IFT

Instituto de Física Teórica
Universidade Estadual Paulista

TESE DE DOUTORAMENTO IFT–T.005/13

Harmonic Superspace from the $AdS_5 \times S^5$ Pure Spinor Formalism

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Agosto de 2013

Agradecimentos

Ao Nathan pela orientação precisa nestes quatro anos e pela grande preocupação com a minha formação.

Ao Mikhailov por várias discussões úteis na elaboração deste trabalho.

Ao Thales pela amizade, por inúmeras discussões e pela leitura cuidadosa desta tese. Ao Renann pela amizade e inúmeras discussões.

Ao Oscar pela amizade, discussões e pela colaboração.

A todo o grupo de cordas do IFT, professores, pós-docs e alunos. A todos os amigos que eu fiz no IFT e que de uma forma ou de outra colaboraram para existência desta tese. Não irei citar todos os nomes, porque a lista seria grande.

À Ana pela amizade, compreensão, carinho e atenção nestes quatro anos. Agradeço a ela também pelo grande apoio em momentos difícies.

À Iara, a minha tia Silvia, ao Ulisses e ao Javier por todo o apoio durante alguns momentos difícies.

Ao Ting e ao Gabriel, dois dos meus grandes incentivadores.

Ao Instituto Perimeter e ao Pedro pela oportunidade de estágio. A todos os amigos e colaboradores que eu fiz lá, entre eles, o João.

Às minhas avós, Diley e Mercedes, pelo amor e apoio.

Aos meus avôs Benito e Luiz Carlos, *in memoriam*, que tanto torceram por mim.

À minha mãe por todo o amparo, estímulo, ajuda e carinho.

Ao meu pai e a Rose por todo o apoio, ajuda, carinho e suporte nestes quatro anos.

À Alessandra e família, pelo amor nestes quatro anos, pelo incentivo e pela compreensão.

À FAPESP pelo apoio financeiro e ao IFT.

Resumo

A conjectura de Maldacena ou (AdS/CFT) desde a sua formulação é um dos tópicos em física de altas energias mais estudados. Uma das versões da conjectura é a dualidade entre a teoria de supercordas do tipo IIB em um *background* $AdS^5 \times S^5$ suportado por um fluxo Ramond-Ramond e a teoria de $\mathcal{N} = 4$ super-Yang-Mills em quatro dimensões. Embora a ação para supercordas neste *background* seja conhecida tanto no formalismo de Green-Schwarz como no formalismo de espinores puros, a construção explícita dos operadores de vértice da teoria em termos de supercampos é um problema em aberto. Nesta tese, os operadores de vértice do formalismo de espinores puros correspondentes aos estados de supergravidade são construídos próximos a fronteira de AdS. A conjectura prevê que todo estado na camada de massa da supercorda é dual a um operador invariante de gauge de $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills, em particular, os estados de supergravidade são duais a operadores *Half-BPS*. Os operadores *Half-BPS* e seus duais podem ser descritos como supercampos em um superespaço harmônico. Os resultados obtidos para os operadores de vértice são descritos em função desses supercampos duais de acordo com o previsto pela conjectura.

Palavras Chaves: Supercordas; Conjectura (AdS/CFT); $\mathcal{N}$=4 super-Yang-Mills; Supersimetria; Superespaço harmônico.

Áreas do conhecimento: Ciências Exatas e da Terra; Física de Partículas e Campos; Física Matemática.

Abstract

The Maldacena's conjecture or (AdS/CFT) has been one of the most studied topics in high energy physics since its formulation. One of the versions of the conjecture is the duality between the theory of type IIB superstrings in the background $AdS^5 \times S^5$ supported by a Ramond-Ramond flux and the theory of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions. Although the action for the superstrings in this background is known both in the Green-Schwarz and in the pure spinor formalisms, an explicit superfield construction of the vertex operators of the theory is an open problem. In this thesis, using the pure spinor formalism, we explicitly construct the vertex operators corresponding to supergravity states close to the boundary of AdS. The conjecture predicts that every on-shell superstring state is dual to a gauge-invariant operator of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills, in particular, the supergravity states are dual to Half-BPS operators. It is possible to describe all the Half-BPS operators and their duals as superfields in harmonic superspace. The results for the vertex operators are described in terms of these dual superfields in agreement with the prediction of the conjecture.

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Chapter 1

Introduction

The Maldacena's conjecture or (AdS/CFT) [1, 2, 3] has been one of the most studied topics in high energy physics since its formulation. One of the versions of the conjecture, and the one relevant for this thesis, is the impressive duality between the theory of type IIB superstrings in the background $AdS^5 \times S^5$ supported by a Ramond-Ramond flux and the theory of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions, two reviews are [4, 5]. The predicted duality is of the "weak-strong" type, or in other words, the strong-coupling regime of one of the theories is mapped into the weak-coupling regime of the dual theory, which makes the conjecture very attractive but hard to prove. The conjecture is the only known tool to perform several computations, for example, one can compute the correlation function of two gauge-invariant operators of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills in the strong-coupling limit from the study of a classical solution of the superstrings equations of motion [6]. Moreover, the conjecture has several applications such as in condensed matter physics and plasma physics, see [7, 8], for example.

In this specific version of the conjecture, the dual theories are related as follows. The beta function of the $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills vanishes in all orders in perturbation theory [9, 10], which implies that the theory is superconformal with the global symmetry group $PSU(2, 2|4)$, and this is precisely the isometry group of the background $AdS^5 \times S^5$. The super-Yang-Mills admits a 't Hooft expansion and, if its gauge group is $SU(N)$ and g_{YM} its coupling constant, the effective coupling constant of the theory is the 't Hooft parameter $\lambda = g_{YM}^2 N$, an introductory book about the N expansion is [11]. In addition, the theory can have a non-zero θ angle. Considering its dual theory, the θ angle is proportional to the VEV of the Ramond-Ramond scalar, the g_s string coupling is $g_s = \lambda/N$ and $\alpha'^2 \sim R^4/\lambda$ where $1/\alpha'^2$ is proportional to the string tension and R is the radius of both AdS^5 and S^5. Furthermore, the conjecture predicts that every on-shell superstring state is dual to a gauge-invariant single-trace operator of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills, its energy corresponding to the dimension of the operator [12]. In the particular case of supergravity states, the dual operators are the Half-BPS operators.

In order to study superstrings in $AdS^5 \times S^5$, it is possible to use both the Green-Schwarz and the pure spinor formalisms. The RNS formalism cannot be used, because it is a Ramond-Ramond background. The superstring action in the Green-Schwarz formalism was constructed by Metsaev and Tseytlin in [13], an excellent review is [14]. The action is κ-invariant and as in the flat space case the theory has first- and second-class constraints. The usual procedure for quantizing the theory is to go to the light-cone gauge, however, in a curved background the procedure of gauge-fixing is more involved than in flat space case, and the resulting Hamiltonian is non-polynomial in the worldsheet variables, which differs from the flat space case where the Hamiltonian is free and straightforward to quantize. In addition, after the procedure of gauge-fixing not all the original symmetries of the theory are kept manifest. The action for the superstrings in the pure spinor formalism was constructed by Berkovits in [15], a recent review is [16]. Unlike in the Green-Schwarz formalism, there are no constraints on the canonical momenta. The action is BRST invariant and the quantization is done preserving all the symmetries of the theory manifest imposing that the physical states are states in the cohomology at +2 ghost number of the BRST operator.

Although the action for superstrings in $AdS^5 \times S^5$ is known both in the Green-Schwarz and pure spinor formalisms, an explicit superfield construction of the vertex operators of the theory is an open problem. The first article about vertex operators in this curved background in the pure spinor formalism was the article by Berkovits and Chandia [17]. In this work, the authors proved the existence of a massless vertex operator by requiring that it preserves all the isometries of the background and reduces to the known flat space result [18] in the flat space limit. Moreover, they proved that the vertex operator is described in terms of an $\mathcal{N} = 2$ bispinor superfield $A_{\bar{\alpha}\hat{\bar{\alpha}}}(x, \theta, \hat{\theta})$ in ten dimensions as

$$V = \lambda^{\bar{\alpha}} \hat{\lambda}^{\hat{\bar{\alpha}}} A_{\bar{\alpha}\hat{\bar{\alpha}}}(x, \theta, \hat{\theta}) \,, \tag{1.1}$$

where $[x, \theta, \hat{\theta}]$ are the $\mathcal{N} = 2$ $d = 10$ superspace coordinates, $\lambda^{\bar{\alpha}}$ and $\lambda^{\hat{\bar{\alpha}}}$ are the left- and right-moving bosonic pure spinor ghosts of the formalism and $\bar{\alpha}, \hat{\bar{\alpha}} = 1, \ldots, 16$. The expansion of the superfield $A_{\bar{\alpha}\hat{\bar{\alpha}}}(x, \theta, \hat{\theta})$ in its component fields was not computed by Berkovits and Chandia, and the connection between this vertex operator and the duals of the Half-BPS operators was not found.

In this thesis, based on the article [19] by the author and Berkovits, a new method for constructing the vertex operators is presented. This method is used for computing the states in the zero mode cohomology at +2 ghost number of the

BRST operator close to the boundary of AdS, which corresponds to the physical supergravity states. Furthermore, the expansion of the superfield $A_{\bar{\alpha}\hat{\bar{\alpha}}}(x,\theta,\hat{\theta})$ is found and the connection of this superfield with the duals of the Half-BPS operators is made clear. Note that another method for constructing the unintegrated massless vertex operator based on symmetry arguments and not emphasizing its boundary behavior exists and was developed by Mikhailov in [20, 21], for the integrated vertex operator see [22].

The first step of the method for constructing the vertex operators used in this thesis consists in expanding the BRST operator as

$$Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + \dots ,$$

where Q_n is proportional to z^n and z the distance from the AdS boundary. The Q_n also depends on the other worldsheet variables and their canonical momenta when restricted to its zero mode terms. As will be explained in the chapter 3, an expansion of the vertex operator in powers of z is also possible close to the boundary of AdS and it has a term with a minimal power of z. After performing both the z expansions, one can use standard methods for computing the cohomology of the BRST operator, one first computes the cohomology of $Q_{-\frac{1}{2}}$, then computes the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$, then computes the cohomology of $Q_{\frac{3}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$, and so on. In fact, it will be argued, making some assumptions, that the cohomology of the complete BRST operator Q is determined by the first two terms $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$ only. The result for the vertex operator constructed using this method is only valid inside the region of validity of the z expansion, or in other words, close to the boundary of AdS. An important result used in the computation is the zero mode cohomology of the operator $Q_{-\frac{1}{2}}$ obtained by Mikhailov and Xu in [23], see also [24].

The resulting supergravity vertex operator computed with this method is described in harmonic superspace. The study of supersymmetric theories using harmonic superspaces was initiated by Galperin, Ivanov, Kalitsyn, Ogievetsky and Sokatchev in [25], where an off-shell formulation of all $\mathcal{N} = 2$ supersymmetric theories was given, an excellent introductory book is [26]. Despite the fact that it is not known how to construct an off-shell superfield formulation of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills, it is possible to solve the constraints obeyed by the $\mathcal{N} = 4$ on-shell vector superfield, or Sohnius superfield [27], keeping the $SU(4)$ R-symmetry manifest using harmonic variables [28, 29]. Moreover, all Half-BPS operators of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills can also be described elegantly as superfields defined in harmonic superspace. The duals to these operators, defined up to a gauge transformation, can also be written as superfields depending on the harmonic variables.

As expected by holography, these dual superfields were related to chiral superfields describing the type IIB supergravity close to the boundary of AdS by Howe and Heslop in [30], an excellent introduction to their work is [31]. These chiral superfields describing the type IIB supergravity were constructed originally by Howe and West in [32] and previous works about holography in superspace are [33, 34, 35, 36]. In this thesis, the dual superfields will be related to the type IIB gauge superfield $A_{\bar{\alpha}\hat{\bar{\alpha}}}(x, \theta, \hat{\theta})$ of (1.1) that appears in the massless vertex operator. As stated above, the duals are defined up to a gauge transformation and we have checked that our results for the vertex operators change by a BRST-trivial quantity under a gauge transformation of the duals, implying that the results are consistent.

This thesis is organized as follows: in the first part of the chapter 2, we briefly review both the minimal and the non-minimal pure spinor formalisms in flat space. Then, the pure spinor formalism in a curved background is explained and the action for superstrings in the $AdS^5 \times S^5$ background with the matter being represented by the unusual supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ together with S^5 variables is constructed. The chapter 3 is devoted to the analysis of the BRST operator of the theory, its expansion in powers of z is performed and the argument that its cohomology is determined only by the first two terms of the expansion is explained. In the chapter 4, we present our results for the supergravity vertex operators close to the boundary of AdS and explain several concepts needed for understanding the results, such as harmonic superspace. Finally, the chapter 5 is devoted to the conclusion and perspectives.

1.1 Notation

In this section, we fix the notation for almost all the indices that are going to appear in this thesis. In addition, the Appendix A has our conventions for the Pauli matrices of $SO(1,3)$ and $SO(6)$ together with several useful properties satisfied by these matrices. The indices are

- $\mu, \nu, \rho, \tau = 0, 1, 2, 3$ — $SO(1,3)$ vector indices
- $\alpha, \beta, \gamma, \delta, \epsilon = 1, 2$ — $SO(1,3)$ chiral spinor indices
- $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta}, \dot{\epsilon} = 1, 2$ — $SO(1,3)$ chiral spinor indices
- $I, J = 1, \ldots, 6$ — $SO(6)$ vector indices
- $i, j, k, l, m, n, p, t = 1, 2, 3, 4$ — $SU(4)$ indices
- $M, N, P, T, S, R = 0, \ldots, 9$ — $SO(1,9)$ vector indices

- $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\epsilon} = 1, \ldots, 16$ — $SO(1,9)$ spinor indices
- $\hat{\bar{\alpha}}, \hat{\bar{\beta}}, \hat{\bar{\gamma}}, \hat{\bar{\delta}}, \hat{\bar{\epsilon}} = 1, \ldots, 16$ — $SO(1,9)$ spinor indices
- $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{e}, \dot{f} = 1, \ldots, 5$ — $SU(5)$ indices
- $a, b, c, d, e, f = 0, 1, \ldots, 4$ — $SO(1,4)$ vector indices
- $A, B, C = (\alpha, \dot{\alpha})$ — $SO(1,4)$ spinor indices
- $a', b', c', d', e', f' = 5, \ldots, 9$ — $SO(5)$ vector indices
- $\dot{I}, \dot{J}, \dot{K} = 1, 2$ — $SU(2)$ harmonic coset indices
- $I', J', K' = 1, 2$ — $SU(2)$ harmonic coset indices

In a section of this thesis, we will Wick rotate $SO(1,9)$ to $SO(10)$ and the indices of $SO(10)$ will be the same of the corresponding ones of $SO(1,9)$. The last comment on notation is that when we need to distinguish between curved and flat indices, the curved indices will be similar to the flat ones except that they will appear with a breve symbol $\breve{}$.

Chapter 2

The Pure Spinor formalism

The pure spinor formalism first appeared in the article [15] by Berkovits. One of its advantages is that it allows the quantization of the superstrings keeping all the symmetries manifest, differently to what happens in other formalisms. In the case of superstrings in flat space background, one can use in addition to the pure spinor formalism, both the RNS or Ramond-Neveu-Schwarz and the Green-Schwarz formalisms. The action in the RNS formalism is worldsheet supersymmetric, however, its spectrum is only supersymmetric after the GSO projection, an excellent introductory book is [37]. The Green-Schwarz formalism has spacetime supersymmetry, but it has first- and second-class constraints that do not allow its covariant quantization, the usual procedure is to go to the light-cone gauge and after gauge-fixing the Lorentz symmetry is not manifest, see [38], for example. The pure spinor formalism has the same spectrum of the Green-Schwarz formalism, as proven by Berkovits and Marchioro in [39]. It was also proven that the results of the scattering amplitudes of superstrings up to two loops are equivalent using either the pure spinor formalism or the RNS formalism [40], nevertheless performing the calculation with the pure spinor formalism is more efficient because all the symmetries are kept manifest in all the steps of the calculation, an excellent introduction is the thesis by Mafra [41], and recent articles are [42, 43, 44]. The pure spinor formalism can also be used to study superstrings in a background supported by a Ramond-Ramond flux such as $AdS^5 \times S^5$.

This chapter is organized as follows: the first section contains a short review of the minimal pure spinor formalism in flat space background which is followed by an introduction to the non-minimal pure spinor formalism. The non-minimal pure spinor variables will be important in the chapter 4, where the main results of this thesis will be presented. The next section has a briefly introduction to the pure spinor formalism in a generic curved background of which $AdS^5 \times S^5$ is an example. The usual $AdS^5 \times S^5$ action with the matter variables represented by the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ is explained in the sequence. Finally, the action with the matter variables represented by the AdS supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ together with S^5 variables is constructed. This action and its BRST charge will be used in the following chapters.

2.1 The Pure Spinor formalism in flat space background

The first step in reviewing the pure spinor formalism or Berkovits formalism is to define what a pure spinor is. The 32 by 32 Gamma matrices Γ^M of $SO(1,9)$ are represented in the Weyl basis as

$$\Gamma^M \;=\; \begin{pmatrix} 0_{16} & (\gamma^M)^{\bar\alpha\bar\beta} \\ (\gamma^M)_{\bar\alpha\bar\beta} & 0_{16} \end{pmatrix}, \tag{2.1}$$

where 0_{16} is the 16 by 16 zero matrix and both $(\gamma^M)^{\bar\alpha\bar\beta}$ and $(\gamma^M)_{\bar\alpha\bar\beta}$ are 16 by 16 symmetric matrices. The Gamma matrices satisfy the Clifford algebra

$$\{\,\Gamma^M\,,\,\Gamma^N\,\} = 2\eta^{MN}\,, \tag{2.2}$$

which is equivalent to

$$\gamma^M_{\bar\alpha\bar\beta}\gamma^{N\bar\beta\bar\gamma} + \gamma^N_{\bar\alpha\bar\beta}\gamma^{M\bar\beta\bar\gamma} = 2\eta^{MN}\delta^{\bar\gamma}_{\bar\alpha}\,, \tag{2.3}$$

and in our conventions the metric is mostly plus, $\eta^{MN} = \mathrm{diag}(-1,1,\ldots,1)$. A chiral spinor $\lambda^{\bar\alpha}$ is called a pure spinor if it satisfies the constraints

$$\lambda^{\bar\alpha}\gamma^M_{\bar\alpha\bar\beta}\lambda^{\bar\beta} = 0\,, \tag{2.4}$$

for all values of M. As will be explained in the chapter 4, it is possible to solve these constraints using $U(5)$ notation and show that a pure spinor has 11 independent components, see [45], for example.

Having defined what a pure spinor is, we can write down the worldsheet action of the formalism, which is

$$S = \frac{1}{2\pi}\int d^2z\,(\frac{1}{2}\partial X^M\bar\partial X_M + p_{\bar\alpha}\bar\partial\theta^{\bar\alpha} + w_{\bar\alpha}\bar\partial\lambda^{\bar\alpha})\,, \tag{2.5}$$

where we have written only its holomorphic part, in the case of closed strings it is necessary to add a similar antiholomorphic part. We have also set the dimensional parameter α' inversely proportional to the string tension to one and this will be done everywhere in this thesis. In the action, the $[X^M,\theta^{\bar\alpha}]$ are superspace coordinates, $p_{\bar\alpha}$ and $w_{\bar\alpha}$ are the conjugate momenta of $\theta^{\bar\alpha}$ and $\lambda^{\bar\alpha}$, respectively. Because

$\lambda^{\bar\alpha}$ is a constrained variable, its conjugate momentum is defined up to the gauge transformation

$$\delta w_{\bar\alpha} = \Lambda^M (\lambda\gamma_M)_{\bar\alpha} , \tag{2.6}$$

for any Λ^M. It is not difficult to see that the action is invariant under this gauge transformation of w. In addition, due to this gauge transformation, w can only appear in gauge-invariant combinations such as

$$N_{MN} = \frac{1}{4}(w\gamma_{MN}\lambda) , \qquad J = w\lambda , \tag{2.7}$$

where N_{MN} are the $SO(1,9)$ ghost Lorentz currents and J is the ghost number current. The action is conformally invariant because the variables have the following conformal weights: $(1,0)$ for $[\partial X^M, p_{\bar\alpha}, w_{\bar\alpha}]$ and $(0,1)$ for $[\bar\partial X^M, \bar\partial\theta^{\bar\alpha}, \bar\partial\lambda^{\bar\alpha}]$. Moreover, it is possible to derive the energy-momentum tensor and prove that the theory does not have a conformal anomaly, or in other words, its central charge is zero. We refer the reader to [46] for further details.

A direct calculation shows that the action is invariant under the global supersymmetry transformations

$$\begin{aligned} \delta X^M = \frac{1}{2}(\varepsilon\gamma^M\theta) , \quad \delta\theta^{\bar\alpha} = \varepsilon^{\bar\alpha} , \quad \delta\lambda^{\bar\alpha} = 0 , \quad \delta w_{\bar\alpha} = 0 , \\ \delta p_{\bar\alpha} = -\frac{1}{2}\varepsilon^{\bar\beta}\gamma^M_{\bar\beta\bar\alpha}\partial X_M + \frac{1}{8}\varepsilon^{\bar\beta}\theta^{\bar\gamma}\partial\theta^{\bar\delta}\gamma^M_{\bar\alpha\bar\delta}\gamma_{M\bar\gamma\bar\beta} , \end{aligned}$$

where ε is a constant spinor and $\theta^{\bar\alpha}$ transforms in the usual way as a translation in superspace. The proof of the invariance of the action follows from the important Fierz identity

$$\gamma^M_{\bar\alpha(\bar\beta}\gamma_{|M|\bar\gamma\bar\delta)} = 0 ,$$

where the parentheses in the indices above mean symmetrization not including the index M inside the $|\,|$. Using the action, one can compute the OPEs involving X and θ using standard methods, for example, as described in [47]. The OPEs are

$$X^M(z)X^N(y) \;\to\; -\eta^{MN}\ln|z-y|^2 , \qquad p_{\bar\alpha}(z)\theta^{\bar\beta}(y) \;\to\; \frac{\delta^{\bar\beta}_{\bar\alpha}}{(z-y)} , \tag{2.8}$$

and defining

$$\Pi^M = \partial X^M + \frac{1}{2}\theta\gamma^M\partial\theta , \quad d_{\bar\alpha} = p_{\bar\alpha} - \frac{1}{2}(\partial X^M + \frac{1}{4}\theta\gamma^M\partial\theta)(\gamma_M\theta)_{\bar\alpha} , \tag{2.9}$$

one can show using the OPEs above that

$$d_{\bar\alpha}(z)\, d_{\bar\beta}(y) \;\to\; -\frac{1}{(z-y)}\gamma^M_{\bar\alpha\bar\beta}\,\Pi_M , \tag{2.10}$$

and these definitions will be important in a moment. The last ingredient of the pure spinor formalism is its BRST charge, which is defined as

$$Q = \int dz\, \lambda^{\bar{\alpha}} d_{\bar{\alpha}}\,, \tag{2.11}$$

and using the pure spinor constraints and the last OPE above, it is straightforward to show that this charge is nilpotent

$$Q^2 \;\propto\; \lambda\gamma^M\lambda\,\Pi_M = 0\,. \tag{2.12}$$

Given the nilpotent operator Q, it is possible to define its cohomology. We call a state A a closed state if A is annihilated by Q, which means $Q \cdot A = 0$. An exact state B is a state that can be written in the form $B = Q \cdot C$ for some C. The cohomology of Q is defined to be the set of closed states that are not exact. Defining the pure spinor variable $\lambda^{\bar{\alpha}}$ to have ghost number +1 and its conjugate momentum $w_{\bar{\alpha}}$ to have ghost number -1, in the pure spinor formalism for open strings the physical states are the states in the cohomology of the BRST operator at +1 ghost number. Similary, in the case of closed strings the physical states are the states in the cohomology of the BRST operator at +2 ghost number.

Note that using the OPEs, one can derive how the BRST operator acts on a generic function $f(X, \theta)$,

$$Q \cdot f(X,\theta) = -\lambda^{\bar{\alpha}} D_{\bar{\alpha}}\, f(X,\theta)\,, \tag{2.13}$$

where $D_{\bar{\alpha}}$ is the ten-dimensional supersymmetric derivative given by

$$D_{\bar{\alpha}} = -\frac{\partial}{\partial\theta^{\bar{\alpha}}} - \frac{1}{2}(\gamma^M)_{\bar{\alpha}\bar{\beta}}\theta^{\bar{\beta}}\frac{\partial}{\partial X^M}\,. \tag{2.14}$$

The zero mode cohomology of this BRST operator is well-known, see for example [48], and the physical states are the gluon and the gluino fields of the $\mathcal{N} = 1$ $d = 10$ super-Yang-Mills as expected. This finishes our short review of the pure spinor formalism in flat space background, and all details omitted in this section can be found in the thesis by Mafra [46].

2.1.1 The non-minimal pure spinor formalism

The non-minimal pure spinor formalism was developed by Berkovits and first appeared in [49], see [41, 50] for reviews. One of the motivations for introducing the non-minimal variables was that they allow the construction of a covariant b ghost which is necessary for computing multiloop superstrings scattering amplitudes. The motivation for us, as will be explained in the chapter 4, is that several of the results of this thesis will depend on these additional non-minimal variables.

The non-minimal formalism has, in addition to the pure spinor variables described previously, a bosonic spinor $\tilde{\lambda}_{\bar{\alpha}}$ and a fermionic spinor $r_{\bar{\alpha}}$ together with their conjugate momenta $\tilde{w}^{\bar{\alpha}}$ and $s^{\bar{\alpha}}$. All these variables are left-moving, and a similar set of right-moving variables has to be introduced in the case of closed strings. The non-minimal variables satisfy the constraints

$$\tilde{\lambda}_{\bar{\alpha}}\gamma^{M\bar{\alpha}\bar{\beta}}\tilde{\lambda}_{\bar{\beta}} = 0\,, \qquad \tilde{\lambda}_{\bar{\alpha}}\gamma^{M\bar{\alpha}\bar{\beta}}r_{\bar{\beta}} = 0\,, \tag{2.15}$$

and the conjugate momenta are defined up to the gauge transformations

$$\delta\tilde{w}^{\bar{\alpha}} = \bar{\Lambda}^M(\gamma_M\tilde{\lambda})^{\bar{\alpha}} - \phi^M(\gamma_M r)^{\bar{\alpha}}\,, \qquad \delta s^{\bar{\alpha}} = \phi^M(\gamma_M\tilde{\lambda})^{\bar{\alpha}}\,, \tag{2.16}$$

for any $\bar{\Lambda}^M$ and ϕ^M. This implies that the variables $\tilde{w}$ and s can only appear in gauge-invariant combinations, such as

$$\bar{N}_{MN} = \frac{1}{4}(\tilde{w}\gamma_{MN}\tilde{\lambda} - s\gamma_{MN}r)\,, \quad \bar{J}_{\tilde{\lambda}} = \tilde{w}\tilde{\lambda} - sr\,, \quad T_{\tilde{\lambda}} = \tilde{w}\partial\tilde{\lambda} - s\partial r\,,$$

etc. The left-moving part of the action is modified to

$$S_{nonmin} = \int d^2z\,(\frac{1}{2}\partial X^M\bar{\partial}X_M + p_{\bar{\alpha}}\bar{\partial}\theta^{\bar{\alpha}} - w_{\bar{\alpha}}\bar{\partial}\lambda^{\bar{\alpha}} - \tilde{w}^{\bar{\alpha}}\bar{\partial}\tilde{\lambda}_{\bar{\alpha}} + s^{\bar{\alpha}}\bar{\partial}r_{\bar{\alpha}})\,, \tag{2.17}$$

and it is conformally invariant because the additional fields have the following conformal weights: $(0,1)$ for $[\bar{\partial}\tilde{\lambda}_{\bar{\alpha}}, \bar{\partial}r_{\bar{\alpha}}]$ and $(1,0)$ for $[\tilde{w}^{\bar{\alpha}}, s^{\bar{\alpha}}]$. It is possible to compute the energy-momentum tensor using standard techniques and show that the non-minimal variables do not give any contribution to the conformal anomaly, which means that the total central charge remains zero.

A very important point for the remaining of this thesis is that the BRST charge is also modified to

$$Q_{nonmin} = \int dz\,(\lambda^{\bar{\alpha}}d_{\bar{\alpha}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,, \tag{2.18}$$

and it is straightforward to see that the additional term is invariant under the gauge transformations of (2.16). As will be explained in the chapter 4, using $U(5)$ notation one can show that both the variables $\tilde{\lambda}$ and r have 11 unconstrained components. Then, one can use the standard quartet mechanism argument of [51, 52] to show that the cohomology of the non-minimal BRST operator is independent of the quartet of variables $(\tilde{\lambda}, \tilde{w})$ and (r, s), which means that it has the same cohomology of the BRST operator constructed only with the minimal variables.

The last comment about the non-minimal pure spinor formalism is that the formalism can be intepreted as a $\hat{c} = 3$ $\mathcal{N} = 2$ critical topological string. We refer the interested reader to [49] for further details.

2.2 The pure spinor formalism in a curved background

Despite the fact that the pure spinor formalism action for superstrings in $AdS^5 \times S^5$ was already constructed in the first article about the formalism by Berkovits in [15], the pure spinor formalism action in a generic supergravity background appeared in a later article by Berkovits and Howe [53], see also [54]. We will briefly review this general action focusing on the case of type IIB superstrings, which has the background $AdS^5 \times S^5$ as a particular case.

The starting point for constructing the Berkovits-Howe action is to write the most general classically conformally invariant action with ghost number zero depending on the variables with the following conformal weights: $(1,0)$ for $[\partial Z^{\check{M}}, d_{\bar{\alpha}}, w_{\bar{\alpha}}]$, $(0,1)$ for $[\bar{\partial} Z^{\check{M}}, \hat{d}_{\hat{\bar{\alpha}}}, \hat{w}_{\hat{\bar{\alpha}}}]$ and $(0,0)$ for $[\lambda^{\bar{\alpha}}, \hat{\lambda}^{\hat{\bar{\alpha}}}]$, where $[\lambda^{\bar{\alpha}}, \hat{\lambda}^{\hat{\bar{\alpha}}}, w_{\bar{\alpha}}, \hat{w}_{\hat{\bar{\alpha}}}]$ are the usual bosonic pure spinor variables introduced in the previous section corresponding to the ghost variables of the theory, $[d_{\bar{\alpha}}, \hat{d}_{\hat{\bar{\alpha}}}]$ are the variables that appear in the BRST operator as in the flat space case of (2.11), and considered as independent variables here and, finally, $Z^{\check{M}} = [X^{\check{M}}, \theta^{\check{\alpha}}, \bar{\theta}^{\check{\alpha}}]$ are curved variables parametrizing the $\mathcal{N} = 2$ superspace in ten dimensions. The most general action is

$$S = \frac{1}{2\pi}\int d^2z\, \frac{1}{2}(G_{\check{M}\check{N}}(Z) + B_{\check{M}\check{N}}(Z))\partial Z^{\check{M}}\bar{\partial} Z^{\check{N}} + w_{\bar{\alpha}}\bar{\partial}\lambda^{\bar{\alpha}} - \hat{w}_{\hat{\bar{\alpha}}}\partial\hat{\lambda}^{\hat{\bar{\alpha}}} \tag{2.19}$$
$$+ e^{\bar{\alpha}}_{\check{M}}(Z) d_{\bar{\alpha}}\bar{\partial} Z^{\check{M}} + e^{\hat{\bar{\alpha}}}_{\check{M}}(Z)\hat{d}_{\hat{\bar{\alpha}}}\partial Z^{\check{M}} + \Omega_{\check{M}\bar{\alpha}}{}^{\bar{\beta}}(Z)\lambda^{\bar{\alpha}} w_{\bar{\beta}}\bar{\partial} Z^{\check{M}} + \hat{\Omega}_{\check{M}\hat{\bar{\alpha}}}{}^{\hat{\bar{\beta}}}(Z)\hat{\lambda}^{\hat{\bar{\alpha}}}\hat{w}_{\hat{\bar{\beta}}}\partial Z^{\check{M}}$$
$$+ P^{\bar{\alpha}\hat{\bar{\beta}}}(Z) d_{\bar{\alpha}}\hat{d}_{\hat{\bar{\beta}}} + C^{\bar{\beta}\hat{\bar{\gamma}}}_{\bar{\alpha}}(Z)\lambda^{\bar{\alpha}} w_{\bar{\beta}}\hat{d}_{\hat{\bar{\gamma}}} + \hat{C}^{\hat{\bar{\beta}}\bar{\gamma}}_{\hat{\bar{\alpha}}}(Z)\hat{\lambda}^{\hat{\bar{\alpha}}}\hat{w}_{\hat{\bar{\beta}}} d_{\bar{\gamma}} + S^{\bar{\beta}\hat{\bar{\delta}}}_{\bar{\alpha}\hat{\bar{\gamma}}}(Z)\lambda^{\bar{\alpha}} w_{\bar{\beta}}\hat{\lambda}^{\hat{\bar{\gamma}}}\hat{w}_{\hat{\bar{\delta}}}\,,$$

where all the superfields that appear in the action above are the background superfields which have geometrical interpretations as will be explained soon. One comment about the action is that in fact it has an additional term, the Fradkin-Tseytlin term, however, this term will not be relevant in the following and it will be omitted, see [53, 54] for details. The superfield e is the supervielbein, the superfield Ω is the spin-connection, $G_{\check{M}\check{N}}$ is the metric related to the flat metric by $G_{\check{M}\check{N}} = e^M_{\check{M}} e^N_{\check{N}}\eta_{MN}$, $B_{\check{M}\check{N}}$ is the two-form potential, the superfields $\hat{C}^{\hat{\bar{\beta}}\bar{\gamma}}_{\hat{\bar{\alpha}}}$ and $C^{\bar{\beta}\hat{\bar{\gamma}}}_{\bar{\alpha}}$ are related to the two gravitini and dilatini field-strengths, the $P^{\bar{\alpha}\hat{\bar{\beta}}}$ is a superfield whose lowest component is related to the Ramond-Ramond field-strengths and, finally, the superfield $S^{\bar{\beta}\hat{\bar{\delta}}}_{\bar{\alpha}\hat{\bar{\gamma}}}$ is related to the curvature.

In the article by Berkovits and Howe [53], it was proven that imposing that the action is both BRST invariant and invariant under the gauge transformations of w and $\hat{w}$ defined in (2.6), all the expected constraints satisfied by the superfields of the type IIB supergravity that appear in the action are reproduced. Mainly, consider the total BRST charge Q_T of the theory, which is

$$Q_T = Q + \hat{Q}\,,$$

where

$$Q = \int dz\, \lambda^{\bar\alpha} d_{\bar\alpha}\,, \qquad \hat Q = \int d\bar z\, \hat\lambda^{\hat{\bar\beta}} d_{\hat{\bar\beta}}\,.$$

The total BRST operator must be nilpotent by consistency of the theory, and

$$\{\, Q_T\,,\, Q_T\,\} = 0\,,$$

implies

$$\{\, Q\,,\, Q\,\} = \{\, Q\,,\, \hat Q\,\} = \{\, \hat Q\,,\, \hat Q\,\} = 0\,, \tag{2.20}$$

which follows from collecting equal powers of λ and $\hat\lambda$. In addition, the charges Q and $\hat Q$ are well defined if the following holomorphy conditions are satisfied

$$\partial(\hat\lambda^{\hat{\bar\beta}} d_{\hat{\bar\beta}}) = 0\,, \quad \bar\partial(\lambda^{\bar\alpha} d_{\bar\alpha}) = 0\,, \tag{2.21}$$

and the consistency conditions (2.20) and (2.21) are only satisfied if the superfields satisfy the expected supergravity constraints.

The action in the pure spinor formalism for superstrings in $AdS^5 \times S^5$ is a particular case of the general action given in (2.19) and it is obtained by replacing the correct expansion of the superfields in this background. This action will be presented using a convenient notation and in great detail in the next section, here we only outline the derivation of the action. The AdS background is described by the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ and for a given coset representative g the supervielbeins and the connections are defined by

$$g^{-1}\partial g = (e^M T_M + \Omega^N H_N)\,, \tag{2.22}$$

where H_N are the generators of the isotropy group $SO(1,4)\times SO(5)$ and T_M the remaining generators of $PSU(2,2|4)$. Moreover, the superfields $\hat C_{\hat{\bar\alpha}}^{\hat{\bar\beta}\bar\gamma}$ and $C_{\bar\alpha}^{\bar\beta\hat{\bar\gamma}}$ related to the gravitini and dilatini are zero in this background. The background is supported by a Ramond-Ramond flux, which implies

$$P^{\bar\alpha\hat{\bar\beta}} \propto F^{a'b'c'd'e'}(\gamma_{a'b'c'd'e'})^{\bar\alpha\hat{\bar\beta}}\,,$$

where $F^{a'b'c'd'e'}$ is the only non-zero constant field-strength of this background. In addition,

$$S_{\bar\alpha\hat{\bar\gamma}}^{\bar\beta\hat{\bar\delta}} \propto (\gamma^{\underline{ab}})_{\bar\alpha}{}^{\bar\beta}(\gamma^{\underline{cd}})_{\hat{\bar\gamma}}{}^{\hat{\bar\delta}} R_{[\underline{ab}][\underline{cd}]}\,,$$

where $R_{[\underline{ab}][\underline{cd}]}$ is the constant Riemann tensor describing the curvature of the background. Finally, the only non-vanishing components of B are [55]

$$B_{\bar\alpha\hat{\bar\beta}} = B_{\hat{\bar\beta}\bar\alpha} \propto (\gamma^{01234})_{\bar\alpha\hat{\bar\beta}}\,.$$

2.3 Superstrings in $AdS^5 \times S^5$

In this section, the usual action for superstrings in the background $AdS^5 \times S^5$ in the pure spinor formalism will be presented in great detail. This action first appeared in the article [15] by Berkovits and a recent review is [16]. The background is described by the supercoset

$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}\,. \tag{2.23}$$

Note that the bosonic subgroup of $PSU(2,2|4)$ is $SU(2,2)\times SU(4)$ which is locally isomorphic to $SO(2,4)\times SO(6)$ and from the bosonic part we have

$$AdS^5 = \frac{SO(2,4)}{SO(1,4)}\,, \qquad S^5 = \frac{SO(6)}{SO(5)}\,. \tag{2.24}$$

In the next subsection, the algebra of $PSU(2,2|4)$ in ten-dimensional notation will be described and several comments will be made, this is important to understand the superstring action which will be reviewed in the sequence.

2.3.1 The $PSU(2,2|4)$ algebra in ten-dimensional notation

The Lie superalgebra of $PSU(2,2|4)$ contains 30 bosonic generators and 32 fermionic generators, a good review of Lie superalgebras for physicists is Kac [56] and an excellent introductory book is [57]. The Lie superalgebra of $PSU(2,2|4)$ has a $\mathcal{Z}_4$-automorphism and this implies that it is possible to organize the generators in a way that the algebra is $\mathcal{Z}_4$-graded, which means that denoting the set of generators with grading $i=0,1,2,3$ as g_i the algebra has the structure

$$[\,g_i\,,\,g_j\,]_{+-} = g_{i+j}\,, \quad \mod 4\,, \tag{2.25}$$

where the subscript $+-$ means commutator or anticommutator. A very good review of this point is [14], see also [55].

In order to write the action for superstrings in the pure spinor formalism in the background $AdS^5\times S^5$ described by the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$, it is convenient to organize the generators of the Lie superalgebra as $[g_0, g_1, g_2, g_3] = [T_{[\underline{ab}]}, T_{\bar\alpha}, T_{\underline{a}}, T_{\hat{\bar\alpha}}]$ and the non-zero structure constants are

$$\begin{aligned}
f^{\underline{a}}{}_{\bar\alpha\bar\beta} &= \frac{1}{2}\gamma^{\underline{a}}_{\bar\alpha\bar\beta}\,, & f^{\underline{a}}{}_{\hat{\bar\alpha}\hat{\bar\beta}} &= \frac{1}{2}\gamma^{\underline{a}}_{\hat{\bar\alpha}\hat{\bar\beta}}\,,\\
f^{\hat{\bar\alpha}}{}_{\bar\beta\underline{a}} &= -\frac{1}{2}\gamma_{\underline{a}\bar\beta\bar\gamma}\,\kappa^{\bar\gamma\hat{\bar\alpha}}\,, & f^{\bar\alpha}{}_{\hat{\bar\beta}\underline{a}} &= \frac{1}{2}\gamma_{\underline{a}\hat{\bar\beta}\hat{\bar\gamma}}\,\kappa^{\bar\alpha\hat{\bar\gamma}}\,,\\
f^{[ab]}{}_{\bar\alpha\hat{\bar\beta}} &= \frac{1}{4}(\gamma^{ab})_{\bar\alpha}{}^{\bar\gamma}\,\kappa_{\bar\gamma\hat{\bar\beta}}\,, & f^{[a'b']}{}_{\bar\alpha\hat{\bar\beta}} &= -\frac{1}{4}(\gamma^{a'b'})_{\bar\alpha}{}^{\bar\gamma}\,\kappa_{\bar\gamma\hat{\bar\beta}}\,,\\
f^{\bar\beta}{}_{[\underline{ab}]\bar\alpha} &= -\frac{1}{2}(\gamma_{\underline{ab}})_{\bar\alpha}{}^{\bar\beta}\,, & f^{\hat{\bar\beta}}{}_{[\underline{ab}]\hat{\bar\alpha}} &= -\frac{1}{2}(\gamma_{\underline{ab}})_{\hat{\bar\alpha}}{}^{\hat{\bar\beta}}\,,
\end{aligned} \tag{2.26}$$

$$f^{\underline{a}}{}_{[bc]\underline{d}} = \eta_{\underline{d}\,[\underline{c}}\delta^{\underline{a}}_{\underline{b}]}\,, \quad f^{[ab]}{}_{cd} = \frac{1}{2}\delta^{a}_{[c}\delta^{b}_{d]}\,, \quad f^{[a'b']}{}_{c'd'} = -\frac{1}{2}\delta^{a'}_{[c'}\delta^{b'}_{d']}\,,$$
$$f^{[\underline{ab}]}{}_{[\underline{cd}][\underline{ef}]} = \frac{1}{2}\eta_{\underline{e}[\underline{d}}\delta^{[\underline{a}}_{\underline{c}]}\delta^{\underline{b}]}_{\underline{f}} + \frac{1}{2}\eta_{\underline{f}[\underline{c}}\delta^{[\underline{a}}_{\underline{d}]}\delta^{\underline{b}]}_{\underline{e}}\,,$$

where $a = 0, 1, \ldots, 4$, $a' = 5, \ldots, 9$, and $\underline{a}$ denotes both a and a'. The $[\,]$ that appears on the right-hand side of the structure constants means antisymmetrization of the indices with no additional factor of half. Furthermore, $(\gamma^{\underline{a}}_{\bar{\alpha}\bar{\beta}}\,, \gamma^{\underline{a}\,\bar{\alpha}\bar{\beta}})$ and $(\gamma^{\underline{a}}_{\hat{\bar{\alpha}}\hat{\bar{\beta}}}\,, \gamma^{\underline{a}\,\hat{\bar{\alpha}}\hat{\bar{\beta}}})$ are two sets of chiral gamma matrices which are related with each other as will be shown soon, and

$$(\gamma_{\underline{ab}})_{\bar{\alpha}}{}^{\bar{\beta}} = \frac{1}{2}(\gamma_{\underline{a}\,\bar{\alpha}\bar{\gamma}}\gamma_{\underline{b}}^{\bar{\gamma}\bar{\beta}} - \gamma_{\underline{b}\,\bar{\alpha}\bar{\gamma}}\gamma_{\underline{a}}^{\bar{\gamma}\bar{\beta}})\,,$$

with a similar definition for the matrices with hatted indices. Moreover,

$$\mathrm{Str}\,(T_{\underline{a}}T_{\underline{b}}) \equiv \eta_{\underline{ab}}\,, \quad \mathrm{Str}\,(T_{\bar{\alpha}}T_{\hat{\bar{\beta}}}) \equiv \kappa_{\bar{\alpha}\hat{\bar{\beta}}}\,, \quad \kappa_{\bar{\alpha}\hat{\bar{\beta}}} = -\kappa_{\hat{\bar{\beta}}\bar{\alpha}}\,, \qquad (2.27)$$
$$\mathrm{Str}\,(T_{[\underline{ab}]}T_{[\underline{cd}]}) \equiv \eta_{[\underline{ab}][\underline{cd}]}\,,$$

where Str denotes the supertrace over the generators. Explicitly $\eta_{\underline{ab}}$ and $\eta_{[\underline{ab}][\underline{cd}]}$ are

$$\eta_{\underline{ab}} = \{\eta_{ab}\,, \eta_{a'b'}\} = \{(-1,1,1,1,1)\,, (1,1,1,1,1)\}\,, \qquad (2.28)$$
$$\eta_{[\underline{ab}][\underline{cd}]} = \{\eta_{[ab][cd]}\,, \eta_{[a'b'][c'd']}\} = \{\eta_{a[d}\eta_{c]b}\,, -\eta_{a'[d'}\eta_{c']b'}\}\,.$$

In order to complete the definitions, we have to define the inverse symbols that appear in the expressions for the structure constants. They are defined implicitly by the relations

$$\kappa^{\bar{\alpha}\hat{\bar{\beta}}}\kappa_{\hat{\bar{\beta}}\bar{\gamma}} = \delta^{\bar{\alpha}}_{\bar{\gamma}}\,, \quad \kappa_{\hat{\bar{\gamma}}\bar{\alpha}}\kappa^{\bar{\alpha}\hat{\bar{\beta}}} = \delta^{\hat{\bar{\beta}}}_{\hat{\bar{\gamma}}}\,, \quad \eta^{\underline{ab}}\eta_{\underline{bc}} = \delta^{\underline{a}}_{\underline{c}}\,, \qquad (2.29)$$
$$\eta^{[\underline{ab}][\underline{ef}]}\eta_{[\underline{ef}][\underline{cd}]} = \delta^{\underline{ab}}_{\underline{cd}}\,, \quad \delta^{\underline{ab}}_{\underline{cd}} = \{\delta^{ab}_{cd}\,, \delta^{a'b'}_{c'd'}\} = \{\frac{1}{2}\delta^{[a}_{c}\delta^{b]}_{d}\,, \frac{1}{2}\delta^{[a'}_{c'}\delta^{b']}_{d'}\}\,.$$

We will make several comments about the superalgebra and its structure constants, not all of them essential for the rest of the thesis but they may be useful to the reader. The first comment is that in order to prove that the structure constants of the algebra given above satisfy all the generalized Jacobi identities

$$(-1)^{\deg A \deg C}[\,T_A\,, [\,T_B\,, T_C\,]_{+-}\,]_{+-} + (-1)^{\deg A \deg B}[\,T_B\,, [\,T_C\,, T_A\,]_{+-}\,]_{+-}$$
$$+(-1)^{\deg C \deg B}[\,T_C\,, [\,T_A\,, T_B\,]_{+-}\,]_{+-} = 0\,,$$

where the value of deg is 1 if the generator is in $[g_1\,, g_3]$, i.e, a fermionic generator or 0 if the generator is in $[g_0\,, g_2]$, i.e, a bosonic generator, the following chiral gamma matrices identities are necessary

$$\gamma^{\underline{a}}_{\bar{\alpha}(\bar{\beta}}\gamma_{|\underline{a}|\bar{\gamma}\bar{\delta})} = \gamma^{a}_{\bar{\alpha}(\bar{\beta}}\gamma_{|a|\bar{\gamma}\bar{\delta})} + \gamma^{a'}_{\bar{\alpha}(\bar{\beta}}\gamma_{|a'|\bar{\gamma}\bar{\delta})} = 0\,,$$

and

$$(\gamma^a)^{\bar\alpha\bar\beta} = -\kappa^{\bar\alpha\hat\gamma}\kappa^{\bar\beta\hat{\bar\delta}}(\gamma^a)_{\hat{\bar\gamma}\hat{\bar\delta}}\,, \qquad (\gamma^a)_{\bar\alpha\bar\beta} = -\kappa_{\bar\alpha\hat\gamma}\kappa_{\bar\beta\hat{\bar\delta}}(\gamma^a)^{\hat{\bar\gamma}\hat{\bar\delta}}\,,$$
$$(\gamma^{a'})^{\bar\alpha\bar\beta} = \kappa^{\bar\alpha\hat\gamma}\kappa^{\bar\beta\hat{\bar\delta}}(\gamma^{a'})_{\hat{\bar\gamma}\hat{\bar\delta}}\,, \qquad (\gamma^{a'})_{\bar\alpha\bar\beta} = \kappa_{\bar\alpha\hat\gamma}\kappa_{\bar\beta\hat{\bar\delta}}(\gamma^{a'})^{\hat{\bar\gamma}\hat{\bar\delta}}\,, \tag{2.30}$$

note that the difference in sign above for a and a' is necessary, for example, the Jacobi identities given below are only satisfied if we have this sign difference:

$$[\,T_{\underline{a}}\,,\,[\,T_{\underline{b}}\,,\,T_{\bar\alpha}\,]\,] + [\,T_{\underline{b}}\,,\,[\,T_{\bar\alpha}\,,\,T_{\underline{a}}\,]\,] + [\,T_{\bar\alpha}\,,\,[\,T_{\underline{a}}\,,\,T_{\underline{b}}\,]\,] = 0\,.$$

Another way to see that there is this sign difference is by studying the second Casimir of the algebra. It is well-known that the dual Coxeter number of $PSU(2,2|4)$ is zero, which implies that its second Casimir vanishes in the adjoint representation. Following Mikhailov and Schafer-Nameki [58], the second Casimir is defined by

$$C = \kappa^{\hat{\bar\alpha}\bar\beta}(T_{\hat{\bar\alpha}}\otimes T_{\bar\beta} - T_{\bar\beta}\otimes T_{\hat{\bar\alpha}}) + \eta^{\underline{ab}}(T_{\underline{a}}\otimes T_{\underline{b}}) + \eta^{[\underline{ab}][\underline{cd}]}(T_{\underline{ab}}\otimes T_{\underline{cd}})\,,$$

and it acts on a generator as

$$C\cdot T_C = (\kappa\eta)^{AB}[\,T_A\,,\,[\,T_B\,,\,T_C\,]_{+-}\,]_{+-} = 0\,, \tag{2.31}$$

where $[A,B,C]$ here can be any of the indices of the generators and $(\kappa\eta)$ is κ or η depending on the value of the indices. Noting that

$$\kappa^{\bar\alpha\hat{\bar\beta}}\{T_{\bar\alpha}\,,\,T_{\hat{\bar\beta}}\} = 0\,, \tag{2.32}$$

which can be verified by direct computation using $(\gamma^{\underline{ab}})_{\bar\alpha}{}^{\bar\alpha} = 0$ or by a group theoretic argument given in [58]. The argument is that if it is not zero it will be an element of the center of g_0, however, the center of g_0 is trivial. The statement “being in the center of g_0” should be understood as that for any $T_{\underline{ab}}$ we must have

$$[\,T_{\underline{ab}}\,,\,\kappa^{\bar\alpha\hat{\bar\beta}}\{T_{\bar\alpha}\,,\,T_{\hat{\bar\beta}}\}\,] = 0\,, \tag{2.33}$$

and one way to prove the result above is by using the generalized Jacobi identities and $\kappa^{\bar\alpha\hat{\bar\beta}}(\gamma_{\underline{ab}})_{\hat{\bar\beta}}{}^{\hat{\bar\gamma}} = -\kappa^{\bar\delta\hat{\bar\gamma}}(\gamma_{\underline{ab}})_{\bar\delta}{}^{\bar\alpha}$. The vanishing of the second Casimir and (2.32) impose many relations among the structure constants, for example,

$$\kappa^{\bar\alpha\hat{\bar\beta}}[\,T_{\hat{\bar\beta}}\,,\,\{\,T_{\bar\alpha}\,,\,T_{\bar\gamma}\}\,] = 0\,, \quad \eta^{\underline{ab}}[\,T_{\underline{a}}\,,\,[T_{\underline{b}}\,,\,T_{\bar\gamma}]\,] = 0\,,$$
$$\kappa^{\hat{\bar\alpha}\bar\beta}\{T_{\hat{\bar\alpha}}\,,\,[\,T_{\bar\beta}\,,\,T_{\underline{a}}]\,\} = \kappa^{\bar\alpha\hat{\bar\beta}}\{\,T_{\bar\alpha}\,,\,[\,T_{\hat{\bar\beta}}\,,\,T_{\underline{a}}]\,\} = -\eta^{\underline{bc}}[\,T_{\underline{b}}\,,\,[\,T_{\underline{c}}\,,\,T_{\underline{a}}\,]\,]\,,$$

and the third relation is only satisfied with the correct assignment of signs of (2.30). The last comment about the structure constants is that it is possible to relate many of them using supertrace identities, for example, we know that the supertrace satisfies

$$\mathrm{Str}\,([\,T_A\,,\,T_B\,]_{+-}\,T_C\,) = \mathrm{Str}(\,T_A\,[\,T_B\,,\,T_C\,]_{+-}\,)\,,$$

in particular,

$$\mathrm{Str}\,(\,\{\,T_{\bar\alpha}\,,\,T_{\bar\beta}\,\}\,,\,T_{\underline{a}}) = \mathrm{Str}\,(\,T_{\bar\alpha}\,,\,[\,T_{\bar\beta}\,,\,T_{\underline{a}}\,]\,)\,,$$

which implies

$$f^{\underline{b}}{}_{\bar\alpha\bar\beta}\,\eta_{\underline{ba}} = f^{\hat{\bar\gamma}}{}_{\bar\beta\underline{a}}\,\kappa_{\bar\alpha\hat{\bar\gamma}}\,.$$

2.3.2 The action

After introducing the $PSU(2,2|4)$ superalgebra in a useful form for understanding the superstrings action in the pure spinor formalism, the action will be presented. We will follow mainly the notation and conventions of [59].

The first step in constructing the action is to define the left-invariant currents. In this direction, we first need a parametrization of the supercoset (2.23). We will choose, for example, the coset representative

$$g = \exp(X_1^{\bar\alpha}T_{\bar\alpha} + X_2^{\underline{a}}T_{\underline{a}} + X_3^{\hat{\bar\alpha}}T_{\hat{\bar\alpha}})\,, \tag{2.34}$$

where as defined in the previous subsection T_A are generators of the Lie superalgebra of $PSU(2,2|4)$ and X_i^A are variables parameterizing the coset. In what follows the precise form of the coset representative will not be necessary, everything will be still valid for any g. The left-invariant currents are defined as

$$\begin{aligned} g^{-1}\partial g &= J^{\underline{ab}}\,T_{\underline{ab}} + J^{\bar\alpha}T_{\bar\alpha} + + J^{\underline{a}}T_{\underline{a}} + J^{\hat{\bar\alpha}}T_{\hat{\bar\alpha}} \\ &= J^0 + J^1 + J^2 + J^3\,, \end{aligned} \tag{2.35}$$

and the currents $\bar J$ are defined similarly, with the replacement of ∂ by $\bar\partial$ on the left-hand side of the expression above. In our conventions, a global $PSU(2,2|4)$ transformation g_P acts on the coset representative by left multiplication, or in other words,

$$g_P\,g(X) = g(X')h' \tag{2.36}$$

where X' are the transformed variables and h' an element of the isotropy group. In its infinitesimal version we can approximate $g_P \sim 1 + \Sigma$ and the formula above reduces to

$$\delta g = \Sigma\,g\,. \tag{2.37}$$

We are now in position to understand why the currents J defined in (2.35) are called left-invariant. Under a global $PSU(2,2|4)$ transformation, we have

$$g^{-1}\partial g \;\to\; g'^{-1}\partial g' = g^{-1}g_P^{-1}\partial\,(g_P g) = g^{-1}\partial g\,,$$

which means that the currents are invariant under these transformations. In addition, local gauge transformations of $SO(1,4)\times SO(5)$ represented by h_P acts on g by right multiplication

$$g(X)\,h_P = g(X')\,, \tag{2.38}$$

or in its infinitesimal form with $h_P \sim 1+\Omega$,

$$\delta g = g\,\Omega\,. \tag{2.39}$$

The currents J transform under gauge transformations, in order to deduce their transformation, note

$$g^{-1}\partial g \;\to\; g'^{-1}\partial g' = h_P^{-1}g^{-1}\partial\,(g\,h_P) = h_P^{-1}\,J^A T_A\,h_P + h_P^{-1}\,\partial h_P\,,$$

and since h_P is an element of the isotropy group with all its generators with 0 grading, projecting the result above onto the subspaces with definite grading under the $\mathcal{Z}_4$, gives

$$J^i \;\to\; h_P^{-1}\,J^i\,h_P, \quad i=1,2,3\,, \quad J^0 \;\to\; h_P^{-1}\,J^0\,h_P + h_P^{-1}\,\partial h_P\,, \tag{2.40}$$

which means, in particular, that J^0 transforms as a connection.

In addition to the currents just defined, the action for the closed superstrings has a pair of bosonic pure spinors, one left-moving $\lambda^{\bar\alpha}$ and one right-moving $\hat\lambda^{\hat{\bar\alpha}}$, satisfying the constraints

$$\lambda\gamma^{\underline{a}}\lambda = 0\,, \qquad \hat\lambda\gamma^{\underline{a}}\hat\lambda = 0\,, \tag{2.41}$$

which implies that each of them has 11 independent components. This result will be explained in great detail in the chapter 4, where the constraints will be explicitly solved using $U(5)$ notation. The conjugate momenta of these variables will be denoted $w_{\bar\alpha}$ and $\hat w_{\hat{\bar\alpha}}$ and they are defined up to the gauge transformation

$$\delta w = (\gamma^{\underline{a}}\lambda)\Lambda_{\underline{a}}\,, \quad \delta\hat w = (\gamma^{\underline{a}}\hat\lambda)\hat\Lambda_{\underline{a}}\,, \tag{2.42}$$

for any $\Lambda_{\underline{a}}$ and $\hat\Lambda_{\underline{a}}$. This implies that they can only appear in the gauge-invariant combinations of either the Lorentz currents

$$N^{\underline{ab}} = \frac{1}{4}(w\gamma^{\underline{ab}}\lambda)\,, \quad \hat N^{\underline{ab}} = \frac{1}{4}(\hat w\gamma^{\underline{ab}}\hat\lambda)\,, \tag{2.43}$$

or the ghost currents $J = (w\lambda)$ and $\hat{J} = (\hat{w}\hat{\lambda})$. The final definitions needed to write down the action are

$$\begin{aligned} \lambda &= \lambda^{\bar{\alpha}} T_{\bar{\alpha}} \,, \quad w = w_{\bar{\alpha}} T_{\hat{\bar{\alpha}}} \kappa^{\bar{\alpha}\hat{\bar{\alpha}}} \,, \quad N = -\{w, \lambda\} \,, \\ \hat{\lambda} &= \hat{\lambda}^{\hat{\bar{\alpha}}} T_{\hat{\bar{\alpha}}} \,, \quad \hat{w} = \hat{w}_{\hat{\bar{\alpha}}} T_{\bar{\alpha}} \kappa^{\bar{\alpha}\hat{\bar{\alpha}}} \,, \quad \hat{N} = -\{\hat{w}\, \hat{\lambda}\} \,. \end{aligned} \tag{2.44}$$

Finally, the worldsheet action is

$$S = \int d^2 z \,\mathrm{Str}\,(\frac{1}{2} J^2 \bar{J}^2 + \frac{3}{4} J^3 \bar{J}^1 + \frac{1}{4} J^1 \bar{J}^3 + w \bar{\nabla} \lambda + \hat{w} \nabla \hat{\lambda} - N \hat{N}) \,, \tag{2.45}$$

where

$$\bar{\nabla}\lambda = \bar{\partial}\lambda + [\, \bar{J}^0 \,,\, \lambda \,] \,, \quad \nabla\hat{\lambda} = \partial\hat{\lambda} + [\, J^0 \,,\, \hat{\lambda} \,] \,. \tag{2.46}$$

Several comments about this action are in order. Firstly, the action is clearly invariant under global $PSU(2,2|4)$ transformations, this follows immediately as a consequence of the invariance of the currents. The action is also gauge-invariant, the currents $[J^1, J^2, J^3]$ transform covariantly under the gauge transformations and the Str ensures the gauge invariance of the terms involving these currents. The pure spinor variables transform as

$$\delta_\Omega \lambda = [\, \lambda \,,\, \Omega \,] \,, \quad \delta_\Omega \hat{\lambda} = [\, \hat{\lambda} \,,\, \Omega \,] \,, \quad \delta_\Omega w = [\, w \,,\, \Omega \,] \,, \quad \delta_\Omega \hat{w} = [\, \hat{w} \,,\, \Omega \,] \,, \tag{2.47}$$

and note that

$$\begin{aligned} \delta \bar{\nabla} \lambda &= \bar{\partial} \delta \lambda + [\, \delta \bar{J}^0 \,,\, \lambda \,] + [\, \bar{J}^0 \,,\, \delta \lambda \,] \\ &= [\, \bar{\partial} \lambda \,,\, \Omega \,] + [\, \lambda \,,\, \bar{\partial} \Omega \,] + [\, [\, \bar{J}^0 \,,\, \Omega \,] \,,\, \lambda \,] + [\, \bar{\partial} \Omega \,,\, \lambda \,] + [\, \bar{J}^0 \,,\, [\, \lambda \,,\, \Omega \,] \,] \\ &= [\, \bar{\nabla} \lambda \,,\, \Omega \,] \,, \end{aligned}$$

where we have used the Jacobi identity. This implies that the remaining terms of the action are also gauge-invariant. One very important comment is that the action is BRST invariant with the BRST transformation generated by the charge

$$\epsilon Q = -\epsilon \int d\sigma \,\mathrm{Str}\,(\, \lambda J^3 + \hat{\lambda} \bar{J}^1 \,) \,, \tag{2.48}$$

and ϵ a fermionic infinitesimal parameter. Under a BRST transformation the coset representative transforms as

$$\epsilon Q \cdot g = g\, (\epsilon \lambda + \epsilon \hat{\lambda}) \,, \tag{2.49}$$

which enables us to find the transformations of the current by varying both sides of the definition (2.35), which implies

$$\delta g^{-1} \partial g + g^{-1} \partial \delta g = \delta J^0 + \delta J^1 + \delta J^2 + \delta J^3 \,, \tag{2.50}$$

and collecting the terms with the same grading, one concludes

$$\begin{aligned}
\delta J^0 &= [\,J^3\,,\,\epsilon\lambda\,] + [\,J^1\,,\,\epsilon\hat{\lambda}\,]\,,\\
\delta J^1 &= \epsilon\partial\lambda + [\,J^0\,,\,\epsilon\lambda\,] + [\,J^2\,,\,\epsilon\hat{\lambda}\,]\,,\\
\delta J^2 &= [\,J^1\,,\,\epsilon\lambda\,] + [\,J^3\,,\,\epsilon\hat{\lambda}\,]\,,\\
\delta J^3 &= \epsilon\partial\hat{\lambda} + [\,J^0\,,\,\epsilon\hat{\lambda}\,] + [\,J^2\,,\,\epsilon\lambda\,]\,.
\end{aligned}$$

In addition, as w and $\hat{w}$ are conjugate to λ and $\hat{\lambda}$, the BRST transformation of these variables are easily deduced from the form of the BRST charge, and they are

$$\epsilon Q \cdot w = -J^3\epsilon\,, \quad \epsilon Q \cdot \hat{w} = -\bar{J}^1\epsilon\,, \tag{2.51}$$

and because w and $\hat{w}$ are defined up to a gauge transformation, the variations above are equally defined up to gauge transformations. To complete the BRST transformations of all fields, the transformations of the pure spinor variables λ and $\hat{\lambda}$ are

$$\epsilon Q \cdot \lambda = \epsilon Q \cdot \hat{\lambda} = 0\,, \tag{2.52}$$

however, there is a subtle detail here. When one performs a BRST transformation of a particular coset representative such as g given in (2.34), it is possible that the final result can only be written as (2.49) after a compensating gauge transformation of $SO(1,4)\times SO(5)$. Since the pure spinor variables transform under local $SO(1,4)\times SO(5)$, these variables will transform under this compensating gauge transformation.

We will now prove that the action is BRST invariant. This is accomplished both by replacing the variations given above and by using the Maurer-Cartan identity that will be defined below. From the definitions of the currents (2.35), it is straightforward to see that they satisfy

$$d\hat{J} + \hat{J}\wedge\hat{J} = 0\,, \tag{2.53}$$

which is the Maurer-Cartan identity. In this formula, $\hat{J}$ is a one-form and $\wedge$ is the usual wedge product of forms, in components

$$\hat{J} = J\,dz + \bar{J}\,d\bar{z}\,, \quad d = dz\frac{\partial}{\partial z} + d\bar{z}\frac{\partial}{\partial \bar{z}}\,,$$

and two useful relations among the currents obtained from this identity after collecting the terms with the same grading are

$$\begin{aligned}
\nabla\bar{J}^1 - \bar{\nabla}J^1 + [\,J^2\,,\,\bar{J}^3\,] + [\,J^3\,,\,\bar{J}^2\,] = 0\,,\\
\nabla\bar{J}^3 - \bar{\nabla}J^3 + [\,J^2\,,\,\bar{J}^1\,] + [\,J^1\,,\,\bar{J}^2\,] = 0\,.
\end{aligned} \tag{2.54}$$

Using these Maurer-Cartan identities and the BRST variation of the currents already given, a straightforward calculation gives the BRST transformation of the matter part of the action

$$\begin{aligned}\delta S_{matter} &= \int d^2z\,\delta\,\mathrm{Str}\,(\frac{1}{2}J^2\bar{J}^2 + \frac{3}{4}J^3\bar{J}^1 + \frac{1}{4}J^1\bar{J}^3) \\ &= \int d^2z\,\mathrm{Str}\,(-\bar{\nabla}J^3\epsilon\lambda - \nabla\bar{J}^1\epsilon\hat{\lambda})\,.\end{aligned} \tag{2.55}$$

The next step in showing that the action is BRST invariant is to compute the BRST variation of the ghost part. Using Jacobi identities and the pure spinor conditions, it is possible to perform a few manipulations such as

$$[\{w\,,\,\lambda\}\,,\,\lambda] = [w\,,\,\{\lambda\,,\,\lambda\}] + [\lambda\,,\,\{\lambda\,,\,w\}] \;\rightarrow\; [\{w\,,\,\lambda\}\,,\,\lambda] = 0\,,$$

where we have used that the first term on the right-hand side vanishes, and show that

$$\delta S_{ghost} = \int d^2z\,\delta\,\mathrm{Str}\,(w\bar{\nabla}\lambda + \hat{w}\nabla\hat{\lambda} - N\hat{N}) = \int d^2z\,\mathrm{Str}\,(\bar{\nabla}J^3\epsilon\lambda + \nabla\bar{J}^1\epsilon\hat{\lambda})\,, \tag{2.56}$$

which precisely cancels the variation of the matter part of (2.55) implying that the action is BRST invariant.

The last comment about the action concerns its equations of motion. Although, we will not need the equations of motion in the remaining of the thesis, the method for computing them will be explained for completeness. Under an infinitesimal variation of the coset representative $\delta g = gY$ with $Y = Y^{\bar{\alpha}}T_{\bar{\alpha}} + Y^{\underline{a}}T_{\underline{a}} + Y^{\hat{\bar{\alpha}}}T_{\hat{\bar{\alpha}}}$, the transformation of the currents can be deduced from (2.50) and they are

$$\delta J = \partial Y + [\,J\,,\,Y\,]\,,$$

which implies, for example,

$$\delta J^1 = \partial Y^1 + [\,J^0\,,\,Y^1\,] + [\,J^2\,,\,Y^3\,] + [\,J^3\,,\,Y^2\,]\,.$$

Replacing these variations in the action and imposing that the variation of the action vanishes for any Y, one concludes, for example, that

$$-\frac{3}{4}\nabla\bar{J}^1 - \frac{1}{4}\bar{\nabla}J^1 + \frac{1}{4}[\,J^3\,,\,\bar{J}^2\,] - \frac{1}{4}[\,\bar{J}^3\,,\,J^2\,] - [\,\bar{J}^1\,,\,N\,] - [\,J^1\,,\,\hat{N}\,] = 0\,,$$

and using the Maurer-Cartan identity, this result can be rewritten as

$$\begin{aligned}\bar{\nabla}J^1 &= [\,J^3\,,\,\bar{J}^2\,] + [\,J^2\,,\,\bar{J}^3\,] + [\,N\,,\,\bar{J}^1\,] + [\,\hat{N}\,,\,J^1\,] \\ \nabla\bar{J}^1 &= [\,N\,,\,\bar{J}^1\,] + [\,\hat{N}\,,\,J^1\,]\,.\end{aligned}$$

The equations of motion for the ghosts can be easily derived using the properties of the supertrace. Varying w, one gets

$$\bar{\nabla}\lambda = [\,\hat{N}\,,\,\lambda\,]\,,$$

and all the remaining equations of motion can be derived in a similar way, see [16], for example.

The action written with a supertrace as in (2.45) is very convenient to prove the BRST invariance and compute its equations of motion in compact notation. However, in the next section, the useful form of the action will be the one with the supertraces evaluated. We will perform these evaluations in the remaining of this section and fix our conventions. The matter part of the action becomes

$$\begin{aligned} S_{matter} &= \int d^2z\,\mathrm{Str}\,(\frac{1}{2}J^2\bar{J}^2 + \frac{3}{4}J^3\bar{J}^1 + \frac{1}{4}J^1\bar{J}^3) \qquad (2.57)\\ &= \int d^2z\,\frac{1}{2}\eta_{\underline{ab}}J^{\underline{a}}J^{\underline{b}} - \frac{1}{2}\kappa_{\bar{\alpha}\hat{\bar{\beta}}}(J^{\bar{\alpha}}\bar{J}^{\hat{\bar{\beta}}} + \bar{J}^{\bar{\alpha}}J^{\hat{\bar{\beta}}}) + \frac{1}{4}\kappa_{\bar{\alpha}\hat{\bar{\beta}}}(J^{\bar{\alpha}}\bar{J}^{\hat{\bar{\beta}}} - \bar{J}^{\bar{\alpha}}J^{\hat{\bar{\beta}}})\,, \end{aligned}$$

where in our conventions, for example,

$$\mathrm{Str}\,(J^3\bar{J}^1) = \mathrm{Str}\,(J^{\hat{\bar{\alpha}}}T_{\hat{\bar{\alpha}}}\bar{J}^{\bar{\beta}}T_{\bar{\beta}}) = -\mathrm{Str}\,(J^{\hat{\bar{\alpha}}}\bar{J}^{\bar{\beta}}T_{\hat{\bar{\alpha}}}T_{\bar{\beta}}) = -J^{\hat{\bar{\alpha}}}\bar{J}^{\bar{\beta}}\mathrm{Str}\,(T_{\hat{\bar{\alpha}}}T_{\bar{\beta}})\,,$$

because both the currents and the generators are fermionic. In order to compute the ghost part, note that using the algebra of $PSU(2,2|4)$, we have

$$\begin{aligned} N &= -\{w\,,\,\lambda\} = -N^{ab}T_{ab} + N^{a'b'}T_{a'b'}\,,\\ \hat{N} &= -\{\,\hat{w}\,,\,\hat{\lambda}\,\} = \hat{N}^{ab}T_{ab} - \hat{N}^{a'b'}T_{a'b'}\,, \end{aligned}$$

and

$$\begin{aligned} S_{ghost} &= \int dz^2\mathrm{Str}\,(w\bar{\nabla}\lambda + \hat{w}\nabla\hat{\lambda} - N\hat{N}) \qquad (2.58)\\ &= w_{\bar{\alpha}}\bar{\nabla}\lambda^{\bar{\alpha}} - \hat{w}_{\hat{\bar{\alpha}}}\nabla\hat{\lambda}^{\hat{\bar{\alpha}}} + \eta_{[\underline{ab}][\underline{cd}]}N^{\underline{ab}}\hat{N}^{\underline{cd}}\,, \end{aligned}$$

where $\bar{\nabla}$ involves the $SO(1,4)\times SO(5)$ connections,

$$\bar{\nabla}\lambda^{\bar{\alpha}} = \bar{\partial}\lambda^{\bar{\alpha}} + \bar{J}^{\underline{ab}}\frac{1}{2}(\gamma_{\underline{ab}})^{\bar{\alpha}}{}_{\bar{\beta}}\lambda^{\bar{\beta}}\,, \qquad (2.59)$$

and similarly for $\nabla\hat{\lambda}^{\hat{\bar{\alpha}}}$.

Finally the BRST charge is

$$\epsilon Q = -\epsilon\int d\sigma\,\mathrm{Str}\,(\,\lambda J^3 + \hat{\lambda}\bar{J}^1\,) = \epsilon\int d\sigma\,\kappa_{\bar{\alpha}\hat{\bar{\beta}}}\lambda^{\bar{\alpha}}J^{\hat{\bar{\beta}}} - \epsilon\int d\sigma\,\kappa_{\bar{\alpha}\hat{\bar{\beta}}}\hat{\lambda}^{\hat{\bar{\beta}}}\bar{J}^{\bar{\alpha}}\,, \qquad (2.60)$$

or, in terms of the usual complex coordinates z and $\bar{z}$,

$$\epsilon Q = \epsilon\int dz\,\kappa_{\bar{\alpha}\hat{\bar{\beta}}}\lambda^{\bar{\alpha}}J^{\hat{\bar{\beta}}} - \epsilon\int d\bar{z}\,\kappa_{\bar{\alpha}\hat{\bar{\beta}}}\hat{\lambda}^{\hat{\bar{\beta}}}\bar{J}^{\bar{\alpha}}\,, \qquad (2.61)$$

where in our conventions $\int dz$ is a short notation for $\int\frac{dz}{2\pi i}$ and $\int d\bar{z}$ is a short notation for $\int\frac{d\bar{z}}{-2\pi i}$.

2.4 Superstrings in $AdS^5 \times S^5$ with a new supercoset

The pure spinor formalism in the background $AdS^5 \times S^5$ with the matter variables represented by the AdS^5 supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ together with $\frac{SO(6)}{SO(5)}$ variables for S^5, instead of being represented by the usual supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ as already reviewed, will be explained in this section. The pure spinor formalism with this new supercoset is the relevant formalism for the rest of the thesis.

Despite the fact that the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)} \times \frac{SO(6)}{SO(5)}$ is related with the previous one by a field redefinition, one of the advantages of working with this supercoset is that the harmonic variables that will be introduced in the chapter 4 transform under $\mathcal{N}=4$ $d=4$ supersymmetry as the $\frac{SO(6)}{SO(5)}$ variables. The method for constructing the worldsheet action and the BRST charge for the pure spinor formalism using this new coset is by comparing with the results of the formalism with the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ after a convenient gauge-fixing.

We will need the four-dimensional version of the superalgebra of $PSU(2,2|4)$ which contains the following generators: the translation generator P_μ, the special conformal generator K_μ, the dilatation generator D, the Lorentz generators $M_{\mu\nu}$, the $SU(4)$ R-symmetry generators U^i_j, the supersymmetry generators $[q_{\alpha i}, \bar{q}^i_{\dot\alpha}]$, and the generators of superconformal transformations $[s^i_\alpha, \bar{s}_{\dot\alpha i}]$. All the non-zero commutators and anticommutators of this superalgebra are given in the Appendix B.

The AdS supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ will be parametrized by 5 bosonic variables denoted $[x^\mu, z]$ and 32 fermionic variables denoted $[\theta^{\alpha j}, \bar\theta^{\dot\alpha}_j, \psi^\alpha_j, \bar\psi^{\dot\alpha j}]$. The coset representative being

$$g = \exp(\, x^\mu P_\mu + i\theta^{\alpha j} q_{\alpha j} + i\bar\theta_{\dot\alpha j}\bar{q}^{\dot\alpha j}\,)\exp(\, i\psi^\alpha_j s^j_\alpha + i\bar\psi^j_{\dot\alpha}\bar{s}^{\dot\alpha}_j\,)\, z^D\,, \tag{2.62}$$

and with this chosen representative the boundary of AdS^5 is located at $z=0$. The way of seeing this is by considering only the bosonic part of the coset representative given above, or in other worlds, excluding the terms with $[\theta, \bar\theta, \psi, \bar\psi]$, and computing the vielbeins e using the definition

$$g^{-1}_{\text{no}\,\theta,\psi}\, dg_{\text{no}\,\theta,\psi} = e^\mu P_\mu + e^z P_z\,,$$

and a straightforward calculation gives

$$e^\mu = \frac{1}{z}\, dx^{\check\mu}\delta^\mu_{\check\mu}\,, \quad e^z = \frac{1}{z}\, dz\,,$$

from where one deduces the metric

$$ds^2 = \eta_{ab}\, e^a\, e^b = \frac{1}{z^2}\,(d\vec{x}^2 + dz^2)\,, \tag{2.63}$$

which is the usual AdS^5 metric in the Poincaré patch with the boundary located at $z = 0$. Two comments about our chosen parametrization, and that will be explained in great detail in the Appendix C, are that it is not consistent to set $\psi = \bar{\psi} = 0$ as a boundary condition and when $z \to 0$ the variables $[x, \theta, \bar{\theta}]$ transform in the usual $\mathcal{N} = 4$ d=4 superconformal manner under global $PSU(2, 2|4)$ transformations.

The S^5 space $\frac{SO(6)}{SO(5)}$ will be parametrized using a unit vector y^J satisfying the constraint $y^J y^J = 1$, where the indices of the vector can be raised and lowered using the usual six-dimensional Euclidean metric. Using the $SO(6)$ Pauli matrices of Appendix A, it is possible to define $y_{jk} = y_J \sigma^J_{jk}$, $y^{jk} = y^J \sigma^{jk}_J$ and the properties of the Pauli matrices together with the constraint imply that these variables satisfy

$$\frac{1}{8}\epsilon^{jklm} y_{jk} y_{lm} = -1\,, \quad y^{jk} = \frac{1}{2}\epsilon^{jklm} y_{lm}\,, \quad y_{jk} y^{kl} = \delta^l_j\,. \tag{2.64}$$

In order to complete the set of necessary variables, we need both the left-moving $[\lambda^{\alpha j}, \bar{\lambda}^{\dot{\alpha}}_j]$ and the right-moving $[\hat{\lambda}^{\alpha j}, \hat{\bar{\lambda}}^{\dot{\alpha}}_j]$ bosonic pure spinor variables together with their respective conjugate momenta $[w_{\alpha j}, \bar{w}^j_{\dot{\alpha}}]$ and $[\hat{w}_{\alpha j}, \hat{\bar{w}}^j_{\dot{\alpha}}]$. These variables were written in four-dimensional notation and they satisfy the constraints that are the dimensional reduction of the ten-dimensional pure spinor constraints of (2.41), which are

$$\begin{aligned} \lambda^{\alpha j}\bar{\lambda}^{\dot{\alpha}}_j = 0\,, \quad \lambda^{\alpha j}\lambda^k_\alpha - \frac{1}{2}\epsilon^{jklm}\bar{\lambda}_{\dot{\alpha} l}\bar{\lambda}^{\dot{\alpha}}_m = 0\,, \\ \hat{\lambda}^{\alpha j}\hat{\bar{\lambda}}^{\dot{\alpha}}_j = 0\,, \quad \hat{\lambda}^{\alpha j}\hat{\lambda}^k_\alpha - \frac{1}{2}\epsilon^{jklm}\hat{\bar{\lambda}}_{\dot{\alpha} l}\hat{\bar{\lambda}}^{\dot{\alpha}}_m = 0\,. \end{aligned} \tag{2.65}$$

This implies, as in the ten-dimensional case, that these variables have 11 independent components and the conjugate momenta have 11 gauge-invariant components.

At several places, we will perform the dimensional reduction of expressions written in ten-dimensional notation, the first example of such a reduction being (2.65). The procedure for performing a reduction is as follows. Consider a vector V^M of $SO(1, 9)$, it decomposes as $[V^\mu, V^{I+3}]$ under its $SO(1, 3) \times SO(6)$ subgroup. In addition, we will use the ansatz for the chiral gamma matrices given below

$$\begin{aligned}
(\gamma^\mu)^{\bar{\alpha}\bar{\beta}} &= \begin{pmatrix} 0_8 & \delta^i_j \otimes i\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\alpha}} \\ \delta^i_j \otimes i\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} & 0_8 \end{pmatrix}, \\
(\gamma^\mu)_{\bar{\alpha}\bar{\beta}} &= \begin{pmatrix} 0_8 & \delta^i_j \otimes i(\sigma^\mu)_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}} \\ \delta^i_j \otimes i(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\epsilon_{\beta\alpha} & 0_8 \end{pmatrix}, \\
(\gamma^{I+3})^{\bar{\alpha}\bar{\beta}} &= \begin{pmatrix} (\sigma^I)^{ij} \otimes \epsilon^{\alpha\beta} & 0_8 \\ 0_8 & (\sigma^I)_{ij} \otimes -\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \\
(\gamma^{I+3})_{\bar{\alpha}\bar{\beta}} &= \begin{pmatrix} (\sigma^I)_{ij} \otimes \epsilon_{\alpha\beta} & 0_8 \\ 0_8 & (\sigma^I)^{ij} \otimes -\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix},
\end{aligned} \tag{2.66}$$

where $\otimes$ means the usual tensor product of matrices. All the γ^M are 16 by 16 matrices, 0_8 are 8 by 8 zero matrices, $[\sigma^\mu, \bar{\sigma}^\mu]$ are 2 by 2 $SO(1,3)$ Pauli matrices and $[\sigma^I_{ij}, \sigma^{Iij}]$ are 4 by 4 $SO(6)$ Pauli matrices. The Pauli matrices are defined in the Appendix A and using their properties that are also given in the Appendix, one can easily show that the ansatz for the chiral gamma matrices given above satisfies the dimensional reduction of

$$(\gamma^M)_{\bar{\alpha}\bar{\beta}}(\gamma^N)^{\bar{\beta}\bar{\gamma}} + (\gamma^N)_{\bar{\alpha}\bar{\beta}}(\gamma^M)^{\bar{\beta}\bar{\gamma}} = 2\eta^{MN}\delta^{\bar{\gamma}}_{\bar{\alpha}} \,.$$

Moreover, any two chiral spinors $A^{\bar{\alpha}}$ and $A_{\bar{\alpha}}$ reduce as

$$A^{\bar{\alpha}} = \begin{pmatrix} A^{\alpha i} \\ \bar{A}_{\dot{\alpha} i} \end{pmatrix} , \qquad A_{\bar{\alpha}} = \begin{pmatrix} A_{\alpha i} \\ \bar{A}^{\dot{\alpha} i} \end{pmatrix} .$$

As an example, we will perform the dimensional reduction of the pure spinor constraints and see that it is in fact given by (2.65). Note first that $\lambda\gamma^M\lambda = 0$ implies $\lambda\gamma^\mu\lambda = 0$ and $\lambda\gamma^{I+3}\lambda = 0$, and using the ansatz for the chiral gamma matrices, one has

$$\begin{aligned} \lambda^{\bar{\alpha}}\gamma^\mu_{\bar{\alpha}\bar{\beta}}\lambda^{\bar{\beta}} &= \lambda^{\alpha i} i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_i + \bar{\lambda}_{\dot{\alpha} i} i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\lambda^i_\alpha \\ &= \lambda^{\alpha i} i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_i + \bar{\lambda}_{\dot{\alpha} i} i(\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}})\lambda^i_\alpha \\ &= 2\lambda^{\alpha i} i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_i = 0 \,, \end{aligned}$$

and one concludes that $\lambda^{\alpha i}\bar{\lambda}^{\dot{\alpha}}_i = 0$. Similarly,

$$\begin{aligned} \lambda^{\bar{\alpha}}\gamma^{I+3}_{\bar{\alpha}\bar{\beta}}\lambda^{\bar{\beta}} &= \lambda^{\alpha i}\sigma^I_{ij}\lambda^j_\alpha - \bar{\lambda}_{\dot{\alpha} i}\sigma^{Iij}\bar{\lambda}^{\dot{\alpha}}_j \\ &= \lambda^{\alpha i}\sigma^I_{ij}\lambda^j_\alpha - \frac{1}{2}\sigma^I_{ij}\epsilon^{ijkl}\bar{\lambda}_{\dot{\alpha} k}\bar{\lambda}^{\dot{\alpha}}_l \\ &\to \lambda^{\alpha i}\lambda^j_\alpha - \frac{1}{2}\epsilon^{ijkl}\bar{\lambda}_{\dot{\alpha} k}\bar{\lambda}^{\dot{\alpha}}_l = 0 \,, \end{aligned}$$

and in this way we have deduced the pure spinor constraints of (2.65).

After introducing all the necessary variables, the next step is to construct the action. The pure spinor formalism with the matter variables represented by the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ together with the S^5 variables y^I has an additional gauge symmetry when compared with the pure spinor formalism with the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$. Note that fixing this additional gauge symmetry the two theories are the same. So the action and the BRST charge of the two theories have to be the same after fixing this additional gauge. This will be our strategy for constructing the action and BRST charge for the formalism with the supercoset $\frac{PSU(2,2|4}{SO(1,4)\times SO(6)} \times \frac{SO(6)}{SO(5)}$, we will check that the results are the correct ones by fixing the gauge $y_{ij} = \sigma^6_{ij}$ and showing that all the results obtained for the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ are correctly reproduced. The first step to compare the two theories in the gauge $y_{ij} = \sigma^6_{ij}$ is relating the $PSU(2,2|4)$ superalgebra written in ten-dimensional notation given in (2.26) with the same algebra in four-dimensional notation as in the Appendix B. This will be the subject of the next subsection.

2.4.1 The $PSU(2,2|4)$ algebra in two different notations

In this subsection, we will relate the superalgebra of $PSU(2,2|4)$ given in ten-dimensional notation in (2.26) with the same algebra written in four-dimensional notation in terms of the generators $[P_\mu, K_\mu, M_{\mu\nu}, D, q_{\alpha i}, \bar{q}^i_{\dot\alpha}, s^i_\alpha, \bar{s}_{\dot\alpha i}]$ which is described in the Appendix B. We will first relate the bosonic generators as

$$T_a = \begin{cases} \frac{1}{2}(P_\mu + K_\mu) & \text{if} \quad a = 0,1,2,3\,, \\ D & \text{if} \quad a = 4\,, \end{cases} \tag{2.67}$$

$$T_{ab} = \begin{cases} M_{\mu\nu} & \text{if} \quad a,b = 0,1,2,3\,, \\ \frac{1}{2}(P_\mu - K_\mu) & \text{if} \quad a = 4\,,\ b = 0,1,2,3\,, \\ -\frac{1}{2}(P_\mu - K_\mu) & \text{if} \quad a = 0,1,2,3\,,\ b = 4\,, \end{cases}$$

and

$$T_{a'} = \frac{1}{2}(\sigma_{(a'-4)6})_l{}^k U^l_k\,, \qquad T_{a'b'} = \frac{1}{2}(\sigma_{(b'-4)(a'-4)})_l{}^k U^l_k\,. \tag{2.68}$$

It is possible to show that all the commutators involving two bosonic generators of (2.26) are reproduced when we organize the generators as above. We will show the details of the calculation for a few examples. Note that using the last structure constant of (2.26), we have

$$[\,T_{4\mu}\,,\,T_{4\nu}\,] = \eta_{4[\mu}T_{4]\nu} + \eta_{\nu[4}T_{\mu]4} = -T_{\mu\nu}\,,$$

and this is precisely reproduced by

$$\begin{aligned} [\,\frac{1}{2}(P_\mu - K_\mu)\,,\,\frac{1}{2}(P_\nu - K_\nu)\,] &= -\frac{1}{4}(2\eta_{\mu\nu}D + 2M_{\mu\nu}) + \frac{1}{4}(2\eta_{\nu\mu}D + 2M_{\nu\mu}) \\ &= -M_{\mu\nu}\,, \end{aligned}$$

where we have used the commutators of the Appendix B. Similarly, note that

$$[\,D\,,\,\frac{1}{2}(P_\mu + K_\mu)\,] = \frac{1}{2}(P_\mu - K_\mu) = \frac{1}{2}\delta^{[a}_4\delta^{b]}_\mu T_{ab}\,, \tag{2.69}$$

with the correct result of (2.26). The final example is

$$
\begin{aligned}
[\,T_{a'}\,,\,T_{b'}\,] &= [\,\frac{1}{2}(\sigma_{(a'-4)6})_l{}^k U^l_k\,,\,\frac{1}{2}(\sigma_{(b'-4)6})_i{}^j U^i_j\,] \\
&= \frac{1}{4}\,(\sigma_{(a'-4)6})_l{}^k\,(\sigma_{(b'-4)6})_i{}^j\,(\delta^l_j U^i_k - \delta^i_k U^l_j) \\
&= \frac{1}{4}(\sigma_{(a'-4)})_{im}(\sigma_{(b'-4)})^{mk} U^i_k - \frac{1}{4}(\sigma_{(b'-4)})_{lm}(\sigma_{(a'-4)})^{mj} U^l_j \\
&= -\frac{1}{2}(\sigma_{(b'-4)(a'-4)})_l{}^j U^l_j \\
&= -\frac{1}{2}\delta^{[c'}_{a'}\delta^{d']}_{b'}\,T_{c'd'}\,,
\end{aligned}
$$

with the expected result. Let us now relate the fermionic generators performing the dimensional reduction of $T_{\bar{\alpha}}$ and of $T_{\hat{\bar{\alpha}}}$, the results are

$$
\begin{aligned}
T^1_{\alpha i} &= \frac{\sqrt{2}}{4}q_{\alpha i} - \frac{\sqrt{2}}{4}(\sigma^6)_{ij}s^j_\alpha, \quad T^{1i}_{\dot{\alpha}} = -\frac{\sqrt{2}}{4}\bar{q}^i_{\dot{\alpha}} - \frac{\sqrt{2}}{4}(\sigma^6)^{ij}\bar{s}_{\dot{\alpha}j}\,, \\
T^3_{\alpha i} &= -\frac{\sqrt{2}}{4}q_{\alpha i} - \frac{\sqrt{2}}{4}(\sigma^6)_{ij}s^j_\alpha, \quad T^{3i}_{\dot{\alpha}} = -\frac{\sqrt{2}}{4}\bar{q}^i_{\dot{\alpha}} + \frac{\sqrt{2}}{4}(\sigma^6)^{ij}\bar{s}_{\dot{\alpha}j}\,,
\end{aligned}
\tag{2.70}
$$

where superscript 1 refers to $T_{\bar{\alpha}}$ and 3 refers to $T_{\hat{\bar{\alpha}}}$.

One way to prove that this is the correct relation is to show that using the algebra of the Appendix B, one gets the same results of performing the dimensional reduction of the algebra (2.26). We will show these for one specific example, and all other cases being similar. Consider the anticommutator

$$
\{\,T_{\bar{\alpha}}\,,\,T_{\bar{\beta}}\,\} = \frac{1}{2}\gamma^{\underline{a}}_{\bar{\alpha}\bar{\beta}}\,T_{\underline{a}}\,, \tag{2.71}
$$

and multiply both sides by the fermionic spinors $\epsilon^{\bar{\alpha}}$ and $\rho^{\bar{\beta}}$, the dimensional reduction of the left-hand side is

$$
\begin{aligned}
&-[\,\epsilon^{\bar{\alpha}}T_{\bar{\alpha}}\,,\,\rho^{\bar{\beta}}T_{\bar{\beta}}\,] = \\
-\frac{1}{8}[\,&\epsilon^{\alpha i}q_{\alpha i} - \epsilon^{\alpha i}(\sigma^6)_{ij}s^j_\alpha - \bar{\epsilon}_{\dot{\alpha}i}\bar{q}^{\dot{\alpha}i} - \bar{\epsilon}_{\dot{\alpha}i}(\sigma^6)^{ij}\bar{s}^{\dot{\alpha}}_j\,, \\
&\rho^{\beta m}q_{\beta m} - \rho^{\beta m}(\sigma^6)_{mp}s^p_\beta - \bar{\rho}_{\dot{\beta}m}\bar{q}^{\dot{\beta}m} - \bar{\rho}_{\dot{\beta}m}(\sigma^6)^{mp}\bar{s}^{\dot{\beta}}_p\,]\,,
\end{aligned}
$$

and using the algebra of the Appendix B and the properties of the Pauli matrices of the Appendix A, the terms proportional to P_μ and K_μ after computing the commutators are

$$
\frac{1}{2}(\epsilon^{\alpha i}\,i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\rho}^{\dot{\alpha}}_i)\,\frac{1}{2}(P_\mu + K_\mu) + \frac{1}{2}(\bar{\epsilon}_{\dot{\alpha}i}\,i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\rho^i_\alpha)\,\frac{1}{2}(P_\mu + K_\mu)\,, \tag{2.72}
$$

and noting that the terms proportional to $M_{\mu\nu}$ cancel among them, the remaining terms proportional to D and U^i_j are

$$
\begin{aligned}
&\frac{1}{2}\epsilon^{\alpha i}(\sigma^6)_{im}\rho^m_\alpha D + \frac{1}{2}\epsilon^{\alpha i}\rho^m_\alpha(\sigma^6)_{mp}U^p_i - \frac{1}{2}\epsilon^{\alpha i}(\sigma^6)_{ij}\rho^m_\alpha U^j_m \\
&-\frac{1}{2}\bar{\epsilon}_{\dot{\alpha}i}(\sigma^6)^{im}\bar{\rho}^{\dot{\alpha}}_m D + \frac{1}{2}\bar{\epsilon}_{\dot{\alpha}i}\bar{\rho}^{\dot{\alpha}}_m(\sigma^6)^{mp}U^i_p - \frac{1}{2}\bar{\epsilon}_{\dot{\beta}i}(\sigma^6)^{ij}\bar{\rho}^{\dot{\beta}}_m U^m_j\,.
\end{aligned}
\tag{2.73}
$$

The next step is to compute the right-hand side of (2.71) and compare with the results of the previous calculations. We have

$$\frac{1}{2}\epsilon^{\bar\alpha}\gamma^{\underline{a}}_{\bar\alpha\bar\beta}\rho^{\bar\beta}\,T_{\underline{a}} = \frac{1}{2}\epsilon^{\bar\alpha}\gamma^{\mu}_{\bar\alpha\bar\beta}\rho^{\bar\beta}\,\frac{1}{2}(P_\mu+K_\mu)+\frac{1}{2}\epsilon^{\bar\alpha}\gamma^{9}_{\bar\alpha\bar\beta}\rho^{\bar\beta}D+\frac{1}{2}\epsilon^{\bar\alpha}\gamma^{(a'-1)}_{\bar\alpha\bar\beta}\rho^{\bar\beta}\frac{1}{2}(\sigma_{(a'-4)6})_l{}^k U^l_k\,,$$

and using the ansatz for the chiral gamma matrices of (2.66), one easily sees that the terms proportional to P_μ and K_μ of (2.72) and the terms proportional to D of (2.73) are reproduced. The terms with U^i_j require a few manipulations before comparison,

$$\begin{aligned}
\frac{1}{2}(\epsilon^{\alpha i}\sigma^{(a'-4)}_{ij}\rho^j_\alpha-\bar\epsilon_{\dot\alpha i}\sigma^{(a'-4)ij}\bar\rho^{\dot\alpha}_j)\frac{1}{2}(\sigma_{(a'-4)6})_l{}^k U^l_k =\\
\frac{1}{4}\epsilon^{\alpha i}\rho^j_\alpha(-2(\delta^k_i\delta^p_j-\delta^p_i\delta^k_j)\sigma^6_{lp}+\sigma^6_{ij}\sigma^{6pk}\sigma_{6lp})U^l_k\\
-\frac{1}{4}\bar\epsilon_{\dot\alpha i}\bar\rho^{\dot\alpha}_j[2(\delta^j_l\delta^i_m-\delta^i_l\delta^j_m)\sigma^{6mk}-\sigma^{6ij}\sigma_{6lm}\sigma^{6mk})U^l_k =\\
\frac{1}{2}\epsilon^{\alpha i}\rho^m_\alpha(\sigma^6)_{mp}U^p_i\ -\ \frac{1}{2}\epsilon^{\alpha i}(\sigma^6)_{ij}\rho^m_\alpha U^j_m\ +\ \frac{1}{2}\bar\epsilon_{\dot\alpha i}\bar\rho^{\dot\alpha}_m(\sigma^6)^{mp}U^i_p\ -\ \frac{1}{2}\bar\epsilon_{\dot\beta i}(\sigma^6)^{ij}\bar\rho^{\dot\beta}_m U^m_j\,,
\end{aligned}$$

where we have used that $U^i_i=0$. The final result is equal to the terms proportional to U^i_j of (2.73). The last comment of this subsection is that in order to perform the dimensional reduction of some of the structure constants it is necessary to know the matrix form of $\kappa_{\bar\alpha\hat{\bar\beta}}$. In our conventions, it is equal to $\kappa_{\bar\alpha\hat{\bar\beta}}=i(\gamma^{01239})_{\bar\alpha\hat{\bar\beta}}$ and using the ansatz for the chiral gamma matrices of (2.66),

$$\kappa_{\bar\alpha\hat{\bar\beta}} = \begin{pmatrix} (\sigma^6)_{ij}\otimes\epsilon_{\alpha\beta} & 0_8 \\ 0_8 & (\sigma^6)^{ij}\otimes\epsilon^{\dot\alpha\dot\beta}\end{pmatrix}\,. \tag{2.74}$$

2.4.2 The action

After relating the two forms of the $PSU(2,2|4)$ superalgebra in the previous subsection, we will construct the action and the BRST charge of the pure spinor formalism with the matter variables represented by the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ together with S^5 variables. It is important to know how the variables transform under a gauge transformation in order to construct a gauge-invariant action. In our conventions, under an infinitesimal local $SO(1,4)\times SO(6)$ gauge transformation the coset representative (2.62) transforms by right multiplication as $\delta g = g\Omega$. The pure spinor variables and the S^5 variables y_{ij} must also transform. Under the $SO(1,3)\times SO(6)$ subgroup of $SO(1,4)\times SO(6)$, these variables transform in the obvious way as

$$\delta y_{jk} \;=\; c^l_j y_{lk}+c^l_k y_{jl}\,, \tag{2.75}$$

and

$$\delta\lambda^j_\alpha = c^\beta_\alpha\lambda^j_\beta - c^j_k\lambda^k_\alpha\,,\qquad \delta\bar\lambda^{\dot\alpha}_j = c^{\dot\alpha}_{\dot\beta}\bar\lambda^{\dot\beta}_j + c^k_j\bar\lambda^{\dot\alpha}_k\,, \tag{2.76}$$

$$\delta w^\alpha_j = -c^\alpha_\beta w^\beta_j + c^k_j w^\alpha_k\,, \qquad \delta \bar w^j_{\dot\alpha} = -c^{\dot\beta}_{\dot\alpha}\bar w^j_{\dot\beta} - c^j_k \bar w^k_{\dot\alpha}\,,$$
$$\delta\hat\lambda^j_\alpha = c^\beta_\alpha\hat\lambda^j_\beta - c^j_k\hat\lambda^k_\alpha\,, \qquad \delta\hat{\bar\lambda}^{\dot\alpha}_j = c^{\dot\alpha}_{\dot\beta}\hat{\bar\lambda}^{\dot\beta}_j + c^k_j\hat{\bar\lambda}^{\dot\alpha}_k,$$
$$\delta\hat w^\alpha_j = -c^\alpha_\beta\hat w^\beta_j + c^k_j\hat w^\alpha_k\,, \qquad \delta\hat{\bar w}^j_{\dot\alpha} = -c^{\dot\beta}_{\dot\alpha}\hat{\bar w}^j_{\dot\beta} - c^j_k\hat{\bar w}^k_{\dot\alpha}\,,$$

where $\Omega = c^j_k U^k_j - \frac{1}{4}(c^\alpha_\beta(\sigma^{\mu\nu})_\alpha{}^\beta + c^{\dot\alpha}_{\dot\beta}(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha})M_{\mu\nu}$. Under a local transformation generated by the four generators of $SO(1,4)$ not contained in $SO(1,3)$, these variables transform as

$$\delta y_{jk} = 0 \tag{2.77}$$

and

$$\begin{aligned}
\delta\lambda^j_\alpha &= -c_{\dot\alpha\alpha}y^{jk}\bar\lambda^{\dot\alpha}_k\,, \qquad & \delta\bar\lambda^{\dot\alpha}_j &= c^{\alpha\dot\alpha}y_{jk}\lambda^k_\alpha\,,\\
\delta\hat\lambda^j_\alpha &= -c_{\dot\alpha\alpha}y^{jk}\hat{\bar\lambda}^{\dot\alpha}_k\,, \qquad & \delta\hat{\bar\lambda}^{\dot\alpha}_j &= c^{\alpha\dot\alpha}y_{jk}\hat\lambda^k_\alpha\,,\\
\delta w^\alpha_j &= c^{\alpha\dot\alpha}y_{jk}\bar w^k_{\dot\alpha}\,, \qquad & \delta\bar w^j_{\dot\alpha} &= -c_{\dot\alpha\alpha}y^{jk}w^\alpha_k\,,\\
\delta\hat w^\alpha_j &= c^{\alpha\dot\alpha}y_{jk}\hat{\bar w}^k_{\dot\alpha}\,, \qquad & \delta\hat{\bar w}^j_{\dot\alpha} &= -c_{\dot\alpha\alpha}y^{jk}\hat w^\alpha_k\,,
\end{aligned} \tag{2.78}$$

where $\Omega = c^{\alpha\dot\alpha}i\sigma^\mu_{\alpha\dot\alpha}T_{4\mu}$.

One comment on how to understand these transformations, specially the last ones, is in order. Note that in the pure spinor formalism with the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ under a gauge transformation of $SO(1,4)\times SO(5)$ the pure spinor variables transform as (2.47). It is possible to show that in the gauge $y_{ij} = \sigma^6_{ij}$ the transformations given above when restricted to $SO(1,4)\times SO(5)$ transformations are the dimensional reduction of (2.47) using the relation among the fermionic generators of (2.70).

Moreover, these transformations imply that

$$\begin{aligned}
\lambda^{Aj} &= [\lambda^j_\alpha, y^{jk}\bar\lambda^{\dot\alpha}_k]\,, \qquad & w_{Aj} &= [w^\alpha_j, -y_{jk}\bar w^k_{\dot\alpha}]\,,\\
\hat\lambda^{Aj} &= [\hat\lambda^j_\alpha, y^{jk}\hat{\bar\lambda}^{\dot\alpha}_k]\,, \qquad & \hat w_{Aj} &= [\hat w^\alpha_j, -y_{jk}\hat{\bar w}^k_{\dot\alpha}]\,,
\end{aligned} \tag{2.79}$$

transform covariantly as $SO(1,4)\times SO(6)$ spinors where $A = (\alpha,\dot\alpha)$ is an $SO(1,4)$ spinor index. In particular, when $y_{ij} = \sigma^6_{ij}$ they transform covariantly as $SO(1,4)\times SO(5)$ spinors where $SO(5)$ is the subgroup of $SO(6)$ that leaves the vectorial index 6 of the Pauli matrix invariant.

The left-invariant currents of the theory are defined by

$$\begin{aligned}
g^{-1}\partial g = {} & J^\mu\frac{1}{2}(P_\mu + K_\mu) + J^4 D + J^{AB}M_{AB} + J^k_j U^j_k\\
& + J^{\alpha j}q_{\alpha j} + J_{\dot\alpha j}\bar q^{\dot\alpha j} + J^\alpha_j s^j_\alpha + J^j_{\dot\alpha}\bar s^{\dot\alpha}_j\,,
\end{aligned} \tag{2.80}$$

where g is the coset representative given in (2.62) and $M_{AB} = M_{BA}$ are the set of $SO(1,4)$ generators with $A, B = (\alpha, \dot{\alpha})$ $SO(1,4)$ spinor indices. Note that in the gauge $y_{ij} = \sigma^6_{ij}$ this expression must be equal to (2.35), in particular, this implies that the fermionic currents must satisfy

$$J^{\bar{\alpha}} T_{\bar{\alpha}} + J^{\hat{\bar{\alpha}}} T_{\hat{\bar{\alpha}}} = J^{\alpha j} q_{\alpha j} + J_{\dot{\alpha} j} \bar{q}^{\dot{\alpha} j} + J^{\alpha}_{j} s^{j}_{\alpha} + J^{j}_{\dot{\alpha}} \bar{s}^{\dot{\alpha}}_{j} \, , \tag{2.81}$$

and performing the dimensional reduction of the left-hand side of the expression above using the relations among the fermionic generators of (2.70), one concludes that

$$\begin{aligned} J^{\alpha j}_{1} &= \sqrt{2}\, J^{\alpha j} + \sqrt{2}\, (\sigma^6)^{ji} J^{\alpha}_{i} \, , \qquad & J^{\dot{\alpha}}_{1j} &= -\sqrt{2}\, J^{\dot{\alpha}}_{j} + \sqrt{2}\, (\sigma^6)_{ji} J^{i\dot{\alpha}} \, , \\ J^{\alpha j}_{3} &= -\sqrt{2} J^{\alpha j} + \sqrt{2} (\sigma^6)^{ji} J^{\alpha}_{i} \, , \qquad & J^{\dot{\alpha}}_{3j} &= -\sqrt{2} J^{\dot{\alpha}}_{j} - \sqrt{2} (\sigma^6)_{ji} J^{\dot{\alpha} i} \, , \end{aligned} \tag{2.82}$$

where the subscript 1 and 3 refer to the currents $J^{\bar{\alpha}}$ and $J^{\hat{\bar{\alpha}}}$, respectively. Similarly, in this gauge, the bosonic currents must satisfy

$$J^{\mu} \frac{1}{2} (P_{\mu} + K_{\mu}) + J^4 D + J^{AB} M_{AB} + J^{k}_{j} U^{j}_{k} = J^{\underline{ab}} T_{\underline{ab}} + J^{\underline{a}} T_{\underline{a}} \, . \tag{2.83}$$

Using the relation among the bosonic generators of (2.68), one concludes that

$$J^{i}_{j} = J^{a'} \frac{1}{2} (\sigma_{(a'-4)6})_{j}{}^{i} + J^{a'b'} \frac{1}{2} (\sigma_{(b'-4)(a'-4)})_{j}{}^{i} \, , \tag{2.84}$$

and from this relation, using the properties of the $SO(6)$ Pauli matrices and the first relation of (2.67), one has $J^{\underline{a}} = [J^{\mu}, J^4, J^{a'}]$ with $J^{a'} = \frac{1}{2} (\sigma^{6(a'-4)})_{k}{}^{j} J^{k}_{j}$. It is left to compare the currents of $SO(1,4)$, note that

$$J^{AB} M_{AB} = J^{\alpha\beta} M_{\alpha\beta} + J^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}} + 2 J^{\alpha\dot{\beta}} M_{\alpha\dot{\beta}} \, ,$$

and

$$J^{ab} T_{ab} = J^{\mu\nu} T_{\mu\nu} + 2 J^{4\mu} T_{4\mu} \, .$$

Equating the right-hand side of the two equations above and replacing $T_{\mu\nu} = M_{\mu\nu}$ and $M_{\alpha\dot{\beta}}$ by

$$M_{\mu\nu} = \frac{1}{2} (\sigma_{\mu\nu})_{\gamma}{}^{\beta} \epsilon^{\gamma\alpha} M_{\alpha\beta} + \frac{1}{2} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}} \, , \quad M_{\alpha\dot{\beta}} = \frac{1}{2} i \sigma^{\mu}_{\alpha\dot{\beta}} T_{4\mu} \, ,$$

it is easy to see that

$$J^{\alpha\beta} = J^{\mu\nu} \epsilon^{\alpha\gamma} \frac{1}{2} (\sigma_{\mu\nu})_{\gamma}{}^{\beta} \, , \quad J^{\dot{\alpha}\dot{\beta}} = J^{\mu\nu} \epsilon^{\dot{\alpha}\dot{\gamma}} \frac{1}{2} (\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\gamma}} \, , \quad J^{\alpha\dot{\alpha}} = J^{4\mu} i \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu} \, . \tag{2.85}$$

We have now all the elements to write down the worldsheet action in the pure spinor formalism with the AdS supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$. The matter part of the action, or in other words, the ghost-independent part is

$$S_{matter} = \int d^2z\,[\frac{1}{2}\eta_{\mu\nu}J^\mu\bar{J}^\nu + \frac{1}{2}J^4\bar{J}^4 - \frac{1}{8}(\nabla y)_{jk}(\bar{\nabla}y)^{jk} \\ -2J^\alpha_j\bar{J}^j_\alpha - 2J^{\alpha j}\bar{J}_{\alpha j} - 2J^j_{\dot\alpha}\bar{J}^{\dot\alpha}_j - 2J_{\dot\alpha j}\bar{J}^{\dot\alpha j} \\ -y_{jk}J^{\alpha j}\bar{J}^k_\alpha - y^{jk}J^\alpha_j\bar{J}_{\alpha k} + y^{jk}J_{\dot\alpha j}\bar{J}^{\dot\alpha}_k + y_{jk}J^j_{\dot\alpha}\bar{J}^{\dot\alpha k}]\,, \tag{2.86}$$

where $(\nabla y)_{jk} = \partial y_{jk} - J^l_j y_{lk} - J^l_k y_{jl}$. In order to check that this is the correct action we will prove that it reproduces the usual pure spinor action of (2.57) in the gauge $y_{jk} = \sigma^6_{jk}$.

It is not difficult to see that in this gauge the term $-\frac{1}{8}(\nabla y)_{jk}(\bar{\nabla}y)^{jk}$ can be rewriten as $\frac{1}{2}\sum_{a'}J^{a'}\bar{J}^{a'}$ where $J^{a'} = \frac{1}{2}(\sigma^{6(a'-4)})_k{}^j J^k_j$. The easiest way to prove this is to compare the two results after a few manipulations using the properties of the Pauli matrices of the Appendix A. Further, note that the third line of the action reduces to

$$-\sigma^6_{jk}J^{\alpha j}\bar{J}^k_\alpha - \sigma^{6jk}J^\alpha_j\bar{J}_{\alpha k} + \sigma^{6jk}J_{\dot\alpha j}\bar{J}^{\dot\alpha}_k + \sigma^6_{jk}J^j_{\dot\alpha}\bar{J}^{\dot\alpha k}\,. \tag{2.87}$$

It is straightforward now to show the equivalence with the action written in terms of the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ supercoset of (2.57) which, after performing the dimensional reduction using the matrix form of $\kappa_{\hat{\bar\alpha}\hat{\bar\beta}}$ given in (2.74), is

$$S_{matter} = \int d^2z[\frac{1}{2}\eta_{\underline{ab}}J^{\underline{a}}_2\bar{J}^{\underline{b}}_2 - \frac{1}{2}\varepsilon_{AB}(\sigma^6)_{JK}(J^{AJ}_1\bar{J}^{BK}_3 + \bar{J}^{AJ}_1 J^{BK}_3) \\ +\frac{1}{4}\varepsilon_{AB}(\sigma^6)_{JK}(J^{AJ}_1\bar{J}^{BK}_3 - \bar{J}^{AJ}_1 J^{BK}_3)]\,, \tag{2.88}$$

where we have used the notation

$$\varepsilon_{AB}(\sigma^6)_{JK}J^{AJ}_1\bar{J}^{BK}_3 = (\sigma^6)_{ij}\epsilon_{\alpha\beta}J^{\alpha i}_1\bar{J}^{\beta j}_3 + (\sigma^6)^{ij}\epsilon^{\dot\alpha\dot\beta}J_{1\dot\alpha i}\bar{J}_{3\dot\beta j}\,,$$

and similarly for the other terms. Substituting $J^{\underline{a}}_2 = [J^\mu, J^4, J^{a'}]$ and the relation among the fermionic currents of (2.82), the two matter actions are equal.

In order to complete the action we need its ghost-dependent contribution, which is

$$S_{ghost} = \int d^2z\,[\,w_{Aj}(\bar{\nabla}\lambda)^{Aj} - \hat{w}_{Aj}(\nabla\hat{\lambda})^{Aj} \\ +\frac{1}{2}y^{jl}(\bar{\nabla}y)_{lk}w_{Aj}\lambda^{Ak} - \frac{1}{2}y^{jl}(\bar{\nabla}y)_{lk}\hat{w}_{Aj}\hat{\lambda}^{Ak} \\ -2N_{\mu\nu}\widehat{N}^{\mu\nu} - 4(y^J N_{J\mu})(y_K\widehat{N}^{K\mu}) + 2N_{JK}\widehat{N}^{JK} - 4(y^L N_{LJ})(y_M\widehat{N}^{MJ})\,]\,, \tag{2.89}$$

where in the first and the second line of the action above, λ^{Aj}, w_{Aj}, $\hat{\lambda}^{Aj}$ and $\hat{w}_{Aj}$ are the $SO(1,4)\times SO(6)$ spinors defined in (2.79) and

$$
\begin{aligned}
w_{Aj}(\bar{\nabla}\lambda)^{Aj} &= w^{\alpha}_{j}\bar{\partial}\lambda^{j}_{\alpha} + \bar{w}^{k}_{\dot{\alpha}}y_{ki}\bar{\partial}(y^{ij}\bar{\lambda}^{\dot{\alpha}}_{j}) - w^{\alpha}_{j}\bar{J}^{\beta}_{\alpha}\lambda^{j}_{\beta} - \bar{w}^{k}_{\dot{\alpha}}\bar{J}^{\dot{\alpha}}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}}_{k} \\
&+2w^{\alpha}_{j}\bar{J}_{\alpha\dot{\alpha}}y^{jk}\bar{\lambda}^{\dot{\alpha}}_{k} - 2\bar{w}^{k}_{\dot{\alpha}}y_{kl}\bar{J}^{\dot{\alpha}\alpha}\lambda^{l}_{\alpha} + w^{\alpha}_{j}\bar{J}^{j}_{k}\lambda^{k}_{\alpha} + \bar{w}^{k}_{\dot{\alpha}}y_{ki}\bar{J}^{i}_{m}y^{mp}\bar{\lambda}^{\dot{\alpha}}_{p}\,,
\end{aligned}
\tag{2.90}
$$

and similarly for $\hat{w}_{Aj}(\nabla\hat{\lambda})^{Aj}$. Note that the covariant derivative above contains the $SO(1,4)\times SO(6)$ connections. In the third line of the action, the $SO(1,9)$ Lorentz currents have been decomposed into their $SO(1,3)\times SO(6)$ components as $[N^{\mu\nu}, N^{\mu J}, N^{JK}]$ and are constructed out of the $SO(1,3)\times SO(6)$ spinors $(\lambda^{\alpha j}, \bar{\lambda}^{\dot{\alpha}}_{j})$ and $(w_{\alpha j}, \bar{w}^{j}_{\dot{\alpha}})$ and similarly for the hatted currents.

This ghost action can be verified by choosing the gauge $y_{jk}=\sigma^{6}_{jk}$ and comparing with the ghost action for the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ supercoset given in (2.58) and reproduced below with a convenient notation

$$
S_{ghost} = \int d^2z[w^{5}_{Aj}\widetilde{\bar{\nabla}}\lambda^{Aj}_{5} - \widehat{w}^{5}_{Aj}\widetilde{\nabla}\hat{\lambda}^{Aj}_{5} + \eta_{[\underline{ab}][\underline{cd}]}\,\widetilde{N}^{\underline{ab}}_{5}\widehat{\widetilde{N}}{}^{\underline{cd}}_{5}]\,,
\tag{2.91}
$$

where $\eta_{[\underline{ab}][\underline{cd}]}$ was defined in (2.27). The meaning of the superscript and subscript 5 in λ^{Aj}_{5}, $\hat{\lambda}^{Aj}_{5}$, w^{5}_{Aj} and $\widehat{w}^{5}_{Aj}$ is that these are $SO(1,4)\times SO(5)$ spinors obtained from the definition of the $SO(1,4)\times SO(6)$ spinors of (2.79) by setting $y_{ij}=\sigma^{6}_{ij}$. Similarly, $\widetilde{N}^{\underline{ab}}_{5}$ is constructed out of these $SO(1,4)\times SO(5)$ spinors. The tilde over the covariant derivative means that $\widetilde{\nabla}$ only involves the $SO(1,4)\times SO(5)$ connections.

Let us compare these two actions. In the gauge $y_{ij}=\sigma^{6}_{ij}$ the first term of the second line of (2.89) reduces to

$$
-\frac{1}{2}w^{\alpha}_{j}\bar{J}^{j}_{k}\lambda^{k}_{\alpha} - \frac{1}{2}w^{\alpha}_{j}\sigma^{6jl}\bar{J}^{m}_{l}\sigma^{6}_{mk}\lambda^{k}_{\alpha} - \frac{1}{2}\bar{w}^{l}_{\dot{\alpha}}\bar{J}^{m}_{l}\bar{\lambda}^{\dot{\alpha}}_{m} - \frac{1}{2}\bar{w}^{l}_{\dot{\alpha}}\sigma^{6}_{lm}\bar{J}^{m}_{k}\sigma^{6ki}\bar{\lambda}^{\dot{\alpha}}_{i}\,.
\tag{2.92}
$$

Consider now the last two terms of (2.90), we have after fixing the gauge

$$
\begin{aligned}
w^{\alpha}_{j}\bar{J}^{j}_{k}\lambda^{k}_{\alpha} + \bar{w}^{k}_{\dot{\alpha}}y_{ki}\bar{J}^{i}_{m}y^{mp}\bar{\lambda}^{\dot{\alpha}}_{p} &\rightarrow w^{\alpha}_{j}\bar{J}^{j}_{k}\lambda^{k}_{\alpha} + \bar{w}^{k}_{\dot{\alpha}}(\sigma^{6})_{ki}\bar{J}^{i}_{m}(\sigma^{6})^{mp}\bar{\lambda}^{\dot{\alpha}}_{p} \\
&= w^{\alpha}_{j}\bar{J}^{a'b'}\frac{1}{2}(\sigma_{(a'-4)(b'-4)})^{j}{}_{k}\lambda^{k}_{\alpha} + \bar{w}^{k}_{\dot{\alpha}}\frac{1}{2}\bar{J}^{a'b'}(\sigma_{(a'-4)(b'-4)})_{k}{}^{p}\bar{\lambda}^{\dot{\alpha}}_{p} \\
&+\frac{1}{2}w^{\alpha}_{j}\bar{J}^{j}_{k}\lambda^{k}_{\alpha} + \frac{1}{2}w^{\alpha}_{j}\sigma^{6jl}\bar{J}^{m}_{l}\sigma^{6}_{mk}\lambda^{k}_{\alpha} + \frac{1}{2}\bar{w}^{l}_{\dot{\alpha}}\bar{J}^{m}_{l}\bar{\lambda}^{\dot{\alpha}}_{m} + \frac{1}{2}\bar{w}^{l}_{\dot{\alpha}}\sigma^{6}_{lm}\bar{J}^{m}_{k}\sigma^{6ki}\bar{\lambda}^{\dot{\alpha}}_{i}\,,
\end{aligned}
\tag{2.93}
$$

where we have used the relation among the currents of (2.84). Note that the terms of the second line of the expression above contain the $SO(5)$ connections of the covariant derivative $\widetilde{\bar{\nabla}}$ of (2.91) and the third line cancels precisely with (2.92). Using the same reasoning of this example, it is not difficult to see that all the terms in the first and second line of (2.89) reproduce in the gauge $y_{ij}=\sigma^{6}_{ij}$ the first and the second term on the right-hand side of (2.91).

The next step is to understand the term with the ghost Lorentz currents. Using the definition of $\eta_{[\underline{ab}][\underline{cd}]}$ of (2.27), we see that

$$\eta_{[\underline{ab}][\underline{cd}]}\,\widetilde{N}_5^{\underline{ab}}\widehat{\widetilde{N}}_5^{\underline{cd}} = -2\widetilde{N}_5^{\mu\nu}\widehat{\widetilde{N}}_{5\mu\nu} - 4\widetilde{N}_5^{4\mu}(\widehat{\widetilde{N}}_5)_{4\mu} + 2\widetilde{N}_5^{a'b'}\widehat{\widetilde{N}}_{5a'b'}\,,$$

and using the ansatz for the chiral gamma matrices of (2.66), one can show that the third line of (2.89) reduces to these terms after gauge-fixing. This shows that the two actions are equal in the gauge $y_{ij} = \sigma^6_{ij}$.

The BRST charge of the theory can be determined in a similar way. In terms of the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ supercoset the BRST operator is given by (2.61) and dimensionally reducing this operator using the matrix form of $\kappa_{\bar{\alpha}\hat{\bar{\beta}}}$ of (2.74) one has

$$Q = \int dz \lambda^{AJ}\varepsilon_{AB}(\sigma^6)_{JK}J_3^{BK} - \int d\bar{z}\hat{\lambda}^{AJ}\varepsilon_{AB}(\sigma^6)_{JK}\bar{J}_1^{BK}\,, \tag{2.94}$$

where we have used the same notation as the one described under (2.88). The BRST operator in terms of the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ supercoset must be equal to the one above when $y_{ij} = \sigma^6_{ij}$ and using the relation among the fermionic currents in this gauge of (2.82), it is easy to see that

$$\begin{aligned} Q = &\int dz[\lambda^{\alpha j}(\sqrt{2}J_{\alpha j} - \sqrt{2}y_{jk}J^k_\alpha) - \bar{\lambda}_{\dot{\alpha}j}(\sqrt{2}J^{\dot{\alpha}j} + \sqrt{2}y^{jk}J^{\dot{\alpha}}_k)] \\ &- \int d\bar{z}\ [\hat{\lambda}^{\alpha j}(\sqrt{2}\bar{J}_{\alpha j} + \sqrt{2}y_{jk}\bar{J}^k_\alpha) + \hat{\bar{\lambda}}_{\dot{\alpha}j}(\sqrt{2}\bar{J}^{\dot{\alpha}j} - \sqrt{2}y^{jk}\bar{J}^{\dot{\alpha}}_k)]. \end{aligned} \tag{2.95}$$

The BRST variation of a representative of the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ is given by

$$\delta g = g\frac{\sqrt{2}}{4}\,[\,i\lambda^{+\alpha j}q_{\alpha j} - i\lambda^{-\alpha j}y_{jk}s^k_\alpha + i\bar{\lambda}^+_{\dot{\alpha}j}\bar{q}^{\dot{\alpha}j} + i\bar{\lambda}^-_{\dot{\alpha}j}y^{jk}\bar{s}^{\dot{\alpha}}_k]\,, \tag{2.96}$$

where we have used the definitions

$$\begin{aligned} \lambda^{-\alpha j} &\equiv -i(\lambda^{\alpha j} + \hat{\lambda}^{\alpha j}), \qquad \lambda^{+\alpha j} \equiv -i(\lambda^{\alpha j} - \hat{\lambda}^{\alpha j}) \\ \bar{\lambda}^-_{\dot{\alpha}j} &\equiv i(\bar{\lambda}_{\dot{\alpha}j} - \hat{\bar{\lambda}}_{\dot{\alpha}j}), \qquad \bar{\lambda}^+_{\dot{\alpha}j} \equiv i(\bar{\lambda}_{\dot{\alpha}j} + \hat{\bar{\lambda}}_{\dot{\alpha}j})\,. \end{aligned} \tag{2.97}$$

Note that in the gauge $y_{ij} = \sigma^6_{ij}$ the BRST variation given above reproduces the BRST variation of a representative of the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ of (2.49) after dimensional reduction and using the relation among the fermionic generators of (2.70).

The BRST variation of $w_{\alpha j}$, $\bar{w}^j_{\dot{\alpha}}$, $\hat{w}_{\alpha j}$ and $\hat{\bar{w}}^j_{\dot{\alpha}}$ can be directly deduced from the BRST operator (2.95) because these variables are conjugate to $\lambda_{\alpha j}$, $\bar{\lambda}_{\dot{\alpha}i}$, $\hat{\lambda}_{\alpha j}$ and $\hat{\bar{\lambda}}_{\dot{\alpha}i}$, respectively. Equivalently, one can compute this variation by noting that in the gauge $y_{ij} = \sigma^6_{ij}$ the result must be equal to the dimensional reduction of (2.51). The variations are

$$\delta w_{\alpha j} = \sqrt{2} J_{\alpha j} - \sqrt{2} y_{jk} J^k_\alpha \,, \qquad \delta \bar{w}^{\dot{\alpha} j} = -\sqrt{2} J^{\dot{\alpha} j} - \sqrt{2} y^{jk} J^{\dot{\alpha}}_k \,,$$
$$\delta \hat{w}_{\alpha j} = \sqrt{2} \bar{J}_{\alpha j} + \sqrt{2} y_{jk} \bar{J}^k_\alpha \,, \qquad \delta \hat{\bar{w}}^{\dot{\alpha} j} = \sqrt{2} \bar{J}^{\dot{\alpha} j} - \sqrt{2} y^{jk} \bar{J}^{\dot{\alpha}}_k \,,$$

where again these variations are defined up to gauge transformations of w and $\hat{w}$. Finally, the BRST transformations of λ^j_α, $\hat{\lambda}^j_\alpha$, $\bar{\lambda}_{\dot{\alpha} i}$, $\hat{\bar{\lambda}}_{\dot{\alpha} i}$ and y^{jk} are zero up to a possible gauge-compensating transformation of $SO(1,4) \times SO(6)$ as already explained in the previous section.

Chapter 3

The BRST operator

In this chapter, the BRST operator of the pure spinor formalism in $AdS^5 \times S^5$ with the worldsheet matter variables represented by the supercoset $\frac{PSU(2.2|4)}{SO(1,4)\times SO(6)}$ together with S^5 variables will be expanded in powers of z. The main motivation for performing this expansion is to compute the cohomology of this operator close to the boundary of AdS where $z \sim 0$.

The worldsheet is parametrized by two coordinates that we will call τ and σ. We will associate τ with its time direction and σ with its space direction. The result of the expansion of the BRST operator will be expressed in terms of the worldsheet variables $[x, \theta, \psi, z, y, \lambda, \hat{\lambda}]$, their canonical momenta $[P_x, P_\theta, P_\psi, P_z, P_y, P_\lambda, P_{\hat{\lambda}}]$ and their σ derivatives. In other words, the dependence of this operator on the time derivatives, or τ derivatives, of the worldsheet variables will be expressed in terms of its canonical momenta defined as $P_X = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X)}$, where X is a shorthand notation for all the variables. This substitution is made for convenience because it will facilitate the computation of the cohomology of the BRST operator. One important remark is that there are no constraints on the canonical momenta in the pure spinor formalism, unlike in the Green-Schwarz formalism which has first- and second-class constraints.

After performing the expansion, the BRST operator Q can be organized in the form

$$Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \dots ,$$

where Q_n is proportional to z^n. As will be explained in this chapter, the vertex operators can also be expanded in powers of z close to the boundary of AdS and they have a term with a minimal power of z. Using both expansions, we will show that the computation of the zero mode cohomology at +2 ghost number of the BRST operator corresponding to the physical supergravity states is equivalent to computing the cohomology of the operator $Q_{-\frac{1}{2}}$, then computing the cohomology of the operator $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$, and so on. The cohomology of the operator $Q_{-\frac{1}{2}}$ was computed in the article [23] by Mikhailov and Xu and their results will be reviewed in the section 3.3. In the last section of this chapter, we will argue that the cohomology of the BRST operator is completely determined by the cohomology of the first two terms of the expansion $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$.

3.1 The expansion of the BRST operator

The first step to performing the expansion of the BRST operator is to compute the canonical momenta of the worldsheet variables. The Lagrangian density can be deduced from the action given in (2.86) and (2.89), in particular, it depends on the left-invariant currents J defined in (2.80). Given our chosen coset representative of (2.62), we can compute these currents using the Hadamard lemma

$$e^{-X}Ye^{X} = Y - [X,Y] + \frac{1}{2}[X,[X,Y]] - \frac{1}{6}[X,[X,[X,Y]]] + \dots\,, \tag{3.1}$$

and the identity

$$e^{-X(t)}\frac{d}{dt}e^{X(t)} = \dot{X} - \frac{1}{2}[X,\dot{X}] + \frac{1}{6}[X,[X,\dot{X}]] + \dots\,,$$

together with the four-dimensional version of the $PSU(2,2|4)$ superalgebra presented in the Appendix B.

One comment is that when one has to compute a commutator of the type $[\theta^{\alpha i}q_{\alpha i}, \bar{\theta}_{\dot{\alpha}j}\bar{q}^{\dot{\alpha}j}]$ where both the generators and the variables are fermionic, there is a minus sign, or in other words, $[\theta^{\alpha i}q_{\alpha i}, \bar{\theta}_{\dot{\alpha}j}\bar{q}^{\dot{\alpha}j}] = -\theta^{\alpha i}\bar{\theta}_{\dot{\alpha}j}\{q_{\alpha i}, \bar{q}^{\dot{\alpha}j}\}$. Defining

$$e^{\mu} = \partial x^{\mu} + i\theta^{\beta i}(\sigma^{\mu})_{\beta\dot{\beta}}\partial\bar{\theta}^{\dot{\beta}}_{i} - i\partial\theta^{\alpha i}(\sigma^{\mu})_{\alpha\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}_{i}\,, \tag{3.2}$$

the currents are

$$J^{\mu}_{P} = \frac{1}{z}e^{\mu}\,, \qquad J^{4} = \frac{\partial z}{z} + 2\partial\theta^{\alpha i}\psi_{\alpha i} + 2\partial\bar{\theta}_{\dot{\alpha}i}\bar{\psi}^{\dot{\alpha}i}\,, \tag{3.3}$$

$$J_{\dot{\alpha}j} = \frac{1}{\sqrt{z}}i\partial\bar{\theta}_{\dot{\alpha}j} + \frac{1}{\sqrt{z}}e^{\mu}\psi^{\alpha}_{j}(\sigma_{\mu})_{\alpha\dot{\alpha}}\,, \quad J^{\alpha j} = \frac{1}{\sqrt{z}}i\partial\theta^{\alpha j} + \frac{1}{\sqrt{z}}e^{\mu}\bar{\psi}^{j}_{\dot{\alpha}}(\bar{\sigma}_{\mu})^{\dot{\alpha}\alpha}\,,$$

$$J^{\alpha}_{j} = \sqrt{z}i\partial\psi^{\alpha}_{j} + 4\sqrt{z}i\psi^{\beta}_{j}\partial\theta^{i}_{\beta}\psi^{\alpha}_{i} - 2\sqrt{z}i\psi^{\alpha}_{k}\bar{\psi}^{k}_{\dot{\alpha}}\partial\bar{\theta}^{\dot{\alpha}}_{j} - 2\sqrt{z}e^{\mu}\psi^{\beta}_{j}(\sigma_{\mu})_{\beta\dot{\alpha}}\bar{\psi}^{\dot{\alpha}i}\psi^{\alpha}_{i}\,,$$

$$J^{j}_{\dot{\alpha}} = \sqrt{z}i\partial\bar{\psi}^{j}_{\dot{\alpha}} - 2\sqrt{z}i\bar{\psi}^{i}_{\dot{\alpha}}\partial\theta^{\alpha j}\psi_{\alpha i} + 4\sqrt{z}i\bar{\psi}^{j}_{\dot{\beta}}\partial\bar{\theta}^{\dot{\beta}}_{i}\bar{\psi}^{i}_{\dot{\alpha}} + 2\sqrt{z}e^{\mu}\psi^{\alpha}_{i}(\sigma_{\mu})_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}j}\bar{\psi}^{i}_{\dot{\alpha}}\,,$$

$$\begin{aligned}J^{j}_{i} = &-4\partial\theta^{\alpha j}\psi_{\alpha i} + 4\partial\bar{\theta}_{\dot{\alpha}i}\bar{\psi}^{\dot{\alpha}j} - 4ie^{\mu}\psi^{\alpha}_{i}(\sigma_{\mu})_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}j}\\ &+\delta^{j}_{i}\partial\theta^{\alpha k}\psi_{\alpha k} - \delta^{j}_{i}\partial\bar{\theta}_{\dot{\alpha}k}\bar{\psi}^{\dot{\alpha}k} + i\delta^{j}_{i}e^{\mu}\psi^{\alpha}_{k}(\sigma_{\mu})_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}k}\,,\end{aligned}$$

$$J^{\mu\nu} = -\partial\theta^{\alpha i}(\sigma^{\mu\nu})_\alpha{}^\gamma\psi_{\gamma i} - \partial\bar{\theta}_{\dot{\alpha}i}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}i}$$
$$+\frac{1}{2}e^\rho i\bar{\psi}^i_{\dot{\alpha}}(\bar{\sigma}_\rho)^{\dot{\alpha}\alpha}(\sigma^{\mu\nu})_\alpha{}^\gamma\psi_{\gamma i} + \frac{1}{2}ie^\rho\psi^\alpha_i(\sigma_\rho)_{\alpha\dot{\alpha}}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}i}\,,$$

$$J^\mu_K = zi\psi^\beta_i(\sigma^\mu)_{\beta\dot{\beta}}\partial\bar{\psi}^{\dot{\beta}i} - iz\partial\psi^\alpha_i(\sigma^\mu)_{\alpha\dot{\gamma}}\bar{\psi}^{\dot{\gamma}i} - 4zi\psi^\alpha_j\partial\theta^i_\alpha\psi^\gamma_i(\sigma^\mu)_{\gamma\dot{\beta}}\bar{\psi}^{\dot{\beta}j}$$
$$+4zi\bar{\psi}^k_{\dot{\alpha}}\partial\bar{\theta}^{\dot{\alpha}}_i\psi^\gamma_k(\sigma^\mu)_{\gamma\dot{\beta}}\bar{\psi}^{\dot{\beta}i} + 2ze^\nu\psi^\alpha_j(\sigma_\nu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}i}\psi^\gamma_i(\sigma^\mu)_{\gamma\dot{\gamma}}\bar{\psi}^{\dot{\gamma}j}\,.$$

It is left to organize the currents in the form that they appear in (2.80). Firstly, note that

$$J^\mu_P\,P_\mu + J^\mu_K\,K_\mu = (J^\mu_P + J^\mu_K)\frac{1}{2}(P_\mu + K_\mu) \,+\, 2\frac{(J^\mu_P - J^\mu_K)}{2}\frac{1}{2}(P_\mu - K_\mu)\,,$$

and from this we conclude

$$J^\mu = (J^\mu_P + J^\mu_K)\,,\quad J^{4\mu} = \frac{(J^\mu_P - J^\mu_K)}{2}\,, \tag{3.4}$$

moreover, recall that in our conventions

$$J^\alpha_\beta = \frac{1}{2}J^{\mu\nu}(\sigma_{\mu\nu})_\beta{}^\alpha\,,\quad J^{\dot{\alpha}}_{\dot{\beta}} = \frac{1}{2}J^{\mu\nu}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\,,$$
$$J^{\alpha\dot{\alpha}} = J^{4\mu}i(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}\,.$$

Having computed the left-invariant currents, we can proceed to compute the canonical momenta. We are going to give a few examples on how to perform these computations and the remaining momenta can be obtained in a similar way. In ours conventions, the z and $\bar{z}$ derivatives, ∂ and $\bar{\partial}$, are expressed in terms of the σ and the τ derivatives as

$$\partial = \frac{1}{2}(\partial_\sigma - \partial_\tau)\,,\quad \bar{\partial} = \frac{1}{2}(\partial_\sigma + \partial_\tau)\,. \tag{3.5}$$

Replacing these relations in the action given in (2.86) and (2.89) and using the expressions for the currents of (3.3), we can collect, for example, all the terms in the Lagrangian density that have derivatives of the type $\partial_\tau z$, which are

$$\mathcal{L}_{\partial_\tau z} = -\frac{1}{8}\Big(\frac{\partial_\tau z}{z} + 2\partial_\tau\theta^{\alpha i}\psi_{\alpha i} + 2\partial_\tau\bar{\theta}_{\dot{\alpha}i}\bar{\psi}^{\dot{\alpha}i}\Big)^2\,,$$

and from this we conclude

$$P_z = \frac{\partial\mathcal{L}}{\partial(\partial_\tau z)} = -\frac{1}{4z}\Big(\frac{\partial_\tau z}{z} + 2\partial_\tau\theta^{\alpha i}\psi_{\alpha i} + 2\partial_\tau\bar{\theta}_{\dot{\alpha}i}\bar{\psi}^{\dot{\alpha}i}\Big)\,.$$

In addition, from the terms

$$\mathcal{L}_{\partial_\tau\lambda} = \frac{1}{2}w^\alpha_j\partial_\tau\lambda^j_\alpha + \frac{1}{2}\hat{w}^\alpha_j\partial_\tau\hat{\lambda}^j_\alpha + \frac{1}{2}\bar{w}^k_{\dot{\alpha}}\partial_\tau\bar{\lambda}^{\dot{\alpha}}_k + \frac{1}{2}\hat{\bar{w}}^k_{\dot{\alpha}}\partial_\tau\hat{\bar{\lambda}}^{\dot{\alpha}}_k\,,$$

we easily compute

$$P_{\lambda^j_\alpha} = \frac{1}{2} w^\alpha_j \,, \quad P_{\hat{\lambda}^j_\alpha} = \frac{1}{2} \hat{w}^\alpha_j \,, \quad P_{\bar{\lambda}^{\dot{\alpha}}_k} = \frac{1}{2} \bar{w}^k_{\dot{\alpha}} \,, \quad P_{\hat{\bar{\lambda}}^{\dot{\alpha}}_k} = \frac{1}{2} \hat{\bar{w}}^k_{\dot{\alpha}} \,. \tag{3.6}$$

The last example is the computation of $P_{y^{ij}}$. The relevant part of the Lagrangian density is

$$\begin{aligned}\mathcal{L}_{\partial_\tau y^{ij}} &= \frac{1}{32} \partial_\tau y_{jk} \partial_\tau y^{jk} + \frac{1}{8} \partial_\tau y_{ij} \partial_\tau J^i_l y^{lj} + \frac{1}{4} w^\alpha_j y^{jk} \partial_\tau y_{km} \lambda^m_\alpha \\ &+ \frac{1}{4} \hat{w}^\alpha_j y^{jk} \partial_\tau y_{km} \hat{\lambda}^m_\alpha + \frac{1}{4} \hat{\bar{w}}^k_{\dot{\alpha}} y_{kl} \partial_\tau y^{lm} \hat{\bar{\lambda}}^{\dot{\alpha}}_m + \frac{1}{4} \bar{w}^k_{\dot{\alpha}} y_{kl} \partial_\tau y^{lm} \bar{\lambda}^{\dot{\alpha}}_m \,,\end{aligned}$$

where we have used the compact notation $\partial_\tau J^i_l$ which should be understood as replacing the ∂ derivatives that appear on the right-hand side of the result of the currents given in (3.3) by ∂_τ. Using

$$\frac{\partial}{\partial y^{ij}} y^{km} = (\delta^k_i \delta^m_j - \delta^k_j \delta^m_i) \,, \qquad \frac{\partial}{\partial y^{ij}} y_{km} = \epsilon_{ijkm} \,,$$

a straightforward calculation gives the answer

$$\begin{aligned} P_{y^{ij}} &= \frac{1}{8} \partial_\tau y_{ij} - \frac{1}{8} y_{kj} \partial_\tau J^k_i + \frac{1}{8} y_{ki} \partial_\tau J^k_j \\ &- \frac{1}{4} w^\alpha_i y_{jk} \lambda^k_\alpha + \frac{1}{4} w^\alpha_j y_{ik} \lambda^k_\alpha - \frac{1}{4} \hat{w}^\alpha_i y_{jk} \hat{\lambda}^k_\alpha + \frac{1}{4} \hat{w}^\alpha_j y_{ik} \hat{\lambda}^k_\alpha \\ &+ \frac{1}{4} \hat{\bar{w}}^k_{\dot{\alpha}} \hat{\bar{\lambda}}^{\dot{\alpha}}_j y_{ki} - \frac{1}{4} \hat{\bar{w}}^k_{\dot{\alpha}} \hat{\bar{\lambda}}^{\dot{\alpha}}_i y_{kj} + \frac{1}{4} \bar{w}^k_{\dot{\alpha}} y_{ki} \bar{\lambda}^{\dot{\alpha}}_j - \frac{1}{4} \bar{w}^k_{\dot{\alpha}} y_{kj} \bar{\lambda}^{\dot{\alpha}}_i \,.\end{aligned}$$

Once we have computed all the canonical momenta by the procedure described above, the next step to performing the expansion of the BRST operator is to rewrite the currents that appear in this operator as a function of the worldsheet variables, their canonical momenta and their σ derivatives. We copy below, for convenience, the BRST operator Q of (2.95),

$$\begin{aligned} Q &= \int dz [\lambda^{\alpha j} (\sqrt{2} J_{\alpha j} - \sqrt{2} y_{jk} J^k_\alpha) - \bar{\lambda}_{\dot{\alpha} j} (\sqrt{2} J^{\dot{\alpha} j} + \sqrt{2} y^{jk} J^{\dot{\alpha}}_k)] \\ &- \int d\bar{z} [\hat{\lambda}^{\alpha j} (\sqrt{2} \bar{J}_{\alpha j} + \sqrt{2} y_{jk} \bar{J}^k_\alpha) + \hat{\bar{\lambda}}_{\dot{\alpha} j} (\sqrt{2} \bar{J}^{\dot{\alpha} j} - \sqrt{2} y^{jk} \bar{J}^{\dot{\alpha}}_k)] \,.\end{aligned}$$

Replacing all ∂ and $\bar{\partial}$ derivatives using (3.5) and noting that in our conventions

$$\int dz \equiv \frac{1}{2\pi i} \int dz \to \int d\sigma \,, \qquad \int d\bar{z} \equiv -\frac{1}{2\pi i} \int d\bar{z} \to \int d\sigma \,, \tag{3.7}$$

the BRST operator becomes

$$Q = \frac{\sqrt{2}}{2} \int d\sigma [(\lambda^{\alpha j} - \hat{\lambda}^{\alpha j}) y_{jk} \partial_\tau J^k_\alpha - (\lambda^{\alpha j} + \hat{\lambda}^{\alpha j}) \partial_\tau J_{\alpha j} \tag{3.8}$$

$$+(\bar\lambda_{\dot\alpha j}-\hat{\bar\lambda}_{\dot\alpha j})\partial_\tau J^{\dot\alpha j}+(\bar\lambda_{\dot\alpha j}+\hat{\bar\lambda}_{\dot\alpha j})y^{jk}\partial_\tau J_k^{\dot\alpha}+(\lambda^{\alpha j}-\hat\lambda^{\alpha j})\partial_\sigma J_{\alpha j}$$

$$-(\lambda^{\alpha j}+\hat\lambda^{\alpha j})y_{jk}\partial_\sigma J_\alpha^k-(\bar\lambda_{\dot\alpha j}+\hat{\bar\lambda}_{\dot\alpha j})\partial_\sigma J^{\dot\alpha j}-(\bar\lambda_{\dot\alpha j}-\hat{\bar\lambda}_{\dot\alpha j})y^{jk}\partial_\sigma J_k^{\dot\alpha}]\,.$$

where we have used the same notation as before, for example, $\partial_\tau J_{\alpha j}$ and $\partial_\sigma J_{\alpha j}$ mean that we replace the ∂ derivatives on the right-hand side of the expressions of the currents (3.3) by ∂_τ and ∂_σ, respectively. It is left to express the currents as a function of the canonical momenta and substitute in the expression above. From the computation of the canonical momenta, we have expressions such as the one given below

$$\begin{aligned}\frac{1}{\sqrt{z}i}P_{\psi_i^\gamma}&=\frac{1}{4}\eta_{\mu\nu}\sqrt{z}(\sigma^\mu)_{\gamma\dot\gamma}\bar\psi^{\dot\gamma i}\partial_\tau J^\nu+\partial_\tau J_\gamma^i-\frac{1}{2}y^{ji}\partial_\sigma J_{\gamma j}+\frac{\sqrt{z}}{2i}w_j^\alpha\epsilon_{\alpha\gamma}\bar\psi_{\dot\alpha}^i y^{jk}\lambda_k^{\dot\alpha}\\&+\frac{\sqrt{z}}{2i}\hat w_j^\alpha\epsilon_{\alpha\gamma}\bar\psi_{\dot\alpha}^i y^{jk}\hat\lambda_k^{\dot\alpha}-\frac{\sqrt{z}}{2i}\bar w_{\dot\alpha}^k y_{kl}\bar\psi^{\dot\alpha i}\lambda_\gamma^l-\frac{\sqrt{z}}{2i}\hat{\bar w}_{\dot\alpha}^k y_{kl}\bar\psi^{\dot\alpha i}\hat\lambda_\gamma^l\,,\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}P_{x^\mu}&=-\frac{1}{4z}\partial_\tau J_\mu+i\frac{1}{4z}(\sigma_\mu)_{\alpha\dot\alpha}w_j^\alpha y^{jk}\lambda_k^{\dot\alpha}+i\frac{1}{4z}(\sigma_\mu)_{\alpha\dot\alpha}\hat w_j^\alpha y^{jk}\hat{\bar\lambda}_k^{\dot\alpha}\\&-i\frac{1}{4z}\bar w_{\dot\alpha}^k y_{kl}\lambda_\alpha^l(\bar\sigma_\mu)^{\dot\alpha\alpha}-i\frac{1}{4z}\hat{\bar w}_{\dot\alpha}^k y_{kl}\hat\lambda_\alpha^l(\bar\sigma_\mu)^{\dot\alpha\alpha}+\dots\,,\end{aligned}\tag{3.10}$$

where $\dots$ means terms that depend on higher powers of z. Using (3.6), (3.9) and (3.10), it is easy to see that

$$\begin{aligned}\partial_\tau J_\gamma^i&=\frac{1}{\sqrt{z}i}P_{\psi_i^\gamma}+z^{\frac{3}{2}}\bar\psi^{\dot\gamma i}(\sigma^\mu)_{\gamma\dot\gamma}P_{x^\mu}+\frac{1}{2}y^{ji}\partial_\sigma J_{\gamma j}+2i\sqrt{z}\epsilon_{\alpha\gamma}\bar\psi_{\dot\alpha}^i y^{jk}\lambda_k^{\dot\alpha}P_{\lambda_\alpha^j}\\&+2i\sqrt{z}\epsilon_{\alpha\gamma}\bar\psi_{\dot\alpha}^i y^{jk}\hat\lambda_k^{\dot\alpha}P_{\hat\lambda_\alpha^j}-2i\sqrt{z}\bar\psi^{\dot\alpha i}y_{kl}\lambda_\gamma^l P_{\bar\lambda_k^{\dot\alpha}}-2i\sqrt{z}\bar\psi^{\dot\alpha i}y_{kl}\hat\lambda_\gamma^l P_{\hat{\bar\lambda}_k^{\dot\alpha}}\,.\end{aligned}\tag{3.11}$$

Similarly, following the same steps, one can compute

$$\begin{aligned}\partial_\tau J_{\alpha i}&=\frac{\sqrt{z}}{i}(P_{\theta^{\alpha i}}+i(\sigma^\mu)_{\alpha\dot\alpha}\theta_i^{\dot\alpha}P_{x^\mu})-2\frac{\sqrt{z}}{i}\psi_{\alpha i}zP_z-4\frac{\sqrt{z}}{i}\psi_{\alpha m}P_{y^{ij}}y^{jm}\\&+\frac{\sqrt{z}}{i}\psi_{\alpha i}P_{y^{mj}}y^{jm}+2\frac{\sqrt{z}}{i}\bar\psi_{\dot\alpha}^m\psi_{\alpha m}\epsilon^{\dot\alpha\dot\gamma}P_{\bar\psi^{\dot\gamma i}}+\frac{1}{2}y_{ji}\partial_\sigma J_\alpha^j-4\frac{\sqrt{z}}{i}\psi_{\alpha j}\psi_i^\beta P_{\psi_j^\beta}\\&+4\frac{\sqrt{z}}{i}\psi_{\alpha j}\lambda_\beta^j P_{\lambda_\beta^i}-\frac{\sqrt{z}}{i}\lambda_\beta^m\psi_{\alpha i}P_{\lambda_\beta^m}+4\frac{\sqrt{z}}{i}\psi_{\alpha j}\hat\lambda_\beta^j P_{\hat\lambda_\beta^i}-\frac{\sqrt{z}}{i}\hat\lambda_\beta^m\psi_{\alpha i}P_{\hat\lambda_\beta^m}\\&-4\frac{\sqrt{z}}{i}\psi_{\alpha j}\bar\lambda_i^{\dot\alpha}P_{\bar\lambda_j^{\dot\alpha}}+\frac{\sqrt{z}}{i}\bar\lambda_j^{\dot\alpha}\psi_{\alpha i}P_{\bar\lambda_j^{\dot\alpha}}-4\frac{\sqrt{z}}{i}\psi_{\dot\alpha j}\hat{\bar\lambda}_i^{\dot\alpha}P_{\hat{\bar\lambda}_j^{\dot\alpha}}+\frac{\sqrt{z}}{i}\hat{\bar\lambda}_j^{\dot\alpha}\psi_{\alpha i}P_{\hat{\bar\lambda}_j^{\dot\alpha}}\\&+2\frac{\sqrt{z}}{i}\lambda_\beta^j(\epsilon_{\delta\alpha}\psi_j^\beta+\delta_\alpha^\beta\psi_{\delta j})P_{\lambda_\delta^i}+2\frac{\sqrt{z}}{i}\hat\lambda_\beta^j(\epsilon_{\delta\alpha}\psi_j^\beta+\delta_\alpha^\beta\psi_{\delta j})P_{\hat\lambda_\delta^i}\,,\end{aligned}\tag{3.12}$$

and similar expressions such as (3.11) and (3.12) can be obtained for the remaining $\partial_\tau J_{\dot\alpha j}$ and $\partial_\tau J_{\dot\alpha}^j$ that appear in the BRST operator of (3.8).

We have everything needed to performing the expansion of the BRST operator and expressing the result in terms of the worldsheet variables, their canonical momenta and their σ derivatives. Replacing the results obtained for $\partial_\tau J^i_\gamma$, $\partial_\tau J_{\alpha i}$, $\partial_\tau J_{\dot\alpha j}$ and $\partial_\tau J^j_{\dot\alpha}$ in (3.8), we first note that the BRST operator splits as

$$Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + ... \tag{3.13}$$

where Q_n is proportional to z^n, and using the definitions of (2.97), we have

$$Q_{-\frac{1}{2}} = (\frac{\sqrt{2}}{2}) z^{-\frac{1}{2}} (\lambda^{+\gamma m} y_{mi} P_{\psi^\gamma_i} - \bar\lambda^{+\dot\alpha}_j y^{ji} P_{\bar\psi^{i\dot\alpha}}) \tag{3.14}$$

$$-i\frac{\sqrt{2}}{4} z^{-1/2} (\lambda^{-\alpha j} y_{jk} (i\partial_\sigma\theta^k_\alpha - \partial_\sigma e^\mu (\sigma_\mu)_{\alpha\dot\alpha} \bar\psi^{\dot\alpha k}) - \bar\lambda^-_{\dot\alpha j} y^{jk} (i\partial_\sigma \bar\theta^{\dot\alpha}_k - \partial_\sigma e^\mu (\sigma_\mu)^{\dot\alpha\alpha} \psi_{\alpha k}))\,,$$

$$Q_{\frac{1}{2}} = (\frac{\sqrt{2}}{2}) z^{\frac{1}{2}} \lambda^{-\alpha i} [-P_{\theta^{\alpha i}} - i(\sigma^\mu)_{\alpha\dot\alpha} \bar\theta^{\dot\alpha}_i P_{x^\mu} + 2\psi_{\alpha i} z P_z + 4\psi_{\alpha k} P_{y^{ij}} y^{jk} - \psi_{\alpha i} P_{y^{jk}} y^{kj}$$

$$-4\psi_{\alpha k} \lambda^k_\beta P_{\lambda^i_\beta} + \psi_{\alpha i} \lambda^k_\beta P_{\lambda^k_\beta} - 4\psi_{\alpha k} \hat\lambda^k_\beta P_{\hat\lambda^i_\beta} + \psi_{\alpha i} \hat\lambda^k_\beta P_{\hat\lambda^k_\beta} - 2\psi^\beta_i \lambda^j_\beta P_{\lambda^{\alpha j}} - 2\psi_{\beta i} \lambda^j_\alpha P_{\lambda^j_\beta}$$

$$-2\psi^\beta_i \hat\lambda^j_\beta P_{\hat\lambda^{\alpha j}} - 2\psi_{\beta i} \hat\lambda^j_\alpha P_{\hat\lambda^j_\beta} + 4\psi_{\alpha k} \bar\lambda^{\dot\alpha}_i P_{\bar\lambda^{\dot\alpha}_k} - \psi_{\alpha i} \bar\lambda^{\dot\alpha}_k P_{\bar\lambda^{\dot\alpha}_k} + 4\psi_{\alpha k} \hat{\bar\lambda}^{\dot\alpha}_i P_{\hat{\bar\lambda}^{\dot\alpha}_k} - \psi_{\alpha i} \hat{\bar\lambda}^{\dot\alpha}_k P_{\hat{\bar\lambda}^{\dot\alpha}_k}$$

$$+4\psi_{\alpha j} \psi^\beta_i P_{\psi^\beta_j} - 2\psi_{\alpha j} \bar\psi^j_{\dot\alpha} P_{\bar\psi^i_{\dot\alpha}}] + (\frac{\sqrt{2}}{2}) z^{\frac{1}{2}} \bar\lambda^{-\dot\alpha}_i [-P_{\bar\theta^{\dot\alpha}_i} - i\theta^{\beta i} (\sigma^\mu)_{\beta\dot\alpha} P_{x^\mu} - 2\bar\psi^i_{\dot\alpha} z P_z$$

$$-4\bar\psi^l_{\dot\alpha} P_{y_{ij}} y_{jl} + \bar\psi^i_{\dot\alpha} P_{y_{kj}} y_{jk} - 4\bar\psi^j_{\dot\alpha} \lambda^i_\alpha P_{\lambda^j_\alpha} + \bar\psi^i_{\dot\alpha} \lambda^k_\alpha P_{\lambda^k_\alpha} - 4\bar\psi^j_{\dot\alpha} \hat\lambda^i_\alpha P_{\hat\lambda^j_\alpha} + \bar\psi^i_{\dot\alpha} \hat\lambda^k_\alpha P_{\hat\lambda^k_\alpha}$$

$$+4\bar\psi^l_{\dot\alpha} \bar\lambda^{\dot\beta}_l P_{\bar\lambda^{\dot\beta}_i} - \bar\psi^i_{\dot\alpha} \bar\lambda^{\dot\beta}_k P_{\bar\lambda^{\dot\beta}_k} + 4\bar\psi^l_{\dot\alpha} \hat{\bar\lambda}^{\dot\beta}_l P_{\hat{\bar\lambda}^{\dot\beta}_i} - \bar\psi^i_{\dot\alpha} \hat{\bar\lambda}^{\dot\beta}_k P_{\hat{\bar\lambda}^{\dot\beta}_k} - 2\bar\psi^i_{\dot\beta} \bar\lambda^{\dot\beta}_k P_{\bar\lambda^{\dot\alpha}_k} + 2\bar\psi^{\dot\beta i} \bar\lambda_{\dot\alpha k} P_{\bar\lambda^{\dot\beta}_k}$$

$$-2\bar\psi^i_{\dot\beta} \hat{\bar\lambda}^{\dot\beta}_k P_{\hat{\bar\lambda}^{\dot\alpha}_k} + 2\bar\psi^{\dot\beta i} \hat{\bar\lambda}_{\dot\alpha k} P_{\hat{\bar\lambda}^{\dot\beta}_k} - 4\bar\psi^m_{\dot\alpha} \bar\psi^{i\dot\beta} P_{\bar\psi^{m\dot\beta}} - 2\psi_{\alpha k} \bar\psi^k_{\dot\alpha} P_{\psi_{\alpha i}}]$$

$$+(\frac{\sqrt{2}}{2}) 2z^{\frac{1}{2}} \lambda^{+\gamma m} y_{mi} (\bar\psi^i_{\dot\alpha} \bar\lambda^{\dot\alpha}_k y^{kj} P_{\lambda^{\gamma j}} + \bar\psi^{i\dot\alpha} y_{kl} \lambda^l_\gamma P_{\bar\lambda^{\dot\alpha}_k} + \bar\psi^i_{\dot\alpha} \hat{\bar\lambda}^{\dot\alpha}_k y^{kj} P_{\hat\lambda^{j\gamma}} + \bar\psi^{i\dot\alpha} y_{kl} \hat\lambda^l_\gamma P_{\hat{\bar\lambda}^{\dot\alpha}_k})$$

$$-(\frac{\sqrt{2}}{2}) 2z^{\frac{1}{2}} \bar\lambda^{+\dot\alpha}_m y^{mi} (\psi_{\alpha i} \bar\lambda_{k\dot\alpha} y^{jk} P_{\lambda^j_\alpha} + \psi^\beta_i \lambda^l_\beta y_{kl} P_{\bar\lambda^{\dot\alpha}_k} + \psi_{\alpha i} \hat{\bar\lambda}_{\dot\alpha k} y^{jk} P_{\hat\lambda^j_\alpha} + \psi^\alpha_i \hat\lambda^l_\alpha y_{kl} P_{\hat{\bar\lambda}^{\dot\alpha}_k})$$

$$+i(\frac{\sqrt{2}}{4}) z^{1/2} (\lambda^{+\alpha j}) [i\partial_\sigma \psi_{\alpha j} + 4i\psi^\beta_j \partial_\sigma \theta^i_\beta \psi_{\alpha i} - 2i\psi_{\alpha k} \bar\psi^k_{\dot\alpha} \partial_\sigma \bar\theta^{\dot\alpha}_j - 2\partial_\sigma e^\mu \psi^\beta_j (\sigma_\mu)_{\beta\dot\beta} \bar\psi^{\dot\beta i} \psi_{\alpha i}]$$

$$+i(\frac{\sqrt{2}}{4}) z^{1/2} (\bar\lambda^+_{\dot\alpha j}) [i\partial_\sigma \bar\psi^{\dot\alpha j} - 2i\bar\psi^{\dot\alpha i} \partial_\sigma \theta^{\alpha j} \psi_{\alpha i} + 4i\bar\psi^j_{\dot\beta} \partial_\sigma \bar\theta^{\dot\beta}_i \bar\psi^{i\dot\alpha} + 2\partial_\sigma e^\mu \psi^\alpha_i (\sigma_\mu)_{\alpha\dot\beta} \bar\psi^{j\dot\beta} \bar\psi^{i\dot\alpha}]\,,$$

$$Q_{\frac{3}{2}} = i(\frac{\sqrt{2}}{2}) z^{\frac{3}{2}} \lambda^{+\gamma m} y_{mi} (\sigma^\mu)_{\gamma\dot\alpha} \bar\psi^{i\dot\alpha} P_{x^\mu} - i(\frac{\sqrt{2}}{2}) z^{\frac{3}{2}} \bar\lambda^{+\dot\alpha}_m y^{mi} \psi^\beta_i (\sigma^\mu)_{\beta\dot\alpha} P_{x^\mu} + \dots\,,$$

and ... are terms which are at least quadratic in ψ. We have suppressed the $\int d\sigma$ in all the expressions above and we have defined $\partial_\sigma e^\mu$ to be equal to the right-hand side of (3.2) but with the ∂ replaced by ∂_σ. Since the conjugate momentum of a variable does not commute with the variable, the BRST operator is only well defined after normal-ordering, however, we will work to lowest order in α' so possible normal-ordering contributions to the operator can be safely ignored.

In this thesis, we are going to compute the zero-mode cohomology of the BRST operator close to the boundary of AdS, which means that all the σ derivatives of the worldsheet variables that appear in the expansion above will be zero. This corresponds to taking the supergravity limit. Moreover, in order to compute the cohomology of the BRST operator using the expansion above, we have to understand how the physical states behave close to the boundary of AdS and this will be explained in the next section.

3.2 Method for computing the BRST cohomology

In order to understand how the physical states behave close to the boundary of AdS, let us study a scalar field ϕ in the background $AdS^5 \times S^5$ in the supergravity limit, this is reviewed, for example, in [4, 5, 60]. The S^5 is a compact space and we can apply the Kaluza-Klein reduction procedure, or in other words, we can expand the scalar field as

$$\phi(x, z, y) = \sum_l \phi_l(x, z) Y_l(y) \tag{3.15}$$

where (x, z) are the AdS^5 coordinates, y are the S^5 coordinates and Y_l is the complete set of spherical harmonics of S^5. The spherical harmonics are eigenfunctions of the Laplacian operator with eigenvalue $m^2 = l(l+4)/R^2$ with R the radius of both AdS^5 and S^5. After performing this expansion, the action for a scalar in AdS is

$$S = \int d^4x\, dz\, \sqrt{-g}\, (g^{ab}\partial_a\phi\, \partial_b\phi + m^2R^2\phi^2)\,,$$

where we have only written the quadratic part of the action because we are interested in the linearized equations of motion. The AdS^5 metric was given in (2.63), and it is

$$ds^2 = \frac{1}{z^2}(d\vec{x}^2 + dz^2)\,,$$

and after substituting this metric in the action, it becomes

$$S = \int d^4x dz\, \frac{1}{z^5}\, (z^2(\partial\phi)^2 + m^2R^2\phi^2)\,,$$

from where one easily deduces the equations of motion

$$z^5\partial_z(\frac{1}{z^3}\partial_z\phi) - z^2(\partial_x\phi)^2 - m^2R^2\phi = 0\,. \tag{3.16}$$

This equation can be solved analytically and the result expressed in terms of Bessel functions. Being a second order differential equation it needs two boundary conditions to determine the solution completely. One of the boundary conditions follows from imposing that the solution is regular in the bulk of AdS which implies that it must vanish when $z \to \infty$. Moreover, close to the boundary of AdS at $z \sim 0$ the solution can be approximated as $\phi \sim z^\alpha$, and replacing it in (3.16), one has

$$\alpha(\alpha - 4) - m^2R^2 = 0\,, \tag{3.17}$$

with the solutions

$$\alpha_\pm = 2 \pm \sqrt{4 + m^2R^2}\,. \tag{3.18}$$

Note that close to the boundary the dominant solution is $\phi \sim z^{\alpha_-}$, and this justifies our second boundary condition, which is

$$\phi(x, z)|_{z=\epsilon} = \epsilon^{\alpha_-}\phi_0(x)\,, \tag{3.19}$$

for a given function $\phi_0(x)$.

The important conclusion of this analysis is that near the AdS boundary the vertex operators V describing physical states can be expanded as $V = \sum_{d\geq d_0} V_d$ where V_d is proportional to z^d and V_{d_0} is the leading behavior near $z = 0$. In this thesis, we will define degree to be the power of z of an expression and with this definition V has a minimum degree d_0. Since the BRST operator splits as a series of terms Q_n with fixed degree, the condition that V is closed under Q,

$$Q \cdot V = 0\,, \tag{3.20}$$

reduces, after collecting the terms with equal powers of z, to

$$\begin{aligned} &Q_{-\frac{1}{2}} \cdot V_{d_0} = 0\,, \\ &Q_{\frac{1}{2}} \cdot V_{d_0} + Q_{-\frac{1}{2}} \cdot V_{d_0+1} = 0\,, \\ &Q_{\frac{3}{2}} \cdot V_{d_0} + Q_{\frac{1}{2}} \cdot V_{d_0+1} + Q_{-\frac{1}{2}} \cdot V_{d_0+2} = 0\,, \\ &\ldots\,. \end{aligned} \tag{3.21}$$

The above conditions mean that the procedure for computing the cohomology of the BRST operator close to the boundary of AdS, or inside the region of validity of the z expansion, is to first compute the cohomology of $Q_{-\frac{1}{2}}$, then compute the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$, then compute the cohomology of $Q_{\frac{3}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$ and $Q_{\frac{1}{2}}$, and so on.

The procedure just described for computing the cohomology of the BRST operator is well defined. The complete BRST operator Q given in (2.95) is nilpotent by construction and this condition implies several relations among the operators Q_n after performing the expansion in z. Starting with

$$\{Q, Q\} = 0\,,$$

and expanding the operator, we have

$$\{Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \ldots, Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \ldots\} = 0\,. \tag{3.22}$$

All the operators Q_n have a fixed degree, or in other words, are proportional to z^n. Therefore, collecting the terms with equal power of z from the expression above, we conclude that

$$\begin{gathered}\{Q_{-\frac{1}{2}}, Q_{-\frac{1}{2}}\} = 0\,, \qquad \{Q_{\frac{1}{2}}, Q_{-\frac{1}{2}}\} = 0\,, \\ \{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\} + 2\{Q_{-\frac{1}{2}}, Q_{\frac{3}{2}}\} = 0\,, \ldots\,.\end{gathered} \tag{3.23}$$

According to the method for computing the BRST cohomology, the operator $Q_{\frac{1}{2}}$ only acts on the states in the cohomology of $Q_{-\frac{1}{2}}$. Consider a state V that belongs to the cohomology of $Q_{-\frac{1}{2}}$ and act on it with the third identity of (3.23), to get

$$\{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\} \cdot V + 2\,Q_{-\frac{1}{2}} \cdot Q_{\frac{3}{2}} \cdot V + 2\,Q_{\frac{3}{2}} \cdot Q_{-\frac{1}{2}} \cdot V = 0\,, \tag{3.24}$$

which implies noting that by assumption $Q_{-\frac{1}{2}} \cdot V = 0$ and the second term above is $Q_{-\frac{1}{2}}$ exact, that

$$\{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\} \cdot V = 0\,, \qquad \text{mod } Q_{-\frac{1}{2}} \text{ exact terms}\,, \tag{3.25}$$

or in other words, the operator $Q_{\frac{1}{2}}$ is nilpotent when acting on states in the cohomology of $Q_{-\frac{1}{2}}$ and it makes sense to compute the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$. A similar argument of nilpotency applies to $Q_{\frac{3}{2}}$, $Q_{\frac{5}{2}}$, ...

In the next section the cohomology of the operator $Q_{-\frac{1}{2}}$ will be presented. As already explained, this is the first step in computing the cohomology of the complete BRST operator.

3.3 The zero mode cohomology of $Q_{-\frac{1}{2}}$

As explained before, the first necessary result in order to compute the cohomology of the complete BRST operator close to the boundary of AdS is the cohomology of the operator $Q_{-\frac{1}{2}}$ of (3.14). The zero mode cohomology of this operator was computed by Mikhailov and Xu in [23], see also [24], by defining a spectral sequence of a bicomplex that converges to the cohomology of this operator, they also used some results from the theory of group representation. One important ingredient in their computation was the result of the cohomology of the operator $Q' = \lambda^{\bar{\alpha}} P_{\theta^{\bar{\alpha}}}$, with $\lambda^{\bar{\alpha}}$ a pure spinor, previously obtained by Berkovits in [48]. Before stating their result of the cohomology, we will prove some identities necessary for its understanding. We reproduce below for convenience the zero mode operator $Q_{-\frac{1}{2}}$ of (3.14),

$$Q_{-\frac{1}{2}} = z^{-\frac{1}{2}}(\lambda^{+\gamma m} y_{mi} P_{\psi_i^\gamma} - \bar{\lambda}_j^{+\dot{\alpha}} y^{ji} P_{\bar{\psi}^{i\dot{\alpha}}})\,,$$

where we have redefined λ in order to adsorb the overall factor of $\frac{\sqrt{2}}{2}$. Note that the spinors $\lambda^{+\bar{\alpha}}$ and $\lambda^{-\bar{\alpha}}$ defined in (2.97) satisfy

$$\lambda^- \gamma^M \lambda^+ = 0\,, \qquad \lambda^- \gamma^M \lambda^- + \lambda^+ \gamma^M \lambda^+ = 0\,, \tag{3.26}$$

which follows from the pure spinor conditions for $\lambda^{\bar{\alpha}}$ and $\hat{\lambda}^{\bar{\alpha}}$ of (2.41). However, we will show it explicitly below. Using the ansatz for the chiral gamma matrices of (2.66), we derive

$$(\lambda^- \gamma^\mu \lambda^+) = (\lambda^{-\alpha j} i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\lambda}_j^{+\dot{\alpha}}) + (\bar{\lambda}^-_{\dot{\alpha}j} i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \lambda^{+j}_\alpha)\,, \tag{3.27}$$

and manipulating the second term on the right-hand side of the expression above, recalling that λ is a bosonic spinor and using the proprieties of the $SO(1,3)$ Pauli matrices given in the Appendix A, we have

$$(\bar{\lambda}^-_{\dot{\alpha}j}\, i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\, \lambda^{+j}_\alpha) = \bar{\lambda}^-_{\dot{\alpha}j}\, i\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\, \lambda^{+j}_\alpha = \lambda^{+\beta j} i\sigma^\mu_{\beta\dot{\beta}} \bar{\lambda}_j^{-\dot{\beta}}\,, \tag{3.28}$$

and substituting (3.28) in (3.27), using the definitions of (2.97) and the pure spinor conditions of (2.65), we conclude

$$i\sigma^\mu_{\alpha\dot{\alpha}}(\lambda^{-\alpha j}\bar{\lambda}_j^{+\dot{\alpha}} + \lambda^{+\alpha j}\bar{\lambda}_j^{\dot{\alpha}-}) = i\sigma^\mu_{\alpha\dot{\alpha}}(\lambda^{\alpha j}\hat{\bar{\lambda}}_j^{\dot{\alpha}} + \hat{\lambda}^{\alpha j}\bar{\lambda}_j^{\dot{\alpha}} - \lambda^{\alpha j}\hat{\bar{\lambda}}_j^{\dot{\alpha}} - \hat{\lambda}^{\alpha j}\bar{\lambda}_j^{\dot{\alpha}}) = 0\,,$$

which proves the first identity of (3.26) when $M = \mu$. Let us now consider the case when $M = I + 3$, we have

$$\lambda^- \gamma^{I+3} \lambda^+ = \lambda^{-\alpha i}\sigma^I_{ij}\lambda^{+j}_\alpha - \bar{\lambda}^-_{\dot{\alpha}i}\sigma^{Iij}\bar{\lambda}_j^{+\dot{\alpha}}\,, \tag{3.29}$$

and after making the substitution of the definitions, we conclude

$$\lambda^{-}\gamma^{I+3}\lambda^{+}=-(\lambda^{\alpha i}+\hat{\lambda}^{\alpha i})\sigma^{I}_{ij}(\lambda^{j}_{\alpha}-\hat{\lambda}^{j}_{\alpha})+(\bar{\lambda}_{\dot{\alpha}i}-\hat{\bar{\lambda}}_{\dot{\alpha}i})\sigma^{Iij}(\bar{\lambda}^{\dot{\alpha}}_{j}+\hat{\bar{\lambda}}^{\dot{\alpha}}_{j})=0\,,$$

finishing the proof of the first identity of (3.26) for all M. The proof of the second one is similar and it will be omitted. After proving these important identities, and their importance will become clear in what follows, let us return to the computation of the zero mode cohomology of $Q_{-\frac{1}{2}}$. This operator annihilates the terms $(\lambda^{-}\gamma^{M}\hat{\psi})$ for all M where $\hat{\psi}^{\bar{\alpha}}\equiv y_{J}(\gamma^{(J+3)\bar{\alpha}\bar{\beta}}\psi_{\bar{\beta}})$. The proof uses again the ansatz for the chiral gamma matrices of (2.66) to obtain when $M=\mu$,

$$(\lambda^{-}\gamma^{\mu}\hat{\psi})=-\lambda^{-\alpha i}i\sigma^{\mu}_{\alpha\dot{\alpha}}y_{ij}\bar{\psi}^{\dot{\alpha}j}+\bar{\lambda}^{-}_{\dot{\alpha}i}i\bar{\sigma}^{\mu\dot{\alpha}\alpha}y^{ij}\psi_{\alpha j}\,,$$

consequently,

$$Q_{-\frac{1}{2}}\cdot(\lambda^{-}\gamma^{\mu}\hat{\psi})\propto\lambda^{-\alpha i}i\sigma^{\mu}_{\alpha\dot{\alpha}}y_{ij}y^{mj}\bar{\lambda}^{+\dot{\alpha}}_{m}+\bar{\lambda}^{-}_{\dot{\alpha}i}i\bar{\sigma}^{\mu\dot{\alpha}\alpha}y^{ij}\epsilon_{\alpha\gamma}\lambda^{+\gamma m}y_{mj}=0\,,$$

which follows from straightforward manipulations after making the substitution of (2.97) and using $y_{ij}y^{jk}=\delta^{k}_{i}$. The case when $M=I+3$ is

$$(\lambda^{-}\gamma^{I+3}\hat{\psi})\ =\ \lambda^{-\alpha i}\sigma^{I}_{ij}y^{jk}\psi_{\alpha k}+\bar{\lambda}^{-}_{\dot{\alpha}i}\sigma^{Iij}y_{jk}\bar{\psi}^{\dot{\alpha}k}\,,$$

and from this, we deduce

$$\begin{aligned}Q_{-\frac{1}{2}}\cdot(\lambda^{-}\gamma^{I+3}\hat{\psi})\ &\propto\ \lambda^{-\alpha i}\sigma^{I}_{ij}y^{jk}\epsilon_{\alpha\gamma}\lambda^{+\gamma m}y_{mk}-\bar{\lambda}^{-}_{\dot{\alpha}i}\sigma^{Iij}y_{jk}\bar{\lambda}^{+\dot{\alpha}}_{m}y^{mk}\\ &\propto\ (\lambda^{\alpha i}+\hat{\lambda}^{\alpha i})\sigma^{I}_{ij}(\lambda^{j}_{\alpha}-\hat{\lambda}^{j}_{\alpha})-(\bar{\lambda}_{\dot{\alpha}i}-\hat{\bar{\lambda}}_{\dot{\alpha}i})\sigma^{Iij}(\bar{\lambda}^{\dot{\alpha}}_{j}+\hat{\bar{\lambda}}^{\dot{\alpha}}_{j})=0\,,\end{aligned}$$

finishing the proof that $(\lambda^{-}\gamma^{M}\hat{\psi})$ is annihilated by $Q_{-\frac{1}{2}}$ for all M. A shorter way to prove this result is by noting that $Q_{-\frac{1}{2}}\cdot(\lambda^{-}\gamma^{M}\hat{\psi})\propto(\lambda^{-}\gamma^{M}\lambda^{+})=0$ and this vanishes because of the first identity of (3.26). After proving these identities, we list below the states in the cohomology of $Q_{-\frac{1}{2}}$ at +2 ghost number found by Mikhailov and Xu with the notation adapted:

1. any function f of λ^{-},
2. $(\lambda^{-}\gamma^{M}\hat{\psi})g(\lambda^{-})$ for any function g,
3. $(\lambda^{-}\gamma^{M}\hat{\psi})(\lambda^{+}\gamma_{M}\hat{\psi})$,
4. $(\hat{\psi}\gamma_{MNP}\hat{\psi})(\lambda\gamma^{TSMNP}\hat{\lambda})-18(\hat{\psi}\gamma^{TSM}\hat{\psi})(\lambda\gamma_{M}\hat{\lambda})$,

where $(\gamma_{MNP})_{\bar\alpha\bar\beta}$ and $(\gamma^{TSMNP})_{\bar\alpha\bar\beta}$ are defined as

$$
\begin{aligned}
(\gamma_{MNR})_{\bar\alpha\bar\beta} &= \frac{1}{3!}\sum_{P'}(-1)^{\operatorname{sgn} P'}(\gamma_{\sigma_{P'}(M)}\gamma_{\sigma_{P'}(N)}\gamma_{\sigma_{P'}(R)})_{\bar\alpha\bar\beta}\,, \qquad (3.30)\\
(\gamma^{TSMNP})_{\bar\alpha\bar\beta} &= \frac{1}{5!}\sum_{P'}(-1)^{\operatorname{sgn} P'}(\gamma^{\sigma_{P'}(T)}\gamma^{\sigma_{P'}(S)}\gamma^{\sigma_{P'}(M)}\gamma^{\sigma_{P'}(N)}\gamma^{\sigma_{P'}(P)})_{\bar\alpha\bar\beta}\,,
\end{aligned}
$$

where P' means sum over all the possible permutations of the indices and sgn P' is the sign of the permutation.

The item 4 of the list above can be rewritten in a more convenient form after a few manipulations as will be described below. From the definitions of λ^+ and λ^- of (2.97), it is not difficult to see that

$$
(\hat\psi\gamma_{MNP}\hat\psi)(\lambda\gamma^{TSMNP}\hat\lambda) = \frac{1}{4}(\hat\psi\gamma_{MNP}\hat\psi)(\lambda^+\gamma^{TSMNP}\lambda^+ - \lambda^-\gamma^{TSMNP}\lambda^-)\,,
$$

and

$$
(\lambda\gamma_M\hat\lambda) = \frac{1}{2}(\lambda^+\gamma_M\lambda^+)\,,
$$

further, using the definition of $(\gamma_{TSMNP})_{\bar\alpha\bar\beta}$ of (3.30) one can show that for λ^- and λ^+,

$$
\begin{aligned}
(\hat\psi\gamma_{MNP}\hat\psi)(\lambda^\pm\gamma^{TSMNP}\lambda^\pm) &= -(\hat\psi\gamma_{MNP}\hat\psi)(\lambda^\pm\gamma^{TMNPS}\lambda^\pm) \qquad (3.31)\\
= -(\hat\psi\gamma_{MNP}\hat\psi)(\lambda^\pm\gamma^T\gamma^{MNP}\gamma^S\lambda^\pm) &+ 6(\hat\psi\gamma^{STM}\hat\psi)(\lambda^\pm\gamma_M\lambda^\pm)\,,
\end{aligned}
$$

where the notation $\lambda^\pm$ means that the identity above is valid for both λ^- and λ^+. Additional manipulations follow from the use of the identity

$$
(\gamma^{MNP})^{\bar\alpha\bar\beta}(\hat\psi\gamma_{MNP}\hat\psi) = 96\,\hat\psi^{\bar\alpha}\hat\psi^{\bar\beta}\,, \qquad (3.32)
$$

where the value of the constant of proportionality can be checked by multiplying both sides of the expression above by $(\gamma_{RST})_{\bar\alpha\bar\beta}$ and using the properties of the chiral gamma matrices. This identity follows from a more general one. Given any two chiral spinors $A^{\bar\alpha}$ and $B^{\bar\beta}$, we have

$$
A^{\bar\alpha}B^{\bar\beta} = A_1(A\gamma^M B)\gamma_M^{\bar\alpha\bar\beta} + A_2(A\gamma^{MNS}B)\gamma_{MNS}^{\bar\alpha\bar\beta} + A_3(A\gamma^{MNSPT}B)\gamma_{MNSPT}^{\bar\alpha\bar\beta}\,,
$$

with A_1, A_2 and A_3 constants. Replacing A and B by $\hat\psi$ we derive (3.32) after fixing the correct value of A_2, because both $(\hat\psi\gamma^M\hat\psi) = 0$ and $(\hat\psi\gamma^{MNSPT}\hat\psi) = 0$ which follows from the fact that $\hat\psi$ is a fermionic spinor and both $(\gamma^M)_{\bar\alpha\bar\beta}$ and $(\gamma^{MNSPT})_{\bar\alpha\bar\beta}$ are symmetric matrices. Replacing (3.32) on the second line of (3.31), we conclude that

$$
\begin{aligned}
&(\hat\psi\gamma_{MNP}\hat\psi)(\lambda^\pm\gamma^{TSMNP}\lambda^\pm) =\\
&6(\hat\psi\gamma^{STM}\hat\psi)(\lambda^\pm\gamma_M\lambda^\pm) - 96(\lambda^\pm\gamma^T\hat\psi)(\lambda^\pm\gamma^S\hat\psi)\,.
\end{aligned}
$$

After performing all these manipulations the item 4 of the list of states in the cohomology of $Q_{-\frac{1}{2}}$ can be rewritten as

$$-12(\hat{\psi}\gamma^{TSM}\hat{\psi})(\lambda^+\gamma_M\lambda^+) - 24(\lambda^+\gamma^T\hat{\psi})(\lambda^+\gamma^S\hat{\psi}) + 24(\lambda^-\gamma^T\psi)(\lambda^-\gamma^S\psi)\,. \quad (3.33)$$

We can proceed further by noting that

$$6Q_{-\frac{1}{2}} \cdot [z^{\frac{1}{2}}(\hat{\psi}\gamma^{TSM}\hat{\psi})(\lambda^+\gamma_M\hat{\psi})] =$$

$$-12(\lambda^+\gamma^{TSM}\hat{\psi})(\lambda^+\gamma_M\hat{\psi}) - 6(\hat{\psi}\gamma^{TSM}\hat{\psi})(\lambda^+\gamma_M\lambda^+) =$$

$$-12(\hat{\psi}\gamma^{TSM}\hat{\psi})(\lambda^+\gamma_M\lambda^+) - 24(\lambda^+\gamma^T\hat{\psi})(\lambda^+\gamma^S\hat{\psi})\,,$$

where we have used the Fierz identity,

$$\gamma^M_{\bar{\alpha}(\bar{\beta}}\gamma_{|M|\bar{\gamma}\bar{\delta})} = 0\,,$$

and

$$(\gamma^{TSM})_{\bar{\alpha}\bar{\beta}} = (\gamma^T\gamma^S\gamma^M)_{\bar{\alpha}\bar{\beta}} + (\gamma^S)_{\bar{\alpha}\bar{\beta}}\,\eta^{TM} - (\gamma^T)_{\bar{\alpha}\bar{\beta}}\,\eta^{SM} - (\gamma^M)_{\bar{\alpha}\bar{\beta}}\,\eta^{TS}\,.$$

The final form of the item 4 is then

$$6Q_{-\frac{1}{2}} \cdot [z^{\frac{1}{2}}(\hat{\psi}\gamma^{TSM}\hat{\psi})(\lambda^+\gamma_M\hat{\psi})] + 24(\lambda^-\gamma^T\hat{\psi})(\lambda^-\gamma^S\hat{\psi})\,, \quad (3.34)$$

which enables us to conclude that it is a function of $\lambda^-\gamma^M\hat{\psi}$ up to a BRST trivial quantity. After this analysis of the item 4 of the states in the cohomology, we can restate the result of the cohomology at +2 ghost number as

1. any function of λ^- and of $\hat{\psi}$ appearing only in the combination $(\lambda^-\gamma^M\hat{\psi})$,

2. $(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma_M\hat{\psi})$.

Note that the only state in the cohomology that depends on λ^+ is the item 2 of the list above. We will show below that allowing dependence on the non-minimal pure spinor variables the cohomology of $Q_{-\frac{1}{2}}$ is independent of λ^+ and as a consequence λ^- can be considered as a pure spinor due to the second identity of (3.26). The operator $Q_{-\frac{1}{2}}$ has one more term once we have the non-minimal variables, the new operator $\hat{Q}_{-\frac{1}{2}}$ is

$$\hat{Q}_{-\frac{1}{2}} = Q_{-\frac{1}{2}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}}\,,$$

and recall from the chapter 2 that $\tilde{w}^{\bar{\alpha}}$ is the conjugate momentum of $\tilde{\lambda}_{\bar{\alpha}}$ which acts on functions of $\tilde{\lambda}_{\bar{\alpha}}$ as $\frac{\partial}{\partial\tilde{\lambda}_{\bar{\alpha}}}$. Consider the action of this new operator on the term given below

$$\hat{Q}_{-\frac{1}{2}}\cdot\Big[\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_{MN}\hat{\psi})\Big] = -\frac{1}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_{MN}\lambda^+)$$
$$-\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^- r)(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_{MN}\hat{\psi}) + \frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(r\gamma_{MN}\hat{\psi}),$$

and using Fierz identity and the properties of the chiral gamma matrices, it is possible to rewrite in a more convenient way all the terms on the right-hand side of the expression above. Consider the first term

$$-\frac{1}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_{MN}\lambda^+) = \frac{1}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_M\gamma_N\lambda^-)$$
$$= 2(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma_M\hat{\psi}) - \frac{1}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_N\gamma_M\lambda^-), \qquad (3.35)$$

and the second and third

$$-\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^- r)(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_{MN}\hat{\psi}) + \frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(r\gamma_{MN}\hat{\psi}) =$$
$$-\frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\lambda^-\gamma^P\hat{\psi})(r\gamma_{MNP}\tilde{\lambda})$$
$$+\frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma_P\gamma^N\tilde{\lambda})(\lambda^-\gamma^P\hat{\psi})(r\gamma_M\gamma_N\hat{\psi})$$
$$+\frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\tilde{\lambda}\gamma_P\gamma^N\hat{\psi})(\lambda^-\gamma^P\hat{\psi})(r\gamma_N\gamma_M\lambda^-), \qquad (3.36)$$

from (3.35) and (3.36), we finally have

$$\hat{Q}_{-\frac{1}{2}}\cdot\Big[\frac{1}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_{MN}\hat{\psi})\Big] = 2(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma_M\hat{\psi})$$
$$-\frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\lambda^-\gamma^P\hat{\psi})(r\gamma_{MNP}\tilde{\lambda})$$
$$-\frac{1}{(\tilde{\lambda}\lambda^-)}(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma^N\hat{\psi})(\tilde{\lambda}\gamma_N\gamma_M\lambda^-)$$
$$+\frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma_P\gamma^N\tilde{\lambda})(\lambda^-\gamma^P\hat{\psi})(r\gamma_M\gamma_N\hat{\psi})$$
$$+\frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\tilde{\lambda}\gamma_P\gamma^N\hat{\psi})(\lambda^-\gamma^P\hat{\psi})(r\gamma_N\gamma_M\lambda^-)$$
$$= 2(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma_M\hat{\psi}) - \frac{1}{2}\frac{z^{\frac{1}{2}}}{(\tilde{\lambda}\lambda^-)^2}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\lambda^-\gamma^P\hat{\psi})(r\gamma_{MNP}\tilde{\lambda})$$
$$+\,\text{Terms that are zero when }\lambda^-\text{ is a pure spinor}\,.$$

The result above shows that the state in the cohomology of $Q_{-\frac{1}{2}}$ which depends on λ^+, precisely the scalar $(\lambda^-\gamma^M\hat{\psi})(\lambda^+\gamma_M\hat{\psi})$, can be represented by the state $z^{\frac{1}{2}}(\tilde{\lambda}\lambda^-)^{-2}(\lambda^-\gamma^M\hat{\psi})(\lambda^-\gamma^N\hat{\psi})(\lambda^-\gamma^P\hat{\psi})(r\gamma_{MNP}\tilde{\lambda})$ up to a $\hat{Q}_{-\frac{1}{2}}$ exact term.

In conclusion, allowing dependence on the non-minimal pure spinor variables the cohomology of $Q_{-\frac{1}{2}}$ is independent of λ^+. The second identity of (3.26) then implies that $\lambda^-\gamma^M\lambda^- = 0$, or in other words, that λ^- is a pure spinor, consequently it has 11 independent components and $(\lambda^-\gamma^M\hat{\psi})$ has only 5 independent components as will be explained in the next chapter when we will rewrite some of the expressions using $U(5)$ notation. Therefore, states in the cohomology of $Q_{-\frac{1}{2}}$ depend on the non-minimal variables, the 21 bosonic variables $[x, z, y, \lambda^-]$ and the 21 fermionic variables $[\theta, \lambda^-\gamma^M\hat{\psi}]$.

3.4 The zero mode cohomology of $Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \dots$

The next step in the calculation of the cohomology of the BRST operator is to compute the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$, then compute the cohomology of $Q_{\frac{3}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$ and $Q_{\frac{1}{2}}$, etc. A vertex operator V is a state in the cohomology of the BRST operator if it satisfies, see (3.21),

$$
\begin{aligned}
&Q_{-\frac{1}{2}} \cdot V_{d_0} = 0\,,\\
&Q_{\frac{1}{2}} \cdot V_{d_0} + Q_{-\frac{1}{2}} \cdot V_{d_0+1} = 0\,,\\
&Q_{\frac{3}{2}} \cdot V_{d_0} + Q_{\frac{1}{2}} \cdot V_{d_0+1} + Q_{-\frac{1}{2}} \cdot V_{d_0+2} = 0\,,\\
&\dots\,,
\end{aligned}
$$

where d_0 is its minimum degree. Let us suppose that the cohomology is non-trivial and there is at least one non-zero state V'. In the next chapter, we will compute the zero mode cohomology at +2 ghost number of the BRST operator and turns out that it is in fact non-trivial. This state V' is a solution of all the equations given above by assumption. The first equation says that V'_{d_0} is a state in the cohomology of $Q_{-\frac{1}{2}}$ and being so is independent of λ^+ which implies that λ^- is a pure spinor up to a BRST trivial quantity and only depend on ψ in the combination $(\lambda^-\gamma^M\hat{\psi})$. The operator $Q_{\frac{1}{2}}$ has terms proportional to λ^+, however, we are expected to compute the cohomology of this operator restricted to states in the cohomology of $Q_{-\frac{1}{2}}$, that does not depend on λ^+, therefore these terms must act as zero. In other words, given that V' is a non-trivial solution of the second equation above, there exists a V_{d_0+1} that removes all terms proportional to λ^+ after the application of $Q_{\frac{1}{2}}$ on V_{d_0}.

Similar arguments can be used to the terms proportional to λ^+ in $Q_{\frac{3}{2}} + \ldots$, all of them act as zero.

Let us now analyze the terms proportional to λ^- of $Q_{\frac{3}{2}} + \ldots$. These operators are linear in λ^- and at least cubic in ψ, which implies that the terms involving λ^- of these operators cannot be expressed in terms of the five $\lambda^-\gamma^M\hat{\psi}$. Further, note that the third equation of (3.23) can be rewritten as

$$Q_{\frac{1}{2}} \cdot Q_{\frac{1}{2}} + \{Q_{-\frac{1}{2}}, Q_{\frac{3}{2}}\} = 0\,,$$

and applying $Q_{\frac{1}{2}}$ to the second equation of (3.21), one has

$$Q_{\frac{1}{2}} \cdot Q_{\frac{1}{2}} \cdot V'_{d_0} + Q_{\frac{1}{2}} \cdot Q_{-\frac{1}{2}} \cdot V'_{d_0+1} = -Q_{-\frac{1}{2}} \cdot (Q_{\frac{3}{2}} \cdot V'_{d_0} + Q_{\frac{1}{2}} \cdot V'_{d_0+1}) = 0\,,$$

where we have also used the second identity of (3.23).

The equation above enables us to conclude that for every V'_{d_0} and V'_{d_0+1} satisfying the second equation of (3.21), the combination $Q_{\frac{3}{2}} \cdot V'_{d_0} + Q_{\frac{1}{2}} \cdot V'_{d_0+1}$ is annihilated by $Q_{-\frac{1}{2}}$. However, recalling that the only term involving ψ and λ^- that is annihilated by $Q_{-\frac{1}{2}}$ is $\lambda^-\gamma^M\hat{\psi}$, the combination $Q_{\frac{3}{2}} \cdot V'_{d_0} + Q_{\frac{1}{2}} \cdot V'_{d_0+1}$ must be a function of $\lambda^-\gamma^M\hat{\psi}$. The operator $Q_{\frac{3}{2}}$ cannot be expressed as a function of ψ in this combination, which means that all terms of $Q_{\frac{3}{2}} \cdot V'_{d_0}$ proportional to λ^- cancel with the terms proportional to λ^- of $Q_{\frac{1}{2}} \cdot V'_{d_0+1}$.

Recall that the third equation of (3.21) is

$$Q_{\frac{3}{2}} \cdot V'_{d_0} + Q_{\frac{1}{2}} \cdot V'_{d_0+1} + Q_{-\frac{1}{2}} \cdot V'_{d_0+2} = 0\,, \tag{3.37}$$

and we have argued that if the second equation of (3.21) is satisfied, this equation is automatically satisfied. In summary, all terms in $Q_{\frac{3}{2}}$ acts as zero when restricted to states in the cohomology of $Q_{\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$. The same argument shows that all the operators $Q_{\frac{5}{2}} + \ldots$ act also as zero.

The conclusion of all this analysis is that the computation of the cohomology of the BRST operator inside the region of validity of the z expansion reduces to computing the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$. This computation will be performed in the next chapter.

Chapter 4

Vertex Operators

As explained in the previous chapter, the computation of the cohomology of the BRST operator close to the boundary of AdS is equivalent, up to a BRST-trivial quantity, to computing the term V_{d_0}, or in other words, the term of lowest degree of a physical vertex operator V. The term V_{d_0} is a state in the cohomology of the $Q_{\frac{1}{2}}$ operator restricted to states in the cohomology of $Q_{-\frac{1}{2}}$. As predicted by the (AdS/CFT) conjecture, every on-shell superstring state in $AdS^5 \times S^5$ is dual to a gauge-invariant operator of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills, in particular, the supergravity states are dual to Half-BPS operators. In this chapter, we will compute the zero mode cohomology of $Q_{\frac{1}{2}}$ at +2 ghost number and show explicitly that every state V_{d_0} is dual to a Half-BPS operator of super-Yang-Mills. The results will be described in terms of superfields defined in harmonic superspace.

The organization of this chapter is as follows: in the first section we will rewrite the operator $Q_{\frac{1}{2}}$ of the previous chapter in a convenient form and a few examples of states in the cohomology of $Q_{\frac{1}{2}}$ will be given. Then, we will review the representation theory of the $\mathcal{N} = 4$ $d = 4$ superconformal algebra and introduce the concept of harmonic superspace. The main result of this thesis will be presented in the section 4.4, where a general expression for the vertex operators will be given. After proving that the vertex operators are states in the cohomology of $Q_{\frac{1}{2}}$, we will evaluate the general expression for the specific example of the dilaton vertex operator. Finally, we will exemplify with a simple term the meaning of the statement "acts as zero" used several times in the previous chapter.

4.1 The $Q_{\frac{1}{2}}$ operator and examples

In the previous chapter, we have argued that states in the BRST cohomology near the boundary of AdS are described by states in the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$ which depend on the non-minimal pure spinor variables and $[x, z, y, \theta, \lambda^-, \lambda^-\gamma^M\hat{\psi}]$ with λ^- a pure spinor $\lambda^-\gamma^M\lambda^- = 0$. In what follows, we are going to suppress most of the times the minus superscript in λ^-. It is possible to write the zero mode terms of the operator $Q_{\frac{1}{2}}$ of (3.14) using a compact ten-dimensional notation as

$$Q_{\frac{1}{2}} = z^{\frac{1}{2}}[\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + 4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}} \tag{4.1}$$
$$+y_{ij}(\lambda\gamma^{ij}\hat{\psi})(2z\frac{\partial}{\partial z} + y^{km}\frac{\partial}{\partial y^{km}} - \lambda^{\bar{\alpha}}\frac{\partial}{\partial \lambda^{\bar{\alpha}}})] + \widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}\,,$$

where we have included the usual non-minimal pure spinor term $\widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}$ that was not present in (3.14). The inclusion of this additional term is necessary because, as will become clear below, the general expression for the vertex operators can only be written as a function of $(\lambda\gamma^M\hat{\psi})$ after introducing the non-minimal variables.

In order to rewrite $Q_{\frac{1}{2}}$ as (4.1), we have performed several manipulations. The first observation is that the overall factor of $\frac{\sqrt{2}}{2}$ was adsorbed by a redefinition of λ. Secondly, the canonical momenta were replaced by derivatives because the canonical momentum of a variable and the variable satisfy a canonical commutation relation. Moreover, in our conventions

$$y_{ij}(\lambda\gamma^{ij}\hat{\psi}) = \lambda^{\alpha i}\psi_{\alpha i} + \bar{\lambda}_{\dot{\alpha} i}\bar{\psi}^{\dot{\alpha} i}\,,$$

and it is easy to organize the relevant terms of (3.14) in the form

$$z^{\frac{1}{2}} y_{ij}(\lambda\gamma^{ij}\hat{\psi})(2z\frac{\partial}{\partial z} + y^{km}\frac{\partial}{\partial y^{km}})\,,$$

upon noticing that $y^{km}\frac{\partial}{\partial y^{km}} \cdot f(y) = y_{km}\frac{\partial}{\partial y_{km}} \cdot f(y)$, where $f(y)$ means any function of y. The term

$$z^{\frac{1}{2}} 4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}} = z^{\frac{1}{2}} 4\lambda^{\alpha i}\psi_{\alpha k}\frac{\partial}{\partial y^{ij}} y^{jk} - z^{\frac{1}{2}} 4\bar{\lambda}^{\dot{\alpha}}_i \bar{\psi}^l_{\dot{\alpha}}\frac{\partial}{\partial y_{ij}} y_{jl}\,, \tag{4.2}$$

follows also from collecting the relevant terms.

There are many terms in the $Q_{\frac{1}{2}}$ of (3.14) that depend on λ derivatives. However, all the terms proportional to λ^+ act as zero and terms proportional to the pure spinor constraints such as

$$z^{\frac{1}{2}}\lambda^{-\alpha i}(4\psi_{\alpha k}\bar{\lambda}^{\dot{\alpha}}_i\frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}_k} + 4\psi_{\alpha k}\hat{\bar{\lambda}}^{\dot{\alpha}}_i\frac{\partial}{\partial \hat{\bar{\lambda}}^{\dot{\alpha}}_k}) \cdot f(\lambda^-) = 0\,,$$

do not contribute. The remaining terms can be easily organized using the Schouten identities (A.5) as

$$-z^{\frac{1}{2}} y_{ij}(\lambda\gamma^{ij}\hat{\psi})(\lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}})\,.$$

In addition, in the term $z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}}$, $D_{\bar{\alpha}}$ is the $d=4$ dimensional reduction of the $d=10$ supersymmetric covariant derivative of (2.14) which is

$$D_{\alpha i} = -\frac{\partial}{\partial\theta^{\alpha i}} - i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}_i\frac{\partial}{\partial x^\mu}\,, \quad \bar{D}^i_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}_i} + i\theta^{\beta i}(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu}\,. \tag{4.3}$$

The final comment is that the term $z^{\frac{1}{2}}4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}}$ of (4.1) is understood to not act on $(\lambda\gamma^M\hat{\psi})$ even though $(\lambda\gamma^M\hat{\psi})$ depends on y_{ij} through the $\hat{\psi}$. This is the case because we have not included in (4.1) the terms

$$z^{\frac{1}{2}}[\,-2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}^j_{\dot{\alpha}}\frac{\partial}{\partial\bar{\psi}^i_{\dot{\alpha}}} - 2\bar{\lambda}^{\dot{\alpha}}_i\psi_{\alpha k}\bar{\psi}^k_{\dot{\alpha}}\frac{\partial}{\partial\psi_{\alpha i}} + 4\lambda^{\alpha i}\psi_{\alpha j}\psi^\beta_i\frac{\partial}{\partial\psi^\beta_j} - 4\bar{\lambda}^{\dot{\alpha}}_i\bar{\psi}^m_{\dot{\alpha}}\bar{\psi}^{i\dot{\beta}}\frac{\partial}{\partial\bar{\psi}^{m\dot{\beta}}}]\,,$$

and it is possible to show that

$$\begin{aligned}\Big(4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}} - 2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}^j_{\dot{\alpha}}\frac{\partial}{\partial\bar{\psi}^i_{\dot{\alpha}}} - 2\bar{\lambda}^{\dot{\alpha}}_i\psi_{\alpha k}\bar{\psi}^k_{\dot{\alpha}}\frac{\partial}{\partial\psi_{\alpha i}}\\ +4\lambda^{\alpha i}\psi_{\alpha j}\psi^\beta_i\frac{\partial}{\partial\psi^\beta_j} - 4\bar{\lambda}^{\dot{\alpha}}_i\bar{\psi}^m_{\dot{\alpha}}\bar{\psi}^{i\dot{\beta}}\frac{\partial}{\partial\bar{\psi}^{m\dot{\beta}}}\Big)(\lambda\gamma^M\hat{\psi}) = 0\,,\end{aligned} \tag{4.4}$$

using the pure spinor conditions for λ. There are at least two different ways to prove the result above. The first one is by direct computation and the second one follows from the nilpotency property of the $Q_{\frac{1}{2}}$ operator. Note that using the ansatz for the chiral gamma matrices given in (2.66), we have in four-dimensional notation

$$(\lambda\gamma^\mu\hat{\psi}) \to -\lambda^{\alpha i}i(\sigma^\mu)_{\alpha\dot{\alpha}}y_{ij}\bar{\psi}^{\dot{\alpha}j} + \bar{\lambda}_{\dot{\alpha}i}i(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}y^{ij}\psi_{\alpha j}\,,$$

$$(\lambda\gamma^{I+3}\hat{\psi}) \to \lambda^{\alpha i}(\sigma^I)_{ij}y^{jk}\psi_{\alpha k} + \bar{\lambda}_{\dot{\alpha}i}(\sigma^I)^{ij}y_{jk}\bar{\psi}^{\dot{\alpha}k}\,,$$

and with these terms written in this form, one can easily evaluate the left-hand side of (4.4). Collecting the terms with two ψ when $M=\mu$, we have

$$4\lambda^{\alpha i}\psi_{\alpha k}y^{jk}\bar{\lambda}_{\dot{\beta}m}(\bar{\sigma}^\mu)^{\dot{\beta}\gamma}\psi_{\gamma n}(\delta^m_i\delta^n_j - \delta^n_i\delta^m_j) + 4\lambda^{\alpha i}\psi_{\alpha j}\psi_{\gamma i}\bar{\lambda}_{\dot{\beta}m}(\bar{\sigma}^\mu)^{\dot{\beta}\gamma}y^{mj} = 0\,,$$

where we have used the pure spinor condition $\lambda^{\alpha i}\bar{\lambda}^{\dot{\alpha}}_i = 0$. Similar arguments can be given to show that the terms proportional to two $\bar{\psi}$ vanish. The terms with one ψ and one $\bar{\psi}$ when $M=\mu$ are

$$\begin{aligned}-4\lambda^{\alpha i}\psi_{\alpha k}y^{jk}\lambda^{\gamma m}(\sigma^\mu)_{\gamma\dot{\gamma}}\epsilon_{mnij}\bar{\psi}^{\dot{\gamma}n} - 4\bar{\lambda}^{\dot{\alpha}}_i\bar{\psi}^l_{\dot{\alpha}}y_{jl}\bar{\lambda}_{\dot{\gamma}m}(\bar{\sigma}^\mu)^{\dot{\gamma}\gamma}\epsilon^{mnij}\psi_{\gamma n}\\ +\,2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}^{\dot{\alpha}j}\lambda^{\gamma m}(\sigma^\mu)_{\gamma\dot{\alpha}}y_{mi} - 2\bar{\lambda}^{\dot{\alpha}}_i\psi_{\alpha k}\bar{\psi}^k_{\dot{\alpha}}\bar{\lambda}_{\dot{\gamma}m}(\bar{\sigma}^\mu)^{\dot{\gamma}\alpha}y^{mi}\\ =\;\lambda^{\alpha i}\psi^\gamma_j\bar{\psi}^{\dot{\alpha}j}\lambda^k_\alpha(\sigma^\mu)_{\gamma\dot{\alpha}}y_{ki} - \bar{\lambda}^{\dot{\alpha}}_i\psi_{\alpha k}\bar{\psi}^k_{\dot{\gamma}}\bar{\lambda}_{\dot{\alpha}j}(\sigma^\mu)^{\dot{\gamma}\alpha}y^{ji}\\ -2\lambda^{\alpha i}\psi^\gamma_k y^{jk}\lambda^m_\alpha(\sigma^\mu)_{\gamma\dot{\gamma}}\epsilon_{mnij}\bar{\psi}^{\dot{\gamma}n} - 2\bar{\lambda}^{\dot{\alpha}}_i\bar{\psi}^l_{\dot{\gamma}}y_{jl}\bar{\lambda}_{\dot{\alpha}m}(\bar{\sigma}^\mu)^{\dot{\gamma}\gamma}\epsilon^{mnij}\psi_{\gamma n}\,,\end{aligned}$$

where we have used the Schouten identities to organize the terms. This combination of terms is equal to zero, because of the pure spinor condition

$$\lambda^{\alpha i}\lambda^j_\alpha - \frac{1}{2}\epsilon^{ijkl}\bar{\lambda}_{\dot{\alpha}k}\bar{\lambda}^{\dot{\alpha}}_l = 0\,,$$

and of the identity

$$y_{ij}y_{kl} + y_{ik}y_{lj} + y_{il}y_{jk} = \epsilon_{jkli}\,,$$

which can be proved by contracting with y_I and y_J both sides of the relation $(\sigma^I)_{i[j}(\sigma^J)_{kl]} = \frac{1}{3}\epsilon_{jklm}(\sigma^I\sigma^J)^m_i$, and using $y_{ij}y^{jk} = \delta^k_i$. This finishes the proof of (4.4) when $M = \mu$. Let us now consider the case $M = I+3$. The terms with two ψ are

$$4\lambda^{\alpha i}\psi_{\alpha k}y^{jk}\lambda^{\gamma p}(\sigma^I)_{pn}(\delta^n_i\delta^m_j - \delta^m_i\delta^n_j)\psi_{\gamma m} + 4\lambda^{\alpha i}\psi_{\alpha j}\psi_{\beta i}\lambda^{\beta k}(\sigma^I)_{kn}y^{nj} = 0\,,$$

and this follows because $\lambda^{\alpha i}(\sigma^I)_{ij}\lambda^{\beta j} \propto \epsilon^{\alpha\beta}\lambda^{\gamma i}(\sigma^I)_{ij}\lambda^j_\gamma$ and $\psi^\alpha_k y^{km}\psi_{\alpha m} = 0$. Similiar arguments can be given to prove that the terms with two $\bar{\psi}$ are zero. The terms with one ψ and one $\bar{\psi}$ when $M = I+3$ are

$$\begin{aligned}&4\lambda^{\alpha i}\psi_{\alpha k}y^{jk}\bar{\lambda}_{\dot{\gamma}l}(\sigma^I)^{ln}\epsilon_{ijnm}\bar{\psi}^{\dot{\gamma}m} - 4\bar{\lambda}^{\dot{\alpha}}_i\bar{\psi}^l_{\dot{\alpha}}y_{jl}\lambda^{\gamma k}(\sigma^I)_{kn}\epsilon^{ijnm}\psi_{\gamma m}\\ &-2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}^{\dot{\gamma}j}\bar{\lambda}_{\dot{\gamma}m}(\sigma^I)^{mn}y_{ni} - 2\bar{\lambda}^{\dot{\alpha}}_i\psi_{\gamma k}\bar{\psi}^k_{\dot{\alpha}}\lambda^{\gamma j}(\sigma^I)_{jm}y^{mi}\,,\end{aligned}$$

and after using the pure spinor condition $\lambda^{\alpha i}\bar{\lambda}^{\dot{\alpha}}_i = 0$,

$$\begin{gathered}y_{ij}(\sigma^I)^{jk} + (\sigma^I)_{ij}y^{jk} = 2\delta^k_i y^I\,, \quad (\sigma^I)^{ij} = \frac{1}{2}\epsilon^{ijkl}(\sigma^I)_{kl}\,,\\ \epsilon^{jkli}y_{im} = -\delta^j_m y^{kl} - \delta^k_m y^{lj} - \delta^l_m y^{jk}\,,\end{gathered}$$

the terms sum to zero. This finishes the proof of (4.4) by direct computation. A different way to prove this identity is by using the nilpotency property of the $Q_{\frac{1}{2}}$ operator when this operator acts on the states in the cohomology of the operator $Q_{-\frac{1}{2}}$. Note first that

$$\begin{aligned}Q_{\frac{1}{2}}\cdot y_{ij} &= z^{\frac{1}{2}}\Big(4(\lambda\gamma^{kl}\hat{\psi})\frac{\partial}{\partial y^{kl}} + y_{mn}(\lambda\gamma^{mn}\hat{\psi})y^{kl}\frac{\partial}{\partial y^{kl}}\Big)\cdot y_{ij}\\ &= z^{\frac{1}{2}}4\lambda^{\alpha m}y^{kl}\psi_{\alpha l}\epsilon_{ijmk} + z^{\frac{1}{2}}4\bar{\lambda}_{\dot{\alpha}m}y_{kl}\bar{\psi}^{\dot{\alpha}l}(\delta^m_i\delta^k_j - \delta^k_i\delta^m_j) + 2z^{\frac{1}{2}}y_{kl}(\lambda\gamma^{kl}\hat{\psi})y_{ij}\,,\end{aligned}$$

and multiplying both sides of this expression by $(\sigma^I)^{ij}$ and using that $(\sigma^I)^{ij}y_{ij} = -4y^I$, we conclude

$$Q_{\frac{1}{2}}\cdot y^I = 2z^{\frac{1}{2}}y_{ij}(\lambda\gamma^{ij}\hat{\psi})\,y^I - 2z^{\frac{1}{2}}(\lambda\gamma^{I+3}\hat{\psi})\,. \tag{4.5}$$

In particular, as an application of this result, note that the $Q_{\frac{1}{2}}$ operator preserves the constraint $y^I y^I = 1$,

$$0 = Q_{\frac{1}{2}} \cdot 1 = Q_{\frac{1}{2}} \cdot (y^I y^I) = 4z^{\frac{1}{2}} y_{ij} (\lambda\gamma^{ij}\hat{\psi})\, y^I y^I - 4z^{\frac{1}{2}} y^I (\lambda\gamma^{I+3}\hat{\psi}) = 0\,.$$

Moreover, we have

$$Q_{\frac{1}{2}} \cdot z^n = z^{\frac{1}{2}} y_{ij} (\lambda\gamma^{ij}\hat{\psi})\, 2z \frac{\partial}{\partial z} \cdot z^n = y_{ij}(\lambda\gamma^{ij}\hat{\psi})\, 2n\, z^{n+\frac{1}{2}}\,, \tag{4.6}$$

and from this result and (4.5), we conclude that

$$Q_{\frac{1}{2}} \cdot (z^{-1} y^I) = -2z^{-\frac{1}{2}} (\lambda\gamma^{I+3}\hat{\psi})\,. \tag{4.7}$$

Applying $Q_{\frac{1}{2}}$ to this equation and using that this operator is nilpotent when acting on the states in the cohomology of $Q_{-\frac{1}{2}}$, we get

$$0 = Q_{\frac{1}{2}} \cdot Q_{\frac{1}{2}} \cdot (z^{-1} y^I) = Q_{\frac{1}{2}} \cdot (-2z^{-\frac{1}{2}} (\lambda\gamma^{I+3}\hat{\psi}))\,, \tag{4.8}$$

and

$$Q_{\frac{1}{2}} \cdot (-2z^{-\frac{1}{2}} (\lambda\gamma^{I+3}\hat{\psi})) = 2y_{ij}(\lambda\gamma^{ij}\hat{\psi})(\lambda\gamma^{I+3}\hat{\psi}) - 2z^{-\frac{1}{2}}\, Q_{\frac{1}{2}} \cdot (\lambda\gamma^{I+3}\hat{\psi})\,. \tag{4.9}$$

It is possible to compute the second term on the right-hand side of the expression above, for this we define

$$\begin{aligned} Q^1_{\frac{1}{2}} = [4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}} + 4\lambda^{\alpha i}\psi_{\alpha j}\psi^\beta_i \frac{\partial}{\partial \psi^\beta_j} - 4\bar{\lambda}^{\dot{\alpha}}_i \bar{\psi}^m_{\dot{\alpha}} \bar{\psi}^{\dot{\beta} i} \frac{\partial}{\partial \bar{\psi}^{\dot{\beta} m}} \\ -2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}^j_{\dot{\alpha}} \frac{\partial}{\partial \bar{\psi}^i_{\dot{\alpha}}} - 2\bar{\lambda}^{\dot{\alpha}}_i \psi_{\alpha k} \bar{\psi}^k_{\dot{\alpha}} \frac{\partial}{\partial \psi_{\alpha i}}]\,, \end{aligned}$$

and, we have

$$\begin{aligned} -2z^{-\frac{1}{2}} Q_{\frac{1}{2}} \cdot (\lambda\gamma^{I+3}\hat{\psi}) &= -2y_{ij}(\lambda\gamma^{ij}\hat{\psi})(y^{mt}\frac{\partial}{\partial y^{mt}} - \lambda^{\bar{\alpha}}\frac{\partial}{\partial \lambda^{\bar{\alpha}}}) \cdot (\lambda\gamma^{I+3}\hat{\psi}) \\ &\quad -2Q^1_{\frac{1}{2}} \cdot (\lambda\gamma^{I+3}\hat{\psi}) \\ &= -2y_{ij}(\lambda\gamma^{ij}\hat{\psi})(\lambda\gamma^{I+3}\hat{\psi}) - 2Q^1_{\frac{1}{2}} \cdot (\lambda\gamma^{I+3}\hat{\psi})\,. \end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) in (4.8), we finally conclude

$$Q^1_{\frac{1}{2}} \cdot (\lambda\gamma^{I+3}\hat{\psi}) = 0\,,$$

and this is precisely the identity (4.4) when $M = I + 3$. This procedure using the nilpotency property of the $Q_{\frac{1}{2}}$ operator proves the identity (4.4) when $M = I + 3$. However, the term $\lambda\gamma^M\hat{\psi}$ has only 5 independent components because λ is a pure spinor, and given that $I = 1, \ldots, 6$, the proof is valid for six of them and so is valid for all M.

4.1.1 Examples of states in the $Q_{\frac{1}{2}}$ cohomology

We will give in this subsection two examples of states in the zero mode cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$. The general expression for all the states will be presented in a future section. Recall that in the pure spinor formalism for closed strings the states in the cohomology at +2 ghost number of the BRST operator corresponds to the physical states of the theory. Our first example is a vertex operator that is independent of y and of the non-minimal variables. The non-zero terms of $Q_{\frac{1}{2}}$ acting on such a vertex operator are

$$Q_{\frac{1}{2}}^{\text{nonzero}} = z^{\frac{1}{2}}[\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + y_{ij}(\lambda\gamma^{ij}\hat{\psi})(2z\frac{\partial}{\partial z} - \lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}})]\,.$$

The cohomology at +2 ghost number of the operator $\lambda^{\bar{\alpha}} D_{\bar{\alpha}}$ is known and it was computed by Berkovits in [48], for example. It corresponds to the antifields of super-Yang-Mills and it is described by the superfield $\lambda^{\bar{\alpha}}\lambda^{\bar{\beta}}A^{*}_{\bar{\alpha}\bar{\beta}}$, which at zero momentum can be gauged to

$$\lambda^{\bar{\alpha}}\lambda^{\bar{\beta}}A^{*}_{\bar{\alpha}\bar{\beta}} = (\lambda\gamma^{M}\theta)(\lambda\gamma^{N}\theta)(\theta\gamma_{MN})^{\bar{\alpha}}\psi^{*}_{\bar{\alpha}} +$$
$$(\lambda\gamma^{M}\theta)(\lambda\gamma^{N}\theta)(\theta\gamma_{MN\mu}\theta)a^{*\mu} + (\lambda\gamma^{M}\theta)(\lambda\gamma^{N}\theta)(\theta\gamma_{MN}{}^{jk}\theta)\phi^{*}_{jk}\,,$$

where $a^{*\mu}$, ϕ^{*}_{jk} and ψ^{*}_{α} are the antifields to the gluon a_{μ}, scalars ϕ^{jk}, and gluino $\psi^{\bar{\alpha}}$, respectively, which is the field content of $\mathcal{N}=4$ $d=4$ super-Yang-Mills. So, the corresponding vertex operator annihilated by $Q_{\frac{1}{2}}$ is

$$V = z\lambda^{\bar{\alpha}}\lambda^{\bar{\beta}}A^{*}_{\bar{\alpha}\bar{\beta}}\,, \tag{4.11}$$

where the factor of z was included in order for it to be annihilated by the second term of $Q_{\frac{1}{2}}^{\text{nonzero}}$ given above. One of the conclusions is that these operators with no dependence on y are the duals to the so-called super-Yang-Mills "singleton" operators, or in other words, the duals to the abelian super-Yang-Mills fields.

The next example is a vertex operator linear in y and independent of $[x, z, \theta, \psi]$. It is

$$V = i\lambda^{\alpha i}\lambda^{k}_{\alpha}y_{ik}\,, \tag{4.12}$$

and note that it is real because of $\lambda^{\alpha i}\lambda^{k}_{\alpha}y_{ik} = -\bar{\lambda}^{\dot{\alpha}}_{i}\bar{\lambda}_{\dot{\alpha}j}y^{ij}$, which follows from the pure spinor conditions for λ. Let us prove that this vertex operator is annihilated by $Q_{\frac{1}{2}}$, the terms of this operator that act non-trivially are

$$Q_{\frac{1}{2}}^{\text{nonzero}} = z^{\frac{1}{2}}[4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}} + y_{ij}(\lambda\gamma^{ij}\hat{\psi})(y^{kl}\frac{\partial}{\partial y^{kl}} - \lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}})]\,,$$

and it is easy to see that the second term on the right-hand side of the expression above annihilates the vertex operator. In order to see that the first one also annihilates it, let us compute it explicitly

$$(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}}\cdot\lambda^{\beta m}\lambda^{n}_{\beta}y_{mn} =$$
$$[\,\lambda^{\alpha j}y^{kl}\psi_{\alpha l}\frac{\partial}{\partial y^{jk}}+\bar{\lambda}_{\dot{\alpha}j}y_{kl}\bar{\psi}^{\dot{\alpha}l}\frac{\partial}{\partial y_{jk}}\,]\cdot\lambda^{\beta m}\lambda^{n}_{\beta}y_{mn} =$$
$$\lambda^{\alpha j}y^{kl}\psi_{\alpha l}\lambda^{\beta m}\lambda^{n}_{\beta}\epsilon_{jkmn}+\bar{\lambda}_{\dot{\alpha}j}y_{kl}\bar{\psi}^{\dot{\alpha}l}\lambda^{\beta m}\lambda^{n}_{\beta}(\delta^{j}_{m}\delta^{k}_{n}-\delta^{k}_{m}\delta^{j}_{n})=0\,,$$

where we have used the pure spinor conditions. Being a scalar under the action of the $PSU(2,2|4)$ group, this vertex operator corresponds to the zero-momentum dilaton that is dual to the linearized super-Yang-Mills action. In a future section, a general formula for the vertex operators that includes these two examples will be presented, however, we will first review some topics relevant for its understanding.

4.2 Representations of the superconformal algebra

The theory of $\mathcal{N}=4$ $d=4$ super-Yang-Mills contains the following fields: the gauge boson A_μ, six real scalars ϕ^I, four chiral fermions and four anti-chiral fermions. In particular, this theory is finite, or in other words, its beta function vanishes to all orders in perturbation theory. This result was proved up to three loops in [61] and to all loops in [9, 10]. A nice argument for its finiteness is that the action of super-Yang-Mills belongs to a Half-BPS multiplet and consequently does not receive quantum corrections, a review is the article [62] by Minahan. Being finite means that the theory of $\mathcal{N}=4$ $d=4$ super-Yang-Mills is superconformal to all orders in perturbation theory.

A conformal field theory does not possess an S-matrix because of the impossibility of defining asymptotic states, however, the theory has well defined operators. In this section, we will briefly review the representations of the $d=4$ conformal group and of the $\mathcal{N}=4$ $d=4$ superconformal group and during the presentation we will define important classes of operators: primary operators, chiral primary operators and Half-BPS operators.

Let us consider first the four-dimensional conformal group. The generators of this group are $[P_\mu, K_\mu, M_{\mu\nu}, D]$ and its algebra is given by the first six commutators of (B.1) with the ones not listed being zero. Local operators in a conformal field theory are eigenstates of the dilatation operator D and, considering $\mathcal{O}$ to be one of such operators, this means

$$[D,\mathcal{O}]=\Delta\mathcal{O}\,, \qquad (4.13)$$

where Δ is its dimension. Let us compute the dimension of the operators $[P_\mu, \mathcal{O}]$ and $[K_\mu, \mathcal{O}]$:

$$
\begin{aligned}
[D, [P_\mu, \mathcal{O}]] &= -[P_\mu, [\mathcal{O}, D]] - [\mathcal{O}, [D, P_\mu]] \\
&= (\Delta + 1)\, [P_\mu, \mathcal{O}]\,,
\end{aligned}
$$

and

$$
\begin{aligned}
[D, [K_\mu, \mathcal{O}]] &= -[K_\mu, [\mathcal{O}, D]] - [\mathcal{O}, [D, K_\mu]] \\
&= (\Delta - 1)[K_\mu, \mathcal{O}]\,,
\end{aligned}
$$

where in both calculations we have used the conformal algebra and the Jacobi identity.

The results above mean that acting with P_μ on an operator increases its dimension by one and acting with K_μ decreases it by one. Requiring the theory to be unitary implies that there is a lower bound on the dimension of the operators [63]. Since acting with K_μ lowers the dimension of an operator, acting sufficiently many times with K_μ will give zero. By definition, a primary operator is an operator that is annihilated by K_μ and its descendants are the operators obtained acting on it with P_μ.

The $\mathcal{N} = 4$ $d = 4$ superconformal algebra in addition to the generators of the conformal algebra has the generators $[U_i^j, q_{\alpha i}, \bar{q}^j_{\dot{\alpha}}, s^i_\alpha, \bar{s}_{\dot{\alpha} j}]$. Its non-zero commutators and anticommutators are given in (B.1). Using the algebra and a given bosonic operator $\mathcal{O}$ with dimension Δ, one can show, using the same manipulations as the ones before, that $[q, \mathcal{O}]$ has dimension $\Delta + \frac{1}{2}$ and $[s, \mathcal{O}]$ has dimension $\Delta - \frac{1}{2}$. A chiral primary operator is, by definition, an operator that is annihilated by all s^i_α and $\bar{s}_{\dot{\alpha} j}$. Note that a chiral primary operator is also a primary operator, because

$$
0 = \{s^i_\alpha, [\bar{s}_{\dot{\alpha} j}, \mathcal{O}]\} + \{\bar{s}_{\dot{\alpha} j}, [s^i_{\dot{\alpha}}, \mathcal{O}]\} = [\{s^i_\alpha, \bar{s}_{\dot{\alpha} j}\}, \mathcal{O}] = 2i\delta^i_j \sigma^\mu_{\alpha\dot{\alpha}} [K_\mu, \mathcal{O}]\,,
$$

and the descendants of a chiral primary operator are obtained by acting on it with $[P_\mu, q_{\alpha i}, \bar{q}^j_{\dot{\alpha}}]$.

A BPS operator is a chiral primary operator that in addition of being annihilated by all the $(s^i_\alpha, \bar{s}^j_{\dot{\alpha}})$ is also annihilated by some of the sixteen operators $(q_{\alpha i}, \bar{q}^j_{\dot{\alpha}})$. A subset of the BPS operators called Half-BPS is composed of the operators that are annihilated by exactly eight of the supercharges. All the gauge-invariants Half-BPS operators of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills have been classified and before stating the result we will motivate it by studying the implications of the $PSU(2, 2|4)$ superalgebra, our arguments will follow closely [62]. Consider a scalar chiral primary operator $\mathcal{O}'$ and act on it with

$$
[\{q_{\alpha i}, s^{\beta j}\}, \mathcal{O}'] = [\delta^j_i (\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} - 2\delta^\beta_\alpha \delta^j_i D + 4\delta^\beta_\alpha U^j_i, \mathcal{O}']\,,
$$

and note that the commutator $[M_{\mu\nu}, \mathcal{O}'] = 0$ because, by assumption, $\mathcal{O}'$ is a Lorentz scalar. Suppose that for a specific value of i and α the operator $\mathcal{O}'$ is also annihilated by $q_{\alpha i}$, then the left-hand side of the expression above is zero and we have

$$2\delta^{\beta}_{\alpha}[U^{j}_{i}, \mathcal{O}'] = \delta^{\beta}_{\alpha}\delta^{j}_{i}\Delta'\mathcal{O}' \,. \tag{4.14}$$

The generators U^j_i are the generators of the $SU(4)$ algebra which is isomorphic to the $SO(6)$ algebra. The relation among the generators is given by $U^j_i = \frac{i}{2}(\sigma^{IJ})^j{}_i R_{IJ}$ where $(\sigma^{IJ})^j{}_i$ is defined in (A.10) and R_{IJ} are the generators of $SO(6)$. The $SO(6)$ algebra has rank 3, which means that its Cartan subalgebra has three generators and we will consider these three generators to be R_{14}, R_{25} and R_{36}. All the highest weight representations of $SO(6)$ are classified according to their charges under these three generators. As an example, suppose that the operator $\mathcal{O}'$ is highest weight carrying R_{14} and R_{25} charge zero and carrying R_{36} charge J. In this case (4.14) becomes

$$2\delta^{\beta}_{\alpha}[U^{j}_{i}, \mathcal{O}'] = i\delta^{\beta}_{\alpha}(\sigma^{36})^{j}{}_{i}J\mathcal{O}' = \delta^{\beta}_{\alpha}\delta^{j}_{i}\Delta'\mathcal{O}' \,, \tag{4.15}$$

and using the representation of the σ^I_{ij} matrices given in the Appendix A, one has

$$i(\sigma^{36})^{j}{}_{i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Considering $\Delta' = J$, the equation (4.15) is satisfied if $i = j = 1, 2$, which implies that the operator $\mathcal{O}'$ is annihilated by $(q_{\alpha 1}, q_{\alpha 2})$. Analogously, one can show using the anticommutator $\{\bar{q}^{\dot{\alpha} i}, \bar{s}_{\beta j}\}$ that this operator is also annihilated by $(\bar{q}^{\dot{\alpha}3}, \bar{q}^{\dot{\alpha}4})$ because one arrives at an equation similar to (4.14) but with the left-hand side multiplied by minus one. The conclusion is that the operator $\mathcal{O}'$ is annihilated by eight of the supercharges $(q_{\alpha i}, \bar{q}^{j}_{\dot{\alpha}})$ which means that it is a Half-BPS operator.

In general, all gauge-invariant Half-BPS operators of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills are of the form [5, 62]

$$\mathcal{O}' = \xi_{I_1,\ldots,I_n}\mathrm{Tr}\,(\phi^{I_1}\ldots\phi^{I_n})\,, \tag{4.16}$$

where $\xi_{I_1,\ldots,I_n}$ is completely symmetric and traceless in all its indices, ϕ^I are the six scalar fields of $\mathcal{N} = 4$ super-Yang-Mills and Tr means trace over the gauge group. All the descendants of these operators are obtained by acting on it with $[P_\mu, q_{\alpha i}, \bar{q}^{i}_{\dot{\alpha}}]$. In the next section, we will show that using harmonic variables all these operators can be elegantly described.

4.3 Harmonic Superspace

The utilization of harmonic superspace techniques for studying theories of extended ($\mathcal{N} > 1$) supersymmetry was first done by Galperin, Ivanov, Kalitsyn, Ogievetsky and Sokatchev in [25], an excellent introductory book is [26]. The basic idea is instead of considering superfields that depend only on the usual Minkowski superspace variables $[x, \theta, \bar{\theta}]$, considering superfields that depend on these variables plus harmonic variables. Using harmonic superspaces an off-shell description of all $\mathcal{N} = 2$ supersymmetric theories was constructed in [25]. Moreover, an off-shell formulation of $\mathcal{N} = 3$ super-Yang-Mills, that on-shell is equivalent to the $\mathcal{N} = 4$ super-Yang-Mills, is possible using an appropriate harmonic superspace [64].

In the case of the theory of $\mathcal{N} = 4$ super-Yang-Mills, an off-shell formulation is not known. However, it is possible to solve all the on-shell constraints of the unique superfield of this theory with spin not higher than one, the so-called Sohnius superfield [27], using harmonic variables and keeping the $SU(4)$ R-symmetry manifest, as will be explained in detail below. Furthermore, all the Half-BPS operators of $\mathcal{N} = 4$ super-Yang-Mills together with their duals can also be elegantly described in harmonic superspace [28, 29]. The plan of this section is to first introduce the Sohnius superfield along with its constraints, then present the relevant harmonic superspace for solving them.

Consider the usual $\mathcal{N} = 4$ $d = 4$ superspace spanned by the variables $[x^\mu, \theta^i_\alpha, \bar{\theta}_{\dot{\alpha}i}]$. In order to construct a supersymmetric gauge theory using superfields defined in this superspace, it is well-known that the usual derivatives must be replaced by the gauge covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu \,, \quad \mathcal{D}_{\alpha i} = D_{\alpha i} + \mathcal{A}_{\alpha i} \,, \quad \bar{\mathcal{D}}^i_{\dot{\alpha}} = \bar{D}^i_{\dot{\alpha}} + \bar{\mathcal{A}}^i_{\dot{\alpha}} \,, \tag{4.17}$$

where $\mathcal{A}_\mu$, $\mathcal{A}_{\alpha i}$ and $\bar{\mathcal{A}}^i_{\dot{\alpha}}$ are gauge connections superfields taking values in the Lie algebra of the gauge group and $(D_{\alpha j}, \bar{D}^i_{\dot{\alpha}})$ were defined in (4.3). From the gauge covariant derivatives we can define the field-strengths $\mathcal{F}$ as

$$[\mathcal{D}_A \,, \mathcal{D}_B]_{+-} = \mathcal{F}_{AB} \,, \tag{4.18}$$

where A and B denote any of the gauge covariant derivatives indices and the subscript $+-$ means commutator or anticommutator depending on the value of A and B. The field-strengths are also superfields taking values in the Lie algebra of the gauge group and under a gauge transformation they transform as

$$\mathcal{F}_{AB} \, \rightarrow \, e^{i\Lambda} \, \mathcal{F}_{AB} \, e^{-i\Lambda} \,, \tag{4.19}$$

where $\Lambda(x,\theta,\bar{\theta}) = \sum_{M'} T_{M'}\lambda^{M'}(x,\theta,\bar{\theta})$, $T_{M'}$ are the Lie algebra generators of the gauge group and $\lambda^{M'}$ is a set of superfields. One comment about this transformation is that when the gauge group is the abelian $U(1)$, we have

$$\mathcal{F}_{AB} \to e^{i\Lambda_{U(1)}}\,\mathcal{F}_{AB}\,e^{-i\Lambda_{U(1)}} = \mathcal{F}_{AB}\,e^{i\Lambda_{U(1)}}\,e^{-i\Lambda_{U(1)}} = \mathcal{F}_{AB}\,, \tag{4.20}$$

or in other words, all the field-strengths are gauge-invariant in this case.

The gauge covariant derivatives satisfy the generalized Jacobi identities given below

$$(-1)^{(\deg A\,\deg C)}[\mathcal{D}_A\,,\,[\mathcal{D}_B\,,\,\mathcal{D}_C\,]_{+-}\,]_{+-} + (-1)^{(\deg B\,\deg A)}[\mathcal{D}_B\,,\,[\mathcal{D}_C\,,\,\mathcal{D}_A\,]_{+-}\,]_{+-} \tag{4.21}$$
$$+(-1)^{(\deg C\,\deg B)}[\mathcal{D}_C\,,\,[\mathcal{D}_A\,,\,\mathcal{D}_B\,]_{+-}\,]_{+-} = 0\,,$$

where, for example, deg$A$ is equal to 0 if A is a bosonic index and 1 if it is fermionic index. Substituting the definitions of the field-strengths (4.18) in the Jacobi identities above, one arrives at the Bianchi-identities that have to be satisfied by these fields.

Imposing suitable constraints on the field-strengths and using the equations of motion, Sohnius showed in [27] that all the Bianchi-identities can be solved in terms of a superfield $W_{ij}(x,\theta,\bar{\theta})$ related to two of the field-strengths as

$$\mathcal{F}_{\alpha i\beta j} = \epsilon_{\alpha\beta}\,W_{ij}\,, \qquad \mathcal{F}^{ij}_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}\,W^{ij}\,,$$

and with the properties

$$W_{ij} = -W_{ji}\,, \qquad (W_{ij})^\dagger = W^{ij} = \frac{1}{2}\epsilon^{ijkl}W_{kl}\,, \tag{4.22}$$

where † means Hermitian conjugation. In addition, this superfield satisfies the constraints

$$\mathcal{D}_{\alpha i}\,W_{jk} = \mathcal{D}_{\alpha[i}W_{jk]}, \qquad \bar{\mathcal{D}}^i_{\dot{\alpha}}\,W_{jk} = -\frac{2}{3}\delta^i_{[j}\bar{\mathcal{D}}^l_{|\dot{\alpha}|}W_{k]l}\,, \tag{4.23}$$

and our notation here is that the indices inside [ ] are antisymmetrized with an additional factor of half and the indices inside | |, as on the last constraint, not being antisymmetrized. The expansion of W_{ij} in components is schematically of the form

$$W_{jk} = \phi_{jk} + \bar{\theta}_{\dot{\alpha}[j}\bar{\xi}^{\dot{\alpha}}_{k]} + \theta^{\alpha l}\xi^m_\alpha\epsilon_{jklm} + \bar{\theta}_{j\dot{\alpha}}\bar{\theta}_{k\dot{\beta}}\bar{F}^{\dot{\alpha}\dot{\beta}} + \epsilon_{jklm}\theta^{\alpha l}\theta^{\beta m}F_{\alpha\beta} + ... \tag{4.24}$$

where ϕ_{jk} are scalars related to the six scalars ϕ^I of $\mathcal{N}=4$ super-Yang-Mills as

$$\phi_{jk} = (\sigma_I)_{jk}\,\phi^I\,, \tag{4.25}$$

and $(\sigma_I)_{jk}$ the Pauli matrices of $SO(6)$ defined in the Appendix A. Moreover, ξ^k_α and $\bar{\xi}^{\dot{\alpha}}_k$ are the chiral and anti-chiral gluinos, and $F_{\alpha\beta}$ and $\bar{F}^{\dot{\alpha}\dot{\beta}}$ are the self-dual and anti-self-dual field-strengths.

The constraints (4.23) of the superfield W_{ij} can be solved using an appropriate harmonic superspace that will be defined below. Instead of defining the superfields to depend only on the variables $[x, \theta, \bar{\theta}]$ of the $\mathcal{N} = 4$ $d = 4$ Minkowski superspace $\mathcal{M}^{4|16}$, we will study superfields on $\mathcal{M}^{4|16} \times \frac{SU(4)}{S(U(2)\times U(2))}$, which means that the superfields can now depend also on the harmonic variables u parameterizing this additional coset. Note that locally the isotropy group of this new coset is

$$S(U(2) \times U(2)) \sim SU(2) \times SU(2) \times U(1)\,,$$

and the coset is parametrized by $15 - 3 - 3 - 1 = 8$ independent variables. An explicit parametrization is

$$\mathbf{A} = (u^j_{\dot{J}}\,,\, i\,\bar{u}^j_{J'}) \;\in\; SU(4)\,, \tag{4.26}$$

where $[u, \bar{u}]$ are the harmonic variables and $\dot{J} = 1, 2$, $J' = 1', 2'$. In our conventions, under the operation of complex conjugation * these variables transform as

$$(u^j_{\dot{J}})^* = \bar{u}^{\dot{J}}_j\,, \qquad (\bar{u}^j_{J'})^* = u^{J'}_j\,, \tag{4.27}$$

where the variables $[\bar{u}^{\dot{J}}_j, u^{J'}_j]$ parametrize the inverse coset. For the matrix (4.26) to be a matrix of $SU(4)$, it is necessary that the harmonic variables and their complex conjugates satisfy the conditions of unitarity

$$\begin{aligned} \mathbf{A}\,\mathbf{A}^\dagger = \mathbf{1} \;\;&\rightarrow\;\; u^j_{\dot{J}}\bar{u}^{\dot{K}}_j = \delta^{\dot{K}}_{\dot{J}}\,, \quad \bar{u}^j_{J'}u^{K'}_j = \delta^{K'}_{J'}\,, \quad u^j_{\dot{J}}u^{K'}_j = 0, \quad \bar{u}^j_{J'}\bar{u}^{\dot{K}}_j = 0, \\ \mathbf{A}^\dagger\mathbf{A} = \mathbf{1} \;\;&\rightarrow\;\; u^j_{\dot{J}}\,\bar{u}^{\dot{J}}_i + \bar{u}^j_{J'}u^{J'}_i = \delta^j_i\,, \end{aligned} \tag{4.28}$$

and we must also have $\det\mathbf{A} = \mathbf{1}$. Recalling that the determinant of a four-dimensional matrix can be written in a compact notation as

$$\det\mathbf{A} = \epsilon_{ijkl}\,A^i{}_1\,A^j{}_2\,A^k{}_3\,A^l{}_4\,,$$

we conclude

$$\det\mathbf{A} = 1 \;\rightarrow\; (i)^2\,\epsilon_{ijkl}\,u^i_1\,u^j_2\,\bar{u}^k_{1'}\,\bar{u}^l_{2'} = -\frac{1}{4}\,\epsilon_{ijkl}\,uu^{ij}\,\overline{uu}^{kl} = 1\,,$$

where we have used two of the important definitions

$$\begin{aligned} uu^{ij} &= \epsilon^{\dot{J}\dot{K}}\,u^i_{\dot{J}}\,u^j_{\dot{K}}\,, \qquad & \overline{uu}^{ij} &= \epsilon^{J'K'}\,\bar{u}^i_{J'}\,\bar{u}^j_{K'}\,, \\ \overline{uu}_{ij} &= \epsilon_{\dot{J}\dot{K}}\,u^{\dot{J}}_i\,u^{\dot{K}}_j\,, \qquad & uu_{ij} &= \epsilon_{J'K'}\,u^{J'}_i\,u^{K'}_j\,, \end{aligned} \tag{4.29}$$

with the antisymmetric tensors $[\epsilon^{\dot{J}\dot{K}}, \epsilon_{\dot{J}\dot{K}}, \epsilon^{J'K'}, \epsilon_{J'K'}]$ having the same non-zero components of the usual $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ defined in (A.1). Note that the condition of unit determinant is equivalent to

$$uu_{ij} = \frac{1}{2}\,\epsilon_{ijkl}\,uu^{kl}\,, \qquad \overline{uu}_{ij} = \frac{1}{2}\,\epsilon_{ijkl}\,\overline{uu}^{kl}\,, \tag{4.30}$$

because $(uu)_{ij}\,\overline{uu}^{ij} = \epsilon_{J'K'}\,u_i^{J'}\,u_j^{K'}\,\epsilon^{T'M'}\,\bar{u}^i_{T'}\,\bar{u}^j_{M'} = -2$.

The $SU(4)$ transformations act on the indices i and it is possible to construct a set of $SU(4)$ invariant derivatives acting on the harmonic variables. Two of such derivatives are going to be useful for us, they are

$$D_{\dot{j}}^{J'} = u^i_{\dot{j}}\frac{\partial}{\partial \bar{u}^i_{J'}}\,, \qquad D_0 = \frac{1}{2}(u^i_{\dot{j}}\frac{\partial}{\partial u^i_{\dot{j}}} - \bar{u}^i_{J'}\frac{\partial}{\partial \bar{u}^i_{J'}})\,. \tag{4.31}$$

Using D_0, we define the charge of the harmonics variables and their inverses under the subgroup $U(1)$ of the isotropy group as the eigenvalue under the action of this operator

$$D_0 \cdot u = \frac{1}{2}u\,, \qquad D_0 \cdot \bar{u} = -\frac{1}{2}\bar{u}\,, \tag{4.32}$$

which means that u has charge $\frac{1}{2}$ and $\bar{u}$ has charge $-\frac{1}{2}$. The first derivative given in (4.31) acts on the harmonic variables and their inverse as

$$D_{\dot{j}}^{J'} \cdot \bar{u}^j_{K'} = \delta^{J'}_{K'} u^j_{\dot{j}}\,, \quad D_{\dot{j}}^{J'} \cdot \bar{u}^{\dot{K}}_j = -\delta^{\dot{K}}_{\dot{j}}\, u_j^{J'}\,, \quad D_{\dot{j}}^{J'} \cdot u = 0\,. \tag{4.33}$$

After introducing the relevant harmonic superspace and fixing the properties satisfied by the bosonic harmonic variables and their inverses, we will solve the constraints of (4.23) and show that all the Half-BPS operators can be elegantly described using the harmonic variables. Consider first that the gauge group is the abelian $U(1)$. In this case, we already know that the superfield $(W_{U(1)})_{ij}$ is gauge-invariant due to the fact that it transforms in the adjoint representation of the gauge group. The connection part of the gauge covariant derivatives that appears in the constraints (4.23) does not give any contribution when acting on a gauge-invariant object, then the constraints become

$$D_{\alpha i}\,(W_{U(1)})_{jk} = D_{\alpha[i}(W_{U(1)})_{jk]}, \qquad \bar{D}^i_{\dot{\alpha}}\,(W_{U(1)})_{jk} = -\frac{2}{3}\delta^i_{[j}\bar{D}^l_{|\dot{\alpha}|}(W_{U(1)})_{k]l}\,, \tag{4.34}$$

and using the definitions

$$D_{\alpha J'} = \bar{u}^j_{J'}\,D_{\alpha j}\,, \quad D_{\alpha \dot{J}} = u^j_{\dot{J}}\,D_{\alpha j}\,, \quad \bar{D}^{\dot{J}}_{\dot{\alpha}} = \bar{u}^{\dot{J}}_j\,\bar{D}^j_{\dot{\alpha}}\,, \quad \bar{D}^{J'}_{\dot{\alpha}} = u^{J'}_j\,\bar{D}^j_{\dot{\alpha}}\,, \tag{4.35}$$

and $(W_{U(1)})^{(1)} = (uu)^{ij}\,(W_{U(1)})_{ij}$, where the superscript (1) indicates that this superfield has charge 1. One can show that the constraints (4.34) are equivalent to

$$D_{\alpha\dot{J}}\,(W_{U(1)})^{(1)} = \bar{D}^{J'}_{\dot{\alpha}}(W_{U(1)})^{(1)} = 0\,. \tag{4.36}$$

It is easy to derive these relations from the constraints obeyed by the superfield W. Note that

$$D_{\alpha \dot{J}}\,(W_{U(1)})^{(1)} = u^j_{\dot{J}} D_{\alpha j} (uu)^{ik} (W_{U(1)})_{ik} = u^j_{\dot{J}}\,\epsilon^{\dot{P}\dot{S}}\,u^i_{\dot{P}}\,u^k_{\dot{S}}\,D_{\alpha[j}(W_{U(1)})_{ik]} = 0\,,$$

which follows because the harmonic variables are bosonic and the indices $[\dot{J},\dot{P},\dot{S}]$ take only the values 1 or 2. Moreover,

$$\bar{D}^{J'}_{\dot{\alpha}}\,(W_{U(1)})^{(1)} = u^{J'}_j\,\bar{D}^j_{\dot{\alpha}}\,(uu)^{ik}\,(W_{U(1)})_{ik} = -\frac{2}{3}\,u^{J'}_j\,(uu)^{ik}\delta^j_{[i}\bar{D}^l_{|\dot{\alpha}|}(W_{U(1)})_{k]l} = 0\,,$$

as a consequence of $u^{J'}_j (uu)^{jm} = 0$.

This example of the abelian $U(1)$ gauge group shows why it is useful to introduce the harmonic variables. They allow the projection of the $SU(4)$ indices on the isotropy group $S(U(2) \times U(2))$ indices without breaking the $SU(4)$, or in other words, using harmonic variables the $SU(4)$ is kept manifest and the constraints (4.34) can be elegantly solved. Let us now not restrict the gauge group to be $U(1)$ and define the gauge-invariant quantity

$$W^{(N)}(u,x,\theta,\bar{\theta}) = (uu)^{i_1 j_1} \dots (uu)^{i_N j_N}\,\mathrm{Tr}\,[\,W_{i_1 j_1}(x,\theta)\dots W_{i_N j_N}(x,\theta)\,]\,, \qquad (4.37)$$

where the superscript (N) indicates that this superfield carries $+N$ charge. Using similar arguments to the ones above, it is easy to show that

$$D_{\alpha \dot{J}}\,W^{(N)} = \bar{D}^{J'}_{\dot{\alpha}}\,W^{(N)} = 0\ , \qquad (4.38)$$

and in what follows a superfield that satisfies these constraints will be called G-analytic. Furthermore, W^N is independent of $\bar{u}$, which implies that

$$\Big(u^j_{\dot{J}}\frac{\partial}{\partial \bar{u}^j_{J'}}\Big) W^{(N)} = 0\,, \qquad (4.39)$$

and a superfield that satisfies these constraints will be called H-analytic. A superfield that is both G-analytic and H-analytic will be called an analytic superfield for short.

The superfield $W^{(N)}$ describes a gauge-invariant Half-BPS operator involving N super-Yang-Mills fields. In order to see this, consider the expansion of W_{ij} given in (4.24) with $\theta = \bar{\theta} = 0$, it implies

$$\begin{aligned} W^{(N)}|_{\theta=\bar{\theta}=0} &= (uu)^{i_1 j_1} \dots (uu)^{i_N j_N}\,\mathrm{Tr}\,[\,\phi_{i_1 j_1} \dots \phi_{i_N j_N}\,] \\ &= (uu)^{i_1 j_1}(\sigma_{I_1})_{i_1 j_1} \dots (uu)^{i_N j_N}(\sigma_{I_N})_{i_N j_N}\,\mathrm{Tr}\,[\,\phi^{I_1} \dots \phi^{I_N}\,]\,, \end{aligned} \qquad (4.40)$$

and noting that for any I_m and I_n

$$(uu)^{i_m j_m}(\sigma_{I_m})_{i_m j_m}(uu)^{i_n j_n}(\sigma_{I_n})_{i_n j_n}\delta^{I_m I_n} = -2\epsilon_{i_m j_m i_n j_n}(uu)^{i_m j_m}(uu)^{i_n j_n} = 0\,, \quad (4.41)$$

which means that $(uu)^{ij}(\sigma_I)_{ij}$ is a null vector and the tensor

$$\xi'_{I_1 \ldots I_N} = (uu)^{i_1 j_1}(\sigma_{I_1})_{i_1 j_1} \ldots (uu)^{i_N j_N}(\sigma_{I_N})_{i_N j_N} ,$$

is completely symmetric and traceless in all its indices. Recalling that all the Half-BPS operators are of the form (4.16) and that the tensor ξ that appears in (4.16) has the same properties of the tensor ξ' defined above, we conclude that $W^{(N)}$ describes a Half-BPS operator. So, Half-BPS operators constructed from N super-Yang-Mills fields are described by analytic superfields of $+N$ $U(1)$ charge.

To construct the duals $T^{(4-N)}(u, \bar{u}, x, \theta, \bar{\theta})$ to these analytic superfields consider the superspace integral

$$\int d^4x \int du \int d^8(u\theta) W^{(N)}(u, x, \theta, \bar{\theta}) T^{(4-N)}(u, \bar{u}, x, \theta, \bar{\theta}) , \tag{4.42}$$

where $\int d^8(u\theta) = D'^4 \bar{D}^4$ with

$$D'^4 = D^{\alpha J'} D^{K'}_{\alpha} D^{\beta}_{J'} D_{\beta K'} , \quad \bar{D}^4 = \bar{D}^{\dot{J}}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}\dot{K}} \bar{D}_{\dot{\beta}\dot{J}} \bar{D}^{\dot{\beta}}_{\dot{K}} , \tag{4.43}$$

and the derivatives above were defined in (4.35). One comment is that both the $SU(2)$ indices $[\dot{J}, J']$ can be raised and lowered with the ϵ symbols, two examples are $\bar{u}_{j\dot{J}} = \epsilon_{\dot{J}\dot{K}} \bar{u}^{\dot{K}}_j$ and $\bar{u}^{J'j} = \epsilon^{J'K'} \bar{u}^j_{K'}$ and this was used in the expression above.

The $\int du$ denotes an integral over the compact space $\frac{SU(4)}{S(U(2)\times U(2))}$. Explicit examples and a more complete explanation on how to compute this integral will be given in the beginning of the section 4.6. The important information for us here is that du is the invariant Haar measure over the group $SU(4)$, which means that the result of the integration is necessary a $SU(4)$ scalar.

For the integral to be non-vanishing, $T^{(4-N)}$ must be a superfield of $U(1)$ charge $(4-N)$ and this is the meaning of the superscript. This follows because the integrand must have total charge equal to zero given that the result of the integral over the compact space is a $SU(4)$ scalar. We know that $W^{(N)}$ carries charge $+N$ and it is easy to see from the definition of $\int d^8(u\theta)$ that it carries charge -4 which implies that $T^{(4-N)}$ must carry charge $(4-N)$. In addition, for the integral to be supersymmetric $T^{(4-N)}$ must be a G-analytic superfield but not necessarily H-analytic. One more comment about the superspace integral is that for a given $W^{(N)}$ and $T^{(4-N)}$ it gives a number, and this is one of the motivations for calling T the dual of W.

Note that T is defined up to a gauge transformation because the integral is invariant under the variation

$$\delta T = (u^i_{\dot{J}} \frac{\partial}{\partial \bar{u}^i_{J'}}) \Lambda^{\dot{J}}_{J'} = D^{J'}_{\dot{J}} \Lambda^{\dot{J}}_{J'} , \tag{4.44}$$

for any G-analytic superfield $\Lambda^{\dot{j}}_{J'}$. Let us prove that the integral is in fact invariant, we will follow [31],

$$\int d^4x \int du \int d^8(u\theta) W^{(N)} \delta T^{(4-N)} = \int d^4x \int du \int d^8(u\theta) W^{(N)} D^{J'}_{\dot{j}} [\, \Lambda^{\dot{j}}_{J'} \,]$$
$$= \int d^4x \int du \int d^8(u\theta) D^{J'}_{\dot{j}} \, [\, W^{(N)} \Lambda^{\dot{j}}_{J'} \,]$$
$$= \int d^4x \int du D^{J'}_{\dot{j}} \, [\, \int d^8(u\theta) \, W^{(N)} \, \Lambda^{\dot{j}}_{J'} \,] = 0 \, , \qquad ,$$

where on the second line we have used that $W^{(N)}$ is H-analytic and on the third line we have used that when $D^{J'}_{\dot{j}}$ acts on $\int d^8(u\theta)$ it gives either terms proportional to $D_{\alpha\dot{j}}$ and $\bar{D}^{J'}_{\dot{\alpha}}$ which annihilate the G-analytic superfields W and Λ, or total x derivatives. Finally, we have used that performing the du integration of a total derivative $D^{J'}_{\dot{j}}$ is zero.

4.4 The vertex operators

After introducing the concept of harmonic superspace in the previous section, in this section, we will present and prove the main result of this thesis, which is the computation of the cohomology at +2 ghost number of the operator $Q_{\frac{1}{2}}$ restricted to states in the cohomology of the operator $Q_{-\frac{1}{2}}$. As argued before, this is equivalent to computing the cohomology of the complete BRST operator of the pure spinor formalism near the boundary of AdS.

The general BRST-closed supergravity vertex operator dual to a Half-BPS operator constructed from N super-Yang-Mills fields is

$$V_N = z^{2-N} \int du \, [\, (yuu)^{N-1} \, \Omega^{(0)} T + 8(N-1)(yuu)^{N-2} \, \Omega^{(1)} T \tag{4.45}$$
$$+ \, 8^2(N-1)(N-2)(yuu)^{N-3} \, \Omega^{(2)} T + 8^3(N-1)(N-2)(N-3)(yuu)^{N-4} \, \Omega^{(3)} T$$
$$+ \, 8^4(N-1)(N-2)(N-3)(N-4)(yuu)^{N-5} \, \Omega^{(4)} T \,] \, ,$$

where T is the dual superfield of (4.42) with the superscript $(4-N)$ omitted, $(yuu) = (y^{ij} uu_{ij})$, and

$$\Omega^{(0)} = \frac{1}{16} (\lambda\tilde{\lambda})^{-2} (\lambda\gamma^M \tilde{D})(\lambda\gamma^N \tilde{D})(\lambda\gamma^P \tilde{D})(\lambda\gamma^S \tilde{D})(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda}) v^T \tag{4.46}$$
$$+ \frac{1}{2} \, z^{-\frac{1}{2}} (\lambda\tilde{\lambda})^{-2} (\lambda\gamma^M \tilde{D})(\lambda\gamma^N \tilde{D})(\lambda\gamma^P \tilde{D})(r\gamma_{PNM}\tilde{\lambda}) \, ,$$

$$\Omega^{(1)} = \frac{1}{4} (\lambda\tilde{\lambda})^{-2} (\lambda\gamma^M \hat{\psi})(\lambda\gamma^N \tilde{D})(\lambda\gamma^P \tilde{D})(\lambda\gamma^S \tilde{D})(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda}) v^T \tag{4.47}$$

$$+\frac{3}{2}z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D})(r\gamma_{PNM}\tilde{\lambda}),$$

$$\Omega^{(2)}=\frac{3}{8}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^P\tilde{D})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda})v^T \tag{4.48}$$

$$+\frac{3}{2}z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^P\tilde{D})(r\gamma_{PNM}\tilde{\lambda}),$$

$$\Omega^{(3)}=\frac{1}{4}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^P\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda})v^T \tag{4.49}$$

$$+\frac{1}{2}z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^P\hat{\psi})(r\gamma_{PNM}\tilde{\lambda}),$$

$$\Omega^{(4)}=\frac{1}{16}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^P\hat{\psi})(\lambda\gamma^S\hat{\psi})(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda})v^T. \tag{4.50}$$

In the above formulas, the vectors v_T and $\bar{v}_T$ are null vectors with non-zero components defined by $v_{J+3}=-\frac{1}{4}\sigma_J^{jk}(uu)_{jk}$ and $\bar{v}_{J+3}=-\frac{1}{4}\sigma_J^{jk}(\overline{uu})_{jk}$ where σ_J^{jk} are the $SO(6)$ Pauli matrices defined in the Appendix A, and $\tilde{D}=\bar{v}_T(\gamma^T D)$.

We will now prove that the vertex operator V_N is BRST-closed and during the proof several features of the result will be explained. Firstly, note that the terms of $Q_{\frac{1}{2}}$ of (4.1) reproduced below,

$$z^{\frac{1}{2}}[y_{ij}(\lambda\gamma^{ij}\hat{\psi})(2z\frac{\partial}{\partial z}+y^{kl}\frac{\partial}{\partial y^{kl}}-\lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}})],$$

annihilate V_N because when they act on the terms of V_N independent of r, they give

$$(2z\frac{\partial}{\partial z})\cdot(V_N)_{\mathrm{no}\,r}=2(2-N)(V_N)_{\mathrm{no}\,r},\quad y^{kl}\frac{\partial}{\partial y^{kl}}\cdot(V_N)_{\mathrm{no}\,r}=2(N-1)(V_N)_{\mathrm{no}\,r},$$
$$-\lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}}\cdot(V_N)_{\mathrm{no}\,r}=-2(V_N)_{\mathrm{no}\,r},$$

and the sum of all these contributions is zero. Similarly, when they act on the terms of V_N that depend on r, they give

$$(2z\frac{\partial}{\partial z})\cdot(V_N)_r=2(2-N-\frac{1}{2})(V_N)_r,\quad y^{kl}\frac{\partial}{\partial y^{kl}}\cdot(V_N)_r=2(N-1)(V_N)_r,$$
$$-\lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}}\cdot(V_N)_r=-(V_N)_r,$$

and again one gets zero. In order to V_N to be also annihilated by the remaining terms of $Q_{\frac{1}{2}}$ given in (4.1), the following equations must be satisfied

$$\begin{aligned}
(z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(0)}T &= 0\,, \\
P_N(1)\,[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(0)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(1)}T] &= 0\,, \\
P_N(2)\,[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(1)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(2)}T] &= 0\,, \\
P_N(3)\,[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(2)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(3)}T] &= 0\,, \\
P_N(4)\,[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(3)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(4)}T] &= 0\,, \\
P_N(5)\,z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(4)}T &= 0\,,
\end{aligned} \tag{4.51}$$

where the factors of $(\lambda\gamma^M\hat{\psi})v_M$ above come from the BRST variation of (yuu) and $P_N(n)$ is defined as

$$P_N(n) = \prod_{m=1}^{n}(N-m)\,.$$

Let us understand this set of equations and make some comments about how the expression for the vertex operator (4.45) should be understood. When $N=1$, for example, only the first equation of (4.51) has to be satisfied and the vertex operator only has the term proportional to $\Omega^{(0)}$, when $N=2$ only the first and the second equations of (4.51) have to be satisfied and the vertex operator only has the terms proportional to $\Omega^{(0)}$ and $\Omega^{(1)}$, etc. When $N<4$ the vertex operator does not depend on $\Omega^{(3)}$ and $\Omega^{(4)}$ and there is a gauge such that $V_{N<4}$ is independent of the non-minimal pure spinor variables $\tilde{\lambda}$ and r. In this gauge, $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$ are replaced with

$$\begin{aligned}
\Omega^{(0)}_{\text{min}} &= -\frac{1}{4}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P\,, \\
\Omega^{(1)}_{\text{min}} &= -(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P + 24(\lambda\gamma^M\hat{\psi})\bar{v}_M(\lambda\gamma^\mu\tilde{D})\frac{\partial}{\partial x^\mu}\,, \\
\Omega^{(2)}_{\text{min}} &= -\frac{3}{2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\tilde{D}\gamma_{MNP}\tilde{D})v^P + 48(\lambda\gamma^M\hat{\psi})\bar{v}_M(\lambda\gamma^\mu\hat{\psi})\frac{\partial}{\partial x^\mu}\,,
\end{aligned} \tag{4.52}$$

however, such a gauge seems not to be possible for $N>3$ because for the fourth equation of (4.51) to be satisfied it would require

$$\Omega^{(3)}_{\text{min}} = -(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\hat{\psi}\gamma_M\gamma_N\gamma_P\tilde{D})v^P + \text{terms with}\,\frac{\partial}{\partial x^\mu}\,,$$

and this expression is not a function of $\lambda\gamma^M\hat{\psi}$, which means that it is not a state in the cohomology of $Q_{-\frac{1}{2}}$, so $\Omega^{(3)}$ and $\Omega^{(4)}$ require non-minimal variables. The final comment about this point is that we know that there always exist a gauge in which the vertex operator is independent of the non-minimal variables, see (1.1), and the fact that $\Omega^{(3)}$ and $\Omega^{(4)}$ require non-minimal variables seems to be a consequence that we need these variables in order to express the result using harmonic variables.

We will proceed to prove that all the equations of (4.51) are satisfied for a generic N. Firstly, note that

$$\begin{aligned} \lambda^{\bar{\alpha}} D_{\bar{\alpha}} &= \bar{v}_M v_N (\lambda\gamma^M\gamma^N D) + v_M \bar{v}_N (\lambda\gamma^M\gamma^N D) \\ &\equiv \lambda D_1 + \lambda D_2 \,, \end{aligned} \tag{4.53}$$

where we have used (4.28). This implies that we can rewrite the first equation of (4.51) as

$$(z^{\frac{1}{2}}\,\lambda D_2 + \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}})\,\Omega^{(0)} T + z^{\frac{1}{2}} [\,\lambda D_1\,,\,\Omega^{(0)}\,]\,T = 0\,, \tag{4.54}$$

where $[\,,]$ means commutator and $(\lambda D_1)\cdot T = 0$ since T is G-analytic. It is easy to see that $[\,\lambda D_1\,,\,\Omega^{(0)}\,] = 0$, because

$$\{\,\lambda D_1\,,\,(\lambda\gamma^N \tilde{D})\,\} = -2(\lambda\gamma^N\gamma^\mu\gamma^S\lambda)\bar{v}_S \frac{\partial}{\partial x^\mu} = 0\,, \tag{4.55}$$

and this vanishes given that

$$(\gamma^N\gamma^\mu\gamma^S)_{\bar{\alpha}\bar{\beta}} = (\gamma^{N\mu S})_{\bar{\alpha}\bar{\beta}} + \text{terms with}\,\eta\,\gamma\,,$$

and λ is a pure spinor. This kind of calculation involving commutators will appear several times in the next section in the study of the gauge invariance of the vertex operators, thus we will illustrate with (4.55) how these calculations are performed. Using the definitions of (4.3), it is straightforward to see that

$$\{D_{\alpha i}\,,\,\bar{D}^j_{\dot{\alpha}}\} = -2i\delta^j_i(\sigma^\mu)_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}\,,$$

and using the ansatz for the chiral gamma matrices of (2.66), we have in four-dimensional notation

$$\begin{aligned} \lambda D_1 &= \lambda^{\alpha i}\overline{uu}_{ij} uu^{jk} D_{\alpha k} + \bar{\lambda}_{\dot{\alpha} i}\overline{uu}^{ij} uu_{jk}\bar{D}^{k\dot{\alpha}}\,, \\ (\lambda\gamma^\mu\tilde{D}) &= -\lambda^{\alpha i} i\sigma^\mu_{\alpha\dot{\alpha}}\overline{uu}_{ij}\bar{D}^{\dot{\alpha} j} + \bar{\lambda}_{\dot{\alpha} i} i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\overline{uu}^{ij} D_{\alpha j}\,, \\ (\lambda\gamma^{I+3}\tilde{D}) &= \lambda^{\alpha i}\sigma^I_{ij}\overline{uu}^{jk} D_{\alpha k} + \bar{\lambda}_{\dot{\alpha} i}\sigma^{Iij}\overline{uu}_{jk}\bar{D}^{k\dot{\alpha}}\,, \end{aligned}$$

and from the expressions above, one can compute the anticommutators

$$\begin{gathered} \{\,\lambda D_1\,,\,(\lambda\gamma^\nu\tilde{D})\,\} = \\ -2i\lambda^{\alpha i}\overline{uu}_{il} i\sigma^\nu_{\beta\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\lambda^{\beta l}\frac{\partial}{\partial x^\mu} + 2i\bar{\lambda}_{\dot{\alpha} i}\overline{uu}^{im} i\bar{\sigma}^{\nu\dot{\beta}\beta}\epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\lambda}_{\dot{\beta} m}\frac{\partial}{\partial x^\mu}\,, \\ \{\,\lambda D_1\,,\,(\lambda\gamma^{I+3}\tilde{D})\,\} = \\ 2i\lambda^{\alpha i}\overline{uu}_{in}\bar{\lambda}_{\dot{\beta} m}\sigma^{Imn}\epsilon^{\dot{\beta}\dot{\gamma}}\sigma^\mu_{\alpha\dot{\gamma}}\frac{\partial}{\partial x^\mu} + 2i\bar{\lambda}_{\dot{\beta} i}\overline{uu}^{ij}\lambda^{\alpha l}\sigma^I_{lj}\epsilon^{\dot{\beta}\dot{\gamma}}\sigma^\mu_{\alpha\dot{\gamma}}\frac{\partial}{\partial x^\mu}\,, \end{gathered}$$

and rewriting the results above in ten-dimensional notation, one gets the right-hand side of (4.55). After showing that the commutator term of (4.54) is zero, it is left to show that

$$(z^{\frac{1}{2}}\lambda D_2 + \widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}})\Omega^{(0)}T = 0\,. \tag{4.56}$$

The proof that this equation is satisfied uses several times the Fierz identity,

$$(\gamma^M)_{\bar{\alpha}(\bar{\beta}}(\gamma_M)_{\bar{\gamma}\bar{\delta})} = 0\,, \tag{4.57}$$

where ( ) means symmetrization of the indices. In fact, every time the words "Fierz identity" appear in the rest of the thesis we mean manipulations using the identity above. It is explicitly

$$(\gamma^M)_{\bar{\alpha}\bar{\beta}}(\gamma_M)_{\bar{\gamma}\bar{\delta}} + (\gamma^M)_{\bar{\alpha}\bar{\gamma}}(\gamma_M)_{\bar{\delta}\bar{\beta}} + (\gamma^M)_{\bar{\alpha}\bar{\delta}}(\gamma_M)_{\bar{\beta}\bar{\gamma}} = 0\,,$$

and an important property of pure spinors follows by multiplying this expression by $\lambda^{\bar{\alpha}}$ and $\lambda^{\bar{\beta}}$, with the conclusion

$$(\lambda\gamma^M)_{\bar{\gamma}}(\lambda\gamma_M)_{\bar{\delta}} = 0\,. \tag{4.58}$$

Another important result for the proof is

$$\widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}} \cdot (\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)(r\gamma_{PNM}\tilde{\lambda}) = 0\,, \tag{4.59}$$

where X, Y and Z are any fermionic spinors. This follows because

$$\begin{aligned}
\widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}} \cdot\ & (\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)(r\gamma_{PNM}\tilde{\lambda}) \\
&= -2(\lambda\tilde{\lambda})^{-3}(\lambda r)(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)(r\gamma_P\gamma_N\gamma_M\tilde{\lambda}) \\
&\quad + (\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)(r\gamma_P\gamma_N\gamma_M r) \\
&= 4(\lambda\tilde{\lambda})^{-2}(\lambda r)(\lambda\gamma^M X)(\lambda\gamma^P Z)(r\gamma_P\gamma_M Y) \\
&\quad -4(\lambda\tilde{\lambda})^{-2}(\lambda r)(\lambda\gamma^M X)(\lambda\gamma^P Z)(r\gamma_P\gamma_M Y) = 0\,,
\end{aligned}$$

where we have used the Fierz identity and that $(\lambda r)\,(\lambda r) = 0$, since r is a fermionic spinor. We are now in a position to show that

$$\widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}} \cdot \Omega^{(0)} = -\frac{1}{2}(\lambda\tilde{\lambda})^{-2}(\lambda D_2)(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D})(r\gamma_{PNM}\tilde{\lambda})\,, \tag{4.60}$$

for the $\Omega^{(0)}$ given in (4.46). This result is a particular case of the general formula

$$\widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}} \cdot (\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)(\lambda\gamma^S W)(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda})v^T \tag{4.61}$$

$$= 2(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)(\lambda\gamma^S W)v_S\,(r\gamma_{PNM}\tilde{\lambda})$$

$$
\begin{aligned}
&- 2(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)(\lambda\gamma^P Z)v_P\,(\lambda\gamma^S W)(r\gamma_{SNM}\tilde{\lambda})\\
&+ 2(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)(\lambda\gamma^N Y)v_N\,(\lambda\gamma^P Z)(\lambda\gamma^S W)(r\gamma_{SPM}\tilde{\lambda})\\
&- 2(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M X)v_M(\lambda\gamma^N Y)(\lambda\gamma^P Z)(\lambda\gamma^S W)\,(r\gamma_{SPN}\tilde{\lambda})\,,
\end{aligned}
$$

which is valid for any fermionic spinors X, Y, Z and W and can be easily proved by manipulating the result of the application of $\widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}$ using Fierz identity.

Note that acting on the r dependent term of $\Omega^{(0)}$ of (4.46) with $z^{\frac{1}{2}}\lambda D_2$ we get the result (4.60) multiplied by minus one, in other words, both terms cancel precisely in (4.56). For the remaining part of the proof, note that the part of $\Omega^{(0)}$ independent of r can be rewritten due to the Fierz identity as

$$
\begin{aligned}
\Omega^{(0)}_{\text{no}\,r} = &-\frac{1}{4}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P\\
&-(\lambda\tilde{\lambda})^{-1}(\lambda D_2)(\lambda\gamma^S\tilde{D})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{PS}\tilde{D})\,,
\end{aligned}
\tag{4.62}
$$

and one can show that $(\lambda D_2)\cdot\Omega^{(0)}_{\text{no}\,r} = 0$. It is easy to see that the second term on the right-hand side is annihilated because $(\lambda D_2)\cdot(\lambda D_2) = 0$, and that the first term is also annihilated will be shown in great detail in the next section. This completes the proof of (4.54).

We will proceed to prove that the second equation of (4.51) is satisfied. This equation can be rewritten in the form

$$
z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(0)}T + (z^{\frac{1}{2}}\lambda D_2 + \widetilde{w}^{\bar{\alpha}} r_{\bar{\alpha}})\Omega^{(1)}T + z^{\frac{1}{2}}[\,\lambda D_1\,,\,\Omega^{(1)}\,]T = 0\,, \tag{4.63}
$$

and it is straightforward to see that $[\,\lambda D_1\,,\,\Omega^{(1)}\,] = 0$ because of (4.55), which means that this term does not give any contribution to the equation above. Consider the term of $\Omega^{(1)}$ given in (4.47) that is independent of r, using the Fierz identity this term is equal to

$$
\begin{aligned}
\Omega^{(1)}_{\text{no}\,r} = &-(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{D}\gamma_{MSP}\tilde{D})v^P\\
&-(\lambda\tilde{\lambda})^{-1}(\lambda\gamma_M\hat{\psi})v^M(\lambda\gamma^S\tilde{D})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{PS}\tilde{D})\\
&-3(\lambda\tilde{\lambda})^{-1}(\lambda D_2)(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D})\,,
\end{aligned}
$$

and for the equation (4.63) to be satisfied, we must have

$$
(\lambda\gamma^M\hat{\psi})v_M\,\Omega^{(0)}_{\text{no}\,r} + (\lambda D_2)\,\Omega^{(1)}_{\text{no}\,r} = 0\,.
$$

Substituting the expression for both $\Omega^{(1)}_{\text{no}\,r}$ given above and $\Omega^{(0)}_{\text{no}\,r}$ of (4.62), many terms trivially cancel and it is left to show that

$$
\begin{aligned}
&-\frac{1}{4}(\lambda\gamma^T\hat{\psi})v_T(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P\\
&-(\lambda D_2)(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{D}\gamma_{MSP}\tilde{D})v^P = 0\,.
\end{aligned}
$$

This equation is in fact satisfied and we will postpone the proof to the next section. Moreover, one can show that

$$\widetilde{w}^{\bar\alpha} r_{\bar\alpha}\cdot\Omega^{(1)} = -\frac{1}{2}(\lambda\tilde\lambda)^{-2}(\lambda\gamma_M\hat\psi)v^M(\lambda\gamma^N\tilde D)(\lambda\gamma^P\tilde D)(\lambda\gamma^S\tilde D)(r\gamma_{SPN}\tilde\lambda) \\ -\frac{3}{2}(\lambda\tilde\lambda)^{-2}(\lambda D_2)(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\tilde D)(\lambda\gamma^P\tilde D)(r\gamma_{PNM}\tilde\lambda)\,, \tag{4.64}$$

where we have used (4.59) and (4.61). Note that this variation cancels with the action of λD_2 on the r dependent term of $\Omega^{(1)}$ given in (4.47) and with the action of $(\lambda\gamma^M\hat\psi)v_M$ on the r dependent term of $\Omega^{(0)}$ given in (4.46). This completes the proof that the second equation of (4.51) is satisfied. Following the same steps, it is possible to show that all the equations of (4.51) are satisfied, which means that the vertex operators V_N are BRST-closed. The last comment of this section is that although it may seem surprising that $\Omega^{(4)}$ given in (4.50) does not depend on r, this is a consequence of

$$\widetilde{w}^{\bar\alpha} r_{\bar\alpha}\cdot\Omega^{(4)} = -\frac{1}{2}(\lambda\tilde\lambda)^{-2}(\lambda\gamma_S\hat\psi)v^S(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\lambda\gamma^P\hat\psi)(r\gamma_{PNM}\tilde\lambda)\,, \tag{4.65}$$

and this variation cancels precisely with the action of $(\lambda\gamma^M\hat\psi)v_M$ on the r dependent term of $\Omega^{(3)}$, implying that the fifth equation of (4.51) is satisfied.

4.4.1 Gauge invariance

As explained in the section 4.3, the dual superfields $T^{(4-N)}$ are defined up to the gauge transformation

$$\delta T = (u\frac{\partial}{\partial\bar u})_j^{J'}\Lambda^j_{J'} = D_j^{J'}\Lambda^j_{J'}\,,$$

where $\Lambda^j_{J'}$ is any G-analytic superfield of $U(1)$ charge $(3-N)$. The BRST-closed vertex operator V_N of (4.45) depend on T and for the result of V_N to be consistent it must change by a BRST-trivial quantity under a gauge transformation of T. This means that $\delta V_N = Q_{\frac{1}{2}}\cdot\Sigma_N$ for some Σ_N when T changes by a gauge transformation. It is possible to show that in fact this is the case. For a generic value of N, we have

$$\Sigma_N = z^{2-\frac{1}{2}-N}\int du[(yuu)^{N-1}(A^{(0)})_j^{J'}\Lambda^j_{J'} + 8(N-1)(yuu)^{N-2}(A^{(1)})_j^{J'}\Lambda^j_{J'} \\ +8^2(N-1)(N-2)(yuu)^{N-3}(A^{(2)})_j^{J'}\Lambda^j_{J'}\,, \tag{4.66}$$

where

$$(A^{(0)})_j^{J'} = 3\,(\lambda\tilde\lambda)^{-1}\,(\lambda\gamma^M\tilde D)(\lambda\gamma^S\tilde D)(\tilde\lambda\gamma_{SM}\tilde D_j^{J'}) \\ +3\,(\lambda\tilde\lambda)^{-1}\,\{\,(\lambda\gamma^M\tilde D_j^{J'})\,,(\lambda\gamma^N\tilde D)\,\}(\tilde\lambda\gamma_{NM}\tilde D)\,, \tag{4.67}$$

$$(A^{(1)})_j^{J'} = 6\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D}_j^{J'}) \qquad (4.68)$$
$$+24\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma_N\hat{\psi})\,\bar{v}_M\,(\lambda\,\gamma_j^{J'}\gamma^\mu\gamma^M\gamma^N\tilde{\lambda})\frac{\partial}{\partial x^\mu}\,,$$

$$(A^{(2)})_j^{J'} = 3\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\hat{\psi})(\tilde{\lambda}\gamma_{SM}\tilde{D}_j^{J'})\,, \qquad (4.69)$$

and $(\tilde{D}_j^{J'})_{\bar{\alpha}} = D_j^{J'}\cdot(\tilde{D}_{\bar{\alpha}})$, $\gamma_j^{J'} = D_j^{J'}\cdot(\bar{v}_M\,\gamma^M)$ and $\{\,,\}$ means anticommutator.

As previously explained, when $N<4$ there is a gauge where the vertex operators $V_{N<4}$ do not depend on the non-minimal pure spinor variables. In this gauge, $\Sigma_{N<4}$ also do not depend on these variables and they have the same form of (4.66) but with $(A^{(0)})_j^{J'}$, $(A^{(1)})_j^{J'}$ and $(A^{(2)})_j^{J'}$ replaced with

$$(A^{(0)}_{\min})_j^{J'} = 4(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_N\tilde{D}_j^{J'})\,,$$

$$(A^{(1)}_{\min})_j^{J'} = 6(\lambda\gamma^N\hat{\psi})(\tilde{D}\gamma_N\tilde{D}_j^{J'})\,,$$

$$(A^{(2)}_{\min})_j^{J'} = 0\,.$$

In what follows we will consider a generic N and prove that Σ_N has the form (4.66). The first step of the proof is the substitution of δT in the expression for the vertex operator V_N of (4.45) to get δV_N. After an integration by parts, one has

$$\delta V_N = z^{2-N}\int du\,[\,(yuu)^{N-1}\,(-D_j^{J'}\Omega^{(0)})\,\Lambda^{\dot{j}}_{J'} + 8(N-1)(yuu)^{N-2}\,(-D_j^{J'}\Omega^{(1)})\,\Lambda^{\dot{j}}_{J'}$$
$$+8^2(N-1)(N-2)(yuu)^{N-3}\,(-D_j^{J'}\Omega^{(2)})\,\Lambda^{\dot{j}}_{J'}$$
$$+8^3(N-1)(N-2)(N-3)(yuu)^{N-4}\,(-D_j^{J'}\Omega^{(3)})\,\Lambda^{\dot{j}}_{J'}]\,,$$

without an $\Omega^{(4)}$ term because $(-D_j^{J'}\Omega^{(4)})=0$. From this expression, we conclude that in order to construct Σ_N satisfying $\delta V_N = Q_{\frac{1}{2}}\cdot\Sigma_N$, we have to solve the following equations

$$(\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + z^{-\frac{1}{2}}\widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,(A^{(0)})_j^{J'}\,\Lambda^{\dot{j}}_{J'} = (-D_j^{J'}\Omega^{(0)})\,\Lambda^{\dot{j}}_{J'}\,, \qquad (4.70)$$
$$(\lambda\gamma^M\hat{\psi})\,v_M\,(A^{(0)})_j^{J'}\,\Lambda^{\dot{j}}_{J'} + (\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + z^{-\frac{1}{2}}\widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,(A^{(1)})_j^{J'}\,\Lambda^{\dot{j}}_{J'} = (-D_j^{J'}\Omega^{(1)})\,\Lambda^{\dot{j}}_{J'}\,,$$
$$(\lambda\gamma^M\hat{\psi})\,v_M\,(A^{(1)})_j^{J'}\,\Lambda^{\dot{j}}_{J'} + (\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + z^{-\frac{1}{2}}\widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,(A^{(2)})_j^{J'}\,\Lambda^{\dot{j}}_{J'} = (-D_j^{J'}\Omega^{(2)})\,\Lambda^{\dot{j}}_{J'}\,,$$
$$(\lambda\gamma^M\hat{\psi})\,v_M\,(A^{(2)})_j^{J'}\,\Lambda^{\dot{j}}_{J'} = (-D_j^{J'}\Omega^{(3)})\,\Lambda^{\dot{j}}_{J'}\,,$$

where again the factors of $(\lambda\gamma^M\hat{\psi})\,v_M$ above come from the BRST variation of (yuu). These equations are the correct ones imposing the additional requirement that

$$y_{ij}(\lambda\gamma^{ij}\hat{\psi})(2z\frac{\partial}{\partial z} + y^{kl}\frac{\partial}{\partial y^{kl}} - \lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}})\cdot\Sigma_N = 0\,,$$

which is satisfied for the Σ_N given in (4.66). The verification of this result is straightforward and the details will be omitted. The proof that $(A^{(0)})_j^{\ J'}$ given in (4.67) satisfies the first equation of (4.70) will be splited in two steps, we will first consider the zero-momentum case (i.e. setting all the anticommutators to zero) and after the general case will be considered. At zero-momentum and using the $\Omega^{(0)}$ of (4.46), it is easy to see that

$$(-D_j^{J'}\Omega^{(0)})\,\Lambda_{J'}^{j} = -\frac{1}{4}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D})(\lambda\gamma^S\tilde{D}_j^{J'})(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda})v^T\,\Lambda_{J'}^{j}$$
$$-\frac{3}{2}\,z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D}_j^{J'})(r\gamma_{PNM}\tilde{\lambda})\,\Lambda_{J'}^{j}\,,$$

and using the Fierz identity and the fact that $\Lambda_{J'}^{j}$ is G-analytic, this expression becomes

$$(-D_j^{J'}\Omega^{(0)})\,\Lambda_{J'}^{j} = 3\,(\lambda\tilde{\lambda})^{-1}(\lambda D_2)\,(\lambda\gamma^M\tilde{D})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D}_j^{J'})\,\Lambda_{J'}^{j}$$
$$-3\,z^{-\frac{1}{2}}\,(\lambda\tilde{\lambda})^{-2}\,(\lambda r)\,(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{PN}\tilde{D}_j^{J'})\,\Lambda_{J'}^{j}$$
$$+3\,z^{-\frac{1}{2}}\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma^M\tilde{D})(\lambda\gamma^P\tilde{D})(r\gamma_{PM}\tilde{D}_j^{J'})\,\Lambda_{J'}^{j}. \tag{4.71}$$

Note that using the $(A^{(0)})_j^{\ J'}$ of (4.67) at zero-momentum, we have

$$(\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + z^{-\frac{1}{2}}\widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,(A^{(0)})_j^{\ J'}\,\Lambda_{J'}^{j} \tag{4.72}$$
$$= (\lambda D_2 + z^{-\frac{1}{2}}\widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,3\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma^M\tilde{D})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D}_j^{J'})\,\Lambda_{J'}^{j}$$
$$= (-D_j^{J'}\Omega^{(0)})\,\Lambda_{J'}^{j},$$

where to go from the second to the third line we have used (4.71). This is precisely the first equation of (4.70), thus at least at zero-momentum this equation is satisfied. Relaxing the condition of zero-momentum, we have that the contribution from the anticommutators is

$$(-D_j^{J'}\Omega^{(0)})_{ac}\,\Lambda_{J'}^{j} = -\frac{3}{2}\,z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\tilde{D})\{(\lambda\gamma^N\tilde{D}_j^{J'}),(\lambda\gamma^P\tilde{D})\}(r\gamma_{PNM}\tilde{\lambda})\,\Lambda_{J'}^{j}$$
$$-\frac{3}{8}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})\{(\lambda\gamma^P\tilde{D}_j^{J'}),(\lambda\gamma^S\tilde{D})\}(\tilde{\lambda}\gamma_{MNPST}\tilde{\lambda})v^T\,\Lambda_{J'}^{j}\,,$$

where the subscript ac means the contribution from the anticommutators. Using the Fierz identity, one gets

$$(-D_j^{J'}\Omega^{(0)})_{ac}\,\Lambda_{J'}^{j} = 3\,(\lambda\tilde{\lambda})^{-1}\,(\lambda D_2)\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j} \tag{4.73}$$
$$+3\,z^{-\frac{1}{2}}\,(\lambda\tilde{\lambda})^{-1}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(r\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j}$$
$$-3\,z^{-\frac{1}{2}}\,(\lambda\tilde{\lambda})^{-2}\,(\lambda r)\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j}$$
$$-\frac{3}{2}\,(\lambda\tilde{\lambda})^{-1}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}v_N(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D})\,\Lambda_{J'}^{j}$$
$$+\frac{3}{2}\,(\lambda\tilde{\lambda})^{-1}\,v_M\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SN}\tilde{D})\,\Lambda_{J'}^{j}$$
$$+\frac{3}{2}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{D}\gamma_{MNT}\tilde{D})v^T\,\Lambda_{J'}^{j}\,,$$

and one of the advantages of having the result written in this form is that the first three terms can be organized as

$$
\begin{aligned}
&3\,(\lambda\tilde{\lambda})^{-1}\,(\lambda D_2)\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j} \\
&+3\,z^{-\frac{1}{2}}\,(\lambda\tilde{\lambda})^{-1}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(r\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j} \\
&-3\,z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}\,(\lambda r)\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j} \\
&=(\lambda D_2+z^{-\frac{1}{2}}\bar{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,3\,(\lambda\tilde{\lambda})^{-1}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j} \\
&=(\lambda^{\bar{\alpha}}D_{\bar{\alpha}}+z^{-\frac{1}{2}}\bar{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,3\,(\lambda\tilde{\lambda})^{-1}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\,\Lambda_{J'}^{j},
\end{aligned}
\tag{4.74}
$$

where we have used that $(\lambda D_1)\cdot\Lambda_{J'}^{j}=0$, and

$$\{(\lambda D_1),\{(\lambda\gamma^M\tilde{D}_j^{J'}),(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\tilde{D})\}=0\,, \tag{4.75}$$

which can be proved by noting that the anticommutator is

$$\propto\ \{(\lambda\gamma^M\tilde{D}_j^{J'}),(\lambda\gamma^N\tilde{D})\}(\tilde{\lambda}\gamma_{NM}\gamma_T\lambda)\bar{v}^T\,,$$

and this vanishes because λ is a pure spinor. The last three terms of (4.73) can also be rewritten in a convenient form after a few manipulations. Note first that

$$\{(\lambda\gamma^S\tilde{D}_j^{J'})\,,\,(\lambda\gamma^P\tilde{D})\}=\,2\,\bar{v}_T\,(\lambda\gamma^S\gamma_j^{J'}\gamma^\mu\gamma^T\gamma^P\lambda)\,\frac{\partial}{\partial x^\mu}\,, \tag{4.76}$$

and the proof of the result above is similar to the one given for (4.55). This enables us to conclude that

$$\{(\lambda\gamma^M\tilde{D}_J^{J'})\,,\,(\lambda D_2)\}\,=-\{(\lambda\gamma^N\tilde{D}_J^{J'})\,,\,(\lambda\gamma^M\tilde{D})\}\,v_N\,, \tag{4.77}$$

because we can freely anticommute the chiral gamma matrices on the right-hand side of (4.76), since when one commutes two of the gamma matrices the term proportional to η has three chiral gamma matrices and vanishes using the pure spinor condition for λ. In addition, we have

$$\{(\lambda\gamma^N\tilde{D})\,,\,(\lambda D)\}=0\,, \tag{4.78}$$

which follows from (4.55) and the trivial fact that $\{(\lambda\gamma^N\tilde{D})\,,\,(\lambda D_2)\}=0$. Deriving both sides of the equation above, the result is

$$\{(\lambda\gamma^N\tilde{D}_J^{J'})\,,\,(\lambda D)\}=0\,, \tag{4.79}$$

thus

$$\{(\lambda\gamma^S\tilde{D}_J^{J'})\,,\,(\lambda D_2)\}=-\{(\lambda\gamma^S\tilde{D}_J^{J'})\,,\,(\lambda D_1)\}. \tag{4.80}$$

The final intermediate result that we need is

$$
\begin{aligned}
3\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma^M\tilde{D}_j^{J'})(\lambda\gamma^S\tilde{D})\{(\lambda D_1)\,,\,(\tilde{\lambda}\gamma_{SM}\tilde{D})\}& \\
+6\{(\lambda D_1)\,,\,(\lambda\gamma^S\tilde{D})(\tilde{D}\gamma_S\tilde{D}_j^{J'})\}& \\
=\frac{3}{2}\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{D}\gamma_{MNT}\tilde{D})v^T,&
\end{aligned}
\tag{4.81}
$$

which can be proved by computing all the anticommutators of the both sides of the equation and comparing the results. Finally, using (4.77), (4.80) and (4.81) the last three terms of (4.73) are equal to

$$
\begin{aligned}
-\frac{3}{2}\,(\lambda\tilde{\lambda})^{-1}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}v_N(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D})\,\Lambda_{J'}^{j}& \\
+\frac{3}{2}\,(\lambda\tilde{\lambda})^{-1}\,v_M\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SN}\tilde{D})\,\Lambda_{J'}^{j}& \\
+\frac{3}{2}\,\{(\lambda\gamma^M\tilde{D}_j^{J'})\,,\,(\lambda\gamma^N\tilde{D})\}(\tilde{D}\gamma_{MNT}\tilde{D})v^T\,\Lambda_{J'}^{j}& \\
=\{(\lambda D_1)\,,\,(A^{(0)})_j^{J'}\},&
\end{aligned}
\tag{4.82}
$$

where, in addition, we have used that the part of $(A^{(0)})_j^{J'}$ given in (4.67) that is independent of the anticommutator can be written in the form

$$
3\,(\lambda\tilde{\lambda})^{-1}\,(\lambda\gamma^M\tilde{D}_j^{J'})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D})+6(\lambda\gamma^M\tilde{D})(\tilde{D}\gamma_M\tilde{D}_j^{J'})\,,
\tag{4.83}
$$

using the Fierz identity. From (4.72), (4.74) and (4.82) we conclude that

$$
(\lambda^{\bar{\alpha}}D_{\bar{\alpha}}+z^{-\frac{1}{2}}\widetilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\,(A^{(0)})_j^{J'}\,\Lambda_{J'}^{j}=(-D_j^{J'}\Omega^{(0)})\,\Lambda_{J'}^{j}\,,
$$

or in other words, that the first equation of (4.70) is satisfied for the $(A^{(0)})_j^{J'}$ given in (4.67).

Using similar arguments and following exactly the same steps, one can show that all the equations given in (4.70) are satisfied for $(A^{(0)})_j^{J'}$, $(A^{(1)})_j^{J'}$ and $(A^{(2)})_j^{J'}$ given in (4.67), (4.68) and (4.69), respectively.

4.5 Proving useful identities

In order to prove that the vertex operator V_N given in the previous section is BRST-closed we had to use several identities. One way to verify these identities is to Wick rotate $SO(1,9)$ to $SO(10)$ and write all the expressions in $U(5)$ notation, an excellent reference on how to perform this change of notation is [45], see also [41]. Another way is by brute force calculation using four-dimensional notation. The identities in question are

$$
(\lambda D_2)\,(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNS}\tilde{D})v^S=0\,,
\tag{4.84}
$$

$$(\lambda\hat\psi)_v\,\frac{1}{4}(\lambda\gamma^M\tilde D)(\lambda\gamma^N\tilde D)(\tilde D\gamma_{MNS}\tilde D)v^S + (\lambda D_2)\,(\lambda\gamma^M\tilde D)(\lambda\gamma^N\tilde D)(\tilde D\gamma_{MNS}\tilde D)v^S = 0\,,$$

$$(\lambda\hat\psi)_v\,(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\tilde D)(\tilde D\gamma_{MNS}\tilde D)v^S + (\lambda D_2)\,\frac{3}{2}(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\tilde D\gamma_{MNS}\tilde D)v^S = 0\,,$$

$$(\lambda\hat\psi)_v\,\frac{3}{2}(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\tilde D\gamma_{MNS}\tilde D)v^S + (\lambda D_2)\,(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\hat\psi\gamma_M\gamma_N\gamma_S\tilde D)v^S = 0\,,$$

$$(\lambda\hat\psi)_v\,(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\hat\psi\gamma_M\gamma_N\gamma_S\tilde D)v^S + (\lambda D_2)\,\frac{1}{4}(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\hat\psi\gamma_{MNS}\hat\psi)v^S = 0\,,$$

$$(\lambda\hat\psi)_v\,(\lambda\gamma^M\hat\psi)(\lambda\gamma^N\hat\psi)(\hat\psi\gamma_{MNS}\hat\psi)v^S = 0\,,$$

where $(\lambda\hat\psi)_v = (\lambda\gamma^M\hat\psi)v_M$ and (λD_2) was defined in (4.53).

4.5.1 The analytic method: $U(5)$ notation

The first step of the proof of the identities (4.84) using this method is to Wick rotate $SO(1,9)$ to $SO(10)$. After performing this rotation, note that a vector V^M of $SO(10)$ splits as $(V'^{\dot a}, V'_{\dot a})$ in $SU(5)\times U(1)$ notation, where $\dot a = 1,\ldots,5$ and in our conventions $V'^{\dot a}$ carries charge $+1$ and $V'_{\dot a}$ carries charge -1 under the $U(1)$, moreover

$$V'^{\dot a} = \frac{1}{2}(V^{2\dot a-1} + iV^{2\dot a})\,,\quad V'_{\dot a} = \frac{1}{2}(V^{2\dot a-1} - iV^{2\dot a})\,.$$

The null vector v_M that appears in the identities (4.84) was defined in the previous section, it has the non-zero components $v_{J+3} = -\frac{1}{4}\sigma_J^{ij}(uu)_{ij}$. In $U(5)$ notation the null condition reduces to

$$0 = v_M v^M = 2\,v_{\dot a}v^{\dot a}\,. \tag{4.85}$$

In addition, we can organize the Gamma matrices Γ^M of $SO(10)$ as

$$b^{\dot a} = \frac{1}{2}(\Gamma^{2\dot a-1} + i\Gamma^{2\dot a})\,,\quad b_{\dot a} = \frac{1}{2}(\Gamma^{2\dot a-1} - i\Gamma^{2\dot a})\,, \tag{4.86}$$

and from the Clifford algebra $\{\Gamma^M,\Gamma^N\} = 2\eta^{MN}$ one can easily deduce the algebra satisfied by the $b^{\dot a}$ and $b_{\dot a}$ defined above

$$\{b^{\dot a}, b_{\dot b}\} = \delta^{\dot a}_{\dot b}\,,\quad \{b^{\dot a}, b^{\dot b}\} = 0\,,\quad \{b_{\dot a}, b_{\dot b}\} = 0\,,$$

which is isomorphic to the algebra of five fermionic creation and annihilation operators. In our conventions, $b^{\dot a} = b^\dagger_{\dot a}$ where † means the adjoint operator. We will define the vacuum state $|0\rangle$ as the state being annihilated by all $b_{\dot a}$,

$$b_{\dot a}|0\rangle = 0\,, \tag{4.87}$$

and defining $\langle 0|$ to be $|0\rangle^\dagger$, we also have

$$\langle 0|b^{\dot a} = 0\,. \tag{4.88}$$

Note that acting with the creation operators on the vacuum, we generate $2^5 = 32$ states that we will call generically as $|A\rangle$, and the same number of states is generated by acting with the annihilation operators on $\langle 0|$, these states we will be called $\langle B|$. One can show that $\langle B|b^{\dot a}|A\rangle = (\Gamma^{\dot a})^B{}_A$ and $\langle B|b_{\dot a}|A\rangle = (\Gamma_{\dot a})^B{}_A$ forms a representation of the Gamma matrices satisfying the Clifford algebra.

The chirality matrix, $\Gamma^{11} = (-i)\Gamma^1 \ldots \Gamma^{10}$, satisfying $(\Gamma^{11})^2 = 1$ and $(\Gamma^{11})^\dagger = \Gamma^{11}$, is written in terms of the operators $b_{\dot a}$ and $b^{\dot a}$ as

$$\Gamma^{11} = (2b_1b^1 - 1)\ldots(2b_5b^5 - 1)\,, \tag{4.89}$$

and it is easy to see that $\Gamma^{11}|0\rangle = |0\rangle$.

The spinors that appear in the identities (4.84) are the bosonic $\lambda^{\bar\alpha}$ and the fermionics $\tilde D^{\bar\alpha}$ and $\hat\psi^{\bar\alpha}$, all of them Weyl spinors of positive chirality. These spinors, by definition, are eigenstates of the chirality matrix Γ^{11} with eigenvalue 1 and they can be described in $SU(5)\times U(1)$ notation as

$$\begin{aligned}
|\lambda\rangle &= \lambda^{++}|0\rangle + \frac{1}{2}\lambda_{\dot a\dot b}b^{\dot b}b^{\dot a}|0\rangle + \frac{1}{24}\lambda^{\dot a}\epsilon_{\dot a\dot b\dot c\dot d\dot e}b^{\dot e}b^{\dot d}b^{\dot c}b^{\dot b}|0\rangle\,, \\
|\tilde D\rangle &= \tilde D^{++}|0\rangle + \frac{1}{2}\tilde D_{\dot a\dot b}b^{\dot b}b^{\dot a}|0\rangle + \frac{1}{24}\tilde D^{\dot a}\epsilon_{\dot a\dot b\dot c\dot d\dot e}b^{\dot e}b^{\dot d}b^{\dot c}b^{\dot b}|0\rangle\,, \\
|\hat\psi\rangle &= \hat\psi^{++}|0\rangle + \frac{1}{2}\hat\psi_{\dot a\dot b}b^{\dot b}b^{\dot a}|0\rangle + \frac{1}{24}\hat\psi^{\dot a}\epsilon_{\dot a\dot b\dot c\dot d\dot e}b^{\dot e}b^{\dot d}b^{\dot c}b^{\dot b}|0\rangle\,,
\end{aligned} \tag{4.90}$$

where $\epsilon_{\dot a\dot b\dot c\dot d\dot e}$ is completely antisymmetric in all its indices and $\epsilon_{12345} = 1$. From the expressions above, it is not difficult to see that $\Gamma^{11}|\lambda\rangle = |\lambda\rangle$ and similarly for the others. In our conventions, the vacuum state $|0\rangle$ does not carry $U(1)$ charge and the operator $b^{\dot a}$ carries charge $+1$, this implies that if we normalize the scalar component of the decomposition of a chiral spinor to have charge $+\frac{5}{2}$, for example λ^{++} in the expression above, $\lambda_{\dot a\dot b}$ carries $U(1)$ charge $\frac{1}{2}$ and $\lambda^{\dot a}$ carries $U(1)$ charge $-\frac{3}{2}$. In conclusion, a generic chiral spinor with positive chirality $S^{\bar\alpha}$ splits as $(S^{++}, S_{\dot a\dot b}, S^{\dot a})$ carrying $U(1)$ charge $(\frac{5}{2}, \frac{1}{2}, -\frac{3}{2})$.

Before starting any computation, we need the charge conjugation matrix C in $U(5)$ notation. This matrix has the property $C\Gamma^M = -(\Gamma^M)^T C$ where ^{T} means matrix transposition and in a particular basis it is expressed in terms of the Gamma matrices as $C = i\,\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^{10}$, which is equivalent to

$$C = \Pi^5_{\dot a=1}(b_{\dot a} - b^{\dot a})\,.$$

Two very important properties of the matrix C that can be verified by explicit computations using the expression above are

$$b_{\dot{a}}C = -Cb^{\dot{a}}\,, \quad b^{\dot{a}}C = -Cb_{\dot{a}}\,. \tag{4.91}$$

In addition, note that the only non-vanishing vacuum matrix element involving one matrix C and creation operators is

$$\langle 0|Cb^{\dot{a}}b^{\dot{b}}b^{\dot{c}}b^{\dot{d}}b^{\dot{e}}|0\rangle = \epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}}\,, \tag{4.92}$$

where $\epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}}$ is completely antisymmetric in all its indices, $\epsilon^{12345} = 1$ and $\langle 0|0\rangle = 1$. The coefficient of proportionality in the expression above can be verified by explicitly performing the computation for $\dot{a} = 1, \dot{b} = 2, \dot{c} = 3, \dot{d} = 4$ and $\dot{e} = 5$ and noting that the left-hand side is completely antisymmetric in $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{e}$.

The pure spinor conditions in $SO(10)$ notation are $\lambda^{\bar{\alpha}}\gamma^M_{\bar{\alpha}\bar{\beta}}\lambda^{\bar{\beta}} = 0$. As a first example we are going to write these conditions in $U(5)$ notation. Note that these conditions can be written as $\lambda\, C\Gamma^M\lambda = 0$ and this implies

$$\langle\lambda|Cb^{\dot{a}}|\lambda\rangle = 0\,, \qquad \langle\lambda|Cb_{\dot{a}}|\lambda\rangle = 0\,,$$

now substituting the expansion of $|\lambda\rangle$ of (4.90) and using the properties of the matrix C given in (4.91), we have

$$\begin{aligned}
\langle\lambda|Cb^{\dot{a}}|\lambda\rangle &= \frac{1}{24}\lambda^{++}\lambda^{\dot{f}}\epsilon_{\dot{f}\dot{b}\dot{c}\dot{d}\dot{e}}\langle 0|Cb^{\dot{a}}b^{\dot{e}}b^{\dot{d}}b^{\dot{c}}b^{\dot{b}}|0\rangle \\
&\quad +\frac{1}{4}\lambda_{\dot{b}\dot{c}}\lambda_{\dot{e}\dot{d}}\langle 0|b_{\dot{b}}b_{\dot{c}}Cb^{\dot{a}}b^{\dot{d}}b^{\dot{e}}|0\rangle \\
&\quad +\frac{1}{24}\lambda^{++}\lambda^{\dot{f}}\epsilon_{\dot{f}\dot{b}\dot{c}\dot{d}\dot{e}}\langle 0|b_{\dot{b}}b_{\dot{c}}b_{\dot{d}}b_{\dot{e}}Cb^{\dot{a}}|0\rangle \\
&= \frac{1}{12}\lambda^{++}\lambda^{\dot{f}}\epsilon_{\dot{f}\dot{b}\dot{c}\dot{d}\dot{e}}\epsilon^{\dot{a}\dot{e}\dot{d}\dot{c}\dot{b}} - \frac{1}{4}\lambda_{\dot{b}\dot{c}}\lambda_{\dot{d}\dot{e}}\epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}} \\
&= 2\lambda^{++}\lambda^{\dot{a}} - \frac{1}{4}\epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}}\lambda_{\dot{b}\dot{c}}\lambda_{\dot{d}\dot{e}} \\
&= 0\,.
\end{aligned} \tag{4.93}$$

Similarly, one can show using the properties of the $b_{\dot{a}}$ and $b^{\dot{a}}$ operators that $\langle\lambda|Cb_{\dot{a}}|\lambda\rangle \propto \lambda^{\dot{b}}\lambda_{\dot{b}\dot{a}} = 0$. The solution of these two equations is well-known and in fact a solution of the first one is automatically a solution of the second. The solution can be parametrized as

$$\lambda^{++} = e^s\,, \quad \lambda_{\dot{a}\dot{b}} = u_{\dot{a}\dot{b}}\,, \quad \lambda^{\dot{a}} = \frac{1}{8}\,e^{-s}\epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}}u_{\dot{b}\dot{c}}u_{\dot{d}\dot{e}}\,, \tag{4.94}$$

where s and $u_{\dot{a}\dot{b}}$ are the 11 independent components of a pure spinor.

We now proceed to prove the identities given in (4.84). In what follows, we are going to work in a Lorentz frame where the only non-zero component of the pure spinor λ is λ^{++}. Consider first the term

$$(\lambda\gamma^M X) \to (\lambda C b^{\dot a} X)\,,\,(\lambda C b_{\dot a} X)\;, \tag{4.95}$$

where X is any chiral spinor and in this Lorentz frame

$$\begin{aligned}(\lambda C b_{\dot a} X) &= \lambda^{++}\langle 0|Cb_{\dot a}|X\rangle \;=\; 0\,, \\ (\lambda C b^{\dot a} X) &= \lambda^{++}\frac{1}{24}\epsilon_{\dot f\dot b\dot c\dot d\dot e}X^{\dot f}\langle 0|Cb^{\dot a}b^{\dot e}b^{\dot d}b^{\dot c}b^{\dot b}|0\rangle \\ &= \lambda^{++}\frac{1}{24}\epsilon_{\dot f\dot b\dot c\dot d\dot e}\epsilon^{\dot a\dot e\dot d\dot c\dot b}X^{\dot f} \;=\; \lambda^{++}X^{\dot a}\,,\end{aligned} \tag{4.96}$$

and from this, we conclude, for example, that $(\lambda\gamma^M\hat\psi) \to \lambda^{++}\hat\psi^{\dot a}$, which explains, in particular, the fact that the term $(\lambda\gamma^M\hat\psi)$ has only 5 independent components. In order to prove the identities (4.84), we need to write the following expression in $U(5)$ notation

$$(\lambda\gamma^M X)(\lambda\gamma^N Y)(Z\gamma_M\gamma_N\gamma_P K)v^P\,, \tag{4.97}$$

for any fermionic chiral spinors X, Y, Z and K and λ a pure spinor. This expression is proportional to

$$(\lambda^{++})^2 X^{\dot a}Y^{\dot b}\langle Z|Cb_{\dot a}b_{\dot b}b_{\dot c}|K\rangle v^{\dot c} + (\lambda^{++})^2 X^{\dot a}Y^{\dot b}\langle Z|Cb_{\dot a}b_{\dot b}b^{\dot c}|K\rangle v_{\dot c}\,, \tag{4.98}$$

where we have used (4.96). It is left to evaluate the matrix element involving $\langle Z|$ and $|K\rangle$. In this direction, we note that

$$\begin{aligned}b_{\dot a}|K\rangle &= b_{\dot a}K^{++}|0\rangle + b_{\dot a}\frac{1}{2}K_{\dot b\dot c}b^{\dot c}b^{\dot b}|0\rangle + b_{\dot a}\frac{1}{24}K^{\dot b}\epsilon_{\dot b\dot c\dot d\dot e\dot f}b^{\dot f}b^{\dot e}b^{\dot d}b^{\dot c}|0\rangle\,, \\ &= K_{\dot b\dot a}b^{\dot b}|0\rangle + \frac{1}{6}K^{\dot b}\epsilon_{\dot b\dot c\dot d\dot e\dot a}b^{\dot e}b^{\dot d}b^{\dot c}|0\rangle\,, \\ b_{\dot b}b_{\dot a}|K\rangle &= K_{\dot b\dot a}|0> + \frac{1}{2}K^{\dot c}\epsilon_{\dot c\dot d\dot e\dot b\dot a}b^{\dot e}b^{\dot d}|0\rangle\,, \\ b_{\dot c}b_{\dot b}b_{\dot a}|K\rangle &= K^{\dot d}\epsilon_{\dot d\dot e\dot c\dot b\dot a}b^{\dot e}|0\rangle\,.\end{aligned}$$

Using the results above, one can easily deduce

$$\langle Z|b_{\dot a}b_{\dot b}b_{\dot c}|K\rangle v^{\dot c} \;=\; Z^{\dot d}K^{\dot e}v^{\dot c}\epsilon_{\dot e\dot d\dot a\dot b\dot c}\,,$$

and

$$\begin{aligned}\langle Z|Cb_{\dot a}b_{\dot b}b^{\dot c}|K\rangle v_{\dot c} &= -\langle Z|Cb_{\dot a}b^{\dot c}b_{\dot b}|K\rangle v_{\dot c} + \langle Z|Cb_{\dot a}|K\rangle v_{\dot b} \\ &= \langle Z|Cb^{\dot c}b_{\dot a}b_{\dot b}|K\rangle v_{\dot c} + \langle Z|Cb_{\dot a}|K\rangle v_{\dot b} - \langle Z|Cb_{\dot b}|K\rangle v_{\dot a} \\ &= Z^{\dot c}K_{\dot a\dot b}v_{\dot c} - Z_{\dot a\dot b}K^{\dot c}v_{\dot c} + Z^{\dot c}K_{\dot c\dot a}v_{\dot b} - Z^{\dot c}K_{\dot c\dot b}v_{\dot a}\,.\end{aligned}$$

Collecting all the terms, one has

$$v_S(\lambda\gamma^S W)(\lambda\gamma^M X)(\lambda\gamma^N Y)(Z\gamma_M\gamma_N\gamma_P K)v^P \propto \qquad (4.99)$$
$$(\lambda^{++})^3 W^{\dot{d}} v_{\dot{d}} X^{\dot{a}} Y^{\dot{b}} Z^{\dot{e}} K^{\dot{f}} v^{\dot{c}} \epsilon_{\dot{f}\dot{e}\dot{a}\dot{b}\dot{c}}$$
$$+ (\lambda^{++})^3 W^{\dot{d}} v_{\dot{d}} X^{\dot{a}} Y^{\dot{b}} (Z^{\dot{c}} K_{\dot{a}\dot{b}} v_{\dot{c}} - Z_{\dot{a}\dot{b}} K^{\dot{c}} v_{\dot{c}} + Z^{\dot{c}} K_{\dot{c}\dot{a}} v_{\dot{b}} - Z^{\dot{c}} K_{\dot{c}\dot{b}} v_{\dot{a}}) \,,$$

where W is any chiral spinor. Setting, for example, $W = X = Y = Z = K = \tilde{D}$ in the expression above,

$$v_S(\lambda\gamma^S \tilde{D})(\lambda\gamma^M \tilde{D})(\lambda\gamma^N \tilde{D})(\tilde{D}\gamma_M\gamma_N\gamma_P \tilde{D})v^P \propto (\lambda^{++})^3 \tilde{D}^{\dot{d}} v_{\dot{d}} \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}^{\dot{e}} \tilde{D}^{\dot{f}} v^{\dot{c}} \epsilon_{\dot{f}\dot{e}\dot{a}\dot{b}\dot{c}}$$
$$+ (\lambda^{++})^3 \tilde{D}^{\dot{d}} v_{\dot{d}} \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} (\tilde{D}^{\dot{c}} \tilde{D}_{\dot{a}\dot{b}} v_{\dot{c}} - \tilde{D}_{\dot{a}\dot{b}} \tilde{D}^{\dot{c}} v_{\dot{c}} + \tilde{D}^{\dot{c}} \tilde{D}_{\dot{c}\dot{a}} v_{\dot{b}} - \tilde{D}^{\dot{c}} \tilde{D}_{\dot{c}\dot{b}} v_{\dot{a}}) = 0 \,,$$

where we have used that

$$\tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}^{\dot{c}} \tilde{D}^{\dot{d}} \tilde{D}^{\dot{e}} = \epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}} \tilde{D}^1 \tilde{D}^2 \tilde{D}^3 \tilde{D}^4 \tilde{D}^5 \,,$$

$v_{\dot{a}} v^{\dot{a}} = 0$ and $(\tilde{D}^{\dot{a}} v_{\dot{a}})(\tilde{D}^{\dot{b}} v_{\dot{b}}) = 0$ due to the fermionic nature of the covariant derivative. This proves analytically the first identity of (4.84) and replacing $\tilde{D}$ by $\hat{\psi}$ the last one as well.

The proof of the remaining identities also follows from the general formula (4.99). In this direction, we will choose $v^{\dot{a}} = (0, 0, 0, 0, a_1)$ where a_1 is a number. The null condition $v^{\dot{a}} v_{\dot{a}} = 0$ implies that $v_{\dot{a}} = (a_2, a_3, a_4, a_5, 0)$. In what follows, we will consider $a_2 = a_3 = a_4 = 0$ in order to simplify the exposition, but the case where these numbers are different from zero is similar. One can also consider the most general case where $v^{\dot{a}} = (a, b, c, d, e)$ and $v_{\dot{a}} = (a', b', c', d', e')$ with the condition $aa' + bb' + cc' + dd' + ee' = 0$, however, the expressions become very long in this case and it is convenient to use the computer program Mathematica.

Using the general formula of (4.99), the expression below

$$X_1\, (\lambda\hat{\psi})_v\, (\lambda\gamma^M \tilde{D})(\lambda\gamma^N \tilde{D})(\tilde{D}\gamma_{MNP} \tilde{D})v^P + X_2\, (\lambda D_2)\, (\lambda\gamma^M \hat{\psi})(\lambda\gamma^N \tilde{D})(\tilde{D}\gamma_{MNP} \tilde{D})v^P \,,$$

with X_1 and X_2 constants, can be rewritten in $U(5)$ notation as

$$(\lambda^{++})^3\, (X_1 a_5 a_1)\, \hat{\psi}^4 \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}^{\dot{e}} \tilde{D}^{\dot{f}} \epsilon_{\dot{f}\dot{e}\dot{a}\dot{b}5} + (\lambda^{++})^3\, (X_2 a_1 a_5)\, \tilde{D}^4 \hat{\psi}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}^{\dot{e}} \tilde{D}^{\dot{f}} \epsilon_{\dot{f}\dot{e}\dot{a}\dot{b}5}$$
$$+ (\lambda^{++})^3\, (4X_1 a_5^2)\, \hat{\psi}^4 \tilde{D}^4 \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}_{\dot{a}\dot{b}} + (\lambda^{++})^3\, (X_2 a_5^2)\, \tilde{D}^4 \hat{\psi}^4 \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}_{\dot{a}\dot{b}}$$
$$= -(\lambda^{++})^3\, (X_1 a_5 a_1)\, 4!\, \hat{\psi}^4 \tilde{D}^1 \tilde{D}^2 \tilde{D}^3 \tilde{D}^4 + (\lambda^{++})^3\, (X_1 a_5 a_1)\, 3!\, \tilde{D}^4 \hat{\psi}^4 \tilde{D}^1 \tilde{D}^2 \tilde{D}^3$$
$$+ (\lambda^{++})^3\, (4X_1 a_5^2)\, \hat{\psi}^4 \tilde{D}^4 \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}_{\dot{a}\dot{b}} + (\lambda^{++})^3\, (X_2 a_5^2)\, \tilde{D}^4 \hat{\psi}^4 \tilde{D}^{\dot{a}} \tilde{D}^{\dot{b}} \tilde{D}_{\dot{a}\dot{b}} \,.$$

It is not difficult to see that the final expression above is equal to zero if $4X_1 = X_2$ and this proves the second identity of (4.84). We will give one more example, consider now the combination of terms

$$X_3\,(\lambda\hat{\psi})_v\,(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P + X_4\,(\lambda D_2)\,(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\tilde{D}\gamma_{MNP}\tilde{D})v^P\,.$$

Using the general formula of (4.99) and performing the appropriate substitutions, these terms are rewritten in $U(5)$ notation as

$$\begin{aligned}(\lambda^{++})^3\,X_3\,\hat{\psi}^{\dot{d}}v_{\dot{d}}\,\hat{\psi}^{\dot{a}}\tilde{D}^{\dot{b}}\tilde{D}^{\dot{e}}\tilde{D}^{\dot{f}}v^{\dot{c}}\epsilon_{\dot{f}\dot{e}\dot{a}\dot{b}\dot{c}} + (\lambda^{++})^3\,X_4\,\tilde{D}^{\dot{d}}v_{\dot{d}}\,\hat{\psi}^{\dot{a}}\hat{\psi}^{\dot{b}}\tilde{D}^{\dot{e}}\tilde{D}^{\dot{f}}v^{\dot{c}}\epsilon_{\dot{f}\dot{e}\dot{a}\dot{b}\dot{c}}\\ + (\lambda^{++})^3\,X_3\,\hat{\psi}^{\dot{d}}v_{\dot{d}}\,\hat{\psi}^{\dot{a}}\tilde{D}^{\dot{b}}\,(\tilde{D}^{\dot{c}}\tilde{D}_{\dot{a}\dot{b}}v_{\dot{c}} - \tilde{D}_{\dot{a}\dot{b}}\tilde{D}^{\dot{c}}v_{\dot{c}} + \tilde{D}^{\dot{c}}\tilde{D}_{\dot{c}\dot{a}}v_{\dot{b}} - \tilde{D}^{\dot{c}}\tilde{D}_{\dot{c}\dot{b}}v_{\dot{a}})\\ + (\lambda^{++})^3\,X_4\,\tilde{D}^{\dot{d}}v_{\dot{d}}\,\hat{\psi}^{\dot{a}}\hat{\psi}^{\dot{b}}\,(\tilde{D}^{\dot{c}}\tilde{D}_{\dot{a}\dot{b}}v_{\dot{c}} - \tilde{D}_{\dot{a}\dot{b}}\tilde{D}^{\dot{c}}v_{\dot{c}} + \tilde{D}^{\dot{c}}\tilde{D}_{\dot{c}\dot{a}}v_{\dot{b}} - \tilde{D}^{\dot{c}}\tilde{D}_{\dot{c}\dot{b}}v_{\dot{a}})\,.\end{aligned}$$

Dropping the terms that are equal to zero, substituting the values for $v^{\dot{a}}$ and $v_{\dot{a}}$ and omitting the overall factor of $(\lambda^{++})^3$, we have

$$\begin{aligned}-(X_3a_5a_1)\,3\,\hat{\psi}^4\tilde{D}^4\hat{\psi}^{\dot{a}}\tilde{D}^{\dot{e}}\tilde{D}^{\dot{f}}\epsilon_{\dot{f}\dot{e}\dot{a}45} - (X_4a_5a_1)\,2\,\tilde{D}^4\hat{\psi}^4\hat{\psi}^{\dot{a}}\tilde{D}^{\dot{e}}\tilde{D}^{\dot{f}}\epsilon_{\dot{f}\dot{e}\dot{a}45}\\ + (X_3a_5^2)\,3\,\hat{\psi}^4\tilde{D}^4\hat{\psi}^{\dot{a}}\tilde{D}^{\dot{b}}\tilde{D}_{\dot{a}\dot{b}} + (X_4a_5^2)\,2\,\tilde{D}^4\hat{\psi}^4\hat{\psi}^{\dot{a}}\tilde{D}^{\dot{b}}\tilde{D}_{\dot{a}\dot{b}}\,,\end{aligned}$$

and it is not difficult to see that the expression above vanishes for the particular choices $X_3 = 1$ and $X_4 = \frac{3}{2}$. This proves the third identity of (4.84) and using similar arguments one can prove analytically all the remaining identities of (4.84).

4.5.2 The brute force procedure

An alternative procedure to show that all the identities of (4.84) are satisfied is using brute force. We will illustrate the method with the first identity and in fact the author of the thesis had developed a Mathematica program that applies it to all the identities. The first step of the method is to write down all possible terms that are scalars with +2 ghost number carrying the correct harmonic $U(1)$ charge and with the correct number of $\hat{\psi}$ and of derivatives $\tilde{D}$, not considering additional terms that can be obtained from the basic ones by the pure spinor conditions. The possible terms with four derivatives are

$$(\lambda^2D^4) = (\lambda^{\alpha i}\overline{uu}_{ij}\lambda^j_\alpha)(D^\beta_k\overline{uu}^{kl}D^\gamma_l D_{\gamma m}\overline{uu}^{mn}D_{\beta n})\,, \tag{4.100}$$

$$(\bar{\lambda}^2\bar{D}^4) = (\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}\bar{\lambda}^{\dot{\alpha}}_j)(\bar{D}^k_{\dot{\beta}}\overline{uu}_{kl}\bar{D}^l_{\dot{\gamma}}\bar{D}^{m\dot{\gamma}}\overline{uu}_{mn}\bar{D}^{\dot{\beta}n})\,,$$

$$(\lambda\bar{\lambda}\bar{D}D^3) = (\lambda^{\alpha i}\overline{uu}_{ij}\bar{D}^j_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_k\overline{uu}^{kl}D^\beta_l D_{\beta m}\overline{uu}^{mn}D_{\alpha n})\,,$$

$$(\lambda\bar{\lambda}\bar{D}^3 D) = (\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij} D_j^\alpha \lambda_\alpha^k \overline{uu}_{kl} \bar{D}_{\dot{\beta}}^l \bar{D}^{\dot{\beta}m} \overline{uu}_{mn} \bar{D}^{\dot{\alpha}n}) \ ,$$

$$(\bar{\lambda}^2 \bar{D}^2 D^2) = (D_i^\alpha \overline{uu}^{ij} \bar{\lambda}_{\dot{\alpha}j} \bar{D}^{\dot{\alpha}k} \overline{uu}_{kl} \bar{D}_{\dot{\beta}}^l \bar{\lambda}_m^{\dot{\beta}} \overline{uu}^{mn} D_{\alpha n}) \,,$$

$$(\lambda^2 D^2 \bar{D}^2) = (\bar{D}_{\dot{\alpha}}^i \overline{uu}_{ij} \lambda^{\alpha j} D_{\alpha k} \overline{uu}^{kl} D_l^\beta \lambda_\beta^m \overline{uu}_{mn} \bar{D}^{\dot{\alpha}n}) \,,$$

and the most general linear combination of these terms with numerical coefficients A_i will be denoted by Ω',

$$\begin{aligned} \Omega' \ &= \ A_1(\lambda^2 D^4) + A_2(\lambda\bar{\lambda}\bar{D}D^3) + A_3(\bar{\lambda}^2\bar{D}^2 D^2) \\ &\quad + A_4(\lambda^2 D^2 \bar{D}^2) + A_5(\lambda\bar{\lambda}\bar{D}^3 D) + A_6(\bar{\lambda}^2 \bar{D}^4) \,. \end{aligned} \tag{4.101}$$

The coefficients above will be fixed by requiring that Ω' is annihilated by (λD_2). Introducing the notation

$$(\lambda D_2) = \lambda^{\alpha i} u_i^{J'} \bar{u}_{J'}^j D_{\alpha j} + \bar{\lambda}_{\dot{\alpha}i} u_j^i \bar{u}_j^{\dot{J}} D^{\dot{\alpha}j} \equiv (\lambda D)_{2D} + (\bar{\lambda}\bar{D})_{2\bar{D}} \,, \tag{4.102}$$

we have to solve the following equations to ensure that Ω' is annihilated

$$\begin{aligned} (\lambda D)_{2D}\, A_1\, (\lambda^2 D^4) = 0\,, \\ (\lambda D)_{2D}\, A_2\, (\lambda\bar{\lambda}\bar{D}D^3) + (\bar{\lambda}\bar{D})_{2\bar{D}}\, A_1\, (\lambda^2 D^4) = 0\,, \\ (\lambda D)_{2D}\, [\, A_3(\bar{\lambda}^2\bar{D}^2 D^2) + A_4(\lambda^2 D^2 \bar{D}^2)\,] + (\bar{\lambda}\bar{D})_{2\bar{D}}\, A_2\, (\lambda\bar{\lambda}\bar{D}D^3) = 0\,, \\ (\lambda D)_{2D}\, A_5\, (\lambda\bar{\lambda}\bar{D}^3 D) + (\bar{\lambda}\bar{D})_{2\bar{D}}\, [\, A_3(\bar{\lambda}^2\bar{D}^2 D^2) + A_4(\lambda^2 D^2 \bar{D}^2)\,] = 0\,, \\ (\lambda D)_{2D}\, A_6\, (\bar{\lambda}^2 \bar{D}^4) + (\bar{\lambda}\bar{D})_{2\bar{D}}\, A_5\, (\lambda\bar{\lambda}\bar{D}^3 D) = 0\,, \\ (\bar{\lambda}\bar{D})_{2\bar{D}}\, A_6\, (\bar{\lambda}^2 \bar{D}^4) = 0\,. \end{aligned} \tag{4.103}$$

The first and the last equation of the set of equations above are trivially satisfied for any A_1 and A_6, because there are only four different derivatives of the type $\bar{u}_{J'}^i D_{\alpha i}$ and four of the type $\bar{u}_i^{\dot{J}} \bar{D}_{\dot{\alpha}}^i$ and the derivatives are fermionic. The second equation is explicitly

$$\begin{aligned} A_1\, (\bar{\lambda}_{\dot{\alpha}i} u_j^i \bar{u}_j^{\dot{J}} \bar{D}^{\dot{\alpha}j})(\lambda^{\alpha k} \overline{uu}_{kl} \lambda_\alpha^l)(\bar{u}D)^4 \\ + A_2\, (\lambda^{\gamma p} u_p^{J'} \bar{u}_{J'}^t D_{\gamma t})(\lambda^{\alpha i} \overline{uu}_{ij} \bar{D}_{\dot{\alpha}}^j \bar{\lambda}_k^{\dot{\alpha}} \overline{uu}^{kl} D_l^\beta D_{\beta m} \overline{uu}^{mn} D_{\alpha n}) = 0\,, \end{aligned} \tag{4.104}$$

where we have used the definition

$$(D_k^\beta \overline{uu}^{kl} D_l^\gamma D_{\gamma m} \overline{uu}^{mn} D_{\beta n}) = (\bar{u}D)^4 \,.$$

It is possible to perform a few manipulations on the second term on the left-hand side of the equation (4.104) in order to rewrite it as the first term. Mainly, one uses the identities

$$\bar{u}_{J'}^j D_{\gamma j}\, \bar{u}_{K'}^l D_l^\beta D_{\beta m} \overline{uu}^{mn} D_{\alpha n} = -\frac{1}{4} \epsilon_{J'K'} \epsilon_{\gamma\alpha} (\bar{u}D)^4 \,, \tag{4.105}$$

which follows because there is only one non-zero possible combination of four derivatives of the type $\bar{u}^i_{J'} D_{\alpha i}$, and

$$0 = \lambda^{\alpha i}\bar{\lambda}^{\dot{\alpha}}_i = \lambda^{\alpha i}\delta^j_i \bar{\lambda}^{\dot{\alpha}}_j = \lambda^{\alpha i} u^{J'}_i \bar{u}^j_{J'} \bar{\lambda}^{\dot{\alpha}}_j + \lambda^{\alpha i}\bar{u}^{\dot{J}}_i u^j_{\dot{J}} \bar{\lambda}^{\dot{\alpha}}_j \,, \tag{4.106}$$

which implies that

$$\lambda^{\alpha i} u^{J'}_i \bar{u}^j_{J'} \bar{\lambda}^{\dot{\alpha}}_j = -\lambda^{\alpha i}\bar{u}^{\dot{J}}_i u^j_{\dot{J}} \bar{\lambda}^{\dot{\alpha}}_j \,. \tag{4.107}$$

Using these identities, we have

$$(\lambda^{\gamma p} u^{J'}_p \bar{u}^t_{J'} D_{\gamma t})(\lambda^{\alpha i}\overline{uu}_{ij}\bar{D}^j_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_k \overline{uu}^{kl} D^\beta_l D_{\beta m}\overline{uu}^{mn} D_{\alpha n}) = \tag{4.108}$$

$$(\lambda^{\gamma p} u^{J'}_p \bar{u}^t_{J'} D_{\gamma t})(\lambda^{\alpha i}\overline{uu}_{ij}\bar{D}^j_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_k \, \bar{u}^k_{I'}\bar{u}^l_{K'}\epsilon^{I'K'} D^\beta_l D_{\beta m}\overline{uu}^{mn} D_{\alpha n}) =$$

$$\frac{1}{4}\lambda^{\gamma p} u^{J'}_p \lambda^{\alpha i}\overline{uu}_{ij}\bar{D}^j_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_k \, \bar{u}^k_{I'} \, \epsilon^{I'K'} \epsilon_{J'K'}\epsilon_{\gamma\alpha} \, (\bar{u}D)^4 =$$

$$-\frac{1}{4}(\lambda^{\gamma p} u^{I'}_p \lambda^i_\gamma \, \bar{u}^{\dot{J}}_i \bar{u}^{\dot{K}}_j \epsilon_{\dot{J}\dot{K}} \, \bar{D}^j_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_k \bar{u}^k_{I'})(\bar{u}D)^4 =$$

$$\frac{1}{4}(\lambda^{\gamma p}\bar{u}^{\dot{I}}_p \lambda^i_\gamma \, \bar{u}^{\dot{J}}_i \bar{u}^{\dot{K}}_j \epsilon_{\dot{J}\dot{K}} \, \bar{D}^j_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_k u^k_{\dot{I}})(\bar{u}D)^4 =$$

$$\frac{1}{8}(\bar{\lambda}_{\dot{\alpha}k} u^k_{\dot{I}} \bar{u}^{\dot{I}}_j \bar{D}^{\dot{\alpha}j})(\lambda^{\gamma p}\overline{uu}_{pi}\lambda^i_\gamma)(\bar{u}D)^4 \,,$$

where in the last line, we have used

$$\lambda^{\gamma i}\bar{u}^{\dot{I}}_i \bar{u}^{\dot{J}}_j \lambda^j_\gamma = \frac{1}{2}\epsilon^{\dot{J}\dot{I}}(\lambda^{\gamma i}\overline{uu}_{ij}\lambda^j_\gamma) \,, \tag{4.109}$$

and substituting the final result of (4.108) in (4.104), we conclude that

$$(A_1 + \frac{1}{8}A_2)\,(\bar{\lambda}\bar{D})_{2\bar{D}}\,(\lambda^4 D^4) = 0 \,, \tag{4.110}$$

which fixes A_2 as a function of A_1. Performing similar manipulations it is possible to fix the value of A_5 as a function of the value of A_6 for the fifth equation of (4.103) to be satisfied. In order to show that the third and the fourth equation are also satisfied for a correct choice of the numerical coefficients, we use the identities

$$D^\alpha_i \bar{u}^i_{\dot{I}} D^\beta_j D_{\beta k}\bar{u}^j_{\dot{J}}\bar{u}^k_{\dot{K}} = \tag{4.111}$$

$$-\frac{1}{3}(\epsilon_{\dot{I}\dot{J}} D^\alpha_i \overline{uu}^{ij} D^\beta_j D_{\beta k}\bar{u}^k_{\dot{K}} + \epsilon_{\dot{I}\dot{K}} D^\alpha_i \overline{uu}^{ik} D^\beta_k D_{\beta j}\bar{u}^j_{\dot{J}}) \,,$$

$$\bar{D}^{\dot{\alpha}i}\bar{u}^{\dot{J}}_i \bar{D}^{\dot{\beta}j}\overline{uu}_{jk}\bar{D}^{\dot{\gamma}k} =$$

$$\frac{1}{3}(\epsilon^{\dot{\alpha}\dot{\beta}}\bar{u}^{\dot{J}}_i \bar{D}^i_{\dot{\delta}}\bar{D}^{\dot{\delta}j}\overline{uu}_{jk}\bar{D}^{\dot{\gamma}k} + \epsilon^{\dot{\alpha}\dot{\gamma}}\bar{u}^{\dot{J}}_i \bar{D}^i_{\dot{\delta}}\bar{D}^{\dot{\delta}j}\overline{uu}_{jk}\bar{D}^{\dot{\beta}k}) \,,$$

the pure spinor conditions, the Schouten identities and

$$\bar{u}^i_{I'} u^j_{\dot{I}} \epsilon_{jikl} = \epsilon_{j\dot{I}} (\bar{u}^{\dot{j}}_l u^{J'}_k \epsilon_{I'J'} + \bar{u}^{\dot{j}}_k u^{J'}_l \epsilon_{J'I'}) \,,$$

which can be derived by multiplying both sides of $\overline{uu}^{ij} = \frac{1}{2}\epsilon^{ijkl}\overline{uu}_{kl}$ by $u^{I'}_j$ then by ϵ_{imnp} and finally by $\epsilon_{I'J'}$. Using all these identities and the previous results, one has

$$(\lambda D)_{2D}\,(\lambda^2 D^4) = 0\,, \qquad (\bar{\lambda}\bar{D})_{2\bar{D}}\,(\bar{\lambda}^2 \bar{D}^4) = 0\,, \tag{4.112}$$

$$(\lambda D)_{2D}\,(\lambda\bar{\lambda}\bar{D}D^3) = \frac{1}{8}\,(\bar{\lambda}\bar{D})_{2\bar{D}}\,(\lambda^2 D^4)\,, \quad (\bar{\lambda}\bar{D})_{2\bar{D}}\,(\lambda\bar{\lambda}\bar{D}^3 D) = \frac{1}{8}\,(\lambda D)_{2D}\,(\bar{\lambda}^2 \bar{D}^4)\,,$$

$$(\lambda D)_{2D}\,(\bar{\lambda}^2 \bar{D}^2 D^2) = -\frac{2}{3}(\bar{\lambda}\bar{D})_{2\bar{D}}\,(\lambda\bar{\lambda}\bar{D}D^3) + \frac{2}{3}\,(\lambda\bar{\lambda}^2\bar{D}^2 D^3)\,,$$

$$(\bar{\lambda}\bar{D})_{2\bar{D}}\,(\bar{\lambda}^2 \bar{D}^2 D^2) = \frac{2}{3}\,(\lambda D)_{2D}\,(\lambda\bar{\lambda}\bar{D}^3 D) - \frac{2}{3}\,(\lambda^2\bar{\lambda}\bar{D}^3 D^2)\,,$$

$$(\bar{\lambda}\bar{D})_{2\bar{D}}\,(\lambda^2 D^2 \bar{D}^2) = -\frac{2}{3}\,(\lambda D)_{2D}\,(\lambda\bar{\lambda}\bar{D}^3 D) - \frac{2}{3}\,(\lambda^2\bar{\lambda}\bar{D}^3 D^2)\,,$$

$$(\lambda D)_{2D}\,(\lambda^2 D^2 \bar{D}^2) = \frac{2}{3}\,(\bar{\lambda}\bar{D})_{2\bar{D}}\,(\lambda\bar{\lambda}\bar{D}D^3) + \frac{2}{3}\,(\lambda\bar{\lambda}^2\bar{D}^2 D^3)\,,$$

where we have introduced the notation

$$\begin{aligned}(\lambda\bar{\lambda}^2\bar{D}^2 D^3) &= \lambda^{\alpha i}\overline{uu}_{ij}\,\bar{D}^j_{\dot{\alpha}}\,\bar{\lambda}^{\dot{\alpha}}_k\,u^k_{\dot{I}}\bar{u}^{\dot{I}}_l\,\bar{D}^l_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}}_m\,\overline{uu}^{mn}D_{\alpha p}\,\overline{uu}^{pt}\,D^{\gamma}_t D_{\gamma n}\,,\\ (\lambda^2\bar{\lambda}\bar{D}^3 D^2) &= \lambda^{\alpha i}u^{I'}_i\bar{u}^j_{I'}D^{\beta}_j\lambda^k_{\beta}\,\overline{uu}_{kl}\,D_{\alpha m}\,\overline{uu}^{mn}\bar{\lambda}_{\dot{\alpha}n}\bar{D}^{\dot{\alpha}p}\,\overline{uu}_{pt}\bar{D}^l_{\dot{\beta}}\bar{D}^{\dot{\beta}t}\,.\end{aligned}$$

Using the set of equations of (4.112), it is not difficult to see that Ω' given in (4.101) is annihilated by (λD_2) if the A_i are replaced with

$$\begin{aligned}\Omega' = (\lambda^2 D^4) - (\bar{\lambda}^2\bar{D}^4) - 8(\lambda\bar{\lambda}\bar{D}D^3) + 8(\lambda\bar{\lambda}\bar{D}^3 D)\\ -6(\bar{\lambda}^2\bar{D}^2 D^2) + 6(\lambda^2 D^2\bar{D}^2),\end{aligned} \tag{4.113}$$

and one can show using the ansatz for the chiral gamma matrices of (2.66) that this result is the four-dimensional reduction of the expression below written in ten-dimensional notation

$$\frac{1}{4}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P\,,$$

which proves the first identity of (4.84) as we wanted. All the remaining identities can be also proved following the same steps.

Another important property of Ω' that can be derived by brute force using four-dimensional notation is

$$[\,(\lambda D_1)\,,\,\Omega'\,] = 0\,, \tag{4.114}$$

where (λD_1) was defined in (4.53). Note that the only non-zero anticommutator involving the covariant derivatives is

$$\{D_{\alpha i}, \bar{D}^j_{\dot{\alpha}}\} = -2i\delta^j_i(\sigma^\mu_{\alpha\dot{\alpha}})\frac{\partial}{\partial x^\mu}\,,$$

and in principle the commutator (4.114) can be non-zero as a consequence of

$$\{u^i_j D_{\alpha i}, \bar{u}^{\dot{K}}_j \bar{D}^j_{\dot{\alpha}}\} = -2i\delta^{\dot{K}}_j(\sigma^\mu_{\alpha\dot{\alpha}})\frac{\partial}{\partial x^\mu}\,,\quad \{\bar{u}^i_{J'} D_{\alpha i}, u^{K'}_j \bar{D}^j_{\dot{\alpha}}\} = -2i\delta^{K'}_{J'}(\sigma^\mu_{\alpha\dot{\alpha}})\frac{\partial}{\partial x^\mu}\,,$$

however, we will give three examples that illustrate how one can show that this commutator vanishes. Note that in four-dimensional notation

$$(\lambda D_1) = \lambda^{\alpha i}\overline{uu}_{ij}uu^{jk}D_{\alpha k} + \bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}uu_{jk}\bar{D}^{\dot{\alpha}k}\,, \tag{4.115}$$

and the first two examples are

$$[\,\lambda^{\alpha i}\overline{uu}_{ij}uu^{jk}D_{\alpha k}\,,\,(\lambda^2 D^4)\,] = 0\,,\qquad [\,\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}uu_{jk}\bar{D}^{\dot{\alpha}k}\,,\,(\bar{\lambda}^2\bar{D}^4)\,] = 0\,,$$

where $(\lambda^2 D^4)$ and $(\bar{\lambda}^2\bar{D}^4)$ were defined in (4.100). A less trivial example is

$$[\,\lambda^{\alpha i}\overline{uu}_{ij}uu^{jk}D_{\alpha k}\,,\,(\bar{\lambda}^2\bar{D}^4)\,] = -8i(\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}\bar{\lambda}^{\dot{\alpha}}_j)(\lambda^{\alpha k}\overline{uu}_{kl}\bar{D}^l_{\dot{\beta}}\bar{D}^{\dot{\beta}m}\overline{uu}_{mn}\bar{D}^{\dot{\gamma}n}\sigma^\mu_{\alpha\dot{\gamma}}\frac{\partial}{\partial x^\mu})\,,$$

$$[\,\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}uu_{jk}\bar{D}^{\dot{\alpha}k}\,,\,(\lambda\bar{\lambda}\bar{D}^3 D)\,] = -i(\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}\bar{\lambda}^{\dot{\alpha}}_j)(\lambda^{\alpha k}\overline{uu}_{kl}\bar{D}^l_{\dot{\beta}}\bar{D}^{\dot{\beta}m}\overline{uu}_{mn}\bar{D}^{\dot{\gamma}n}\sigma^\mu_{\alpha\dot{\gamma}}\frac{\partial}{\partial x^\mu})\,,$$

where we have used the pure spinor conditions for λ. From the expressions above it is easy to see that with the correct value of the coefficients that we can read from (4.113), one has

$$[\,\lambda^{\alpha i}\overline{uu}_{ij}uu^{jk}D_{\alpha k}\,,\,-(\bar{\lambda}^2\bar{D}^4)\,] + [\,\bar{\lambda}_{\dot{\alpha}i}\overline{uu}^{ij}uu_{jk}\bar{D}^{\dot{\alpha}k}\,,\,8(\lambda\bar{\lambda}\bar{D}^3 D)\,] = 0\,.$$

As illustrated by these examples, the way to prove that the commutator (4.114) vanishes using four-dimensional notation is by computing the commutators, then collecting the terms with equal number of D and $\bar{D}$ and after a judicious use of the pure spinor conditions showing that they vanish. The calculation is tedious but straightforward and further details will be omitted.

4.6 An example: the dilaton vertex operator

In this section, we are going to give an example on how the general formula for the vertex operators V_N given in (4.45) can be evaluated. We will show that when a particular dual superfield T, that will be defined below, is replaced in the general formula, the vertex operator for the zero-momentum dilation of (4.12) is recovered.

In order to do this we will need to evaluate the integrals $\int du$ over the group $SU(4)$ and these integrals can be evaluated indirectly using group theoretic arguments as we will explain. Firstly, we need to fix our normalization condition which is

$$\int du \; = 1 \,. \tag{4.116}$$

The measure du is the Haar measure over the compact group $SU(4)$ [65] and it is invariant under the transformations of this group. From the group theory, we know that the only three invariant tensors under this group are δ^i_j, ϵ_{ijkl} and ϵ^{ijkl}, which means that the result of the integration must be a combination of terms that depend on these tensors. We will illustrate this idea with three examples. The first one is the integral

$$\int du \, (u^j_j \bar{u}^j_k) \; \propto \; \delta^j_k \,, \tag{4.117}$$

where the symbol $\propto$ is because the only invariant tensor that can be formed with the $SU(4)$ indices appearing on the left-hand side is the δ tensor. In order to fix the constant of proportionality we contract both sides of the expression with δ^k_j and the final result is

$$\int du \, (u^j_j \bar{u}^j_k) = \frac{1}{2} \delta^j_k \,. \tag{4.118}$$

Similarly, we have

$$\int du \, \overline{uu}_{ij} \, uu_{kl} \propto \epsilon_{ijkl} \,, \tag{4.119}$$

and contracting both sides of this expression with ϵ^{ijkl} we find that the constant of proportionality is $-\frac{1}{6}$. The last example is

$$\int du \, \bar{u}^i_{I'} u^{I'}_j \, \bar{u}^k_{J'} u^{J'}_l \; = A \, \delta^i_j \, \delta^k_l + B \, \delta^k_j \, \delta^i_l \,, \tag{4.120}$$

and we can get two conditions for A and B by contracting both sides of this expression with δ^j_i and with δ^j_k. The two conditions are

$$\times \, \delta^j_i \; \rightarrow \; \delta^k_l = 4A\delta^k_l + B\delta^k_l \,, \quad \times \, \delta^j_k \; \rightarrow \; \frac{1}{2}\delta^i_l = A\delta^i_l + 4B\delta^i_l \,, \tag{4.121}$$

and the solution of the system of equations above is $A = \frac{7}{30}$ and $B = \frac{1}{15}$.

After this short explanation about harmonic integration, let us return to the main purpose of this subsection which is the derivation of the dilaton vertex operator of (4.12) from a particular choice of $T^{(2)}(u, \bar{u}, x, \theta, \bar{\theta})$. The superfield T in question is

$$T^{(2)}(u, \bar{u}, x, \theta, \bar{\theta}) = i \prod_{J'=1,2} \prod_{\alpha=1,2} (u^{J'}_j \theta^j_\alpha) = \frac{i}{12} (\theta^{\alpha i} \, uu_{ij} \, \theta^{\beta j} \theta^k_\beta \, uu_{kl} \, \theta^l_\alpha) \,. \tag{4.122}$$

As a consistent check, we know that the dilaton vertex operator is dual to the linearized super-Yang-Mills action. This implies that when we replace this T in (4.42) we must get the action. In fact using

$$(D_i^\alpha\,\overline{uu}^{ij}\,D_j^\beta D_{\beta k}\,\overline{uu}^{kl}\,D_{\alpha l})\cdot(\theta^{\gamma m}\,uu_{mn}\,\theta^{\delta n}\theta_\gamma^p\,uu_{pt}\,\theta_\gamma^t) = 12^2\,, \qquad (4.123)$$

one has

$$\int d^4x\int du\int d^8(u\theta)W^{(2)}(u,x,\theta,\bar\theta)T^{(2)}(u,\bar u,x,\theta,\bar\theta)\propto\int d^4x\int du\,\bar D^4\,\mathrm{Tr}\,(W^2)\,,$$

which is the linearized super-Yang-Mills action.

The general formula for the vertex operators V_N given in (4.45) reduces in the case of $N=2$ to

$$V=\int du\,[(y_{ij}uu^{ij})\,\Omega^{(0)}_{min}\,T^{(2)}+8\,\Omega^{(1)}_{min}\,T^{(2)}]\,, \qquad (4.124)$$

where $\Omega^{(0)}_{min}$ and $\Omega^{(1)}_{min}$ were defined in (4.52). Replacing the expression of $T^{(2)}$ of (4.122) on the formula above, we note that the only non-zero contribution is from the terms in $\Omega^{(0)}_{min}$ and $\Omega^{(1)}_{min}$ with derivatives of the type $\bar u^i_{J'}D_{\alpha i}$ described in four-dimensional notation. We will compute these terms in great detail for $\Omega^{(0)}_{min}$ and give the result for $\Omega^{(1)}_{min}$ since the procedure to obtain these terms on both cases is the same. We first note that

$$\begin{aligned}\Omega^{(0)}_{min} &= -\frac{1}{4}(\lambda\gamma^M\tilde D)(\lambda\gamma^N\tilde D)(\tilde D\gamma_{MNP}\tilde D)\,v^P = \qquad (4.125)\\ &-\frac{1}{4}(\lambda\gamma^\mu\tilde D)(\lambda\gamma^\nu\tilde D)(\tilde D\gamma_{\mu\nu I+3}\tilde D)\,v^{I+3}-\frac{1}{2}(\lambda\gamma^{I+3}\tilde D)(\lambda\gamma^\mu\tilde D)(\tilde D\gamma_{I+3}\gamma_\mu\gamma_{J+3}\tilde D)v^{J+3}\\ &-\frac{1}{4}(\lambda\gamma^{I+3}\tilde D)(\lambda\gamma^{J+3}\tilde D)(\tilde D\gamma_{(I+3)(J+3)(K+3)}\tilde D)\,v^{K+3}\,,\end{aligned}$$

and using the ansatz for the chiral gamma matrices of (2.66) and keeping only the terms with $\bar u^i_{J'}D_{\alpha i}$ derivatives, we have

$$\begin{aligned}(\lambda\gamma^\mu\tilde D) &\to (\bar\lambda_{\dot\alpha i}\,i\sigma^{\mu\dot\alpha\alpha}\,\overline{uu}^{ij}\,D_{\alpha j})\,,\\ (\lambda\gamma^{I+3}\tilde D) &\to (\lambda^{\alpha i}\,\sigma^I_{ij}\,\overline{uu}^{jk}\,D_{\alpha k})\,,\\ (\tilde D\gamma_\mu\gamma_\nu\gamma_{I+3}\tilde D)\,v^{I+3} &\to -(D_i^\alpha\overline{uu}^{ij}i(\sigma_\mu)_{\alpha\dot\alpha}i(\bar\sigma_\nu)^{\dot\alpha\beta}uu_{jk}\overline{uu}^{kl}D_{\beta l})\,,\\ (\tilde D\gamma_{I+3}\gamma_{J+3}\gamma_{K+3}\tilde D)\,v^{K+3} &\to -(D_i^\alpha\overline{uu}^{ij}(\sigma_I)_{jk}(\sigma_J)^{kl}uu_{lm}\overline{uu}^{mn}D_{\alpha n})\,,\\ (\tilde D\gamma_{I+3}\gamma_\mu\gamma_{J+3}\tilde D)v^{J+3} &\to 0\,,\end{aligned}$$

where the last term is equal to zero because it necessarily has a $\bar u_i^J\bar D^i_{\dot\alpha}$ derivative. Using the properties of the Pauli matrices given in the Appendix A and the pure spinor conditions for λ, one can contract all the terms and after a few manipulations the final result is

$$\Omega^{(0)}_{minD^4} = -(\lambda^{\alpha i}\,\overline{uu}_{ij}\lambda^j_\alpha)(D^\beta_k\,\overline{uu}^{kl}\,D^\gamma_l D_{\gamma m}\,\overline{uu}^{mn}\,D_{\beta n})\,, \tag{4.126}$$

where the subscript D^4 means that we have kept only the terms with four derivatives of the type $\bar{u}^i_{J'}D_{\alpha i}$. Using (4.122), (4.123) and the result above, one concludes

$$\Omega^{(0)}_{min}T^{(2)} = -12i(\lambda^{\alpha i}\,\overline{uu}_{ij}\lambda^j_\alpha)\,. \tag{4.127}$$

Following the same steps, one can show that

$$\begin{gathered}\Omega^{(1)}_{minD^3} = \\ -4\lambda^{\alpha i}\bar{\psi}^{\dot\alpha}_i\bar{\lambda}_{\dot\alpha j}\overline{uu}^{jk}D^\beta_k D_{\alpha l}\overline{uu}^{lm}D_{\beta m} - 4\bar{\lambda}^{\dot\alpha}_i\psi^i_\alpha\bar{\lambda}_{\dot\alpha j}\overline{uu}^{jk}D^\beta_k D^\alpha_l\overline{uu}^{lm}D_{\beta m} \\ -4\epsilon_{ijkl}\lambda^{\alpha i}\psi^j_\alpha\lambda^{\beta m}\overline{uu}^{ln}D_{\beta n}D^\gamma_p\overline{uu}^{pk}u^{I'}_m\bar{u}^t_{I'}D_{t\gamma} \\ +4\epsilon_{ijkl}\lambda^{\alpha i}\psi^j_\alpha\lambda^{\beta l}\overline{uu}^{mn}D_{\beta n}D^\gamma_p\overline{uu}^{pk}u^{I'}_m\bar{u}^t_{I'}D_{\gamma t} \\ +4\bar{\lambda}_{\dot\alpha i}\bar{\psi}^{\dot\alpha}_j\lambda^{\alpha k}\overline{uu}^{il}D_{\alpha l}D^\beta_m\overline{uu}^{mj}uu_{kn}\overline{uu}^{np}D_{\beta p} - 4\bar{\lambda}_{\dot\alpha i}\bar{\psi}^{\dot\alpha}_j\lambda^{\alpha k}\overline{uu}^{jl}D_{\alpha l}D^\beta_m\overline{uu}^{mi}uu_{kn}\overline{uu}^{np}D_{\beta p} \\ -4\bar{\lambda}_{\dot\alpha i}\bar{\psi}^{\dot\alpha}_j\lambda^{\alpha j}\overline{uu}^{kl}D_{\alpha l}D^\beta_m\overline{uu}^{mi}uu_{kn}\overline{uu}^{np}D_{\beta p}\,,\end{gathered}$$

where the subscript D^3 means that we have kept only the terms with three derivatives of the type $\bar{u}^i_{J'}D_{\alpha i}$. We can proceed by computing the action of this operator on the $T^{(2)}$ given in (4.122), mainly one uses

$$\begin{gathered}D^\gamma_k\,D^\beta_j\,D^\alpha_i\cdot(\theta^{\delta l}uu_{lm}\theta^{\epsilon m}\theta^n_\epsilon uu_{np}\theta^p_\delta) = \\ 4\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}uu_{ij}uu_{kl}\theta^l_\delta - 4\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}uu_{ij}\theta^l_\delta uu_{lk} \\ +4\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}uu_{ik}uu_{jl}\theta^l_\delta - 4\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}uu_{il}\theta^l_\delta uu_{jk} \\ +4\epsilon^{\alpha\beta}\epsilon^{\delta\gamma}uu_{ik}uu_{jl}\theta^l_\delta - 4\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}uu_{il}uu_{jk}\theta^l_\delta\,,\end{gathered} \tag{4.128}$$

and after a tedious calculation, the final result is

$$\begin{gathered}\Omega^{(1)}_{min}T^{(2)} = \\ -12i\,\bar{\lambda}^{\dot\alpha}_i\psi^i_\alpha\bar{\lambda}_{\dot\alpha j}\overline{uu}^{jk}uu_{kl}\theta^{\alpha l} - 12i\,\psi^i_\alpha\overline{uu}_{ij}\lambda^{\beta j}\lambda^{\alpha k}uu_{kl}\theta^l_\beta \\ -12i\,\lambda^{\alpha i}\overline{uu}_{ij}\lambda^{\beta j}\psi^k_\beta uu_{kl}\theta^l_\alpha - 32i\,\lambda^{\alpha i}\bar{\psi}^{\dot\alpha}_i\bar{\lambda}_{\dot\alpha j}\overline{uu}^{jk}uu_{kl}\theta^l_\alpha \\ +8i\,\bar{\lambda}_{\dot\alpha i}\,\overline{uu}^{ij}\,\bar{\psi}^{\dot\alpha}_j\lambda^{\alpha k}uu_{kl}\theta^l_\alpha + 4i\,\bar{\lambda}_{\dot\alpha i}\overline{uu}^{ij}uu_{jk}\bar{\psi}^{\dot\alpha}_l\overline{uu}^{lm}uu_{mn}\lambda^{\alpha n}\theta^k_\alpha \\ -4i\bar{\lambda}_{\dot\alpha i}\overline{uu}^{ij}uu_{jk}\bar{\psi}^{\dot\alpha}_l\overline{uu}^{lm}uu_{mn}\theta^n_\alpha\lambda^{\alpha k}\,.\end{gathered} \tag{4.129}$$

The final step of the computation is to replace (4.127) and (4.129) on (4.124) and perform the harmonic integrals. Noting that

$$(\lambda\gamma^{I+3}\hat{\psi})(\lambda^{\alpha i}\sigma_{Iij}\theta^j_\alpha) = -2\epsilon_{ijkl}\lambda^{\alpha i}y^{jm}\psi_{\alpha m}\lambda^{\beta k}\theta^l_\beta + 2\bar{\lambda}_{\dot\alpha i}y_{jk}\bar{\psi}^{\dot\alpha k}\lambda^{\alpha j}\theta^i_\alpha\,, \tag{4.130}$$

one can organize the final result in the form

$$V = 4i\lambda^{\alpha i} y_{ij} \lambda^j_\alpha - 40i(\lambda\gamma^{I+3}\hat{\psi})(\lambda^{\alpha i}\sigma_{Iij}\theta^j_\alpha)\,. \tag{4.131}$$

The vertex operator is defined up to a BRST-trivial quantity. The two terms on the right-hand side of the expression above are equal up to a BRST-trivial quantity as one can show that

$$Q_{\frac{1}{2}} \cdot (z^{-\frac{1}{2}} y^I \lambda^{\alpha i}\sigma_{Iij}\theta^j_\alpha) = -2(\lambda\gamma^{I+3}\hat{\psi})(\lambda^{\alpha i}\sigma_{Iij}\theta^j_\alpha) - (\lambda^{\alpha i} y_{ij}\lambda^j_\alpha)\,,$$

where we have used (4.5) and (4.6). So, up to a BRST-trivial quantity, the vertex operator is

$$V = -24i(\lambda^{\alpha i} y_{ij}\lambda^j_\alpha)\,, \tag{4.132}$$

and this is, apart from a numerical factor, the dilaton vertex operator of (4.12) as we wanted to show.

4.7 Making the statement "acts as zero" precise

This section is devoted to making the statement that the terms proportional to λ^+ of $Q_{\frac{1}{2}}$ acts as zero when we restrict this operator to act on the states in the cohomology of $Q_{-\frac{1}{2}}$. The idea is to study one example and show how it works. The example consists of acting with $Q_{\frac{1}{2}}$ on $z^{-\frac{1}{2}}(\lambda^-\gamma^M\hat{\psi})$, however, it is easy to generalize it to any function $f(\lambda^-\gamma^M\hat{\psi})$. The operator $Q_{\frac{1}{2}}$ after several manipulations was presented in its final form in (4.1), note that the terms proportional to the pure spinor conditions for λ^- and the terms proportional to λ^+ were excluded because we have argued that they act as zero. These terms are

$$\begin{aligned} Q^{zero}_{\frac{1}{2}} = z^{\frac{1}{2}}[&4\lambda^{-\alpha i}\psi_{\alpha j}\bar{\lambda}^{-\dot{\alpha}}_i P_{\bar{\lambda}^{-\dot{\alpha}}_j} - 4\bar{\lambda}^{-\dot{\alpha}}_i \bar{\psi}^j_{\dot{\alpha}}\lambda^{-i}_\alpha P_{\lambda^{-j}_\alpha} - 2\lambda^{+\gamma i} y_{ij}\bar{\psi}^{j\dot{\alpha}} y_{kl}\lambda^{+l}_\gamma P_{\bar{\lambda}^{-\dot{\alpha}}_k} \\ &+ 2\bar{\lambda}^{+\dot{\alpha}}_i y^{ij}\psi^\alpha_j \lambda^{+l}_\alpha y_{kl} P_{\bar{\lambda}^{-\dot{\alpha}}_k} - 2\lambda^{+\alpha i} y_{ij}\bar{\psi}^j_{\dot{\alpha}}\bar{\lambda}^{+\dot{\alpha}}_k y^{kj} P_{\lambda^{-\alpha j}} + 2\bar{\lambda}^{+\dot{\alpha}}_i y^{ij}\psi_{\alpha j}\bar{\lambda}^{+}_{k\dot{\alpha}} y^{jk} P_{\lambda^{-j}_\alpha}]\,, \end{aligned} \tag{4.133}$$

where the superscript *zero* means the terms of $Q_{\frac{1}{2}}$ that act as zero.

Let us act with the operator $Q_{\frac{1}{2}}$ on $z^{-\frac{1}{2}}(\lambda^-\gamma^M\hat{\psi})$. Considering λ^- a pure spinor and excluding the terms that act as zero, we already know that $z^{-\frac{1}{2}}(\lambda^-\gamma^M\hat{\psi})$ is annihilated because of (4.4). Instead, let us act on $z^{-\frac{1}{2}}(\lambda^-\gamma^M\hat{\psi})$ with the complete $Q_{\frac{1}{2}}$, i.e. $Q_{\frac{1}{2}}$ of (4.1) plus $Q^{zero}_{\frac{1}{2}}$, and without considering λ^- a pure spinor. Firstly, the case when $M=\mu$ will be considered,

$$Q_{\frac{1}{2}} \cdot z^{-\frac{1}{2}}(\lambda^-\gamma^\mu\hat{\psi}) = \{Q^{zero}_{\frac{1}{2}} + z^{\frac{1}{2}}[4\lambda^{-\alpha i}\psi_{\alpha k} P_{y^{ij}} y^{jk} \tag{4.134}$$

$$-4\bar\lambda_i^{-\dot\alpha}\bar\psi^l_{\dot\alpha}P_{y_{ij}}y_{jl}+4\lambda^{-\alpha i}\psi_{\alpha j}\psi_i^\beta P_{\psi_j^\beta}-4\bar\lambda_i^{-\dot\alpha}\bar\psi^m_{\dot\alpha}\bar\psi^{\dot\beta i}P_{\bar\psi^{\dot\beta m}}-2\lambda^{-\alpha i}\psi_{\alpha j}\bar\psi^j_{\dot\alpha}P_{\bar\psi^i_{\dot\alpha}}$$
$$-2\lambda_i^{-\dot\alpha}\psi_{\alpha k}\bar\psi^k_{\dot\alpha}P_{\psi_{\alpha i}}]\}\cdot z^{-\frac{1}{2}}(\lambda^-\gamma^\mu\hat\psi)=$$
$$-2\lambda^{+\alpha i}y_{ij}\bar\psi^j_{\dot\alpha}\lambda^{+k}_\alpha\sigma^{\mu\dot\alpha\beta}\psi_{\beta k}-2\bar\lambda^+_{\dot\alpha i}y^{ij}\psi^\alpha_j\lambda^{+k}_\alpha\sigma^{\mu\dot\alpha\beta}\psi_{\beta k}+(\lambda^{+\alpha i}y_{ij}\lambda^{+j}_\alpha)\bar\psi^k_{\dot\alpha}\sigma^{\mu\dot\alpha\beta}\psi_{\beta k}$$
$$+2\bar\lambda^+_{\dot\alpha i}\bar\lambda^{+\dot\alpha}_j\bar\psi^i_{\dot\beta}\sigma^{\mu\dot\beta\alpha}\psi_{\alpha k}y^{jk}+2\lambda^{+i}_\alpha y_{ij}\bar\psi^j_{\dot\alpha}\bar\lambda^{+\dot\alpha}_k\sigma^{\mu\dot\alpha\alpha}\bar\psi^k_{\dot\alpha}+(\bar\lambda^+_{\dot\alpha i}y^{ij}\bar\lambda^{+\dot\alpha}_j)\bar\psi^k_{\dot\beta}\sigma^{\mu\dot\beta\alpha}\psi_{\alpha k}\,.$$

Note that the result above depends on λ^+. The statement "acts as zero" in fact means that these terms are $Q_{-\frac{1}{2}}$ exact. It is not difficult to see that the first three terms of the final result of (4.134) are equal to

$$-2Q_{-\frac{1}{2}}\cdot(z^{\frac{1}{2}}\psi^\alpha_i\bar\psi^i_{\dot\alpha}\lambda^{+j}_\alpha\sigma^{\mu\dot\alpha\beta}\psi_{\beta j})\,,$$

similarly, the last three terms are

$$2Q_{-\frac{1}{2}}\cdot(z^{\frac{1}{2}}\psi_{\alpha i}\bar\psi^i_{\dot\alpha}\bar\lambda^{+\dot\alpha}_j\sigma^{\mu\dot\beta\alpha}\bar\psi^j_{\dot\beta})\,,$$

and this proves that all the terms are $Q_{-\frac{1}{2}}$ exact.

We can repeat the same steps of the calculation above for the case when $M=I+3$ with the same conclusion, and for completeness we will present the details,

$$Q_{\frac{1}{2}}\cdot z^{-\frac{1}{2}}(\lambda^-\gamma^{I+3}\hat\psi)= \tag{4.135}$$
$$2\lambda^{+\alpha i}\bar\lambda^+_{\dot\alpha i}\psi_{\alpha j}y^{jk}\sigma^I_{kl}\bar\psi^{\dot\alpha l}-2\lambda^{+\alpha i}y_{ij}\bar\psi^j_{\dot\alpha}\lambda^{+k}_\alpha\sigma^I_{kl}\bar\psi^{\dot\alpha l}-2\lambda^{+\alpha i}y_{ij}\bar\psi^j_{\dot\alpha}\bar\lambda^{+\dot\alpha}_k y^{kl}\sigma^I_{lm}y^{mn}\psi_{\alpha n}$$
$$-2\bar\lambda^+_{\dot\alpha i}y^{ij}\psi^\alpha_j\lambda^{+k}_\alpha y_{kl}\sigma^{Ilm}y_{mn}\bar\psi^{\dot\alpha n}+2\bar\lambda^{+\dot\alpha}_i y^{ij}\psi^\alpha_j\bar\lambda^+_{\dot\alpha k}\sigma^{Ikl}\psi_{\alpha l}-2\lambda^{+\alpha i}\bar\lambda^+_{\dot\alpha i}\psi_{\alpha j}\sigma^{Ijk}y_{kl}\bar\psi^{\dot\alpha l}\,,$$

and the first three lines are equal to

$$2Q_{-\frac{1}{2}}\cdot(z^{\frac{1}{2}}\lambda^{+\alpha i}y_{ij}\bar\psi^j_{\dot\alpha}\psi_{\alpha k}y^{kl}\sigma^I_{lm}\bar\psi^{\dot\alpha m})\,,$$

moreover, the last three lines can be written as

$$2Q_{-\frac{1}{2}}\cdot(z^{\frac{1}{2}}\bar\lambda^+_{\dot\alpha i}y^{ij}\psi^\alpha_j\psi_{\alpha k}\sigma^{Ikl}y_{lm}\bar\psi^{\dot\alpha m})\,,$$

finally, one concludes that when $M=I+3$, all the terms are $Q_{-\frac{1}{2}}$ exact.

In this section, we showed that all the terms proportional to λ^+ resulting from the action of $Q_{\frac{1}{2}}$ on the term $z^{-\frac{1}{2}}(\lambda^-\gamma^M\hat\psi)$ are $Q_{-\frac{1}{2}}$ exact. It is straightforward to generalize this result for a function $f(\lambda^-\gamma^M\hat\psi)$ due to the fact that the operator $Q_{\frac{1}{2}}$ acts as a derivative operator. The aim of presenting this example was to illustrate how the statement "acts as zero" used at several points in the chapter 3 should be understood. The case of $Q_{\frac{1}{2}}$ was considered here, nevertheless similar arguments can be used for all $Q_{\frac{3}{2}}+\ldots$.

Chapter 5

Conclusion

In this thesis, we computed the zero mode cohomology at +2 ghost number of the BRST operator of the pure spinor formalism in the background $AdS^5 \times S^5$ close to the boundary of AdS. The states in this cohomology correspond to on-shell supergravity states. The first step of the method used for the computation consists in expanding both the BRST operator and the physical vertex operators V in powers of z, where z is the distance from the AdS boundary. Since the expansion of V has a term of minimum degree, where degree is defined to be the power of z, and all the terms in the BRST operator expansion have a fixed degree, it was possible to use standard methods to compute the cohomology of the BRST operator. Note that our results are valid inside the region of the validity of the z expansion.

The conjecture (AdS/CFT) predicts that every on-shell physical superstring state is dual to a single-trace gauge-invariant operator of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills, in particular, supergravity states are dual to Half-BPS operators. All the Half-BPS operators of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills and their duals can be elegantly described as superfields defined in a specific harmonic superspace, as explained in the chapter 4. The vertex operators in the BRST cohomology corresponding to the on-shell physical supergravity states constructed in this thesis were described in terms of these dual superfields as expected by holography. The results were proved to be consistent, because under a gauge transformation of the dual superfields, the vertex operators change by a BRST-trivial quantity.

In principle, the method for constructing the supergravity vertex operators used in this thesis can be generalized to construct the massive vertex operators of the theory. In the supergravity limit, the worldsheet variables only depend on the worldsheet coordinate τ and all their σ derivatives are zero. To construct the massive vertex operators, it is necessary to consider that the worldsheet variables depend on both τ and σ. So, the first step in generalizing the method would be the computation of the cohomology of the complete operator $Q_{-\frac{1}{2}}$ defined in (3.14), i.e, not dropping the terms that contain σ derivatives of the variables. As explained in the chapter 3, the zero mode cohomology of this operator was already computed by Mikhailov and Xu in [23], and their results used in this thesis. The only known massive vertex operator in the pure spinor formalism is the one proposed by Mazzucato and Vallilo in [66] for the Konishi state and it would be interesting to compare their result with the one resulting from the generalization of the method used in this thesis. It would be also interesting to compare with the RNS vertex operators in a specific limit proposed by Minahan in [67].

Generalizing the method for computing the massive vertex operators will also allow the computation of the spectrum. It is not known how to compute the spectrum of the superstrings in $AdS^5 \times S^5$ from first principles. Recent progresses in this direction are the articles by Benichou [68, 69, 70] and a previous article by Mikhailov and Schafer-Nameki [58]. Benichou computed the fusion of a class of line operators in the lowest order in perturbation theory using the pure spinor formalism and derived a Hirota equation which allows the computation of the spectrum using integrability techniques. The proof that the theory of superstrings in $AdS^5 \times S^5$ is integrable, at least at the classical level, was given by Bena, Roiban and Polchinski in [71] using the Green-Schwarz formalism and by Vallilo in [72] in the case of the pure spinor formalism. The energy of an on-shell excitation of a superstring is equal by holography to the dimension of the dual gauge-invariant operator of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills. Using integrability techniques and making some assumptions, the value of the dimension of all single-trace gauge-invariant operators of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills was computed for any value of the coupling constant, a recent review is [12]. The resulting spectrum of the superstrings could be compared with these results.

The prescription for computing scattering amplitudes of superstrings in the pure spinor formalism in the background $AdS^5 \times S^5$ exists and it was proposed by Berkovits, see [73], for example. However, a superstring scattering amplitude has never been computed because of the lack of a vertex operator. In this thesis, the supergravity vertex operators were constructed, but the results are only valid close to the boundary of AdS. In principle, the computation of a scattering amplitude requires the knowledge of the vertex operator for any value of z and not only its leading term close to the boundary. It is possible to compute more terms in the z expansion of the vertex operator, as illustrated with a simple example in the section 4.7, where an additional term was explicitly computed. Nevertheless, the operators $Q_{\frac{3}{2}} + \ldots$ that appear in the expansion of the BRST operator are of increasing complexity and it is not trivial to compute many more terms in the expansion of V. Knowing the boundary behavior, however, should be enough to compute disc amplitudes for one operator in the bulk and N vertex operators corresponding to open strings located on D-branes close to the boundary of AdS. These open strings vertex operators can be found, for example, in [41]. The resulting amplitudes are expected to contain the term given in (4.42) and reproduced below

$$\int d^4x \int du \int d^8(u\theta) W^{(N)}(u, x, \theta, \bar{\theta}) T^{(4-N)}(u, \bar{u}, x, \theta, \bar{\theta}) \,.$$

Appendix A

Pauli Matrices and Spinors

In this thesis, spinors and the Pauli matrices of $SO(1,3)$ and $SO(6)$ appear frequently. This Appendix is devoted to fixing our conventions and enumerating several useful properties.

A.1 $SO(1,3)$

Our conventions follow closely the conventions of the book by Wess and Bagger [74]. The metric is mostly plus $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and the 4 by 4 Gamma matrices satisfying the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ are

$$\Gamma^\mu = \begin{pmatrix} 0_2 & i\sigma^\mu_{\alpha\dot{\alpha}} \\ i\bar{\sigma}^{\mu\dot{\alpha}\alpha} & 0_2 \end{pmatrix} ,$$

where 0_n is a zero n by n matrix, $\sigma^\mu_{\alpha\dot{\alpha}}$ are

$$\sigma^0_{\alpha\dot{\alpha}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \sigma^1_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\sigma^2_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and as matrices

$$\bar{\sigma}^0 = \sigma^0 , \qquad \bar{\sigma}^i = -\sigma^i , \qquad i = 1,2,3 .$$

The completely antisymmetric tensors $\epsilon^{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ have the non-zero components

$$\epsilon_{21} = \epsilon^{12} = 1 , \quad \epsilon_{12} = \epsilon^{21} = -1 , \quad \epsilon_{\dot{2}\dot{1}} = \epsilon^{\dot{1}\dot{2}} = 1 , \quad \epsilon_{\dot{1}\dot{2}} = \epsilon^{\dot{2}\dot{1}} = -1 . \qquad \text{(A.1)}$$

Useful identities involving σ matrices and ϵ tensors are

$$\begin{aligned}(\bar\sigma^\mu)^{\dot\alpha\alpha} &= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}\,, \qquad (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}\,, \\ (\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)_{\beta\dot\beta} &= -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,, \qquad \mathrm{Tr}\,\sigma^\mu\bar\sigma^\nu = -2\eta^{\mu\nu}\,, \\ (\sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta_\alpha^\beta\,, \qquad (\bar\sigma^\mu\sigma^\nu+\bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} = -2\eta^{\mu\nu}\delta_{\dot\beta}^{\dot\alpha}\,.\end{aligned} \tag{A.2}$$

Defining

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{2}(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta}-\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta})\,, \quad (\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha} = \frac{1}{2}(\bar\sigma^{\mu\dot\beta\alpha}\sigma^\nu_{\alpha\dot\alpha}-\bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha})\,, \tag{A.3}$$

it is not difficult to see using (A.2) that

$$\begin{aligned}(\sigma^{\mu\nu})_\alpha{}^\beta\,\epsilon_{\beta\gamma} &= (\sigma^{\mu\nu})_\gamma{}^\beta\,\epsilon_{\beta\alpha}\,, \qquad (\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\,\epsilon^{\dot\alpha\dot\gamma} = (\bar\sigma^{\mu\nu})^{\dot\gamma}{}_{\dot\alpha}\,\epsilon^{\dot\alpha\dot\beta}\,, \\ (\sigma^{\mu\nu})_\alpha{}^\alpha &= 0\,, \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\alpha} = 0\,,\end{aligned} \tag{A.4}$$

and

$$\begin{gathered}(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\delta{}^\gamma = -4(\epsilon_{\alpha\delta}\epsilon^{\beta\gamma}+\delta_\alpha^\gamma\delta_\delta^\beta)\,, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\gamma}{}_{\dot\delta} = -4(\epsilon^{\dot\alpha\dot\gamma}\epsilon_{\dot\beta\dot\delta}+\delta_{\dot\delta}^{\dot\alpha}\delta_{\dot\beta}^{\dot\gamma})\,, \\ (\sigma^{\mu\nu})_\alpha{}^\beta(\bar\sigma_{\mu\nu})^{\dot\gamma}{}_{\dot\delta} = 0\,.\end{gathered}$$

Important identities involving the ϵ tensors are the Schouten identities,

$$\begin{aligned}\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta}+\epsilon_{\alpha\gamma}\,\epsilon_{\delta\beta}+\epsilon_{\alpha\delta}\,\epsilon_{\beta\gamma} &= 0\,, \\ \epsilon_{\dot\alpha\dot\beta}\,\epsilon_{\dot\gamma\dot\delta}+\epsilon_{\dot\alpha\dot\gamma}\,\epsilon_{\dot\delta\dot\beta}+\epsilon_{\dot\alpha\dot\delta}\,\epsilon_{\dot\beta\dot\gamma} &= 0\,,\end{aligned} \tag{A.5}$$

which can be easily derived from the relation

$$\epsilon^{\alpha\beta}\,\epsilon_{\gamma\delta} = -(\delta_\gamma^\alpha\delta_\delta^\beta-\delta_\gamma^\beta\delta_\delta^\alpha)\,,$$

and similarly for the dotted indices. The spinorial indices can be raised and lowered using the ϵ tensors, for any two chiral spinors A^α and $\bar A^{\dot\alpha}$ our convention is

$$A_\beta = \epsilon_{\beta\alpha}A^\alpha\,, \quad \bar A_{\dot\beta} = \epsilon_{\dot\beta\dot\alpha}\bar A^{\dot\alpha}\,,$$

and note that there is an important detail, the spinorial indices of the derivatives are raised and lowered with an additional minus sign

$$\frac{\partial}{\partial A_\beta} = -\epsilon^{\beta\alpha}\frac{\partial}{\partial A^\alpha}\,, \quad \frac{\partial}{\partial \bar A_{\dot\beta}} = -\epsilon^{\dot\beta\dot\alpha}\frac{\partial}{\partial \bar A^{\dot\alpha}}\,.$$

A very good review about two-component spinors techniques and the use of the dot and undot notation is [75].

A.2 $SO(6)$

Our conventions follow closely the conventions of the book by Green, Schwarz and Witten [76]. The 8 by 8 Gamma matrices in the Weyl basis satisfying the Clifford algebra $\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}$, where $\delta^{IJ} = 1$ if $I = J$ and zero otherwise, are

$$\Gamma^I = \begin{pmatrix} 0_4 & \sigma^I_{ij} \\ \sigma^{Iij} & 0_4 \end{pmatrix},$$

where, adapting the results of [77],

$$\sigma^1_{ij} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma^2_{ij} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \tag{A.6}$$

$$\sigma^3_{ij} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \sigma^4_{ij} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma^5_{ij} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \sigma^6_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$(\sigma^I)^{ij} = \frac{1}{2}\epsilon^{ijkl}(\sigma^I)_{kl}, \qquad (\sigma^I)_{ij} = \frac{1}{2}\epsilon_{ijkl}(\sigma^I)^{kl}, \tag{A.7}$$

where ϵ^{ijkl} and ϵ_{ijkl} are completely antisymmetric in all their indices with $\epsilon^{1234} = 1$ and $\epsilon_{1234} = 1$. Note from (A.6) that $\sigma^I_{ij} = -\sigma^I_{ji}$ and similarily for σ^{Iij}. Note also that

$$(\sigma^I)^{ij}(\sigma^J)_{jk} + (\sigma^J)^{ij}(\sigma^I)_{jk} = 2\delta^i_k\delta^{IJ},$$

which ensures that the Gamma matrices Γ^I satisfy the Clifford algebra. Additional useful identities are

$$(\sigma^I)_{ij}(\sigma_I)^{kl} = 2(\delta^l_i\delta^k_j - \delta^k_i\delta^l_j), \tag{A.8}$$

$$(\sigma^I)_{ij}(\sigma_I)_{kl} = -2\epsilon_{ijkl}, \qquad (\sigma^I)^{ij}(\sigma_I)^{kl} = -2\epsilon^{ijkl},$$

$$(\sigma^I)_{i[j}(\sigma^J)_{kl]} = \frac{1}{3}\epsilon_{jklm}(\sigma^I\sigma^J)_i{}^m, \qquad (\sigma^I)_{[ij}(\sigma^J)_{kl]} = -\frac{1}{3}\epsilon_{ijkl}\delta^{IJ},$$

where [ ] means antisymmetrization of the indices, and

$$(\sigma^I)^\dagger_{ij} = (\sigma^I)^{ij}\,, \qquad (\sigma^I)^{ij\dagger} = (\sigma^I)_{ij}\,, \tag{A.9}$$

with † meaning Hermitian conjugation. Finally, we define

$$(\sigma^{IJ})_i{}^j = \frac{1}{2}(\sigma^I_{ik}\sigma^{Jkj} - \sigma^J_{ik}\sigma^{Ikj})\,, \qquad (\sigma^{IJ})^i{}_j = \frac{1}{2}(\sigma^{Iik}\sigma^J_{kj} - \sigma^{Jik}\sigma^I_{kj})\,. \tag{A.10}$$

Appendix B

The $PSU(2,2|4)$ algebra in four-dimensional notation

In this Appendix, we present the $PSU(2,2|4)$ superalgebra in four-dimensional notation and fix our conventions. The generators of this superalgebra are: the translation generator P_μ, the special conformal generator K_μ, the dilatation generator D, the Lorentz generators $M_{\mu\nu}$, the $SU(4)$ R-symmetry generators U^i_j, the supersymmetry generators $[q_{\alpha i}, \bar{q}^i_{\dot{\alpha}}]$ and the generators of superconformal transformations $[s^i_\alpha, \bar{s}_{\dot{\alpha}i}]$. Using the conventions of the Pauli matrices of Appendix A, the non-zero commutators and anticommutators of the superalgebra are

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\rho[\nu}M_{\mu]\tau} + \eta_{\tau[\mu}M_{\nu]\rho}\,, \\
[M_{\mu\nu}, P_\rho] = \eta_{\rho[\nu}P_{\mu]}\,, &\qquad [M_{\mu\nu}, K_\rho] = \eta_{\rho[\nu}K_{\mu]}\,, \\
[D, P_\mu] = P_\mu\,, &\qquad [D, K_\mu] = -K_\mu\,, \\
[P_\mu, K_\nu] &= 2\eta_{\mu\nu}D + 2M_{\mu\nu}\,, \\
[D, q_{\alpha i}] = \frac{1}{2}q_{\alpha i}\,, \quad [D, \bar{q}^i_{\dot{\alpha}}] = \frac{1}{2}\bar{q}^i_{\dot{\alpha}}\,, &\qquad [D, s^i_\alpha] = -\frac{1}{2}s^i_\alpha\,, \quad [D, \bar{s}_{\dot{\alpha}i}] = -\frac{1}{2}\bar{s}_{\dot{\alpha}i}\,, \\
\{q_{\alpha i}, \bar{q}^j_{\dot{\alpha}}\} = 2i\delta^j_i(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu\,, &\qquad \{s^i_\alpha, \bar{s}_{\dot{\alpha}j}\} = 2i\delta^i_j(\sigma^\mu)_{\alpha\dot{\alpha}}K_\mu\,, \\
[M_{\mu\nu}, q_{\alpha i}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta q_{\beta i}\,, &\qquad [M_{\mu\nu}, s^i_\alpha] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta s^i_\beta\,, \\
[M_{\mu\nu}, \bar{q}^{\dot{\beta}i}] = \frac{1}{2}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{q}^{\dot{\alpha}i}\,, &\qquad [M_{\mu\nu}, \bar{s}^{\dot{\beta}}_i] = \frac{1}{2}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{s}^{\dot{\alpha}}_i\,, \\
[q_{\alpha i}, K_\mu] = i\,(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{s}^{\dot{\alpha}}_i\,, &\qquad [s^i_\alpha, P_\mu] = i\,(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{q}^{\dot{\alpha}i}\,, \\
[\bar{q}^{\dot{\beta}i}, K_\mu] = i(\bar{\sigma}_\mu)^{\dot{\beta}\beta}s^i_\beta\,, &\qquad [\bar{s}^{\dot{\beta}}_i, P_\mu] = i(\bar{\sigma}_\mu)^{\dot{\beta}\beta}q_{\beta i}\,, \\
[U^i_j, U^k_l] &= \delta^i_l U^k_j - \delta^k_j U^i_l\,, \\
[U^i_j, q_{\alpha k}] = \delta^i_k q_{\alpha j} - \frac{1}{4}\delta^i_j q_{\alpha k}\,, &\qquad [U^i_j, s^{\alpha k}] = -(\delta^k_j s^{\alpha i} - \frac{1}{4}\delta^i_j s^{\alpha k})\,, \\
\{q_{\alpha i}, s^{\beta j}\} = \delta^j_i(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} &- 2\delta^\beta_\alpha\delta^j_i D + 4\delta^\beta_\alpha U^j_i\,, \\
\{\bar{q}^{\dot{\alpha}i}, \bar{s}_{\dot{\beta}j}\} = \delta^i_j(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}M_{\mu\nu} &- 2\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^i_j D - 4\delta^{\dot{\alpha}}_{\dot{\beta}}U^i_j\,,
\end{aligned}
\tag{B.1}
$$

and, for example, $[\nu\,\mu] = \nu\mu - \mu\nu$ means antisymmetrization of the indices with no additional factor of half. Under Hermitian conjugation † the generators transform as

$$(P_\mu)^\dagger = -P_\mu\,, \quad (K_\mu)^\dagger = -K_\mu\,, \quad (D)^\dagger = -D\,, \quad (M_{\mu\nu})^\dagger = -M_{\mu\nu}\,, \tag{B.2}$$
$$(U^i_j)^\dagger = U^j_i\,, \quad (q_{\alpha i})^\dagger = \bar{q}^i_{\dot\alpha}\,, \quad (s^i_\alpha)^\dagger = \bar{s}_{\dot\alpha i}\,.$$

Appendix C

Boundary transformations of the variables

In this Appendix, we will show that with our chosen coset representative given in (2.62) the variables x^μ, $\theta^{\alpha j}$ and $\bar\theta^{\dot\alpha}_j$ transform in the usual $\mathcal{N}=4$ $d=4$ superconformal manner when $z\to 0$. Part of this Appendix is based on the article [30] by Heslop and Howe.

Recall that the $\mathcal{N}=4$ superspace is described by the supercoset

$$\mathcal{M}^{4|16} = \frac{\{P_\mu, K_\mu, D, M_{\mu\nu}, q_{\alpha i}, \bar q^i_{\dot\alpha}, U^i_j, s^i_\alpha, \bar s_{\dot\alpha i}\}}{\{M_{\mu\nu}, D, U^i_j, K_\mu, s^i_\alpha, \bar s_{\dot\alpha i}\}}\,,$$

a possible coset representative being

$$g(x,\theta,\bar\theta) = \exp\left(x^\mu P_\mu + i\,\theta^{\alpha j} q_{\alpha j} + i\,\bar\theta_{\dot\alpha j}\bar q^{\dot\alpha j}\right). \qquad \text{(C.1)}$$

Recall, also, that the AdS^5 space is described by the coset

$$AdS_5 \;=\; \frac{SO(2,4)}{SO(1,4)} = \frac{\{\frac12(P_\mu+K_\mu), D, \frac12(P_\mu-K_\mu), M_{\mu\nu}\}}{\{\frac12(P_\mu-K_\mu), M_{\mu\nu}\}}\,, \qquad \text{(C.2)}$$

and the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ is

$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)} = \frac{\{\frac12(P_\mu+K_\mu), D, \frac12(P_\mu-K_\mu), M_{\mu\nu}, q_{\alpha i}, \bar q^i_{\dot\alpha}, U^i_j, s^i_\alpha, \bar s_{\dot\alpha i}\}}{\{\frac12(P_\mu-K_\mu), M_{\mu\nu}, U^i_j\}}\,.$$

In this thesis, our chosen coset representative for this last supercoset was given in (2.62) and is reproduced below

$$g = g(x,\theta,\bar\theta)\exp\left(i\,\psi^\alpha_j s^j_\alpha + i\,\bar\psi^j_{\dot\alpha}\bar s^{\dot\alpha}_j\right) z^D\,. \qquad \text{(C.3)}$$

In general, a coset is of the form G/H where G is a group and H one of its subgroups called the isotropy group in this context. The generators of the Lie algebra of G will be denoted here as $\{Y_A,\, X_B\}$. We have organized the generators in two sets Y_A and X_B where X_B is the set of generators of the subgroup H and Y_A

the remaining generators of G. Using this notation, for a given coset representative g the vielbeins e^A and the connections w^B are defined by

$$g^{-1}dg \;=\; (e^A Y_A + w^B X_B)\,.$$

As an example, we will compute the vielbeins and connections in the case of the $\mathcal{N}=4$ superspace using the coset representative $g(x,\theta,\bar\theta)$ of (C.1),

$$g(x,\theta,\bar\theta)^{-1}dg(x,\theta,\bar\theta) = e^\mu P_\mu + e^{\alpha j}\, q_{\alpha j} + e_{\dot\alpha j}\bar q^{\dot\alpha j}\,, \tag{C.4}$$

where all connections are zero in this example and using the $PSU(2,2|4)$ algebra of the Appendix B the non-zero components of the one-form vielbeins are

$$\begin{gathered} e^\mu_{\breve\mu} = \delta^\mu_{\breve\mu}\,, \quad e^\mu_{\breve\alpha\breve i} = -i(\sigma^\mu)_{\breve\alpha\breve{\dot\gamma}}\bar\theta^{\breve{\dot\gamma}}_{\breve i}\,, \quad e^{\mu,\breve{\dot\beta}\breve i} = -i(\bar\sigma^\mu)^{\breve{\dot\beta}\breve\beta}\theta^{\breve i}_{\breve\beta}\,, \\ e^{\breve{\dot\alpha}\breve j}_{\dot\alpha k} = i\delta^{\breve j}_k\delta^{\breve{\dot\alpha}}_{\dot\alpha}\,, \quad e^{\beta k}_{\breve\alpha\breve i} = i\delta^\beta_{\breve\alpha}\delta^k_{\breve i}\,. \end{gathered} \tag{C.5}$$

In this Appendix, we will sometimes distinguish curved indices from flat indices, one example being the results above. We remind the reader that in our conventions curved indices are similar to the flat indices except that they appear with a breve symbol. Under a global transformation of $PSU(2,2|4)$ with an element g_P of this group, the coset representative $g(x,\theta,\bar\theta)$ transforms as

$$g_P\, g(x,\theta,\bar\theta) \;=\; g(x',\theta',\bar\theta')h\,, \tag{C.6}$$

where h is an element of the isotropy group and the variables with prime are the transformed variables.

In the case of an infinitesimal transformation parametrized by ζ^A, we can write $X'^{\breve M} = X^{\breve M} + \delta X^{\breve M}$ where $X^{\breve M}$ is a shorthand notation for all the variables, and (C.6) becomes

$$(1+\zeta^A T_A)\, g(X) \;=\; (g(X) + \delta X^{\breve M}\frac{\partial g(X)}{\partial X^{\breve M}})\,(1+\delta h)\,, \tag{C.7}$$

where T_A means all the generators of the group. Multiplying from the left both sides of this equation with $g(X)^{-1}$, we conclude

$$g(X)^{-1}\,(\zeta^A T_A)\, g(X) = \delta X^{\breve M}\, e^A_{\breve M}\, Y_A + \text{isotropy}\,. \tag{C.8}$$

The above equation is the main formula that we will use in order to understand how the variables transform close to the boundary of AdS. We will illustrate its use with simple examples. Consider that the only non-zero ζ^A is ζ^i_j with $\zeta^i_i = 0$ and replacing $g(X)$ by $g(x,\theta,\bar\theta)$, the left-hand side of (C.8) becomes

$$\begin{gathered} g^{-1}(x,\theta,\bar\theta)\,(\zeta^i_j U^j_i)\, g(x,\theta,\bar\theta) = \\ \zeta^i_j U^j_i + i\theta^{\gamma k}\zeta^i_k q_{\gamma i} - i\bar\theta_{\dot\gamma i}\zeta^i_j\bar q^{\dot\gamma j} - 2i\zeta^i_k\theta^{\gamma k}(\sigma^\mu)_{\gamma\dot\gamma}\bar\theta^{\dot\gamma}_i P_\mu = \\ i\theta^{\gamma k}\zeta^i_k q_{\gamma i} - i\bar\theta_{\dot\gamma i}\zeta^i_j\bar q^{\dot\gamma j} - 2i\zeta^i_k\theta^{\gamma k}(\sigma^\mu)_{\gamma\dot\gamma}\bar\theta^{\dot\gamma}_i P_\mu + \text{isotropy} \end{gathered} \tag{C.9}$$

where we have used the Hadamard lemma of (3.1) for performing the computation. Equating this result with the right-hand side of (C.8) we have

$$i\theta^{\gamma k}\zeta^i_k q_{\gamma i} - i\bar\theta_{\dot\gamma i}\zeta^i_j\bar q^{\dot\gamma j} - 2i\zeta^i_k\theta^{\gamma k}(\sigma^\mu)_{\gamma\dot\gamma}\bar\theta^{\dot\gamma}_i P_\mu =$$
$$(\delta x^\mu e^\nu_\mu + \delta\theta^{\alpha j}e^\nu_{\alpha j} + \delta\bar\theta_{\dot\gamma i}e^{\nu,\dot\gamma i})P_\nu + (\delta\theta^{\alpha j}e^{ak}_{\alpha j})q_{ak} + (\delta\bar\theta_{\dot\alpha j}e^{\dot\alpha j}_{\dot\gamma k})\bar q^{\dot\gamma k}\,,$$

and solving for δx^μ, $\delta\theta^{\alpha j}$ and $\delta\bar\theta_{\dot\alpha j}$, taking into account the expressions for the vielbeins given in (C.5), one easily concludes

$$\delta x^\mu = 0\,,\quad \delta\theta^{\alpha j} = \zeta^j_k\theta^{\alpha k}\,,\quad \delta\bar\theta_{\dot\alpha j} = -\zeta^i_j\bar\theta_{\dot\alpha i}\,. \tag{C.10}$$

In the next example, let us consider that the only non-zero ζ^A is ζ^μ, then

$$g^{-1}(x,\theta,\bar\theta)\,(\zeta^\mu P_\mu)\,g(x,\theta,\bar\theta) = \zeta^\mu P_\mu\,, \tag{C.11}$$

and following the same steps of the first example, it is easy to see that

$$\delta x^\mu = \zeta^\mu\,,\quad \delta\theta^{\alpha j} = 0\,,\quad \delta\bar\theta_{\dot\alpha j} = 0\,.$$

We will give two more examples where the only non-zero ζ^A are ζ and $[\zeta^{\alpha j}, \bar\zeta_{\dot\alpha j}]$. The first case is

$$\begin{aligned} g^{-1}(x,\theta,\bar\theta)\,(\zeta D)\,g(x,\theta,\bar\theta) &= \zeta D + \zeta x^\nu P_\nu + \frac{i}{2}\zeta\theta^{\gamma k}q_{\gamma k} + \frac{i}{2}\zeta\bar\theta_{\dot\gamma j}\bar q^{\dot\gamma j} \\ &= \zeta x^\nu P_\nu + \frac{i}{2}\zeta\theta^{\gamma k}q_{\gamma k} + \frac{i}{2}\zeta\bar\theta_{\dot\gamma j}\bar q^{\dot\gamma j} + \text{isotropy}\,, \end{aligned}$$

and

$$\delta x^\mu = \zeta x^\mu\,,\quad \delta\theta^{\alpha j} = \frac{1}{2}\zeta\theta^{\alpha j}\,,\quad \delta\bar\theta_{\dot\alpha j} = \frac{1}{2}\zeta\bar\theta_{\dot\alpha j}\,,$$

finally,

$$g^{-1}(x,\theta,\bar\theta)\,(i\zeta^{\alpha j}q_{\alpha j} + i\bar\zeta_{\dot\alpha j}\bar q^{\dot\alpha j})\,g(x,\theta,\bar\theta) =$$
$$i\zeta^{\alpha j}q_{\alpha j} + i\bar\zeta_{\dot\alpha j}\bar q^{\dot\alpha j} + 2i\theta^{\gamma k}(\sigma^\mu)_{\gamma\dot\gamma}\bar\zeta^{\dot\gamma}_k P_\mu - 2i\zeta^{\alpha j}(\sigma^\mu)_{\alpha\dot\delta}\bar\theta^{\dot\delta}_j P_\mu\,,$$

which implies

$$\delta x^\mu = -i\zeta^{\alpha j}(\sigma^\mu)_{\alpha\dot\gamma}\bar\theta^{\dot\gamma}_j - i\bar\zeta_{\dot\alpha j}(\bar\sigma^\mu)^{\dot\alpha\alpha}\theta^j_\alpha\,,\quad \delta\theta^{\alpha j} = \zeta^{\alpha j}\,,\quad \delta\bar\theta_{\dot\alpha j} = \bar\zeta_{\dot\alpha j}.$$

Using a similar reasoning, it is not difficult to compute the remaining $\mathcal{N} = 4$ $d = 4$ superconformal transformations of the variables not computed above. We will proceed to study the global $PSU(2,2|4)$ transformations of the variables that parametrize the AdS supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(6)}$ with the coset representative being g

of (C.3). Firstly, let us consider the case $\psi = \bar{\psi} = 0$ and compute the vielbeins and connections for this case, the results are

$$g^{-1}dg|_{\psi=\bar{\psi}=0} = \frac{1}{z}e^{\mu}P_{\mu} + \frac{1}{\sqrt{z}}e^{\alpha j}q_{\alpha j} + \frac{1}{\sqrt{z}}e_{\dot{\alpha}j}\bar{q}^{\dot{\alpha}j} + \frac{dz}{z}D + \sqrt{z}id\psi^{\alpha}_{j}s^{j}_{\alpha} + \sqrt{z}id\bar{\psi}^{j}_{\dot{\alpha}}\bar{s}^{\dot{\alpha}}_{j} ,$$

where $e^{\mu}, e^{\alpha j}, e_{\dot{\alpha}j}$ were defined in (C.5). Analyzing these results, we conclude that

$$e^{\mu}_{(AdS_0)} = \frac{1}{z}e^{\mu}_{(\mathcal{N}=4)} , \quad e^{\alpha j}_{(AdS_0)} = \frac{1}{\sqrt{z}}e^{\alpha j}_{(\mathcal{N}=4)} , \quad e_{\dot{\alpha}j(AdS_0)} = \frac{1}{\sqrt{z}}e_{\dot{\alpha}j(\mathcal{N}=4)} , \tag{C.12}$$
$$e^{z} = \frac{1}{z} , \quad e^{\alpha\breve{k}}_{j\breve{\alpha}} = i\sqrt{z}\delta^{\alpha}_{\breve{\alpha}}\delta^{\breve{k}}_{j} , \quad e^{j\breve{\dot{\alpha}}}_{\dot{\alpha}\breve{k}} = i\sqrt{z}\delta^{j}_{\breve{k}}\delta^{\breve{\dot{\alpha}}}_{\dot{\alpha}} ,$$

and the subscript (AdS_0) means that the vielbeins were computed with the AdS supercoset representative of (C.3) with $\psi = \bar{\psi} = 0$ and the subscript $(\mathcal{N} = 4)$ means computation performed with the $\mathcal{N} = 4$ coset representative of (C.1). Using these results, one can compute the inverse vielbeins that are defined by the relations $e^{\breve{M}}_{N} e^{N}_{\breve{P}} = \delta^{\breve{M}}_{\breve{P}}$ and $e^{M}_{\breve{N}} e^{\breve{N}}_{P} = \delta^{M}_{N}$. After a straightforward calculation, one has

$$e_{\mu(AdS_0)} = ze_{\mu(\mathcal{N}=4)} , \quad e_{\alpha j(AdS_0)} = \sqrt{z}e_{\alpha j(\mathcal{N}=4)} , \quad e^{\dot{\alpha}j}_{(AdS_0)} = \sqrt{z}e^{\dot{\alpha}j}_{(\mathcal{N}=4)} , \tag{C.13}$$
$$e_{z} = z , \quad e^{j\breve{\alpha}}_{\alpha\breve{k}} = -i\frac{1}{\sqrt{z}}\delta^{\breve{\alpha}}_{\alpha}\delta^{j}_{\breve{k}} , \quad e^{\dot{\alpha}\breve{k}}_{j\breve{\dot{\alpha}}} = -i\frac{1}{\sqrt{z}}\delta^{\breve{k}}_{j}\delta^{\dot{\alpha}}_{\breve{\dot{\alpha}}} .$$

In order to use the formula (C.8), we first compute its left-hand side

$$g^{-1}\left(\zeta^{A}T_{A}\right)g = \tag{C.14}$$
$$\frac{1}{z}\zeta^{\mu}_{(\mathcal{N}=4)}P_{\mu} + \frac{1}{\sqrt{z}}\zeta^{\alpha j}_{(\mathcal{N}=4)}q_{\alpha j} + \frac{1}{\sqrt{z}}\zeta_{\dot{\alpha}j(\mathcal{N}=4)}\bar{q}^{\dot{\alpha}j} + \hat{\zeta}D + \sqrt{z}i\hat{\zeta}^{\alpha}_{j}s^{j}_{\alpha} + \sqrt{z}i\hat{\zeta}^{j}_{\dot{\alpha}}\bar{s}^{\dot{\alpha}}_{j}$$
$$+\ldots ,$$

and after comparing with the right-hand side of (C.8), we conclude, for example, that $\delta x^{\breve{\mu}} = \zeta^{\breve{\mu}}$ with $\zeta^{\breve{\mu}} = z^{-1}\zeta^{\nu}_{(\mathcal{N}=4)}e^{\breve{\mu}}_{\nu} + z^{-\frac{1}{2}}\zeta^{\alpha j}_{(\mathcal{N}=4)}e^{\breve{\mu}}_{\alpha j} + z^{-\frac{1}{2}}\zeta_{\dot{\alpha}j(\mathcal{N}=4)}e^{\breve{\mu},\dot{\alpha}j}$ where $e^{\breve{\mu}}_{A}$ are the inverse vielbeins. Using the expressions of (C.13) and the result of (C.14), it is not difficult to see that

$$\delta_{ads}x|_{\psi=\bar{\psi}=0} = \delta_{sc}x + O(z^{2}) , \quad \delta_{ads}\theta^{\alpha i}|_{\psi=\bar{\psi}=0} = \delta_{sc}\theta^{\alpha i} , \tag{C.15}$$
$$\delta_{ads}\psi^{i}_{\alpha}|_{\psi=\bar{\psi}=0} = \hat{\zeta}^{i}_{\alpha} , \quad \delta_{ads}z|_{\psi=\bar{\psi}=0} = z\hat{\zeta} ,$$

with similar results for $\bar{\theta}$ and $\bar{\psi}$. The subscript sc above means the usual $\mathcal{N} = 4$ $d = 4$ superconformal transformations. The presence of the factor $O(z^2)$ on the x transformation will be explained in more detail in what follows, but one briefly explanation is that the correct basis of the Lie algebra generators for describing the AdS space is $\frac{1}{2}(P_{\mu} + K_{\mu})$ and $\frac{1}{2}(P_{\mu} - K_{\mu})$ as in (C.2) instead of the basis P_{μ} and K_{μ}, and the factor in question comes from a basis rotation.

The transformations (C.15) show that, at least in the case $\psi = \bar{\psi} = 0$, the variables x and θ transform in the usual $\mathcal{N} = 4$ $d = 4$ superconformal manner close to the boundary of AdS at $z \sim 0$. Note that we cannot set $\psi = \bar{\psi} = 0$ as a boundary condition at $z \sim 0$ because making a $PSU(2,2|4)$ transformation the values of these variables change. It is interesting to note that if instead of using the coset representative g of (C.3), one uses the coset representative g' given below

$$g' = g(x,\theta,\bar{\theta})\, z^D \exp(i\,\psi_j^\alpha s_\alpha^j + i\,\bar{\psi}_{\dot{\alpha}}^j \bar{s}_j^{\dot{\alpha}})\,,$$

the transformations of ψ and $\bar{\psi}$ are changed to

$$\delta_{ads}\psi_\alpha^i|_{\psi=\bar{\psi}=0} = \sqrt{z}\hat{\zeta}_\alpha^i\,, \quad \delta_{ads}\bar{\psi}_{\dot{\alpha}i}|_{\psi=\bar{\psi}=0} = \sqrt{z}\hat{\zeta}_{\dot{\alpha}i}\,, \tag{C.16}$$

and it is now consistent to set $\psi = \bar{\psi} = 0$ as a boundary condition because these variables do not transform when $z \sim 0$.

All the analysis of the transformations of the variables of the AdS supercoset were performed with the simplying assumption $\psi = \bar{\psi} = 0$. Let us now consider the general case where these variables have arbitrary value, we expect that (C.15) changes to

$$\delta_{ads}x = \delta_{sc}x + O(z^2) + f(z,\psi,X)\,, \quad \delta_{ads}\theta = \delta_{sc}\theta + g(z,\psi,X)\,,$$
$$\delta_{ads}\psi = \zeta_{\tilde{w}} + w(z,\psi,X)\,, \quad \delta_{ads}z = z\zeta + j(z,\psi,X)\,,$$

where f, g, w and j are functions that vanish when $\psi = \bar{\psi} = 0$.

We are going to focus on the transformations of the variables $[x,\theta,\bar{\theta}]$, because similar arguments can be used for understanding the transformations of $[z,\psi,\bar{\psi}]$. We start by considering the left-hand side of (C.8),

$$\begin{aligned} g^{-1}\,(\zeta^A T_A)\,g = z^{-D}\,e^{-(\psi\cdot s)}\,g(x,\theta,\bar{\theta})^{-1}\,(\zeta^A T_A)\,g(x,\theta,\bar{\theta})\,e^{(\psi\cdot s)}\,z^D = \\ z^{-D}\,e^{-(\psi\cdot s)}\,(\zeta^\mu_{(\mathcal{N}=4)}P_\mu + \zeta^{\alpha j}_{(\mathcal{N}=4)}q_{\alpha j} + \zeta_{\dot{\alpha}j(\mathcal{N}=4)}\bar{q}^{\dot{\alpha}j} + \ldots)\,e^{(\psi\cdot s)}\,z^D \end{aligned} \tag{C.17}$$

where again the subscript ($\mathcal{N} = 4$) means the result of a computation performed with the coset representative (C.1) describing the $\mathcal{N} = 4$ $d = 4$ superspace. The next step is to compute the veilbeins,

$$\begin{aligned} g^{-1}dg &= z^{-D}\,e^{-(\psi\cdot s)}\,g(x,\theta,\bar{\theta})^{-1}(dg(x,\theta,\bar{\theta}))\,e^{(\psi\cdot s)}\,z^D + z^{-D}\,e^{-(\psi\cdot s)}d(e^{(\psi\cdot s)}\,z^D) \\ &= z^{-D}\,e^{-(\psi\cdot s)}\,(e^\mu_{(\mathcal{N}=4)}P_\mu + e^{\alpha j}_{(\mathcal{N}=4)}q_{\alpha j} + \bar{e}_{\dot{\alpha}j(\mathcal{N}=4)}\bar{q}^{\dot{\alpha}j})\,e^{(\psi\cdot s)}\,z^D + \ldots \end{aligned} \tag{C.18}$$

where $\ldots$ above only contains terms proportional to $d\psi$ and dz. The formula (C.8) implies that we have to equate (C.17) and (C.18) after contracting the second one which is a one-form with the vector $\delta\vec{X}$. The result after multiplying from the left by $e^{(\psi\cdot s)}\,z^D$ and from the right by $z^{-D}\,e^{-(\psi\cdot s)}$ is

$$\begin{aligned}(\zeta^\mu_{(\mathcal{N}=4)}P_\mu + \zeta^{\alpha j}_{(\mathcal{N}=4)}q_{\alpha j} + \zeta_{\dot\alpha j(\mathcal{N}=4)}\bar{q}^{\dot\alpha j} + \ldots) = \\ (\delta X^{\check{M}} e^\mu_{\check{M}(\mathcal{N}=4)}P_\mu + \delta X^{\check{M}} e^{\alpha j}_{\check{M}(\mathcal{N}=4)}q_{\alpha j} + \delta X^{\check{M}}\bar{e}_{\check{M}\dot\alpha j(\mathcal{N}=4)}\bar{q}^{\dot\alpha j}) + \ldots\end{aligned} \tag{C.19}$$

where the ... on the right-hand side of the expression above only contains terms proportional to $[\delta\psi, \delta\bar{\psi}, \delta z]$. Apart from ... this is the same equation obtained for the case of the $\mathcal{N}=4$ supercoset and it seems to imply that the transformations of $[x, \theta, \bar{\theta}]$ would be the superconformal $\mathcal{N}=4$ $d=4$ for any value of $[z, \psi, \bar{\psi}]$. However, there is a subtlety that may alter the transformations, recall that we have to reorganize both sides of the result above in the correct basis of the Lie algebra for describing AdS, consider for example,

$$\begin{aligned}\delta X^{\check{M}} e^\mu_{\check{M}}P_\mu + \delta X^{\check{M}} e^\mu_{\check{M}(K)}K_\mu = \\ \delta X^{\check{M}}(e^\mu_{\check{M}} + e^\mu_{\check{M}(K)})\frac{1}{2}(P_\mu + K_\mu) + \delta X^{\check{M}}(e^\mu_{\check{M}} - e^\mu_{\check{M}(K)})\frac{1}{2}(P_\mu - K_\mu)\,,\end{aligned} \tag{C.20}$$

and from (C.18), it is not difficult to see that $e^\mu_{\check{M}} \sim z^{-1}$ and $e^\mu_{\check{M}(K)} \sim z$, which implies that changing the basis may give corrections of order z^2 to the result. We finally have

$$\begin{aligned}\delta_{ads}x = \delta_{sc}x + O(z^2)f(\psi, X)\,, \quad \delta_{ads}\theta = \delta_{sc}\theta + O(z^2)g(\psi, X) \\ \delta_{ads}\psi = \zeta_{\tilde{w}} + w(z,\psi)\,, \quad \delta_{ads}z = z\zeta + j(z,\psi)\,.\end{aligned}$$

The conclusion which follows from the results above is that when $z \sim 0$ at the AdS boundary the variables $[x, \theta, \bar{\theta}]$ transform in the usual $\mathcal{N}=4$ $d=4$ superconformal manner.

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