

Duality in $N = 2$ Supersymmetric Yang-Mills Theories

Cand. Scient. Thesis

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1 Introduction

Elementary particles and their interactions are described to a good accuracy within the framework of relativistic quantum field theory. At least this is true for energies up to the order of 100 GeV. In relativistic quantum field theory the underlying symmetry group is the Poincaré group, and from this we can classify the elementary particles according to their spin. Furthermore, the bosons, with integral spin, obey canonical commutation relations while the fermions, with half-integral spin, obey canonical anti-commutation relations. The bosons and the fermions thus constitute two distinct sectors of the Hilbert space. Supersymmetry is a symmetry that relates bosons and fermions.

It is appealing to have symmetries that relates different particles. For example, Grand Unifying symmetries are appealing because they relate particles that are different in the standard model (and which have the same spin). In this sense, supersymmetry is more “reasonable” than no supersymmetry.

If supersymmetry has anything to do with nature, it has to be broken at the low energies of about 100 GeV that we have probed in experiments to date. This is because supersymmetry predicts degenerate multiplets of particles with the same mass and quantum numbers, but with different spins. Such multiplets have not been observed.

However, this discrepancy between observation and exact supersymmetry could be a blessing rather than a curse since it offers an explanation to the so-called “gauge hierarchy problem” [1]. This is the problem of why the scale of the standard model (~ 100 GeV) is many orders of magnitude smaller than the GUT scale of about 10^{15} GeV (or the Planck scale of about 10^{19} GeV). Supersymmetry “explains” this by protecting the masses of scalar particles from receiving quantum corrections in perturbation theory. Thus, if the mass term of the Higgs field – which is related to the electroweak scale – is set to zero at the tree level, it will stay zero to all orders of perturbation theory. If supersymmetry is spontaneously broken by non-perturbative effects (perhaps instantons) this could give a small Higgs mass and thereby a small electroweak scale, as well as lifting the degeneracy of the supersymmetry multiplets. The point is that non-perturbative effects typically would involve numerical factors like $e^{-1/\alpha_{\text{GUT}}}$, where α_{GUT} is the GUT fine structure constant. The gauge hierarchy problem was one of the original motivations for studying supersymmetric theories.

In this report we will be interested in supersymmetry for a different reason, namely because it provides us with *toy models*. There are two aspects of supersymmetry that are interesting in this connection. First, supersymmetric theories often have better solvability properties than non-supersymmetric theories. This has to do with the fact that supersymmetric Lagrangians can be expressed in terms of holomorphic functions of the fields and coupling constants of the theory. In some cases one is able to determine these functions *exactly* [2]. And second, there exists certain *duality symmetries* in many supersymmetric theories [3]. These symmetries are similar to the duality between electricity and magnetism in the free Maxwell’s equations (i.e. without matter).

We will be occupied with one theory in particular. This is the so-called ‘ $N = 2$ supersymmetric Yang-Mills theory’, in our case with gauge group $SU(2)$. The ‘ N ’ refers to the “size” of the degenerate supersymmetry multiplets, so ‘ $N = 2$ ’ means that they contain more particles than a theory with simple – or ‘ $N = 1$ ’ – supersymmetry. It is a toy model which has both of the properties described above. The exact solution of this model and a description of the duality symmetry it possesses is treated in the already famous paper by Seiberg and Witten: “Electric-Magnetic Duality, Monopole Condensation, and Confinement in $N = 2$ Supersymmetric Yang-Mills theory” [4]. The purpose of this report is to explain the important concepts of this paper and to lay the necessary groundwork in order to do so.

The organization of the report is the following. In Chapter 2 we introduce the basic concepts of supersymmetry: the supersymmetric extension of the Poincaré algebra, its representations, and superspace. We then work out some supersymmetric field theories. This will all be at tree level.

In Chapter 3 we discuss quantum effects and renormalization. It turns out there are certain “non-renormalization” theorems in supersymmetric theories, which refers to the fact that various terms in the Lagrangian (e.g. potential terms) does not receive quantum corrections from renormalization. There are the “old” non-renormalization theorems which applies in perturbation theory (and which is relevant to the gauge hierarchy problem). Then there are the “new” non-renormalization theorems which extends to non-perturbative effects as well. This is where holomorphy and the exact solutions come in. We also introduce various “advanced topics”, such as moduli spaces and complex Kähler geometry, and start our discussion of the $N = 2$ supersymmetric Yang-Mills theory.

By Chapter 4 we break our line of development of the $N = 2$ theory to discuss duality. We recall the electric-magnetic duality of Maxwell’s equations, and we explain a famous conjecture by Montonen and Olive about a duality of the Georgi-Glashow model. They stated that the magnetic monopoles of positive and negative charge, which are solitons (the ’t Hooft-Polyakov monopoles), are duals of the W^+ - and W^- -bosons. There would then be a dual Lagrangian with the exact same form as the original one (at the tree level) where the heavy gauge bosons were the magnetic monopoles and where the solitons were the W^+ - and W^- -bosons. The conjecture actually does not hold in the original formulation of Montonen and Olive, but in supersymmetric theories this situation is improved, as we will explain.

Finally, in Chapter 5, we show how the two concepts of holomorphy and duality join together to give an elegant “solution” of the $N = 2$ theory, which is both unique and exact. Moreover, as a bonus, we see the phenomenon of confinement of electric charge in a perturbed version of the $N = 2$ theory, which we are then able to explain in terms of condensation of magnetic monopoles. This explanation of confinement coincides with ideas developed in the seventies. The $N = 2$ supersymmetric Yang-Mills theory is thus the first example of a quantum field theory where confinement is explained in an exact sense.

In Appendix A we review some basic representation theory of the Lorentz group, Appendix B contains some background on spinors, in Appendix C we give the notational conventions, and in Appendix D some useful formulae.

2 Basics of supersymmetry

2.1 The supersymmetry algebra

We will start our investigations of supersymmetric theories¹ by writing down the extensions of the Poincaré algebra. Once we have done that, we can find the irreducible representations and go on to construct invariant Lagrangians.

Supersymmetry transformations are generated by operators Q on the Hilbert space, which map bosons into fermions and fermions into bosons:

$$Q|boson\rangle = |fermion\rangle, \quad Q|fermion\rangle = |boson\rangle.$$

The Q ’s must be fermionic, i.e. they form sets of operators which transform among themselves as spinor representations of the Lorentz group, and they obey anticommutation relations. By the spin-statistics theorem this is the same thing. It is clear that an extension of the Poincaré

¹The general references on supersymmetry that has been used throughout this report, and in particular in this chapter are Refs. [5]–[8].

algebra with such Q 's cannot be a Lie-algebra, which is characterized by *commutation* relations only. It must be a structure that is characterized by both commutation and anticommutation relations:

$$[B_i, B_j] = if_{ij}^k B_k, \quad (1)$$

$$[Q_\alpha, B_i] = s_{\alpha i}^\beta Q_\beta, \quad (2)$$

$$\{Q_\alpha, Q_\beta\} = \gamma_{\alpha\beta}^i B_i, \quad (3)$$

Such a structure is a *graded* Lie algebra. The B 's are the *even* elements and the Q 's are the *odd* elements of the algebra. They generate elements of a *super Lie group* in the following way:

$$G(\zeta, c) = e^{-i\zeta_\alpha Q_\alpha - ic_i B_i}$$

where ζ_α are anticommuting (or Grassmann) parameters,

$$\{\zeta_\alpha, \zeta_\beta\} = 0, \quad \{\zeta_\alpha, Q_\beta\} = 0$$

The fact that this group is a consequence of the Baker-Campbell-Hausdorff formula. We will return to this point in the next section.

The minimal extension of the Poincaré algebra consists of the Poincaré generators and one spinorial generator² Q_α . We also include its hermitean adjoint $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$ to make the algebra stable under hermitean conjugation. The Q 's and \bar{Q} 's transform as the $(\frac{1}{2}, 0)$ - and $(0, \frac{1}{2})$ -representations of the Lorentz group³, respectively:

$$\begin{aligned} [Q_\alpha, M_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \\ [\bar{Q}_{\dot{\alpha}}, M_{\mu\nu}] &= \frac{1}{2}\bar{Q}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}}. \end{aligned}$$

This relation belongs under eq. (2) since the B 's corresponds to the P 's and M 's of the Poincaré algebra

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [P_\mu, M_{\rho\sigma}] &= i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho} - g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho}), \end{aligned} \quad (4)$$

which corresponds to eq. (1). Eq. (3) and the rest of eq. (2) is

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu, \quad (5)$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \quad (6)$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0. \quad (7)$$

This is the *supersymmetry algebra*.

Eqs. (5-7) can be deduced from transformation properties. For example, the anticommutator (5) transforms as $(\frac{1}{2}, \frac{1}{2})$, i.e. as a four-vector. By eq. (3) it must be an even element of the graded Lie algebra, that is, it must be one of the Poincaré generators. Since the only four-vector of the Poincaré algebra is the energy-momentum vector P_μ , the anticommutator must be proportional to this. This gives us eq. (5), with only the factor '2' to explain. The numerical value of this coefficient is just a matter of normalization of the Q 's, so we are left with the question of the sign. However, by treating eq. (5) as a matrix equation in $(\alpha, \dot{\beta})$ -space, we can take the trace and obtain

$$\sum_\alpha \{Q_\alpha, (Q_\alpha)^\dagger\} = 2\text{Tr}(\sigma^\mu P_\mu) = 4P_0. \quad (8)$$

²We are using two-component spinor notation, see Appendix B.

³Some elementary representation theory of the Lorentz group is given in Appendix A.

The left-hand side is positive definite and so must the energy P_0 be. Hence the sign in eq. (5). By a similar argument we can see that eq. (6) must hold, while the most general expression for the commutator (7) is

$$[Q_\alpha, P_\mu] = c(\sigma_\mu)_\alpha^{\dot{\beta}} \bar{Q}_{\dot{\beta}}.$$

However, the constant c can be shown to be zero by commuting this expression with P_μ , and using the generalized Jacobi identities that hold for a graded Lie algebra, i.e. identities of the type

$$\{A, \{B, C\}\} \pm \{B, \{C, A\}\} \pm \{C, \{A, B\}\} = 0,$$

where $\{, \}$ means either a commutator or an anticommutator, according to whether A, B and C are even or odd.

Although these arguments are not very rigorous, they are in fact fragments of a proof of a general theorem by Haag, Łopuszański and Sohnius [9]. This theorem is an extension of an earlier theorem by Coleman and Mandula about the symmetries of the S -matrix [10]. Coleman and Mandula proved under some reasonable assumptions⁴ that the symmetries of the S -matrix is the direct product of the Poincaré group with some internal symmetry group. That is, the Poincaré algebra is expanded by

$$\begin{aligned} [B_r, B_s] &= ic_{rs}^t B_t, \\ [B_r, P_\mu] &= [B_r, M_{\mu\nu}] = 0, \end{aligned} \tag{9}$$

where the B 's are generators of the internal symmetry. However, the Coleman-Mandula theorem only deals with ordinary Lie groups. It is this situation that is generalized by the Haag-Łopuszański-Sohnius theorem, which states that if we allow graded Lie algebras, then the symmetries of the S -matrix are given by the Poincaré algebra (4), the internal symmetry algebra (9), and the *extended supersymmetry algebra*

$$[M_{\mu\nu}, Q_{\alpha i}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_{\beta i}, \tag{10}$$

$$\{Q_{\alpha i}, \bar{Q}_{\dot{\beta}}^j\} = 2\delta_i^j (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \tag{11}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\epsilon_{\alpha\beta} Z_{ij}, \tag{12}$$

$$[Q_{\alpha i}, P_\mu] = 0, \tag{13}$$

$$[Q_{\alpha i}, B_r] = (b_r)_i^j Q_{\alpha j}. \tag{14}$$

In addition come the hermitean adjoint expressions. The $i = 1, \dots, N$ means that the spinor operators Q come in N copies, which are rotated by the internal symmetry generators B_r (eq. (14)). The Q 's still commute with the P 's, but the anticommutators of the Q 's with themselves, are now not necessarily zero. $\{Q_{\alpha i}, Q_{\beta j}\}$ transforms as a $(0,0) \oplus (1,0)$, where the $(1,0)$ -representation is an antisymmetric, selfdual tensor. The only tensor of this type in the even part of the Lie algebra is the selfdual part of $M_{\mu\nu}$, but $\{Q_{\alpha i}, Q_{\beta j}\}$ cannot contain a part that is proportional to this since it would violate eq. (13). Hence we get an equation like (12), where Z_{ij} is antisymmetric in i and j and in general a linear combination of the B 's:

$$Z_{ij} = \Leftrightarrow Z_{ji}, \quad Z_{ij} = a_{ij}^r B_r.$$

Furthermore, it can be shown that such Z 's commute with all the other generators,

$$[Z_{ij}, \text{anything}] = 0,$$

⁴This includes Lorentz invariance and analytic dependence of elastic scattering amplitudes on the center-of-mass energy s and the momentum transfer t .

making them *central charges*. The possibility of having central charges, which requires $N \geq 2$ because of the antisymmetry of i and j , will become important for us later when we discuss duality. We shall find such an operator by explicit construction in the $N = 2$ Yang-Mills theory, where there is a single, complex central charge, and where the real and imaginary parts have the interpretation of an electric and a magnetic charge, respectively. Eqs. (4), (9) and (10-14) exhaust all the possible algebras of a supersymmetric theory if we exclude conformal symmetry.

Now that we have the supersymmetry algebra, we turn to find its representations of the one-particle states. We will deal with representations on fields in the next section. Let us first recall how the irreducible representations of the Poincaré group are found [11]. This was worked out by Wigner in his famous paper from 1939 [12] by the method of “induced representations”. The Casimir operators of the Poincaré group are $P_\mu P^\mu$ and $W_\mu W^\mu$ where W^μ is the Pauli-Lubanski vector:

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}.$$

For an irreducible representation they are given by

$$P_\mu P^\mu = M^2, \quad W_\mu W^\mu = \pm M^2 J(J+1),$$

where M is the mass and J is the spin – or helicity – that characterizes the representation. The procedure is then to choose a frame of reference so that the momentum four-vector becomes, say, $P_\mu = (M, 0, 0, 0)$ in the massive case or $P_\mu = (E, 0, 0, E)$, where E is the energy, in the massless case. The subgroup of the Poincaré group that leaves P_μ invariant, the “little group”, is represented irreducibly, and the full representation is given by applying all possible boosts on these states.

We can find the irreducible representations of the supersymmetry algebra in a similar way. $P_\mu P^\mu$ is still a Casimir operator since P_μ commutes with the Q ’s. This is the reason why all the particles of an irreducible supersymmetry multiplet have the same mass. However, $W_\mu W^\mu$ is no longer a Casimir operator because the algebra contains the operators $Q_{\alpha i}$ which change the spin of a state. Let us deal separately with the massive and massless cases without central charges, and the massive case with central charges.

Massive case, no central charge. First we choose the rest frame where $P_\mu = (M, 0, 0, 0)$. The algebra to be represented – the “little algebra” – is given by eqs. (11) and (12) (and the hermitean adjoint of (12)). In the chosen frame and in the absence of central charges this is

$$\begin{aligned} \{Q_{\alpha i}, \bar{Q}_{\dot{\beta} j}\} &= 2M \delta_{\alpha\dot{\beta}} \delta_{ij}, \\ \{Q_{\alpha i}, Q_{\beta j}\} &= \{\bar{Q}_{\dot{\alpha} i}, \bar{Q}_{\dot{\beta} j}\} = 0. \end{aligned}$$

If we rescale the Q ’s into

$$a_{\alpha i} = \frac{1}{\sqrt{2M}} Q_{\alpha i}, \quad a_{\alpha i}^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha} i},$$

the algebra becomes

$$\begin{aligned} \{a_{\alpha i}, a_{\beta j}^\dagger\} &= \delta_{\alpha\beta} \delta_{ij}, \\ \{a_{\alpha i}, a_{\beta j}\} &= \{a_{\alpha i}^\dagger, a_{\beta j}^\dagger\} = 0, \end{aligned}$$

which we recognize as a *Clifford algebra* of $2N$ creation and annihilation operators. Now we choose a state $|M, J, J_z\rangle$ which corresponds to the rest frame, has spin J and spin component J_z in the z -direction, and which further satisfies

$$a_{\alpha i} |M, J, J_z\rangle = 0, \quad \text{all } \alpha, i.$$

This last property makes $|M, J, J_z\rangle$ a *Clifford ground state*. The supersymmetry multiplet is now constructed by applying the a^\dagger 's on this Clifford ground state:

$$|M, J, J_z\rangle, \quad a_{\alpha i}^\dagger |M, J, J_z\rangle, \quad \dots, \quad a_{\alpha i}^\dagger \cdots a_{\beta j}^\dagger |M, J, J_z\rangle, \dots$$

The full irreducible representation is obtained by applying boosts and rotations to these states.

The maximum number of a^\dagger 's we can apply on the Clifford ground state is $2N$ because of the Fermi statistics, and by working out the combinatorics the dimension of the representation is:

$$d = \sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N}.$$

The number of spin states is then $(2J+1)2^{2N}$. If the Clifford ground state has spin J , the largest spin in the multiplet is $J+N/2$, while the smallest spin is $J \Leftrightarrow N/2$, or 0 if $J \Leftrightarrow N/2 < 0$.

As an example, the simplest case is $N=1, J=0$, where the multiplet is

$$\begin{aligned} |M, 0, 0\rangle & \quad 1 \text{ state of spin } 0 \\ a_\alpha^\dagger |M, 0, 0\rangle & \quad 2 \text{ states of spin } \frac{1}{2} \\ a_1^\dagger a_2^\dagger |M, 0, 0\rangle & \quad 1 \text{ state of spin } 0 \end{aligned} \tag{15}$$

In fact, this is the only massive multiplet that will interest us, because all the other values of N and J will give multiplets which contains spin 1 or more. In the corresponding field theory we would then have massive vector fields. Such a theory is not renormalizable unless there is a Higgs mechanism, in which case the vector field is massless in the fundamental Lagrangian.

Massless case. When we deal with central charges in the next paragraph, we shall see that these must be represented by zero in the massless sector. We can therefore assume that there are no central charges in the superalgebra.

We now choose the frame of reference where $P_\mu = (E, 0, 0, E)$. Because

$$\sigma_{\alpha\dot{\beta}}^\mu P_\mu = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix},$$

the little algebra is

$$\begin{aligned} \{Q_{\alpha i}, \bar{Q}_{\dot{\beta} j}\} &= 2 \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \delta_{ij}, \\ \{Q_{\alpha i}, Q_{\beta j}\} &= \{\bar{Q}_{\dot{\alpha} i}, \bar{Q}_{\dot{\beta} j}\} = 0. \end{aligned}$$

The operators Q_{2i} and $\bar{Q}_{\dot{2}i}$ have vanishing anticommutators and must therefore be represented by zero. This leaves us only with the Q_1 's and $\bar{Q}_{\dot{1}}$'s to represent. If we define

$$a_i = \frac{1}{2\sqrt{E}} Q_{1i}, \quad a_i^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}i},$$

the algebra becomes:

$$\begin{aligned} \{a_i, a_j^\dagger\} &= \delta_{ij}, \\ \{a_i, a_j\} &= \{a_i^\dagger, a_j^\dagger\} = 0, \end{aligned}$$

which is a Clifford algebra of N creation and annihilation operators. Now the Clifford ground state is characterized by a helicity h :

$$a_i |E, h\rangle = 0, \quad \text{all } i$$

The multiplet is again built by applying the a^\dagger 's:

$$|E, h\rangle, \quad a_i^\dagger |E, h\rangle, \quad \dots, \quad a_i^\dagger \cdots a_j^\dagger |E, h\rangle,$$

where the full representation is obtained by boosting. By the same combinatorics as in the massive case, we find the dimension now to be $d = 2^N$.

Some examples are:

1. $N = 1, h = 0$:

$$\begin{array}{ll} \text{helicity:} & 0 \quad \frac{1}{2} \\ \text{states:} & 1 \quad 1 \end{array}$$

As it stands, this representation is not PCT -symmetric as it should be in a relativistic field theory. Therefore we must also use the representation $N = 1, h = \Leftrightarrow \frac{1}{2}$:

$$\begin{array}{ll} \text{helicity:} & \Leftrightarrow \frac{1}{2} \quad 0 \\ \text{states:} & 1 \quad 1 \end{array}$$

and "add" them together into the multiplet

$$\begin{array}{lll} \text{helicity:} & \Leftrightarrow \frac{1}{2} & 0 \quad \frac{1}{2} \\ \text{states:} & 1 & 2 \quad 1 \end{array}$$

It describes a scalar and a pseudoscalar with a massless fermion as supersymmetric partner.

2. $N = 1, h = \frac{1}{2}$:

$$\begin{array}{ll} \text{helicity:} & \frac{1}{2} \quad 1 \\ \text{states:} & 1 \quad 1 \end{array}$$

To get a PCT -invariant multiplet we must add

$$\begin{array}{ll} \text{helicity:} & \Leftrightarrow 1 \quad \Leftrightarrow \frac{1}{2} \\ \text{states:} & 1 \quad 1 \end{array}$$

so that we get

$$\begin{array}{lllll} \text{helicity:} & \Leftrightarrow 1 & \Leftrightarrow \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \text{states:} & 1 & 1 & 0 & 1 & 1 \end{array}$$

This describes a vector with a fermion as superpartner.

3. $N = 2, h = 0$: By adding PCT -conjugate states, we get

$$\begin{array}{lllll} \text{helicity:} & \Leftrightarrow 1 & \Leftrightarrow \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \text{states:} & 1 & 2 & 2 & 2 & 1 \end{array}$$

Note that the multiplets of Examples 1 and 2 added together give the same helicity content as this. This reflects the fact that one can always decompose high- N multiplets into lower- N multiplets.

4. $N = 2, h = \Leftrightarrow \frac{1}{2}$:

$$\begin{array}{lll} \text{helicity:} & \Leftrightarrow \frac{1}{2} & 0 \quad \frac{1}{2} \\ \text{states:} & 1 & 2 \quad 1 \end{array}$$

This is a *PCT-self-conjugate* representation.

5. $N = 4, h = \Leftrightarrow 1$:

$$\begin{array}{lllll} \text{helicity:} & \Leftrightarrow 1 & \Leftrightarrow \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \text{states:} & 1 & 4 & 6 & 4 & 1 \end{array}$$

This representation is also *PCT-self-conjugate*.

Massive case, with central charges. The central charges Z_{ij} commute with everything, and it is therefore possible to find a basis in representation space where the Z 's are diagonal, so that they can be represented by complex *numbers* Z_{ij} . We assume that N is even. Odd N 's can be worked out similarly, but only even N 's will be of interest to us. Since the Z_{ij} forms an antisymmetric $N \times N$ matrix, it is possible to use a unitary transformation $Z_{ij} \rightarrow U_i^k U_j^l Z_{kl}$ to bring this into the standard form

$$Z = \begin{pmatrix} 0 & D \\ \Leftrightarrow D & 0 \end{pmatrix}, \quad (16)$$

where D is a real, positive $N/2 \times N/2$ matrix with eigenvalues Z_r , $r = 1, \dots, N/2$. Without loss of generality we can assume that this has been done, and that the Q 's have been rotated by the same transformation. In accordance with eq. (16) we now break the index i down to $i = (a, r)$, where $a = 1, 2$, $r = 1, \dots, N/2$, and the superalgebra in the rest frame becomes⁵:

$$\begin{aligned} \{Q_{\alpha ar}, \bar{Q}_{\dot{\beta} bs}\} &= 2M\delta_{ab}\delta_{rs}\delta_{\alpha\dot{\beta}}, \\ \{Q_{\alpha ar}, Q_{\beta bs}\} &= 2\epsilon_{\alpha\beta}\epsilon_{ab}\delta_{rs}Z_r, \\ \{\bar{Q}_{\dot{\alpha} ar}, \bar{Q}_{\dot{\beta} bs}\} &= \Leftrightarrow 2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab}\delta_{rs}Z_r. \end{aligned}$$

We need to disentangle these anticommutation relations. To do this, we note that the part of the Poincaré group we are supposed to represent is the spatial rotations, so that $Q_{\alpha ar}$ and $\bar{Q}^{\dot{\alpha} ar}$ have the same transformation properties⁶. Then the linear combinations

$$\begin{aligned} a_{\alpha r} &= \frac{1}{2}(Q_{\alpha 1r} + \bar{Q}^{\dot{\alpha} 2r}), \\ b_{\alpha r} &= \frac{1}{2}(Q_{\alpha 1r} \Leftrightarrow \bar{Q}^{\dot{\alpha} 2r}), \end{aligned}$$

have well defined transformation properties. Using these, one can rewrite the algebra into

$$\begin{aligned} \{a_{\alpha r}, a_{\beta s}\} &= \{b_{\alpha r}, b_{\beta s}\} = \{a_{\alpha r}, b_{\beta s}\} = 0, \\ \{a_{\alpha r}, a_{\dot{\beta} s}^\dagger\} &= \delta_{\alpha\dot{\beta}}\delta_{rs}(M + Z_r), \\ \{b_{\alpha r}, b_{\dot{\beta} s}^\dagger\} &= \delta_{\alpha\dot{\beta}}\delta_{rs}(M \Leftrightarrow Z_r). \end{aligned}$$

Viewed as matrices in (α, r) -space the last two anticommutators are positive. This means that

$$Z_r \leq M, \quad \text{all } r. \quad (17)$$

⁵Recall that $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

⁶See the comment in Appendix B

Furthermore, if n Z_r 's are equal to M , n b 's must be represented by zero. So we get a Clifford algebra of $2(N \Leftrightarrow n)$ creation and annihilation operators, which we now know how to handle. Eq. (17) shows that in the massless case all the Z_r must be represented by zero.

Let us close this section by showing that a representation must contain an equal number of fermionic and bosonic states. Consider the operator $(\Leftrightarrow 1)^{N_F}$, where N_F is the fermion number operator. $(\Leftrightarrow 1)^{N_F}$ gives $+1$ on a bosonic state and $\Leftrightarrow 1$ on a fermionic state. Because Q_α^i changes the fermion number, we have that

$$(\Leftrightarrow 1)^{N_F} Q_\alpha^i = \Leftrightarrow Q_\alpha^i (\Leftrightarrow 1)^{N_F}.$$

For a finite dimensional representation the trace is well-defined, so that we have

$$\begin{aligned} \text{Tr}[(\Leftrightarrow 1)^{N_F} \{Q_\alpha^i, \bar{Q}_\beta^j\}] &= \text{Tr}[\Leftrightarrow Q_\alpha^i (\Leftrightarrow 1)^{N_F} \bar{Q}_\beta^j + (\Leftrightarrow 1)^{N_F} \bar{Q}_\beta^j Q_\alpha^i] \\ &= \text{Tr}[\Leftrightarrow Q_\alpha^i (\Leftrightarrow 1)^{N_F} \bar{Q}_\beta^j + Q_\alpha^i (\Leftrightarrow 1)^{N_F} \bar{Q}_\beta^j] \\ &= 0, \end{aligned}$$

where we have used the cyclic property of the trace. From this follows

$$2\delta^{ij} \sigma_{\alpha\dot{\beta}}^\mu \text{Tr}[(\Leftrightarrow 1)^{N_F} P_\mu] = \text{Tr}[(\Leftrightarrow 1)^{N_F} \{Q_\alpha^i, Q_{\dot{\beta}}^j\}] = 0.$$

Thus, for fixed non-zero momentum P_μ we have

$$\text{Tr}(\Leftrightarrow 1)^{N_F} = 0.$$

Since $\text{Tr}(\Leftrightarrow 1)^{N_F} = n_B \Leftrightarrow n_F$, where n_B is the number of bosons and n_F is the number of fermions in the representation, we get the desired result that $n_B = n_F$.

2.2 Superspace

In this report we will mostly deal with $N = 1$ and $N = 2$ supersymmetry, although we will occasionally mention the $N = 4$ case. These N 's are the most relevant ones to four-dimensional renormalizable field theories. $N = 2$ will be discussed in the next chapter, but for now we restrict ourselves to $N = 1$. This will be sufficient to introduce the basic concepts.

The Poincaré generators can be represented by the differential operators

$$P_\mu = i\partial_\mu, \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

which operate on scalar fields on Minkowski space. In a similar way, we would like the supersymmetry generators to be represented by differential operators as well. This can not be done in Minkowski space because the Q 's satisfy anticommutation relations while spacetime derivatives do not. Therefore we will expand Minkowski space into a *superspace* with coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, where θ_α and $\bar{\theta}_{\dot{\alpha}}$ are anticommuting spinors:

$$\{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0.$$

We shall see that the Q 's can be written as differential operators in this space.

Let us first investigate the group of finite supersymmetry transformations that corresponds to the superalgebra. We introduce the anticommuting spinor parameter ξ_α :

$$\{\xi_\alpha, \xi_\beta\} = \{\xi_\alpha, Q_\beta\} = \cdots = [P_\mu, \xi_\alpha] = 0$$

Then, with c^μ a four-vector parameter, an element G of this group is

$$G(c, \xi, \bar{\xi}) = e^{i(cP + \xi Q + \bar{\xi} \bar{Q})},$$

where $cP = c^\mu P_\mu$, $\xi Q = \xi^\alpha Q_\alpha$, $\bar{\xi}\bar{Q} = \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$, and where $\bar{\xi}_{\dot{\alpha}} = (\xi_\alpha)^*$. The Q 's enter in the hermitean combination $\xi Q + \bar{\xi}\bar{Q}$ so that G is unitary. The P 's appear because they are needed to close the algebra of the Q 's. Products of such elements can be found by using the Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \quad (18)$$

By using spinor parameters ξ_α and η_α we can write the superalgebra in terms of commutators:

$$\begin{aligned} [\xi Q, \bar{\eta}\bar{Q}] &= \xi Q \bar{\eta}\bar{Q} \Leftrightarrow \bar{\eta}\bar{Q} \xi Q \\ &= \xi^\alpha \bar{\eta}^{\dot{\beta}} Q_\alpha \bar{Q}_{\dot{\beta}} \Leftrightarrow \bar{\eta}^{\dot{\beta}} \xi^\alpha \bar{Q}_{\dot{\beta}} Q_\alpha \\ &= 2\xi^\mu \bar{\eta} P_\mu \\ [\xi Q, \eta Q] &= [\bar{\xi}\bar{Q}, \bar{\eta}\bar{Q}] = 0 \\ [P_\mu, \xi Q] &= [P_\mu, \bar{\xi}\bar{Q}] = 0 \end{aligned}$$

From this we see that the dots in the exponent of formula (18) are actually zero: the next term should be

$$\frac{1}{2}[[A, B], B] + \frac{1}{2}[A, [A, B]],$$

but this vanishes because $[A, B]$ vanishes or is proportional to P_μ . We can now find the product of two G 's:

$$\begin{aligned} G(c, \xi, \bar{\xi}) G(d, \eta, \bar{\eta}) &= e^{i(cP+\xi Q+\bar{\xi}\bar{Q})} e^{i(dP+\eta Q+\bar{\eta}\bar{Q})} \\ &= e^{i((c+d)P+(\xi+\eta)Q+(\bar{\xi}+\bar{\eta})\bar{Q}) - \frac{1}{2}[\xi Q+\bar{\xi}\bar{Q}, \eta Q+\bar{\eta}\bar{Q}]} \\ &= e^{i((c+d)P+(\xi+\eta)Q+(\bar{\xi}+\bar{\eta})\bar{Q}) + \frac{1}{2}i[\xi Q, \bar{\eta}\bar{Q}] + \frac{1}{2}i[\bar{\xi}\bar{Q}, \eta Q]} \\ &= e^{i((c+d+i\xi\sigma\bar{\eta}-i\eta\sigma\bar{\xi})P+(\xi+\eta)Q+(\bar{\xi}+\bar{\eta})\bar{Q})} \\ &= G(c+d+i\xi\sigma\bar{\eta} \Leftrightarrow i\eta\sigma\bar{\xi}, \xi+\eta, \bar{\xi}+\bar{\eta}) \end{aligned} \quad (19)$$

This shows that the supersymmetry transformations are a group.

Let us now return to superspace. Technically, we can think of superspace as the quotient of the super-Poincaré group and the Lorentz group, *super-Poincaré/Lorentz*, which is a manifold that is parametrized by one four-vector and a (complex) two-spinor. In fact, we can think of Minkowski space as the quotient *Poincaré/Lorentz*. Loosely we have:

$$Poincaré/Lorentz = e^{i(xP+\frac{1}{2}\omega M)} / e^{\frac{1}{2}i\omega M} = e^{ixP},$$

so the quotient is parametrized by the four-vector x^μ . In the supersymmetry case this is (equally loose):

$$super-Poincaré/Lorentz = e^{i(xP+\theta Q+\bar{\theta}\bar{Q}+\frac{1}{2}\omega M)} / e^{\frac{1}{2}i\omega M} = e^{i(xP+\theta Q+\bar{\theta}\bar{Q})}.$$

Note that by the last exponential we have given the prescription for how we parametrize the quotient manifold (i.e. superspace). We could have used the parametrizations $e^{i(yP+\theta Q)} e^{i\bar{\theta}\bar{Q}}$ or $e^{i(y^\dagger P+\bar{\theta}\bar{Q})} e^{i\theta Q}$, where the relations between the different parametrizations are $y^\mu = x^\mu \Leftrightarrow i\theta\sigma^\mu\bar{\theta}$ and $y^\dagger = x^\mu + i\theta\sigma^\mu\bar{\theta}$, respectively⁷. A translation in Minkowski space is induced by left multiplication on the quotient manifold:

$$e^{icP} e^{ixP} = e^{ix'P},$$

⁷Note that these definitions of y and y^\dagger are interchanged with respect to the ones of Ref. [5].

where y is the translation and $x' = x + c$. Supersymmetry transformations in superspace are induced in the same way:

$$G(0, \xi, \bar{\xi})G(x, \theta, \bar{\theta}) = G(x', \theta', \bar{\theta}'),$$

where ξ and $\bar{\xi}$ are the parameters of the transformation and

$$\begin{aligned} x' &= x + i\xi\sigma\bar{\theta} \Leftrightarrow i\theta\sigma\bar{\xi} \\ \theta' &= \theta + \xi \\ \bar{\theta}' &= \bar{\theta} + \bar{\xi} \end{aligned} \tag{20}$$

as follows from eq. (19).

Functions $F(x, \theta, \bar{\theta})$ on superspace are called *superfields*. They may have spinor or vector indices and carry some representation of the Lorentz group. If we expand a superfield in a power series in the spinor coordinates, it will terminate after a finite number of terms, because terms of the fifth power or higher in θ and $\bar{\theta}$ vanish on account of their anticommuting nature. A scalar superfield in its most general form is thus:

$$\begin{aligned} F(x, \theta, \bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) \\ &\quad + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &\quad + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x), \end{aligned}$$

The fields f , m , n and d are (in general complex) scalars, ϕ , ψ , $\bar{\chi}$ and $\bar{\lambda}$ are spinors, and v_μ is a vector. These fields are called component fields. It is clear that sums and products of superfields are again superfields.

We are now ready to find the representations of the Q 's as differential operators on superspace. That is, we want to find Q and \bar{Q} such that an infinitesimal supersymmetry transformation with parameter ξ is given by

$$\delta_\xi F = i(\xi Q + \bar{\xi}\bar{Q})F.$$

Because of eqs. (20), a scalar superfield F regarded as an operator on the Hilbert space has the transformation property:

$$G(0, \xi, \bar{\xi})F(x, \theta, \bar{\theta})G^{-1}(0, \xi, \bar{\xi}) = F(x \Leftrightarrow i\xi\sigma\bar{\theta} + i\theta\sigma\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}).$$

For infinitesimal ξ this means:

$$\delta_\xi F = i(\xi\sigma^\mu\bar{\theta} \Leftrightarrow \theta\sigma^\mu\bar{\xi})\partial_\mu F + \xi\frac{\partial}{\partial\theta}F + \bar{\xi}\frac{\partial}{\partial\bar{\theta}}F,$$

and so we can read off the representations of the Q 's to be:

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\bar{\theta})_\alpha\partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \Leftrightarrow (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu.$$

The operators $\partial/\partial\theta^\alpha$ and $\partial/\partial\bar{\theta}^{\dot{\alpha}}$ are anticommuting spinors that operates to the right:

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha^\beta, \quad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}}.$$

One can check that the Q 's satisfy the supersymmetry algebra:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2i\sigma_{\alpha\dot{\beta}}^\mu\partial_\mu, \\ \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\ [P_\mu, Q_\alpha] &= [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \end{aligned}$$

where we also have represented P_μ by $i\partial_\mu$.

The infinitesimal supersymmetry transformations of the component fields can be found from the relation:

$$\begin{aligned}\delta_\xi F(x, \theta, \bar{\theta}) &= i(\xi Q + \bar{\xi} \bar{Q})F(x, \theta, \bar{\theta}) \\ &= \delta_\xi f(x) + \theta \delta_\xi \phi(x) + \bar{\theta} \delta_\xi \bar{\lambda}(x) \\ &\quad + \theta \theta \delta_\xi m(x) + \bar{\theta} \bar{\theta} \delta_\xi n(x) + \theta \sigma^\mu \bar{\theta} \delta_\xi v_\mu(x) \\ &\quad + \theta \theta \bar{\theta} \delta_\xi \bar{\lambda}(x) + \bar{\theta} \bar{\theta} \theta \delta_\xi \psi(x) + \theta \theta \bar{\theta} \bar{\theta} \delta_\xi d(x).\end{aligned}$$

The component fields of a superfield does not in general form an irreducible representation of the supersymmetry transformations. If this is to be the case, we need to impose “super-covariant” constraints on the superfield, i.e. constraints that are preserved under supersymmetry transformations.

One covariant constraint is the reality constraint:

$$V = V^\dagger.$$

A superfield that satisfies this can in general be written as:

$$\begin{aligned}V(x, \theta, \bar{\theta}) &= C(x) + i\theta\chi(x) \Leftrightarrow i\bar{\theta}\bar{\chi}(x) \\ &\quad + \frac{1}{2}i\theta\theta[M(x) + iN(x)] \Leftrightarrow \frac{1}{2}i\bar{\theta}\bar{\theta}[M(x) \Leftrightarrow iN(x)] \\ &\quad + \theta\sigma^\mu\bar{\theta}V_\mu(x) \Leftrightarrow i\theta\theta\bar{\theta}[\bar{\lambda}(x) \Leftrightarrow \frac{1}{2}i\partial_\mu\chi(x)\sigma^\mu] \\ &\quad + i\bar{\theta}\bar{\theta}\theta[\lambda(x) + \frac{1}{2}i\sigma^\mu\partial_\mu\bar{\chi}(x)] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(x) \Leftrightarrow \frac{1}{2}\square C(x)],\end{aligned}\tag{21}$$

where the fields C , D , M and N are real scalars, V_μ is a real vector, and λ and χ are spinors. The presence of the vector field V_μ makes V the natural superfield to use if we want to find supersymmetric extensions of gauge theories. V is called the vector superfield because of V_μ . However, V is not irreducible. Some of the fields are redundant and can be gauged away in a certain sense. The particular combination of fields in eq. (21) is chosen in light of this, and we will return to the vector superfield when we consider supersymmetric gauge theories.

Other types of constraints are effectuated by super-covariant differential operators with respect to the spinor coordinates. Such operators D_α and $\bar{D}_{\dot{\alpha}}$ should obey the anticommutation relations:

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0,\tag{22}$$

or equivalently

$$[D_\alpha, \xi Q + \bar{\xi} \bar{Q}] = [\bar{D}_{\dot{\alpha}}, \xi Q + \bar{\xi} \bar{Q}] = 0,$$

for anticommuting parameters ξ and $\bar{\xi}$. In other words: the supercovariant derivatives commute with supersymmetry transformations. This can be achieved by taking the D ’s to be:

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} \Leftrightarrow i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = \Leftrightarrow \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu.$$

We could also have found these D ’s as the generators of supersymmetry transformations induced by right multiplication on the superspace quotient manifold. Left multiplication commutes with right multiplication, so this would automatically give eqs. (22). This means that the D ’s satisfy an algebra that is isomorphic to the one the Q ’s satisfy:

$$\begin{aligned}\{D_\alpha, \bar{D}_{\dot{\beta}}\} &= 2i\sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \\ \{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \\ [P_\mu, D_\alpha] &= [P_\mu, \bar{D}_{\dot{\beta}}] = 0,\end{aligned}$$

which can be checked by explicit calculation. Only a finite number of covariant differential operators can be formed by multiplying the D 's together because of their anticommuting nature.

A constraint that will be useful to us is the so-called *chiral* constraint:

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \quad (23)$$

If we use the coordinates $y^\mu = x^\mu \Leftrightarrow i\theta\sigma^\mu\bar{\theta}$ we see that

$$\bar{D}_{\dot{\alpha}}y^\mu = \bar{D}_{\dot{\alpha}}(x^\mu \Leftrightarrow i\theta\sigma^\mu\bar{\theta}) = 0, \quad \bar{D}_{\dot{\alpha}}\theta = 0$$

Thus, any function of y and θ will satisfy eq. (23):

$$\begin{aligned} \Phi &= A(y) + \theta\psi(y) + \theta\theta F(y) \\ &= A(x) \Leftrightarrow i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) \Leftrightarrow \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square A(x) \\ &\quad + \theta\psi(x) + \frac{1}{2}i\theta\theta(\sigma^\mu\bar{\theta})\partial_\mu\psi(x) + \theta\theta F(x), \end{aligned}$$

where we have expanded in x and used spinor identities like $\theta^\alpha\theta^\beta = \Leftrightarrow\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta$ (see Appendix C). Here, A and F are complex scalars and ψ is a two-spinor. That this is the most general function that solves (23) can be seen by writing the D 's in terms of y , θ and $\bar{\theta}$:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} \Leftrightarrow 2i(\sigma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial y^\mu}, \quad \bar{D}_{\dot{\alpha}} = \Leftrightarrow \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}.$$

The expression for $\bar{D}_{\dot{\alpha}}$ shows that Φ cannot contain any $\bar{\theta}$'s. The fact that, in the y -coordinates, Φ is a function only of θ_α , which is a spinor of definite chirality, is the reason that it is called a chiral superfield.

Analogously, the complex conjugate of Φ , the superfield Φ^\dagger , can be written as:

$$\begin{aligned} \Phi^\dagger &= A^*(y^\dagger) + \bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^*(y^\dagger) \\ &= A^*(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A^*(x) \Leftrightarrow \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square A^*(x) \\ &\quad + \bar{\theta}\bar{\psi}(x) \Leftrightarrow \frac{1}{2}i\bar{\theta}\bar{\theta}(\theta\sigma^\mu)\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x), \end{aligned}$$

where $y^\dagger = x^\mu + i\theta\sigma^\mu\bar{\theta}$. This is the most general field that satisfies:

$$D_\alpha\Phi^\dagger = 0,$$

as can be seen from the form of the D 's in terms of y^\dagger , θ and $\bar{\theta}$.

We can also find the supersymmetry transformations of the component fields. The easiest way is again to use the y -coordinates, and write down the corresponding Q 's:

$$Q_\alpha = \Leftrightarrow i\frac{\partial}{\partial\theta^\alpha}, \quad \bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \Leftrightarrow 2(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu,$$

where $\partial_\mu = \partial/\partial y^\mu$. An infinitesimal transformation is then:

$$\begin{aligned} \delta_\xi\Phi &= i(\xi Q + \bar{\xi}\bar{Q})\Phi \\ &= \xi\frac{\partial}{\partial\theta}\Phi + \bar{\xi}\frac{\partial}{\partial\bar{\theta}}\Phi \Leftrightarrow 2i\theta\sigma^\mu\bar{\xi}\partial_\mu\Phi \\ &= \xi\psi + 2\theta\xi F \Leftrightarrow 2i\theta\sigma^\mu\bar{\xi}\partial_\mu A \Leftrightarrow 2i(\theta\sigma^\mu\bar{\xi})\theta\partial_\mu\psi \\ &= \xi\psi + \theta 2\xi F \Leftrightarrow \theta 2i(\sigma^\mu\bar{\xi})\partial_\mu A \Leftrightarrow \theta\theta i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi, \end{aligned} \quad (24)$$

and so the component fields transform as:

$$\begin{aligned} \delta_\xi A &= \xi\psi \\ \delta_\xi\psi &= \Leftrightarrow 2i(\sigma^\mu\bar{\xi})\partial_\mu A + 2\xi F \\ \delta_\xi F &= \Leftrightarrow i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi \end{aligned} \quad (25)$$

This is equally true if we use the x -coordinates, as can be checked by expanding eq. (24).

From eq. (25) we see that the highest component, or F -component, of the superfield Φ transforms into a space-time derivative of the field ψ . Actually, it is a general feature of the supersymmetry transformations that the highest component field transforms into space-time derivatives of the other fields. This can be seen from the differential form of the supersymmetry generators Q_α and $\bar{Q}^{\dot{\alpha}}$: the first term reduces the power of the θ 's while the second term – which is a space-time derivative – increases it. So the only increments in the highest component are space-time derivatives of next-to-highest components. This fact will be useful to us when we construct manifestly supersymmetric Lagrangians out of superfields.

We can make the mass dimensions of the component fields coincide with the usual quantum field theory dimensions by assigning appropriate dimensions to the superfields. From the anticommutator $\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$, we deduce that the Q 's must have dimension $\frac{1}{2}$. Parameters of supersymmetry transformations, ξ_α and $\bar{\xi}_{\dot{\alpha}}$, must therefore have dimensions $\Leftrightarrow \frac{1}{2}$ in order to make the expression $\xi Q + \bar{\xi} \bar{Q}$ (which appears in an exponent) dimensionless. The same goes for θ^α and $\bar{\theta}_{\dot{\alpha}}$. Then, if the scalar superfield Φ has dimension 1, the scalar component A also has dimension 1 and the spinor ψ dimension $\frac{3}{2}$. The scalar field F , which is also a scalar, thus has dimension 2, but this is all right, since, as we shall see, it will play the role of an auxiliary or external field.

2.3 The Wess-Zumino model

We will now construct the Lagrangian of the simplest supersymmetric model – the Wess-Zumino model [13]. It involves the massive multiplet (15), with one fermion and two real scalars. We shall see that these fields are components of the chiral superfield Φ from the previous section. This consists of the two-spinor ψ and the complex scalars A and F , where F is auxiliary. We are going to use both four-component and two-component notation for the spinors. On the mass-shell – where the equations of motion are satisfied – this gives a field content of two real scalars and one Majorana spinor.

Let us see what kind of superfields we can build out of the chiral fields Φ_i . First of all we have the products

$$\begin{aligned} \Phi_i \Phi_j &= A_i(y) A_j(y) + \theta[\psi_i(y) A_j(y) + A_i(y) \psi_j(y)] \\ &\quad + \theta\theta[A_i(y) F_j(y) + A_j(y) F_i(y) \Leftrightarrow \frac{1}{2} \psi_i(y) \psi_j(y)], \end{aligned} \quad (26)$$

and

$$\begin{aligned} \Phi_i \Phi_j \Phi_k &= A_i(y) A_j(y) A_k(y) \\ &\quad + \theta[\psi_i A_j A_k + \psi_j A_i A_k + \psi_k A_i A_j] \\ &\quad + \theta\theta[F_i A_j A_k + F_j A_k A_i + F_k A_i A_j] \\ &\quad \Leftrightarrow \frac{1}{2} \psi_i \psi_j A_k \Leftrightarrow \frac{1}{2} \psi_j \psi_k A_i \Leftrightarrow \frac{1}{2} \psi_k \psi_i A_j]. \end{aligned} \quad (27)$$

These products are also chiral. Products of four or more chiral fields will not give renormalizable terms when we build Lagrangians, as we will see later. We can also multiply a chiral field with an anti-chiral one:

$$\begin{aligned} \Phi_i^\dagger \Phi_j &= A_i^*(x) A_j(x) + \theta \psi_j(x) A_i^*(x) + \bar{\theta} \bar{\psi}_i(x) A_j(x) \\ &\quad + \theta\theta A_i^*(x) F_j(x) + \bar{\theta}\bar{\theta} F_i^*(x) A_j(x) \\ &\quad + \theta^\alpha \bar{\theta}^{\dot{\beta}} [i\sigma_{\alpha\dot{\beta}}^\mu (\partial_\mu A_i^* A_j \Leftrightarrow A_i^* \partial_\mu A_j) \Leftrightarrow \bar{\psi}_{i\dot{\beta}} \psi_{j\alpha}] \\ &\quad + \theta\theta \bar{\theta}^{\dot{\alpha}} [\frac{1}{2} i\sigma_{\beta\dot{\alpha}}^\mu (\partial_\mu A_i^* \psi_j^\beta \Leftrightarrow A_i^* \partial_\mu \psi_j^\beta) \Leftrightarrow F_j \bar{\psi}_{i\dot{\alpha}}] \end{aligned} \quad (28)$$

$$\begin{aligned}
& +\bar{\theta}\bar{\theta}\theta^\alpha[\Leftrightarrow\frac{1}{2}i\sigma_{\alpha\dot{\beta}}^\mu(\partial_\mu\bar{\psi}_i^{\dot{\beta}}A_j\Leftrightarrow\bar{\psi}_i^{\dot{\beta}}\partial_\mu A_j)+F_i^*\psi_{j\alpha}] \\
& +\theta\theta\bar{\theta}\bar{\theta}[\Leftrightarrow\frac{1}{4}A_i^*\square A_j\Leftrightarrow\frac{1}{4}\square A_i^*A_j+\frac{1}{2}\partial_\mu A_i^*\partial^\mu A_j \\
& \Leftrightarrow\frac{1}{4}i\partial_\mu\bar{\psi}_i\bar{\sigma}^\mu\psi_j+\frac{1}{4}i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j+F_i^*F_j]
\end{aligned}$$

This superfield is not chiral.

We have already noted that the highest components of superfields transform into space-time derivatives under the supersymmetry transformations. By looking at the highest components of the product fields, we see that the $\theta\theta\bar{\theta}\bar{\theta}$ -coefficient – called the “ D -term” – of $\Phi_i^\dagger\Phi_j$ looks like a kinetic term of a Lagrangian, the $\theta\theta$ -term, or “ F -term”, of $\Phi_i\Phi_j$ looks like a mass term, while the F -term of $\Phi_i\Phi_j\Phi_k$ is an interaction. Then the most general renormalizable and supersymmetric Lagrangian built from chiral superfields is of the form

$$\mathcal{L} = \Phi_i^\dagger\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} + \left[(\lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k)|_{\theta\theta} + h.c. \right], \quad (29)$$

where λ_i , m_{ij} and g_{ijk} are (in general complex) coupling constants symmetric in their indices. We have added the hermitean conjugates of the chiral terms, in order to make the Lagrangian real. The mass dimension of a Lagrangian is four, so in the light of the previous discussion, we see that λ_i , m_{ij} and g_{ijk} have dimensions 2, 1 and 0, respectively. Now we see why products of four or more chiral superfields give non-renormalizable interactions: they would require coupling coefficients with negative mass dimensions in the Lagrangian.

We can write manifestly super-invariant Lagrangians – i.e. Lagrangians that involves superfields – in a more elegant (and, as it turns out, useful) way by introducing integration over superspace. This can be done by the usual Berezin integral, well known from the path integral treatment of Fermi fields. By definition, we have for a single Grassmann variable θ :

$$\int d\theta = 0, \quad \int d\theta\theta = 1.$$

Then, for a function $f(\theta) = a + b\theta$ of θ :

$$\int d\theta f(\theta) = b, \quad \int d\theta f(\theta)\theta = a.$$

For the Grassmann variables θ_α and $\bar{\theta}_{\dot{\alpha}}$ we define

$$\begin{aligned}
d^2\theta &= \frac{1}{4}\epsilon_{\alpha\beta}d\theta^\alpha d\theta^\beta, \\
d^2\bar{\theta} &= \frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}d\bar{\theta}_{\dot{\alpha}}d\bar{\theta}_{\dot{\beta}},
\end{aligned}$$

so that

$$\int d^2\theta\theta\theta = 1, \quad \int d^2\bar{\theta}\bar{\theta}\bar{\theta} = 1. \quad (30)$$

Furthermore, we define

$$d^4\theta \equiv d^2\theta d^2\bar{\theta}.$$

Recall that integration over Grassmann variables is the same as differentiation. Indeed,

$$\begin{aligned}
\frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta &= \int d^2\theta\theta\theta = 1, \\
\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} &= \int d^2\bar{\theta}\bar{\theta}\bar{\theta} = 1.
\end{aligned} \quad (31)$$

The Lagrangian (29) now becomes:

$$\mathcal{L} = \int d^4\theta \Phi_i^\dagger \Phi_i + \int d^2\theta (\lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k) + h.c.$$

By eq. (30) the infinitesimal volume elements $d\theta_\alpha$ and $d\bar{\theta}_{\dot{\alpha}}$ have dimension $+\frac{1}{2}$. Note that the Lagrangian contains integration over the whole of superspace in one term and over just half the superspace in other terms. This general structure appears in other supersymmetric models as well. The integral that goes over just half the superspace is called a *chiral integral* for obvious reasons. If the corresponding integrand is a super-generalization of a potential, then it is called a *superpotential*.

Let us consider the simplest case of a single chiral field Φ . We decompose the complex fields A and F into real and imaginary parts by writing

$$\Phi(y, \theta) = \frac{1}{2}(A(y) \Leftrightarrow iB(y)) + \theta\psi(y) + \frac{1}{2}\theta\theta(F(y) + iG(y)) \quad (32)$$

(hoping there will be no confusion between the complex fields and their real parts). Here, A and F are scalars while B and G are pseudoscalars. To see this, recall that a parity transformation takes the $(0, \frac{1}{2})$ -representation of the Lorentz group into the $(\frac{1}{2}, 0)$ -representation and vice versa. The chiral field (32) is then transformed into the anti-chiral field $\Phi^\dagger(y^\dagger, \bar{\theta})$ because the chiral constraint equation $\bar{D}_{\dot{\alpha}}\Phi = 0$ is transformed into $D_\alpha\Phi^\dagger = 0$. This means that a parity transformation changes the signs of B and G , that is, they are pseudoscalars. We also introduce the Majorana spinor

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix},$$

so that

$$\begin{aligned} \bar{\Psi}\gamma^\mu\partial_\mu\Psi &= \psi\sigma^\mu\partial_\mu\bar{\psi} + \bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi, \\ \bar{\Psi}\Psi &= \bar{\psi}\bar{\psi} + \psi\psi, \\ \bar{\Psi}\gamma_5\Psi &= \bar{\psi}\bar{\psi} \Leftrightarrow \psi\psi, \end{aligned}$$

see Appendix C. Then we have:

$$\begin{aligned} \mathcal{L}_{kin} &= 2 \int d^4\theta \Phi^\dagger \Phi = \Leftrightarrow \frac{1}{2} A^* \square A \Leftrightarrow \frac{1}{2} \square A^* A + \partial_\mu A^* \partial^\mu A \\ &\quad + \frac{1}{2} i \psi \sigma^\mu \partial_\mu \bar{\psi} + \frac{1}{2} i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + 2 F^* F \\ &= \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 + \frac{1}{2} i \bar{\Psi} \not{\partial} \Psi + \frac{1}{2} (F^2 + G^2), \\ \mathcal{L}_{mass} &= m \left(\int d^2\theta \Phi^2 + \int d^2\bar{\theta} \Phi^{\dagger 2} \right) = 2mFA \Leftrightarrow \frac{1}{2} m \psi\psi + 2mF^* A^* \Leftrightarrow \frac{1}{2} m \bar{\psi}\bar{\psi} \\ &= m(FA + GB) \Leftrightarrow \frac{1}{2} m \bar{\Psi}\Psi, \\ \mathcal{L}_{int} &= \frac{4}{3}g \left(\int d^2\theta \Phi^3 + \int d^2\bar{\theta} \Phi^{\dagger 3} \right) = 4gFA^2 \Leftrightarrow 2g\psi\psi A + 4gF^* A^{*2} \Leftrightarrow 2g\bar{\psi}\bar{\psi} A^* \\ &= g(FA^2 \Leftrightarrow FB^2 + 2GAB) \Leftrightarrow g\bar{\Psi}(A + i\gamma_5 B)\Psi, \end{aligned}$$

where we have thrown away total derivatives. The numerical factors have been chosen to make the expressions look more like the ones in the literature. The complete Lagrangian of the Wess-Zumino model in components is then:

$$\begin{aligned} \mathcal{L}_{WZ} &= \mathcal{L}_{kin} + \mathcal{L}_{mass} + \mathcal{L}_{int} \\ &= \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 + \frac{1}{2} i \bar{\Psi} \not{\partial} \Psi + \frac{1}{2} (F^2 + G^2) \\ &\quad + m(FA + GB) \Leftrightarrow \frac{1}{2} m \bar{\psi}\psi \\ &\quad + g(A^2 F \Leftrightarrow B^2 F + 2ABG) \Leftrightarrow g\bar{\Psi}(A + i\gamma_5 B)\Psi. \end{aligned} \quad (33)$$

One can also calculate the infinitesimal transformations under supersymmetry of these components, that is, the counterparts of eqs. (25). Using relations in Appendix C, we get:

$$\begin{aligned}
\delta A &= \bar{\alpha}\Psi \\
\delta B &= \Leftrightarrow i\bar{\alpha}\gamma_5\Psi \\
\delta\Psi &= \Leftrightarrow i\cancel{\not{D}}(A \Leftrightarrow i\gamma_5 B)\alpha + (F \Leftrightarrow i\gamma_5 G)\alpha \\
\delta F &= \Leftrightarrow i\bar{\alpha}\cancel{\not{D}}\Psi \\
\delta G &= \Leftrightarrow \bar{\alpha}\gamma_5\cancel{\not{D}}\Psi
\end{aligned} \tag{34}$$

where α is an infinitesimal Majorana parameter.

The fields F and G are auxiliary since they have no derivative terms. They can be eliminated by using their equations of motion:

$$\begin{aligned}
F &= \Leftrightarrow mA \Leftrightarrow g(A^2 \Leftrightarrow B^2) \\
G &= \Leftrightarrow mB \Leftrightarrow 2gAB
\end{aligned} \tag{35}$$

We have

$$\begin{aligned}
\frac{1}{2}(F^2 + G^2) &= \frac{1}{2}m^2(A^2 + B^2) + \frac{1}{2}g^2(A^2 + B^2)^2 + mgA(A^2 + B^2) \\
m(AF + BG) &= \Leftrightarrow m^2(A^2 + B^2) \Leftrightarrow mgA(A^2 + B^2) \\
g(A^2F \Leftrightarrow B^2F + 2ABG) &= \Leftrightarrow mgA(A^2 + B^2) \Leftrightarrow g^2(A^2 + B^2)^2
\end{aligned}$$

and the component Lagrangian becomes:

$$\begin{aligned}
\mathcal{L}_{WZ} &= \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 \Leftrightarrow \frac{1}{2}m^2A^2 \Leftrightarrow \frac{1}{2}m^2B^2 \\
&+ \frac{1}{2}i\bar{\Psi}\cancel{\not{D}}\Psi \Leftrightarrow \frac{1}{2}m\bar{\Psi}\Psi \Leftrightarrow g\bar{\Psi}(A + i\gamma_5 B)\Psi \\
&\Leftrightarrow mgA(A^2 + B^2) \Leftrightarrow \frac{1}{2}g^2(A^2 + B^2)^2
\end{aligned} \tag{36}$$

Note that the scalar fields have gotten mass terms with the same mass as the fermion. The Lagrangian (33) is invariant under supersymmetry regardless of whether the equations of motion (35) are satisfied or not, i.e. it is off-shell invariant. For eq. (36) to be invariant, the eqs. (35) must hold, i.e. it is invariant only on-shell. What's more, the number of bosonic and fermionic degrees of freedom match in both cases: four real scalar fields and four components of a fermion off-shell, and two real fields and two components of a Majorana fermion on-shell.

Lagrangians with chiral superfields may have additional symmetries known as R -symmetries. An R -symmetry acts on a chiral field with R -charge n as:

$$\begin{aligned}
\Phi(x, \theta) &\rightarrow e^{in\alpha}\Phi(x, e^{-i\alpha}\theta) \\
\Phi^\dagger(x, \bar{\theta}) &\rightarrow e^{-in\alpha}\Phi^\dagger(x, e^{i\alpha}\bar{\theta}).
\end{aligned} \tag{37}$$

In components this is:

$$\begin{aligned}
A &\rightarrow e^{in\alpha}A \\
\psi &\rightarrow e^{i(n-1)\alpha}\psi \\
F &\rightarrow e^{i(n-2)\alpha}F.
\end{aligned}$$

Because the R -transformations acts on the θ 's we also have:

$$\begin{aligned}
d^2\theta &\rightarrow e^{-2i\alpha}d^2\theta \\
d^2\bar{\theta} &\rightarrow e^{2i\alpha}d^2\bar{\theta}.
\end{aligned}$$

From eq. (37) we see that the product of two chiral fields with R -charges n_1 and n_2 has R -charge $n_1 + n_2$. Thus, a chiral term in a Lagrangian is R -invariant if it is a product of fields with R -charges that adds up to $n = 2$.

One example is a model with three chiral fields Φ_0 , Φ_1 and Φ_2 with R -charges $n_0 = 2$, $n_1 = 0$ and $n_2 = 2$, respectively [14]. The most general renormalizable R -invariant interaction that we can build from these fields is:

$$\begin{aligned} \mathcal{L}_{int} = & \int d^2\theta (\lambda_0 \Phi_0 + \lambda_2 \Phi_2 + (m_0 \Phi_0 + m_2 \Phi_2) \Phi_1 \\ & + (g_0 \Phi_0 + g_2 \Phi_2) \Phi_1 \Phi_1) + h.c. \end{aligned}$$

The Lagrangian can be restricted further if we require invariance under the discrete symmetry:

$$\Phi_{1,2} \rightarrow \Leftrightarrow \Phi_{1,2}, \quad \Phi_0 \rightarrow \Phi_0.$$

The result is:

$$\mathcal{L}_{int} = \int d^2\theta (\lambda_0 \Phi_0 + m_2 \Phi_2 \Phi_1 + g_0 \Phi_0 \Phi_1 \Phi_1) + h.c. \quad (38)$$

This is known as the O’Raifeartaigh model.

2.4 The supercurrent

We started this chapter by taking the supersymmetry algebra to be the underlying symmetry algebra of our theory. Then, from representations of this algebra on fields we were able to build invariant Lagrangians. By standard field theoretical arguments it is also possible to go the other way. We start with a Lagrangian that depends on some fields ϕ_i and their derivatives: $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$. A symmetry of the theory defined by this Lagrangian is a set of transformations $\delta \phi_i$ on the fields such that

$$\delta \mathcal{L} = \partial_\mu K^\mu,$$

where K^μ is a function of the fields that vanishes sufficiently fast at infinity. This ensures that the action is invariant:

$$\delta S = \int d^4x \delta \mathcal{L} = 0.$$

At the same time the increment in the Lagrangian can be written as

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \delta \phi_i \right)$$

by using the equations of motion, which enables us to define a conserved current

$$J_\mu^r = \frac{\partial}{\partial \alpha^r} \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi_i)} \delta \phi_i \Leftrightarrow K^\mu \right),$$

where α^r are infinitesimal (possibly Grassmann) parameters of the transformation. The generators of the symmetry is given by the charges

$$Q^r = \int d^3x J_0^r.$$

One can then go on to calculate the algebra of the generators by using the canonical equal-time commutation relations

$$[\phi(x), \partial_0 \phi(y)] = i \delta^{(3)}(x \Leftrightarrow y)$$

for a scalar field ϕ , and the anti-commutation relations

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta} \delta^{(3)}(x \leftrightarrow y)$$

for a (four) spinor ψ .

Let us try to calculate the supersymmetry current (the “supercurrent”) J_α^μ of the Wess-Zumino model. The Lagrangian is given by eq. (33) and the fields transform by eqs. (34). First let us find $\delta\mathcal{L}_{kin}$ ⁸:

$$\begin{aligned} \delta\mathcal{L}_{kin} &= \partial^\mu A \partial_\mu \delta A + \partial^\mu B \partial_\mu \delta B \\ &\quad + \frac{1}{2} i \delta \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{1}{2} i \bar{\psi} \gamma^\mu \partial_\mu \delta \psi \\ &\quad + F \delta F + G \delta G. \end{aligned}$$

The expression for $\delta\bar{\psi}$ is obtained from $\delta\psi$ by using the Majorana flip properties (see Appendix B):

$$\delta\bar{\psi} = i\bar{\alpha} \partial_\lambda (A \leftrightarrow i\gamma_5 B) \gamma^\lambda + \bar{\alpha} (F \leftrightarrow i\gamma_5 G).$$

Inserting this along with the other transformations we get

$$\begin{aligned} &\frac{1}{2} \bar{\alpha} \gamma^\mu \gamma^\lambda \partial_\lambda (A \leftrightarrow i\gamma_5 B) \partial_\mu \psi + \frac{1}{2} \bar{\alpha} \gamma^\mu \gamma^\lambda \partial_\mu \partial_\lambda (A \leftrightarrow i\gamma_5 B) \psi \\ &\quad \Leftrightarrow \frac{1}{2} i \bar{\alpha} (F \leftrightarrow i\gamma_5 G) \gamma^\mu \partial_\mu \psi \Leftrightarrow \frac{1}{2} i \bar{\alpha} \gamma^\mu \partial_\mu (F + i\gamma_5 G) \psi, \end{aligned}$$

where we have used some more flip identities to write the expression entirely in terms of $\bar{\alpha}$. We have also used $\{\gamma^\mu, \gamma^\lambda\} = 2g^{\mu\lambda}$ and the fact that $\gamma^\mu \gamma^\lambda \partial_\mu \partial_\lambda = \partial_\mu \partial^\mu$ because of the symmetry in μ and λ . All in all we get

$$\delta\mathcal{L}_{kin} = \frac{1}{2} \bar{\alpha} \partial_\mu (\gamma^\mu \gamma^\lambda \partial_\lambda (A \leftrightarrow i\gamma_5 B) \psi \Leftrightarrow i(F \leftrightarrow i\gamma_5 G) \gamma^\mu \psi).$$

$\delta\mathcal{L}_{mass}$ and $\delta\mathcal{L}_{int}$ can be calculated in a similar way. We will only state the results, which are⁹:

$$\begin{aligned} \delta\mathcal{L}_{mass} &= \Leftrightarrow i m \bar{\alpha} \partial_\mu (\gamma^\mu (A + i\gamma_5 B) \psi), \\ \delta\mathcal{L}_{int} &= \Leftrightarrow i g \bar{\alpha} \partial_\mu (\gamma^\mu (A + i\gamma_5 B)^2 \psi). \end{aligned}$$

Thus we see that the Lagrangian transforms as a total divergence:

$$\delta\mathcal{L} = \partial_\mu K^\mu,$$

where

$$\begin{aligned} K^\mu &= \frac{1}{2} \bar{\alpha} \gamma^\mu \gamma^\lambda \partial_\lambda (A \leftrightarrow i\gamma_5 B) \psi \Leftrightarrow \frac{1}{2} i \bar{\alpha} (F \leftrightarrow i\gamma_5 G) \gamma^\mu \psi \\ &\quad \Leftrightarrow i m \gamma^\mu (A + i\gamma_5 B) \psi \Leftrightarrow i g \gamma^\mu (A + i\gamma_5 B)^2 \psi. \end{aligned}$$

We can now go on to calculate the supercurrent:

$$\begin{aligned} J^\mu &= \frac{\partial}{\partial \bar{\alpha}} (\partial^\mu A \delta A + \partial^\mu B \delta B + \frac{1}{2} i \bar{\psi} \gamma^\mu \delta \psi \Leftrightarrow K^\mu) \\ &= \gamma^\lambda \partial_\lambda (A + i\gamma_5 B) \gamma^\mu \psi + i m \gamma^\mu (A + i\gamma_5 B) \psi + i g \gamma^\mu (A + i\gamma_5 B)^2 \psi. \end{aligned} \tag{39}$$

This is a spinor, as we have already mentioned.

An important fact about the current (39) is that it exist in an *interacting* theory. It is easy to construct conserved supersymmetry currents for free theories, but it is difficult to preserve this when interactions are added, unless we are dealing with “real” supersymmetry and

⁸The Majorana spinor Ψ from the previous section is now just called ‘ ψ ’.

⁹To verify the expression for $\delta\mathcal{L}_{int}$, we need the identity $\psi(\bar{\psi}\psi) = -\gamma_5 \psi(\bar{\psi}\gamma_5 \psi)$.

supersymmetry preserving interactions. An example of this is the free field toy model defined by

$$\mathcal{L}_{toy} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}i\bar{\psi}\not{\partial}\psi.$$

This theory has the supersymmetry

$$\begin{aligned}\delta A &= \bar{\alpha}\psi, \\ \delta\psi &= i\partial_\mu A\gamma^\mu\alpha,\end{aligned}$$

with the conserved supercurrent

$$J^\mu = \frac{1}{2}\gamma^\lambda\partial_\lambda A\gamma^\mu\psi,$$

which can readily be verified by using the equations of motion $\square A = 0$ and $\gamma^\mu\partial_\mu\psi = 0$. The supersymmetry is a trivial one in the sense that it cannot be maintained if we add interaction terms to the Lagrangian. This should come as no surprise since the number of fermionic and bosonic degrees of freedom are not the same.

2.5 Supersymmetric gauge theories

In this section we will generalize gauge theories to also have supersymmetry. This has the consequence of introducing particles that are supersymmetric partners to the usual gauge bosons. In the abelian case this is the “photino”, and in the non-abelian case it is the “gluinos”. These particles are spin $\frac{1}{2}$ fermions. Let us start with abelian gauge theories.

We want to find the supersymmetric generalization of the gauge transformation

$$V_\mu \rightarrow V_\mu + \partial_\mu\alpha, \tag{40}$$

where V_μ is the gauge field and α is a local phase. We have already noted that the real vector superfield V , which satisfies $V = V^\dagger$, generalizes the four-vector field V_μ . In order to maintain invariance under supersymmetry, we must also generalize the field α to a superfield. This can be done with the chiral superfield Λ , in which case the super-generalized gauge transformation can be taken to be

$$V \rightarrow V + i(\Lambda \Leftrightarrow \Lambda^\dagger) \tag{41}$$

This will also be called a gauge transformation. To see that this generalizes the transformation (40), let us expand the field Λ , which is chiral, on its components. We have

$$\begin{aligned}\Lambda &= \frac{1}{2}(A \Leftrightarrow iB) + \theta\psi + \frac{1}{2}\theta\theta(F + iG) \Leftrightarrow \frac{1}{2}i\theta\sigma^\mu\bar{\theta}\partial_\mu(A \Leftrightarrow iB) \\ &\quad + \frac{1}{2}i\theta\theta\bar{\theta}(\partial_\mu\psi\sigma^\mu) \Leftrightarrow \frac{1}{8}\theta\theta\bar{\theta}\bar{\theta}\square(A \Leftrightarrow iB),\end{aligned}$$

so that

$$\begin{aligned}i(\Lambda \Leftrightarrow \Lambda^\dagger) &= B + i\theta\psi \Leftrightarrow i\bar{\theta}\bar{\psi} + \frac{1}{2}i\theta\theta(F + iG) \Leftrightarrow \frac{1}{2}i\bar{\theta}\bar{\theta}(F \Leftrightarrow iG) \\ &\quad + \theta\sigma^\mu\bar{\theta}\partial_\mu A + i\theta\theta\bar{\theta}(\frac{1}{2}i\partial_\mu\psi\sigma^\mu) + i\bar{\theta}\bar{\theta}\theta(\frac{1}{2}i\sigma^\mu\partial_\mu\bar{\psi}) \Leftrightarrow \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square B.\end{aligned}$$

Since we were so clever in our choice of component fields in eq. (21), we can now read off the gauge transformations in components:

$$\begin{aligned}C &\rightarrow C + B \\ \chi &\rightarrow \chi + \psi \\ M &\rightarrow M + F \\ N &\rightarrow N + G \\ V_\mu &\rightarrow V_\mu + \partial_\mu A \\ \lambda &\rightarrow \lambda \\ D &\rightarrow D\end{aligned}$$

Now we see that the field V_μ transforms just like a gauge field should. The transformation leaves λ and D invariant. C , χ , M and N , however, can be gauged away by choosing

$$\begin{aligned} B &= \Leftrightarrow C \\ \psi &= \Leftrightarrow \chi \\ F &= \Leftrightarrow M \\ G &= \Leftrightarrow N \end{aligned}$$

This choice of gauge is called the Wess-Zumino gauge, or WZ-gauge. In the WZ-gauge the superfield V becomes:

$$V_{WZ} = \theta\sigma^\mu\bar{\theta}V_\mu \Leftrightarrow i\theta\theta\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D.$$

Note that after we have fixed the WZ-gauge, we still have the freedom of transforming the V_μ -field by the usual gauge transformations.

Now that we have the superfield that generalizes a gauge field, we must use this to find the superfield that generalizes a gauge invariant field strength. We will then be close to having a Lagrangian that is both gauge invariant and super-invariant. One systematic way of finding this field would be to set up differential geometry in superspace and introduce fiber bundles. The vector superfield V would then be a connection and the field strength would be a curvature, just like in the ordinary fiber bundle formulation of gauge theories. We will only give the result of this investigation since we will anyhow see that the field strength superfield contains the usual field strength as a component. The field we are looking for is:

$$W_\alpha = \Leftrightarrow \frac{1}{4}\bar{D}\bar{D}D_\alpha V. \quad (42)$$

It is chiral because of the anticommutativity of the \bar{D} 's:

$$\bar{D}_{\dot{\beta}}W_\alpha = 0.$$

It is also gauge invariant, since, under gauge transformations:

$$\begin{aligned} W_\alpha \rightarrow W'_\alpha &= \Leftrightarrow \frac{1}{4}\bar{D}\bar{D}D_\alpha(V + i(\Lambda \Leftrightarrow \Lambda^\dagger)) \\ &= \Leftrightarrow \frac{1}{4}\bar{D}\bar{D}D_\alpha V \Leftrightarrow \frac{1}{4}i\bar{D}\bar{D}D_\alpha \Lambda \\ &= \Leftrightarrow \frac{1}{4}\bar{D}\bar{D}D_\alpha V \Leftrightarrow \frac{1}{4}i\bar{D}\{\bar{D}, D_\alpha\}\Lambda \\ &= \Leftrightarrow \frac{1}{4}\bar{D}\bar{D}D_\alpha V = W_\alpha \end{aligned}$$

The component form can be calculated most easily in the y -coordinates (and in the WZ-gauge), where the field V is

$$V = \theta\sigma^\mu\bar{\theta}V_\mu \Leftrightarrow i\theta\theta\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D \Leftrightarrow i\partial_\mu V^\mu).$$

Now we have

$$\begin{aligned} D_\alpha V &= \left(\frac{\partial}{\partial\theta^\alpha} \Leftrightarrow 2i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \right) V \\ &= (\sigma^\mu\bar{\theta})_\alpha V_\mu \Leftrightarrow 2i\theta_\alpha\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\lambda_\alpha + \theta_\alpha\bar{\theta}\bar{\theta}(D \Leftrightarrow i\partial_\mu V^\mu) \\ &\quad \Leftrightarrow i\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha\partial_\mu V_\nu + \theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\ &= (\sigma^\mu\bar{\theta})_\alpha V_\mu \Leftrightarrow 2i\theta_\alpha\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\lambda_\alpha + \theta_\alpha\bar{\theta}\bar{\theta}D \\ &\quad + \frac{1}{2}\bar{\theta}\bar{\theta}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha, \end{aligned}$$

where $F_{\mu\nu} = \partial_\mu V_\nu \Leftrightarrow \partial_\nu V_\mu$ is the usual field strength tensor, and we finally get:

$$\begin{aligned} W_\alpha &= \Leftrightarrow \frac{1}{4}\bar{D}\bar{D}D_\alpha V = \Leftrightarrow \frac{1}{4}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\bar{\theta}}D_\alpha V \\ &= \Leftrightarrow i\lambda_\alpha \Leftrightarrow \theta_\alpha D \Leftrightarrow \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} \Leftrightarrow \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha. \end{aligned}$$

The Lagrangian is now found from

$$W^\alpha W_\alpha|_{\theta\theta} = 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 \Leftrightarrow \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \Leftrightarrow \frac{1}{2}iF_{\mu\nu}\tilde{F}^{\mu\nu},$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The result is:

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\int d^2\theta W^\alpha W_\alpha + h.c. \\ &= \Leftrightarrow \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}i\bar{\Lambda}\not{\partial}\Lambda + \frac{1}{2}D^2,\end{aligned}\tag{43}$$

where we have integrated by parts and introduced the Majorana spinor Λ which is built from λ . The integration by parts gets rid of the $F\tilde{F}$ -term, which is fine in an abelian theory, but in non-abelian theories non-trivial field configurations exist which may not allow us to do this. The spinor superfield W_α is gauge invariant, so using other gauges than the WZ-gauge will not introduce any other components from the field V than the fields $F_{\mu\nu}$, λ and D already present. This reflects the fact that these fields only transforms into each other under supersymmetry and so constitute an irreducible supermultiplet. While V_μ describes the photon and Λ the photino, the field D is auxiliary, as we can see from the Lagrangian (43). In the pure gauge theory it vanishes by the equations of motion, but it will couple to other fields when we include matter.

We also note that W_α and $\bar{W}_{\dot{\alpha}}$ satisfy

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}},$$

which follows from the definition (42) and its conjugate since V is real. This equation also appears in the fiber bundle treatment, where it is the super-generalized Bianchi-identity.

Matter is included by using two chiral fields S_1 and S_2 in the complex combinations

$$S = \frac{1}{\sqrt{2}}(S_1 + iS_2) \quad \text{and} \quad T = \frac{1}{\sqrt{2}}(\bar{S}_1 + i\bar{S}_2),$$

where we have introduced the notation \bar{S}_i for the complex conjugate of S_i . Note that T is not just the conjugate of S . It is needed in order to get a sensible gauge invariant mass term. The fact that we use complex superfields is analogous to ordinary field theory where e.g. charged scalar fields are complex. Global phase transformations by a parameter λ is then given by:

$$S \rightarrow e^{-2ig\lambda}S, \quad T \rightarrow e^{-2ig\lambda}T,$$

where g is the charge of the fields. The Lagrangian

$$\mathcal{L}_{matter} = 2\int d^4\theta(\bar{S}S + \bar{T}T) + m\int d^2\theta\bar{T}S + h.c.\tag{44}$$

is invariant under this symmetry, which we now want to gauge. But making λ a function of space-time is not enough: if S' and T' are to remain chiral superfields, λ must be promoted to a chiral superfield Λ in the following way:

$$\begin{aligned}S &\rightarrow e^{-2ig\Lambda}S, & \bar{D}_{\dot{\alpha}}\Lambda &= 0 \\ T &\rightarrow e^{-2ig\Lambda^\dagger}T, & D_\alpha\Lambda^\dagger &= 0.\end{aligned}$$

With this definition of a gauge transformation, the mass terms of the Lagrangian (44) are still invariant. The kinetic terms are not, however. Instead they transform by

$$\begin{aligned}\bar{S}S &\rightarrow \bar{S}S e^{-2ig(\Lambda-\Lambda^\dagger)} \\ \bar{T}T &\rightarrow \bar{T}T e^{2ig(\Lambda-\Lambda^\dagger)}.\end{aligned}\tag{45}$$

It is now clear that an invariant kinetic term is constructed by coupling this to the vector superfield V in the combination

$$\bar{S}e^{2gV}S \quad \text{and} \quad \bar{T}e^{-2gV}T.$$

These terms look non-linear because of the exponentials. However, in the WZ-gauge,

$$V^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}V_\mu V^\mu, \quad V^n = 0, \quad n \geq 3,$$

so the expansion of the exponential will terminate and we have:

$$\begin{aligned} \bar{S}e^{2gV}S + \bar{T}e^{-2gV}T &= \bar{S}S + \bar{T}T + 2gV(\bar{S}S \Leftrightarrow \bar{T}T) \\ &\quad + 2g^2V^2(\bar{S}S + \bar{T}T) \\ &= \bar{S}_1S_1 + \bar{S}_2S_2 + 2igV(\bar{S}_1S_2 \Leftrightarrow \bar{S}_2S_1) \\ &\quad + 2g^2V^2(\bar{S}_1S_1 + \bar{S}_2S_2) \end{aligned}$$

(in the WZ-gauge). The mass terms are:

$$m\bar{T}S = \frac{1}{2}m(S_1^2 + S_2^2)$$

and its hermitean conjugate. The Lagrangian for the matter is obtained by taking the D -terms of the kinetic terms and the F -terms of the mass terms,

$$\mathcal{L}_{matter} = 2 \int d^4\theta (\bar{S}e^{2gV}S + \bar{T}e^{-2gV}T) + m \int d^2\theta \bar{T}S + h.c.$$

In components this is:

$$\begin{aligned} \mathcal{L}_{matter} &= \frac{1}{2}(\partial_\mu A_1)^2 + \frac{1}{2}(\partial_\mu A_2)^2 + \frac{1}{2}(\partial_\mu B_1)^2 + \frac{1}{2}(\partial_\mu B_2)^2 \\ &\quad + \frac{1}{2}i\bar{\psi}_1\bar{\not{D}}\psi_1 + \frac{1}{2}i\bar{\psi}_2\bar{\not{D}}\psi_2 + \frac{1}{2}(F_1^2 + F_2^2 + G_1^2 + G_2^2) \\ &\quad \Leftrightarrow g(A_1 \overset{\leftrightarrow}{\partial}_\mu A_2 + B_1 \overset{\leftrightarrow}{\partial}_\mu B_2)V^\mu \Leftrightarrow ig\bar{\psi}_1\gamma^\mu\psi_2V_\mu \\ &\quad \Leftrightarrow \frac{1}{2}g\bar{\lambda}(A_2 \Leftrightarrow i\gamma_5 B_2)\psi_1 + \frac{1}{2}g\bar{\lambda}(A_1 \Leftrightarrow i\gamma_5 B_1)\psi_2 \\ &\quad + g(A_1B_2 \Leftrightarrow B_1A_2)D \Leftrightarrow \frac{1}{2}g^2(A_1^2 + A_2^2 + B_1^2 + B_2^2)V_\mu^2 \\ &\quad + m(A_1F_1 + B_1G_1 + A_2F_2 + B_2G_2) \Leftrightarrow \frac{1}{2}m\bar{\psi}_1\psi_1 \Leftrightarrow \frac{1}{2}m\bar{\psi}_2\psi_2, \end{aligned}$$

where $A \overset{\leftrightarrow}{\partial}_\mu B \equiv A(\partial_\mu B) \Leftrightarrow (\partial_\mu A)B$. If we put this together with the gauge field Lagrangian (43) and use the equations of motion:

$$\begin{aligned} D &= \Leftrightarrow g(A_1B_2 \Leftrightarrow B_1A_2) \\ F_i &= \Leftrightarrow mA_i, \quad G_i = \Leftrightarrow mB_i, \quad i = 1, 2 \end{aligned}$$

to eliminate the auxiliary fields, we get:

$$\begin{aligned} \mathcal{L} &= \Leftrightarrow \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}i\bar{\lambda}\not{D}\lambda + \frac{1}{2}(\partial_\mu A_1)^2 + \frac{1}{2}(\partial_\mu A_2)^2 \\ &\quad + \frac{1}{2}(\partial_\mu B_1)^2 + \frac{1}{2}(\partial_\mu B_2)^2 + \frac{1}{2}i\bar{\psi}_1\bar{\not{D}}\psi_1 + \frac{1}{2}i\bar{\psi}_2\bar{\not{D}}\psi_2 \\ &\quad \Leftrightarrow g(A_1 \overset{\leftrightarrow}{\partial}_\mu A_2 + B_1 \overset{\leftrightarrow}{\partial}_\mu B_2 + i\bar{\psi}_1\gamma_\mu\psi_2)V^\mu \\ &\quad + \frac{1}{2}g\bar{\lambda}(A_1 \Leftrightarrow i\gamma_5 B_1)\psi_2 \Leftrightarrow \frac{1}{2}g\bar{\lambda}(A_2 \Leftrightarrow i\gamma_5 B_2)\psi_1 \\ &\quad \Leftrightarrow \frac{1}{2}g^2(A_1^2 + A_2^2 + B_1^2 + B_2^2)V_\mu^2 \Leftrightarrow g^2(A_1B_2 \Leftrightarrow B_1A_2)^2 \\ &\quad \Leftrightarrow \frac{1}{2}m^2(A_1^2 + A_2^2 + B_1^2 + B_2^2) \Leftrightarrow \frac{1}{2}m\bar{\psi}_1\psi_1 \Leftrightarrow \frac{1}{2}m\bar{\psi}_2\psi_2 \end{aligned}$$

The Majorana fermions ψ_1 and ψ_2 can be thought of as the left- and right-handed parts of a Dirac fermion – the electron. The supersymmetric scalar partners are the “selectrons”.

One can easily generalize all this to non-abelian gauge theories with compact gauge groups. The transformation laws for the matter superfields are now:

$$\begin{aligned} S &\rightarrow e^{-ig\Lambda} S, & \Lambda &= \Lambda_a T_a, \\ T &\rightarrow e^{ig\Lambda^\dagger} T, \end{aligned}$$

where Λ is Lie algebra valued and T_a are the generators of the gauge group in some representation. In this case, if the kinetic terms

$$\bar{S}e^{gV}S + \bar{T}e^{-gV}T$$

are to be invariant, the following transformation law for the (Lie-algebra valued) superfield V must hold:

$$e^{gV} \rightarrow e^{-ig\Lambda^\dagger} e^{gV} e^{ig\Lambda},$$

or, for infinitesimal Λ :

$$V \rightarrow V + i(\Lambda \Leftrightarrow \Lambda^\dagger) \Leftrightarrow \frac{1}{2}[\Lambda + \Lambda^\dagger, V].$$

The field strength superfield is now:

$$W_\alpha = \Leftrightarrow \frac{1}{4} \bar{D} \bar{D} e^{-gV} D_\alpha e^{gV},$$

which transforms covariantly:

$$W_\alpha \rightarrow e^{-ig\Lambda} W_\alpha e^{ig\Lambda}.$$

A supersymmetric Lagrangian of the gauge field is then¹⁰:

$$\mathcal{L} = \frac{1}{2} \int d^2\theta \text{Tr} W^\alpha W_\alpha + h.c.$$

If the representation of the group is real in the sense that the generators T^a are antisymmetric, then

$$\Lambda^T = \Leftrightarrow \Lambda \quad \text{and} \quad V^T = \Leftrightarrow V,$$

and we can make the identification

$$\bar{T} = S.$$

We have then the kinetic term $\bar{S}e^{gV}S$ and the mass term mS^2 :

$$\mathcal{L} = \int d^4\theta \bar{S}e^{gV}S + \int d^2\theta mS^2 + h.c.$$

If, for example, the matter field S transforms as the adjoint representation, we have the Lagrangian:

$$\begin{aligned} \mathcal{L} = & \Leftrightarrow \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} i \bar{\lambda}^a \not{D} \lambda^a + \frac{1}{2} (D_\mu A^a)^2 + \frac{1}{2} (D_\mu B^a)^2 + \frac{1}{2} i \bar{\psi}^a \not{D} \psi^a \\ & + \frac{1}{2} g \text{Tr} \bar{\lambda} [A \Leftrightarrow i\gamma_5 B, \psi] + \frac{1}{2} g^2 \text{Tr} [A, B]^2 \Leftrightarrow \frac{1}{2} m^2 A^a A^a \Leftrightarrow \frac{1}{2} m^2 B^a B^a \Leftrightarrow \frac{1}{2} m \bar{\psi}^a \psi^a. \end{aligned}$$

In the massless limit of this theory we have a new symmetry between λ and ψ . If we define $\psi_1 \equiv \lambda$ and $\psi_2 \equiv \psi$, we get

$$\begin{aligned} \mathcal{L} = & \Leftrightarrow \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} i \bar{\psi}_i^a \not{D} \psi_i^a + \frac{1}{2} (D_\mu A^a)^2 + \frac{1}{2} (D_\mu B^a)^2 \\ & + \frac{1}{2} g \epsilon_{ij} \text{Tr} ([\bar{\psi}^i, (A \Leftrightarrow i\gamma_5 B) \psi^j] + \frac{1}{2} g^2 \text{Tr} [A, B]^2, \end{aligned}$$

¹⁰We will always use a normalization of the generators so that $\text{Tr} AB = \frac{1}{2} A^a B^a$.

where the new symmetry is realized as the $SU(2)$ transformations of ϕ_i . This symmetry is “another” supersymmetry so we really have $N = 2$. The $SU(2)$ symmetry acts on the two spinor supersymmetry generators of $N = 2$. In fact, with the gauge group $SU(2)$ this is the theory which plays a leading role in this report, and we will examine it in detail later on. Let us just record the supersymmetry current in this model, which is

$$S_i^\mu = \text{Tr}(F_{\rho\sigma}\sigma^{\rho\sigma}\gamma^\mu\Psi_i + \epsilon_{ij}\mathcal{D}(A \Leftrightarrow i\gamma_5 B)\gamma^\mu\Psi^j + ig\gamma_5[A, B]\Psi_i).$$

From this current one can calculate the $N = 2$ supersymmetry algebra.

Let us also record the Lagrangian of the pure $N = 4$ Yang-Mills theory for the sake of completeness. In components it is [15]:

$$\begin{aligned}\mathcal{L} = & \Leftrightarrow\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}i\bar{\lambda}^a\mathcal{D}\lambda^a + \frac{1}{2}i\bar{\psi}_i^a\mathcal{D}\psi_i^a + \frac{1}{2}(D_\mu A_i^a)^2 + \frac{1}{2}(D_\mu B_i^a)^2 \\ & \Leftrightarrow g\text{Tr}\bar{\psi}_i[i\alpha_j A_i + i\beta_j B_j i\gamma_5, \psi_i] + \frac{1}{2}g^2\text{Tr}[A_i, A_j][A_i, A_j] \\ & + \frac{1}{2}g^2\text{Tr}[B_i, B_j][B_i, B_j] + \frac{1}{2}g^2\text{Tr}[A_i, b_j][A_i, B_j],\end{aligned}$$

where $i, j = 1, 2, 3$, and α_i and β_i are 4×4 matrices that satisfy a certain algebra. All fields are in the adjoint of the gauge group. The field content of this theory corresponds to Ex. 5 in Sec. 2.1. In terms of $N = 1$ multiplets, it consists of a gauge multiplet and three chiral multiplets.

3 Non-renormalization theorems and exact potentials

3.1 Renormalization and perturbation theory

So far we have only considered supersymmetry at the classical level. We will now discuss renormalization. It turns out that supersymmetric theories has “nice” renormalization properties. Historically this was discovered in perturbation theory where one found that divergences were at most logarithmic. Linear and higher order divergences were absent. This enabled people to formulate perturbative non-renormalization theorems [16]. Eventually, however, one realized that non-renormalization could be seen as a consequence of the fact that supersymmetric actions are written in terms of holomorphic, or analytic, functions of the fields and the parameters of the theory [17]. This means that we can make statements of the renormalization properties of a supersymmetric theory also non-perturbatively¹¹. Let us start by discussing perturbation theory.

Generally, if a Lagrangian possesses a symmetry, this improves its renormalization properties. For instance, in QED, where the Lagrangian has a gauge symmetry, the Ward identities make Z_2 , the wave function renormalization factor of the electron (the term $Z_2\bar{\psi}\not{\partial}\psi$), equal to Z_1 , the renormalization factor of the gauge coupling (the term $Z_1 i\bar{\psi}\not{A}\psi$), just as it should be in order to preserve the gauge invariance of the combination $\bar{\psi}\not{D}\psi = \bar{\psi}\not{\partial}\psi + i\bar{\psi}\not{A}\psi$ under renormalization. A similar thing is true when the symmetry in question is a supersymmetry.

In perturbation theory, we can understand the nice renormalization properties of a supersymmetric theory in the following heuristic way. In Feynman diagrams, closed fermion loops have negative signs attached to them, whereas boson loops have positive signs. Because of supersymmetry the number of fermionic degrees of freedom is the same as the number of bosonic, so the contributions to, say, Green’s functions from loops tend to cancel out.

It is possible to set up perturbation theory in superspace in the sense that the propagating fields in the Feynman diagrams are superfields. Thus one single Feynman supergraph “contains” several ordinary diagrams. The machinery that is needed for this super-perturbation theory is

¹¹A short review on these matters are Ref. [2].

complicated and will not be dealt with in this report, but the lesson to be learned is that quantum corrections in the effective action¹² are of the form

$$\int d^4\theta \int d^4x_1 \cdots d^4x_n F_1(x_1, \theta, \bar{\theta}) \cdots F_n(x_n, \theta, \bar{\theta}) G(x_1, \dots, x_n),$$

i.e. one single integration over the entire superspace [16]. This means that the parts of the classical, or tree level, action that are given by chiral integrals over superspace is not renormalized in perturbation theory. If we consider the Wess-Zumino model with a single chiral superfield Φ ,

$$\mathcal{L}_{WZ} = \int d^4\theta \Phi^\dagger \Phi + \int d^2\theta \left(\frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3 \right) + h.c.,$$

we see that the mass and self-coupling term is expressed by such a chiral integration. When we include the quantum corrections, this goes into

$$Z_\Phi \int d^4\theta \Phi^\dagger \Phi + \int d^2\theta \left(\frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3 \right) + h.c.,$$

where Z_Φ is the wave function renormalization factor, so by rescaling the field Φ we have

$$m \rightarrow Z_\Phi^{-1} m \quad \text{and} \quad g \rightarrow Z_\Phi^{-3/2} g.$$

From standard renormalization theory we know that wave function renormalizations are only logarithmically divergent, and so the renormalized mass and coupling constant diverges as a logarithm (or as a power of a logarithm).

3.2 The Wilsonian effective action

As we have already mentioned, the renormalization properties of supersymmetric theories, e.g. the non-renormalization theorems in perturbation theory, can be seen in a much wider perspective. This has to do with the structure of supersymmetric theories, in the sense that Lagrangians often can be expressed in terms of holomorphic functions [17, 2]. For example, in the Wess-Zumino model, the superpotential

$$\int d^2\theta \left(\frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3 \right) \tag{46}$$

is a function of Φ only and not a function of the conjugate field Φ^\dagger . The converse is true for the “h.c.”-term. But this is the definition of a holomorphic function:

$$\frac{\partial}{\partial \bar{z}} f(z, \bar{z}) = 0 \Leftrightarrow f \text{ is holomorphic.}$$

Eq. (46) can therefore be written as

$$\int d^2\theta f(\Phi),$$

where

$$f(z) = \frac{1}{2} m z^2 + \frac{1}{3} g z^3.$$

These circumstances form the basis of powerful non-renormalization theorems which are not restricted to perturbation theory but contains the perturbative theorems as special cases. It is important to understand that the non-renormalization theorems are statements about the so-called Wilsonian effective action, which we will call S_W , rather than the more familiar effective

¹²‘Effective action’ is to be understood as Wilson’s effective action, see below.

action, the one-particle irreducible (1PI) action, usually called Γ , [18]. We will therefore explain what the Wilson action is, and what the difference between that and the 1PI-action is.

First of all, we are interested in calculating physical quantities such as scattering amplitudes, i.e. Green's functions. We do this by regularizing our theory by introducing a large momentum cutoff Λ . The Green's functions are then given in perturbation theory by expanding on all possible Feynman diagrams and loop integrations are carried out only for momenta p up to the cutoff: $0 \leq p \leq \Lambda$. If we use the Feynman rules derived from the classical (or tree level) Lagrangian for this, we are going to get results that depend on Λ in such a way that they diverge as Λ is taken to infinity. However, if we are dealing with renormalizable theories, it is possible to add counterterms to the classical Lagrangian so that we can define the *bare* Lagrangian

$$\mathcal{L}_{bare}(\Lambda) = \mathcal{L}_{tree} + \text{counterterms}$$

This new, bare Lagrangian has the same form as the tree level Lagrangian, but the coefficients in \mathcal{L}_{bare} , such as coupling constants, are functions of Λ in such a way that the Green's functions calculated from the corresponding Feynman rules are independent of Λ . Λ can then be taken to infinity without the Green's functions diverging. The bare action $S(\Lambda)$ is defined as the spacetime integral of $\mathcal{L}_{bare}(\Lambda)$:

$$S(\Lambda) = \int d^4x \mathcal{L}_{bare}(\Lambda).$$

We should recall that an action is really a functional of a field configuration: $S(\Lambda)[\eta]$, where η is the field¹³. A field η has a decomposition in terms of modes, or frequencies, but when the theory is completely specified by giving a cutoff Λ , the field η must contain only modes with frequencies less than or equal to Λ . Alternatively, a field that is composed of a single mode of frequency greater than Λ has action $S(\Lambda)$ equal to zero. The Wilsonian effective action S_W is defined at a low energy μ . Usually we have $\mu \ll \Lambda$. It is related to S by the path integral

$$e^{iS_W(\mu)[\phi]} = \int_{\mu < p \leq \Lambda} d\eta e^{iS(\Lambda)[\phi+\eta]} \quad (47)$$

where p is the energy modes of the paths in the path integral and ϕ is the field configuration on which S_W is evaluated. ϕ contains only modes with energy less than μ . η is a field which contains modes between μ and Λ , and thus the high energy modes have been “integrated out” to give an effective description of the physics at low energies. At low energies, $S_W(\mu)$ contains the same amount of information as $S(\Lambda)$.

Perturbatively we can think of S_W as having an expansion in terms of Feynman diagrams, just like Γ , but loop momenta run only from μ to Λ . Thus the Wilsonian effective action contains the same “structures” as the 1PI effective action, but the loop momenta of the latter run from *zero* to Λ . The 1PI-action is also a function of a low energy scale μ , namely the renormalization point. The 1PI-action contains physical quantities such as coupling constants directly. If we want to extract this information from S_W we still need to do loop integrals where $0 \leq p \leq \mu$. Equivalently, this amounts to doing the “residual” path integral over the low energy modes.

The great benefit of working with S_W rather than Γ , for the low energy effective description, is the following: if S can be expressed in terms of holomorphic functions so can S_W . This is true because the process of extracting the Wilsonian action from the original action, as in (47), can be done in a continuous way. The range of high energy modes from μ to Λ can be divided into infinitesimal parts and then be path integrated over one by one. Clearly this conserves properties of the action such as holomorphy. The same thing is not true for Γ , where infrared effects can

¹³Technically speaking, one should distinguish between the concept of a ‘field’ and the concept of a ‘field configuration’. We use the symbol ‘ η ’ for both in the text, trusting it will not cause any confusion.

introduce “holomorphic anomalies” [18]. Such effects can be traced back to the presence of massless interacting particles. In fact, in the absence of massless interacting particles the two effective actions are the same. Since holomorphic functions are much more restricted than non-holomorphic functions, we may sometimes be able to determine the effective action S_W *exactly*. This means that we have found all the quantum corrections to the tree level action to all orders in perturbation theory and otherwise (i.e. non-perturbative effects).

3.3 Holomorphy and non-renormalization

We are now in condition to state the non-perturbative equivalent to the non-renormalization theorem. This is a statement about the effective superpotential W_{eff} in the Wilsonian action. The Wilsonian action S_W can be defined at any energy scale $\mu \leq \Lambda_{cutoff}$, but usually μ will be some dynamical energy scale that exists in the theory: $\mu = \Lambda_{dynamical}$. For example this can be the scale set by the expectation value of a Higgs field. We will simply call this scale ‘ Λ ’, as the cutoff-energy is considered to be infinitely large. W_{eff} is then a function of the fields Φ_i of the theory, the coupling parameters λ_I and the dynamical scale Λ :

$$W_{eff} = W_{eff}(\Phi_i, \lambda_I, \Lambda).$$

Here, i and I are indices numbering the various fields and coupling constants. We use the terminology that ‘coupling constants’ includes masses. Other examples are gauge coupling constants or θ -angles that measure CP -violation. We will find it useful to think of coupling constants as (possibly complex) background fields. These fields will be full dynamical fields if we embed our theory in some hypothetical high energy theory. At the energies where our theory is a relevant description of the physics, these fields are “frozen” in their “vacuum” expectation values, and thus appear to be constants. A well known example of a coupling constant that is sometimes treated as a field is the axion, or Peccei-Quinn field, whose vacuum expectation value is a θ -angle. The statement of the theorem is now that W_{eff} possesses the following properties [17]:

1. Holomorphy: W_{eff} is holomorphic in Φ_i , λ_I and Λ .
2. Symmetries: We can assign transformation laws to the coupling constants such that the tree level Lagrangian gets an enlarged symmetry. The effective Lagrangian must then also be invariant under this enlarged symmetry. Anomalies can be taken care of in a similar way by treating the scale Λ as a background field and assign appropriate transformation laws to it.
3. Various limits: The behaviour of W_{eff} can be determined for asymptotic values of its arguments, e.g. the weak coupling limit.

Note that this “new” non-renormalization theorem works in a positive way, in the sense that it narrows down the possible form of the effective superpotential, and it is often possible to determine this *exactly*. Note also that although we have formulated it that way, the non-renormalization theorem works for *any* chiral structure of the Lagrangian, like the gauge kinetic terms, and not just the superpotential.

Let us use this on the simplest version of the Wess-Zumino model, that is, the one with a single chiral field Φ [17]. In the previous section we saw that W is not renormalized in perturbation theory (except for the common wave function factor). The tree level superpotential is

$$W_{tree} = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3.$$

The theory has an R -symmetry if the parameter g is charged under this symmetry. The point is that W must have R -charge 2 if $\int d^2\theta W$ is to be invariant. We can also identify another $U(1)$ symmetry. All in all, the various charges are as follows:

	$U(1)$	\times	$U(1)_R$
Φ	1		1
m	$\Leftrightarrow 2$		0
g	$\Leftrightarrow 3$		$\Leftrightarrow 1$

The fact that m and g appear as coupling *constants* in our theory means that the symmetries under which they are charged are “spontaneously broken”. It is easy to see that the most general form of the effective superpotential that satisfies the symmetries and holomorphy is

$$W_{eff} = \frac{1}{2}m\Phi^2 f\left(\frac{g\Phi}{m}\right),$$

where f is a holomorphic function. Since f is analytic, we can expand it in a power series. This cannot contain negative powers of g because then the weak coupling limit of W_{eff} would not be well behaved. Thus we have

$$W_{eff} = \sum_{n=0}^{\infty} a_n \frac{1}{m^{n-1}} g^n \Phi^{n+2}.$$

The n 'th term of this expansion has the interpretation of a tree diagram with $n+2$ external legs, n vertices and $n \Leftrightarrow 1$ propagators. For $n > 1$ this is not 1PI and it should not be included in the effective superpotential. Note that structures that are absent from , must also be absent from S_W , while the converse is in general not true as a consequence of the possible “holomorphic anomalies” previously mentioned. To conclude, the effective superpotential is

$$W_{eff} = \frac{1}{2}m'\Phi^2 + \frac{1}{3}g'\Phi^3 = W_{tree},$$

and so the superpotential is not renormalized neither perturbatively nor non-perturbatively.

3.4 SUSY QCD and moduli spaces

Supersymmetric QCD, or SUSY QCD for short, is the supersymmetric version of a theory with a local color symmetry and a global flavor symmetry. It offers a more complicated example of the use of the non-renormalization theorem [19, 20]. It will also serve to introduce the concept of ‘moduli space’.

In SUSY QCD it is customary to leave the number of colors N_c and the number of flavors N_f unspecified. Expressing the quantities of the theory as functions of these parameters will then give us information on the theory. We will see an example of this below, where we also will realize that SUSY QCD has nothing to do with real QCD – it should be considered as a toy model.

The theory has $SU(N_c)$ as gauge group, with N_f chiral superfields in the fundamental representation and N_f in the antifundamental representation, i.e. the representation that is carried by the fields T from the section on gauge theories in the last chapter:

$$\begin{aligned} W_\alpha &= \Leftrightarrow i\lambda_\alpha \Leftrightarrow \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} \Leftrightarrow \cdots, \\ Q_i &= q_i + \theta\psi_i + \cdots, \\ \tilde{Q}_{\tilde{i}} &= \tilde{q}_{\tilde{i}} + \theta\tilde{\psi}_{\tilde{i}} + \cdots, \quad i, \tilde{i} = 1, \dots, N_f \end{aligned}$$

The color indices are not displayed. The superfield W_α consists of the gluinos λ_α and the gluon field strength $F_{\mu\nu}$, while the superfield Q_i consists of the “squarks” q_i and quarks ψ_i . The

superfield \tilde{Q}_i has the corresponding antisquarks and antiquarks as components. In the absence of explicit mass terms the classical Lagrangian is

$$\mathcal{L} = \int d^4\theta (\bar{Q} e^V Q + \bar{\tilde{Q}} e^{-V} \tilde{Q}) + \frac{1}{4} \int d^2\theta W^{\alpha\alpha} W_\alpha^\alpha + h.c.$$

Our task is now to determine the effective superpotential W_{eff} of this theory by using the non-renormalization theorem.

Before we do that, however, let us make a digression about scalar potentials and spontaneous symmetry breaking in supersymmetric theories. In a sense this is a continuation of Chapter 2 on the elements of supersymmetry. From kinetic Lagrangians of the type $\int d^4\theta \bar{\Phi}_i e^V \Phi_i$ we get the term $F_i^{a*} F_i^a$, where F_i^a are the auxiliary fields in Φ_i^a , see e.g. eq. (28). F_i^a also enters in the superpotential $\int d^2\theta W(\Phi_i)$, if there is one. Let us analyze this. Suppose that W is *any* analytic function of Φ_i so that it has the power series expansion

$$W(\Phi_i) = \sum_n c_{i_1 \dots i_N} \Phi_{i_1} \cdots \Phi_{i_N}.$$

We are thinking of effective Lagrangians and not just bare Lagrangians. Two of the terms in the expansion are given by eqs. (26) and (27). We can from this induce that the general form of W is

$$W(\Phi_i) = W(A_i) + \theta \psi_i \frac{\partial W(A_i)}{\partial A_i} + \theta \theta \left[F_i \frac{\partial W(A_i)}{\partial A_i} \Leftrightarrow \frac{1}{4} \psi_i \psi_j \frac{\partial^2 W(A_i)}{\partial A_i \partial A_j} \right],$$

when expanded on the spinor coordinates. What we then learn is that the equation of motion for F_i^{a*} is

$$F_i^* = \Leftrightarrow \frac{\partial W(A_i)}{\partial A_i},$$

so that the contribution to the scalar potential $V(A_i)$ from the superpotential W is:

$$V(A_i) = \sum_i \left| \frac{\partial W}{\partial A_i} \right|^2.$$

There is also a contribution that originates from the gauge kinetic term $\int d^2\theta W^\alpha W_\alpha$, namely the component

$$\frac{1}{2} \sum_a D^{a2},$$

see e.g. eq. (43). The auxiliary fields D^a also appear in the matter kinetic terms. This has the result that the equation of motion is

$$D^a = \Leftrightarrow g \sum_i A_i^* T_{(i)}^a A_i, \tag{48}$$

where $T_{(i)}^a$ are the generators of the gauge symmetry in the representation of the field A_i . All in all the complete scalar potential is

$$V(A_i) = \sum_i \left| \frac{\partial W}{\partial A_i} \right|^2 + \frac{1}{2} g^2 \sum_a \left(\sum_i A_i^* T_{(i)}^a A_i \right)^2.$$

The presence of such a scalar potential in a theory may break the gauge or global symmetries, and we need to know how this affects the supersymmetry. In this report we are interested in effective theories with manifest supersymmetry at low energies. It is therefore necessary that

supersymmetry is not spontaneously broken. Spontaneously broken supersymmetry means that the vacuum is *not* annihilated by some of the generators of supersymmetry:

$$Q_\alpha|0\rangle \neq 0, \quad \bar{Q}_{\dot{\alpha}}|0\rangle \neq 0, \quad \text{some } \alpha \text{ or } \dot{\alpha}.$$

The point is now that spontaneous breaking of supersymmetry does not happen if and only if the minimum of the scalar potential is *exactly zero*. This can be seen from eq. (8). When sandwiched between two vacuum states this gives

$$\langle E \rangle = \frac{1}{4} \sum_{\alpha} \|Q_\alpha|0\rangle\|^2,$$

which proves the statement. Thus the spontaneous breaking of a gauge or global internal symmetry and the spontaneous breaking of supersymmetry are independent phenomena. This ends our digression.

We return now to SUSY QCD. At the tree level there is no superpotential so the only contribution to the potential for the scalar fields q_i and \tilde{q}_i is the gauge D -terms of eq. (48). They turn out to be

$$D^a = g\bar{q}_i T^a q^i \Leftrightarrow g\bar{\tilde{q}}_i T^a \tilde{q}^i.$$

The potential that is built from this is positive, so a zero of the potential is a minimum. Thus we need to solve the equation $D^a = 0$ to find the minima of the potential. To cut a long story short we will only give the results. By using our freedom to perform gauge and global symmetry transformations the results can be written in the form

$$q = \tilde{q} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_f} \end{pmatrix},$$

for $N_f < N_c$, and

$$q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} \tilde{a}_1 & & & \\ & \tilde{a}_2 & & \\ & & \ddots & \\ & & & \tilde{a}_{N_c} \end{pmatrix},$$

for $N_f \geq N_c$. The notation here is that the color indices run along the columns of the matrices and the flavor indices run along the rows. Blank spaces means zero and the a 's and \tilde{a} 's are *arbitrary* complex numbers. The fact that the zeros of the potential – i.e. the vacua of the theory – are characterized by arbitrary numbers is very important. Note that the degeneracy of the ground states of the theory that is a consequence of this arbitrariness has nothing to do with the symmetries of the theory. This space of physically inequivalent ground states parametrized by the a 's and \tilde{a} 's is the *moduli space*, sometimes called the *flat directions* of the potential. The actual ground state that the system chooses is not decided within the theory. In this sense the moduli space is part of the *parameter space* of the theory.

Thus far we have been working at the classical level. We now turn to the quantum theory. We are in particular interested in what happens to the degeneracy of the classical moduli space when quantum corrections are included. To find out we must use the non-renormalization theorem to see if a superpotential is generated that lifts this degeneracy (which would make the degeneracy “accidental”). In the spirit of our non-renormalization theorem we must identify

the symmetries of the theory. Classically, the global symmetry is $U(N_f)_L \times U(N_f)_R \times U(1)_X$, where the $U(N_f)_L \times U(N_f)_R$ is a QCD-type symmetry that rotates separately the Q and \tilde{Q} fields. The two $U(1)$'s in this are a vector and axial vector symmetry, respectively. $U(1)_X$ is an R -symmetry that acts by

$$\begin{aligned} W_\alpha(\theta) &\rightarrow e^{-i\alpha} W_\alpha(e^{i\alpha}\theta) \\ Q(\theta) &\rightarrow Q(e^{i\alpha}\theta) \\ \tilde{Q}(\theta) &\rightarrow \tilde{Q}(e^{i\alpha}\theta) \end{aligned}$$

At the level of the component fermions, this is

$$\begin{aligned} \lambda &\rightarrow e^{-i\alpha} \lambda \\ \psi &\rightarrow e^{i\alpha} \psi \\ \tilde{\psi} &\rightarrow e^{i\alpha} \tilde{\psi} \end{aligned}$$

The point is now that this is a γ_5 -symmetry in the language of four-spinors. For instance, in the case of the Majorana spinor Λ built from λ :

$$\Lambda = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\alpha} \lambda_\alpha \\ e^{i\alpha} \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} = e^{-i\alpha\gamma_5} \Lambda.$$

This is a chiral symmetry, which is anomalous. The divergence of the current that belongs to this symmetry is given by the standard expression

$$\partial_\mu J_5^\mu = \frac{g^2 N}{16\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu},$$

where N is the number of fermions that contributes to the anomaly. For the $U(1)_X$ -symmetry this number is $N_f \Leftrightarrow N_c$. The ' N_f ' is the number of Dirac spinors that is constructed from the ψ_i and $\tilde{\psi}_{\tilde{i}}$. The ' N_c ' is associated with the gluinos and is found by a proper investigation of the relevant triangle graphs. There is a ' \Leftrightarrow ' in front of ' N_c ' because the λ 's and the ψ 's and $\tilde{\psi}$'s are oppositely charged under $U(1)_X$. A similar anomaly exists for the axial vector symmetry $U(1)_A$. We can actually make an anomaly free $U(1)_R$ -symmetry from $U(1)_X$ and $U(1)_A$ by taking an appropriate linear combination of them so that

$$\partial_\mu J_R^\mu = 0.$$

Without going into details, one then finds that this symmetry acts by

$$\begin{aligned} W_\alpha &\rightarrow e^{-i\alpha} W_\alpha(e^{i\alpha}\theta) \\ Q(\theta) &\rightarrow e^{i\alpha(N_c - N_f)/N_f} Q(e^{i\alpha}\theta) \\ \tilde{Q}(\theta) &\rightarrow e^{i\alpha(N_c - N_f)/N_f} \tilde{Q}(e^{i\alpha}\theta) \end{aligned}$$

Thus the global symmetry of the quantum theory is $SU(N_f) \times SU(N_f) \times U(1)_V \times U(1)_R$.

By the non-renormalization theorem we must now construct the most general superpotential that is holomorphic in Q_i and $\tilde{Q}_{\tilde{i}}$ and is invariant under the symmetries. This turns out to be [19, 20]:

$$W_{eff} = (N_c \Leftrightarrow N_f) \frac{\Lambda^{\frac{3N_c - N_f}{N_c - N_f}}}{(\det \tilde{Q} Q)^{\frac{1}{N_c - N_f}}}, \quad (49)$$

where Λ is the dynamical scale (like the familiar Λ_{QCD}) and the determinant is over the flavor indices. Only the numerical factor in front of this expression involves a choice, which corresponds

to a choice of renormalization scheme. The functional form of W_{eff} is deduced in the following way. The combination $\tilde{Q}_i Q_i$ (with contraction in the color indices) is needed for gauge invariance, while the determinant is needed to get invariance under flavor-rotations. W_{eff} should have $U(1)_R$ -charge 2, and this determines the power of $\det \tilde{Q} Q$. We are only left with the power of Λ , which must then be chosen so that the mass dimension is correct. From the non-renormalization theorem we now understand that if a superpotential is generated, it must have the form of eq. (49), or it is not generated at all.

If $N_f = N_c$, the expression (49) makes no sense because of the powers in the exponents. If $N_f > N_c$, the determinant vanishes identically. Either way, the potential (49) is not generated, and there is a quantum moduli space. For a generic point in the *classical* moduli space, all the symmetries of the theory are broken. In particular, there are massive gluons and gluinos because of the Higgs mechanism. These particles can then be “integrated out” of the low energy description by solving their equations of motion and inserting the solutions in the effective Lagrangian. At some points (or hypersurfaces) of the moduli space, however, like the origin, broken gauge symmetries are restored. The effective description where the gauge particles are integrated out then becomes singular at these points. These points are therefore singularities of the classical moduli space. One can investigate if similar properties exist for the *quantum* moduli space. If one does this, one finds that for $N_f = N_c$, the moduli space is smooth, or singularity free. For $N_f > N_c$ there are singularities, which in this case has the interpretation that some composite objects become massless. The fact that it is not the gauge particles that becomes massless means that the $SU(N_c)$ gauge symmetry is completely broken in the quantum theory!

If $N_f < N_c$ the superpotential *is* generated. For $N_f = N_c \Leftrightarrow 1$ it is generated by instantons, and for $N_f < N_c \Leftrightarrow 1$ by gluino condensation, where ‘gluino condensation’ means that $\langle \lambda^\alpha \lambda_\alpha \rangle \neq 0$. Supersymmetry is then broken, but the potential does not have a minimum: it slopes to zero at infinity. Then there are no ground states, which does not make sense in quantum field theory! This concludes our discussion of SUSY QCD.

3.5 $N = 2$ supersymmetry

In our pursuit of increasing complexity we have now come to the point where it is appropriate to discuss the $N = 2$ supersymmetric theory in more detail. This is the theory of our main interest. We have previously mentioned that the $N = 1$ gauge theory with massless matter fields in the adjoint has a “second supersymmetry”. This turns out to be a pure $N = 2$ gauge theory without matter.

The $N = 2$ theory can be developed in close analogy to the $N = 1$. We will not go into any details – merely state some results that will be relevant to us. Superspace is introduced with coordinates $(x^\mu, \theta^{\alpha i}, \bar{\theta}_{\dot{\alpha} j})$, where $i, j = 1, 2$ and the θ ’s anticommute

$$\{\theta^{\alpha i}, \theta^{\beta j}\} = \{\bar{\theta}_{\dot{\alpha} i}, \bar{\theta}_{\dot{\beta} j}\} = \{\theta^{\alpha i}, \bar{\theta}_{\dot{\beta} j}\} = 0.$$

In the general case, we would also have needed a complex coordinate corresponding to the central charges, but in the situations where the $N = 2$ superspace formalism is relevant these are always represented by zero. The supersymmetry generators Q_α^i and $\bar{Q}_{\dot{\alpha}}^i$ are then represented by

$$\begin{aligned} Q_\alpha^i &= \Leftrightarrow i \frac{\partial}{\partial \theta_i^\alpha} + (\sigma^\mu \bar{\theta})_\alpha^i \partial_\mu, \\ \bar{Q}_{\dot{\alpha}}^i &= i \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}^i} \Leftrightarrow (\theta \sigma^\mu)_{\dot{\alpha}}^i \partial_\mu \end{aligned}$$

and the differential operators D_α^i and $\bar{D}_{\dot{\alpha}}^i$ by

$$\begin{aligned} D_\alpha^i &= \frac{\partial}{\partial \theta_i^\alpha} \Leftrightarrow i(\sigma^\mu \bar{\theta})_\alpha^i \partial_\mu \\ \bar{D}_{\dot{\alpha}}^i &= \Leftrightarrow \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}^i} + i(\theta \sigma^\mu)_{\dot{\alpha}}^i \partial_\mu \end{aligned}$$

A superfield, which is a function on the $N = 2$ superspace, is in general not irreducible and should therefore be constrained. One possible constraint is the chiral constraint

$$\bar{D}_{\dot{\alpha}}^i \Psi(x^\mu, \theta^{\alpha i}, \bar{\theta}_{\dot{j}}^\beta) = 0. \quad (50)$$

A superfield Ψ which satisfies this is a function of the variables $y^\mu = x^\mu \Leftrightarrow i\theta^{\alpha i} \sigma_{\alpha\beta}^\mu \bar{\theta}_{\dot{i}}^\beta$ and θ_α^i because

$$\bar{D}_{\dot{\alpha}}^i y^\mu = 0 \quad \text{and} \quad \bar{D}_{\dot{\alpha}}^i \theta_\beta^j = 0.$$

Unlike the $N = 1$ case, however, an $N = 2$ chiral superfield is still reducible and we need to impose more constraints to make it irreducible. A good constraint turns out to be

$$D_i^\alpha D_{\alpha j} \Psi = \bar{D}_{\dot{\alpha} i} \bar{D}_{\dot{j}}^{\dot{\alpha}} \Psi^\dagger \quad (51)$$

in which case Ψ corresponds to the multiplet of Ex. 3 in the massless case of Sec. 2.1. By the word ‘corresponds’ we mean the following. The multiplet contains a vector field from which we can build a field strength. Ψ is the $N = 2$ supersymmetric generalization of this field strength. The particle content of Ψ is thus the vector A_μ , two spinors λ^i and one complex scalar C . In addition comes a triplet of auxiliary scalars C_A . Furthermore, Ψ , being a field strength will in general belong to the adjoint representation of a gauge group. In other words, it is Lie algebra valued. As a side-remark, let us mention that if we had followed a geometrical approach, we would have obtained eq. (51) as a Bianchi identity.

We can write a chiral superfield as a power series expansion in θ_α^i . Because the θ ’s are anticommuting, there are only sixteen non-vanishing products of θ ’s (including unity). The non-trivial ones can be organized as [21]:

$$\begin{aligned} b_A(\theta) &= \Leftrightarrow \frac{1}{2} \theta_i^\alpha (\tau_A)_j^i \theta_\alpha^j, \quad A = 1, 2, 3 \\ a_{\mu\nu}(\theta) &= \frac{1}{4} \theta_i^\alpha (\sigma_{\mu\nu})_\alpha^\beta \theta_\beta^i, \\ \chi_\alpha^i(\theta) &= \frac{\partial}{\partial \theta_i^\alpha} u(\theta), \\ u(\theta) &= \theta_1^1 \theta_2^1 \theta_1^2 \theta_2^2. \end{aligned}$$

The τ_A -matrices are the Pauli-matrices which generate the $SU(2)$ -rotations of the Q ’s and θ ’s. The products b_A and $a_{\mu\nu}$ are obtained by tensor decomposition. The most general superfield Ψ which satisfies the constraints has the expansion

$$\begin{aligned} \Psi(y, \theta) &= C(y) + \theta_i^\alpha \lambda_\alpha^i(y) \Leftrightarrow b_A(\theta) C_A(y) \\ &\quad + a_{\mu\nu}(\theta) F^{\mu\nu}(y) + \chi_i^\alpha(\theta) \phi_\alpha^i(y) + u(\theta) D(y) \end{aligned}$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ C_A^* &= \Leftrightarrow C_A \\ \phi_\alpha^i &= \Leftrightarrow 2i(\sigma^\mu \partial_\mu \bar{\lambda})_\alpha^i \\ D &= \Leftrightarrow 4\Box C^* \end{aligned}$$

when the gauge group is abelian, and

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\
C_A^\dagger &= \bar{C}_A \\
\phi_\alpha^i &= \bar{\epsilon} 2i(\sigma^\mu D_\mu \bar{\lambda})_\alpha^i + i2g[\lambda_\alpha^i, C^\dagger] \\
D &= \bar{\epsilon} 4D_\mu D^\mu C^\dagger \bar{\epsilon} i2g\{\bar{\lambda}_\alpha^i, \bar{\lambda}_i^\alpha\} \bar{\epsilon} 4g^2[C^\dagger, [C^\dagger, C]] \\
D_\mu &= \partial_\mu + ig[A_\mu, \quad]
\end{aligned}$$

when it is non-abelian. The Lagrangians for the two cases is now obtained from the highest component, or u -component, of the superfields $\Psi\Psi$ and $\text{Tr}(\Psi\Psi)$, respectively, and their hermitean conjugates. In the non-abelian case it is

$$\mathcal{L} = \frac{1}{4}\text{Tr}(\Psi\Psi|_u + \bar{\Psi}\bar{\Psi}|_{\bar{u}}).$$

If we introduce volume elements in superspace so that¹⁴

$$\int d^4\theta u = 1, \quad \int d^4\bar{\theta} \bar{u} = 1,$$

we have

$$\mathcal{L} = \frac{1}{4} \int d^4\theta \text{Tr}(\Psi\Psi) + \frac{1}{4} \int d^4\bar{\theta} \text{Tr}(\bar{\Psi}\bar{\Psi}).$$

To find the component form we must go through a rather long calculation. We will only state the intermediate results¹⁵

$$\begin{aligned}
\Psi\Psi|_u &= 2CD + 2C_A C_A \bar{\epsilon} F_{\mu\nu} F^{\mu\nu} \bar{\epsilon} iF_{\mu\nu} \tilde{F}^{\mu\nu} + 2\lambda_i^\alpha \phi_\alpha^i, \\
\bar{\Psi}\bar{\Psi}|_{\bar{u}} &= 2C^\dagger D^\dagger + 2C_A C_A \bar{\epsilon} F_{\mu\nu} F^{\mu\nu} \bar{\epsilon} iF_{\mu\nu} \tilde{F}^{\mu\nu} \bar{\epsilon} 2\bar{\lambda}_{\dot{\alpha}i} \bar{\phi}^{\dot{\alpha}i}.
\end{aligned} \tag{52}$$

The final Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= \bar{\epsilon} \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + 2(D_\mu C^*)^a (D^\mu C)^a + i\bar{\lambda}_i^a \bar{\sigma}^\mu D_\mu \lambda^{ai} + \frac{1}{2} C_A^a C_A^a \\
&\quad \bar{\epsilon} 2ig \text{Tr} C \{\bar{\lambda}_i, \bar{\lambda}^i\} \bar{\epsilon} 2ig \text{Tr} C^\dagger \{\lambda_i, \lambda^i\} \bar{\epsilon} 4g^2 \text{Tr}[C, C^\dagger]^2
\end{aligned} \tag{53}$$

The fields C_A are auxiliary and they vanish by their equations of motion in the pure Yang-Mills theory. In the abelian case we have

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8} \int d^4\theta \Psi\Psi + \frac{1}{8} \int d^4\bar{\theta} \bar{\Psi}\bar{\Psi} \\
&= \bar{\epsilon} \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\lambda}^i \bar{\sigma}^\mu \partial_\mu \lambda_i + 2\partial_\mu C^* \partial^\mu C + \frac{1}{2} C_A C_A
\end{aligned} \tag{54}$$

Note that none of these fields are charged under the $U(1)$ group, whereas in the non-abelian case, the fields carry “adjoint charges” and are minimally coupled to the vector field.

Let us record how this looks in the $N = 1$ formalism. If we organize the component fields into the super-field strength $W^a = (\lambda^a, F_{\mu\nu}^a, \dots)$ and the chiral “matter” field $\Phi^a = (\phi^a, \psi^a, \dots)$, where we have defined the spinors $\lambda^a \equiv \lambda_1^a$ and $\psi^a \equiv \lambda_2^a$ and the scalar $\phi^a \equiv \sqrt{2}C^a$, then the Lagrangian is

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \Phi^a (e^{gV})_{ab} \bar{\Phi}^b + \frac{1}{4} \int d^2\theta W^a W^a + h.c. \tag{55}$$

¹⁴There are now two different meanings of ‘ $d^4\theta$ ’ depending on whether we are dealing with $N = 1$ or $N = 2$ superspace. We hope this will not cause any confusion.

¹⁵More details are given in Appendix A of Ref. [22].

Of course, the Φ 's are not matter superfields in the $N = 2$ sense: they are the “superpartners” of the W 's. This is reflected in the absence of a mass term and superpotential for the Φ 's.

We must also mention the $N = 2$ massive multiplet, or the *hypermultiplet* as it is sometimes called. It will suffice to use the $N = 1$ formalism, in which case the hypermultiplet is given by two chiral superfields:

$$\begin{aligned} Q &= q + \theta\psi_q + \theta\theta F, \\ \tilde{Q} &= \tilde{q}^\dagger + \theta\psi_q^\dagger + \theta\theta\tilde{F}^\dagger. \end{aligned} \tag{56}$$

As usual, the q 's are complex scalars, the ψ 's are two component fermions and the F 's are complex auxiliary scalars. There are two comments to be made about this. First, the hypermultiplet always appears in the context of gauge theories. Q then carries a representation of the gauge group and \tilde{Q} carries the conjugate representation, hence the daggers in eq. (56). It has canonical kinetic terms,

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left(\bar{Q}^a (e^{gV})_{ab} Q^b + \bar{\tilde{Q}}^a (e^{-gV})_{ab} \tilde{Q}^b \right),$$

and it couples to the $N = 2$ gauge superfield Ψ through the $N = 1$ chiral field Φ by the superpotential

$$W = g\tilde{Q}^a \Phi_{ab} Q^b.$$

Φ transforms in the adjoint of the gauge group so this works for any representation. A mass term is given by

$$m\tilde{Q}^a Q^a.$$

Second, there are eight particle states in the multiplet. In Sec. 2.1 we saw that “ordinary” massive $N = 2$ multiplets have $2^4 = 16$ states. The explanation can only be that the hypermultiplet carries a representation of a central charge and so must belong to a *small representation*. This means that $Z = M$ in eq. (17) (there is only one ‘ r ’ for $N = 2$), the algebra becomes effectively that of the massless case, and thus the multiplet we are dealing with is actually the one of Ex. 4. This multiplet has four helicity states, but to represent the central charge we need two such multiplets, giving a total of eight states.

Let us finally say that the $N = 2$ chiral multiplet Ψ does not have a central charge. This follows directly from the chiral constraint (50) and the anticommutation relations for the \bar{D} 's, which are isomorphic to the ones for the \bar{Q} 's. That is, we get an equation of the type

$$D_z \Psi = 0,$$

where D_z is a differential operator that represents the central charges. Ψ is the only superfield we will see in the $N = 2$ superspace formalism, which is why we did not need to represent the central charges by differential operators.

3.6 The $SU(2)$ Yang-Mills theory

Let us now concentrate on the gauge group $SU(2)$. Our aim is to extract information from it by the methods developed in the previous sections. Before we do that, however, let us rewrite the Lagrangian so that it becomes

$$\mathcal{L} = \frac{1}{8g^2} \int d^4\theta \Psi^a \Psi^a + h.c. \tag{57}$$

That is, we have taken the factor $1/g^2$ outside. This can be achieved by rescaling the gauge fields so that

$$D_\mu = \partial_\mu + i[A_\mu, \quad],$$

and the spinors and scalars so that $\lambda_i \rightarrow \lambda_i/g$ and $C \rightarrow C/g$. In the corresponding $N = 1$ formalism this is

$$\mathcal{L} = \frac{1}{g^2} \left[\int d^2\theta d^2\bar{\theta} \Phi^a (e^V)_{ab} \bar{\Phi}^b + \frac{1}{4} \int d^2\theta W^a W^a + h.c. \right],$$

and in components:

$$\begin{aligned} \mathcal{L} = & \frac{1}{g^2} \left[\Leftrightarrow \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + (D_\mu \phi^*)^a (D^\mu \phi)^a + i \bar{\lambda}_i \bar{\sigma}^\mu D_\mu \lambda^i \right. \\ & \left. \Leftrightarrow \sqrt{2} i \text{Tr} \phi \{ \bar{\lambda}_i, \bar{\lambda}^i \} \Leftrightarrow \sqrt{2} i \text{Tr} \phi^\dagger \{ \lambda_i, \lambda^i \} \Leftrightarrow \text{Tr} [\phi, \phi^\dagger]^2 \right], \end{aligned}$$

where we have used $\phi^a = \sqrt{2} C^a$. The point in writing $1/g^2$ as a factor outside will become clear in a moment.

We want to find the quantum corrections to eq. (57), that is, we want to find the effective Lagrangian at low energies. The crucial observation is now that the Lagrangian (57) involves a chiral integral over $N = 2$ superspace and that the integrand is a holomorphic function of the fields, namely

$$\frac{1}{8g^2} z^a z^a, \quad z^a \in \mathbf{C}^3.$$

Thus, by the definition of the Wilsonian action, the effective Lagrangian must have the form [23]

$$\mathcal{L}_{eff} = \int d^4\theta \mathcal{F}(\mathcal{A}^a) + h.c., \quad (58)$$

where \mathcal{A}^a is the $N = 2$ super-field strength that corresponds to Ψ^a at low energies, and \mathcal{F} is a holomorphic function. This function “describes” the theory, and is the object we are going to determine. We can also write this in the $N = 1$ formalism in the following way:

$$\mathcal{L}_{eff} = \int d^2\theta d^2\bar{\theta} 2K(A^a, \bar{A}^a, V^a) + \int d^2\theta \frac{1}{2} f_{ab}(A^a) W^a W^b + h.c., \quad (59)$$

where A^a , V^a and W^a are the low energy chiral, vector and field strength superfields, respectively, and

$$\begin{aligned} K &= 2\partial^a \mathcal{F}(e^V)_{ab} \bar{A}^b, \\ f_{ab}(A^a) &= 2\partial_a \partial_b \mathcal{F}. \end{aligned}$$

This is trivially true in the special case when the Wilson action cutoff μ is equal to the bare cutoff Λ_{cutoff} . Then we have

$$\mathcal{F}(\Psi^a) = \frac{1}{8g^2} \Psi^a \Psi^a$$

and so

$$\begin{aligned} K &= \frac{1}{2g^2} \Phi^a (e^V)_{ab} \bar{\Phi}^b, \\ f_{ab} &= \frac{1}{2g^2} \delta_{ab}. \end{aligned}$$

To see that it is true in the general case, we could expand the two Lagrangians (58) and (59) on their component fields, like we did with the superpotential W in Sec. 3.4, and then compare coefficients. Note that the coefficient f_{ab} of the term $W^a W^b$ is related to the holomorphic function \mathcal{F} of the fields of the theory. In the bare Lagrangian the coefficient is $(1/g^2)\delta_{ab}$. This means that the effective coupling constant at low energies is related to the holomorphic function \mathcal{F} . In fact, this was the reason for writing $1/g^2$ as an overall factor in the bare Lagrangian.

To proceed, we must determine the symmetries of the theory. Apart from the $SU(2)$ gauge symmetry there is a global $SU(2)_R \times U(1)_\mathcal{R}$ R -symmetry. The $SU(2)_R$ acts on the two θ 's by rotating them, while the $U(1)_\mathcal{R}$ acts by multiplying them with a phase. In the $N = 1$ formalism only one of the three generators of $SU(2)_R$ is manifest. The symmetry it generates will be called $U(1)_J$. In terms of superfields $U(1)_J$ and $U(1)_\mathcal{R}$ acts by

$$\begin{aligned} U(1)_J : \quad \Phi(\theta) &\rightarrow \Phi(e^{-i\alpha}\theta), & W_\alpha(\theta) &\rightarrow e^{i\alpha}W_\alpha(e^{-i\alpha}\theta) \\ U(1)_\mathcal{R} : \quad \Phi(\theta) &\rightarrow e^{2i\alpha}\Phi(e^{-i\alpha}\theta), & W_\alpha(\theta) &\rightarrow e^{i\alpha}W_\alpha(e^{-i\alpha}\theta) \end{aligned}$$

and at the component level this is

$$\begin{aligned} U(1)_J : \quad \phi &\rightarrow \phi \\ \psi &\rightarrow e^{-i\alpha}\psi \\ \lambda &\rightarrow e^{i\alpha}\lambda \\ \\ U(1)_\mathcal{R} : \quad \phi &\rightarrow e^{2i\alpha}\phi \\ \psi &\rightarrow e^{i\alpha}\psi \\ \lambda &\rightarrow e^{i\alpha}\lambda \end{aligned}$$

One can show from triangle graphs that the $U(1)_\mathcal{R}$ symmetry is anomalous. The divergence of the corresponding current is¹⁶

$$\partial_\mu J^\mu = \frac{1}{4\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}, \quad (60)$$

and non-perturbative effects breaks $U(1)_\mathcal{R}$ down to \mathbf{Z}_8 . The $U(1)_J$ and $U(1)_\mathcal{R}$ transformations with $\alpha = \pi$ are the same. Thus the global symmetry of the quantum theory is $(SU(2)_\mathcal{R} \times \mathbf{Z}_8)/\mathbf{Z}_2$.

In the classical Lagrangian, the scalar field ϕ^a in Ψ^a has the potential

$$V(\phi) = \frac{1}{g^2} \text{Tr}[\phi, \phi^\dagger]^2. \quad (61)$$

This is positive because¹⁷

$$\text{Tr}[\phi, \phi^\dagger]^2 = \text{Tr} \left(\phi^a \phi^{*b} \left[\frac{1}{2}\sigma^a, \frac{1}{2}\sigma^b \right] \right)^2 = \Leftrightarrow \frac{1}{2}(\epsilon^{abc} \phi^b \phi^{*c})^2 = \frac{1}{2}|\epsilon^{abc} \phi^b \phi^{*c}|^2 \geq 0$$

The ground state of the system is thus given by a constant ϕ that makes the potential (61) vanish, that is, for which

$$[\phi, \phi^\dagger] = 0. \quad (62)$$

¹⁶The familiar number of flavors N_f in the expression

$$\partial_\mu J^\mu = \frac{N_f}{16\pi^2} F \tilde{F}$$

is replaced by $2N_c$ when there are two “flavors” of fermions in the adjoint of $SU(N_c)$. In the $SU(2)$ case this number is 4.

¹⁷Recall that for $SU(2)$ we have $[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = i\epsilon^{abc}\frac{\sigma^c}{2}$ and $\text{Tr}(\frac{\sigma^a}{2})(\frac{\sigma^b}{2}) = \frac{1}{2}\delta^{ab}$, where the $\frac{\sigma^a}{2}$'s are the generators.

One value of ϕ which satisfies this is

$$\phi = \frac{1}{2}a\sigma^3 = (0, 0, a), \quad (63)$$

where a is an arbitrary complex number. Any other ϕ that satisfy eq. (62) can be brought on the form (63) by a gauge transformation. This leaves us with a sign ambiguity in a because the transformation that acts on a in (63) by $a \rightarrow \pm a$ is a gauge transformation. Therefore, the complex number u , defined by

$$u = \frac{1}{2}a^2 = \text{Tr}\phi^2$$

is a gauge invariant parameter which labels the physically inequivalent ground states of the system. Thus the complex plane, parametrized by u , is a classical moduli space. The $U(1)_{\mathcal{R}}$ charge of a is 2 so that the $U(1)_{\mathcal{R}}$ charge of u is 4. \mathbf{Z}_8 acts then on u as $u \rightarrow \pm u$, i.e. as a \mathbf{Z}_2 . For non-zero values of a , both this \mathbf{Z}_2 and the $SU(2)$ gauge symmetry is spontaneously broken. The $SU(2)$ gauge group is broken to $U(1)$ because we have a ‘‘Higgs field’’ in the adjoint. This means that we have a Higgs mechanism where two of the three vector fields become massive along with their corresponding fermion superpartners. The ground state is a zero of the potential so by an extension of the argument in Sec. 3.4, $N = 2$ supersymmetry is not broken. Therefore the massless spectrum must be described by an abelian $N = 2$ multiplet $\Psi = \sqrt{\Psi^a \Psi^a}$, containing a photon A_μ , two uncharged fermions ψ and λ , and an uncharged scalar ϕ . If $a = 0$, the gauge symmetry is intact and all the particles of the theory are massless, so there is a ‘‘singularity’’ at the classical moduli space at $u = 0$.

For the quantum theory the most general form of the Lagrangian is given by eq. (59), which does not contain a superpotential. Therefore, the scalar potential is not renormalized (in the sense that only the coefficient $1/g^2$ receive quantum corrections), and there is a quantum moduli space. At a generic point in this moduli space the gauge symmetry is again spontaneously broken with a Higgs mechanism as a consequence. At low energies well below the masses of the gauge bosons, the effective Lagrangian describes a massless abelian $N = 2$ multiplet \mathcal{A} . The Lagrangian is that of eq. (54). Note that the Lagrangian does not contain massless interacting particles, and so the Wilsonian action is the same as the 1PI one.

Let us make a remark about the Lagrangian (59) which is expressed by the two functions K and f_{ab} . In the low energy theory, the Higgs field have picked out a direction in gauge space so that all quantities with an adjoint index points in this same direction. This means that $(e^V)_{ab} = \delta_{ab}$ because the vector superfield couples together different components in gauge space. As a consequence we get

$$\begin{aligned} K &= 2\partial_a \mathcal{F} \bar{A}_a \\ f_{ab} &= 2\partial_a \partial_b \mathcal{F}. \end{aligned}$$

The point is now that we get the direct relation

$$f_{ab} = \partial_a \bar{\partial}_b K \quad (64)$$

between the two. This is a useful property for the following reason. If we expand eq. (59) on component fields we find that the kinetic term of the complex scalar field ϕ^a is

$$f_{ab}(\partial_\mu \phi^*)^a (\partial^\mu \phi)^b.$$

This is a number and can be thought of as a *scalar product* in the sense that the ϕ^a takes values in a complex manifold on which f_{ab} is a *metric*. The fact that there is a function K such that f_{ab} is given by the relation (64) makes the complex manifold a *Kähler manifold* and the metric f_{ab} a *Kähler metric*. The function K , which is not holomorphic, is called a *Kähler potential*. In

Sec. 3.4 when we investigated SUSY QCD, we were able to discuss the singularities of the moduli space, which are properties of the *topology* of the moduli space. In the $N = 2$ Yang-Mills theory we are able to discuss the metric on the moduli space as well, which is a *geometric* property.

We now return to the determination of the holomorphic function \mathcal{F} . We will not finish the job in this section. In this section we will only find its *asymptotic form* for small values of the coupling g , which means we are considering perturbation theory. We must wait until the last chapter, where we will investigate the *singularities* of \mathcal{F} , to find the exact solution. \mathcal{F} is a function of the low energy effective fields, and in particular it is a function of the vacuum expectation value a of ϕ . In other words it is a function on the moduli space, where a is a local coordinate. Before we carry out the perturbative evaluation of \mathcal{F} , let us argue that small g is equivalent to large $|a|$. First of all, the theory has asymptotic freedom. Intuitively this is clear because there are too few fermions to turn the asymptotic behaviour of the theory¹⁸, but we will explicitly demonstrate this by calculating the β -function at the end of this section. The theory also has a Higgs mechanism and therefore we have dimensional transmutation in the sense that the vacuum expectation value of the Higgs field sets the scale of the low energy effective coupling constant. The point is that if we probe the physics of the bare, or microscopic, theory well above the scale of the Higgs field, the asymptotic freedom is operative and the coupling is small. As we go down in energy, the coupling constant grows until we reach energies of the order of the vacuum expectation value of the Higgs field. At this scale the heavy fields decouple, so the only fields that are left in the infrared are the ones of the neutral $N = 2$ multiplet. Because these fields are neutral, the coupling constant does not run in this range and is therefore “frozen” in its value at the scale of the Higgs field. Thus, if $|a|$ is much larger than the dynamical scale Λ of the theory, the coupling is small in the infrared. One might say we have asymptotic freedom without infrared slavery.

Now to the perturbative evaluation of \mathcal{F} . The $U(1)_{\mathcal{R}}$ symmetry is broken down to \mathbf{Z}_8 by non-perturbative effects, but as long as we only consider perturbation theory it is still intact. This symmetry acts on \mathcal{A}^a by

$$\mathcal{A} \rightarrow e^{2i\alpha} \mathcal{A}(e^{-i\alpha}\theta).$$

Now we can write down the most general form of the perturbative \mathcal{F}^{pert} that is holomorphic and has the correct symmetry properties. It is

$$\mathcal{F}^{pert} = \frac{1}{8g^2} \mathcal{A}^2 \left(A_1 + A_2 \ln \frac{\mathcal{A}^2}{\Lambda^2} \right),$$

where A_1 and A_2 are two constants to be determined. We can choose A_1 to be what we want by suitably defining the scale Λ . We will pick Λ so that $A_1 = 0$. A_2 is determined in the following way. The effective action is invariant under $U(1)_{\mathcal{R}}$, but the effective Lagrangian is not because there is an anomaly. Under a $U(1)_{\mathcal{R}}$ transformation we get

$$\begin{aligned} \mathcal{F}^{pert} &\rightarrow \frac{A_2}{8g^2} e^{4i\alpha} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda^2} + \frac{i\alpha A_2}{2g^2} e^{4i\alpha} \mathcal{A}^2, \\ \bar{\mathcal{F}}^{pert} &\rightarrow \frac{A_2}{8g^2} e^{-4i\alpha} \bar{\mathcal{A}}^2 \ln \frac{\bar{\mathcal{A}}^2}{\Lambda^2} \Leftrightarrow \frac{i\alpha A_2}{2g^2} e^{-4i\alpha} \bar{\mathcal{A}}^2. \end{aligned}$$

The effective Lagrangian thus transforms by

$$\delta \mathcal{L}_{eff}^{pert} = \frac{i\alpha A_2}{2g^2} \left(\int d^4\theta \mathcal{A}^2 \Leftrightarrow \int d^4\bar{\theta} \bar{\mathcal{A}}^2 \right).$$

¹⁸Recall that it takes 16.5 flavors of quarks to turn the asymptotic behaviour of QCD.

Because of the minus sign in these brackets, and because of eq. (52), we have

$$\delta\mathcal{L}_{eff}^{pert} = \frac{\alpha A_2}{g^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}.$$

This is supposed to give $\alpha\partial_\mu J^\mu$, so comparing with eq. (60) we get

$$A_2 = \frac{g^2}{4\pi^2},$$

and we obtain the result

$$\mathcal{F}^{pert}(\mathcal{A}) = \frac{1}{32\pi^2} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda^2}.$$

Finally, we will compute the perturbative β -function, as promised. By definition the β -function contains information about the running coupling constant as a function of the scale at which we probe the physics. In our case, however, the same information is encoded in the low energy effective coupling constant as a function of the scale set by the Higgs field. By differentiating \mathcal{F}^{pert} twice, we get

$$f^{pert} = 2 \frac{\partial^2 \mathcal{F}^{pert}(a)}{\partial a^2} = \frac{3}{8\pi^2} + \frac{1}{8\pi^2} \ln \frac{a^2}{\Lambda^2}.$$

From this we get that

$$\frac{1}{g_{eff}^{2,pert}(a)} = 2f^{pert} = \frac{3}{4\pi^2} + \frac{1}{4\pi^2} \ln \frac{a^2}{\Lambda^2}. \quad (65)$$

Differentiation with respect to $\ln(a/\Lambda)$ gives

$$\Leftrightarrow \frac{2}{g_{eff}^{3,pert}} \beta(g) = \frac{1}{2\pi^2},$$

so that

$$\beta(g) = \Leftrightarrow \frac{1}{4\pi^2} g^3 + \text{non-perturbative effects}.$$

The sign of the β -function is negative, and so the asymptotic freedom is demonstrated.

4 Duality

4.1 Maxwell duality

In Sec. 3.6 we brought the $N = 2$ Yang-Mills theory as far as we could by using “naïve” non-renormalization techniques. To take the final step towards an exact solution we need some more input by an interpretation of the physics of the theory. This is where duality comes in. We will be more specific about what we mean by duality later, but the kind of duality we are thinking about is a “high tech” version of the standard duality between electricity and magnetism in Maxwell’s equations without matter. We will start by discussing this relatively simple case.

Maxwell’s equations in vacuum are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, & \nabla \times \mathbf{B} &\Leftrightarrow \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{E} &+ \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \end{aligned} \quad (66)$$

They are invariant under the transformation

$$(\mathbf{E}, \mathbf{B}) \rightarrow (\mathbf{B}, \Leftrightarrow \mathbf{E}) \quad (67)$$

of the electric and magnetic fields. In other words, the transformation (67) is a *symmetry* of (66). (67) is a duality transformation since if it is applied twice we end up with what we started with¹⁹. For this reason the vacuum equations (66) are selfdual.

If we include electrically charged matter in the system, the situation is more complicated. For example, an electrical point charge of strength e at rest at the origin of the coordinate system has an electric field

$$\mathbf{E} = \frac{e}{4\pi} \frac{\mathbf{r}}{r^3}, \quad r = |\mathbf{r}|$$

which means that²⁰

$$\nabla \cdot \mathbf{E} = e\delta^{(3)}(\mathbf{r}).$$

This system is selfdual only if there are magnetically charged point particles of magnetic strength g :

$$\mathbf{B} = \frac{g}{4\pi} \frac{\mathbf{r}}{r^3},$$

such that

$$\nabla \cdot \mathbf{B} = g\delta^{(3)}(\mathbf{r})$$

in the rest frame of the particle, and (67) is accompanied by

$$e \rightarrow g, \quad g \rightarrow \Leftrightarrow e.$$

In relativistic notation the electric and magnetic fields are described by the Maxwell tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & \Leftrightarrow E^1 & \Leftrightarrow E^2 & \Leftrightarrow E^3 \\ E^1 & 0 & \Leftrightarrow B^3 & B^2 \\ E^2 & B^3 & 0 & \Leftrightarrow B^1 \\ E^3 & \Leftrightarrow B^2 & B^1 & 0 \end{pmatrix},$$

so that

$$E^i = F^{0i}, \quad B^i = \Leftrightarrow \frac{1}{2} \epsilon^{ijk} F^{jk}.$$

The Maxwell equations are

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \Leftrightarrow j^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= \Leftrightarrow k^\nu \end{aligned} \quad (68)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual of $F^{\mu\nu}$, obtained by the substitution (67) in $F^{\mu\nu}$, j^ν and k^ν are the electric and magnetic current, respectively. Under duality transformations we have

$$F^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} \rightarrow \Leftrightarrow F^{\mu\nu},$$

and

$$j^\mu \rightarrow k^\mu, \quad k^\mu \rightarrow \Leftrightarrow j^\mu,$$

¹⁹This holds up to a sign difference, which is not important because the Maxwell equations (66) are insensitive to that. Alternatively we could define the duality transformations to include multiplication with i .

²⁰We recall the identity $\nabla \cdot \frac{\mathbf{r}}{r^3} = 4\pi\delta^{(3)}(\mathbf{r})$.

which is a symmetry of eqs. (68).

Note that if we introduce a vector potential A_μ so that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the duality is lost, because the Bianchi identity is

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

This is relevant in the context of quantum mechanics, since the vector potential is then the natural field to use in the description of the physics. Nevertheless, a quantum theory which includes magnetically charged particles can be constructed. An attractive feature of such a theory is the famous Dirac quantization condition [24]:

$$eg = 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots \quad (69)$$

which implies quantization of electric charge. This formula, and its generalization to dyons, which are particles of both electric and magnetic charge, will be discussed in the next section.

4.2 Magnetic monopoles and dyons in quantum mechanics

In this section we will mainly consider the quantum mechanics of monopoles and dyons. That is to say, only at the end of the section will we refer to any internal structure that these particles might have²¹.

Let us begin by deriving the Dirac quantization condition (69) in a heuristic way. We will use an argument that originates from a paper by Saha in 1936 [26]. Suppose we have a magnetic charge of strength g at the origin and an electric charge of strength e at \mathbf{r} . The strategy will be to calculate the total angular momentum of the electromagnetic field according to Maxwell's theory. Then, by the further quantum mechanical requirement that angular momentum is quantized in half-integer units, we will obtain the quantization condition.

The momentum density of the electromagnetic field is given by the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

The total angular momentum of the system is then given by the space integral

$$\mathbf{L} = \int d^3x \mathbf{x} \times (\mathbf{E} \times \mathbf{B}).$$

By using the expression

$$\mathbf{B} = \frac{g}{4\pi} \frac{\mathbf{x}}{x^3}$$

for the magnetic field and the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}),$$

we get

$$\begin{aligned} \mathbf{L} &= \int d^3x \frac{g}{4\pi x^3} [\mathbf{E}(\mathbf{x} \cdot \mathbf{x}) - \mathbf{x}(\mathbf{x} \cdot \mathbf{E})] \\ &= \Leftrightarrow \int d^3x \frac{g}{4\pi} \mathbf{E} \cdot \nabla \left(\frac{\mathbf{x}}{x} \right) = \int d^3x \nabla \cdot \mathbf{E} \frac{g\mathbf{x}}{4\pi x} \\ &= \frac{eg}{4\pi} \frac{\mathbf{r}}{r}, \end{aligned} \quad (70)$$

²¹A useful review on magnetic monopoles that we have used throughout this chapter is [25].

where we have used that

$$\partial^i \hat{x}^j = \frac{\delta^{ij} \Leftrightarrow \hat{x}^i \hat{x}^j}{x}, \quad \hat{x}^i \equiv \frac{x^i}{x},$$

and

$$\nabla \cdot \mathbf{E} = e\delta^{(3)}(\mathbf{r}).$$

From the quantum mechanical requirement that \mathbf{L} is quantized in half-integer units along, say, the \mathbf{r} -axis:

$$\hat{\mathbf{r}} \cdot \mathbf{L} = \frac{1}{2}n, \quad n = 0, \pm 1, \pm 2, \dots$$

we obtain Dirac's condition:

$$eg = 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots \quad (71)$$

Note that if we make the somewhat bolder assumption that the angular momentum should be quantized in *integer* units (after all we are dealing with electromagnetism), then the quantization condition is

$$eg = 4\pi n. \quad (72)$$

This is a more restrictive constraint than eq. (71), which follows from quantum mechanics only.

An immediate consequence of eq. (71) is that the existence of magnetic monopoles would imply that electric charge was quantized. Even if there was only one single monopole in the entire universe, it would mean that *any* electric charge would be a multipum of $2\pi/g$, where g was the magnetic charge of this one monopole.

Suppose now that we introduced dyons into the theory, i.e. particles with electric and magnetic charge (e, g) . It is then possible to generalize the quantization condition (71). If we have two particles with electric and magnetic charges (e_1, g_1) and (e_2, g_2) , respectively, then a similar argument that led to Dirac's quantization condition would give us the dyonic quantization condition

$$e_1 g_2 \Leftrightarrow e_2 g_1 = 2\pi n. \quad (73)$$

If there are particles of electric charge $(e, 0)$ in the theory, then eq. (73) restricts the possible magnetic charge of a dyon of charges (q, g) by $eg = 2\pi n$. The electric charge q of the dyon is on the other hand not subjected to any restrictions. There is, however, a restriction on the *difference* between the electric charges of two dyons. By the existence of a particle with $(e, 0)$, there is a minimum allowed magnetic charge $g = 2\pi/e$. The electric charges of two dyons with (q, g) and (q', g) must then satisfy $(q \Leftrightarrow q')g = 2\pi n$, or

$$q \Leftrightarrow q' = ne, \quad (74)$$

i.e. the difference is quantized in units of e .

We can restrict the possible electric charges of a dyon further by the assumption that there is CP conservation in the theory. Then, because magnetic fields are even under a CP transformation and electric fields are odd, the respective charges are even and odd, too. Thus, a $(q, 2\pi/e)$ dyon must have a CP mirror image dyon with $(\Leftrightarrow q, 2\pi/e)$. For these two particles, the quantization condition gives $4\pi q/e = 2\pi n$, so that

$$q = ne \quad \text{or} \quad q = (n + \frac{1}{2})e. \quad (75)$$

Therefore, a dyon can have an electric charge which is an integer or half-integer multiple of the fundamental charge e , but not both possibilities are realized at the same time because of eq. (74).

A more interesting situation is that when CP is violated, and in particular when the violation is measured by a θ -angle. This means that we are thinking about gauge theories where the gauge symmetry is spontaneously broken down to the $U(1)$ of electromagnetism, and where the CP violation is given by the term

$$\frac{\theta e^2}{32\pi^2} F_{\mu\nu}^a F^{a\mu\nu}$$

in the Lagrangian. This violates CP because it is proportional to $\mathbf{E} \cdot \mathbf{B}$, which is odd under a CP transformation. The existence of magnetic monopoles and dyons in such a theory will be discussed in the next section. We will close this one by recording the consequences of the CP violating θ -term [27]. The result is that the electric charge q of a dyon is quantized by

$$q = ne \Leftrightarrow \frac{\theta e}{2\pi}.$$

This is known as the ‘‘Witten effect’’. Note that for a gauge theory, in the absence of a θ -term, it is the first one of the two possibilities in eq. (75) that is realized.

4.3 Magnetic monopoles and dyons in field theory

The existence of magnetic monopoles in a theory where electromagnetism is embedded in a larger gauge group has been known since their explicit construction by 't Hooft [28] and Polyakov [30] in 1974. The dyons were found by Julia and Zee [29] shortly afterwards. In this section we will review these objects in the Georgi-Glashow model. This is the simplest model in which these objects occur, and also the one in which they were first discovered. The fact that electromagnetism is embedded in a larger theory is clearly relevant with respect to duality. One reason for looking at the Georgi-Glashow model, apart from simplicity, is that the $N = 2$, $SU(2)$ Yang-Mills theory is a ‘‘minimal extension’’ of this model.

We first recall some basic facts about the Georgi-Glashow model. It is an $SU(2)$ gauge theory with a scalar field (a Higgs field) in the adjoint, or triplet, representation. The Lagrangian is

$$\mathcal{L} = \Leftrightarrow \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2} (D_\mu \phi)^a (D^\mu \phi)^a \Leftrightarrow \frac{1}{8} \lambda (\phi^2 \Leftrightarrow \phi_0^2)^2 \quad (76)$$

where

$$\begin{aligned} G_{\mu\nu}^a &= \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + g\epsilon^{abc} V_\mu^b V_\nu^c \\ D_\mu \phi^a &= \partial_\mu \phi^a + g\epsilon^{abc} V_\mu^b \phi^c \end{aligned}$$

We can also write this in terms of an expansion around the spontaneously broken vacuum by using the unitary gauge

$$\phi^a(x) = (0, 0, \phi_0 + \sigma(x)).$$

If we also define the photon and the W -bosons by

$$A_\mu = V_\mu^3 \quad \text{and} \quad W_\mu = \frac{1}{\sqrt{2}}(V_\mu^1 + iV_\mu^2),$$

we get

$$\begin{aligned} \mathcal{L} &= \Leftrightarrow \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \sigma)^2 \Leftrightarrow \frac{1}{2} m_H^2 \sigma^2 \\ &\Leftrightarrow \frac{1}{2} |\check{D}_\mu W_\nu - \check{D}_\nu W_\mu|^2 \Leftrightarrow m_W^2 |W_\mu|^2 + \mathcal{L}_{int} \end{aligned}$$

where

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\
\tilde{D}_\mu &= \partial_\mu \Leftrightarrow ig A_\mu, \\
\mathcal{L}_{int} &= \Leftrightarrow ig F_{\mu\nu} F^{\mu\nu} W^{*\mu} W^\nu \Leftrightarrow \frac{1}{4} g^2 (W_\mu^* W_\nu \Leftrightarrow W_\mu W_\nu^*)^2 \\
&\quad \Leftrightarrow \frac{1}{2} (2g\phi_0\sigma + g^2\sigma^2) |W_\mu|^2 \Leftrightarrow \frac{1}{8} \lambda (4\phi_0\sigma^3 + \sigma^4),
\end{aligned}$$

and the masses of the Higgs particle σ and the W -bosons are

$$m_H^2 = \lambda\phi_0^2 \quad \text{and} \quad m_W^2 = g^2\phi_0^2,$$

respectively. The equations of motion found from eq. (76) are

$$\begin{aligned}
D_\mu G^{a\mu\nu} &= \Leftrightarrow g\epsilon^{abc}\phi^b D^\nu\phi^c \\
D_\mu D^\mu\phi^a &= \Leftrightarrow \frac{1}{2}\lambda(\phi^2 \Leftrightarrow \phi_0^2)\phi^a
\end{aligned} \tag{77}$$

The magnetic monopoles and dyons are then static solutions to these equations. The equations are difficult to solve in the general case because they are non-linear, second order equations, so we need some kind of strategy. We will follow the approach of Bogomol'nyi [31], who managed to rewrite the equations in the static case into first order equations by using some tricks. These solutions can then be solved by using a sensible *ansatz*.

First of all, let us recall that the existence of monopoles and dyons is connected with the possibility of having “unshrinkable” maps from the sphere at spatial infinity into the vacuum manifold of the Higgs field, which is a sphere in field space of radius ϕ_0 . By ‘unshrinkable’ we mean that the maps are characterized by a winding number n_m , so that the boundary conditions of the Higgs field at infinity can be specified by

$$\phi^a \rightarrow n^a \phi_0 \quad \text{for} \quad x \rightarrow \infty,$$

where n^a is a unit vector in field space defined by $(\theta, \phi = \text{spherical angles})$:

$$n^a = (\sin\theta \cos n_m\phi, \sin\theta \sin n_m\phi, \cos\theta).$$

For example, if $n_m = 1$, we have a “hedgehog” solution [30]. The number n_m is a *topological quantum number* and the fact that it is an integer ensures the stability under decay of a configuration of one value of n_m into a configuration of another value of n_m .

Let us find the energy of a given field configuration. It is given by the space integral of T_{00} , the 00-component of the energy momentum tensor $T_{\mu\nu}$:

$$E = \int d^3x T_{00}$$

If we calculate the energy momentum tensor in the canonical way, we get (η_i denotes all the fields in the Lagrangian, that is, ϕ^a and V_μ^a):

$$\begin{aligned}
T_{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu \eta_i)} \partial_\nu \eta_i \Leftrightarrow g_{\mu\nu} \mathcal{L} \\
&= G_{\mu\rho}^a (\partial^\rho V_\nu)^a + (D_\mu \phi)^a \partial_\nu \phi^a \Leftrightarrow g_{\mu\nu} \mathcal{L}.
\end{aligned}$$

There is a problem with this form of the energy-momentum, however, as it is neither symmetric nor gauge invariant [29, 32]. We must make use of our freedom to add improvement-terms to it. Alternatively we can calculate $T_{\mu\nu}$ as one would in general relativity. We then let the metric

$g_{\mu\nu}$ be an independent field, find the variation of the action $S = \int d^4x \sqrt{\det g} \mathcal{L}$ with respect to the metric, and then take the flat space limit of the result. That is,

$$T_{\mu\nu} = \frac{2}{\sqrt{\det g}} \frac{\partial(\mathcal{L} \sqrt{\det g})}{\partial g^{\mu\nu}}.$$

This will automatically give a symmetric and gauge invariant energy-momentum tensor. Either way, the result is

$$T_{\mu\nu} = G_{\mu\rho}^a G_{\nu}^{a\rho} + (D_\mu \phi)^a (D_\nu \phi)^a \Leftrightarrow g_{\mu\nu} \mathcal{L}$$

The 00-component of this is

$$\begin{aligned} T_{00} &= (G_{0i}^a)^2 + (D_0 \phi)^2 \Leftrightarrow \mathcal{L} \\ &= \frac{1}{4}(G_{ij}^a)^2 + \frac{1}{2}(G_{0i}^a)^2 + \frac{1}{2}(D_0 \phi)^2 + \frac{1}{2}(D_i \phi)^2 + \frac{1}{8}\lambda(\phi^2 \Leftrightarrow \phi_0^2)^2 \end{aligned}$$

This expression would simplify if the last term, the Higgs potential, was not present. At the same time, if we take the limit $\lambda \rightarrow 0$, we obtain a lower bound on the energy because λ is the coefficient of a positive term. It is not clear what this limit means because if there is no potential, there is not a well-defined vacuum value ϕ_0 of the Higgs field. We will in this section, however, assume that the limit $\lambda \rightarrow 0$ is taken in such a way that ϕ_0 remains unchanged. This limit is called the BPS limit (after Bogomol'nyi, Prasad and Sommerfield [31, 32]). It is usually formulated in terms of the parameter $\beta \equiv \lambda/g^2 = M_H^2/M_W^2$, by $\beta \rightarrow 0$. We will say more about the BPS limit later. Now the energy is

$$E = \int d^3x \left[\frac{1}{4}(G_{ij}^a)^2 + \frac{1}{2}(G_{0i}^a)^2 + \frac{1}{2}(D_0 \phi)^2 + \frac{1}{2}(D_i \phi)^2 \right],$$

which we are supposed to minimize.

Let us first consider the magnetic monopole. This corresponds to boundary conditions

$$\begin{aligned} \phi_0 &\rightarrow n^a \phi_0 \quad \text{for} \quad x \rightarrow \infty, \\ V_i^a &\rightarrow \frac{1}{g} \epsilon^{abc} n^a \partial_i n^c \quad \text{for} \quad x \rightarrow \infty, \end{aligned}$$

and where the time component V_0^a of the gauge field is zero. $V_0^a \neq 0$ corresponds to a dyon, as we shall see in a moment. Because we are considering static, or time independent, solutions, V_0^a means that G_{0i}^a and $D_0 \phi^a = 0$ so that the energy is

$$E = \int d^3x \left[\frac{1}{4}(G_{ij}^a)^2 + \frac{1}{2}(D_i \phi)^2 \right] \tag{78}$$

Now comes Bogomol'nyi's trick: we rewrite (78) as

$$E = \int d^3x \left[\frac{1}{4}(G_{ij}^a \Leftrightarrow \epsilon_{ijk} D_k \phi^a)^2 + \frac{1}{2} \epsilon_{ijk} G_{ij}^a D_k \phi^a \right],$$

where the last term is equal to the divergence of a vector:

$$\begin{aligned} \frac{1}{2} \epsilon_{ijk} G_{ij}^a D_k \phi^a &= \partial_i S_i, \\ S_i &= \frac{1}{2} \epsilon_{ijk} G_{jk}^a \phi^a \end{aligned}$$

The interpretation of this divergence term is the following. We consider the electromagnetic field strength tensor given by 't Hooft [28], i.e. the field strength of the $U(1)$ gauge field:

$$F_{\mu\nu} = \frac{1}{|\phi|} \phi^a G_{\mu\nu}^a \Leftrightarrow \frac{1}{|\phi|^3} \epsilon^{abc} \phi^a D_\mu \phi^b D_\nu \phi^c$$

This is gauge invariant and reduces to

$$F_{\mu\nu} = \frac{1}{\phi_0} \phi_0^a G_{\mu\nu}^a$$

in a region where ϕ^a is equal to the vacuum value $\phi_0^a = n^a \phi_0$, for some unit vector n^a . When we recall the definition of a magnetic field:

$$B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$$

we see that

$$S_i = \phi_0 B_i.$$

The space integral over the divergence of B^i is equal to the magnetic charge of the monopole. Put differently, the magnetic charge is given by the surface integral

$$G = \int B^i dS^i = \frac{4\pi}{g} n_m. \quad (79)$$

If we use this and the fact that $M_W = g\phi_0$, we get for the energy:

$$E = \frac{4\pi M_W}{g^2} n_m + \int d^3x \left[\frac{1}{4} (G_{ij}^a \Leftrightarrow \epsilon_{ijk} D_k \phi^a)^2 \right] \quad (80)$$

Note that the winding number n_m is a measure of the magnetic charge of the monopole in units of $4\pi/g$.

The form (80) of the energy is a great progress, since for a given n_m , what minimizes it is the vanishing of the square in the space integral. That is, a minimum is a solution to the equation

$$G_{ij}^a = \epsilon_{ijk} D_k \phi^a \quad (81)$$

which is a first order equation. If we consider the configuration with magnetic charge $n_m = 1$, we could search for a solution of (81) of the form

$$\begin{aligned} \phi^a &= n^a s(r) \phi_0 \\ V_i^a &= \frac{1}{g} \epsilon^{abc} n^b \partial_i n^c v(r) \end{aligned} \quad (82)$$

By insertion of this ansatz in (81) we obtain the following system of equations

$$\begin{aligned} \frac{1}{M_W} \frac{dv}{dr} &= s(1 \Leftrightarrow v), \\ \frac{ds}{dr} &= \frac{v(2 \Leftrightarrow v)}{M_W r^2}. \end{aligned}$$

We also have the boundary conditions $v \rightarrow 1, s \rightarrow 1$ for $r \rightarrow \infty$. The solutions are

$$\begin{aligned} s(r) &= \frac{\cosh r}{\sinh r} \Leftrightarrow \frac{1}{r}, \\ v(r) &= 1 \Leftrightarrow \frac{r}{\sinh r}, \end{aligned}$$

where r is the distance measured in units of $1/M_W$.

Let us remark on the stability of the monopoles. If the magnetic charge n_m is greater than 1, the energy is minimized by

$$E_m = \frac{4\pi M_W}{g^2} n_m$$

in the BPS limit. Thus a monopole with charge n_m has the same mass as n_m monopoles of charge 1 and so is marginally unstable under decay into n_m monopoles.

We will now investigate the dyon. A dyon appears if we allow the time components of the vector field to be different from zero. The fields still do not depend on time. If we separate the space and time components in the equations of motion (77), we get

$$\begin{aligned} D_i D^i V^{a0} &= \Leftrightarrow g \epsilon^{abc} \phi^b D^0 \phi^c, \\ D_i G^{a ij} &= \Leftrightarrow g \epsilon^{abc} \phi^b D^j \phi^c + g \epsilon^{abc} V^{b0} D^j V^{c0}, \\ D_0 D^0 \phi^a + D_i D^i \phi^a &= \Leftrightarrow \frac{1}{2} \lambda (\phi^2 \Leftrightarrow \phi_0^2) \phi^a. \end{aligned}$$

We have used that the time derivatives vanish. In the BPS limit $\beta = \lambda/g^2 \rightarrow 0$ the terms involving λ disappear and the equations gets easier to solve. Let us look for a solution where

$$V_0^a = C \phi^a \quad (83)$$

where C is a constant. This means that

$$D_0 \phi^a = \partial_0 \phi^a + g \epsilon^{abc} C \phi^b \phi^c = 0.$$

The equations for V_0^a and ϕ^a are then the same and the equations we must solve is

$$\begin{aligned} D_i G^{a ij} &= \Leftrightarrow g (1 \Leftrightarrow C^2) \epsilon^{abc} \phi^b D^j \phi^c \\ D_i D^i \phi^a &= 0 \end{aligned} \quad (84)$$

Had it not been for the factor $(1 \Leftrightarrow C^2)$ this would have been the same equations as for the monopole case. We can make it disappear if we put

$$x^\mu = \frac{y^\mu}{\sqrt{1 \Leftrightarrow C^2}}.$$

Then we have

$$\frac{\partial}{\partial x^i} = \sqrt{1 \Leftrightarrow C^2} \frac{\partial}{\partial y^i}.$$

In order to get a covariant derivative with respect to y , we define $\tilde{V}_i^a(y)$ by the expression

$$\begin{aligned} (D_i)^{ab} &= \partial_i^x \delta^{ab} + i g \epsilon^{abc} V_i^c(x) \\ &= \sqrt{1 \Leftrightarrow C^2} \partial_i^y \delta^{ab} + i g \sqrt{1 \Leftrightarrow C^2} \epsilon^{abc} \tilde{V}_i^c(y) \\ &= \sqrt{1 \Leftrightarrow C^2} (\tilde{D}_i)^{ab}, \end{aligned}$$

where we also have defined \tilde{D}_i . The corresponding field strength is related by

$$G_{ij}^a(x) = (1 \Leftrightarrow C^2) \tilde{G}_{ij}^a(y).$$

Finally we set

$$\tilde{\phi}^a(y) = \phi^a(x)$$

and so eqs. (84) becomes equal to the monopole equations which has the solutions (82) and (83). We can calculate the electric charge Q of the dyon from the 't Hooft electromagnetic tensor:

$$\begin{aligned} Q &= \int dS_i F_{0i} = \int dS_i C (D_i \phi)^a n^a \\ &= \frac{C}{\sqrt{1 \Leftrightarrow C^2}} \int d\tilde{S}_i (\tilde{D}_i \tilde{\phi}) n^a = \frac{4\pi n_m}{g} \frac{C}{\sqrt{1 \Leftrightarrow C^2}}. \end{aligned}$$

Note that Q is not necessarily an integer (in the classical theory) because C is arbitrary.

It is possible to obtain a lowest bound on the mass of a dyon of magnetic charge G and electric charge Q . The bound is saturated in the BPS limit $\beta \rightarrow 0$. To find this bound we use the energy in the form

$$E = \int d^3x \left(\frac{1}{4}(G_{ij}^a)^2 + \frac{1}{2}(D_i\phi)^2 + (G_{0i}^2)^2 \right)$$

and the form (83) of V_0 . The latter relation implies

$$(G_{0i}^a)^2 = C^2(D_i\phi)^2.$$

Now the energy at the bound is

$$E = \int d^3x \left(\frac{1}{4}(G_{ij}^a)^2 + \frac{1+C^2}{2}(D_i\phi)^2 \right).$$

In the y -coordinates this is

$$E = \sqrt{1 \Leftrightarrow C^2} \int d^3y \frac{1}{4}(\tilde{G}_{ij}^a)^2 + \frac{1+C^2}{\sqrt{1 \Leftrightarrow C^2}} \int d^3y \frac{1}{2}(\tilde{D}_i\tilde{\phi})^2$$

The quantities with tildes are solutions of the monopole equations. In particular, they satisfy

$$\tilde{G}_{ij}^a = \epsilon_{ijk} \tilde{D}_k \tilde{\phi}^a$$

which implies that

$$\frac{1}{4}(\tilde{G}_{ij}^a)^2 = \frac{1}{2}(\tilde{D}_i\tilde{\phi})^2 \tag{85}$$

If E_m denotes the energy of a monopole of charge n_m (recall that n_m is an integer) then eq. (85) means that

$$\int d^3y \frac{1}{4}(\tilde{G}_{ij}^a)^2 = \int d^3y \frac{1}{2}(\tilde{D}_i\tilde{\phi})^2 = \frac{E_m}{2}.$$

Thus the dyon energy becomes

$$E = \sqrt{1 \Leftrightarrow C^2} \frac{E_m}{2} + \frac{1+C^2}{\sqrt{1 \Leftrightarrow C^2}} \frac{E_m}{2} = E_m \frac{1}{\sqrt{1 \Leftrightarrow C^2}}.$$

We have already found that

$$E_m = \frac{4\pi M_W}{g^2} n_m = \frac{4\pi n_m}{g} \phi_0.$$

Using the expression (85) for Q and $G = 4\pi/g$, can calculate that

$$E = \phi_0 \sqrt{G^2 + Q^2},$$

which is the desired lower bound, known as the BPS bound. Because the energy of a static solution is also the mass this solution, we get an inequality that any configuration must satisfy:

$$M \geq \phi_0 \sqrt{G^2 + Q^2}. \tag{86}$$

If this bound holds in the quantum theory, a state that satisfy the equality in eq. (86) is called a BPS saturated state.

4.4 The Olive-Montonen duality conjecture

We return now to the question of duality. The fact that objects with electric and magnetic charges exist in non-abelian gauge theories is interesting, because such theories have proven to be relevant to the description of nature. The Georgi-Glashow model is relatively simple, so it seems to be a natural question to ask if there is some kind of electric-magnetic duality in this theory. Montonen and Olive conjectured that there is [33].

The conjecture is the following. The dual quantum field theory is described by a Lagrangian of the exact same form as eq. (76) but where the monopoles of positive and negative magnetic charge play the same roles as the electrically charged heavy gauge bosons. Furthermore, the dual theory would have the gauge bosons as solitons. This is a duality that bears some resemblance to the equivalence between the Thirring and sine-Gordon models.

Montonen and Olive give three arguments in favour of their conjecture. 1) The “elementary” states in the model have the only possible magnetic charges 0 and $\pm G$. This follows from the requirement of spherical symmetry of the soliton solutions. A solution that was spherically non-symmetric would be connected with a tower of rotational states in the quantum theory, much like in molecular spectroscopy. Clearly, such a solution would violate any reasonable definition of “elementarity”. 2) The mass formula (86) is also valid for the W -bosons in the classical theory. This might suggest that the W -bosons could be taken to be solitons. This argument is wrong as we will see below. 3) The force between monopoles can be calculated in some idealized situations to be equal the corresponding force between gauge bosons.

Montonen and Olive also recognized some problems with the conjecture. These are features of the classical theory which they proposed would be different in the quantized theory. First, there is the problem of the dyons. By the “elementarity” requirement they have magnetic charges of $\pm G$, but they also have electrical charges characterized by arbitrary integers. The problem is that a dyon cannot decay into a magnetic monopole and electrically charged gauge bosons and so does not fit into the duality scheme. The second problem was that the spin of a gauge boson is 1, while the monopole solution has classically spin 0 since it is spherically symmetric. They believed that this problem would disappear upon proper quantization.

There is actually a few other problems with the Olive-Montonen conjecture, that was not recognized by them. Let us discuss these problems in some details, since we shall see that they are “solved” by supersymmetry²². The four problems with the duality of the Georgi-Glashow model in the Olive-Montonen sense are the following:

1. What is the scalar potential

$$V(\phi) = \frac{1}{8}\lambda(\phi^2 \Leftrightarrow \phi_0^2)^2$$

in the quantum theory? This is a version of the well known problems of Higgs fields and renormalization. The point is that the renormalization point is arbitrary and consequently the zeros of the potential are not well defined.

2. What is the gauge coupling constant g ? Duality acts on the coupling constant by $g \rightarrow 4\pi/g$ because of eq. (79), but the theory is known to have $\beta(g) < 0$, i.e. asymptotic freedom. This means that g gets smaller as we probe the physics at larger energies and so $4\pi/g$ gets larger. This violates duality because $4\pi/g$ is supposed to enter the dual Lagrangian in the same way as g enters the original one.

²²The following is essentially an account of one of Witten’s lectures at the Jerusalem Winter School in Physics, 1994/95 [3].

3. Why is the mass formula $M = \phi_0 \sqrt{G^2 + Q^2}$ for BPS saturated states valid quantum mechanically? The point is that in the original formulation of the theory, this formula is valid for the semiclassical approximation (i.e. for small g) both for the “fundamental” heavy gauge bosons and for the magnetic monopoles, which are the solitons. The latter have the mass $M_m = 4\pi M_W/g^2$ where $M_W = \phi_0 g$. Montonen and Olive then claims that if one transforms $g \rightarrow m \equiv 4\pi/g$, then, since the W -bosons now are the solitons, we should have $M_W = 4\pi M_m/m^2$ where $M_m = \phi_0 m$. This argument is not correct! The reason is that the mass formulae are valid for small g (it is a semiclassical approximation) not for small $m = 4\pi/g$. For large coupling, the soliton mass gets a relevant renormalization.
4. What about the spin of the monopole? The spin of the monopole is zero, because of the spherical symmetry. The “elementarity” argument previously mentioned is not relevant here. Lorentz transformation properties must show up at the classical level, like e.g. a four-vector index, or not at all.

Let us now see how these problems are overcome. 1 and 3 is eliminated by going to $N \geq 2$ supersymmetry. Let us first address problem 1, the quantum meaning of $V = 0$. The minimal extension of the Georgi-Glashow model to $N = 2$ SUSY is just the “pure” $SU(2)$ Yang-Mills theory. There is a complex scalar ϕ^a in the adjoint representation, or equivalently there are two real scalars, related to each other by chirality transformations. There is a unique scalar potential

$$V = \text{Tr}[\phi, \phi^\dagger]^2,$$

which is not changed by renormalization. A zero of this potential at tree level remains therefore a zero in the quantum theory. Also, there is a flat direction in the sense that

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & \Leftrightarrow a \end{pmatrix}$$

gives $V = 0$ for arbitrary complex a . This means that magnetic monopoles and dyons are automatically BPS saturated.

Problem 3 is solved in the $N \geq 2$ theory because it is a consequence of the supersymmetry algebra [34]. When one calculates the proper form of this algebra from the current (46), one finds the commutator (in Majorana notation)

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} \gamma_{\alpha\beta}^\mu P_\mu + \epsilon_{ij} \delta_{\alpha\beta} U + i\epsilon_{ij} (\gamma_5)_{\alpha\beta} V, \quad (87)$$

where

$$\begin{aligned} U &= \int d^3x \partial_i (A^a F_{0i}^a + B^a \frac{1}{2} \epsilon_{ijk} F_{jk}^a), \\ V &= \int d^3x \partial_i (A^a \frac{1}{2} \epsilon_{ijk} F_{jk}^a + B^a F_{0i}^a). \end{aligned} \quad (88)$$

A^a and B^a are the real and imaginary part of the Higgs field ϕ^a . U and V are central charges because of eq. (87). They are also linear combinations of the electric and magnetic charges of a state, as can be seen from their expressions (88). From the discussion of Sec. 2.1 on central charges, we learn that we have the inequality

$$M^2 \geq \phi_0^2 (Q^2 + G^2) \quad (89)$$

We also know that the representations that satisfy the equality in eq. (89) are “small” representations with four helicity states. From the discussion in the last section, we get that these are

precisely the BPS saturated states. Conversely, representations that does not satisfy the equality has sixteen helicity states. When we include quantum corrections, we expect the parameters of the theory to change, but we do not expect the number of states in a representation to change. Therefore, the equality in eq. (89) must hold for BPS saturated states in the quantum theory.

Problems 2 and 4 are eliminated in the $N = 4$ theory. Problem 2, which has to do with the coupling constant g , is not a problem in the $N = 4$ Yang-Mills theory. It turns out that this model has the right composition of fields to make $\beta(g) = 0$. The theory is finite and g is a natural dimensionless parameter. There is also a θ -parameter in the theory. By constructing the “complex coupling constant”

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$$

it is possible to redefine the action of the inversion of the coupling constant g by the transformation

$$\tau \rightarrow \Leftrightarrow 1/\tau$$

This coincides with $g \rightarrow 4\pi/g$ for $\theta = 0$.

Neither is the fourth point a problem in the $N = 4$ theory. The Yang-Mills or gauge multiplet is the smallest multiplet of this theory, with spins ≤ 1 . There are three scalar and three pseudoscalar fields in the theory, which all can form soliton configurations with the gauge fields. Such a soliton must, in the quantum theory, be contained in a multiplet of spins ≤ 1 since there are no smaller multiplets in the theory. All the fields in the same multiplet have the same quantum numbers, and so there must be spin 1 particles with magnetic charge $\pm G$. These are the possible duals to the W -bosons.

These circumstances suggests that the $N = 4$ theory possesses electric-magnetic duality more or less in the sense of Montonen and Olive [15]. The two latter problems are not overcome in the $N = 2$ case. Nevertheless, there is a duality of the $N = 2$ theory which bears some resemblance to the Olive-Montonen duality. This shall be the subject of the last chapter.

5 The solution of Seiberg and Witten

5.1 Coordinates on the moduli space

We have now developed enough machinery to give convincing arguments for an exact solution to the $N = 2$, $SU(2)$ Yang-Mills theory [4]. We start by examining introducing a complex coupling constant:

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2},$$

where g is the gauge coupling constant and θ is the “vacuum angle” that measures the amount of CP violation from non-trivial configurations of the gauge fields. Duality will act on this object, rather than just the gauge coupling g . From the theory of θ -angles we know that θ can always be rotated to zero if there are massless fermions in the theory. We can now write the classical action of the theory as

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \int d^4\theta \frac{1}{2} \tau_{cl} \Psi^a \Psi^a,$$

where Ψ^a , $a = 1, 2, 3$, is the $N = 2$ chiral field strength superfield and τ_{cl} contains the classical values of θ and g .

In this notation, the Wilsonian effective action at low energy becomes

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A^a} (e^V)_{ab} \bar{A}^b + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial A^a \partial A^b} W^a W^b \right].$$

We can simplify this further since we know that in general the gauge symmetry is broken. Let us define the function \mathcal{H} so that

$$\mathcal{F}(\sqrt{A \cdot A}) = \mathcal{H}(A \cdot A)$$

This makes sense because it is always the gauge invariant combination $A \cdot A$ that appears in \mathcal{F} . Then we have that

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial A^a} &= \frac{\partial \mathcal{H}}{\partial A^a} = \mathcal{H}'(A \cdot A) \cdot 2A^a, \\ \frac{\partial^2 \mathcal{F}}{\partial A^a \partial A^b} &= + \frac{\partial^2 \mathcal{H}}{\partial A^a \partial A^b} = \mathcal{H}''(A \cdot A) \cdot 4A^a A^b + \mathcal{H}'(A \cdot A) \cdot \delta^{ab}, \end{aligned}$$

and so

$$\mathcal{L} = \frac{1}{2\pi} \text{Im} \left[\int d^4\theta \mathcal{H}' A^a (e^V)_{ab} \bar{A}^b + \int d^2\theta \frac{1}{2} (\mathcal{H}' \delta^{ab} + 2\mathcal{H}'' A^a A^b) W^a W^b \right] \quad (90)$$

By exposing the index structure like this it is clear what happens when we keep only the component of the adjoint vectors A^a and W^a that corresponds to the massless component of \mathcal{A}^a , the low energy equivalent of Ψ^a . We can then drop the adjoint indices, and make the replacements

$$\begin{aligned} A^a (e^V)_{ab} \bar{A}^b &\rightarrow A \bar{A} \\ \delta^{ab} W^a W^b &\rightarrow W W \\ A^a A^b W^a W^b &\rightarrow A A W W \end{aligned} \quad (91)$$

In terms of \mathcal{F} we then have the Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A} \bar{A} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial A^2} W W \right]. \quad (92)$$

By making the substitutions (91) we throw away mass and coupling terms for the massive fields. The effects of these terms in the low energy theory are taken into account by the function \mathcal{F} except for virtual processes with momenta below the Wilsonian action cutoff μ , but the contributions from these processes can be made arbitrarily small because of the smooth behaviour in the infrared.

Now we note that the chiral superfield A is the field whose scalar component is a – the vacuum expectation value of ϕ . Since a takes its values in the complex manifold that is the moduli space, the Kähler potential that we can read off from (92),

$$K = \text{Im} \left(\frac{\partial \mathcal{F}(A)}{\partial A} \bar{A} \right),$$

is the Kähler potential on the moduli space. Thus the metric on the moduli space is

$$ds^2 = \text{Im} \frac{\partial^2 \mathcal{F}(a)}{\partial a^2} da d\bar{a}.$$

At the same time, from the second term of (92) we can read off the effective coupling constant

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2}.$$

This means that the low energy effective coupling constant is the metric on the moduli space:

$$ds^2 = \text{Im}\tau(a) da d\bar{a}. \quad (93)$$

Now, $\tau(a)$, being the effective coupling constant is known to us for large values of $|a|$. We calculated it in Sec. 3.6 to be²³

$$\tau(a) \approx \frac{i}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right), \quad (94)$$

This means that $\text{Im}\tau(a)$ is single valued for large $|a|$. But then it is a harmonic function, which does not have a minimum. Thus a cannot be a good coordinate everywhere on the moduli space as it would give regions of negative metric. It is therefore necessary to operate with other coordinate systems besides a . Let us define

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}.$$

It is then possible to write the metric as

$$ds^2 = \text{Im} da_D d\bar{a} = \Leftrightarrow \frac{i}{2} (da_D d\bar{a} \Leftrightarrow da d\bar{a}_D). \quad (95)$$

This expression is symmetric in a and a_D which implies that it is possible to use a_D as a local coordinate on the moduli space with another holomorphic function replacing τ in eq. (93). If u is an *arbitrary* local coordinate on the moduli space, we can write (95) as

$$ds^2 = \text{Im} \frac{da_D}{du} \frac{d\bar{a}}{d\bar{u}} du d\bar{u} = \Leftrightarrow \frac{i}{2} \left(\frac{da_D}{du} \frac{d\bar{a}}{d\bar{u}} \Leftrightarrow \frac{da}{du} \frac{d\bar{a}_D}{d\bar{u}} \right) du d\bar{u}. \quad (96)$$

For instance, we can pick $u = a$ and so we get (93). We can also pick u to be $\langle \text{Tr}\phi^2 \rangle$, which is defined globally on the moduli space. This coincides at the classical level with the previously defined u -parameter, viz. $\text{Tr}\langle \phi \rangle^2$. By bringing both $a(u)$ and $a_D(u)$ into play, it will be possible to ensure the positivity of the metric. In the following we will use this last definition of u as the global coordinate.

There is a set of symmetries of the metric. We can make them manifest by introducing the notation $a^\alpha = (a_D, a)$, $\alpha = 1, 2$. If $\epsilon_{\alpha\beta}$ is the usual antisymmetric tensor, then

$$ds^2 = \Leftrightarrow \frac{i}{2} \epsilon_{\alpha\beta} \frac{da^\alpha}{du} \frac{d\bar{a}^\beta}{d\bar{u}} du d\bar{u}.$$

as can be obtained from (96). The invariance group of the metric is now seen to be $SL(2, \mathbf{R})$ (it cannot be $SL(2, \mathbf{C})$ because it does not commute with complex conjugation). Soon we will see that, for physical reasons, the symmetry group is actually $SL(2, \mathbf{Z})$.

5.2 Duality transformations

We have seen that the metric is invariant under transformations of the group $SL(2, \mathbf{R})$, which is generated by the matrices

$$T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ \Leftrightarrow 1 & 0 \end{pmatrix},$$

where b is real. Any element of $SL(2, \mathbf{R})$ can be written as a product of powers of such matrices. A matrix of the first type acts on the vector (a_D, a) by

$$T_b : \quad a_D \rightarrow a_D + ba, \quad a \rightarrow a.$$

²³We did not consider the θ -angle in chapter 3, but it is automatically included in (94) since a is complex.

Since $\tau = \partial a_D / \partial a = \theta / 2\pi + i4\pi/g^2$ we have that $\tau \rightarrow \tau + b$, or $\theta \rightarrow \theta + 2\pi b$. We are only changing coordinates on the moduli space and this should not affect the physics. This means that b must be an integer as θ , being an angle, is only defined modulo 2π . Thus the invariance group of the metric is really $SL(2, \mathbf{Z})$.

We now turn our attention to the transformation S . We shall see that this corresponds to an electric-magnetic duality transformation. What we would like to do is to describe the physics at low energies in terms of the gauge field A_μ , which is “handed down” to us from the microscopic theory. Then by definition we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and so

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

is identically true, being the Bianchi identity. This means that if we include electrically charged matter in the system, duality is lost as a symmetry, since all magnetic charges would be identically zero. Duality transformations are still possible, however, if they are considered as a mapping of one theory to another. That is, the theory with A_μ as gauge field, where $\partial_\mu F^{\mu\nu} = 0$ is the equation of motion and $\partial_\mu \tilde{F}^{\mu\nu} = 0$ the Bianchi identity, is mapped into a theory with another gauge field $V_{D\mu}$ such that $\tilde{F}_{\mu\nu} = \partial_\mu V_{D\nu} - \partial_\nu V_{D\mu}$, $\partial_\mu \tilde{F}^{\mu\nu} = 0$ is the equation of motion and $\partial_\mu F^{\mu\nu} = 0$ is the Bianchi identity.

The duality transformation is effectuated at the path integral level as a change in variables [35]. Recall that we have scaled the gauge fields by absorbing the coupling constant such that the covariant derivative is given by

$$D_\mu = \partial_\mu + iA_\mu,$$

i.e. the gauge field couples to electrically charged matter with unit strength. The part of the Lagrangian which involves the photon A_μ is then²⁴

$$\Leftrightarrow \frac{1}{32\pi} \text{Im}[\tau(F + i\tilde{F})^2] = \Leftrightarrow \frac{1}{4g^2} F^2 \Leftrightarrow \frac{\theta}{32\pi^2} F\tilde{F},$$

and the corresponding part of the path integral is

$$\int \mathcal{D}A_\mu e^{i[-\frac{1}{32\pi} \text{Im} \int \tau(F + i\tilde{F})^2]}. \quad (97)$$

Charged fields are heavy and therefore does not appear in the low energy theory. We can write (97) as a path integral over $F^{\mu\nu}$ instead of A_μ as long as we integrate only over those F 's that satisfy the Bianchi identity²⁵:

$$\int \mathcal{D}F_{\mu\nu} e^{i[-\frac{1}{32\pi} \text{Im} \int \tau(F + i\tilde{F})^2]} \delta(\partial_\mu \tilde{F}^{\mu\nu}). \quad (98)$$

The Bianchi constraint δ -functional makes (98) equivalent to (97). The δ -functional can be represented by a path integral as

$$\delta(\partial_\mu \tilde{F}^{\mu\nu}) = \int \mathcal{D}V_{D\mu} e^{i[-\frac{1}{4\pi} \int V_{D\mu} \partial_\nu \tilde{F}^{\nu\mu}]}. \quad (99)$$

²⁴Some facts about the dual tensors are $\tilde{\tilde{F}} = -F$ and $\tilde{F}^2 = -F^2$.

²⁵We are ignoring gauge fixing problems. In the abelian theory gauge fixing terms are just (infinite) factors that can be brought outside the path integral.

which is a functional version of

$$\delta(x) = \int \frac{dp}{\sqrt{2\pi}} e^{ipx}.$$

The normalization of (99) is such that a magnetic monopole²⁶

$$\partial_\mu \tilde{F}^{\mu 0} = \Leftrightarrow 4\pi \delta^{(3)}(\mathbf{r}) = \Leftrightarrow 4\pi k^0$$

couples to the field $V_{D\mu}$ with charge one. $V_{D\mu}$ can be thought of as a Lagrange multiplier field. The exponent in (99) can be written

$$\begin{aligned} \Leftrightarrow \frac{1}{4\pi} \int V_{D\mu} \cdot \frac{1}{2} \epsilon^{\nu\mu\rho\sigma} \partial_\nu F_{\rho\sigma} &= \frac{1}{8\pi} \int F_{D\mu\nu} \cdot \frac{1}{2} \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma} \\ &= \frac{1}{8\pi} \int \tilde{F}_D F = \frac{1}{16\pi} \text{Re} \int (\tilde{F}_D \Leftrightarrow i F_D)(F + i\tilde{F}), \end{aligned}$$

where $F_{D\mu\nu} = \partial_\mu V_{D\nu} - \partial_\nu V_{D\mu}$ is equal to $\tilde{F}_{\mu\nu}$ on shell. The path integral (98) now becomes

$$\int \mathcal{D}F_{\mu\nu} \mathcal{D}V_{D\mu} e^{i[-\frac{1}{32\pi} \text{Im} \int \tau (F + i\tilde{F})^2 + \frac{1}{16\pi} \text{Re} \int (\tilde{F}_D - i F_D)(F + i\tilde{F})]}.$$

We can now integrate over the $F_{\mu\nu}$, obtaining the path integral in terms of the dual gauge field $V_{D\mu}$. To do this we note that the antisymmetric tensor $F_{\mu\nu}$ can be written as the sum

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{2}(F_{\mu\nu} + i\tilde{F}_{\mu\nu}) + \frac{1}{2}(F_{\mu\nu} - i\tilde{F}_{\mu\nu}) \\ &\equiv (F_+)_{\mu\nu} + (F_-)_{\mu\nu}, \end{aligned}$$

where $F_+^* = F_-$, $F_-^* = F_+$ and F_+ and F_- are orthogonal:

$$(F_+)_{\mu\nu} (F_-)^{\mu\nu} = 0.$$

Therefore, $\mathcal{D}F_{\mu\nu} = \mathcal{D}(F_+)_{\mu\nu} \mathcal{D}(F_-)_{\mu\nu}$, and the integral, because it is a gaussian, becomes straightforward:

$$\begin{aligned} &\int \mathcal{D}F \mathcal{D}V_D e^{i[-\frac{1}{8\pi} \text{Im} \tau F_+^2 + \frac{1}{4\pi} \text{Re}(-i F_{D+}) F_+]} \\ &= \int \mathcal{D}F \mathcal{D}V_D e^{-\frac{1}{2}(\frac{1}{8\pi} \tau) F_+^2 - \frac{1}{2}(-\frac{1}{8\pi} \bar{\tau}) F_-^2 + (\frac{1}{8\pi} F_{D+}) F_+ + (-\frac{1}{8\pi} F_{D-}) F_-} \\ &= \int \mathcal{D}V_D e^{\frac{1}{2}(\frac{1}{8\pi})^2 (\frac{1}{8\pi})^{-1} \frac{1}{\tau} F_{D+}^2 + \frac{1}{2}(-\frac{1}{8\pi})^2 (-\frac{1}{8\pi})^{-1} \frac{1}{\bar{\tau}} F_{D-}^2} \\ &= \int \mathcal{D}V_D e^{i[-\frac{1}{8\pi} \text{Im} \frac{-1}{\tau} F_{D+}^2]}. \end{aligned}$$

We have used the functional analogue of

$$\int dx e^{-\frac{1}{2}ax^2 + bx} = N e^{\frac{1}{2}(\frac{1}{a})b^2}$$

and the fact that

$$\text{Im} z = \frac{1}{2i}(z - \bar{z}), \quad \text{Re} z = \frac{1}{2}(z + \bar{z}).$$

Thus the dual “photonic” action is

$$\Leftrightarrow \frac{1}{32\pi} \text{Im} \frac{\Leftrightarrow 1}{\tau(a)} (F_D + i\tilde{F}_D)^2.$$

²⁶We are now using the convention that point charges are normalized to be $\mathbf{B} = \mathbf{r}/r$, which differs from the convention of Chap. 4 by a factor $g/4\pi$.

Note that the “dual coupling constant” is minus the inverse of the original one:

$$\tau_D = \Leftrightarrow 1/\tau.$$

This means that the dual of a weakly coupled theory is a strongly coupled theory and vice versa.

So far we have only been concentrating on the photonic part of the low energy action. What we really want to do is to transform all of it by a duality transformation. This can be done by first repeating the transformation of the action in (97) with its $N = 1$ generalization

$$\int \mathcal{D}V e^{i[\frac{1}{8\pi}\text{Im} \int d^2\theta \tau(A)W^2]},$$

where the chiral superfield W_α is defined from the real superfield V by

$$W_\alpha = \Leftrightarrow \frac{1}{4} \bar{D}^2 D_\alpha V,$$

and satisfies the identity

$$DW = \bar{D}\bar{W} \quad \text{or} \quad \text{Im}DW = 0,$$

which is the super-generalized Bianchi identity. We now path integrate over the unconstrained (but chiral) W and \bar{W} while we implement the Bianchi identity by a δ -functional in the shape of a path integral over the real Lagrange multiplier superfield V_D :

$$\begin{aligned} & \int \mathcal{D}W \mathcal{D}\bar{W} e^{i[\frac{1}{8\pi}\text{Im} \int d^2\theta \tau W^2]} \delta(\text{Im}DW) \\ &= \int \mathcal{D}W \mathcal{D}\bar{W} \mathcal{D}V_D e^{i[\frac{1}{8\pi}\text{Im} \int d^2\theta \tau W^2 + \frac{1}{4\pi}\text{Im} \int d^4\theta V_D DW]}. \end{aligned} \quad (100)$$

The superspace integral in the last term in the exponent can be made to go over only half of the superspace in the following way:

$$\begin{aligned} \int d^4\theta V_D DW &= \Leftrightarrow \int d^4\theta D V_D W \\ &= \int d^2\theta \left(\Leftrightarrow \frac{1}{4} \bar{D}^2 \right) D V_D W \\ &= \int d^2\theta W_D W, \end{aligned}$$

where we have used that W is chiral, $\bar{D}W = 0$, and the fact that Grassmann integration is equivalent to differentiation:

$$\int d^2\bar{\theta} = \Leftrightarrow \frac{1}{4} \bar{D}^2.$$

Thus the expression (100) becomes

$$\begin{aligned} & \int \mathcal{D}W \mathcal{D}\bar{W} \mathcal{D}V_D e^{i[\frac{1}{8\pi}\text{Im} \int d^2\theta \tau W^2 + \frac{1}{4\pi}\text{Im} \int d^2\theta W_D W]} \\ &= \int \mathcal{D}W \mathcal{D}\bar{W} \mathcal{D}V_D e^{-\frac{1}{2}(-\frac{1}{8\pi})\tau \int d^2\theta W^2 - \frac{1}{2}(\frac{1}{8\pi})\bar{\tau} \int d^2\bar{\theta} \bar{W}^2} \\ & \quad \times e^{(\frac{1}{8\pi}) \int d^2\theta W_D W + (-\frac{1}{8\pi}) \int d^2\bar{\theta} \bar{W}_D \bar{W}} \\ &= \int \mathcal{D}V_D e^{\frac{1}{2}(\frac{1}{8\pi})^2 (-\frac{1}{8\pi})^{-1} \frac{1}{\tau} \int d^2W_D^2 + \frac{1}{2}(-\frac{1}{8\pi})^2 (\frac{1}{8\pi})^{-1} \frac{1}{\bar{\tau}} \int d^2\bar{W}_D^2} \\ &= \int \mathcal{D}V_D e^{i[\frac{1}{8\pi}\text{Im} \int d^2\theta \frac{-1}{\tau} W_D^2]}, \end{aligned}$$

and so the dual action is

$$\frac{1}{8\pi} \text{Im} \int d^2\theta \frac{\Leftrightarrow 1}{\tau(A)} W_D^2.$$

What remains now is to find a chiral field A_D which has a_D as scalar component, and then use this to rewrite the coupling coefficient:

$$\frac{\Leftrightarrow 1}{\tau(A)} = \tau_D(A_D). \quad (101)$$

Furthermore, if we define

$$h(A) = \frac{\partial \mathcal{F}(A)}{\partial A},$$

then the kinetic term of the chiral field is

$$\text{Im} \int d^4\theta h(A) \bar{A}, \quad (102)$$

which we also need to express this in terms of A_D . We can implement these changes by taking

$$\begin{aligned} A_D &= h(A), \\ h_D(A_D) &= h_D(h(A)) = \Leftrightarrow A, \end{aligned}$$

Note that, from this, h_D is *minus* the inverse of h . For the term (102) this gives

$$\begin{aligned} \text{Im} \int d^4\theta h(A) \bar{A} &= \text{Im} \int d^4\theta A_D \bar{A} \\ &= \text{Im} \int d^4\theta (\Leftrightarrow A) \bar{A}_D \\ &= \text{Im} \int d^4\theta h_D(A_D) \bar{A}_D. \end{aligned} \quad (103)$$

Eq. (101) is true because

$$\tau_D(A_D) = h'_D(A_D) = \Leftrightarrow \frac{1}{h'(A)} = \frac{\Leftrightarrow 1}{\tau(A)}.$$

Hence, from the second equality in (103), we see that the act of rewriting the action in terms of A_D (and in the same form) is the same as transforming (A_D, A) with the matrix S .

We emphasize that the duality transformation is not a symmetry. It maps one description of the low energy physics into another. Because of (101) we see that if one theory is weakly coupled the other one is strongly coupled and vice versa.

Let us also remark on the way in which the coupling constant τ is transformed under $SL(2, \mathbf{Z})$. The action of the generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ \Leftrightarrow 1 & 0 \end{pmatrix}$$

has the following effect on τ :

$$\begin{aligned} T : \quad \tau &\rightarrow \tau + 1, \\ S : \quad \tau &\rightarrow \Leftrightarrow 1/\tau. \end{aligned}$$

In conclusion, the $SL(2, \mathbf{Z})$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on τ by

$$A : \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d}.$$

This coincides with the group of transformations that acts on the modular parameter τ which characterizes a torus, and which leaves the torus unchanged. This is one of the key mathematical ingredients of finding the exact expression for the metric. In a sense, it means that every physically distinct ground state of our theory corresponds to a torus. The physical interpretation of this is not known.

5.3 The mass formula

Classically the BPS saturated states of the theory, including monopoles and dyons, satisfy the mass formula

$$M^2 = |Z|^2, \tag{104}$$

where classically

$$Z_{cl} = a(n_e + \tau_{cl}n_m). \tag{105}$$

n_e and n_m are the electric and magnetic charges of the state. They have integer values. We have seen that the mass formula (104) is a consequence of the fact that we have central charges in the supersymmetry algebra, which implies that it also should hold in the quantum theory. The question is then what the “charge vector” Z will look like.

One way to find Z would be to calculate the supersymmetry algebra using the Wilsonian action (90). This would give the result

$$Z = an_e + a_D n_m, \tag{106}$$

which reduces to (105) in the classical case. Another way of seeing this is to couple an $N = 2$ hypermultiplet with electric charge n_e (and no magnetic charge) to the theory. In $N = 1$ language this is described by two chiral superfields E and \tilde{E} . This corresponds to a “small representation” appropriate for BPS-saturated states. By $N = 2$ supersymmetry the superpotential must have the form

$$n_e A E \tilde{E} + m_E E \tilde{E}. \tag{107}$$

The last term is a mass term which is connected to an ambiguity in the definition of A in the low energy theory, i.e. it can be created by a shift in the field A . m_E must be zero if we take into account that (107) is the low energy limit of the full $SU(2)$ theory and if A in (107) is to be the massless remnant of the three Φ^a (a =adjoint index). This is because an explicit mass term for the fields E_a and \tilde{E}_a will lift the degeneracy of the moduli space and so would give a completely different theory. Thus the superpotential for E and \tilde{E} is $n_e A E \tilde{E}$, and the charge vector is $Z = an_e$. A state with magnetic charge n_m will then have $Z = a_D n_m$, as implied by the duality transformation. The charge vector is then given by (106) for a general state with electric and magnetic charges (n_e, n_m)

5.4 Singularities and monodromies

So far we have established that the complex plane labeled by the coordinate u is the manifold of ground states of the theory – the moduli space. The theories that are built on ground states at large u , $|u| \gg 1$, are weakly coupled because large u means large a and thus small g . In this

region, therefore, quantum corrections to tree level quantities are given to a good approximation by their one-loop results. Also, $u = \langle \text{Tr} \phi^2 \rangle \approx \frac{1}{2} a^2$. One such quantity is the function \mathcal{F} :

$$\mathcal{F}_{\text{one loop}} = i \frac{\mathcal{A}^2}{2\pi} \ln \frac{\mathcal{A}^2}{\Lambda^2}.$$

This implies that

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \approx \frac{2ia}{\pi} \ln \frac{a}{\Lambda} + \frac{ia}{\pi}, \quad \text{large } a. \quad (108)$$

Written like this it is clear that a_D is not a single valued function on the u -plane (or the a -plane) – the logarithm has infinitely many branches. This ambiguity is in agreement with the $SL(2, \mathbf{Z})$ invariance of the metric on the moduli space in the following sense. Let us go around the u -plane in a closed path in the large u region, circling the origin once in the counterclockwise direction. Then physically nothing has changed since we start and end in the same vacuum. For the coordinate u we have $u \rightarrow e^{2\pi i} u$, so $\ln u \rightarrow \ln u + 2\pi i$, and therefore $\ln a \rightarrow \ln a + \pi i$. By (108) this means that we have

$$\begin{aligned} a_D &\rightarrow \Leftrightarrow a_D + 2a, \\ a &\rightarrow \Leftrightarrow a, \end{aligned} \quad (109)$$

This transformation can be written as a matrix multiplied onto the vector (a_D, a) :

$$M_\infty = \begin{pmatrix} \Leftrightarrow 1 & 2 \\ 0 & \Leftrightarrow 1 \end{pmatrix}. \quad (110)$$

An effect of this type, i.e. a change in some functions on a manifold, induced by going around a closed path on the manifold, is known as a monodromy. When a monodromy is written like a matrix, it represents an element of the first homotopy group – the “fundamental group” – on the manifold, which is the group of mappings of the circle to the manifold with the product being “cutting and gluing” two mappings together. In this case the manifold in question is the complex u -plane (or the Riemann sphere) with (at least) the point at infinity removed. The monodromy (110) is connected with this point and this is why we have indexed M_∞ with ‘ ∞ ’.

It is obvious that the identity is represented by the matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, because M_∞ is non-trivial, it is impossible to shrink the “large” closed path that led to (109) into a point, and so there must be one or more holes in the u -plane. The holes must be points. Larger regions are forbidden by holomorphy. We will speak of such points as “singularities” because it can be shown that the u -plane parametrizes a one-complex parameter family of curves (in fact, tori of the usual doughnut type) which becomes singular at these points.

There must be at least two singularities in the interior of the u -plane. If there was only one, then any closed path that circled around it once could be continuously deformed into a big circle at infinity with the monodromy M_∞ . This means that any closed path would leave a^2 invariant, which by definition means that a^2 would be a well defined function and consequently a good global coordinate. But this would mean that the metric is given globally by (93), which it is not as we have already argued. Hence we need two or more singularities. Moreover, the remains of the chiral symmetry acts on the u -plane by $u \leftrightarrow \Leftrightarrow u$, so the singularities must come in pairs in order to obey this symmetry. We will make the assumption that there are exactly two singularities. Although this is an assumption, it will be argued that this leads to a very

non-trivial and unique exact solution of the theory (in the sense that a_D and a will be given as exact functions of u).

The singularities on the u -plane must have a physical interpretation. For example, in the classical theory – the theory without quantum corrections – the point $u = 0$ is a singularity in the sense that at this point we have $a = 0$ and so the full $SU(2)$ gauge symmetry is restored. Therefore all three gauge bosons are massless and the low energy description, where we have integrated out two of the gauge bosons, breaks down. A singular point is a property of the low energy description of the physics which means that quantum states that generically are massive become massless at these points. What, then, are the states that become massless in the quantum theory? It is possible to argue in a rigorous way that it is not the gauge bosons in this case. By appealing to the $N = 2$ superconformal algebra, which is the extension of the 15 dimensional conformal algebra. The point is that massless gauge bosons means conformal invariance in the infrared. We will not go into this here, but intuitively one might say that conformal invariance is in conflict with the fact that we have a spontaneously broken chiral symmetry²⁷ $u \rightarrow \Leftrightarrow u$ for $u \neq 0$.

If it is not the gauge bosons that become massless, it can not be any of the “fundamental” fields in the Lagrangian. Therefore it must only be some kind of bound states or collective excitations. Two such possibilities are the monopoles and dyons. Indeed, we will take this to be the case. We can not prove this rigorously. In fact, the statement that it is the monopoles and dyons that become massless has the status of an assumption that passes many (non-trivial) tests.

This leads us to the following strategy:

1. By using the information about the masslessness of monopoles and dyons, we determine the fundamental group on the u -plane in terms of the monodromy matrices. The fact that this can be done in a consistent way is an argument in favour of the monopole/dyon interpretation of the singularities.

2. The monodromies and asymptotic values of the functions a_D and a on the u -plane allows us to determine unique and exact expressions for a_D and a (and thereby, implicitly, the function \mathcal{F}). This is highly non-trivial.

3. We add a perturbation to the theory which has the effect of causing the monopoles and dyons to develop vacuum expectation values. In the case of the monopoles this leads to confinement of electric charge by some standard arguments. This is in agreement with previous investigations of the perturbed theory, where one has found indications of confinement without consideration of the monopole topological solutions of the field equations.

5.5 The monodromies

We will now see that the assumption that the two singular points in the interior of the moduli space are points where monopoles and dyons become massless makes sense at the level of the monodromies of the moduli space. We will do this by first using the duality transformation to calculate explicitly the monodromy connected to the point where the monopoles become massless. Then it is possible to obtain the monodromy at the other singularity, and it is checked that this corresponds to massless dyons.

A magnetic monopole has $n_e = 0$ and $n_m = 1$, so if $M = 0$ then $a_D = 0$ by the mass formula. Let us call the point where this happens u_0 . The singular nature of u_0 is due to the fact that the monopoles are not included in the low energy theory. If we look at a small region around the point u_0 , where the mass of the monopoles is very small, we can include them in the low

²⁷For example, in QCD, chiral symmetry breaking appears together with mass terms, which breaks conformal invariance.

energy theory and all is fine. This corresponds to path integrating out all high energy modes that are larger than the monopole mass, including the ones that describes the heavy particles such as W -bosons. If we also perform a duality transformation so that the physics is described by the gauge vector field \mathcal{A}_D then we get an abelian gauge theory where the gauge field couples to magnetic monopoles just like an ordinary photon would couple to electrons. In other words, we have supersymmetric QED.

The fact that all things are magnetic does not make any difference. A person who lives in a world with a ground state close to u_0 would not know that the world was “magnetic”. Only someone who knew the full microscopic theory, including the entire moduli space, would know that the electric low energy variables – by definition those that are “handed down” from the microscopic theory – were dual to the magnetic ones.

The great virtue of having supersymmetric (in fact $N = 1$ supersymmetric) QED is that we know many properties of this theory including the running of the Wilsonian coupling constant²⁸ as a function of the low energy cutoff μ [18]:

$$\frac{1}{g_D^2} = \frac{1}{g_{D0}^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda}{\mu}.$$

Here Λ is a large cutoff momentum and g_{D0} is the bare coupling constant at that scale. Since we are using the monopole mass as the low energy cutoff and this is proportional to a_D , we get for the effective magnetic coupling constant $\tau_D(a_D)$ when $a_D \approx 0$:

$$\tau_D \approx \Leftrightarrow \frac{i}{\pi} \ln \frac{a_D}{\Lambda}. \quad (111)$$

The term $1/g_{D0}^2$ is omitted because it is small compared to the other terms. (111) also involves a choice of θ_D -parameter. Note that when $u \rightarrow u_0$, and thereby $a_D \rightarrow 0$, τ_D diverges. This means that the magnetic coupling constant g_D vanishes at this point. Conversely, since $\tau = \Leftrightarrow 1/\tau_D$, the electric coupling constant g diverges.

Near u_0 $\text{Im}\tau_D$ is a positive function of a_D . Since this is the metric on the moduli space expressed in the dual variables, it means that a_D is a good complex coordinate near that point and we can set

$$a_D \approx c_0(u \Leftrightarrow u_0), \quad (112)$$

with some complex constant c_0 . If we use the fact that $\tau_D = dh_D/da_D$ and $a(u) = \Leftrightarrow h_D(u)$, we can also find

$$a(u) \approx a_0 + \frac{i}{\pi} a_D \ln a_D \approx a_0 + \frac{i}{\pi} c_0(u \Leftrightarrow u_0) \ln(u \Leftrightarrow u_0) \quad (113)$$

for some $a_0 = a(u_0)$. This constant must be different from zero otherwise the electrically charged particles also become massless at u_0 which would invalidate our expression (111) for τ_D .

From (112) and (113) we can now find the monodromy matrix at the point u_0 where monopoles become massless. When $u \Leftrightarrow u_0 \rightarrow e^{2\pi i}(u \Leftrightarrow u_0)$ we have $\ln(u \Leftrightarrow u_0) \rightarrow \ln(u \Leftrightarrow u_0) + 2\pi i$, and so

$$\begin{aligned} a_D &\rightarrow a_D, \\ a &\rightarrow a \Leftrightarrow 2a_D. \end{aligned}$$

In matrix notation this is

$$M_1 = \begin{pmatrix} 1 & 0 \\ \Leftrightarrow 2 & 1 \end{pmatrix}$$

²⁸This coupling constant is not the same as the 1PI one, as we have previously said, but it coincides with it at one loop.

The reason for the subscript ‘1’ is that we now normalize the u -plane so that the two singularities occur at ± 1 .

From this monodromy we can find the last monodromy M_{-1} and then check if it corresponds to a dyon. For the monodromies to match, one turn in the counterclockwise sense at infinity must equal first one turn around $u = \Leftrightarrow 1$ and then one turn around $u = 1$ (see fig. (1)). That is,

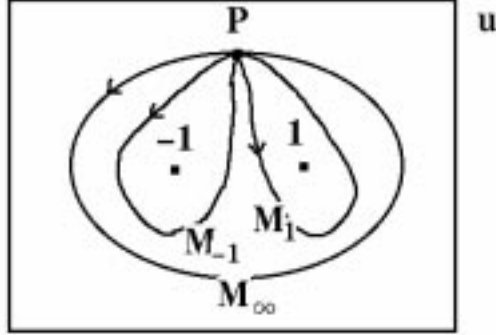


Figure 1: The monodromies of the u -plane

$$M_\infty = M_1 M_{-1},$$

and so²⁹

$$M_{-1} = M_1^{-1} M_\infty = \begin{pmatrix} \Leftrightarrow 1 & 2 \\ \Leftrightarrow 2 & 3 \end{pmatrix}.$$

We note that M_{-1} can be obtained from M_1 by conjugation with the matrix

$$A = \begin{pmatrix} \Leftrightarrow 1 & 1 \\ \Leftrightarrow 2 & 1 \end{pmatrix},$$

namely

$$M_{-1} = A M_1 A^{-1}.$$

Now we make an observation about the magnetic monopole. If we write this state, which becomes massless at $u = 1$, as $q_1 = (1, 0)$, then a property of this state is that

$$q_1 M_1 = q_1$$

Conjugation with A gives

$$q_1 A^{-1} A M_1 A^{-1} = q_1 A^{-1}$$

that is

$$q_{-1} M_{-1} = q_{-1}, \quad q_{-1} = q_1 A^{-1}$$

Since we have that $q_{-1} = (1, \Leftrightarrow 1)$ this confirms that dyons become massless at $u = \Leftrightarrow 1$.

²⁹Note that a choice of representation of the monodromies involves a choice of base point P in the moduli space.

5.6 The exact solution

Let us collect the information that we have so far. We have the u -plane with singularities at $\Leftrightarrow 1$, 1 and ∞ . The u -plane is the base manifold of a vector bundle, where the vectors take values in \mathbf{C}^2 modulo $SL(2, \mathbf{Z})$. The vectors $(a_D(u), a(u))$ are a section of this bundle. It has the asymptotic values

$$\begin{aligned} a &\approx \sqrt{2u}, \\ a_D &\approx i \frac{\sqrt{2u}}{\pi} \ln u, \end{aligned}$$

near $u = \infty$, and

$$\begin{aligned} a_D &\approx c_0(u \Leftrightarrow 1), \\ a &\approx a_0 + \frac{i}{\pi} a_D \ln a_D, \end{aligned}$$

with some constants a_0 and c_0 near $u = 1$. The behaviour near $u = \Leftrightarrow 1$ is similar and determined in principle by the symmetry $u \leftrightarrow \Leftrightarrow u$. We have the monodromies around ∞ , 1 and $\Leftrightarrow 1$:

$$\begin{aligned} M_\infty &= \begin{pmatrix} \Leftrightarrow 1 & 2 \\ 0 & \Leftrightarrow 1 \end{pmatrix}, \\ M_1 &= \begin{pmatrix} 1 & 0 \\ \Leftrightarrow 2 & 1 \end{pmatrix}, \\ M_{-1} &= \begin{pmatrix} \Leftrightarrow 1 & 2 \\ \Leftrightarrow 2 & 3 \end{pmatrix}. \end{aligned}$$

Besides this, the metric on the u -plane is

$$ds^2 = \text{Im}\tau |da|^2,$$

with

$$\tau = \frac{da_D/du}{da/du},$$

and so $\text{Im}\tau$ must be positive definite. With this information it is possible to find the unique and exact expression for $a_D(u)$ and $a(u)$. This also gives the function $\mathcal{F}(a)$ implicitly so in this sense we have therefore “solved” the theory exactly!

In order to obtain the exact expressions for a_D and a it is necessary to appeal to complicated mathematics such as complex curve theory. It is beyond the scope of this report to do so. Instead we will just state the exact expressions and then show that they have the desired asymptotic behaviour. a and a_D are given by the integrals

$$\begin{aligned} a &= \frac{\sqrt{2}}{\pi} \int_{-1}^1 \frac{dx \sqrt{x \Leftrightarrow u}}{\sqrt{x^2 \Leftrightarrow 1}}, \\ a_D &= \frac{\sqrt{2}}{\pi} \int_1^u \frac{dx \sqrt{x \Leftrightarrow u}}{\sqrt{x^2 \Leftrightarrow 1}}. \end{aligned} \tag{114}$$

Let us check the asymptotic values of (114) near $u = \infty$. We get

$$a \approx \frac{\sqrt{2u}}{\pi} \int_{-1}^1 \frac{dx}{\sqrt{1 \Leftrightarrow x^2}} = \frac{\sqrt{2u}}{\pi} [\arcsin x]_{-1}^1 = \sqrt{2u}.$$

For a_D we make the substitution $x = uz$:

$$a_D = \frac{\sqrt{2u}}{\pi} \int_{1/u}^1 \frac{dz \sqrt{z \Leftrightarrow 1}}{\sqrt{z^2 \Leftrightarrow u^{-2}}}.$$

The integral has a logarithmic divergence in $z = 0$ as $u \rightarrow \infty$. The divergent part is

$$a_D \approx \frac{\sqrt{2u}}{\pi} [\Leftrightarrow i \ln z]_{1/u} = i \frac{\sqrt{2u} \ln u}{\pi}.$$

These are the desired expressions for a_D and a . What about $u = 1$?

$$\begin{aligned} a_D &= \frac{\sqrt{2u}}{\pi} \int_{1/u}^1 \frac{dz \sqrt{z \Leftrightarrow 1}}{\sqrt{z \Leftrightarrow \frac{1}{u}} \sqrt{z + \frac{1}{u}}} \approx \frac{1}{\pi} \int_{1/u}^1 \frac{dz \sqrt{z \Leftrightarrow 1}}{\sqrt{z \Leftrightarrow \frac{1}{u}}} \\ &= \frac{i}{2} (1 \Leftrightarrow \frac{1}{u}) \approx \frac{i}{2} (u \Leftrightarrow 1). \end{aligned}$$

a is finite at $u = 1$:

$$a(u = 1) = \frac{\sqrt{2}}{\pi} \int_{-1}^1 \frac{dx}{\sqrt{x+1}} = \frac{\sqrt{2}}{\pi} [2\sqrt{x+1}]_{-1}^1 = \frac{4}{\pi}.$$

This gives us a_0 . To get the u -dependence of a near $u = 1$, we differentiate to get the next term in the Taylor expansion:

$$\frac{da}{du} = \Leftrightarrow \frac{\sqrt{2}}{2\pi} \int_{-1}^1 \frac{dx}{\sqrt{x+1} \sqrt{x \Leftrightarrow 1} \sqrt{x \Leftrightarrow u}}. \quad (115)$$

We have a logarithmic divergence at $x = 1$ when $u \rightarrow 1$. In this region, the integrand is nearly

$$\frac{\Leftrightarrow 1}{\sqrt{2}(u \Leftrightarrow x)}.$$

(115) thus becomes

$$\frac{da}{du} \approx \frac{\sqrt{2}}{2\pi} \left[\frac{\Leftrightarrow 1}{\sqrt{2}} \ln(u \Leftrightarrow x) \right]_{x=1} = \Leftrightarrow \frac{\ln(u \Leftrightarrow 1)}{2\pi},$$

and so the expression for a near $u = 1$ is

$$a = \frac{4}{\pi} \Leftrightarrow \frac{(u \Leftrightarrow 1) \ln(u \Leftrightarrow 1)}{2\pi} + \dots$$

This verifies that (114) are indeed the exact expressions.

Let us also recall that a holomorphic function – or a holomorphic section of a bundle – is essentially uniquely determined by its behaviour, so eqs. (114) are essentially unique.

5.7 Confinement of electric charge

Our last consistency check concerns confinement. Actually, we will perturb our $N = 2$ Yang-Mills theory to an $N = 1$ theory by adding a matter term for the Higgs field Φ , and it is this theory which has confinement. The point is that this conclusion can be reached in two different ways. The first one was used by Witten in Ref. [36]. Here one puts the system in a box with finite volume and the gauge field is required to obey the so-called 't Hooft's twisted boundary conditions. The topologically non-trivial configurations of the gauge field in such a set-up are an integer number of color-electric and color-magnetic flux lines in the x -, y - or z -direction. One can then show that the energy connected to a single electric flux line is finite – that is, greater

than zero. When the volume of the box is taken to infinity, the energy of a single flux line also goes to infinity, which implies confinement as will be explained below. Note that no mention of magnetic monopoles has been made.

The second way has to do with the condensation of the magnetic monopoles. By writing the low energy Lagrangian in terms of the dual, magnetic variables, it is a matter of solving the equations for the vacuum expectation values of the monopoles in the presence of the perturbation to see that they condense. Before we do that, however, let us see why monopole condensation leads to confinement of electric charge.

The prototypical problem of confinement is that of quarks in QCD³⁰. As a consequence of their having a color charge, they are the sources of color electric flux lines. These flux lines radiate inwards or outwards depending on whether one is dealing with a quark or an antiquark. This is very similar to QED. However, it is believed that color electric flux lines are always bunched together into thin stringlike tubes. One indication that it is in fact so is the observed occurrence of the so-called Regge trajectories. Let us consider mesons for simplicity, i.e. bound states of a quark and an antiquark. If we considered all the mesonic states that were observed with exactly the same quantum numbers (such as isospin, strangeness etc.) except mass and spin, they would fall on a straight line in a plot like the one in fig. (2). This is in agreement with what one would calculate from a classical system consisting of a rotating “rod” with a uniform

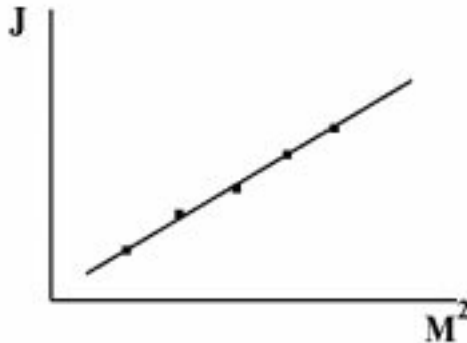


Figure 2: Regge trajectory

energy density and two massless point particles at each end. Intuitively one sees that the faster this (relativistic) system rotates the longer is the rod because of the centrifugal force on its ends, and thus the mass increases with increasing spin. This tells us that color electric flux lines are bunched together into tubes with a uniform energy density.

Color electric flux strings with uniform energy density implies confinement because a single free quark would sit at the end of an infinitely long string which would then have an infinitely large energy. This does not make sense and must be ruled out. Alternatively we could start with a bound state of a quark and an antiquark and then try to separate them by pulling them away from each other. As we pulled, the flux string between them would grow longer and the energy that resided in the string would increase. At one point, however, we would reach the point where the energy of the string would be equal to the threshold of the creation of a new quark-antiquark pair from the vacuum. Thus a new quark would pop out and stick to the old quark and a new antiquark would pop out and stick to the old antiquark, forming two new bound meson states. We could therefore separate a quark from an antiquark but we could not make

³⁰The following account of confinement is close to that of Huang, Ref. [37].

them free.

So far we have established that quark confinement is equivalent to the fact that color electric flux lines are collected into thin strings. We still need to know the mechanism behind this. We can get a hint to this by looking at another system with similar properties, namely a type II superconductor. ‘Type II’ means that if we take a sample of such a superconductor and place it in a magnetic field, then the magnetic field is not altogether expelled from the sample, as it would be in the type I case. Instead the magnetic field is confined into thin stringlike regions – the Abrikosov vortices. Hence we have a system that bunches magnetic flux lines into tubes and thus leads to confinement of magnetic charge. There is a slight problem, of course, that it is magnetic charge that is confined rather than electric³¹, but we shall see that the electric-magnetic duality we have in our supersymmetric theory solves the problem. A progress lies in the fact that we actually know the mechanism that collects the magnetic flux lines in a superconductor.

It turns out that we can describe a superconductor by a relativistic quantum field theory. More precisely it is an abelian Higgs theory with an abelian gauge field coupled to a complex scalar Higgs field. In the superconductor, the gauge field is just the photon, and the Higgs field is the Cooper-pair condensate. The Higgs field feels a ‘Mexican hat’ potential, i.e. $V(|\phi|) = (|\phi|^2 - \phi_0^2)^2$. A well known topologically stable (and non-trivial) solution to the equations of motion is a magnetic vortex line – a Nielsen-Olesen vortex. Obviously, since it is stable this represents a local minimum of the energy, and so magnetic flux lines are bunched together because it is energetically favourable. From this we deduce that magnetic charge is confined in a (type II) superconductor. Note that confinement is a property of the “vacuum”, or in this case the superconducting medium, and not a property of the dynamical quarks or charges themselves. In a sense it is a statement about (possibly hypothetical) test charges.

Now we turn to the problem of the desired electrical confinement. This has a natural solution when we suitably perturb the $N = 2$ theory. In the unperturbed theory there are two dual and equivalent descriptions of the physics at generic points of the moduli space. One is the ‘electrical’ one with the gauge superfield W and chiral superfield A that are remnants of the three W^a and Φ^a in the microscopical theory. The other one is the dual ‘magnetic’ one with gauge superfield W_D and a chiral field A_D . By construction, the gauge field $V_{D\mu}$ in W_D couples to magnetic charge just like an ordinary photon field would couple to electric charge. It is the electric charge in the first description that we want to confine. The point is that we have a gauge field that couples to magnetic charge, and near the point $u = 1$ we also have fields that describes the magnetic monopoles *including two complex scalars of magnetic charge*. Here we have used that near the point $u = 1$ there must be a sensible field description of the monopoles, since at exactly this point they become massless and so can be pair created *en masse*. These fields must constitute an $N = 2$ matter multiplet, or in $N = 1$ language, the chiral superfields M and \tilde{M} with one complex scalar each. All we need now in order to actually have the same situation as in the abelian Higgs model, is a non-vanishing vacuum expectation value for one (or both) of the scalars. This is where the perturbation to $N = 1$ comes in.

A mass term for the chiral fields Φ is $m\text{Tr}\Phi^2$. Since $\text{Tr}\Phi^2$ is the gauge invariant chiral field whose scalar component expectation value is u , we will call it U . We add mU to the tree level Lagrangian. The low energy effective theory expressed in the dual variables before we add the perturbation has the Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left[A_D e^{V_D} \bar{A}_D + M e^{V_D} \bar{M} + \tilde{M} e^{-V_D} \bar{\tilde{M}} \right]$$

³¹The fact that it is a $U(1)$ (magnetic) charge rather than a, say, $SU(2)$, charge that is confined in a superconductor will not bother us. We will take the confinement of the $U(1)$ electric charge to be the problem we are interested in.

$$+ \int d^2\theta \left[W_D^2 + A_D M \tilde{M} \right] + h.c. \quad (116)$$

The term $A_D M \tilde{M}$ is the superpotential and is required by $N = 2$. By adding the term mU at the tree level this effective superpotential turns into

$$\hat{W} = A_D M \tilde{M} + mU(A_D) \quad (117)$$

The term ‘ mU ’ is in other words unchanged after the quantum corrections. Note that the superfield U is to be regarded as a (complicated) function of A_D . U is a gauge independent field and should thus not be sensitive to the gauge dependent fields that are taken as “fundamental” in the description of the (gauge independent) physics. (117) is found by considering the symmetries, holomorphy and the small m limit, that is, by the non-renormalization theorem.

We are now interested in the ground state of this perturbed theory. We must therefore solve

$$\frac{d\hat{W}}{dA_D} = 0, \quad \text{and} \quad \frac{d\hat{W}}{dM} = \frac{d\hat{W}}{d\tilde{M}} = 0$$

(cf. the discussion of potentials in Sec. 3.4). In addition, M and \tilde{M} , the scalar components of the superfields of the same name (!) satisfy

$$|M|^2 \Leftrightarrow |\tilde{M}|^2 = 0$$

from the D -terms. In other words $|M| = |\tilde{M}|$. We now get

$$\begin{aligned} \sqrt{2}M\tilde{M} + m\frac{du}{da_D} &= 0, \\ a_DM &= a_D\tilde{M} = 0. \end{aligned}$$

These are the scalar field equations, and $u'(a_D) = du/da_D$ is a component of a “derivative superfield” $U'(A_D)$. Thus, when $m \neq 0$, we get

$$M = \tilde{M} = \sqrt{\frac{\Leftrightarrow mu'(0)}{\sqrt{2}}}, \quad a_D = 0.$$

So M and \tilde{M} has a non-vanishing expectation value and we have confinement. Note that $a_D = 0$ in the ground state which means that only the point $u = 1$ remains of the moduli space (in the region close to this point, where (116) is a relevant Lagrangian).

This concludes our discussion of the $N = 2$ supersymmetric Yang-Mills theory.

A Representations of the Lorentz group

We use the metric $g_{\mu\nu} = \text{diag}(+1, \Leftrightarrow 1, \Leftrightarrow 1, \Leftrightarrow 1)$. The generators $M_{\mu\nu}$ of the Lorentz group obey the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\nu\rho}M_{\mu\sigma} \Leftrightarrow g_{\nu\sigma}M_{\mu\rho} \Leftrightarrow g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho})$$

This is also the algebra of the group $SL(2, \mathbf{C})$ which is the covering group of the Lorentz group. All fields must transform according to a representation of this algebra, so we want to find these representations starting with the irreducible ones³².

We introduce the generators J_i of spatial rotations and K_i of boosts by

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, \quad K_i = M_{0i}.$$

³²See e.g. Ref. [11]

They obey the commutation relations

$$\begin{aligned}[J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\epsilon_{ijk}K_k, \\ [K_i, K_j] &= \Leftrightarrow i\epsilon_{ijk}J_k,\end{aligned}$$

If we now define

$$A_i = \frac{1}{2}(J_i + iK_i), \quad B_i = \frac{1}{2}(J_i \Leftrightarrow iK_i),$$

we get the decoupled commutation relations

$$\begin{aligned}[A_i, A_j] &= i\epsilon_{ijk}A_k, \\ [B_i, B_j] &= i\epsilon_{ijk}B_k, \\ [A_i, B_j] &= 0.\end{aligned}$$

The J_i and the K_i are hermitean operators so the A_i and the B_i are not – they are each others adjoints:

$$A_i^\dagger = B_i, \quad B_i^\dagger = A_i.$$

Nevertheless, we have managed to write the algebra of $SL(2, C)$ as the direct sum of two $SU(2)$ algebras, of which we know the irreducible representations. These are characterized by the eigenvalues of the Casimir operators

$$\begin{aligned}\mathbf{A}^2 &= a(a+1), \quad a = 0, \frac{1}{2}, 1, \dots \\ \mathbf{B}^2 &= b(b+1), \quad b = 0, \frac{1}{2}, 1, \dots\end{aligned}$$

I.e. an irreducible representation can be labelled by (a, b) , and \mathbf{A}^2 and \mathbf{B}^2 acts within this representation as $a(a+1)$ and $b(b+1)$ times the identity, respectively. Since $J_i = A_i + B_i$, the spin of a representation is $a + b$.

The simplest representations are:

1. $(0, 0)$: The scalar representation.
2. $(\frac{1}{2}, 0)$: A left-handed spinor (the left-handedness is a convention). We would like to have $J^i = \frac{1}{2}\sigma^i$, because the spin is $\frac{1}{2}$. This can be done if we take

$$A^i = \frac{1}{2}\sigma^i, \quad B^i = 0.$$

We then have

$$\begin{aligned}M^{0i} &\equiv \frac{1}{2}\sigma^{0i} = \Leftrightarrow i(A^i \Leftrightarrow B^i) = \Leftrightarrow \frac{1}{2}i\sigma^i, \\ M^{ij} &\equiv \frac{1}{2}\sigma^{ij} = \epsilon^{ijk}(A^k + B^k) = \frac{1}{2}\epsilon^{ijk}\sigma^k,\end{aligned}$$

where we also have defined the matrices $\sigma^{\mu\nu}$. The $\sigma^{\mu\nu}$ are antisymmetric in μ and ν , and satisfy the selfduality relations

$$\tilde{\sigma}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\sigma^{\rho\sigma} = i\sigma^{\mu\nu}, \quad \epsilon^{0123} = 1.$$

A spinor ψ that transforms in this representation has two components labelled by an index α, β, \dots , etc. (for the index structure, see Appendix B). A Lorentz transformation Λ with the infinitesimal parameters $\omega_{\mu\nu} = \Leftrightarrow \omega_{\nu\mu}$ is given by

$$U(\Lambda) \approx I + \frac{1}{2}i\omega_{\mu\nu}M^{\mu\nu},$$

and so from the expression

$$U(\Lambda)\psi_\alpha U^{-1}(\Lambda) \approx (\delta_\alpha^\beta + \frac{1}{2}i\omega_{\mu\nu} \cdot \frac{1}{2}(\sigma^{\mu\nu})_\alpha^\beta)\psi_\beta$$

we find the transformation property of a left-handed spinor:

$$[M^{\mu\nu}, \psi_\alpha] = \frac{1}{2}(\sigma^{\mu\nu})_\alpha^\beta \psi_\beta.$$

3. $(0, \frac{1}{2})$: A right-handed spinor. Here we take

$$A^i = 0, \quad B^i = \frac{1}{2}\sigma^i,$$

so that we have

$$\begin{aligned} M^{0i} &\equiv \frac{1}{2}\bar{\sigma}^{0i} = \Leftrightarrow i(A^i \Leftrightarrow B^i) = \frac{1}{2}i\sigma^i, \\ M^{ij} &\equiv \frac{1}{2}\bar{\sigma}^{ij} = \epsilon^{ijk}(A^k + B^k) = \frac{1}{2}\epsilon^{ijk}\sigma^k. \end{aligned}$$

We record that $\bar{\sigma}^{\mu\nu} = (\sigma^{\mu\nu})^\dagger$, which could have been anticipated because the $(0, \frac{1}{2})$ is the hermitean adjoint representation of $(\frac{1}{2}, 0)$. We also have that $\bar{\sigma}^{\mu\nu}$ satisfy the selfduality relations

$$\bar{\sigma}^{\tilde{\mu}\nu} = \Leftrightarrow i\bar{\sigma}^{\mu\nu}.$$

A spinor $\bar{\psi}$ that transforms in this representation has dotted indices $\dot{\alpha}, \dot{\beta}, \dots$, etc. Its transformation properties are

$$[M^{\mu\nu}, \bar{\psi}_{\dot{\alpha}}] = \Leftrightarrow \frac{1}{2}\bar{\psi}_{\dot{\beta}}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}}.$$

Note that a parity transformation acts by

$$J^i \rightarrow J^i \quad \text{and} \quad K^i \rightarrow \Leftrightarrow K^i.$$

Thus the $(\frac{1}{2}, 0)$ is the parity transform of $(0, \frac{1}{2})$ and vice versa.

Other representations can be built from the spinor representations by addition or multiplication:

4. $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$: A Dirac spinor. The four components of the Dirac spinor Ψ is organized as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix},$$

where ψ and $\bar{\chi}$ are its constituent left- and right-handed two-spinors, respectively. Its four components are also labelled by (undotted) letters from the beginning of the Greek alphabet. We more to say about four-spinors in Appendix B.

5. $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$: A four-vector. This is true because it is a four dimensional irreducible representation. The spin is also 1. The four vector $V_{\alpha\dot{\beta}}$ can be written in the conventional (Lorentz indexed) way by

$$V_{\alpha\dot{\beta}} = V_\mu \sigma^\mu_{\alpha\dot{\beta}},$$

where $\sigma^\mu \equiv (1, \vec{\sigma})$.

6. $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0) \oplus (1, 0)$: This is a sum of a scalar and an antisymmetric tensor with definite selfduality properties. The tensor $T_{\alpha\beta}$ has the expansion

$$T_{\alpha\beta} = T\epsilon_{\alpha\beta} + T_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}.$$

Similarly, if $T_{\dot{\alpha}\dot{\beta}}$ is an antisymmetric, anti-selfdual tensor in the representation $(0, 1)$, then

$$T_{\dot{\alpha}\dot{\beta}} = T_{\mu\nu}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}.$$

A second rank, antisymmetric Lorentz tensor without any selfduality properties thus transforms in the $(1, 0) \oplus (0, 1)$ representation. An example of this is the Maxwell tensor $F_{\mu\nu}$.

B Spinors

Two-spinors are anticommuting objects that transform under the group $SL(2, \mathbf{C})$, which is the covering group of the Lorentz group. The elements of $SL(2, \mathbf{C})$ are the 2×2 complex matrices with determinant 1. There are four equivalent representations of $SL(2, \mathbf{C})$ in the sense that if M is a matrix that represents an element, then the hermitian conjugate M^\dagger , the transpose inverse $(M^T)^{-1}$, and the hermitian conjugate inverse $(M^\dagger)^{-1}$, represents the same element equivalently. Upper or lower and dotted or undotted indices distinguish the various representations:

$$\psi'_\alpha = M_\alpha^\beta \psi_\beta, \quad \bar{\psi}'_{\dot{\alpha}} = M^*_{\dot{\alpha}}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \quad (118)$$

$$\psi'^\alpha = M^{-1}{}^\alpha_\beta \psi^\beta, \quad \bar{\psi}'^{\dot{\alpha}} = (M^*)^{-1}{}^{\dot{\alpha}}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}. \quad (119)$$

This is in contrast to e.g. the unitary group $SU(2)$, where there are only two equivalent representations:

$$x'_i = U_i^j x_j, \quad x'^i = U^{-1}{}^i_j x^j.$$

Just like for the unitary groups, we can raise and lower the two spinor indices by using ϵ -tensors:

$$\begin{aligned} \psi^\alpha &= \epsilon^{\alpha\beta} \psi_\beta, & \bar{\psi}^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \\ \psi_\alpha &= \epsilon_{\alpha\beta} \psi^\beta, & \bar{\psi}_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}. \end{aligned}$$

The ϵ -tensors are defined by

$$e_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & \epsilon 1 \\ 1 & 0 \end{pmatrix},$$

and similarly for the dotted ϵ -tensors. With these definitions we have $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$. Scalar products between two two-spinors are defined by contraction of upper and lower indices:

$$\begin{aligned} \psi\chi &\equiv \psi^\alpha \chi_\alpha = \epsilon\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi, \\ \bar{\psi}\bar{\chi} &\equiv \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \epsilon\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}. \end{aligned}$$

We can go from undotted to dotted indices and vice versa by complex conjugation:

$$(\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}}, \quad (\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}},$$

so that undotted spinors transform in the $(\frac{1}{2}, 0)$ -representation and dotted spinors transform in the (conjugate) $(0, \frac{1}{2})$ -representation of the Lorentz group. The σ -matrices $\sigma^\mu = (I, \sigma^i)$ and $\bar{\sigma}^\mu = (I, \epsilon\sigma^i)$ have the index structure

$$\sigma^\mu_{\alpha\dot{\beta}} \quad \text{and} \quad \bar{\sigma}^{\mu\dot{\alpha}\beta},$$

so that $\psi_\alpha \bar{\chi}_{\dot{\beta}}$ and $\bar{\psi}_{\dot{\beta}} \chi_\alpha$ can be written into “manifest” four-vectors by

$$\psi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \chi^{\dot{\beta}} \quad \text{and} \quad \bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \chi_\beta.$$

There is also the relation

$$\bar{\sigma}^{\mu\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\beta\delta} \sigma^\mu_{\delta\dot{\gamma}}.$$

The generators $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ of the Lorentz transgormations, defined in Appendix A, can be given in terms of the σ -matrices by

$$\begin{aligned} \sigma^{\mu\nu}{}^\beta{}_\alpha &= \frac{1}{2}i(\sigma^\mu_{\alpha\dot{\gamma}} \bar{\sigma}^{\nu\dot{\gamma}\beta} \Leftrightarrow \sigma^\nu_{\alpha\dot{\gamma}} \bar{\sigma}^{\mu\dot{\gamma}\beta}), \\ \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{2}i(\bar{\sigma}^{\mu\dot{\alpha}\gamma} \sigma^\nu_{\gamma\dot{\beta}} \Leftrightarrow \bar{\sigma}^{\nu\dot{\alpha}\gamma} \sigma^\mu_{\gamma\dot{\beta}}). \end{aligned}$$

Four-spinors are given in terms of two-spinors by

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

to give the $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ -representation of the Lorentz group. We define γ -matrices by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \Leftrightarrow I & 0 \\ 0 & I \end{pmatrix}.$$

They satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma_5^2 = I$. A barred four-spinor is defined in the usual way by

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0 = (\chi_\alpha, \bar{\psi}^{\dot{\alpha}}).$$

Thus, if

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \phi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix},$$

we have the product

$$\bar{\Psi}\Phi = \chi\phi + \bar{\psi}\bar{\eta}.$$

Other bilinears are straightforward to work out. We can define a charge conjugation matrix by

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix},$$

and thereby a charge conjugate spinor

$$\Psi^c = C\bar{\Psi}^T,$$

which is then the charge-conjugated of Ψ . A Majorana spinor is a self-conjugate spinor:

$$\Psi = \Psi^c.$$

It contains only one two-spinor in the following way:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}.$$

Bilinear expressions in Majorana spinors have some properties which bilinear expressions in non-Majorana spinors does not have, which is often useful in calculations. These are the so-called

‘Majorana flip’ properties. If γ denotes $I, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5$ or $\gamma^\mu \gamma^\nu$, ($\mu \neq \nu$), then the majorana flip properties are that

$$\bar{\Psi}, \Phi = \bar{\Psi}, \tilde{\Psi},$$

where

$$\begin{aligned} \tilde{\gamma} &= +, & \text{for } \gamma &= I, \gamma_5 \text{ or } \gamma^\mu \gamma_5, \\ \tilde{\gamma} &= \Leftrightarrow, & \text{for } \gamma &= \gamma^\mu \text{ or } \gamma^\mu \gamma^\nu \ (\mu \neq \nu) \end{aligned}$$

(see also eqs. (149-153) in Appendix D).

C Notation and conventions

$$\text{Metric:} \quad g_{\mu\nu} = (1, \Leftrightarrow 1, \Leftrightarrow 1, \Leftrightarrow 1) \quad (120)$$

$$\text{Levi-Civita tensor:} \quad \epsilon^{0123} = 1 \quad (121)$$

$$\epsilon\text{-tensor:} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ \Leftrightarrow 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & \Leftrightarrow 1 \\ 1 & 0 \end{pmatrix} \quad (122)$$

$$\text{Spinor product:} \quad \psi\chi = \psi^\alpha \chi_\alpha, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \quad (123)$$

$$\text{Hermitean conjugation of spinors:} \quad (\psi^\alpha \chi_\alpha)^\dagger = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \quad (124)$$

$$\sigma\text{-matrices:} \quad \sigma^\mu = (I, \sigma^i), \quad \bar{\sigma}^\mu = (I, \Leftrightarrow \sigma^i) \quad (125)$$

$$\gamma\text{-matrices:} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} \Leftrightarrow I & 0 \\ 0 & I \end{pmatrix} \quad (126)$$

$$\text{Dirac spinor:} \quad \Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (127)$$

$$\text{Charge conjugation matrix:} \quad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \quad (128)$$

$$\text{Majorana spinor:} \quad \Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad \Psi = \Psi^c \equiv C\bar{\Psi}^T \quad (129)$$

$$\begin{aligned} \text{Index raising and lowering:} \quad \psi^\alpha &= \epsilon^{\alpha\beta} \psi_\beta, & \psi_\alpha &= \epsilon_{\alpha\beta} \psi^\beta, \\ \bar{\psi}^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}, & \bar{\psi}_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} \end{aligned}$$

D Useful fomulae

$$\theta^\alpha \theta^\beta = \Leftrightarrow \frac{1}{2} \epsilon^{\alpha\beta} \theta\theta, \quad \theta_\alpha \theta_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \theta\theta, \quad (130)$$

$$\bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \Leftrightarrow \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}\bar{\theta}, \quad \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}\bar{\theta} \quad (131)$$

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma \quad (132)$$

$$\theta^\alpha \theta_\beta = \frac{1}{2} \delta_\beta^\alpha \theta\theta \quad (133)$$

$$\bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \Leftrightarrow \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}\bar{\theta} \quad (134)$$

$$\frac{\partial}{\partial \theta^\alpha} \theta\theta = 2\theta_\alpha, \quad \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \bar{\theta}\bar{\theta} = 2\bar{\theta}_{\dot{\alpha}} \quad (135)$$

$$\frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \theta\theta = \Leftrightarrow 4 \quad (136)$$

$$\frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \bar{\theta}\bar{\theta} = 4 \quad (137)$$

$$\bar{\sigma}^{\mu\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\beta\delta} \sigma_{\delta\dot{\gamma}}^\mu \quad (138)$$

$$\sigma^{\mu\nu} = \frac{1}{2} i (\sigma^\mu \bar{\sigma}^\nu \Leftrightarrow \sigma^\nu \bar{\sigma}^\mu) \quad (139)$$

$$(\sigma^\mu \bar{\sigma}^\nu)_\alpha{}^\beta = g^{\mu\nu} \delta_\alpha{}^\beta \Leftrightarrow i (\sigma^{\mu\nu})_\alpha{}^\beta \quad (140)$$

$$\epsilon_{\gamma\beta} (\sigma^{\mu\nu})_\alpha{}^\beta = \epsilon_{\alpha\beta} (\sigma^{\mu\nu})_\gamma{}^\beta \quad (141)$$

$$\text{Tr} \sigma^\mu \bar{\sigma}^\nu = (\sigma^\mu \bar{\sigma}^\nu)_\alpha{}^\alpha = 2g^{\mu\nu} \quad (142)$$

$$\text{Tr} \sigma^{\mu\nu} \sigma^{\rho\sigma} = 2(g^{\mu\rho} g^{\nu\sigma} \Leftrightarrow g^{\mu\sigma} g^{\nu\rho}) + 2i \epsilon^{\mu\nu\rho\sigma} \quad (143)$$

$$(\theta \sigma^\mu \bar{\theta})(\theta \sigma^\nu \bar{\theta}) = \frac{1}{2} \theta\theta \bar{\theta}\bar{\theta} g^{\mu\nu} \quad (144)$$

$$(\theta\phi)(\theta\psi) = \Leftrightarrow \frac{1}{2} (\phi\psi)(\theta\theta) \quad (145)$$

$$(\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = \Leftrightarrow \frac{1}{2} (\bar{\phi}\bar{\psi})(\bar{\theta}\bar{\theta}) \quad (146)$$

$$\chi \sigma^\mu \bar{\psi} = \Leftrightarrow \bar{\psi} \bar{\sigma}^\mu \chi \quad (147)$$

$$(\chi \sigma^\mu \bar{\psi})^\dagger = \psi \sigma^\mu \bar{\chi} \quad (148)$$

$$\bar{\Psi} \Phi = \bar{\Phi} \Psi = \bar{\psi} \bar{\phi} + \psi \phi \quad (149)$$

$$\bar{\Psi} \gamma_5 \Phi = \bar{\Phi} \gamma_5 \Psi = \bar{\psi} \bar{\phi} \Leftrightarrow \psi \phi \quad (150)$$

$$\bar{\Psi} \gamma^\mu \Phi = \Leftrightarrow \bar{\Phi} \gamma^\mu \Psi = \psi \sigma^\mu \bar{\phi} + \bar{\psi} \bar{\sigma}^\mu \phi \quad (151)$$

$$\bar{\Psi} \gamma^\mu \gamma_5 \Phi = \bar{\Phi} \gamma^\mu \gamma_5 \Psi = \psi \sigma^\mu \bar{\phi} \Leftrightarrow \bar{\psi} \bar{\sigma}^\mu \phi \quad (152)$$

$$\bar{\Psi} \gamma^\mu \gamma^\nu \Phi = \Leftrightarrow \bar{\Phi} \gamma^\mu \gamma^\nu \Psi = \Leftrightarrow 2 \psi \sigma^\mu \bar{\sigma}^\nu \phi \Leftrightarrow 2 \bar{\psi} \bar{\sigma}^\mu \sigma^\nu \phi \quad (153)$$

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