T-duality relations between hyperkähler and bi-hypercomplex structures

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Abstract. We investigate the T-duality relations between hyperkähler and bi-hypercomplex structures using the doubled formalism. In generalized geometry, both the hyperkähler and bi-hypercomplex structures are embedded in generalized hyperkähler structures that satisfy the split-bi-quaternion algebra. We write down the analogue of the Buscher rule, which is the T-duality transformation of the hyperkähler and bi-hypercomplex structures. As a practical example, we construct the bi-hypercomplex structure of the 5^2_2-brane, known as a T-fold, from the hyperkähler structure of the Taub-NUT space using the T-duality transformation. The bi-hypercomplex structures of the T-fold have non-trivial monodromies. This results in the fact that the worldsheet instantons on the T-fold are multi-valued. We comment on the resolution of this issue using the Born sigma model.

1. Introduction

Duality is an important notion to understand string theories [1]. T-duality is a distinctive feature in string theories that does not appear in theories based on point particles and is hence useful to understand the stringy nature of spacetime.

T-duality among different spacetime geometries is investigated in various contexts. The Buscher rule [2,3] is derived in the two-dimensional string sigma model as a transformation rule for the target space. In supersymmetric theories, T-duality is realized by replacing the chiral and twisted chiral multiplets in the two-dimensional $\mathcal{N} = (2,2)$ sigma model [4–6]. The target space of $\mathcal{N} = (2,2)$ theory only with chiral multiplet is a Kähler geometry [7,8]. On the other hand, the target space of $\mathcal{N} = (2,2)$ theory with twisted chiral multiplets admits two commutative complex structures (J_+, J_-) which are compatible with metric [9,10]. This space is called bi-hermitian, and thus Kähler and bi-hermitian geometries are T-dual with each other. Similarly, $\mathcal{N} = (4,4)$ theory requires that the target space is generally a bi-hypercomplex geometry. The bi-hypercomplex geometry has a set of complex structures $(J_{a,+}, J_{a,-})$ $(a = 1,2,3)$ where $J_{a,+}$ and $J_{a,-}$ are commutative and satisfy the $\mathfrak{su}(2)$-algebra independently. When $J_{a,+}$ and $J_{a,-}$ coincide, this structure becomes a hyperkähler structure.

Generalized geometry [11,12] plays an important role in the T-duality relation of these target spaces M. The bi-hermitian structure (J_+, J_-) and the bi-hypercomplex structure $(J_{a,+}, J_{a,-})$ on the tangent bundle TM are equivalent to the generalized Kähler structure $(\mathcal{J}_+, \mathcal{J}_-)$ and the generalized hyperkähler structure $(\mathcal{J}_{a,+}, \mathcal{J}_{a,-})$ on the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, respectively. This equivarence is so-called Gualtieri map [12].

Double field theory (DFT) [13] is a reformulation of supergravity which makes T-duality be a manifest symmetry. A doubled space $\mathcal{M}$ is defined as a $2D$-dimensional space with global $O(D, D)$ symmetry that linearly realizes T-duality, and DFT is defined on it. The geometry of $\mathcal{M}$ is called Born geometry since it has Born structure $(\eta, \mathcal{H}, \omega)$. Here, η is an $O(D, D)$-invariant metric, $\mathcal{H}$ is a generalized metric, and ω is a fundamental two-form. Through the $O(D, D)$-invariant metric, the tangent bundle of the doubled space $T\mathcal{M}$ and the generalized tangent bundle $\mathbb{T}M$ are related. It is possible to comprehensively discuss the T-duality of geometric structures by considering the Born geometry with the generalized hyperkähler structure.

In the following, we systematically derive the T-duality transformation rule for the bi-hypercomplex structure using the doubled formalism of DFT. As a practical example, we write down the bi-hypercomplex structure of the 5^2_2-brane known as a T-fold from the hyperkähler structure of the Taub-NUT space. Furthermore, we explicitly show the non-trivial monodromies of the geometric structures in the T-fold. We finally discuss the worldsheet instantons on spaces with such non-trivial monodromies by using the Born sigma model. The main results in this contribution are based on the original works [14, 15].

2. Hyperkähler, bi-hypercomplex and generalized hyperkähler structures

In this section, we define Kähler, bi-hermitian, hyperkähler and bi-hypercomplex structures, and also give generalized Kähler and generalized hyperkähler structures as their embedding into generalized geometry.

Let M be a smooth manifold. A *hermitian structure* on M is a pair (J, g) consisting of an integrable almost complex structure J on M and a hermitian metric g in the tangent bundle TM. Here, a hermitian metric is a Riemannian metric invariant under J, such that $g(JX, JY) = g(X, Y)$ for any vector fields X and Y on M. The hermitian structure gives a fundamental two-form satisfying $\omega(X, Y) = g(JX, Y)$. For this reason, the hermitian structure is often denoted by the triple (J, ω, g). If the fundamental two-form is closed $\mathrm{d}\omega = 0$, the triple (J, ω, g) is called a *Kähler structure* on M.

We then give bi-hermitian, hyperkähler and bi-hypercomplex structures using the above definitions. A *bi-hermitian structure* on M is defined by two hermitian structures $(J_\pm, \omega_\pm, g)$ such that J_+ and J_- are commutative. When J_+ and J_- coincide and $\omega_\pm$ are closed, the bi-hermitian structure is Kähler. A *hyperkähler structure* on M is defined by three Kähler structures (J_a, ω_a, g) $(a = 1, 2, 3)$ that satisfy the quaternionic relation $J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -1$. A *bi-hypercomplex structure* on M is defined by three bi-hermitian structures $(J_{a,\pm}, \omega_{a,\pm}, g)$ where $J_{a,+}$ and $J_{a,-}$ satisfy $\mathfrak{su}(2)$-algebra independently. The bi-hypercomplex structure is the hyperkähler structure when $J_{a,+}$ and $J_{a,-}$ coincide and $\omega_{a,\pm}$ are closed.

In generalized geometry of a manifold M, the T-duality relation of various quantities is discussed in the setting of a generalized tangent bundle $\mathbb{T}M$ given by the direct sum of the tangent bundle TM and the cotangent bundle T^*M. The almost complex structure J and the fundamental two-form ω on M are embedded in endomorphisms on $\mathbb{T}M$ as

$$\mathcal{I}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \qquad \mathcal{I}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{1}$$

and these are called generalized almost complex structures. Here $J^* : T^*M \to T^*M$ is the adjoint of J. When J is integrable and ω is closed, $\mathcal{I}_J$ and $\mathcal{I}_\omega$ are Courant integrable [12]. In this case, the generalized almost complex structures become the generalized complex structures.

A *generalized Kähler structure* is defined by a pair of commutative generalized complex structures $(\mathcal{J}_+, \mathcal{J}_-)$ whose product defines a positive-definite metric on $\mathbb{T}M$. If we use the Kähler structure (J, ω, g) on M and the embedding (1) to give the generalized complex structure as

$\mathcal{J}_+ = \mathcal{I}_J$ and $\mathcal{J}_- = \mathcal{I}_\omega$, one easily finds that $\mathcal{J}_\pm$ are commutative and their product gives the positive-definite metric on $\mathbb{T}M$.

The bi-hermitian structure $(J_\pm, \omega_\pm)$ on M defines generalized complex structures as

$$\mathcal{J}_\pm = \frac{1}{2}\Big(\mathcal{I}_{J_+} \pm \mathcal{I}_{J_-} + \mathcal{I}_{\omega_+} \mp \mathcal{I}_{\omega_-}\Big). \tag{2}$$

Since they are commutative with each other $[\mathcal{J}_+, \mathcal{J}_-] = 0$, the pair is the generalized Kähler structure. The map (2) is sometimes called the Gualtieri map.

The discussion is parallel in the case of the bi-hypercomplex. We can show that the bi-hypercomplex structure $(J_{a,\pm}, \omega_{a,\pm})$ on M defines generalized complex structures as in

$$\mathcal{J}_{a,\pm} = \frac{1}{2}\Big(\mathcal{I}_{J_{a,+}} \pm \mathcal{I}_{J_{a,-}} + \mathcal{I}_{\omega_{a,+}} \mp \mathcal{I}_{\omega_{a,-}}\Big). \tag{3}$$

One can see that $\mathcal{J}_{a,\pm}$ in (3) satisfy the following relation:

$$\begin{aligned} &\mathcal{J}_{a,+}\mathcal{J}_{b,+} = -\delta_{ab}\mathbf{1}_{2D} + \varepsilon_{abc}\mathcal{J}_{c,+}, \qquad &&\mathcal{J}_{a,-}\mathcal{J}_{b,-} = -\delta_{ab}\mathbf{1}_{2D} + \varepsilon_{abc}\mathcal{J}_{c,+}, \\ &\mathcal{J}_{a,+}\mathcal{J}_{b,-} = \delta_{ab}\mathcal{G} + \varepsilon_{abc}\mathcal{J}_{c,-}, &&\mathcal{J}_{a,-}\mathcal{J}_{b,+} = \delta_{ab}\mathcal{G} + \varepsilon_{abc}\mathcal{J}_{c,-}, \end{aligned} \tag{4}$$

where $\mathcal{G}$ is a positive-definite metric on $\mathbb{T}M$ given by $\mathcal{G} = \mathcal{I}_{J_{a,\pm}}\mathcal{I}_{\omega_{a,\pm}}$ (no summation over a). The pair of the generalized complex structures satisfying the relation (4) is known as the *generalized hyperkähler structure* [16]. We find that the relation (4) is a split-bi-quaternion algebra.

3. T-duality transformation between two different bi-hypercomplex structures

To derive the formulae for the T-duality transformation of the bi-hypercomplex structure defined in the previous section, we first review how the Buscher rule is realized in doubled formalism. Double field theory (DFT) is an $O(D,D)$-covariant reformulation of supergravity defined on a $2D$-dimensional doubled space $\mathcal{M}$. The fields of the NSNS sector of type II supergravities, the metric g, the B-field, and the dilaton ϕ, are organized into the generalized metric $\mathcal{H}$ and the generalized dilaton d:

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}, \qquad \mathrm{e}^{-2d} = \sqrt{-\det g}\,\mathrm{e}^{-2\phi}. \tag{5}$$

The T-duality transformations of g, B and ϕ are read off from the $O(D,D)$ transformations of $\mathcal{H}$ and d, that is, $\mathcal{H}' = \mathcal{O}^\mathsf{T}\mathcal{H}\mathcal{O}$ and $\mathrm{e}^{-2d'} = \mathrm{e}^{-2d}$, where $\mathcal{O}$ is an $O(D,D)$ transformation matrix. The T-duality transformation along the isometry direction y is represented by the factorized T-duality transformation:

$$h_y = \begin{pmatrix} 1 - t_y & t_y \\ t_y & 1 - t_y \end{pmatrix}, \qquad (t_y)^\mu{}_\nu = \delta^\mu_y \delta^y_\nu, \tag{6}$$

where $\mu, \nu = 1, \ldots, D$. By using h_y to transform the generalized metric as $\mathcal{H}' = h_y^\mathsf{T}\mathcal{H}h_y$ and reading off the parametrization in (5) for $\mathcal{H}'$, we obtain the following formulae:

$$\begin{aligned} g'_{ij} &= g_{ij} - \frac{g_{iy}g_{jy} - B_{iy}B_{jy}}{g_{yy}}, \qquad && g'_{iy} = \frac{B_{iy}}{g_{yy}}, \qquad && g'_{yy} = \frac{1}{g_{yy}}, \\ B'_{ij} &= B_{ij} - \frac{B_{iy}g_{jy} - g_{iy}B_{jy}}{g_{yy}}, && B'_{iy} = \frac{g_{iy}}{g_{yy}}, && \phi' = \phi - \frac{1}{2}\log g_{yy}, \quad i, j \neq y. \end{aligned} \tag{7}$$

These T-duality transformation formulae are well known as the Buscher rule [2,3]. The Buscher rule is usually derived using the string sigma model, although in doubled formalism it can be easily shown by using the linear transformation (6).

In a manner similar to deriving the Buscher rule in doubled formalism, we derive the T-duality transformation formulae for the bi-hypercomplex structure. Since the generalized tangent bundle $\mathbb{T}M$ and the doubled tangent bundle $T\mathcal{M}$ are equivalent by the $O(D,D)$-invariant metric η of DFT, the generalized complex structures can be interpreted as also the endomorphisms on $T\mathcal{M}$. The set $(\eta, \mathcal{H}, \omega)$ is called the Born structure on $\mathcal{M}$, where ω is a fundamental two-form associated with the para-complex structure on $\mathcal{M}$. In order that the Born structure and the generalized hyperkähler structure are compatible through the positive-definite metric on $\mathbb{T}M \simeq T\mathcal{M}$, the generalized hyperkähler structure (3) needs to be modified as

$$\mathrm{e}^{-B}\,\mathcal{J}_{a,\pm}\,\mathrm{e}^{B} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}\begin{pmatrix} J_{a,+} \pm J_{a,-} & -(\omega_{a,+}^{-1} \mp \omega_{a,-}^{-1}) \\ \omega_{a,+} \mp \omega_{a,-} & -(J_{a,+}^{*} \pm J_{a,-}^{*}) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \tag{8}$$

The $O(D,D)$ transformation using the factorized T-duality (6) for the generalized hyperkähler structure is $\mathrm{e}^{-B'}\mathcal{J}'_{a,\pm}\,\mathrm{e}^{B'} = h_y \mathrm{e}^{-B}\mathcal{J}_{a,\pm}\,\mathrm{e}^{B} h_y$. From this equation, by reading off the T-duality transformation rule for the bi-hypercomplex structure from $(J_{a,\pm}, \omega_{a,\pm})$ to $(J'_{a,\pm}, \omega'_{a,\pm})$, we obtain the following formulae:

$$\begin{aligned}
(J'_{a,\pm})^i{}_j &= (J_{a,\pm})^i{}_j - \frac{(J_{a,\pm})^i{}_y}{g_{yy}}(g_{yj} \mp B_{yj}), & (J'_{a,\pm})^i{}_y &= \mp\frac{(J_{a,\pm})^i{}_y}{g_{yy}}, \\
(J'_{a,\pm})^y{}_j &= B_{yk}\left((J_{a,\pm})^k{}_j - \frac{(J_{a,\pm})^k{}_y}{g_{yy}}(g_{yj} \mp B_{yj})\right) \pm (\omega_{a,\pm})_{yj}, & (J'_{a,\pm})^y{}_y &= \mp\frac{B_{yk}(J_{a,\pm})^k{}_y}{g_{yy}}, \\
(\omega'_{a,\pm})_{ij} &= (\omega_{a,\pm})_{ij} - \frac{(\omega_{a,\pm})_{iy}(g_{yj} \mp B_{yj}) + (g_{iy} \pm B_{iy})(\omega_{a,\pm})_{yj}}{g_{yy}}, & (\omega'_{a,\pm})_{iy} &= \mp\frac{(\omega_{a,\pm})_{iy}}{g_{yy}}.
\end{aligned} \tag{9}$$

This is the Buscher-like T-duality rule between two different bi-hypercomplex structures. The formulae derived from the doubled formalism are exactly the same as the ones discussed in the context of worldsheet supersymmetry [17–20].

We focus on the relation between hyperkähler and bi-hypercomplex structures. In the hyperkähler structure, we have $B = 0$, and $J_{a,+}$ coincides with $J_{a,-}$. This means $J_{a,+} = J_{a,-} = J_a$ and $\omega_{a,+} = \omega_{a,-} = \omega_a$. By applying these conditions to the T-duality transformation rule (9), we obtain the Buscher-like rule from the hyperkähler structure (J_a, ω_a) to the bi-hypercomplex structure $(J'_{a,\pm}, \omega'_{a,\pm})$. See the original paper [14] for details.

4. T-duality from KK-vortex to T-fold geometries

We obtained the formulae for the T-duality transformation of a metric, B-field, dilaton and bi-hypercomplex structures in the previous sections. In this section, we consider the T-duality transformation of the KK-monopole in type II string theories.

The four-dimensional transverse geometry of the KK-monopole is the (Euclidean) Taub-NUT space, which is a hyperkähler manifold. The metric of the Taub-NUT space is given by

$$\mathrm{d}s^2 = H\,\mathrm{d}x_{123}^2 + H^{-1}\big(\mathrm{d}x^4 + A\big)^2, \tag{10}$$

where H is a harmonic function in the flat (x^1, x^2, x^3) space and $A = A_i\,\mathrm{d}x^i$ $(i = 1, 2, 3)$ is a one-form defined by a vector potential A_i. The one-form A satisfies the monopole equation $\mathrm{d}A = \hat{*}_3\mathrm{d}H$. Here $\hat{*}_3$ is the Hodge star operator defined in the flat (x^1, x^2, x^3) space. The three fundamental two-forms on the Taub-NUT space are given by

$$\begin{aligned}
\omega_1 &= \mathrm{d}x^1 \wedge \big(\mathrm{d}x^4 + A\big) + H\,\mathrm{d}x^2 \wedge \mathrm{d}x^3, \\
\omega_2 &= \mathrm{d}x^2 \wedge \big(\mathrm{d}x^4 + A\big) + H\,\mathrm{d}x^3 \wedge \mathrm{d}x^1, \\
\omega_3 &= \mathrm{d}x^3 \wedge \big(\mathrm{d}x^4 + A\big) + H\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned} \tag{11}$$

The metric (10) and the fundamental two-forms (11) give the three complex structures equipped in the Taub-NUT space from the condition $J_a = -g^{-1}\omega_a$. The detailed expressions of J_a can be found in the original paper [14].

The Taub-NUT space has the isometry in the x^4-direction. The T-duality transformation (7) along this isometry yields the four-dimensional transverse geometry of the H-monopole (smeared NS5-brane). By using the formulae (9), the bi-hypercomplex structure of the H-monopole can be obtained (see the original paper [14] for details).

For T-duality transformation along another direction, we introduce an additional isometry by smearing the x^3-direction. The harmonic function is then $H = h_0 + \sigma \log(\mu/\rho)$, which is codimension two. Here, h_0, σ and μ are constants and $\rho = (x^1)^2 + (x^2)^2$. The solution to the monopole equation $\mathrm{d}A = \hat{*}_3 \mathrm{d}H$ is $A_1 = A_2 = 0$, $A_3 = -\sigma\theta$, where $\theta = \arctan(x^2/x^1)$. The geometry whose metric is given by (10) of codimension two is called the KK-vortex. The T-duality transformation (7) along the x^3-direction of the KK-vortex yields

$$\mathrm{d}s'^2 = H\,\mathrm{d}x_{12}^2 + HK^{-1}\,\mathrm{d}x_{34}^2, \qquad B' = -A_3K^{-1}\,\mathrm{d}x^3 \wedge \mathrm{d}x^4, \qquad \mathrm{e}^{-2\phi'} = \mathrm{e}^{-2\phi_0}H^{-1}K, \qquad (12)$$

where $K = H^2 + A_3^2$. This is the four-dimensional transverse geometry of the 5_2^2-brane [21, 22]. Furthermore, the T-duality transformation (9) along the x^3-direction with respect to the hyperkähler structure (J_a, ω_a) of the KK-vortex yields the bi-hypercomplex structure $(J'_{a,\pm}, \omega'_{a,\pm})$ of the 5_2^2-brane. We find the resulting six complex structures

$$J'_{1,+} = \begin{pmatrix} 0 & 0 & A_3K^{-1} & -HK^{-1} \\ 0 & 0 & HK^{-1} & A_3K^{-1} \\ -A_3 & -H & 0 & 0 \\ H & -A_3 & 0 & 0 \end{pmatrix}, \quad J'_{1,-} = \begin{pmatrix} 0 & 0 & -A_3K^{-1} & -HK^{-1} \\ 0 & 0 & -HK^{-1} & A_3K^{-1} \\ A_3 & H & 0 & 0 \\ H & -A_3 & 0 & 0 \end{pmatrix},$$

$$J'_{2,+} = \begin{pmatrix} 0 & 0 & -HK^{-1} & -A_3K^{-1} \\ 0 & 0 & A_3K^{-1} & -HK^{-1} \\ H & -A_3 & 0 & 0 \\ A_3 & H & 0 & 0 \end{pmatrix}, \quad J'_{2,-} = \begin{pmatrix} 0 & 0 & HK^{-1} & -A_3K^{-1} \\ 0 & 0 & -A_3K^{-1} & -HK^{-1} \\ -H & A_3 & 0 & 0 \\ A_3 & H & 0 & 0 \end{pmatrix},$$

$$J'_{3,+} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad J'_{3,-} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad (13)$$

and the associated fundamental two-forms

$$\omega'_{1,+} = HK^{-1}\begin{pmatrix} 0 & 0 & -A_3 & H \\ 0 & 0 & -H & -A_3 \\ A_3 & H & 0 & 0 \\ -H & A_3 & 0 & 0 \end{pmatrix}, \quad \omega'_{1,-} = HK^{-1}\begin{pmatrix} 0 & 0 & A_3 & H \\ 0 & 0 & H & -A_3 \\ -A_3 & -H & 0 & 0 \\ -H & A_3 & 0 & 0 \end{pmatrix},$$

$$\omega'_{2,+} = HK^{-1}\begin{pmatrix} 0 & 0 & H & A_3 \\ 0 & 0 & -A_3 & H \\ -H & A_3 & 0 & 0 \\ -A_3 & -H & 0 & 0 \end{pmatrix}, \quad \omega'_{2,-} = HK^{-1}\begin{pmatrix} 0 & 0 & -H & A_3 \\ 0 & 0 & A_3 & H \\ H & -A_3 & 0 & 0 \\ -A_3 & -H & 0 & 0 \end{pmatrix},$$

$$\omega'_{3,+} = \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & -HK^{-1} \\ 0 & 0 & HK^{-1} & 0 \end{pmatrix}, \quad \omega'_{3,-} = \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & HK^{-1} \\ 0 & 0 & -HK^{-1} & 0 \end{pmatrix}. \qquad (14)$$

By the same discussion of T-duality from Kähler to bi-hermitian structures in [14], we see that $J'_{a,\pm}$ given by (13) satisfy $(J'_{a,\pm})^2 = -1$ and $[J'_{a,+}, J'_{a,-}] = 0$. We also confirm that these structures satisfy the compatibility condition $\omega'_{a,\pm}J'_{a,\pm} = g'$.

The fields (g, B, ϕ) of the 5^2_2-brane do not return to their original values after moving around the brane center from $\theta = 0$ to $\theta = 2\pi$, hence they are multi-valued. While the monodromy is characteristic of codimension two branes, the monodromy of the metric g and B-field is easily evaluated by using the generalized metric $\mathcal{H}$. The monodromy of the generalized metric of the 5^2_2-brane is expressed as $\mathcal{H}^{(2\pi)} = \mathcal{O}_\beta^{\mathsf{T}} \mathcal{H}^{(0)} \mathcal{O}_\beta$, where $\mathcal{H}^{(2\pi)}$ and $\mathcal{H}^{(0)}$ are the generalized metric at $\theta = 2\pi$ and $\theta = 0$, respectively, and $\mathcal{O}_\beta$ is the $O(4,4)$ matrix given by

$$\mathcal{O}_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 \\ 0 & -2\pi\sigma\boldsymbol{\epsilon} \end{pmatrix}, \qquad \boldsymbol{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15}$$

The $O(D,D)$ monodromy matrices are generally classified as

$$\mathcal{O}_\Lambda = \begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^{-1})^{\mathsf{T}} \end{pmatrix}, \qquad \mathcal{O}_\Theta = \begin{pmatrix} 1 & 0 \\ \Theta & 1 \end{pmatrix}, \qquad \mathcal{O}_\Pi = \begin{pmatrix} 1 & \Pi \\ 0 & 1 \end{pmatrix}, \tag{16}$$

where $\mathcal{O}_\Lambda$ is the $O(D,D)$ matrix for the diffeomorphism Λ, $\mathcal{O}_\Theta$ for the shift Θ of the B-field, and $\mathcal{O}_\Pi$ for the T-duality transformation called the β-shift which is neither of them. The monodromy of the generalized metric of the defect NS5-brane (of codimension two) is the B-shift $\mathcal{O}_\Theta$, and that of the KK-vortex is the diffeomorphism $\mathcal{O}_\Lambda$. On the other hand, the monodromy of the 5^2_2-brame $\mathcal{O}_\beta$ is the β-shift. The space in which local charts are patched together with the β-shift is called *T-fold* [23, 24]. The 5^2_2-brane is one example of T-fold.

We show that the bi-hypercomplex structures of the 5^2_2-brane (13) and (14) also have monodromies. When we evaluate the monodromies using the parametrization of the generalized complex structure (8) and $(J'_{a,\pm}, \omega'_{a,\pm})$, we find

$$\begin{aligned} \mathcal{J}^{(2\pi)}_{1,+} &= \mathcal{O}_{-\beta} \mathcal{J}^{(0)}_{1,+} \mathcal{O}_\beta, & \mathcal{J}^{(2\pi)}_{2,+} &= \mathcal{O}_{-\beta} \mathcal{J}^{(0)}_{2,+} \mathcal{O}_\beta, & \mathcal{J}^{(2\pi)}_{3,+} &= \mathcal{O}_{-\beta} \mathcal{J}^{(0)}_{3,+} \mathcal{O}_\beta, \\ \mathcal{J}^{(2\pi)}_{1,-} &= \mathcal{O}_{-\beta} \mathcal{J}^{(0)}_{1,-} \mathcal{O}_\beta, & \mathcal{J}^{(2\pi)}_{2,-} &= \mathcal{O}_{-\beta} \mathcal{J}^{(0)}_{2,-} \mathcal{O}_\beta, & \mathcal{J}^{(2\pi)}_{3,-} &= \mathcal{O}_{-\beta} \mathcal{J}^{(0)}_{3,-} \mathcal{O}_\beta, \end{aligned} \tag{17}$$

where $\mathcal{J}^{(2\pi)}_{a,\pm}$ and $\mathcal{J}^{(0)}_{a,\pm}$ are the generalized hyperkähler structures modified by (8) at $\theta = 2\pi$ and $\theta = 0$, respectively.

5. Worldsheet instantons in T-fold

The worldsheet instantons are configurations that minimize the Euclidean action of the fundamental string in a given topological sector [25]. The worldsheet instanton equation requires a complex structure on the target space. The worldsheet instantons on T-fold are ill-defined because the complex structure has a T-duality monodromy. To resolve this issue, we introduce a string sigma model that is compatible with the doubled formalism.

The Born sigma model is a two-dimensional string sigma model with manifest T-duality of the target space. The target space of the Born sigma model is a $2D$-dimensional Born manifold $\mathcal{M}$ equipped with the Born structure $(\eta, \mathcal{H}, \omega)$. The action of the Born sigma model is given by

$$S = \frac{1}{4} \int_\Sigma \Big(\mathcal{H}_{MN} \, \mathrm{d}\mathbb{X}^M \wedge * \mathrm{d}\mathbb{X}^N - \Omega_{MN} \, \mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N \Big), \tag{18}$$

where Σ is the two-dimensional string worldsheet, $\mathbb{X}^M = (X^\mu, \tilde{X}_\mu)$ is the local coordinate of the Born manifold $\mathcal{M}$ and $*$ is the Hodge star operator in the worldsheet Σ. We follow [23, 24] and employ the topological term $\Omega_{MN} \, \mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N = -2 \, \mathrm{d}X^\mu \wedge \mathrm{d}\tilde{X}_\mu$.

Since the Born sigma model has the doubled degrees of freedom, it is necessary to impose a constraint that removes a half of the degrees of freedom. This constraint is called the chirality condition and is given by $\mathrm{d}\mathbb{X}^M = \eta^{MK} \mathcal{H}_{KN} * \mathrm{d}\mathbb{X}^N$. If we impose the chirality condition to select a D-dimensional subspace spanned by X^μ from the $2D$-dimensional target space, the extra degrees of freedom $\tilde{X}_\mu$ can be eliminated, and the action (18) gives the action of the ordinary string sigma model:

$$S = \frac{1}{2}\int_\Sigma \Big(g_{\mu\nu}\,\mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu + B_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu\Big). \tag{19}$$

We consider the instantons in the Born sigma model. In the following, the D-dimensional spacetime metric and the worldsheet metric have the Euclidean signature and $*^2 = -1$. The Bogomol'nyi bound of the generalized metric term in the action (18) is obtained as

$$\begin{aligned} S_{\mathrm{E}} &= \frac{1}{8}\int \Big[\mathcal{H}_{MN}\Big(\mathrm{d}\mathbb{X}^M \pm \mathcal{J}_\pm{}^M{}_P * \mathrm{d}\mathbb{X}^P\Big) \wedge *\Big(\mathrm{d}\mathbb{X}^N \pm \mathcal{J}_\pm{}^N{}_Q * \mathrm{d}\mathbb{X}^Q\Big) \pm 2(\omega_\pm)_{MN}\,\mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N\Big] \\ &\geq \pm\frac{1}{4}\int (\omega_\pm)_{MN}\,\mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N, \end{aligned} \tag{20}$$

where $\mathcal{J}_\pm$ are the generalized complex structures given by $\mathcal{J}_\pm = \mathrm{e}^{-B}\,\mathcal{J}^0_\pm\,\mathrm{e}^B$ with (2) as $\mathcal{J}^0_\pm$ and $\omega_\pm = \mathcal{H}\mathcal{J}_\pm$ are the fundamental two-forms associated with $\mathcal{J}_\pm$. The Bogomol'nyi bound (20) is saturated when the mapping $\mathbb{X} : \Sigma \to \mathcal{M}$ satisfies

$$\mathrm{d}\mathbb{X}^M \pm \mathcal{J}_\pm{}^M{}_N * \mathrm{d}\mathbb{X}^N = 0. \tag{21}$$

We call (21) the doubled instanton equations. Under the chirality condition, the doubled instanton equations are reduced to the ordinary worldsheet instanton equations associated with $J_\pm$, and the action bound is also the same. See the original paper [15] for details.

We now discuss the worldsheet instantons on the T-fold. The bi-hypercomplex structures on the 5^2_2-brane (13) and (14) are embedded in the generalized complex structures $\mathcal{J}_\pm$. By using (17), the monodromy of the doubled instanton equations with these $\mathcal{J}_\pm$ is

$$\begin{aligned} 0 &= (\mathrm{d}\mathbb{X}^{(2\pi)})^M \pm (\mathcal{J}^{(2\pi)}_\pm)^M{}_N * (\mathrm{d}\mathbb{X}^{(2\pi)})^N \\ &= (\mathrm{d}\mathbb{X}^{(2\pi)})^M \pm (\mathcal{O}_{-\beta})^M{}_K(\mathcal{J}^{(0)}_\pm)^K{}_L(\mathcal{O}_\beta)^L{}_N * (\mathrm{d}\mathbb{X}^{(2\pi)})^N, \end{aligned} \tag{22}$$

where $\mathcal{J}^{(2\pi)}_\pm$ and $\mathcal{J}^{(0)}_\pm$ are the generalized complex structures on the 5^2_2-brane at $\theta = 2\pi$ and $\theta = 0$, respectively. It is necessary for the doubled instanton equation to be $O(D,D)$-covariant, which requires that the mapping $\mathbb{X}^M$ also have a non-trivial T-duality monodromy:

$$(\mathrm{d}\mathbb{X}^{(2\pi)})^M = (\mathcal{O}_{-\beta})^M{}_N(\mathrm{d}\mathbb{X}^{(0)})^N. \tag{23}$$

Then the doubled instanton equations become

$$0 = (\mathcal{O}_{-\beta})^M{}_K\Big[(\mathrm{d}\mathbb{X}^{(0)})^K \pm (\mathcal{J}^{(0)}_\pm)^K{}_L * (\mathrm{d}\mathbb{X}^{(0)})^L\Big], \tag{24}$$

and the worldsheet instantons on T-folds are well-defined through the doubled instantons.

6. Conclusion

In this contribution, we discussed the T-duality relations between two different bi-hypercomplex structures. The bi-hypercomplex structure is embedded in the generalized hyperkähler structure of generalized geometry, which is T-duality covariant. By using the generalized hyperkähler structure and the factorized T-duality transformation, we derived the Buscher-like formula of the T-duality transformation for the bi-hypercomplex structure. Although the formula itself was already known, the laborious procedure of derivation using the sigma model is highly simplified by using our doubled formalism.

The hyperkähler structure is a special case of the bi-hypercomplex structure in which $J_{a,+}$ and $J_{a,-}$ coincide and $\omega_{a,\pm}$ are closed. We explicitly showed the bi-hypercomplex structure of the 5^2_2-brane from the hyperkähler structure of the KK-vortex as a concrete example of the T-duality transformation. Since the 5^2_2-brane is a T-fold, the geometric structures are expected to have non-trivial T-duality monodromies. We found that the monodromy of the bi-hypercomplex structure of the 5^2_2-brane in the doubled formalism is described by the same β-shift as in the generalized metric.

There is an issue in T-fold that the worldsheet instantons are multi-valued and ill-defined because the bi-hypercomplex structure has the monodromy. We introduced the idea of doubled instantons in Born sigma models to resolve this issue. Since doubled instanton equations are written in T-duality $O(D,D)$-covariant, the worldsheet instantons on T-fold are well-defined through the doubled formalism. Furthermore, the doubled instantons include all the worldsheet instantons in different T-duality frames, for example, the defect NS5-brane and the KK-vortex.

Some details of the contents in this contribution are found in the original papers [14, 15].

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