a \wedge \left(\mathcal{D}\imath_\xi e^a + P_\xi{}^a{}_b e^b\right) + \mathbf{E}_B \wedge \left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) \\ & + \tilde{\mathbf{E}}_I \wedge \left(\imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\right) + \mathbf{E}_\phi \imath_\xi d\phi + \mathbf{E}_x \imath_\xi d\phi^x \\ & - d\mathbf{\Theta}(\varphi, \delta_\xi\varphi)\Big\} \,,\end{aligned} \tag{4.62}$$

while, from Eq. (4.16), we get

$$\begin{aligned}\mathbf{\Theta}(\varphi, \delta_\xi \varphi) = {} & e^{-2\phi} \star (e^a \wedge e^b) \wedge (\imath_\xi R_{ab} + \mathcal{D}P_{\xi\, ab}) \\ & - 2\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge (\mathcal{D}\xi_b + P_{\xi\, bc}e^c) \\ & + 8e^{-2\phi} \star d\phi \imath_\xi d\phi - e^{-2\phi} g_{xy} \star d\phi^y \imath_\xi d\phi^x \\ & - e^{-2\phi} M_{IJ} \star \mathcal{F}^J \wedge \left(\imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\right) \\ & - e^{-2\phi} \star H \wedge \left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) \,.\end{aligned} \tag{4.63}$$

Next, we consider the terms in $\delta_\xi S$ that contain momentum maps, integrating by parts those which involve their derivatives:

$$
\begin{aligned}
&\mathbf{E}_a \wedge P_\xi{}^a{}_b e^b + \tilde{\mathbf{E}}_I \wedge d\mathcal{P}_\xi{}^I + \mathbf{E}_B \wedge \left(\mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) \\
&= \mathbf{E}^{[a} \wedge e^{b]} P_{\xi\, ab} + P_\xi d\mathbf{E}_B + (-1)^d \mathcal{P}_{\xi\, I}\left[d\tilde{\mathbf{E}}^I + (-1)^d \mathbf{E}_B \wedge \mathcal{F}^I\right] \\
&+ d\left(P_\xi \wedge \mathbf{E}_B + (-1)^{d-1}\mathcal{P}_\xi{}^I \tilde{\mathbf{E}}_I\right) .
\end{aligned}
\tag{4.64}
$$

The terms in the first line vanish as a consequence of the Noether identities Eqs. (4.21a)-(4.21c) and we are left with the total derivative which will be added to $\mathbf{\Theta}(\varphi, \delta_\xi\varphi)$. Thus, the variation of the action takes the form

$$
\begin{aligned}
\delta_\xi S = -\int \Big\{ &\mathbf{E}_a \wedge \mathcal{D}\imath_\xi e^a + \mathbf{E}_B \wedge \imath_\xi H + \tilde{\mathbf{E}}_I \wedge \imath_\xi \mathcal{F}^I + \mathbf{E}_\phi \imath_\xi d\phi + \mathbf{E}_x \imath_\xi d\phi^x \\
&-d\left[\mathbf{\Theta}(\varphi, \delta_\xi\varphi) - P_\xi \wedge \mathbf{E}_B + (-1)^d \mathcal{P}_\xi{}^I \tilde{\mathbf{E}}_I\right]\Big\} .
\end{aligned}
\tag{4.65}
$$

Integrating the first term of Eq. (4.65) by parts we get another total derivative to add to $\mathbf{\Theta}(\varphi, \delta_\xi\varphi)$ and ($\imath_\xi e^a = \xi^a$)

$$
(-1)^d \mathcal{D}\mathbf{E}_a \xi^a + \mathbf{E}_B \wedge \imath_\xi H + \tilde{\mathbf{E}}_I \wedge \imath_\xi \mathcal{F}^I + \mathbf{E}_\phi \imath_\xi d\phi + \mathbf{E}_x \imath_\xi d\phi^x = 0 \,, \tag{4.66}
$$

by virtue of the Noether identity associated to the invariance under diffeomorphisms and, therefore,

$$
\delta_\xi S = \int d\mathbf{\Theta}'(\varphi, \delta_\xi\varphi) \,, \tag{4.67}
$$

where

$$
\mathbf{\Theta}'(\varphi, \delta_\xi\varphi) = \mathbf{\Theta}(\varphi, \delta_\xi\varphi) + (-1)^d \mathbf{E}_a \xi^a - P_\xi \wedge \mathbf{E}_B + (-1)^d \mathcal{P}_\xi{}^I \tilde{\mathbf{E}}_I \,. \tag{4.68}
$$

Usually, the last three terms, which are proportional to equations of motion and vanish on-shell, are ignored for this very reason. However, we have found that keeping them is actually quite useful for finding the Wald-Noether charge, because they are exactly what is needed to write $\mathbf{J}$ as a total derivative. Without them, we would have had to guess which combinations of the equations of motion should be added to achieve that goal. Furthermore, the result that we will obtain will be valid off-shell.

Since the action is exactly invariant under the gauge transformations Eq. (4.18), but it is only invariant up to a total derivative under standard infinitesimal diffeomorphisms, under the combined transformations Eqs. (4.61)

$$
\delta_\xi S = -\int d\imath_\xi \mathbf{L} \,, \tag{4.69}
$$

which, combined with Eq. (4.67), leads to the identity

$$d\mathbf{J} = 0\,, \tag{4.70}$$

which holds off-shell for arbitrary ξ with

$$\mathbf{J} \equiv \mathbf{\Theta}'(\varphi, \delta_\xi \varphi) + \imath_\xi \mathbf{L}\,. \tag{4.71}$$

Eq. (4.70) implies the local existence of a $(d-2)$-form $\mathbf{Q}[\xi]$ such that

$$\mathbf{J} = d\mathbf{Q}[\xi]\,. \tag{4.72}$$

Using the previous results we find that, up to total derivatives and up to the overall factor $(g_s^{(d)\,2} 16\pi G_N^{(d)})^{-1}$ that we are suppressing to get simpler expressions

$$\begin{aligned}\mathbf{Q}[\xi] = & (-1)^d \star (e^a \wedge e^b) \left[e^{-2\phi} P_{\xi\,ab} - 2\imath_a d e^{-2\phi} \xi_b\right] \\ & + (-1)^{d-1} \mathcal{P}_\xi{}^I \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J\right) - P_\xi \wedge \left(e^{-2\phi} \star H\right)\,.\end{aligned} \tag{4.73}$$

4.4 Zeroth laws

The zeroth law and its generalizations, ensuring that the surface gravity and the electrostatic potential are constant over the event (Killing) horizon $\mathcal{H}$ are important ingredients in the standard derivation of the first law of black-hole mechanics in the context of the Einstein-Maxwell theory [20]. In presence of higher-rank p-form fields, it is not clear how these laws should be further generalized. However, it is possible to prove the first law using Wald's formalism working on the bifurcation sphere $\mathcal{BH}$, where the Killing vector k associated to the horizon vanishes. This restricts the validity of the proof to bifurcate horizons but, on the other hand, it makes it possible to carry out the proof using a more restricted form of the (generalized) zeroth laws which states the closedness of the electrostatic potential and its higher-rank generalizations on $\mathcal{BH}$. Since the electrostatic potential is a scalar, its closedness implies that it is constant on $\mathcal{BH}$, which is a restricted version of the generalized zeroth law. For higher-rank potentials closedness is, actually, all we need, as we will see in the next section.

We start by assuming that all the field strengths of the theory are regular on the horizon.[9] This implies that

$$\imath_k \mathcal{F}^I \stackrel{\mathcal{BH}}{=} 0\,, \tag{4.74a}$$

$$\imath_k H \stackrel{\mathcal{BH}}{=} 0\,. \tag{4.74b}$$

The first equation directly implies the closedness of the components of the momentum map $\mathcal{P}_k^I$ on $\mathcal{BH}$ on account of its definition Eq. (4.43), and, hence, its constancy on $\mathcal{BH}$, a

[9] Observe that in this theory in which all the field strengths are gauge-invariant, this is a gauge-invariant statement that should be valid in a regular coordinate patch.

statement that we can call *restricted generalized zeroth law* after the natural identification of $\mathcal{P}_k^I$ with the electrostatic black-hole potential Φ^I. Observe that, our gauge-invariant definition of the electrostatic black-hole potential guarantees that it is fully defined up to an additive constant that can be determined by setting the value of the potential at infinity to zero.

Using Eq. (4.74b) and the constancy of $\mathcal{P}_k^I$ on on $\mathcal{BH}$ in the definition of the KR momentum map Eq. (4.53) we find that

$$0 \stackrel{\mathcal{BH}}{=} -\imath_k H = dP_k + \mathcal{P}_{k\,I}\mathcal{F}^I \stackrel{\mathcal{H}}{=} d\left(P_k + \mathcal{P}_{k\,I}\mathcal{A}^I\right) . \tag{4.75}$$

We can call the combination $P_k + \mathcal{P}_{k\,I}\mathcal{A}^I$ that is closed on $\mathcal{BH}$ the KR black-hole potential Φ and its closedness can be understood as another restricted generalized zeroth law of black-hole mechanics in this theory. Observe that Φ is not gauge-invariant, but P_k is only defined up to shifts by exact 1-forms anyway and, when we use Φ as the 1-form Λ in the calculation of the KR charge Eq. (4.26), the addition of exact 1-forms does not change the value of the associated KR charge Eq. (4.27). The fact that this Φ occurs in the expressions leading to the first law precisely plays this role is quite a non-trivial check of the consistency of our results.

4.5 The first law

We start by defining the *pre-symplectic* $(d-1)$*-form* [27]

$$\omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1\mathbf{\Theta}(\varphi, \delta_2\varphi) - \delta_2\mathbf{\Theta}(\varphi, \delta_1\varphi)\,, \tag{4.76}$$

and the *symplectic form* relative to the Cauchy surface Σ

$$\Omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \int_\Sigma \omega(\varphi, \delta_1\varphi, \delta_2\varphi)\,. \tag{4.77}$$

Now, following Ref. [22], when φ solves the equations of motion $\mathbf{E}_\varphi = 0$ if $\delta_1\varphi = \delta\varphi$ is an arbitrary variation of the fields and $\delta_2\varphi = \delta_\xi\varphi$ is their variation under diffeomorphisms, we have that

$$\omega(\varphi, \delta\varphi, \delta_\xi\varphi) = \delta\mathbf{J} + d\imath_\xi\mathbf{\Theta}' = \delta d\mathbf{Q}[\xi] + d\imath_\xi\mathbf{\Theta}'\,, \tag{4.78}$$

where, in our case, $\mathbf{J} = d\mathbf{Q}$, where $\mathbf{Q}$ is given by Eq. (4.73) and $\mathbf{\Theta}'$ is given in Eq. (4.68). Since, on-shell, $\mathbf{\Theta} = \mathbf{\Theta}'$, we have that, if $\delta\varphi$ satisfies the linearized equations of motion, $\delta d\mathbf{Q} = d\delta\mathbf{Q}$. Furthermore, if the parameter $\xi = k$ generates a transformation that leaves invariant the field configuration, $\delta_k\varphi = 0$,[10] linearity implies that $\omega(\varphi, \delta\varphi, \delta_k\varphi) = 0$, and

$$d\left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right) = 0\,. \tag{4.79}$$

Integrating this expression over a hypersurface Σ with boundary $\delta\Sigma$ and using Stokes' theorem we arrive at

[10] We have constructed variations of the fields δ_ξ for which this is possible.

$$\int_{\delta\Sigma} \left(\delta\mathbf{Q}[k] + \imath_k \mathbf{\Theta}'\right) = 0\,. \tag{4.80}$$

We are interested in asymptotically flat, stationary, black-hole spacetimes and we choose k as the Killing vector whose Killing horizon coincides with the event horizon $\mathcal{H}$, which we assume to be a bifurcate horizon. This Killing vector k is assumed to be linear combination with constant coefficients Ω^n of the timelike Killing vector associated to stationarity, $t^\mu\partial_\mu$ and the $[\frac{1}{2}(d-1)]$ inequivalent rotations $\phi_n^\mu\partial_\mu$

$$k^\mu = t^\mu + \Omega^n \phi_n^\mu\,. \tag{4.81}$$

Furthermore, we choose the hypersurface Σ to be the space between infinity and the bifurcation sphere ($\mathcal{BH}$) on which $k = 0$. Then, its boundary $\delta\Sigma$ has two disconnected pieces: a $(d-2)$-sphere at infinity, S^{d-2}_∞, and the bifurcation sphere $\mathcal{BH}$. Then, taking into account that $k = 0$ on $\mathcal{BH}$, we obtain the relation

$$\delta \int_{\mathcal{BH}} \mathbf{Q}[k] = \int_{\mathrm{S}^{d-2}_\infty} \left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right)\,. \tag{4.82}$$

As explained in Ref. [22,52], the right-hand side can be identified with $\delta M - \Omega^m \delta J_n$, where M is the total mass of the black-hole spacetime and J_n are the independent components of the angular momentum.

Using the explicit form of $\mathbf{Q}[k]$, Eq. (4.73), and restoring the overall factor $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$, we find

$$\begin{aligned}
\delta\int_{\mathcal{BH}} \mathbf{Q}[k] = {}& \frac{(-1)^{d-1} g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}} \mathcal{P}_k{}^I \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J\right) \\
& - \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}} P_k \wedge \left(e^{-2\phi}\star H\right) \\
& + \frac{(-1)^d g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}} \star(e^a\wedge e^b)\left[e^{-2\phi}P_{k\,ab} - 2\imath_a d e^{-2\phi} k_b\right]\,.
\end{aligned} \tag{4.83}$$

The last term vanishes over the bifurcation sphere and will be removed from now on.

As it is, this expression has two problems that make it difficult for us to obtain the kind of terms that occur in the first law. In the first line, we have an expression that we should be able to interpret in terms of the electric charges $\mathcal{Q}_I$. However, when we compare this with Eq. (4.32) we see that the second term in the integrand is missing. Without that term, the charge is not conserved. On the other hand, in the second line, we have an expression that we should be able to interpret in terms of the KR charge using Eq. (4.26). However, the 1-form P_k is not closed on $\mathcal{BH}$.

The solution to these two problems is unique: the addition and subtraction of the term $\mathcal{P}_{k\,I}\mathcal{A}^I \wedge \left(e^{-2\phi}\star H\right)$ in the integrand, so that the integral to evaluate on $\mathcal{BH}$ takes the form

$$\begin{aligned}
\delta\int_{\mathcal{BH}}\mathbf{Q}[k] = {} & \frac{(-1)^{d-1}g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}}\mathcal{P}_k{}^I\left[e^{-2\phi}M_{IJ}\star\mathcal{F}^J + e^{-2\phi}\star H\wedge\mathcal{A}_I\right] \\
& -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}}\left(P_k+\mathcal{P}_{k\,I}\mathcal{A}^I\right)\wedge\left(e^{-2\phi}\star H\right) \\
& +\frac{(-1)^d g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}}e^{-2\phi}\star(e^a\wedge e^b)P_{k\,ab}\,.
\end{aligned} \tag{4.84}$$

Now, using the generalized zeroth law that ensures that $\mathcal{P}_k{}^I \equiv \Phi^I$ is constant over $\mathcal{H}$, in particular on $\mathcal{BH}$, and the definition of electric charge Eq. (4.32), the first term in the right-hand side takes the form

$$\Phi^I\delta\mathcal{Q}_I\,. \tag{4.85}$$

Next, from the closedness of the combination $\Phi = P_k + \mathcal{P}_{k\,I}\mathcal{A}^I$ on $\mathcal{BH}$, (the restricted generalized zeroth law) using the Hodge decomposition

$$P_k + \mathcal{P}_{k\,I}\mathcal{A}^I \stackrel{\mathcal{BH}}{=} de + \Phi^i\Lambda_{h\,i}\,, \tag{4.86}$$

where the $\Lambda_{h\,i}$ are harmonic 1-forms on $\mathcal{BH}$ and the Φ^1 are constants that have the interpretation of potentials associated to the charge of the KR field (the dipole charge of Ref. [136] in particular), and using the definition Eq. (4.27), we find that the second term in the right-hand side takes the form

$$\Phi^i\delta Q_i\,, \qquad Q_i \equiv Q[\Lambda_{h\,i}]\,. \tag{4.87}$$

Observe that the addition and subtraction of the term $\mathcal{P}_{k\,I}\mathcal{A}^I\wedge\left(e^{-2\phi}\star H\right)$ has been crucial to recover the correct definition of the charges which, in particular, demands the occurrence of the closed 1-form $P_k + \mathcal{P}_{k\,I}\mathcal{A}^I$.

Now, let us consider the third integral. Before we compute it explicitly, we notice that the integrand is identical, up to a sign, to the Lorentz charge Eq. (4.37) computed for the Lorentz parameter $P_k{}^a{}_b$ which is covariantly constant over the bifurcation surface. This coincidence is very intriguing and will be further explored in Ref. [53].

Using Eq. (4.50)

$$\begin{aligned}\frac{(-1)^d\kappa}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}} e^{-2(\phi-\phi_\infty)}\star(e^a\wedge e^b)n_{ab} &= -\frac{\kappa}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}} e^{-2(\phi-\phi_\infty)}n^{ab}n_{ab}\\ &= T\delta\frac{\mathcal{A}_{\mathcal{H}}}{4G_N^{(d)}}\,,\end{aligned} \tag{4.88}$$

where we have used the normalization of the binormal $n_{ab}n^{ab} = -2$, $T = \kappa/2\pi$ is the Hawking temperature and

$$\mathcal{A}_{\mathcal{H}} \equiv \int_{\mathcal{B}} d^{d-2}S e^{-2(\phi-\phi_\infty)}\,, \tag{4.89}$$

is the area of the horizon measured with the *modified Einstein frame metric* [141] which is obtained from the string one by multiplying by the conformal factor $e^{-4(\phi-\phi_\infty)/(d-2)}$, and computed using the spatial section $\mathcal{BH}$.

We finally get the following expression for the first law of black hole mechanics in the Heterotic Superstring effective action to leading order in α':

$$\delta M = T\delta\frac{\mathcal{A}_{\mathcal{H}}}{4G_N^{(d)}} + \Omega^m\delta J_m + \Phi^i\delta Q_i + \Phi^I\delta\mathcal{Q}_I\,, \tag{4.90}$$

which leads to the interpretation of the area of the horizon divided by $4G_N^{(d)}$ as the black-hole entropy.

4.6 Momentum Maps for Black Rings in $d = 5$

In this section we are goin to illustrate how the definitions made and the properties proven in the previous sections work in an explicit example. In particular, we are going to determine the values of the momentum maps, checking the restricted generalized zeroth laws.

The solution we are going to consider is a non-extremal, charged, black ring solution of pure $\mathcal{N} = 1, d = 5$ supergravity which can be easily embedded in the toroidally-compactified Heterotic Superstring effective field theory using the results in Appendix B. This embedding is necessary because all the definitions and formulae that we have developed are adapted to that theory. In Appendix B we show how the action Eq. (4.4), for $d = 5$ can be consistently truncated to that of pure $\mathcal{N} = 1, d = 5$ supergravity Eq. B.26 in two steps:

1. A direct truncation of some fields of the Heterotic theory, to obtain a model of $\mathcal{N} = 1, d = 5$ supergravity coupled to two vector multiplets. The Kalb-Ramond 2-form has to be dualized into a 1-form in order to obtain the supergravity theory in the standard form, with 3 1-forms which can be treated on the same footing and which may be linearly combined.

2. A consistent truncation of the two vector supermultiplets. In this truncation, rather than setting two of the vector fields to zero, they are identified with the surviving vector, up to numerical factors. This allows the scalars in the vecort supermultiplets to take their vacuum values.

Given a solution of pure $\mathcal{N} = 1, d = 5$ supergravity, one can easily retrace those steps, restoring, first, the two "matter" vector fields so the solution becomes now a solution of $\mathcal{N} = 1, d = 5$ supergravity coupled to two vector multiplets. Then, dualizing the vector in the supergravity multiplet to recover the Kalb-Ramond 2-form, the solution can immediately be interpreted as a solution of the Heterotic Superstring effective field theory in which many other fields simply take their vacuum values.

The non-extremal, charged, black ring solution that we are going to consider is the one given in Section 4 of Ref. [140]. This solution belongs to a more general family of non-supersymmetric black rings with three charges α_i, three dipoles μ_i, with $i = 1, 2, 3$, and two angular momenta J_φ and J_ψ in the theory with two vector supermultiplets. The solution above corresponds to setting all three charges and three dipoles equal, $\alpha_i = \alpha$ and $\mu_i = \mu$ for all i. This identification of the charges and dipoles coprresponds to the identification between the vector fields that leads from the supergravity theory with matter to the theory of pure supergravity. Let us review the solution and its main features.

The physical fields of the solution (the metric and the Abelian connection A) can be written in terms of the five parameters $(R, \alpha, \mu, \lambda, \nu)$ (all of them dimensionless except for the length scale R) and the three functions, $F(\xi), H(\xi)$ and $G(\xi)$, given by

$$H(\xi) = 1 - \mu\xi\,, \qquad F(\xi) = 1 + \lambda\xi\,, \qquad G(\xi) = (1 - \xi^2)(1 + \nu\xi)\,. \tag{4.91}$$

The line element is

$$\begin{aligned} ds^2 = &\frac{U(x,y)}{h_\alpha^2(x,y)}\left(dt + \omega_\psi(y)d\psi + \omega_\varphi(x)d\varphi\right)^2 - h_\alpha(x,y)F(x)H(x)H(y)^2\times \\ &\times \frac{R^2}{(x-y)^2}\left[-\frac{G(y)}{F(y)H(y)^3}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3}d\varphi^2\right]\,, \end{aligned} \tag{4.92}$$

where we use the shorthand notation $s = \sinh\alpha$ and $c = \cosh\alpha$, the following combinations of the fundamental parameters

$$C_\lambda = \epsilon_\lambda\sqrt{\lambda(\lambda - \nu)\frac{1+\lambda}{1-\lambda}}\,, \quad C_\mu = \epsilon_\mu\sqrt{\mu(\mu + \nu)\frac{1-\mu}{1+\mu}}\,, \quad \epsilon_{\lambda,\mu} = \pm 1\,, \tag{4.93}$$

and the following combinations of the fundamental functions in Eq. (4.91)

$$U(x,y) = \frac{H(x)}{H(y)}\frac{F(y)}{F(x)}\,, \tag{4.94a}$$

$$h_\alpha(x,y) = 1 + \frac{(\lambda+\mu)(x-y)}{F(x)H(y)}s^2\,, \tag{4.94b}$$

$$\omega_\psi(y) = R(1+y)\left[\frac{1}{F(y)}C_\lambda c^3 - \frac{3}{H(y)}C_\mu c s^2\right]\,, \tag{4.94c}$$

$$\omega_\varphi(x) = -R(1+x)s\left(\frac{1}{F(x)}C_\lambda s^2 - \frac{3}{H(x)}C_\mu c^2\right)\,. \tag{4.94d}$$

Finally, the gauge field reads

$$\begin{aligned} -A/\sqrt{3} = {} & \frac{U(x,y)-1}{h_\alpha(x,y)}csdt \\ & + \frac{R(1+y)}{h_\alpha(x,y)}\left[\frac{U(x,y)}{F(y)}C_\lambda c^2 s - \frac{U(x,y)}{H(y)}C_\mu s^3 - \frac{2}{H(y)}C_\mu c^2 s\right]d\psi \\ & + \frac{R(1+x)}{h_\alpha(x,y)}\left[2\frac{U(x,y)}{H(x)}C_\mu c s^2 - \frac{1}{F(x)}C_\lambda c s^2 + \frac{1}{H(x)}C_\lambda c^3\right]d\varphi\,. \end{aligned} \tag{4.95a}$$

The parameters of the solution must satisfy the constraints

$$0 < \nu \leq \lambda < 1\,, \qquad\qquad 0 \leq \mu < 1\,, \tag{4.96}$$

to avoid naked singularities. Additional constraints arise from the codition of absence of Dirac-Misner strings and conical sigularities, as we are going to see.

The coordinates x, y take values in

$$-\infty < y \leq -1\,, \qquad\qquad -1 \leq x \leq 1\,. \tag{4.97}$$

The surfaces of constant y have the topology $\mathrm{S}^2\times\mathrm{S}^1$. x is a polar coordinate on the S^2 (essentially, $x \sim \cos\theta$), which is also parametrized by φ, which plays the role of azymuthal angle. ψ parametrizes the S^1, see Fig. 4.1. Spatial infinity is approached when both x and y go to -1, although the coordinates are ill-defined in that limit.[11] The orbits of the vector ∂_φ close off at $x=-1$, but do not do the same at $x=1$ unless $\omega_\varphi(x=+1)=0$, which can forces us to require

$$\frac{C_\lambda}{1+\lambda}s^2 = \frac{3C_\mu}{1-\mu}c^2\,, \tag{4.98}$$

[11] Good coordinates at infinity can be found in Ref. [140].

which removes any possible Dirac-Misner strings. (The same constraint makes $A_\varphi(x=+1)$ independent of y.) Then, the fixed point sets of ∂_ψ and ∂_φ are, respectively, $y=-1$ (axis of the ring) and $x=1,-1$ (inner and outer axes of the S^2).

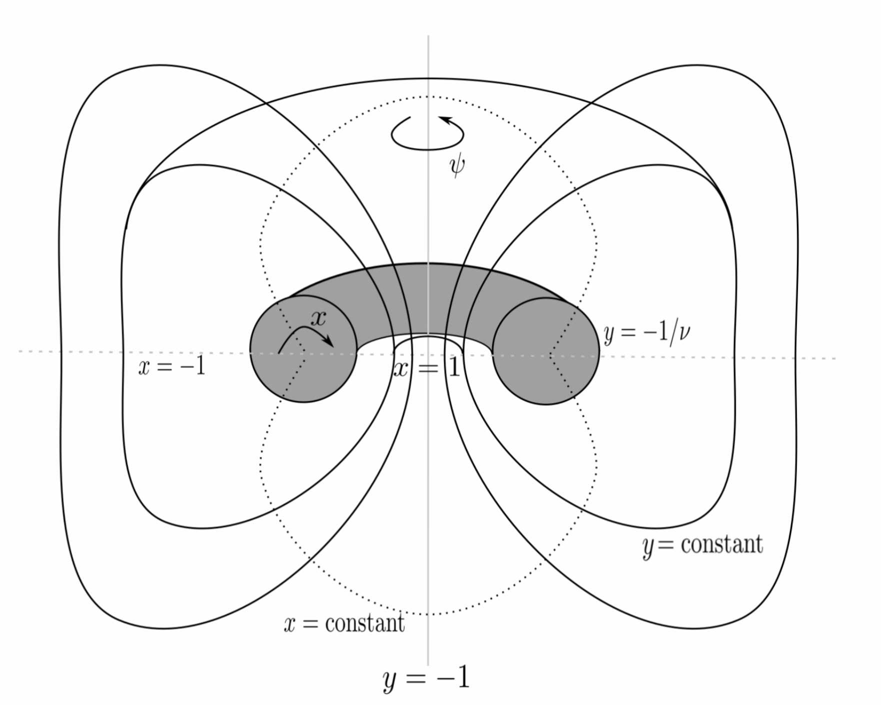

Figure 4.1: Sketch of a section of constant t and φ of the black ring (figure based on Ref. [136]). The disc at $x=1$ and infinite annulus at $x=-1$ are the axes (fixed points) of ∂_φ, while the axis of the ring is at $y=-1$ (fixed points of ∂_ψ). Surfaces of constant y have topology $S^1\times S^2$. $y=-1/\nu$ corresponds to the horizon (shaded surface) while surfaces of constant $y\in(-1/\nu,-1)$ are fatter rings containing the horizon in their interior.

Finally, the periods of ψ and φ must be chosen appropriately so as to avoid conical singularities. The axes $y=-1$ and $x=-1$ (which extend to infinity) are regular for the periods

$$\Delta\psi=\Delta\varphi=2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\,(1+\mu)^{3/2}\,. \tag{4.99}$$

For generic values of the parameters, though, the period of φ required by smoothness at the inner axis, $x=1$, differs from the above $\Delta\varphi$. Making both periods coincide ("balancing" the ring) is possible only when the following constraint holds

$$\left(\frac{1-\nu}{1+\nu}\right)^2=\frac{1-\lambda}{1+\lambda}\left(\frac{1+\mu}{1-\mu}\right)^3\,. \tag{4.100}$$

Henceforth we shall assume that Eqs. (4.98) and (4.100) hold, so that, effectively, we will be dealing with a three-parameter family of solutions. As shown in Ref. [140], the mass, the two independent angular momenta and the area of the event horizon of the solution read

$$M = \frac{3\pi R^2}{4G_N^{(5)}} \frac{(\lambda+\mu)(1+\mu)^2}{1-\nu} \cosh 2\alpha\,, \tag{4.101a}$$

$$J_\psi = \frac{\pi R^3}{2G_N^{(5)}} \frac{(1-\lambda)^{3/2}(1+\mu)^{9/2}}{(1-\nu)^2} \left[\frac{C_\lambda}{1-\lambda} c^3 - \frac{3C_\mu}{1+\mu} s^2 c\right]\,, \tag{4.101b}$$

$$J_\varphi = -\frac{3\pi R^3}{G_N^{(5)}} \frac{\sqrt{1-\lambda}\,(1+\mu)^{7/2}(\lambda+\mu)}{(1-\nu)^2(1-\mu)} C_\mu c^2 s\,, \tag{4.101c}$$

$$\mathcal{A}_\mathcal{H} = 8\pi^2 R^3 \,\frac{(1-\lambda)(\lambda-\nu)^{1/2}(1+\mu)^3(\nu+\mu)^{3/2}}{(1-\nu)^2(1+\nu)} \quad \frac{C_\lambda}{\lambda-\nu} c^3 + \frac{3C_\mu}{\nu+\mu} s^2 c \quad . \tag{4.101d}$$

There is an ergosurface at $y = -1/\lambda$, where the norm of ∂_t vanishes, and the event horizon lies at $y = -1/\nu$. It is a Killing horizon of

$$k = \partial_t + \Omega \partial_\psi, \tag{4.102}$$

where Ω, the angular velocity of the horizon in the direction ψ, can be conveniently written as $\Omega = -1/\omega_\psi(-1/\nu)$.[12] A rather unusual property of this solution is that the horizon has no angular velocity in the direction φ even though $J_\varphi \neq 0$. Finally, the horizon temperature is

$$T_\mathcal{H}^{-1} = 4\pi R \frac{\sqrt{\lambda-\nu}(\mu+\nu)^{3/2}}{\nu(1+\nu)} \quad \frac{C_\lambda}{\lambda-\nu} c^3 + \frac{3C_\mu}{\nu+\mu} s^2 c \quad . \tag{4.103}$$

This solution of pure $\mathcal{N}=1, d=5$ supergravity corresponds to a following solution of the Heterotic Superstring effective field theory compactified on $\mathrm{T}^4 \times \mathrm{S}^1$ with the same metric and the non-trivial matter fields given by[13]

[12] Notice we work with coordinates φ, ψ whose periods are not the standard ones, but those given in Eq. (4.99).

[13] The fields that arise in the compactification over T^4 and which are set to their vacuum values (they are trivial) have not been considered. In particular, the index I takes only two values because the fields corresponding to the other values are trivial.

$$\phi = \phi_\infty \,, \tag{4.104a}$$

$$M_{IJ} = \begin{pmatrix} k_\infty^2 & 0 \\ 0 & k_\infty^{-2} \end{pmatrix} \,, \tag{4.104b}$$

$$\mathcal{A}^I = \begin{pmatrix} k_\infty^{-1} \\ k_\infty \end{pmatrix} \mathcal{A} \,, \tag{4.104c}$$

$$H = dB - \tfrac{1}{2}\mathcal{A}_I \wedge \mathcal{F}^I = \star\mathcal{F} \tag{4.104d}$$

where, for convenience, we have introduced $\mathcal{A} = -A/\sqrt{3}$ and its field strength $\mathcal{F} = d\mathcal{A}$. Let us obtain the vector and KR momentum maps asociated to the Killing vector k in Eq. (4.102) for this solution, denoted, respectively, as ${\mathcal{P}_k}^I$ and P_k. In the following we consider a constant t surface Σ defined by which extends from the bifurcate surface (here, a ring) $\mathcal{BH}$ at $y = -1/\nu$ to infinity (analogously to one leaf of the Einstein–Rosen bridge). The vector momentum maps $\mathcal{P}_k^I$ can be written as

$$\mathcal{P}_k^I = \begin{pmatrix} k_\infty^{-1} \\ k_\infty \end{pmatrix} \mathcal{P}_k \,, \tag{4.105}$$

where $\mathcal{P}_k$ satisfies the equation

$$d\mathcal{P}_k = -\imath_k \mathcal{F} \,. \tag{4.106}$$

Since in our gauge $\pounds_k \mathcal{A} = 0$ it is clear that a solution (as a matter of fact, any solution) of the above equation is provided by

$$\mathcal{P}_k = \imath_k \mathcal{A} + C \,, \tag{4.107}$$

for some constant C. Notice, though, that this is not the definition of the momentum map, but rather a particular form of $\mathcal{P}_k$ which is available in the gauge in which the black-ring solution is given. The momentum map is, by definition, gauge invariant. The constant C is determined by demanding $\mathcal{P}_k$ (which will be interpreted as the black ring's electrostatic potential Φ) to vanish at infinity, and it is not difficult to see that $C = 0$.

This solution admits an analytic prolongation to the bifurcate ring $\mathcal{BH}$ at $y = -1/\nu$ (and actually beyond that) and, in agreement with the generalised zeroth law, it is a constant over the whole event horizon $\mathcal{H}$ that we will denote by $\Phi_\mathcal{H}$

$$\begin{aligned} \mathcal{P}_k &\overset{\mathcal{H}}{=} \mathcal{P}_k(x, -1/\nu) \\ &= -\frac{\cosh 2\alpha \left[C_\lambda(\mu+\nu) + 3C_\mu(\lambda-\nu)\right] + C_\lambda(\mu+\nu) + C_\mu(\lambda-\nu)}{\cosh 2\alpha \left[C_\lambda(\mu+\nu) + 3C_\mu(\lambda-\nu)\right] + C_\lambda(\mu+\nu) - 3C_\mu(\lambda-\nu)} \tanh\alpha \\ &\equiv \Phi_\mathcal{H} \,. \end{aligned} \tag{4.108}$$

Observe that, in the gauge in which the solution is given, the potential $\mathcal{A}$ is ill-defined over $\mathcal{BH}$: $\imath_k\mathcal{A}$ is a non-vanishing constant there and k vanishes, which implies that $\mathcal{A}$ must diverge there. It is worth stressing that the momentum map is unaffected by such gauge pathologies since the solution Eq. (4.107) extends from infinity all the way down to $\mathcal{BH}$ (and beyond). This is a consequence of the fact that, although the momentum maps may only exist locally, they are defined by a gauge invariant equation.

The KR momentum map 1-form, P_k, is defined by Eq. (4.53), and, for this particular solution

$$dP_k = -\left(\imath_k H + \mathcal{P}_k{}^I \mathcal{F}_I\right) = -\left(\imath_k \star \mathcal{F} + 2\mathcal{P}_k \mathcal{F}\right) . \tag{4.109}$$

If we knew the KR potential B in a gauge in which $\pounds_k B = 0$, using $\mathcal{P}_k = \imath_k\mathcal{A}$, we would obtain the KR momentum map 1-form

$$P_k = \imath_k B - \mathcal{P}_k \mathcal{A} + \alpha\,, \tag{4.110}$$

where α is an arbitrary closed 1-form, $d\alpha = 0$, that could be determined by imposing regularity: smoothness of P_k both at the axis of the ring, $P_\psi(x, y = -1) = 0$, and at the outer axis of the spheres, $P_\varphi(x = -1, y) = 0$, so that it is well defined when approaching infinity). Finding B is, however, as hard as finding P_k directly from Eq. (4.109), which is what we are going to do, taking into account that we are only interested in the pullback of P_k to the constant-t surface Σ, which must be of the form

$$P_k \stackrel{\Sigma}{=} P^{\Sigma}_{k\,\varphi}(x, y) d\varphi + P^{\Sigma}_{k\,\psi}(x, y) d\psi\,, \tag{4.111}$$

because of the general form of the solution.

The two functions $P^{\Sigma}_{k\,\varphi}(x, y)$ and $P^{\Sigma}_{k\,\psi}(x, y)$ are given by

$$\begin{aligned} P^{\Sigma}_{k\,\varphi}(x, y) &= -\int^y \left(\imath_k \star \mathcal{F} + 2\mathcal{P}_k \mathcal{F}\right)_{y\varphi} dy + f_\varphi(x) \\ &= -2\mathcal{P}_k \mathcal{A}_\varphi + \int^y I_\varphi(x, y) dy + f_\varphi(x)\,, \end{aligned} \tag{4.112a}$$

$$\begin{aligned} P^{\Sigma}_{k\,\psi}(x, y) &= -\int^y \left(\imath_k \star \mathcal{F} + 2\mathcal{P}_k \mathcal{F}\right)_{y\psi} dy + f_\psi(x) \\ &= -2\mathcal{P}_k \mathcal{A}_\psi + \int^y I_\psi(x, y) dy + f_\psi(x)\,, \end{aligned} \tag{4.112b}$$

where

$$I_\varphi(x,y) = 2\mathcal{A}_\varphi\left(\partial_y\mathcal{A}_t + \Omega\partial_y\mathcal{A}_\psi\right)$$
$$+ \partial_x\mathcal{A}_t\left(\frac{R^2\Omega F(x)G(x)H(y)h(x,y)^2}{F(y)H(x)(x-y)^2} + \frac{F(y)G(x)H(y)\omega_\psi(y)(\Omega\omega_\psi(y)+1)}{F(x)G(y)H(x)h(x,y)}\right)$$
$$- \partial_x\mathcal{A}_t\left(\frac{\Omega H(x)^2\omega_\varphi(x)^2}{H(y)^2h(x,y)}\right)$$
$$- \partial_x\mathcal{A}_\psi\frac{F(y)G(x)H(y)(\Omega\omega_\psi(y)+1)}{F(x)G(y)H(x)h(x,y)} + \partial_x\mathcal{A}_\varphi\frac{\Omega H(x)^2\omega_\varphi(x)}{H(y)^2h(x,y)}\,, \tag{4.113a}$$

$$I_\psi(x,y) = \frac{H(x)^2\left(\omega_\varphi(x)\partial_x\mathcal{A}_t - \partial_x\mathcal{A}_\varphi\right)}{H(y)^2h(x,y)} + 2\mathcal{A}_\psi\left(\partial_y\mathcal{A}_t + \Omega\partial_y\mathcal{A}_\psi\right)\,, \tag{4.113b}$$

for some functions $f_\varphi(x)$ and $f_\psi(x)$ to be determined.

In this form, the functions are well defined at $y=-1/\nu$ (and beyond), and we can analytically prolongate P_k there.

The functions $f_\varphi(x)$ and $f_\psi(x)$ can be readily fixed from the fact that the combination $P_k + 2\mathcal{P}_k\mathcal{A}$ is closed on $\mathcal{BH}$ (the restricted generalized zeroth law). Indeed, pulling back on $\mathcal{BH}$ the KR momentum map Eq. (4.109), one has

$$d\left(P_k + 2\Phi_{\mathcal{H}}\mathcal{A}\right) \overset{\mathcal{BH}}{=} 0\,. \tag{4.114}$$

Thus, a solution of the form (4.111) that is well defined at $y=-1/\nu$ must satisfy the boundary condition

$$P_k \overset{\mathcal{BH}}{=} -2\Phi_{\mathcal{H}}\mathcal{A} + C_\varphi d\varphi + C_\psi d\psi \tag{4.115}$$

for some constants C_φ and C_ψ. This implies that our solution reads

$$P^{\Sigma}_{k\,\varphi}(x,y) = -2\mathcal{P}_k\mathcal{A}_\varphi + \int_{-1/\nu}^{y} I_\varphi(x,y)dy + C_\varphi\,, \tag{4.116a}$$

$$P^{\Sigma}_{\psi}(x,y) = -2\mathcal{P}_k\mathcal{A}_\psi + \int_{-1/\nu}^{y} I_\psi(x,y)dy + C_\psi\,. \tag{4.116b}$$

Remarkably,

$$\int_{-1/\nu}^{y} I_{\varphi}(-1,y)dy = 0\,, \qquad \forall y \neq -1\,, \tag{4.117a}$$

$$\int_{-1/\nu}^{-1} I_{\psi}(x,y)dy = \frac{\cosh 2\alpha\,[C_\lambda(\mu+\nu)+C_\mu(\nu-\lambda)]+C_\lambda(\mu+\nu)+C_\mu(\lambda-\nu)}{\cosh 2\alpha\,[C_\lambda(\mu+\nu)+3C_\mu(\lambda-\nu)]+C_\lambda(\mu+\nu)-3C_\mu(\lambda-\nu)}\times$$

$$\times\frac{\nu-1}{\mu+\nu}C_\mu R\,\mathrm{sech}\,\alpha\,, \qquad \forall x\,, \tag{4.117b}$$

so regularity at $y=-1$ and $x=-1$ is achieved by setting

$$C_\varphi = 0\,, \tag{4.118}$$

$$C_\psi = \frac{\cosh 2\alpha\,[C_\lambda(\mu+\nu)+C_\mu(\nu-\lambda)]+C_\lambda(\mu+\nu)+C_\mu(\lambda-\nu)}{\cosh 2\alpha\,[C_\lambda(\mu+\nu)+3C_\mu(\lambda-\nu)]+C_\lambda(\mu+\nu)-3C_\mu(\lambda-\nu)}\frac{1-\nu}{\mu+\nu}C_\mu R\,\mathrm{sech}\,\alpha$$

$$\equiv C(\lambda,\mu,\nu,\alpha)\frac{1-\nu}{\mu+\nu}C_\mu R\,\mathrm{sech}\,\alpha, \tag{4.119}$$

which completes the solution.

We conclude by noticing that the associated KR potential 1-form at $\mathcal{BH}$ is *purely harmonic* and given by,

$$\Phi_{KR} = P_k + 2\mathcal{P}_k\mathcal{A} \overset{\mathcal{BH}}{=} \Phi_{KR\tilde{\psi}}d\tilde{\psi}\,, \tag{4.120}$$

where $\tilde{\psi} = (2\pi/\Delta\psi)\psi$ is the angular coordinate with canonical period $\tilde{\psi}\sim\tilde{\psi}+2\pi$ and

$$\Phi_{KR\tilde{\psi}} = C_\psi\frac{\Delta\psi}{2\pi} = C(\lambda,\mu,\nu,\alpha)\frac{\sqrt{1-\lambda}(1+\mu)^{3/2}}{\mu+\nu}C_\mu R\,\mathrm{sech}\,\alpha\,. \tag{4.121}$$

For $\alpha = 0$, Φ_{KR} coincides with the potential given in Ref. [136] up to (parameter-independent) numerical prefactors.

4.7 Discussion

In this paper we have derived the first law of black hole mechanics in the context of the effective action of the Heterotic Superstring compactified on a torus at leading order in α'. The first law includes the variations of the conserved charges of the 1-forms, $\mathcal{Q}_I$, and of the charges associated to the KR field, $\mathcal{Q}_i$, multiplied by the potentials Φ^I and Φ^i which are constants that we have computed on the bifurcation surface.[14]

[14]It is not hard to prove that the potentials Φ^I, defined as the momentum maps $\mathcal{Q}_k{}^I$ are constant over the complete event horizon using the dominant energy condition and the Einstein equations as it is done

The main ingredients in this proof are the identification of the parameters of the gauge transformations that generate symmetries of the complete field configurations, the careful definitions of the associated charges and the corresponding potentials through what we have called restricted generalized zeroth laws. Due to the interactions between 1-forms and the KR 2-form induced by the Chern-Simons terms, all the terms involving charges and potentials in the first law are interrelated and all their definitions are either right or wrong simultaneously. This can be seen as a test of our definitions and of the final result.

In the theory considered in this chapter we have arrived at the well-known result that the entropy is one quarter of the area. In theories of higher order in the curvature it is known that there are additional contributions from the terms that contain the curvature, as the Iyer-Wald prescription makes manifest. However, as explained in the introduction, in the case of the Heterotic Superstring effective action at first order in α', we also expect that the need to have well-defined charges and, simultaneously, closed forms over the bifurcation sphere will result in the need to include additional terms in the "gravitational charge" that, in the end, will give us the entropy. Work in this direction is well under way [53].

Finally, we would like to comment upon two apparent shortcomings of Wald's formalism: it is not clear how to include the variation of the scalar charges and the moduli [142, 143] in the first law. In 5 dimensions, for instance, the KR field is dual to a 1-form and black-hole solutions electrically charged with respect to this dual 1-form exist. If we describe the theory in terms of the KR 2-form, it is not clear how to make the variation of this electric charge appear in the first law following this procedure. In this particular case, the electric charge of the 1-forms would be associated to S5-branes wrapped on T^5 and it would be very interesting to see the precise definition of this kind of charge to try to solve the ambiguities detected in Ref. [107].

for a single 1-form field in Ref. [56]. It is not clear, though, how definition of the potentials Φ^i may be extended using other sections of the event horizon different from the bifurcation sphere because the closedness of $P_k + \mathcal{P}_{k\,I}\mathcal{F}^I$ is based on the property $\imath_k H \stackrel{\mathcal{BH}}{=} 0$. It is not clear how to extend this property to other sections of the event horizon different from the bifurcation surface $\mathcal{BH}$.

5

Wald entropy formula for Heterotic Superstring effective action at first-order in α'

5.1 Introduction

The interpretation of the black-hole entropy in terms of the degeneracy of string microstates is, beyond any doubt, one of the main achievements of String Theory [89]. This interpretation relies, on the one hand, on the correct identification of the black-hole charges in terms of branes whose presence affects the quantization of the string. On the other, it depends on a correct calculation of the macroscopic entropy. In more complicated cases, the couplings can make the identification of the brane sources through the charges more complicated [107] and, beyond leading order in α', the presence of terms of higher order in the curvature and, in the Heterotic Superstring case, of complicated Yang-Mills (YM) and Lorentz Chern-Simons terms [73], can also make the calculation of the macroscopic entropy very difficult. This is the problem we will deal with in this chapter.

The standard method to calculate the black-hole entropy in theories of higher order in the curvature is to use Wald's formalism [27,28], usually applying directly the Iyer-Wald prescription [22]. As we have discussed in the previous two chapters (see also Refs. [33,34, 88] and the references therein), the Iyer-Wald prescription was derived assuming that all the fields of the theory behave as tensors under diffeomorphisms which, as matter of fact, is only true for the metric and uncharged scalars. All the fields of the Standard Model, except for the metric, have some kind of gauge freedom and do not transform as tensors under diffeomorphisms. Even the gravitational field, if it is described by a Vielbein instead of by a metric, has a gauge freedom, as it transforms under local Lorentz transformations. In theories with fermions, Vielbeins are necessary to work with the spinorial fields in curved space time.

In the previous two chapters (see also Ref. [88]), we proposed a simple solution, based on the construction of covariant Lie derivatives of all the fields with gauge freedom. This construction is based on the introduction of *momentum maps* [42,122] which play a crucial role in this chapter and which we will define later. The Lie-Lorentz derivative can also be seen as based on the definition of a Lorentz momentum map.[1]

In Chapter 4, we have shown how to use momentum maps to construct covariant Lie derivatives in the Heterotic Superstring Effective action compactified in a torus at zeroth order in α'. The KR field of that theory contains Abelian Chern-Simons terms[2]

[1] In Refs. [120,121], momentum maps emerge as "improved gauge transformations".

[2] Only the Kaluza-Klein and winding vector fields appear there at zeroth order in α'.

which induce Nicolai-Townsend transformations of the 2-form [144]. These terms modify the definitions of the conserved charges which ultimately appear in the first law of black hole mechanics along the lines of the classical Refs. [36, 132–134].

In this chapter, we are going to use the same technique quite extensively to deal with the variety of fields and couplings that occur in the Heterotic Superstring effective action at first order in α' and prove the first law of black hole mechanics, identifying the entropy. As we are going to see, the entropy formula obtained is manifestly gauge-invariant and contains only terms which are known and can be computed explicitly. This is the first entropy formula proposed for this theory that satisfies all these properties. It allows us to compute reliably the entropy of black hole solutions to first order in α' and compare the result with the entropy computed through microstate counting. As we will show in the last section, it gives the same results as the non-gauge-invariant formulae used in Refs. [32–34] in certain basis.[3] This confirms the values of the entropies obtained in those references, and shows why, in spite of the manifest deficiencies of the entropy formulae used, we obtained the right result.

A very interesting aspect of the momentum maps is that they are related to the zeroth law of black hole mechanics and its generalizations.[4] For higher-rank fields, Copsey and Horowitz [131] and, afterwards, Compère [52] proved a restricted form of the generalized zeroth law (restricted because it refers only to the bifurcation sphere) which follows from the closedness of certain differential forms on it. In Chapter 4, we proved that these closed forms are related to the momentum maps and we will call these statements *restricted generalized zeroth laws*. Here we will extend the results of Chapter 4 to YM and KR fields and to the more complicated couplings of the Heterotic Superstring effective action at first order in α'.[5]

The restricted generalized zeroth laws play a crucial role in the proof of the first law and in the identification of the entropy, and they are intimately related to the definitions of conserved charges. In Wald's formalism, the entropy is identified only after the terms $\sim \Phi\delta\mathcal{Q}$ have been identified in the first law. As in Chapter 4, this identification requires the addition and subtraction of several terms as demanded by the definitions of the charges $\mathcal{Q}$ and the potentials Φ on account of the restricted generalized zeroth laws. However, in this case, some of the terms added and subtracted will be shown to contribute to the entropy.

This chapter is organized as follows: in Section 5.2 we introduce the effective action of the Heterotic Superstring to first order in α' and find how it changes under an arbitrary variation of the fields, which allows us to determine the equations of motion. In Section 5.3 we study how the fields change under gauge and general coordinate transformations. We construct variations of the fields that vanish when the parameters of the transformations generate a symmetry of the field configuration and we find the integrals that give the associated conserved charges. The conserved charges associated to the invariance under diffeomorphisms are the Wald-Noether charges. As we have discussed, the correct identi-

[3] These results differ slightly from the results obtained in Refs. [29,30] using the Iyer-Wald prescription in the higher-dimensional action before dimensional reduction. As pointed out in Ref. [107], the dependence on the Riemann tensor changes after dimensional reduction and the formulae in Refs. [32–34] have been found using the dimensionally-reduced action. The formula that we give here does not suffer of any of these problems. See the discussion in Section 5.7.

[4] This was first noticed by Prabhu, albeit in a completely different language [51].

[5] Some of these couplings have been discussed before in the literature, specially in Ref. [129] (see also references therein). See the discussion in Section 5.7.

fication of the conserved charges is essential to obtain for the correct identification of the entropy in the first law. In Section 5.4 we discuss the restricted generalized zeroth laws of this theory, which also play an essential role in the proof of the first law. In Section 5.5 we prove the first law using the results obtained in the previous sections, which leads us to identify the Wald entropy formula in Section 5.6. Section 5.7 contains a discussion of our results, comparing them with the existing literature.

5.2 The HST effective action at first order in α'

The Heterotic Superstring effective action can be described at first order in α' as follows [73]:[6] we start by defining the zeroth-order KR field strength $H^{(0)}$ and its components $H^{(0)}{}_{\mu\nu\rho}$ as

$$H^{(0)} \equiv dB = \tfrac{1}{3!}H_{\mu\nu\rho}dx^{\mu} \wedge dx^{\mu} \wedge dx^{\rho}\,, \tag{5.1}$$

where $B = \frac{1}{2}B_{\mu\nu}dx^{\mu} \wedge dx^{\mu}$ is the KR 2-form potential. Then, if $\omega^{ab} = \omega_{\mu}{}^{ab}dx^{\mu}$ is the Levi-Civita spin connection,[7] we define the zeroth-order torsionful spin connections[8]

$$\Omega^{(0)}_{(\pm)\,ab} = \omega_{ab} \pm \tfrac{1}{2}\imath_b\imath_a H^{(0)}\,, \tag{5.2}$$

and their corresponding zeroth-order curvature 2-forms and Chern-Simons 3-forms

$$R^{(0)}_{(\pm)}{}^{ab} \equiv d\Omega^{(0)}_{(\pm)}{}^{ab} - \Omega^{(0)}_{(\pm)}{}^{a}{}_{c} \wedge \Omega^{(0)}_{(\pm)}{}^{cb}\,, \tag{5.3a}$$

$$\omega^{(0)}_{(\pm)} = R^{(0)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(0)}_{(\pm)}{}^{b}{}_{a} + \tfrac{1}{3}\Omega^{(0)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(0)}_{(\pm)}{}^{b}{}_{c} \wedge \Omega^{(0)}_{(\pm)}{}^{c}{}_{a}\,. \tag{5.3b}$$

Next, we define the gauge field strength 2-form and the Chern-Simons 3-forms for the YM field $A^A = A^A{}_{\mu}dx^{\mu}$ by

$$F^A = dA^A + \tfrac{1}{2}f_{BC}{}^{A}A^B \wedge A^C\,, \tag{5.4}$$

$$\omega^{\rm YM} = F_A \wedge A^A - \tfrac{1}{6}f_{ABC}A^A \wedge A^B \wedge A^C\,, \tag{5.5}$$

where we have lowered the adjoint group indices $A, B, C, \ldots$ in the structure constants $f_{AB}{}^{C}$ and gauge fields using the Killing metric.

Then, we can define the first-order KR field strength 3-form as

$$H^{(1)} \equiv H^{(0)} + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{(0)}_{(-)}\right)\,. \tag{5.6}$$

[6]We use the conventions of Ref. [42], reviewed for the zeroth-order case in Ref. [74]. In particular, the relation with the fields in Ref. [73] can be found in Ref. [75].

[7]If $e^a = e^a{}_{\mu}dx^{\mu}$ are the Vielbein, the spin connection is defined to satisfy the Cartan structure equation $\mathcal{D}e^a \equiv de^a - \omega^a{}_b \wedge e^b = 0$.

[8]We denote by $\imath_a A$ the inner product of $e_a \equiv e_a{}^{\mu}\partial_{\mu}$ ($e_a{}^{\mu}e^b{}_{\mu} = \delta^a{}_b$) with the differential form A. If A is a p-form with components $A_{\mu_1\cdots\mu_p}$, $\imath_a A$ is the $(p-1)$ form with components $e_a{}^{\nu}A_{\nu\mu_1\cdots\mu_{p-1}}$.

Its Bianchi identity takes the well-known form

$$dH^{(1)} = \frac{\alpha'}{4}\left(F_A \wedge F^A + R_{(-)}^{(0)\,a}{}_b \wedge R_{(-)}^{(0)\,b}{}_a\right) . \tag{5.7}$$

Having made these definitions and adding the dilaton field ϕ, we can write the Heterotic Superstring effective action to first-order in α' as

$$\begin{aligned} S^{(1)}[e^a, B, A^A, \phi] = {} & \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int e^{-2\phi}\left[(-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} - 4d\phi \wedge \star d\phi\right. \\ & \left. + \tfrac{1}{2} H^{(1)} \wedge \star H^{(1)} + (-1)^d \frac{\alpha'}{4}\left(F_A \wedge \star F^A + R_{(-)}^{(0)\,a}{}_b \wedge \star R_{(-)}^{(0)\,b}{}_a\right)\right] \\ \equiv {} & \int \mathbf{L}^{(1)} . \end{aligned} \tag{5.8}$$

Although this action is defined in 10 dimensions, we have left the dimension arbitrary (d) because that allows us to use the results in other dimensions after trivial dimensional reduction on a torus. In this action, ${G_N}^{(d)}$ is the d-dimensional Newton constant and $g_s^{(d)}$ is the d-dimensional string coupling constant, identified with the vacuum expectation value of the exponential of the d-dimensional dilaton field $g_s^{(d)} =< e^\phi >$. In solutions such as black holes that asymptote to a vacuum solution at infinity $e^\phi \to e^{\phi_\infty} =< e^\phi >= g_s^{(d)}$.

This is a very complex action. Due to this complexity and to the lemma proven in Ref. [73] which we will explain later, it is convenient to perform a general variation of the action in two steps: first, we only vary the action with respect to the *explicit* occurrences of the fields, where we define "explicit occurrences" as those which do not take place in the torsionful spin connection $\Omega_{(-)}^{(0)}$. Then, we vary the action with respect to the occurrences of the fields via $\Omega_{(-)}^{(0)}$ using the chain rule. All the occurrences of the dilaton and YM fields are explicit, but those of the Vielbein and KR field are not, because they (and only they) are present in $\Omega_{(-)}^{(0)}$.

Thus, setting $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1} = 1$ for the time being in order to simplify the formulae, we find that under a general variation of the "explicit" occurrences of the fields, the action transforms as follows:

$$\begin{aligned} \delta_{\exp} S^{(1)} = \int \Big\{ & \mathbf{E}^{(1)}_{\exp\,a} \wedge \delta e^a + \mathbf{E}^{(1)}_{\exp\,B} \wedge \delta B + \mathbf{E}^{(1)}_{\phi}\delta\phi + \mathbf{E}^{(1)}_{A}\delta A^A \\ & + d\mathbf{\Theta}^{(1)}_{\exp}(\varphi, \delta\varphi)\Big\} , \end{aligned} \tag{5.9}$$

where φ stands for all the fields of the theory,

$$
\begin{aligned}
\mathbf{E}^{(1)}_{\exp\, a} = {} & e^{-2\phi}\imath_a \star (e^c \wedge e^d) \wedge R_{cd} - 2\mathcal{D}(\imath_b d e^{-2\phi}) \wedge \star(e^b \wedge e^c) g_{ca} \\
& + (-1)^{d-1} 4 e^{-2\phi} \left(\imath_a d\phi \star d\phi + d\phi \wedge \imath_a \star d\phi\right) \\
& + \frac{(-1)^d}{2} e^{-2\phi} \left(\imath_a H^{(1)} \wedge \star H^{(1)} + H^{(1)} \wedge \imath_a \star H^{(1)}\right) \\
& + \frac{\alpha'}{4} e^{-2\phi} \big(\imath_a F_A \wedge \star F^A - F_A \wedge \imath_a \star F^A \\
& + \imath_a R_{(-)}^{(0)\,b}{}_c \wedge \star R_{(-)}^{(0)\,c}{}_b - R_{(-)}^{(0)\,b}{}_c \wedge \imath_a \star R_{(-)}^{(0)\,c}{}_b \big)
\end{aligned} \tag{5.10a}
$$

$$
\mathbf{E}^{(1)}_{\exp\, B} = -d\left(e^{-2\phi} \star H^{(1)}\right) , \tag{5.10b}
$$

$$
\mathbf{E}^{(1)}_{\phi} = 8d\left(e^{-2\phi} \star d\phi\right) - 2\mathbf{L}^{(1)} , \tag{5.10c}
$$

$$
\mathbf{E}^{(1)}_{A} = -\frac{\alpha'}{2} \left\{\mathcal{D}\left(e^{-2\phi} \star F_A\right) + (-1)^d e^{-2\phi} \star H^{(0)} \wedge F_A\right\} - \frac{\alpha'}{4} \mathbf{E}^{(1)}_{\exp\, B} \wedge A_A , \tag{5.10d}
$$

and

$$
\begin{aligned}
\mathbf{\Theta}^{(1)}_{\exp}(\varphi, \delta\varphi) = {} & -e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta\omega_{ab} + 2\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta e_b - 8 e^{-2\phi} \star d\phi \delta\phi \\
& + e^{-2\phi} \star H^{(1)} \wedge \delta B + \frac{\alpha'}{2} e^{-2\phi} \left(\star F_A - \tfrac{1}{2} \star H^{(1)} \wedge A_A\right) \wedge \delta A^A .
\end{aligned} \tag{5.11}
$$

An alternative form of the YM equations that arises in the calculations is

$$
\mathbf{E}^{(1)}_{A} = -\frac{\alpha'}{2} \mathcal{D}\left(e^{-2\phi} \star F_A - e^{-2\phi} \star H^{(0)} \wedge A_A\right) + (-1)^{d-1} \frac{\alpha'}{4} e^{-2\phi} \star H^{(0)} \wedge dA_A . \tag{5.12}
$$

Observe that neither the YM equations of motion transform covariantly nor $\mathbf{\Theta}^{(1)}_{\exp}$ is invariant under YM gauge transformations. For the YM equations this is not a big problem since the troublesome term is proportional to the KR equation of motion, but there is no obvious fix for the pre-symplectic potential. Nevertheless, we will see that, in the end, we will get gauge-invariant charges and, in particular a gauge-invariant Wald-Noether charge.

An important property of the HST effective action is that the YM fields and the torsionful spin connection occur in it exactly on the same footing [90]. The variation of the action with respect to the torsionful spin connection takes exactly the same form as the YM equation, the only difference being the group indices and their contractions. Thus,

$$\delta S^{(1)} = \int \Big\{ \mathbf{E}^{(1)}_{\exp\, a} \wedge \delta e^{a} + \mathbf{E}^{(1)}_{\exp\, B} \wedge \delta B + \mathbf{E}^{(1)}_{\phi}\delta\phi + \mathbf{E}^{(1)}_{A} \wedge \delta A^{A} + \mathbf{E}^{(1)\,b}{}_{a} \wedge \delta\Omega^{(0)}_{(-)}{}^{a}{}_{b} \\ + d\mathbf{\Theta}^{(1)}(\varphi,\delta\varphi) \Big\}\,, \tag{5.13}$$

where the variation with respect to the torsionful spin connection is given by

$$\mathbf{E}^{(1)\,b}{}_{a} = -\frac{\alpha'}{2}\Big\{\mathcal{D}_{(-)}\left(e^{-2\phi}\star R^{(0)}_{(-)}{}^{b}{}_{a}\right) + (-1)^{d} e^{-2\phi}\star H^{(0)} \wedge R^{(0)}_{(-)}{}^{b}{}_{a}\Big\} - \frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp\, B} \wedge \Omega^{(0)}_{(-)}{}^{b}{}_{a}\,, \tag{5.14}$$

or

$$\mathbf{E}^{(1)\,b}{}_{a} = -\frac{\alpha'}{2}\mathcal{D}_{(-)}\left(e^{-2\phi}\star R^{(0)}_{(-)}{}^{b}{}_{a} - e^{-2\phi}\star H^{(0)} \wedge \Omega^{(0)}_{(-)}{}^{b}{}_{a}\right) + (-1)^{d-1}\frac{\alpha'}{4}\star H^{(0)} \wedge d\Omega^{(0)}_{(-)}{}^{b}{}_{a}\,, \tag{5.15}$$

and the pre-symplectic $(d-1)$-form is given by

$$\mathbf{\Theta}^{(1)}(\varphi,\delta\varphi) = \mathbf{\Theta}^{(1)}_{\exp}(\varphi,\delta\varphi) + \frac{\alpha'}{2}e^{-2\phi}\left(\star R^{(0)}_{(-)}{}^{b}{}_{a} - \tfrac{1}{2}\star H^{(1)} \wedge \Omega^{(0)}_{(-)}{}^{b}{}_{a}\right) \wedge \delta\Omega^{(0)}_{(-)}{}^{a}{}_{b}\,, \tag{5.16}$$

with $\mathbf{\Theta}^{(1)}_{\exp}(\varphi,\delta\varphi)$ given in Eq. (5.11).

The parallelism between the YM and torsionful spin connection terms also leads to the same problems of non-covariance of $\mathbf{E}^{(1)\,b}{}_{a}$ and non-invariance of the additional term in $\mathbf{\Theta}^{(1)}$.

An important difference between the equations of motion of these two connections is that, according to the lemma proven in Ref. [73], $\mathbf{E}^{(1)\,a}{}_{b}$ is proportional to α' and to a combination of the zeroth-order equations $\mathbf{E}^{(0)}_{a}, \mathbf{E}^{(0)}_{B}$ and $\mathbf{E}^{(0)}_{\phi}$. This means that field configurations that solve the equations $\mathbf{E}^{(1)}_{\exp\, a} = 0$, $\mathbf{E}^{(1)}_{\exp\, B} = 0$, $\mathbf{E}^{(1)}_{\phi} = 0$ and $\mathbf{E}^{(1)}_{A} = 0$ are solutions of the complete first-order equations, to that order in α'. This crucial property effectively reduces the degree of the differential equations to 2, avoiding the problems that arise with dynamical equations that involve derivatives of the fields of higher order.

5.3 Variations of the fields

It is convenient to start by describing the gauge transformations of the fields and the associated Noether identities to be able to compute the associated conserved charges. Afterwards, we will discuss the transformations of the fields under diffeomorphisms and the associated Wald-Noether charge.

5.3.1 Gauge transformations

The fields occurring in the effective action Eq. (5.8) transform under 3 kinds of gauge transformations:

1. KR gauge transformations with 1-form parameter Λ, δ_Λ, which only act on B.
2. YM gauge transformations with parameter χ^A, δ_χ, which act on the YM fields and on B as Nicolai-Townsend transformations.
3. Local Lorentz transformations with parameter σ^{ab}, δ_σ, which act on the Vielbein and induce transformations of spin connections and curvature and which also act on B as Nicolai-Townsend transformations.

The transformation rules are

$$\delta_\sigma e^a = \sigma^a{}_b e^b\,, \tag{5.17a}$$

$$\delta_\chi A^A = \mathcal{D}\chi^A \equiv d\chi^A + f_{BC}{}^A A^B \chi^C\,, \tag{5.17b}$$

$$\delta B = (\delta_\Lambda + \delta_\chi + \delta_\sigma)B = d\Lambda - \frac{\alpha'}{4}\chi_A dA^A - \frac{\alpha'}{4}\sigma^a{}_b d\Omega^{(0)}_{(-)}{}^b{}_a\,. \tag{5.17c}$$

The induced local Lorentz transformations of the connections are

$$\delta_\sigma \omega^{ab} = \mathcal{D}\sigma^{ab} = d\sigma^{ab} - 2\omega^{[a|}{}_c\sigma^{c|b]}\,, \tag{5.18a}$$

$$\delta_\sigma \Omega^{(0)}_{(-)}{}^{ab} = \mathcal{D}^{(0)}_{(-)}\sigma^{ab} = d\sigma^{ab} - 2\Omega^{(0)}_{(-)}{}^{[a|}{}_c\sigma^{c|b]}\,, \tag{5.18b}$$

and the transformations of the curvatures are

$$\delta_\chi F^A = -\chi^B f_{BC}{}^A F^C \tag{5.19a}$$

$$\delta_\sigma R^{ab} = 2\sigma^{[a|}{}_c R^{c|b]}\,. \tag{5.19b}$$

$$\delta_\sigma R^{(0)}_{(-)}{}^{ab} = 2\sigma^{[a|}{}_c R^{(0)}_{(-)}{}^{c|b]}\,. \tag{5.19c}$$

Finally, for the sake of completeness and their later use, we quote the gauge transformations of the Chern-Simons 3-forms

$$\delta_\chi \omega^{\text{YM}} = \frac{\alpha'}{4} d\left(\chi_A dA^A\right)\,, \tag{5.20a}$$

$$\delta_\sigma \omega^{(0)}_{(-)} = +\frac{\alpha'}{4} d\left(\sigma^a{}_b d\Omega^{(0)}_{(-)}{}^b{}_a\right)\,, \tag{5.20b}$$

and the Ricci identities

$$\mathcal{D}\mathcal{D}\chi^A = -f_{BC}{}^A\chi^B F^C = \delta_\chi F^A\,, \tag{5.21a}$$

$$\mathcal{D}^{(0)}_{(-)}\mathcal{D}^{(0)}_{(-)}\sigma^{ab} = -2R^{(0)}_{(-)}{}^{[a|}{}_c\sigma^{c|b]} = \delta_\sigma R^{(0)}_{(-)}{}^{ab}\,. \tag{5.21b}$$

The exact invariance of the action $S^{(1)}$ in Eq. (5.8) under the above gauge transformations leads, in a rather trivial way, to the following Noether identities [75]

$$d\mathbf{E}^{(1)}_{\exp B} = 0\,, \tag{5.22a}$$

$$\mathcal{D}\mathbf{E}^{(1)}_A + (-1)^{d-1}\frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp B}\wedge dA_A = 0\,, \tag{5.22b}$$

$$\mathcal{D}^{(0)}_{(-)}\mathbf{E}^{(1)}{}_b{}^a + (-1)^{d-1}\frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp B}\wedge d\Omega^{(0)}_{(-)\,b}{}^a = 0\,, \tag{5.22c}$$

$$\mathbf{E}^{(1)\,[a}_{\exp}\wedge e^{b]} + \frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp B}\wedge d\Omega^{(0)\,ab} + (-1)^{d-1}\mathcal{D}^{(0)}_{(-)}\mathbf{E}^{(1)\,ab} = 0\,. \tag{5.22d}$$

Eq. (5.22c) is just a particular case of Eq. (5.22b) with adjoint Lorentz indices. Furthermore, the last two identities imply the symmetry of the Einstein equation, which in the language f differential forms and Vielbeins, is expressed in the form

$$\mathbf{E}^{(1)\,[a}_{\exp}\wedge e^{b]} = 0\,. \tag{5.23}$$

5.3.2 Gauge charges

For ths sake of simplicity, we are going to start by the charge associated to the δ_Λ transformations, that we are going to call Kalb-Ramond charge.

Kalb-Ramond charge

Let us consider the transformation of the action Eq. (5.8) under the gauge transformations δ_Λ. Taking into account that this symmetry only acts on B,[9] Eqs. (5.13) and (5.16) we get

$$\delta_\Lambda S^{(1)} = \int\left\{\mathbf{E}^{(1)}_{\exp B}\wedge d\Lambda + d\left[e^{-2\phi}\star H^{(1)}\wedge d\Lambda\right]\right\}\,. \tag{5.24}$$

Integrating by parts the first term and using the Noether identity Eq. (5.22a)

$$\delta_\Lambda S^{(1)} = \int d\left\{(-1)^d\mathbf{E}^{(1)}_{\exp B}\wedge\Lambda + e^{-2\phi}\star H^{(1)}\wedge d\Lambda\right\} \equiv \int d\mathbf{J}[\Lambda]\,. \tag{5.25}$$

[9] We consider the variation of the torsionful spin connection to be zero under this transformation.

Since $\delta_\Lambda S^{(1)} = 0$, the integrand must vanish, which means that $\mathbf{J}[\Lambda]$ must be locally exact. Indeed,

$$\mathbf{J}[\Lambda] = d\mathbf{Q}[\Lambda]\,, \quad \text{with} \quad \mathbf{Q}[\Lambda] = \Lambda \wedge \left(e^{-2\phi} \star H^{(1)}\right)\,. \tag{5.26}$$

Integrating the $(d-2)$-form $\mathbf{Q}[\Lambda]$ over $(d-2)$-dimensional compact surfaces $\mathcal{S}_{d-2}$ for Λs that leave invariant the KR field B we get conserved charges associated to those Λs. These Λs are simply closed 1-forms.[10] The Hodge decomposition theorem allows us to write each of them as the sum of an exact and a harmonic form that we denote by Λ_e and Λ_h, respectively. On-shell, the exact form $\Lambda_e = d\lambda$ will not contribute to the integral and the charge will be given by

$$\mathcal{Q}(\Lambda_h) = \int_{\mathcal{S}_{d-2}} \Lambda_h \wedge \left(e^{-2\phi} \star H\right)\,. \tag{5.27}$$

Now we can use duality between homology and cohomology: if C_{Λ_h} is the $(d-3)$-cycle dual to Λ_h we arrive at the charges

$$\mathcal{Q}(\Lambda_h) = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{C_{\Lambda_h}} e^{-2\phi} \star H\,, \tag{5.28}$$

where we have recovered the factor of $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$ and added a conventional sign.

Yang-Mills charge

Now, let us consider the charges associated to the YM gauge transformations δ_χ. Again, from Eqs. (5.13) and (5.16), taking into account that this symmetry acts on the YM fields A^A but also on the KR 2-form B, we have

$$\begin{aligned}\delta_\chi S^{(1)} = \int \Big\{ &\mathbf{E}^{(1)}_{\exp B} \wedge \delta_\chi B + \mathbf{E}^{(1)}_A \wedge \delta_\chi A^A \\ &+ d\left[e^{-2\phi} \star H^{(1)} \wedge \delta_\chi B + \frac{\alpha'}{2} e^{-2\phi}\left(\star F_A - \tfrac{1}{2} \star H^{(1)} \wedge A_A\right) \wedge \delta_\chi A^A\right]\Big\}\,.\end{aligned} \tag{5.29}$$

The parameters χ^A that we will use are those that preserve the field configuration, leaving A^A and B invariant. The YM fields are left invariant by covariantly constant χ^As, *i.e.* χ^As that we will denote by κ^A satisfying

$$\mathcal{D}\kappa^A = 0\,. \tag{5.30}$$

We can call these parameters *vertical Killing vector fields* from he principal bundle point of view, with the standard Killing vectors of the base manifold playing the rôle of *horizontal Killing vector fields.*

[10] Here we follow Refs. [52,131]. This discussion is identical to the discussion we made for the zeroth-order case in Ref. [74].

The integrability condition of the vertical Killing vector equation is, according to Eq. (5.21a),

$$\delta_\kappa F^A = -f_{BC}{}^A \kappa^B F^C = 0\,, \tag{5.31}$$

so they also leave invariant the field strengths, as expected.

The vertical Killing vector fields κ^As will not leave B invariant, though, but we can rewrite the transformation in the form

$$\delta_\chi B = -\frac{\alpha'}{4}\kappa_A dA^A = -\frac{\alpha'}{2}\kappa_A F^A + d\left(\frac{\alpha'}{4}\kappa_A A^A\right)\,. \tag{5.32}$$

Now we observe that, due to the YM Bianchi identity $\mathcal{D}F^A = 0$, $\kappa_A F^A$ is a closed 2-form and, locally, there is a 1-form Ψ_κ such that

$$d\Psi_\kappa = -\kappa_A F^A\,, \tag{5.33}$$

and which we will call *vertical YM momentum map.*[11]

Then, we define the parameter of a compensating Λ transformation

$$\Lambda_\chi = -\frac{\alpha'}{2}\Psi_\chi - \frac{\alpha'}{4}\chi_A A^A\,, \tag{5.34}$$

where Ψ_χ is a 1-form such that, when $\chi^A = \kappa^A$ (*i.e.* when it is a vertical Killing vector field), it satisfies Eq. (5.33). Combining the original δ_χ transformation with the compensating δ_{Λ_χ} transformation we find a new $\delta_\chi B$ that vanishes for covariantly constant χ^As:

$$\delta_\chi B \equiv -\frac{\alpha'}{2}\left(d\Psi_\chi + \chi_A F^A\right) - \frac{\alpha'}{4}\mathcal{D}\chi_A \wedge A^A\,. \tag{5.35}$$

The vanishing of $\delta_\chi B$ for covariantly constant χ^As is gauge invariant because

$$\delta_{\chi'}\delta_\chi \sim \mathcal{D}\chi\,. \tag{5.36}$$

Substituting the transformation Eq. (5.35) and the standard gauge transformation of the YM fields into Eq. (5.29) we get

$$\begin{aligned}\delta_\chi S^{(1)} = \int \Bigg\{ &\mathbf{E}^{(1)}_A \wedge \mathcal{D}\chi^A + \mathbf{E}^{(1)}_{\exp B} \wedge \left[-d\left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) - \frac{\alpha'}{4}\chi_A dA^A\right] \\ &+ d\Bigg\{ e^{-2\phi} \star H^{(1)} \wedge \left[-d\left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) - \frac{\alpha'}{4}\chi_A dA^A\right] \\ &+ \frac{\alpha'}{2}e^{-2\phi}\left(\star F_A - \tfrac{1}{2}\star H^{(1)} \wedge A_A\right) \wedge \mathcal{D}\chi^A \Bigg\}\Bigg\}\,.\end{aligned} \tag{5.37}$$

[11] Compare this equation with the equation satisfied by the standard (horizontal) YM momentum map Eq. (5.59).

Integrating by parts the first terms and combining the different terms in an appropriate way we can rewrite the variation in the form

$$
\begin{aligned}
\delta_\chi S^{(1)} = \int \Bigg\{ & (-1)^d \chi^A \left(\mathcal{D}\mathbf{E}_A^{(1)} + (-1)^{d-1} \frac{\alpha'}{4} \mathbf{E}^{(0)}_{\exp B} \wedge dA_A \right) \\
& - \left(\frac{\alpha'}{2} \Psi_\chi + \frac{\alpha'}{4} \chi_A A^A \right) \wedge d\mathbf{E}^{(0)}_{\exp B} \\
& + d \Bigg\{ (-1)^{d-1} \chi^A \left(\mathbf{E}_A^{(1)} + (-1)^d \frac{\alpha'}{4} e^{-2\phi} \star H^{(0)} \wedge dA_A \right) \\
& - \left(\frac{\alpha'}{2} \Psi_\chi + \frac{\alpha'}{4} \chi_A A^A \right) \wedge \mathbf{E}^{(0)}_{\exp B} \\
& + e^{-2\phi} \star H^{(1)} \wedge \left[-d \left(\frac{\alpha'}{2} \Psi_\chi + \frac{\alpha'}{4} \chi_A A^A \right) \right] \\
& + \frac{\alpha'}{2} e^{-2\phi} \left(\star F_A - \tfrac{1}{2} \star H^{(1)} \wedge A_A \right) \wedge \mathcal{D}\chi^A \Bigg\} \Bigg\} \, .
\end{aligned}
\tag{5.38}
$$

The terms in the first and second lines vanish identically because of the Noether identities Eqs. (5.22b) and (5.22a), respectively, and we arrive to

$$
\begin{aligned}
\delta_\chi S^{(1)} = \int d \Bigg\{ & (-1)^{d-1} \chi^A \left(\mathbf{E}_A^{(1)} + (-1)^d \frac{\alpha'}{4} e^{-2\phi} \star H^{(0)} \wedge dA_A \right) \\
& - \left(\frac{\alpha'}{2} \Psi_\chi + \frac{\alpha'}{4} \chi_A A^A \right) \wedge \mathbf{E}^{(0)}_{\exp B} \\
& - d \left(\frac{\alpha'}{2} \Psi_\chi + \frac{\alpha'}{4} \chi_A A^A \right) \wedge \left(e^{-2\phi} \star H^{(0)} \right) \\
& + \frac{\alpha'}{2} e^{-2\phi} \left(\star F_A - \tfrac{1}{2} \star H^{(1)} \wedge A_A \right) \wedge \mathcal{D}\chi^A \Bigg\} \\
\equiv \int & d\mathbf{J}[\chi] \, .
\end{aligned}
\tag{5.39}
$$

The same arguments we made in the previous case lead to the existence of a $(d-2)$-form $\mathbf{Q}[\chi]$ such that $\mathbf{J}[\chi] = d\mathbf{Q}[\chi]$. The $(d-2)$-form is given by

$$
\mathbf{Q}[\chi] = -(-1)^d \frac{\alpha'}{2} \left\{ e^{-2\phi} \star \left(-\chi^A F_A \right) + (-1)^d \Psi_\chi \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right\} \, . \tag{5.40}
$$

For Abelian vector fields the κ^As are constant and $\Psi_\kappa = \kappa_A A^A$ (up to a total derivative) and we recover immediately the $\mathbf{Q}[\chi]$ found in Ref. [74]. On the other hand, when we change Ψ_κ by a total derivative, $\mathbf{Q}[\kappa]$ is invariant on-shell up to a total derivative which will not contribute to the charge which is now given by the integral

$$\mathcal{Q}[\kappa] = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{S^{d-2}} (-1)^d \frac{\alpha'}{2} \left\{ e^{-2\phi} \star d\Psi_\kappa + (-1)^d \Psi_\kappa \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right\} , \tag{5.41}$$

where we have made use of the definition of the vertical momentum map Ψ_κ in Eq. (5.33).

Lorentz charge

Let us now consider local Lorentz transformations. As we have stressed repeatedly we can treat the local Lorentz transformations and the torsionful spin connection in parallel to the YM gauge transformations and the gauge fields. The only difference is the presence of one additional term in the Lorentz case: the Einstein-Hilbert case. If we follow the same steps as in the YM case we arrive to

$$\mathbf{Q}[\sigma] = (-1)^{d-1} e^{-2\phi} \star (e^a \wedge e^b) \sigma_{ab} - (-1)^d \frac{\alpha'}{2} \left\{ e^{-2\phi} \star \left(-\sigma^a{}_b R^{(0)\,b}{}_a \right) + (-1)^d \Pi_\sigma \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right\} , \tag{5.42}$$

where Π_σ is a 1-form that becomes a *vertical Lorentz momentum map* whan the Lorentz parameter $\sigma^a{}_b = \kappa^a{}_b$, a Lorentz parameter that generates a symmetry of the field configuration, *i.e. a vertical Killing vector.* This happens when the Vielbein and the spin connection are left invariant

$$\kappa^a{}_b e^b = 0 \,, \tag{5.43a}$$

$$\mathcal{D}\kappa^a{}_b = 0 \,. \tag{5.43b}$$

These two conditions imply the invariance of the torsion $\frac{1}{2}\imath_b \imath_a H^{(0)}$ Hence, they also implies the invariance of the torsionful spin connection $\Omega_{(-)}^{(0)\,a}{}_b$,

$$\mathcal{D}_{(-)}^{(0)} \kappa^a{}_b = 0 \,. \tag{5.44}$$

These conditions can be used to modify the transformation of the KR field so that it is also left invariant, as we did in the YM case. We just quote the final form:

$$\delta_\sigma B = -\frac{\alpha'}{2} \left(d\Pi_\sigma + \kappa^a{}_b R_{(-)}^{(0)\,b}{}_a \right) - \frac{\alpha'}{4} \mathcal{D}_{(-)}^{(0)} \sigma^a{}_b \wedge \Omega_{(-)}^{(0)\,b}{}_a \,, \tag{5.45}$$

where the vertical Lorentz momentum map Π_σ is such that, when $\sigma^a{}_b = \kappa^a{}_b$

$$d\Pi_\kappa = \kappa^a{}_b R_{(-)}^{(0)\,b}{}_a \,. \tag{5.46}$$

The conserved charge is the integral of the $(d-2)$-form Eq. (5.42) for vertical Killing vector fields $\kappa^a{}_b$ satisfying Eqs. (5.43) and (5.43b). The first condition annihilates the first term, corresponding to the Einstein-Hilbert term in the action but the rest of the terms survive in this case and we get the non-vanishing Lorentz charge

$$\mathcal{Q}[\kappa] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{S^{d-2}} \left\{ (-1)^d \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Pi_\kappa + (-1)^d \Pi_\kappa \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right] \right\} . \tag{5.47}$$

In the proof of the first law we will find the integral of $(d-2)$-form Eq. (5.42) for a Lorentz parameter that satisfies Eq. (5.43b) only. This integral give, precisely, the entropy.

5.3.3 The transformations under diffeomorphisms

Now we turn our attention to the diffeomorphisms. Our treatment is similar to the treatment of the δ_χ gauge transformations, although the use of compensating gauge transformations admits a more general justification in terms of the gauge covariance of the modified transformations (covariant Lie derivatives). Since we have discussed at length these modifications in Refs. [74, 88] we will only discuss the aspects not covered there: torsionful spin connections, non-Abelian gauge fields and the more complicated transformations of the KR 2-form.

In this section k will always be a (horizontal) Killing vector which generates a symmetry of the complete field configuration.

Lie-Lorentz derivatives

The transformations of the Vielbeins, the Levi-Civita spin connection and its curvature 2-form have been discussed in Refs. [74, 88], but it is convenient to adapt some of the formulae to the torsionful spin connection. They are generically given in terms of the Lie-Lorentz (or Lorentz-covariant Lie derivative Refs. [41, 42, 44–47]) by $\delta_\xi = -\mathbb{L}_\xi$. Therefore, we will continue this discussion in terms of the latter.

The parameter of the compensating local Lorentz transformation that appears in the Lie-Lorentz derivative of $\Omega_{(-)}^{(0)\,ab}$ is still given by

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - \nabla^{[a}\xi^{b]} \, , \tag{5.48}$$

but it is useful to rewrite it using $\Omega_{(-)}^{(0)\,ab}$ in the covariant derivatives. Due to the complete antisymmetry of the torsion, it takes the simple form

$$\sigma_\xi{}^{ab} = \imath_\xi \Omega_{(-)}^{(0)\,ab} - \mathcal{D}_{(+)}^{(0)\,[a}\xi^{b]} \, . \tag{5.49}$$

Observe that the presence of fully antisymmetric torsion does not modify the Killing equation[12]

$$2\mathcal{D}^{(0)}_{(\pm)\,(a}\xi_{b)} = 0 \, . \tag{5.50}$$

[12] The presence of generic torsion does modify the Killing equation.

Notice that Eqs. (5.49) and (5.50) are completely independent of $H^{(0)}$ even if we have formally rewritten them in terms of the torsionful spin connection $\Omega^{(0)}_{(-)}$.

The Lie-Lorentz derivative of the torsion $\imath_b \imath_a H^{(0)}$ follows the general formula while that of the Levi-Civita connection ω^{ab} is given by

$$\mathbb{L}_\xi \omega^{ab} = \pounds_\xi \omega^{ab} - \mathcal{D}\sigma_\xi{}^{ab} \,, \tag{5.51}$$

and, therefore, it is easy to see that

$$\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)} = \pounds_\xi \Omega^{(0)\,ab}_{(-)} - \mathcal{D}^{(0)}_{(-)}\sigma_\xi{}^{ab} \,, \tag{5.52}$$

and it is equally easy to see that it can be rewritten in the form

$$\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)} = \imath_\xi R^{(0)\,ab}_{(-)} + \mathcal{D}_{(-)} P_{(-)\xi}{}^{ab} \,, \tag{5.53}$$

with

$$P_{(-)\xi}{}^{ab} \equiv \mathcal{D}^{(0)\,[a}_{(+)} \xi^{b]} \,, \tag{5.54}$$

The identity

$$\xi^\nu R^{(0)}_{(-)\,\nu\mu}{}^{ab} + \mathcal{D}^{(0)}_{(-)\,\mu} P_{(-)\xi}{}^{ab} = \mathcal{D}^{(0)\,[a}_{(-)} \left(\nabla^{b]}\xi_\mu + \nabla_\mu \xi^{b]}\right) - \tfrac{3}{2}\nabla_{[\mu|}\left(\xi^\nu H^{(0)}_{\nu|\rho\sigma]}\right) e^{a\rho} e^{b\sigma} \,, \tag{5.55}$$

proves that $\delta_\xi \Omega^{(0)\,ab}_{(-)} = -\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)}$ vanishes when $\xi^\mu = k^\mu$, because, in that case,

$$-\imath_k R^{(0)\,ab}_{(-)} = \mathcal{D}^{(0)}_{(-)} P_{(-)k}{}^{ab} \,. \tag{5.56}$$

Because $P_{(-)k}{}^{ab}$ satisfies this equation, we will call it the *horizontal Lorentz momentum map associated to the torsionful spin connection.*

k, then, generates a diffeomorphism that leaves invariant the metric and the KR 3-form field strength.

Again, $P_{(-)\xi}{}^{ab}$ is a Lorentz tensor and $\delta_\xi \Omega^{(0)\,ab}_{(-)} = -\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)}$ is a Lorentz tensor although $\Omega^{(0)\,ab}_{(-)}$ is a connection. When it vanishes, it vanishes in all Lorentz frames.

Lie-Yang-Mills derivatives

Since the spin connection is just the connection of the Lorentz group, this case is very similar to the previous one, the main difference being that the YM fields are fundamental fields while the spin connection is a composite field. Apart from this, in many (but not all, because of the absence of a YM analogue of the Vielbein) instances we may just apply the same formulae with the sole change of the adjoint group indices, as we are going to see.

In order to find the gauge-covariant Lie derivative of YM fields it is convenient to consider the Lie-Lorentz derivative of the curvature tensor first. In this case, since we do not know the form of the parameter of the compensating gauge transformation, we can simply consider the standard Lie derivative of the gauge field strength 2-form defined in Eq. (5.4):

$$\pounds_\xi F^A = (\imath_\xi d + d\imath_\xi)F^A = \mathcal{D}\imath_\xi F^A - f_{BC}{}^A \imath_\xi A^B F^C\,, \tag{5.57}$$

where we have used the Bianchi identity $\mathcal{D}F^A = 0$.

When $\xi = k$ this expression should vanish up to an infinitesimal gauge transformation with some parameter that we denote by $\tilde{\chi}_k{}^A$. Then,

$$\mathcal{D}\imath_\xi F^A = f_{BC}{}^A \left(\imath_\xi A^B + \tilde{\chi}_k{}^B\right) F^C \equiv f_{BC}{}^A P_k{}^B F^C\,, \tag{5.58}$$

which, upon use of the Ricci identity Eq. (5.21a), can be solved by a $P_k{}^A$ that we call the *(horizontal) Yang-Mills momentum map* satisfying the equation

$$-\imath_k F^A = \mathcal{D}P_k{}^A\,. \tag{5.59}$$

Eq. (5.56) is nothing by a particular case of this equation for which the momentum map is explicitly known. This happens because we know how to express the gauge field in terms of a more fundamental field (the Vielbein). In general, the general form of $P_k{}^A$ is not known but is determined up to a covariantly-constant gauge parameter. We will use a $P_\xi{}^A$ which is undetermined except for the fact that it reduces to $P_k{}^A$ satisfying Eq. (5.59) for Killing vectors.

Now, we can use as definition of the Lie-Yang-Mills derivative of F^A the following expression which is guaranteed to vanish when $\xi = k$ on account of Eq. (5.58):

$$\mathbb{L}_\xi F^A = \mathcal{D}\imath_\xi F^A - f_{BC}{}^A P_\xi{}^B F^C = \pounds_\xi F^A - \delta_{\chi_\xi} F^A\,, \tag{5.60}$$

where the gauge compensating parameter $\chi_\xi{}^A$ is given by the (now usual) expression

$$\chi_\xi{}^A = \imath_\xi A^A - P_\xi{}^A\,. \tag{5.61}$$

The Lie-Yang-Mills derivative of the gauge field is, then

$$\mathbb{L}_\xi A^A \equiv \pounds_\xi A^A - \mathcal{D}\chi_\xi{}^A = \imath_\xi F^A + \mathcal{D}P_\xi{}^A\,, \tag{5.62}$$

and, by construction, it vanishes automatically when ξ is a Killing vector field k^μ and $P_k{}^A$ is the momentum map satisfying Eq. (5.59).

The Kalb-Ramond field

The parameters of the compensating YM and local Lorentz transformations of the KR field are the same transformations $\chi_\xi{}^A$ and $\sigma_\xi{}^{ab}$ that we perform on other fields with YM and Lorentz indices, given by Eqs. (5.61) and (5.48). Thus, if we want to construct a transformation of this field under diffeomorphisms that annihilates it when $\xi = k$ by combining its standard Lie derivative with gauge transformations, the only gauge parameter we can still play with is the 1-form Λ because the rest are already completely determined. We have

$$
\begin{aligned}
\delta_\xi B =& -\pounds_\xi B + (\delta_{\Lambda_\xi} + \delta_{\chi_\xi} + \delta_{\sigma_\xi})B \\
=& -\pounds_\xi B + d\Lambda_\xi - \frac{\alpha'}{4}\chi_{\xi\,A} dA^A - \frac{\alpha'}{4}\sigma_\xi{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a \,.
\end{aligned} \tag{5.63}
$$

Again, it is convenient to start by considering the transformation of the 3-form field strength $H^{(1)}$ defined in Eq. (5.6) under diffeomorphisms, because it is gauge invariant:

$$
\begin{aligned}
\delta_\xi H^{(1)} =& -\pounds_\xi H^{(1)} \\
=& -\imath_\xi dH^{(1)} - d\imath_\xi H^{(1)} \\
=& -d\imath_\xi H^{(1)} - \frac{\alpha'}{2}\left(\imath_\xi F_A \wedge F^A + \imath_\xi R_{(-)}^{(0)\,a}{}_b \wedge R_{(-)}^{(0)\,b}{}_a\right) ,
\end{aligned} \tag{5.64}
$$

where we have used the Bianchi identity Eq. (5.7).

When $\xi = k$ we can use Eqs. (5.56) and (5.59), integrate by parts, and use now the Bianchi identities for the curvatures, getting:

$$
\begin{aligned}
\delta_k H^{(1)} =& -d\imath_k H^{(1)} + \frac{\alpha'}{2}\left(\mathcal{D}P_{k\,A} \wedge F^A + \mathcal{D}_{(-)}P_{(-)\,k}{}^a{}_b \wedge R_{(-)}^{(0)\,b}{}_a\right) \\
=& -d\left[\imath_k H^{(1)} - \frac{\alpha'}{2}\left(P_{k\,A}F^A + P_{(-)\,k}{}^a{}_b R_{(-)}^{(0)\,b}{}_a\right)\right] .
\end{aligned} \tag{5.65}
$$

By assumption, the above expression must vanish identically. Therefore, locally, there must exist a gauge-invariant 1-form, the *horizontal Kalb-Ramond momentum map* P_k, satisfying

$$
-\imath_k H^{(1)} + \frac{\alpha'}{2}\left(P_{k\,A}F^A + P_{(-)\,k}{}^a{}_b R_{(-)}^{(0)\,b}{}_a\right) = dP_k \,. \tag{5.66}
$$

Then, if we apply the rule of thumb that the parameter of the compensating gauge transformation is the inner product of the vector that generates the diffeomorphisms with the "connection" (here B) minus the momentum map (here some 1-form P_ξ that in this case satisfies Eq. (5.66) when $\xi = k$)

$$
\Lambda_\xi = \imath_\xi B - P_\xi \,, \tag{5.67}
$$

we arrive at the following candidate to $\delta_\xi B$:

$$
\begin{aligned}
\delta_\xi B =& - \pounds_\xi B + d\Lambda_\xi - \frac{\alpha'}{4}\left(\chi_{\xi\,A} dA^A + \sigma_\xi{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right) \\
=& - \imath_\xi H^{(1)} - \frac{\alpha'}{4}\left(A_A \wedge \imath_\xi F^A + \Omega_{(-)}^{(0)\,a}{}_b \wedge \imath_\xi R_{(-)}^{(0)\,b}{}_a\right) \\
& - dP_\xi + \frac{\alpha'}{4}\left(P_{\xi\,A} dA^A + P_{(-)\,\xi}{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right) .
\end{aligned}
\tag{5.68}
$$

Let us see if, with this definition, $\delta_k B = 0$. Using Eqs. (5.66), (5.59) and (5.56) we get, instead of zero, a total derivative

$$
\delta_k B = - \frac{\alpha'}{4} d\left(P_{k\,A} A^A + P_{(-)\,k}{}^a{}_b \Omega_{(-)}^{(0)\,b}{}_a\right) , \tag{5.69}
$$

which we can simple absorb in redefinition of Λ_ξ in Eq. (5.67):

$$
\Lambda_\xi \equiv \imath_\xi B - P_\xi + \frac{\alpha'}{4} d\left(P_{\xi\,A} A^A + P_{(-)\,\xi}{}^a{}_b \Omega_{(-)}^{(0)\,b}{}_a\right) . \tag{5.70}
$$

With this new parameter,

$$
\begin{aligned}
\delta_\xi B =& - \pounds_\xi B + d\Lambda_\xi - \frac{\alpha'}{4}\chi_{\xi\,A} dA^A - \frac{\alpha'}{4}\sigma_\xi{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a \\
=& - \left[\imath_\xi H^{(1)} - \frac{\alpha'}{2}\left(P_{\xi\,A} F^A + P_{(-)\,\xi}{}^a{}_b R_{(-)}^{(0)\,b}{}_a\right) + dP_k\right] \\
& + \frac{\alpha'}{4}\left(A_A \wedge \delta_\xi A^A + \Omega_{(-)}^{(0)\,a}{}_b \wedge \delta_\xi \Omega_{(-)}^{(0)\,b}{}_a\right) \\
\equiv& -\mathbb{L}_\xi B ,
\end{aligned}
\tag{5.71}
$$

that vanishes identically when $\xi = k$ by virtue of the definition of the KR momentum map Eq. (5.66) and of $\delta_\xi A^A = \delta_\xi \Omega_{(-)}^{(0)\,b}{}_a = 0$.

The behavior of this variation under gauge transformations is far from obvious. A direct calculation gives

$$
\delta_{\text{gauge}} \delta_\xi B = \frac{\alpha'}{4}\left(d\chi_A \wedge \delta_\xi A^A + d\sigma^a{}_b \wedge \delta_\xi \Omega_{(-)}^{(0)\,b}{}_a\right) , \tag{5.72}
$$

with $\delta_\xi A^A = -\mathbb{L}_\xi A^A$ with the Lie-Yang-Mills covariant derivative given by Eq. (5.62) and with $\delta_\xi \Omega_{(-)}^{(0)\,ab} = -\mathbb{L}_\xi \Omega_{(-)}^{(0)\,ab}$, with the Lie-Lorentz derivative given by Eq. (5.53). Therefore, although the $\delta_\xi B$ defined above is not gauge-invariant, $\delta_k B$ vanishes in a gauge-invariant way.

5.3.4 The Wald-Noether charge

Now we consider the variation of the action $S^{(1)}$ given in Eq. (5.8) under the transformations $\delta_\xi = -\mathbb{L}_\xi$ for all the fields, where $\mathbb{L}_\xi$ is the gauge-covariant derivative which, for the Vielbein is given by [88]

$$\mathbb{L}_\xi e^a = \mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\,, \tag{5.73}$$

for the torsionful spin connection in Eq. (5.53), for the YM fields in Eq. (5.62) and for the KR field in Eq. (5.71).

From Eq. (5.13)

$$\begin{aligned}
\delta_\xi S^{(1)} = -\int \Big\{ & \mathbf{E}^{(1)}_{\exp a} \wedge \left(\mathcal{D}\imath_\xi e^a + P_\xi{}^a{}_b e^b\right) + \mathbf{E}^{(1)}_\phi \imath_\xi d\phi \\
& + \mathbf{E}^{(1)}_A \wedge \left(\imath_\xi F^A + \mathcal{D}P_\xi{}^A\right) + \mathbf{E}^{(1)\,b}{}_a \wedge \left(\imath_\xi R_{(-)}^{(0)\,a}{}_b + \mathcal{D}_{(-)}P_{(-)\xi}{}^a{}_b\right) \\
& + \mathbf{E}^{(1)}_{\exp B} \wedge \Big[\imath_\xi H^{(1)} + \frac{\alpha'}{4}\left(A_A \wedge \imath_\xi F^A + \Omega_{(-)}^{(0)\,a}{}_b \wedge \imath_\xi R_{(-)}^{(0)\,b}{}_a\right) \\
& - \frac{\alpha'}{4}\left(P_{\xi\,A} dA^A + P_{(-)\,\xi}{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right) + d\left[P_\xi - \frac{\alpha'}{4}\left(P_{\xi\,A}A^A + P_{(-)\,\xi}{}^a{}_b \Omega_{(-)}^{(0)\,b}{}_a\right)\right]\Big] \\
& - d\mathbf{\Theta}^{(1)}(\varphi, \delta_\xi\varphi)\Big\}\,,
\end{aligned} \tag{5.74}$$

where $\mathbf{\Theta}^{(1)}(\varphi, \delta_\xi\varphi)$ is given by

$$\begin{aligned}
\mathbf{\Theta}^{(1)}(\varphi, \delta_\xi\varphi) =& e^{-2\phi} \star (e^a \wedge e^b) \wedge (\imath_\xi R_{ab} + \mathcal{D}P_{\xi\,ab}) - 2\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge (\mathcal{D}\imath_\xi e_b + P_{\xi\,bc}e^c) \\
& + 8e^{-2\phi} \star d\phi \imath_\xi d\phi \\
& - e^{-2\phi} \star H^{(1)} \wedge \Big\{\imath_\xi H^{(1)} + \frac{\alpha'}{4}\left(A_A \wedge \imath_\xi F^A + \Omega_{(-)}^{(0)\,a}{}_b \wedge \imath_\xi R_{(-)}^{(0)\,b}{}_a\right) \\
& - \frac{\alpha'}{4}\left(P_{\xi\,A} dA^A + P_{(-)\,\xi}{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right) + d\left[P_\xi - \frac{\alpha'}{4}\left(P_{\xi\,A}A^A + P_{(-)\,\xi}{}^a{}_b \Omega_{(-)}^{(0)\,b}{}_a\right)\right]\Big\} \\
& - \frac{\alpha'}{2} e^{-2\phi}\left(\star F_A - \tfrac{1}{2} \star H^{(0)} \wedge A_A\right) \wedge \left(\imath_\xi F^A + \mathcal{D}P_\xi{}^A\right)\,. \\
& - \frac{\alpha'}{2} e^{-2\phi}\left(\star R_{(-)}^{(0)\,b}{}_a - \tfrac{1}{2} \star H^{(0)} \wedge \Omega_{(-)}^{(0)\,b}{}_a\right) \wedge \left(\imath_\xi R_{(-)}^{(0)\,a}{}_b + \mathcal{D}_{(-)}P_{(-)\xi}{}^a{}_b\right)\,.
\end{aligned} \tag{5.75}$$

Integrating by parts and using the Noether identities Eqs. (5.22a), (5.22b), (5.22c), (5.23) and the Noether identity associated to the invariance under diffeomorphisms

$$
\begin{aligned}
&(-1)^d \mathcal{D}\mathbf{E}^{(1)}_{\exp\, a} \imath_\xi e^a + \mathbf{E}^{(1)}_{\exp\, B} \wedge \imath_\xi H^{(1)} + \mathbf{E}^{(1)}_{\phi} \imath_\xi d\phi \\
&+ \left(\mathbf{E}^{(1)}_{A} + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\, B} \wedge A_A\right) \wedge \imath_\xi F^A + \left(\mathbf{E}^{(1)\, b}{}_{a} + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\, B} \wedge \Omega^{(0)\, b}_{(-)\;\; a}\right) \wedge \imath_\xi R^{(0)\, a}_{(-)\;\; b} \\
&= 0\,,
\end{aligned}
\tag{5.76}
$$

we can see that the volume term in the variation of the action Eq. (5.74) reduces to another total derivative

$$
\delta_\xi S^{(1)} = \int d\mathbf{\Theta}^{(1)\prime}(\varphi, \delta_\xi \varphi)\,, \tag{5.77}
$$

with

$$
\begin{aligned}
\mathbf{\Theta}^{(1)\prime}(\varphi, \delta_\xi\varphi) = \mathbf{\Theta}^{(1)}(\varphi, \delta_\xi\varphi) \\
&+ (-1)^d \mathbf{E}^{(1)}_{\exp\, a} \imath_\xi e^a + (-1)^{d-1}\mathbf{E}^{(1)}_{\exp\, B} \wedge P_\xi \\
&+ (-1)^d \left(\mathbf{E}^{(1)}_{A} + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\, B} \wedge A_A\right) P_\xi{}^A \\
&+ (-1)^d \left(\mathbf{E}^{(1)\, b}{}_{a} + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\, B} \wedge \Omega^{(0)\, b}_{(-)\;\; a}\right) P_{(-)\xi}{}^a{}_b\,.
\end{aligned}
\tag{5.78}
$$

The usual reasoning leads us to the off-shell identity

$$
d\mathbf{J}^{(1)}[\xi] = 0\,, \tag{5.79}
$$

where

$$
\mathbf{J}^{(1)}[\xi] \equiv d\mathbf{\Theta}^{(1)\prime}(\varphi, \delta_\xi\varphi) + \imath_\xi \mathbf{L}^{(1)}\,, \tag{5.80}
$$

and to the local existence of a $(d-2)$-form $\mathbf{Q}^{(1)}[\xi]$ such that $\mathbf{J}^{(1)}[\xi] = d\mathbf{Q}^{(1)}[\xi]$.

A straightforward calculation leads to the fully gauge-invariant Wald-Noether charge

$$
\begin{aligned}
\mathbf{Q}^{(1)}[\xi] =&(-1)^d \star (e^a \wedge e^b)\left[e^{-2\phi} P_{\xi\, ab} - 2\imath_a d e^{-2\phi}\xi_b\right] \\
&+ (-1)^{d-1}\frac{\alpha'}{2}\left[P_{\xi\, A} e^{-2\phi} \star F^A + P_{(-)\xi}{}^a{}_b \left(e^{-2\phi} \star R^{(0)\, b}_{(-)\;\; a}\right)\right] \\
&- P_\xi \wedge \left(e^{-2\phi} \star H^{(1)}\right)\,,
\end{aligned}
\tag{5.81}
$$

which is one of the main results of this chapter.

5.4 Restricted generalized zeroth laws

One of the main ingredients in Wald's approach to the first law of black hole mechanics is the zeroth law stating that κ is constant over the horizon [20]. Originally, this law was proved using the Einstein equations and the dominant energy condition (see, for instance, Ref. [124]) but a completely geometrical proof was presented in Ref. [125].

In presence of an electromagnetic field one also needs to use the *generalized zeroth law* that guarantees that the electrostatic potential is also constant over the whole horizon. There is no purely geometrical proof of this law, though, and the standard proof also makes use of the Einstein equations and of the dominant energy condition. In Ref. [74] we have explained how this proof can be extended to a theory containing an arbitrary number of Abelian vector fields and the KR field coupled to them via Chern-Simons terms. Essentially one gets a sum of non-negative terms containing the contribution of each field, and each of them has to vanish. Extending this proof to the non-Abelian case, as long as we restrict ourselves to a gauge group with definite positive Killing metric because one gets sums of non-negative terms. However, the $R^{(0)\,2}_{(-)}$ term of our theory is of YM type, but with non-definite Killing metric because of the non-compactness of the Lorentz group and the proof cannot be extended to this case in a streightforward manner.

It is, however, possible to proof the first law in bifurcate horizons if one can proof generalized zeroth laws for the matter fields restricted to the bifurcation sphere $\mathcal{BH}$ where the Killing vector associated to the event horizon, k, vanishes identically. These *restricted generalized zeroth laws* state the closedness of certain differential forms on $\mathcal{BH}$. The definitions of the potentials as certain constants follow from them as we are going to explain.

Assuming all the fields are regular over the horizon, it is clear that the inner products of their field strengths with k must vanish on $\mathcal{BH}$:

$$\imath_k d\phi \stackrel{\mathcal{BH}}{=} 0\,, \tag{5.82a}$$

$$\imath_k H \stackrel{\mathcal{BH}}{=} 0\,, \tag{5.82b}$$

$$\imath_k F^A \stackrel{\mathcal{BH}}{=} 0\,, \tag{5.82c}$$

$$\imath_k R_{(-)}^{(0)\,a}{}_{b} \stackrel{\mathcal{BH}}{=} 0\,. \tag{5.82d}$$

(5.82e)

Eq. (5.82a) is actually true over the whole spacetime, by assumption. From Eq. (5.82c) and the definition of the YM momentum map $P_k{}^A$ we find that

$$\mathcal{D}P_k{}^A \stackrel{\mathcal{BH}}{=} 0\,, \tag{5.83}$$

which tells us that the horizontal YM momentum map ${P_k}^A$ is, at the same time, a vertical Killing vector field on $\mathcal{BH}$. This is what we need in order to have an associated conserved charge there (see the discussion in Section 5.3.2).

Analogously, from Eq. (5.82d) and the definition of the momentum map ${P_{(-)k}}^a{}_b$ Eq. (5.56) we get

$$\mathcal{D}^{(0)}_{(-)} {P_{(-)k}}^a{}_b \stackrel{\mathcal{BH}}{=} 0\,, \tag{5.84}$$

which tells us that the horizontal Lorentz momentum map ${P_k}^A$ is, also, a vertical Killing vector field on $\mathcal{BH}$.

Observe that the last two equations have as consequence the existence of the gauge-invariant 1-forms Ψ_{P_k} and Π_{P_k} defined by

$$d\Pi_{P_k} \stackrel{\mathcal{BH}}{=} {P_{(-)k}}^a{}_b R^{(0)}_{(-)}{}^b{}_a\,, \tag{5.85a}$$

$$d\Psi_{P_k} \stackrel{\mathcal{BH}}{=} P_{k\,A}F^A\,. \tag{5.85b}$$

The closedness of the right-hand sides of these equations on $\mathcal{BH}$, which guarantee the local existence of Ψ_{P_k} and Π_{P_k} there are the restricted generalized zeroth laws for the YM and torsionful spin connecton fields.

Finally, from Eq. (5.82b) and the definition of the KR momentum map Eq. (5.66) plus the above two equations that define Ψ_{P_k} and Π_{P_k} we get

$$d\left[P_k - \frac{\alpha'}{2}\left(\Psi_{P_k} + \Pi_{P_k}\right)\right] \stackrel{\mathcal{BH}}{=} 0\,, \tag{5.86}$$

which is the restricted generalized zeroth law of the KR field.

5.5 The first law

Following Wald [28], we start by defining the *pre-symplectic* $(d-1)$*-form* [27]

$$\omega^{(1)}(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1\mathbf{\Theta}^{(1)}(\varphi, \delta_2\varphi) - \delta_2\mathbf{\Theta}^{(1)}(\varphi, \delta_1\varphi)\,, \tag{5.87}$$

and the *symplectic form* relative to the Cauchy surface Σ

$$\Omega^{(1)}(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \int_\Sigma \omega^{(1)}(\varphi, \delta_1\varphi, \delta_2\varphi)\,. \tag{5.88}$$

When φ is a solution of the equations of motion $\mathbf{E}_\varphi = 0$, $\delta_1\varphi = \delta\varphi$ is an arbitrary variation of the fields and $\delta_2\varphi = \delta_\xi\varphi$ is their variation under diffeomorphisms [22]

$$\omega^{(1)}(\varphi, \delta\varphi, \delta_\xi\varphi) = \delta\mathbf{J}^{(1)} + d\imath_\xi\mathbf{\Theta}^{(1)\,\prime} = \delta d\mathbf{Q}^{(1)}[\xi] + d\imath_\xi\mathbf{\Theta}^{(1)\,\prime}\,, \tag{5.89}$$

where, in our case, the Noether-Wald $(d-2)$-form charge $\mathbf{Q}^{(1)}$ is given by Eq. (5.81) and $\mathbf{\Theta}'$ is given in Eq. (5.78). Since, on-shell, $\mathbf{\Theta}^{(1)} = \mathbf{\Theta}^{(1)\,\prime}$, we have that, if $\delta\varphi$ satisfies the linearized equations of motion, $\delta d\mathbf{Q}^{(1)} = d\delta\mathbf{Q}^{(1)}$. Furthermore, if the parameter $\xi = k$ generates a transformation that leaves invariant the field configuration, $\delta_k\varphi = 0$,[13] linearity implies that $\omega^{(1)}(\varphi, \delta\varphi, \delta_k\varphi) = 0$, and

$$d\left(\delta\mathbf{Q}^{(1)}[k] + \imath_k\mathbf{\Theta}^{(1)\,\prime}\right) = 0\,. \tag{5.90}$$

Integrating this expression over a hypersurface Σ with boundary $\delta\Sigma$ and using Stokes' theorem we arrive at

$$\int_{\delta\Sigma}\left(\delta\mathbf{Q}^{(1)}[k] + \imath_k\mathbf{\Theta}^{(1)\,\prime}\right) = 0\,. \tag{5.91}$$

We consider field configurations that describe asymptotically flat, stationary, black-hole spacetimes with bifurcate horizons $\mathcal{H}$ and the Killing vector k is the one whose Killing horizon is the black hole's event horizon. k, then, will be given by a linear combination with constant coefficients Ω^n of the timelike Killing vector associated to stationarity, $t^\mu\partial_\mu$ and the $[\frac{1}{2}(d-1)]$ generators of inequivalent rotations in d spacetime dimensions $\phi_n^\mu\partial_\mu$

$$k^\mu = t^\mu + \Omega^n\phi_n^\mu\,. \tag{5.92}$$

The constant coefficients Ω^n are the angular velocities of the horizon.

The hypersurface Σ to be the space bounded by infinity and the bifurcation sphere $\mathcal{BH}$ on which $k = 0$, so $\delta\Sigma$ has two disconnected pieces: a $(d-2)$-sphere at infinity, S^{d-2}_∞, and the bifurcation sphere $\mathcal{BH}$. Then, taking into account that $k = 0$ on $\mathcal{BH}$, we obtain the relation

$$\delta\int_{\mathcal{BH}}\mathbf{Q}^{(1)}[k] = \int_{\mathrm{S}^{d-2}_\infty}\left(\delta\mathbf{Q}^{(1)}[k] + \imath_k\mathbf{\Theta}^{(1)\,\prime}\right)\,. \tag{5.93}$$

As explained in Ref. [22,52], the right-hand side can be identified with $\delta M - \Omega^m\delta J_n$, where M is the total mass of the black-hole spacetime and J_n are the independent components of the angular momentum.[14]

Using the explicit form of $\mathbf{Q}^{(1)}[k]$, Eq. (5.81), noticing that $-2\imath_a de^{-2\phi}k_b \stackrel{\mathcal{BH}}{=} 0$ and

[13] Notice that our goal in Section 5.3.3 was, precisely, to construct variations of the fields δ_ξ with that property.

[14] When the spacetime has compact dimensions, the d-dimensional mass M is a combination of the lower-dimensional mass and Kaluza-Klein charges. The details depend on the compactification and will be studied elsewhere.

restoring the overall factor $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$, we find

$$
\begin{aligned}
\delta\int_{\mathcal{BH}}\mathbf{Q}^{(1)}[k] = {} & \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{\mathcal{BH}}(-1)^d e^{-2\phi}\star(e^a\wedge e^b)P_{k\,ab} \\
& + \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{\mathcal{BH}}(-1)^{d-1}\frac{\alpha'}{2}P_{(-)k}{}^a{}_b\left(e^{-2\phi}\star R_{(-)}^{(0)\,b}{}_a\right) \\
& + \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{\mathcal{BH}}(-1)^{d-1}\frac{\alpha'}{2}P_{k\,A}e^{-2\phi}\star F^A \\
& - \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{\mathcal{BH}}P_k\wedge\left(e^{-2\phi}\star H^{(1)}\right)\,.
\end{aligned}
\tag{5.94}
$$

The right-hand side of this identity is expected to be of the form $T\delta S+\Phi\delta\mathcal{Q}$ for some charges $\mathcal{Q}$ and potentials Φ. However, when we compare the third and fourth integrals in the right-hand side with the definitions of the YM and KR charges Eqs. (5.41) and (5.28) we see that some terms are missing in the integrand of the first and that, in the second, there is no closed or harmonic form in the integrand, since the horizontal KR momentum map is not necessarily closed on $\mathcal{BH}$. We found a similar problem in Ref. [74] and the solution is essentially the same: add and subtract the same term in different integrals in order to complete the integrand of the definition of YM charge and in order to construct a 1-form which is closed in $\mathcal{BH}$.

The 1-form shich is closed on $\mathcal{BH}$ and which contains P_k follows from the restricted generalized zeroth law of the KR field, Eq. (5.86). We must add a term $-\frac{\alpha'}{2}\Psi_{P_k}$ to the fourth integral and substract the same term to the third, which now contains all the terms associated to the YM charge becuase of the restricted generalized zeroth law Eq. (5.83). However, Eq. (5.86) also tells us to add another term $-\frac{\alpha'}{2}\Pi_{P_k}$ to the fourth integral and we can only compensate by subtracting it to the second. This completes the closed 1-form in the fourth integral and completes the integrand of the Lorentz charge according to Eq. (5.47) and thanks to the restricted generalized zeroth law Eq. (5.84).

The result of these additions and subtractions is

$$
\begin{aligned}
\delta \int_{\mathcal{BH}} \mathbf{Q}^{(1)}[k] = {} & \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^d e^{-2\phi} \star (e^a \wedge e^b) P_{k\,ab} \\
& + \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Pi_{P_k} + (-1)^d \Pi_{P_k} \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right] \\
& + \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Psi_{P_k} + (-1)^d \Psi_{P_k} \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right] \\
& - \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} \left[P_k - \frac{\alpha'}{2} (\Psi_{P_k} + \Pi_{P_k}) \right] \wedge \left(e^{-2\phi} \star H^{(1)} \right) .
\end{aligned}
\tag{5.95}
$$

where Ψ_{P_k} and Π_{P_k} satisfy Eqs. (5.85b) and (5.85a), respectively, whose integrability is guaranteed by the fact that the YM and Lorentz momentum maps are covariantly constant on $\mathcal{BH}$ (the restricted generalized zeroth laws).

Now, let us assume that the particular field configuration under consideration admits a set of covariantly constant YM parameters on $\mathcal{BH}$ that we label with an index I, ${\kappa_I}^A$

$$
\mathcal{D}\kappa_I^A \stackrel{\mathcal{BH}}{=} 0 \,, \quad \Rightarrow \quad {P_k}^A \stackrel{\mathcal{BH}}{=} \Phi^I \kappa_I^A \,, \tag{5.96}
$$

where the constants Φ^I will be interpreted as the potentials associated to the YM charges $\mathcal{Q}_I$ computed with the parameter ${\kappa_I}^A$ Eq. (5.41)

$$
\mathcal{Q}_I \equiv \mathcal{Q}[\kappa_I] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Psi_I + (-1)^d \Psi_I \wedge \left(e^{-2\phi} \star H^{(0)} \right) \right] \,, \tag{5.97}
$$

where

$$
d\Psi_I = -\kappa_{I\,A} F^A \,. \tag{5.98}
$$

As a result, the third line in Eq. (5.95) becomes $\Phi^I \delta \mathcal{Q}_I$.

Now, following Refs. [52, 131], as a consequence of the KR restricted generalized zeroth law Eq. (5.86), we can write (Hodge decomposition)

$$
P_k - \frac{\alpha'}{2} \left(\Psi_{P_k} + \Pi_{P_k} \right) \stackrel{\mathcal{BH}}{=} de + \Phi^i \Lambda_{h\,i} \,, \tag{5.99}
$$

where e is some function, the $\Lambda_{h\,i}$ are the harmonic 1-forms of the bifurcation sphere and the Φ^i are constants that can be interpreted as the potentials associated to the KR charges $\mathcal{Q}_i = \mathcal{Q}(\Lambda_{h\,i})$ Eq. (5.28)

$$
\mathcal{Q}_i = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{C_{\Lambda_{h\,i}}} e^{-2\phi} \star H \,, \tag{5.100}
$$

where $C_{\Lambda_{h\,i}}$ is the $(d-3)$-cycle dual to the harmonic 1-form $\Lambda_{h\,i}$ in $\mathcal{BH}$.

As a result, the fourth line in Eq. (5.95) becomes $\Phi^i \delta \mathcal{Q}_i$ and we are left with the first two, which are linear in the Lorentz momentum map $P_k{}^{ab}$, which, on $\mathcal{BH}$, is given by κn^{ab}, where n^{ab} is the binormal to the horizon. The terms in those two lines must, therefore, be interpreted as those giving rise to the term $T\delta S$ in the first law

$$\delta M = T\delta S + \Phi^I \delta \mathcal{Q}_I + \Phi^i \delta \mathcal{Q}_i + \Omega^n \delta J_n \,. \tag{5.101}$$

5.6 Wald entropy

It follows from the results of the previous section that the entropy is given by

$$S = (-1)^d \frac{g_s^{(d)\,2}}{8G_N^{(d)}} \int_{\mathcal{BH}} e^{-2\phi} \left\{ \left[\star(e^a \wedge e^b) + \frac{\alpha'}{2} e^{-2\phi} \star R_{(-)}^{(0)\,ab} \right] n_{ab} + (-1)^d \frac{\alpha'}{2} \Pi_n \wedge \star H^{(0)} \right\} , \tag{5.102}$$

where we have the defined the 1-form Π_n (vertical Lorentz momentum map associated to the binormal) on the bifurcation sphere

$$d\Pi_n \stackrel{\mathcal{BH}}{=} R_{(-)}^{(0)\,ab} n_{ab} \,. \tag{5.103}$$

This is the main result of this chapter, and the thesis as a whole, which we will discuss in the next section. It is worth stressing that the term that involves Π_n, and which has been shown to given an important contribution to the entropy of well-known black-hole solutions Refs. [29, 30, 32–34] occurs in the entropy formula just to cancel an equivalent term that we had to add to get the correct definition of the KR charge and the associated potential. Without a detailed knowledge of the conserved charges, the restricted generalized zeroth laws and the potentials associated, the presence of that term in the entropy formula could not have been guessed.

5.7 Discussion

In this chapter we have derived an entropy formula for the black-hole solutions of the Heterotic Superstring effective action to first order in α' using Wald's formalism [27, 28] taking carefully into account all the symmetries of the theory. As a result, our entropy formula Eq. (5.102) is manifestly gauge invariant. In particular, it is manifestly invariant under local Lorentz transformations.

It is interesting to compare this result with the one that would follow form the direct (and naive) application of the Iyer-Wald prescription [22]. The first two terms in Eq. (5.102) can be obtained from Eq. (5.8) by varying the Einstein-Hilbert term and the $R^2_{(-)}$ term with respect to the Riemann curvature tensor, but the third term cannot be obtained in that way from the H^2 term. As stressed in Refs. [32–34], the variation of this term with respect to the Riemann tensor gives a term of the form

$$\frac{\alpha'}{4} e^{-2\phi} \left(\Omega_{(-)}^{(0)\,ab} n_{ab} \right) \wedge \star H^{(0)} \,, \tag{5.104}$$

which is not Lorentz-covariant. The coefficient of this term differs from the last term in Eq. (5.102) if we associate Π_n to $\Omega^{(0)}_{(-)}{}^{ab} n_{ab}$, which is the right thing to do as we are going to show. But this coefficient changes after dimensional reduction, as observed in Ref. [107]. The explicit calculation in Ref. [32] shows that the right coefficient is the one that arises after dimensional reduction,[15] but, certainly, there are ambiguities in the way in which the Chern-Simons terms are defined in lower dimensions.

It is interesting to observe that because $\mathcal{D} n_{ab} \overset{\mathcal{BH}}{=} 0$,

$$d\Pi_n \overset{\mathcal{BH}}{=} d\left(\Omega^{(0)}_{(-)}{}^{ab} n_{ab}\right) + \Omega^{(0)}_{(-)}{}^{a}{}_{c} \wedge \Omega^{(0)}_{(-)}{}^{cb} n_{ab} \,. \tag{5.106}$$

For the non-extremal Reissner-Nordström black hole of Ref. [108], whose α' corrections were computed in Ref. [32], the second term vanishes identically in the tangent space basis used (see Appendix C). This shows that, in that basis, our entropy formula and the entropy formula obtained via the Iyer-Wald prescription (after dimensional reduction) give the same result. Of course, our formula is valid in any basis.

Our entropy formula seems to differ from the entropy formula obtained in Ref. [31], but a detailed comparison is not possible since that formula contains undetermined parameters that guarantee its invariance under Lorentz transformations. In Ref. [31] it was argued that those undetermined parameters do not contribute to the entropy in certain cases but, without an explicit expression, it is difficult to understand why or when this may happen. Furthermore, as we have shown, the identification of the entropy formula can only be made after the first law of black hole mechanics has been proven and this requires a careful identification of the conserved charges of the theory: some terms (the one involving Π_n) occur in the entropy formula only because they are needed to compensate other terms that have to appear in the correct definition of the KR charge. This analysis was simply not carried out in Ref. [31].

Our entropy formula (the contribution due to the presence of Lorentz- or gravitational Chern-Simons terms in $H^{(1)}$) also differs from the one found in Ref. [129]. Observe that Eq. (40) in Ref. [129], similar to the terms contains in the formulae derived in Refs. [33, 34] and to Eq. (5.104) is not covariant. Thus, it may give the right result in certain basis, if at all.[16] The problems in the derivation of Ref. [129] are having overlooked the KR conserved charge and the determination of the gauge parameters that generate symmetries of the complete field configuration.

Finally, it is interesting to notice that the entropy formula looks like the charge associated to the Lorentz transformations generated by the binormal to the horizon. These transformations preserve the connections ω and $\Omega^{(0)}_{(-)}$ on the bifurcation sphere, but they do not preserve the Vielbein, as we assumed in Section 5.3.2 (Eq. (5.43)), which produces an additional term associated to the Einstein-Hilbert term.

The main use of the entropy formula that we have found is to put in solid ground

[15] The entropy calculated in this way satisfies the first law or, equivalently, the thermodynamic relation

$$\frac{\partial S}{\partial M} = \frac{1}{T} \,. \tag{5.105}$$

[16] the non-covariance of Tachikawa's entropy formula was observed in Ref. [130], where an alternative method was devised to deal with this problem. Nevertheless, the formula obtained in Ref. [130] reduces to Tachikawa's in $\mathcal{BH}$, apparently losing the covariance, while ours does not.

the calculations of the macroscopic entropies of α'-corrected black holes, an ineluctable condition for a fair comparison with the microscopic ones. More α'-corrected solutions will be available to this end [145]. As mentioned in the introduction, another necessary ingredient for this comparison is the correct identification of the relation between the charges of the black hole and the branes in the string background. These results and those of our previous work [74] single out a very precise definition of the conserved charges, which turn out to be of *Page type*, conserved and gauge-invariant under the assumptions made. This fact should shed light on this problem and we intend to pursue this line of research in future work.

6 Conclusions

The main objective of this thesis has been better our understanding of black hole entropy in theories beyond General Relativity, specifically in the case of the heterotic string effective action up to first order in α'.

The first part of the thesis focused on dimensionally reducing the heterotic string theory up to α' using the formulation based on the supersymmetry completion of the Lorentz Chern-Simons terms that occur in the Kalb-Ramond field strength. We have found a transformation $\mathbb{Z}_2$ of the fields left invariant by the action dimensionally reduced to first order in α' and that generalize and, in the limit $\alpha' \to 0$, reduces to the transformations of Standard T duality (*Buscher's rules* [82, 83, 109]), which exchange the Kaluza-Klein vectors and *winding* vectors and invert the Kaluza-Klein scalar. These transformations had been proposed by [85] but here we give the explicit form of the action and prove its invariance.

Then, we used the dimensionally-reduced action to find, following the Iyer-Wald prescription, an entropy formula for stringy black holes that can be obtained from a 10-dimensional solution by a single non-trivial compactification on a circle, supplemented by a trivial compactification on a torus, which we applied to the α'-corrected heterotic version of the Strominger-Vafa black hole, obtaining an entropy that matched the microscopic entropy result previously calculated. An important point is that the entropy calculated, apparently corrected, differs by a factor of 2 in a term of the one obtained by applying the prescription from Iyer-Wald to 10-dimensional action. This factor of 2 is necessary to obtain an entropy that satisfies the thermodynamic relation,

$$\frac{\partial S}{\partial M} = \frac{1}{T}\,, \tag{6.1}$$

as had been checked in [32]. Besides this problem, the value of the entropy depends on the choice of base of 1-forms in cotangent space. Removing these ambiguities were the main motivations for the rest of the thesis.

In the second section, we focused on the main goal of this thesis: the proof of the first law and calculate the Iyer-Wald entropy for the Heterotic string action. This was done in a piecewise fashion, focusing first on a toy model of the Reissner-Nordström-Tangherlini black hole in the Einstein-Maxwell d-dimensional theory, and then the effective theory of the compactified heterotic superstring to zeroth-order in α' on a torus, before moving to the much more complex case we were interested in.

To deal with fields with gauge freedoms, we define the gauge covariant Lie derivatives as combinations of the standard Lie derivatives and compensating gauge transformations built with the *momentum maps*. This has allowed us demonstrate the first laws of black hole mechanics, including terms of work that do not appear in the treatment of Iyer-Wald, and identify a Wald entropy formula manifestly invariant under gauge transformations (including local Lorentz transformations).

To reach these results, it was necessary to develop a generalization valid for differential forms of order greater than 1 for the generalized zeroth law, valid for the Maxwell field, which states that the electrostatic potential is constant over the horizon. This generalization states that certain differential forms that generalize the electrostatic potential are closed. These differential forms are closely related (or coincide) with the *momentum maps*. However, we have only been able to test them on bifurcation surfaces, so we refer to them as *generalized zero laws constrained* (to the bifurcation surface). The restriction did not prevent reaching the final results. We have studied how these laws hold in the non-trivial case of supergravity black rings pure $\mathcal{N} = 1$, $d = 5$.

Comparing our entropy formula with the one we obtained in our first chapter from the Iyer-Wald prescription we have seen that, in the chosen 1-form basis, our formula gives the same result as the Iyer-Wald prescription, except for the factor of 2 that the latter only includes if we work with the action compactified. Our formula therefore leads to entropy macroscopic that coincide with the microscopic entropy and that they satisfy the thermodynamic relation Eq. (6.1).

Comparing the terms of work that appear in the first law that we have obtained with those appearing in, for example, [142], we see that our treatment recovers the work terms proportional to the variations of electric-type charges, but not those proportional to the variations of the magnetic charges, the variations of the *moduli* or the cosmological constant [146] [147], because in the theories we consider there is no gauge symmetries associated with them. These absences can be considered as an inadequacy of the methods proposed in this thesis. Nevertheless, in two recent papers [148] [149], it has been shown that the techniques developed in this thesis can be used to find the work terms proportional to the variations of the cosmological constant if one describes is as the electric charge of a $(d-1)$-form potential and the terms proportional to the magnetic charges in Smarr formulae. Although more work is needed to understand and repair the absence of the terms of work associated with the variations of the magnetic charges and those of the *moduli* in the first law, we believe that the ideas and methods presented in this thesis lay a foundation on which fund the necessary advances to solve these problems.

A
Relationship between 10 and 9 dimensional fields

In this appendix, we demonstrate how the 10 dimensional fields in our heterotic theory, introduced in Chapter 2, can be decomposed into 9 dimensional fields. Section A.1 corresponds to the zeroth order case, while section *A.2* corresponds to the $\mathcal{O}(\alpha')$ case.

A.1 Relation between 10- and 9-dimensional fields at zeroth order in α'

At zeroth order in α', the 10-dimensional fields can be expressed in terms of the 9-dimensional ones as follows:

$$
\begin{aligned}
\hat{g}_{\mu\nu} &= g_{\mu\nu} - k^2 A_\mu A_\nu \,, \\
\hat{g}_{\mu\underline{z}} &= -k^2 A_\mu \,, \\
\hat{g}_{\underline{zz}} &= -k^2 \,, \\
\hat{B}_{\mu\nu} &= B^{(0)}{}_{\mu\nu} - A_{[\mu} B^{(0)}{}_{\nu]} \,, \\
\hat{B}_{\mu\underline{z}} &= B^{(0)}{}_{\mu} \,, \\
\hat{\phi} &= \phi + \tfrac{1}{2} \log k \,.
\end{aligned}
\tag{A.1}
$$

The inverse relations are

$$
\begin{aligned}
g_{\mu\nu} &= \hat{g}_{\mu\nu} - \hat{g}_{\underline{z}\mu}\hat{g}_{\underline{z}\nu}/\hat{g}_{\underline{zz}}\,, \\
A_\mu &= \hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, \\
k &= |\hat{g}_{\underline{zz}}|^{1/2}\,, \\
B^{(0)}{}_{\mu\nu} &= \hat{B}_{\mu\nu} + \hat{g}_{\underline{z}[\mu}\hat{B}_{\nu]\underline{z}}/\hat{g}_{\underline{zz}}\,, \\
B^{(0)}{}_{\mu} &= \hat{B}_{\mu\underline{z}}\,, \\
\phi &= \hat{\phi} - \tfrac{1}{4}\log\left(-\hat{g}_{\underline{zz}}\right).
\end{aligned}
\tag{A.2}
$$

A.2 Relation between 10- and 9-dimensional fields at $\mathcal{O}(\alpha')$

At first order in α', the 10-dimensional fields can be expressed in terms of the 9-dimensional ones as follows:

$$
\begin{aligned}
\hat{g}_{\mu\nu} &= g_{\mu\nu} - k^2 A_\mu A_\nu\,, \\
\hat{g}_{\mu\underline{z}} &= -k^2 A_\mu\,, \\
\hat{g}_{\underline{zz}} &= -k^2\,, \\
\hat{B}_{\mu\nu} &= B^{(1)}{}_{\mu\nu} - A_{[\mu}\left[B^{(1)}{}_{\nu]} + \frac{\alpha'}{2}k\left(\varphi_A A^A{}_{|\nu]} + \tfrac{1}{2}\Omega^{(0)}_{(-)\,|\nu]}{}^{ab}K^{(+)}{}_{ab} - K^{(-)}{}_{|\nu]}{}^{a}\partial_a \log k\right)\right]\,, \\
\hat{B}_{\mu\underline{z}} &= B^{(1)}{}_{\mu} + \frac{\alpha'}{4}k\left(\varphi_A A^A{}_\mu + \tfrac{1}{2}\Omega^{(0)}_{(-)\,\mu}{}^{ab}K^{(+)}{}_{ab} - K^{(-)}{}_{\mu}{}^{a}\partial_a \log k\right)\,, \\
\hat{\phi} &= \phi + \tfrac{1}{2}\log k\,, \\
\hat{A}^A{}_\mu &= A^A{}_\mu + k\varphi^A A_\mu\,, \\
\hat{A}^A{}_{\underline{z}} &= k\varphi^A\,.
\end{aligned}
\tag{A.3}
$$

The inverse relations are

$$
\begin{aligned}
g_{\mu\nu} &= \hat{g}_{\mu\nu} - \hat{g}_{\underline{z}\mu}\hat{g}_{\underline{z}\nu}/\hat{g}_{\underline{zz}}\,, \\
A_{\mu} &= \hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, \\
k &= |\hat{g}_{\underline{zz}}|^{1/2}\,, \\
B^{(1)}{}_{\mu\nu} &= \hat{B}_{\mu\nu} + \hat{g}_{\underline{z}[\mu}\left[\hat{B}_{|\nu]\,\underline{z}} + \frac{\alpha'}{4}\left(\hat{A}^{A}{}_{|\nu]}\hat{A}_{A\,\underline{z}} + \hat{\Omega}^{(0)}_{(-)\,|\nu]}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{b}}{}_{\hat{a}}\right)\right]/\hat{g}_{\underline{zz}}\,, \\
B^{(1)}{}_{\mu} &= \hat{B}_{\mu\,\underline{z}} - \frac{\alpha'}{4}\left[\hat{A}^{A}{}_{\mu}\hat{A}_{A\,\underline{z}} + \hat{\Omega}^{(0)}_{(-)\,\mu}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{b}}{}_{\hat{a}}\right. \\
&\quad \left. -\hat{g}_{\mu\underline{z}}\left(\hat{A}^{A}{}_{\underline{z}}\hat{A}_{A\,\underline{z}} + \hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{a}}{}_{\hat{b}}\hat{\Omega}^{(0)}_{(-)\,\underline{z}}{}^{\hat{b}}{}_{\hat{a}}\right)/\hat{g}_{\underline{zz}}\right]\,, \\
\phi &= \hat{\phi} - \tfrac{1}{4}\log\left(-\hat{g}_{\underline{zz}}\right)\,, \\
A^{A}{}_{\mu} &= \hat{A}^{A}{}_{\mu} - \hat{A}^{A}{}_{\underline{z}}\hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, \\
\varphi^{A} &= \hat{A}^{A}{}_{\underline{z}}/(-\hat{g}_{\underline{zz}})^{1/2}\,.
\end{aligned}
\tag{A.4}
$$

B

A truncation of the $d = 5$ theory to a $\mathcal{N} = 1, d = 5$ supergravity

A very useful, almost algorithmic, procedure has been developed in Refs. [150–152] to construct supersymmetric solutions (black holes and black rings, in particular) of $\mathcal{N} = 1, d = 5$ supergravity coupled to vector supermultiplets.[1] We can use this procedure in the context of the Heterotic Superstring Effective action compactified on a T^5 if we find a consistent truncation that produces a model $\mathcal{N} = 1, d = 5$ supergravity. A very simple truncation with this property has been used, for instance, in Ref. [29]. It can be described more conveniently as a trivial dimensional reduction on a T^4 (with all the fields that arise in the reduction set to their vacuum values) followed by a non-trivial compactification on a circle. The only fields that survive are the KR 2-form (which can be dualized into a vector field), the KK and winding vectors and the dilaton and KK scalars. This field content fits into $\mathcal{N} = 1, d = 5$ supergravity (metric and graviphoton vector field) coupled to two vector multiplets (one vector and one real scalar field each).

In order to profit from the solution-generating techniques developed for $\mathcal{N} = 1, d = 5$ supergravity theories, we need to rewrite this truncated version of the Heterotic Superstring effective action in the appropriate form: first, we rewrite the action in the Einstein frame and then we will dualize the KR field into a vector. After that, we will identify the scalar manifold etc.

The action of the truncated theory is

$$
\begin{aligned}
S[e^a, B, \phi, k, A, B] = \frac{g_s^{(5)\,2}}{16\pi G_N^{(5)}} \int e^{-2\phi} \Big[&\star(e^a \wedge e^b) \wedge R_{ab} - 4d\phi \wedge \star d\phi \\
&+\tfrac{1}{2}k^{-2}dk \wedge \star dk - \tfrac{1}{2}k^2 F \wedge \star F - \tfrac{1}{2}k^{-2} G \wedge \star G + \tfrac{1}{2} H \wedge \star H \Big] ,
\end{aligned}
\tag{B.1}
$$

where H is simply

$$
H = dB - \tfrac{1}{2} A \wedge G - \tfrac{1}{2} B \wedge F . \tag{B.2}
$$

The string-frame Vielbein e^a is related to the (modified) Einstein-frame Vielbein $\tilde{e}^a$ by

[1] These are supergravities invariant under 8 independent supersymmetry transformations, which are combined in a minimal 5-dimensional spinor. Often, they are referred to as $\mathcal{N} = 2, d = 5$ supergravities.

$$e^a = e^{2(\phi-\phi_\infty)/3}\tilde{e}^a\,, \qquad g_s = e^{\phi_\infty}\,, \tag{B.3}$$

and the action in the (modified) Einstein frame takes the form (removing the tildes for simplicity)

$$\begin{aligned} S[e^a, B, \phi, k, A, B] = \frac{1}{16\pi G_N^{(5)}} \int \Big[&\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{4}{3} d\phi \wedge \star d\phi + \tfrac{1}{2}k^{-2}dk \wedge \star dk \\ &-\tfrac{1}{2}k^2 e^{-4\phi/3} F \wedge \star F - \tfrac{1}{2}k^{-2}e^{-4\phi/3} G \wedge \star G + \tfrac{1}{2}e^{-8\phi/3} H \wedge \star H \Big]\,. \end{aligned} \tag{B.4}$$

The next step is the dualization of the KR 2-form. As usual, we consider the above action as a functional of the 3-form field strength H and add a Lagrange-multiplier term to enforce its Bianchi identity $dH = -\frac{1}{2}\mathcal{F}_I \wedge \mathcal{F}^I$

$$\begin{aligned} S[e^a, H, \phi, k, A, B] = \frac{1}{16\pi G_N^{(5)}} \int \Big[&\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{4}{3} d\phi \wedge \star d\phi + \tfrac{1}{2}k^{-2}dk \wedge \star dk \\ &- \tfrac{1}{2}k^2 e^{-4\phi/3} F \wedge \star F - \tfrac{1}{2}k^{-2}e^{-4\phi/3} G \wedge \star G + \tfrac{1}{2}e^{-8\phi/3} H \wedge \star H \\ &-C \wedge (dH + F \wedge G)\big]\,, \end{aligned} \tag{B.5}$$

where C is the 1-form dual to the 2-form B. Varying this action with respect to H, we get

$$\frac{\delta S}{\delta H} = e^{-8\phi/3} \star H - dC = 0\,, \tag{B.6}$$

which is solved by

$$H = e^{8\phi/3} \star K\,, \qquad K \equiv dC\,. \tag{B.7}$$

Substituting this solution into the action Eq. (B.5) we find the dual action

$$\begin{aligned} S[e^a, \phi, k, A, B, C] = \frac{1}{16\pi G_N^{(5)}} \int \Big[&\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{4}{3} d\phi \wedge \star d\phi + \tfrac{1}{2}k^{-2}dk \wedge \star dk \\ &- \tfrac{1}{2}k^2 e^{-4\phi/3} F \wedge \star F - \tfrac{1}{2}k^{-2}e^{-4\phi/3} G \wedge \star G - \tfrac{1}{2}e^{8\phi/3} K \wedge \star K \\ &-F \wedge G \wedge C\big]\,. \end{aligned} \tag{B.8}$$

The final step consists in finding the relation between the fields of this action and those of a $\mathcal{N} = 1, d = 5$ theory with two vector supermultiplets written in the standard form[2]

$$S[e^a, \phi^x, A^I] = \frac{1}{16\pi G_N^{(5)}} \int \Big[\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} g_{xy} d\phi^x \wedge \star d\phi^y - \tfrac{1}{2} a_{IJ} F^I \wedge \star F^J$$
$$+ \tfrac{1}{3^{3/2}} C_{IJK} F^I \wedge F^J \wedge A^K \Big] \, , \tag{B.9}$$

where the indices $I, J, \ldots = 0, 1, 2$ and the indices $x, y, \ldots = 1, 2$. The metrics $g_{xy}(\phi), a_{IJ}(\phi)$ are defined in terms of the symmetric, constant tensor C_{IJK} which fully characterizes the theory and the *real special geometry* of the scalar manifold as follows: we start by defining 3 combinations of the 2 scalars $h^I(\phi)$ that satisfy the constraint

$$C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi) = 1 \, . \tag{B.10}$$

Next, we define

$$h_I \equiv C_{IJK} h^J h^K, \quad \Rightarrow \quad h^I h_I = 1, \tag{B.11}$$

and

$$h^I_x \equiv -\sqrt{3} h^I{}_{,x} \equiv -\sqrt{3} \frac{\partial h^I}{\partial \phi^x}, \qquad h_{Ix} \equiv +\sqrt{3} h_{I,x}, \quad \Rightarrow \quad h_I h^I_x = h^I h_{Ix} = 0. \tag{B.12}$$

Then, a_{IJ} is defined implicitly by the relations

$$h_I = a_{IJ} h^I, \qquad h_{Ix} = a_{IJ} h^J{}_x. \tag{B.13}$$

It can be checked that

$$a_{IJ} = -2 C_{IJK} h^K + 3 h_I h_J. \tag{B.14}$$

The metric of the scalar manifold $g_{xy}(\phi)$, which we will use to raise and lower x, y indices is (proportional to) the pullback of a_{IJ}

$$g_{xy} \equiv a_{IJ} h^I{}_x h^J{}_y = -2 C_{IJK} h^I_x h^J_y h^K. \tag{B.15}$$

If we make the identifications

$$A^0 = -\sqrt{3} C \, , \qquad A^1 = -\sqrt{3} A \, , \qquad A^2 = -\sqrt{3} B \, , \tag{B.16}$$

we find that

$$C_{012} = 1/6 \, , \quad a_{00} = e^{8\phi/3}/3 \, , \quad a_{11} = k^2 e^{-4\phi/3}/3 \, , \quad a_{22} = k^{-2} e^{-4\phi/3}/3 \, . \tag{B.17}$$

[2]Here we are using the notation and conventions of Ref. [153] with minor changes explained in Appendix A of Ref. [154]. See also Ref. [42].

Since, for this C_{IJK}, the only non-vanishing components of a_{IJ} are the diagonal ones with $a_{II} = 3(h_I)^2$ we find that

$$h_0 = e^{4\phi/3}/3\,, \qquad h_1 = ke^{-2\phi/3}/3\,, \qquad h_2 = k^{-1}e^{-2\phi/3}/3\,, \tag{B.18}$$

which, in its turn, implies that

$$h^0 = e^{-4\phi/3}\,, \qquad h^1 = k^{-1}e^{2\phi/3}\,, \qquad h^2 = ke^{2\phi/3}\,. \tag{B.19}$$

Finally, the non-vanishing components of the scalar metric are

$$g_{\phi\phi} = 8/3\,, \qquad g_{kk} = k^{-2}\,. \tag{B.20}$$

The equations of motion of a general $\mathcal{N} = 1, d = 5$ theory are (up to a global factor of $(16\pi G_N^{(5)})^{-1}$ that we omit for simplicity)

$$\begin{aligned}
\mathbf{E}_a = {} & \imath_a \star (e^c \wedge e^d) \wedge R_{cd} - \tfrac{1}{2} g_{xy} \left(\imath_a d\phi^x \star d\phi^y + d\phi^x \wedge \imath_a \star d\phi^y \right) \\
& + \tfrac{1}{2} a_{IJ} \left(\imath_a F^I \wedge \star F^J - F^I \wedge \imath_a \star F^J \right) \,,
\end{aligned} \tag{B.21a}$$

$$\mathbf{E}_x = -g_{xy} \left\{ d \star d\phi^y + \Gamma_{zw}{}^y d\phi^z \wedge \star d\phi^w + \tfrac{1}{2} \partial^y a_{IJ} F^I \wedge \star F^J \right. \,, \tag{B.21b}$$

$$\mathbf{E}_I = -d \left(a_{IJ} \star F^J \right) + \tfrac{1}{\sqrt{3}} C_{IJK} F^J \wedge F^K\,. \tag{B.21c}$$

In this action, ϕ stands, actually, for $\phi - \phi_\infty$. In other words: the field ϕ is constrained to vanish at infinity.

For the particular model that we have obtained as a truncation of the compactified Heterotic Superstring effective action in $d = 5$ dimensions, these equations take the particular form

$$\mathbf{E}_a = \imath_a \star (e^c \wedge e^d) \wedge R_{cd} - \tfrac{4}{3}\left(\imath_a d\phi \star d\phi + d\phi \wedge \imath_a \star d\phi\right)$$

$$- \tfrac{1}{2}k^{-2}\left(\imath_a dk \star dk + dk \wedge \imath_a \star dk\right) + \tfrac{1}{6}e^{8\phi/3}\left(\imath_a F^0 \wedge \star F^0 - F^0 \wedge \imath_a \star F^0\right)$$

$$+ \tfrac{1}{6}e^{-4\phi/3}k^2\left(\imath_a F^1 \wedge \star F^1 - F^1 \wedge \imath_a \star F^1\right) + \tfrac{1}{6}e^{-4\phi/3}k^{-2}\left(\imath_a F^2 \wedge \star F^2 - F^2 \wedge \imath_a \star F^2\right)\,, \tag{B.22a}$$

$$\mathbf{E}_\phi = -\tfrac{8}{3}\left\{d \star d\phi + \tfrac{1}{6}e^{8\phi/3}F^0 \wedge \star F^0 - \tfrac{1}{12}e^{-4\phi/3}k^2 F^1 \wedge \star F^1 - \tfrac{1}{12}e^{-4\phi/3}k^{-2}F^2 \wedge \star F^2\right\}\,, \tag{B.22b}$$

$$\mathbf{E}_k = -k^{-2}\left\{d \star dk - k^{-1}dk \wedge \star k + e^{-4\phi/3}k^3 F^1 \wedge \star F^1 - k^{-1}e^{-4\phi/3}F^2 \wedge \star F^2\right\}\,, \tag{B.22c}$$

$$\mathbf{E}_0 = -\tfrac{1}{3}d\left(e^{8\phi/3} \star F^0\right) + \frac{1}{3^{3/2}}F^1 \wedge F^2\,, \tag{B.22d}$$

$$\mathbf{E}_1 = -\tfrac{1}{3}d\left(e^{-4\phi/3}k^2 \star F^1\right) + \frac{1}{3^{3/2}}F^0 \wedge F^2\,, \tag{B.22e}$$

$$\mathbf{E}_2 = -\tfrac{1}{3}d\left(e^{-4\phi/3}k^{-2} \star F^2\right) + \frac{1}{3^{3/2}}F^0 \wedge F^1\,. \tag{B.22f}$$

B.1 Further truncation to pure $\mathcal{N} = 1, d = 5$ supergravity

We can truncate this theory further, to minimal (*pure*) supergravity as follows: if the two scalars are constant, taking into account that for ϕ this constant value must be $\phi = 0$, (we call k_∞ the constant value of k) their equations become the constraints

$$0 = F^0 \wedge \star F^0 - \tfrac{1}{2}k_\infty^2 F^1 \wedge \star F^1 - \tfrac{1}{2}k_\infty^{-2}F^2 \wedge \star F^2\,, \tag{B.23a}$$

$$0 = k_\infty^3 F^1 \wedge \star F^1 - k_\infty^{-1}F^2 \wedge \star F^2\,, \tag{B.23b}$$

whose simplest solution is this relation between vector field strengths

$$F^0 = k_\infty F^1 = k_\infty^{-1}F^2 \equiv F\,. \tag{B.24}$$

Substituting this solution into the Einstein and vector equations we get only these two independent equations

$$\mathbf{E}_a = \imath_a \star (e^c \wedge e^d) \wedge R_{cd} + \tfrac{1}{2}\left(\imath_a F \wedge \star F - F \wedge \imath_a \star F\right) \tag{B.25a}$$

$$-\tfrac{2}{3}\mathbf{E} = -\tfrac{1}{3} d \star F + \frac{1}{3^{3/2}} F \wedge F\,, \tag{B.25b}$$

which follow from the action of minimal $d=5$ supergravity [127]

$$S[e^a, A] = \frac{1}{16\pi G_N^{(5)}} \int \left[\star(e^a \wedge e^b) \wedge R_{ab} - \tfrac{1}{2} F \wedge \star F + \tfrac{1}{6\sqrt{3}} F \wedge F \wedge A\right]\,. \tag{B.26}$$

The truncation procedure we have followed to arrive to this action starting from the 10-dimensional Heterotic Superstring effective action can be easily reversed to embed solutions of pure $\mathcal{N}=1, d=5$ supergravity into the 10-dimensional Heterotic Superstring effective theory. In particular, we apply this recipe to the charged, non-extremal, black ring solution of Ref. [140] in Section 4.6.

C
Resumen

La termodinámica de los agujeros negros es probablemente uno de los campos de investigación más activos de la Física Teórica. Interconecta áreas de la Física tan aparentemente dispares como la Gravedad, la Teoría Cuántica de Campos y la Teoría de la Información, proporcionando una visión profunda de todas ellas. Si bien inicialmente solo era válida para la Relatividad General, Wald y sus colaboradores desarrollaron un nuevo enfoque para demostrar la primera ley de la mecánica de los agujeros negros en teorías generales invariantes bajo difeomorfismos más generales que la Relatividad General. Como subproducto, este enfoque condujo a la identificación de una expresión que juega el papel de entropía (*entropía de Wald*) en la primera ley en teorías más allá de la Relatividad General.

Sin embargo, las primeras leyes y las fórmulas de entropía derivadas en la literatura con este formalismo (la prescripción de Iyer-Wald, en concreto) presentan graves deficiencias en ciertas teorías de cuerdas, como la falta de términos de trabajo en las primeras leyes y la falta de invariancia de gauge de la fórmula de entropía. Esto impide una comparación justa con la entropía microscópica calculada utilizando otras técnicas (correspondencia AdS/CFT, etc.). El objetivo principal de esta tesis es identificar las raíces de estos problemas y solucionarlos. Como veremos, la raíz de estos problemas es el tratamiento inadecuado de los campos que exhiben algún tipo de libertad de gauge. Estos son, de hecho, todos los campos excepto los escalares y la métrica (si no se usa el formalismo de tétradas).

Esta tésis está dividida en dos partes. En la primera sección se realiza la compactificación de la acción efectiva de la cuerda heterótica en S^1 a primer orden en α', lo que nos permitirá volver a calcular las reglas de Buscher y demostrar que es invariante bajo T dualidad. Luego usaremos la fórmula de Iyer-Wald en la acción del modelo dimensionalmente reducido para derivar una fórmula de entropía que se puede aplicar a soluciones de agujeros negros que pueden ser obtenidos por una sola compactifición no trivial en un círculo y discutiremos su invariancia bajo las transformaciones de T dualidad corregidas por α'. En concreto, lo aplicaremos al agujero negro extremo de Strominger-Vafa. Demostraremos que, además de la falta de invariancia de gauge, existe una ambigüedad en la aplicación de la fórmula, ya que al aplicarla a $d = 10$ y $d = 5$ produce dos resultados diferentes que difieren por un factor de 2.

Como se mencionó anteriormente, la fórmula de Iyer-Wald no se puede aplicar sin ambigüedades en el caso de la cuerda heterótica, ya que una de las suposiciones principales en su derivación era que todos los campos se comportaban como tensores y todos los campos, excepto el métrico y el escalar, poseen libertades de gauge y sus transformaciones bajo difeomorfismos siempre están acoplados a transformaciones de gauge. Esto sirve de motivación para la segunda sección de la tesis en la que probamos la primera ley de la termodinámica de agujeros negros de una manera invariante de gauge, introduciendo transformaciones bajo difeomorfismos covariantes de gauge (derivadas de Lie covariantes de gauge). La construcción de estas transformaciones implica la definición de *momentum maps* asociados a los campos y a los vectores que generan sus simetrías. Estos objetos juegan el papel de potenciales termodinámicos generalizados en la primera ley y satisfacen las "leyes cero generalizadas restringidas".

Después de haber puesto a prueba nuestras ideas sobre el agujero negro Reissner-Nordström-Tangherlini en el contexto de la teoría de Einstein-Maxwell d-dimensional, nos centramos en el caso de la cuerda heterótica. Inicialmente, examinamos el caso de la teoría efectiva de la cuerda heterótica hasta orden cero en α' compactificada sobre un toro. Esta teoría es interesante debido a las soluciones de agujeros negros que admite, y debido a los términos abelianos de Chern-Simons presentes en la intensidades de campo de la 3 forma de Kalb-Ramond. La presencia de esos términos induce las llamadas *transformaciones de gauge de Nicolai-Townsend* de la 2-forma de Kalb-Ramond. Estos términos y transformaciones de gauge aparecen en la teoría de 10 dimensiones a primer orden en α' de una manera mucho más complicada (no-abeliana, gravitacional) y este modelo puede usarse como un modelo de juguete para poner a prueba nuestras ideas. Así, explicamos cómo hay que tratar todas estas simetrías de gauge y derivamos la primera ley en términos de cantidades manifiestamente invariantes de gauge. Explícitamente, demostraremos esto en el caso de una solución de anillo negro cargada no-extrema de supergravedad pura $\mathcal{N} = 1$, $d = 5$ que se puede ver tambi'en como solución de la teoría efectiva de supercuerda heterótica.

En el capítulo final, llegamos a nuestro resultado principal, basado en el trabajo de los capítulos anteriores. En él demostramos la primera ley de la mecánica de los agujeros negros en el contexto de la acción efectiva de la supercuerda heterótica a primer orden en α' utilizando el formalismo de Wald, teniendo en cuenta correctamente todas las simetrías de la teoría. Esto requiere un cuidado adicional debido a la presencia de los términos no-abelianos de Lorentz y Yang-Mills Chern-Simons que se encuentran en la intensidad de campo de Kalb-Ramond. Como resultado, obtenemos una fórmula de entropía manifiestamente invariante de gauge (incluyendo transformaciones de Lorentz locales) en la que todos los términos puede calcularse explícitamente. Una fórmula de entropía con estas propiedades permite cálculos inambiguos de entropías de agujeros negros macroscópicos de primer orden en α' que pueden usarse de forma fiable en una comparación con los microscópicos. Tal fórmula aún faltaba en la literatura.

D
Conclusiones

El principal objetivo de esta tesis ha sido mejorar nuestra comprensión de la entropía de los agujeros negros en teorías más allá de la Relatividad General y, en particular, en el caso de la acción efectiva de la supercuerda heterótica a primer orden en α'.

La primera parte de la tesis se centró en reducir dimensionalmente la acción de la teoría de la supercuerda heterótica a primer orden en α' usando la formulación de Bersghoeff-de Roo basada en la supersimetrización de los términos de Lorentz Chern-Simons presentes en la intensidad de campo de Kalb-Ramond. Hemos encontrado una transformación $\mathbb{Z}_2$ de los campos que deja invariante la acción dimensionalmente reducida a primer orden en α' y que generalizan y en el límite $\alpha' \to 0$ se reducen a las transformaciones de T dualidad estándar (*reglas de Buscher* [82, 83, 109]) que intercambian los vectores de Kaluza-Klein y *winding* e invierten el escalar de Kaluza-Klein. Estas transformaciones habían sido propuestas por [85] pero aquí nosotros damos la forma explícita de la acción y demostramos su invariancia.

Después, usamos la acción dimensionalmente reducida obtenida anteriormente para encontrar, siguiendo la prescripción de Iyer-Wald, una fórmula de entropía para aquellos agujeros negros que se pueden obtener a partir de una solución 10-dimensional mediante una sola compactación no trivial en un círculo, complementada mediante una compactación trivial sobre un toro. Aplicando esta fórmula a la versión heterótica con correcciones de primer orden en α' del agujero negro de Strominger-Vafa obtuvimos una entropía que coincidía con el resultado de entropía microscópica calculado en la literatura. Un punto importante es que la entropía así calculada, aparentemente correcta, difiere por un factor de 2 en un término de la que se obtiene al aplicar la prescripción de Iyer-Wald a la acción 10-dimensional. Que este factor de 2 es necesario para obtener una entropía que satisfaga la relación termodinámica

$$\frac{\partial S}{\partial M} = \frac{1}{T}\,, \tag{D.1}$$

había sido comprobado en [32]. Además de este problema, el valor de la entropía depende de la elección de base de 1-formas en el espacio cotangente. Eliminar estas ambigüedades fue la motivación principal del resto de la tesis.

En la segunda parte, nos centramos en el objetivo principal de esta tesis: la demostración de la primera ley y el cálculo de la entropía de Wald en la teoría efectiva de la supercuerda heterótica a primer orden en α'. Esto se hizo por partes, tratando primero un caso muy simple: el agujero negro de Reissner-Nordström-Tangherlini en la teoría de Einstein-Maxwell d-dimensional y luego la teoría efectiva de la supercuerda heterótica compactificada en un toro a orden cero en α' antes de pasar al caso que nos interesaba, mucho más complejo.

Para lidiar con los campos con libertades gauge definimos las derivadas de Lie covariantes de gauge como combinaciones de las derivadas de Lie estándar y transformaciones de gauge compensatorias construidas con los *momentum maps*. Esto nos ha permitido demostrar las primeras leyes de la mecánica de los agujeros negros incluyendo términos de trabajo que no aparecen en el tratamiento de Iyer-Wald e identificar una fórmula de entropía de Wald manifiestamente invariante bajo transformaciones de gauge (incluyendo las transformaciones Lorentz locales).

Para llegar a estos resultados fue necesario elaborar una generalización válida para formas diferenciales de orden superior a 1 de la ley cero generalizada, válida para el campo de Maxwell, que establece que el potencial electrostático es constante sobre el horizonte. Esta generalización establece que ciertas formas diferenciales que generalizan el potencial electrostático son cerradas. Estas formas diferenciales están estrechamente relacionadas (o coinciden) con los *momentum maps*. Sin embargo, sólo hemos podido probarlas en las superficies de bifurcación, por lo que nos referimos a ellas como *leyes cero generalizadas restringidas* (a la superficie de bifurcación). La restricción no nos ha impedido llegar a los resultados finales. Hemos estudiado cómo estas leyes se cumplen en el caso no trivial de los anillos negros de supergravedad pura $\mathcal{N} = 1$, $d = 5$.

Al comparar nuestra fórmula de entropía con la que obtuvimos en nuestro primer artículo a partir de la prescripción de Iyer-Wald hemos visto que, en la base de 1-formas elegida, nuestra fórmula da el mismo resultado que la prescripción de Iyer-Wald, excepto por el factor de 2 que esta última sólo incluye si trabajamos con la acción compactificada. Nuestra fórmula, por lo tanto, lleva a entropías macroscópicas que coinciden con la entropía microscópica y que satisfacen la relación termodinámica Eq. (D.1).

Al comparar los términos de trabajo que aparecen en la primera ley que hemos obtenido con los que aparecen, por ejemplo, en [142], vemos que en la nuestra aparecen los términos proporcionales a las variaciones de las cargas de tipo eléctrico, pero no los proporcionales a las variaciones de cargas de tipo magnético ni los proporcionales a las variaciones de los *moduli* o a la constante cosmológica [146,147], debido a que en la teorías que consideramos no hay simetrías de gauge asociadas a ellos. Estas ausencias pueden considerarse como una insuficiencia de los métodos propuestos en esta tesis. Sin embargo, en dos trabajos recientes [148,149], se ha demostrado que las técnicas desarrolladas en esta tesis pueden usarse para encontrar el término de trabajo proporcional a las variaciones de la constante cosmológica si ésta se describe como la carga eléctrica de una $(d-1)$-forma y que los términos proporcionales a las cargas magnéticas sí que aparecen de forma natural en las fórmulas de Smarr. Aunque se necesita más trabajo para comprender y reparar la ausencia de los términos de trabajo asociados a las variaciones de las cargas magnéticas y las de los *moduli* en la primera ley, creemos que las ideas y métodos presentados en esta tesis sientan una base sobre la que se pueden fundar los avances necesarios para resolver estos problemas.

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