

Topical Review

Gauge theoretic approach to (ordinary) gravity and its fuzzy extensions in three and four dimensions

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Abstract

In the prospect to discuss the construction of fuzzy gravity theories based on the gauge-theoretic approach of ordinary gravity, in the present article we review first the latter in three and four dimensions and then, after recalling the formulation of gauge theories on noncommutative spaces, we present in detail the construction of fuzzy gravity theories in three and four dimensions, as matrix models.

Keywords: gauge theories, gauge theories of gravity, four-dimensional gravity, noncommutative spaces, fuzzy de Sitter, noncommutative gravity, spontaneous symmetry breaking

1. Introduction

One of the main research areas addressing the problem of the lack of knowledge of the spacetime quantum structure is based on the idea that at extremely small distances (Planck

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length) the coordinates might exhibit a noncommutative structure. Then a natural aim is the construction of a noncommutative generalization of the General theory of Relativity (GR), which becomes essentially noncommutative, particularly in regions where the commutative limit would be singular. In this framework, the description of space-time using a set of commuting coordinates would only be valid at length scales larger than some fundamental one. At smaller scales it would be impossible to localize a point and a new geometry should be used. Accepting this picture we can think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by ‘coarse-graining’ over regions whose size is determined by a fundamental area scale, which is presumably but not necessarily, of the order of the Planck area $G\hbar$. Their usual notion breaks down and should be replaced by elements of a noncommutative algebra when one considers phenomena at higher scales. On the other hand at more ordinary (say LHC) distances the Strong, Weak and Electromagnetic interactions are successfully formulated using gauge theories, while at much smaller distances the Grand Unified Gauge Theories provide a very attractive unification scheme of the three interactions. The gravitational interaction is not part of this picture, admitting a geometric formulation, that of GR. However, there exists a gauge-theoretic approach to gravity besides the geometric one [1–12]. This approach started with the pioneering work of Utiyama [1] and was refined by other authors [3] as a gauge theory of the de Sitter $SO(1,4)$ group, spontaneously broken by a scalar field to the Lorentz $SO(1,3)$ group. Similarly using the gauge-theoretic approach, Weyl gravity has been constructed as a gauge theory of the four-dimensional conformal group [7, 8]. Also, three-dimensional gravity is translated in the gauge-theoretic picture as a Chern–Simons gauge theory of the $ISO(1,2)$ group in an exact way [12], contrary to the rest of the cases that require further actions (symmetry breaking) to achieve complete correspondence.

It is worth mentioning that in the process of specifying the associated gauge group in each case, a part of the set of the gauge fields that are assigned are identified to be the vielbein and the spin connection. In this way the correspondence among the geometric and gauge-theoretic approaches is achieved through the first order formulation of gravity which is a geometric interpretation of GR, however, the dynamical degrees of freedom are described in terms of the vierbein and spin connection instead of the metric.

Then returning to the noncommutative framework and taking into account the gauge-theoretic description of gravity, the well-established formulation of gauge theories on noncommutative spaces leads to the construction of models of noncommutative gravity [13–22]. In these treatments the authors use the constant noncommutativity (Moyal–Weyl), the formulation of the star-product and the Seiberg and Witten map [23].

In addition to the above treatments, noncommutative gravitational models can be constructed using the noncommutative realization of matrix geometries [24–36], while it should also be noted that there exist alternative approaches [37–39]. Both the latter directions will not be discussed further here.

The construction of quantum field theories on noncommutative spaces is a difficult task though and, furthermore, problematic ultraviolet features have appeared [40–46]. However, noncommutative geometry has been proposed as a framework to construct particle physics models with noncommutative gauge theories [47–50] (see also [51–61]). It is worth noting that a very interesting development in the framework of noncommutative geometry is the programme in which extra dimensions of higher-dimensional theories are considered to be noncommutative (fuzzy) [62–73]. This programme overcomes the ultraviolet/infrared problematic behaviors of theories defined on noncommutative spaces. A very welcome feature of such theories is that they are renormalizable, contrary to all known higher-dimensional theories, which gives serious hopes for a similar behavior of the four-dimensional theory of gravity that will

be discussed later here. Also, it is very appealing that this programme results in phenomenologically promising four-dimensional unified theories.

Our orientation is towards the matrix-realized models, although equivalent descriptions using the star product can be studied. Specifically, we focus on a particular class of noncommutative spaces, called covariant [74–80], which are characterized by the very important property for our purposes that they preserve Lorentz covariance [27, 81–83]. In particular, we focus on a very interesting class of models that can be constructed on the so-called fuzzy spaces, which is a subclass of noncommutative spaces which preserve the isometries of their commutative analogues. The most typical example of such a space is the fuzzy 2-sphere [76], whose isometry group is $SO(3)$ and in the commutative limit the ordinary 2-sphere is recovered.

It is worth recalling that a fuzzy two-sphere [76] (see also [84, 85]) is constructed from finite-dimensional matrices and the size of matrices represents the number of quanta on the noncommutative manifold. The fuzzy sphere, S_F^2 , at fuzziness level $N-1$, is the noncommutative manifold whose coordinate functions are $N \times N$ matrices proportional to the generators of the N -dimensional representation of $SU(2)$. Introducing a cutoff parameter $N-1$ for angular momentum in a two-sphere, the number of independent functions is N^2 . Then one can replace the functions defined on this noncommutative manifold by $N \times N$ matrices, and therefore algebras on the sphere become noncommutative.

However, a generalization to a higher-dimensional sphere is not straightforward. In particular, in the case of a four-dimensional sphere, the same procedure leads to a number of independent functions which is not a square of an integer. Therefore, one cannot construct a map from functions to matrices. One can restate this difficulty algebraically. Algebras of a fuzzy four-sphere have been constructed in [81] and the difference from the fuzzy two-sphere case is that the commutators of the coordinates do not close. This is the source of the difficulties to analyze field theories on the fuzzy four-sphere (see [81] and references therein; for more details about fuzzy four-sphere see [86, 87]). In [88] (see also [89, 90]), we started a programme realizing gravity as noncommutative gauge theory in three dimensions. Specifically, we considered three-dimensional noncommutative spaces based on $SU(2)$ and $SU(1,1)$, as foliations of fuzzy two-spheres [77, 91–94] and fuzzy two-hyperboloids [95], respectively. This onion-like construction led to a matrix model, which was analyzed in a straightforward way.

It should be also stressed that the formulation of noncommutative gravity implies, in general, noncommutative deformations which break the Lorentz invariance. However, ‘covariant noncommutative spaces’ have been constructed too, preserving the Lorentz invariance [75, 78].

Another feature that should be particularly noted is that on these spaces the gauge theories that are built use as gauge groups their isometry groups, which however are eventually enlarged due to the inclusion of more operators as generators. The extension of the set of the generators is due to the necessity of the closure of the anticommutators, which is of high importance in the noncommutativity framework, where they appear naturally. Besides the enlargement, in order that the anticommutators of the generators stop producing operators that are not generators of the algebra, their representation must be specified. One more feature that appears in developing the noncommutative gauge theory is that covariant coordinate is introduced, that is the analogue of the covariant derivative in the ordinary ones, and the gauge fields are also involved. Among the various gauge fields, the vielbein and the spin connection are introduced and their transformation rules are determined. Consequently, noncommutative deformations of field theories have been constructed [14, 96–99] (see also [79–83]).

In our next contribution to the subject, we worked on the more realistic, four-dimensional case [96–98, 100–103]. First, motivated by [78], we got involved with the construction of a

suitable -for our purposes- four-dimensional covariant noncommutative space, which served as the background space on which the gauge theory was developed. The space we formulated was a fuzzy version of the four-dimensional dS space obtaining, among other features, the defining commutation relation of the coordinates of the space. Being a fuzzy space, its isometry group is that of the four-dimensional dS space, i.e. the $SO(1,4)$. As noted earlier, due to the involvement of the anticommutators, the gauge group expands to the $SO(2,4) \times U(1)$ and the representation is fixed, which means that the generators are represented by 4×4 matrices. The development of the gauge theory leads to the transformation rules of the gauge fields introduced and to the expressions of the various tensors. For the field strength tensor to transform covariantly, an auxiliary antisymmetric 2-form field was introduced in the theory and eventually, we proposed a gauge-invariant action of the constructed gravity theory.

Since our aim was to result with a theory respecting the Lorentz symmetry, we imposed first certain constraints in order to break the initial symmetry. This symmetry breaking was achieved in a subsequent work [100] using the usual Higgs mechanism and a Lagrange multiplier. After the symmetry breaking, the action took its final form and its variation led to the equations of motion. It should be also noted that, before the symmetry breaking, the results of the above construction reduce to the ones of the conformal gravity in the commutative limit.

Following the above, the outlook of this article is as follows: First, we remind briefly the gauge-theoretic analogue of the three- and four-dimensional GR and then, after a brief reminder of the toolkit for noncommutative gauge theories, we write down the extensions of the above gauge-theoretic description of gravity on noncommutative spaces, as viewed from our perspective.

2. Gauge-theoretic approach to gravity theories

In the following section we will make a brief review of the formulation of gravity in three and four dimensions as gauge theories of their respective isometry groups. This formulation provides an alternative description of gravity, apart from the geometric one.

2.1. Three-Dimensional case

In three dimensions, it is possible to recover the first order formulation of gravity as a gauge theory of the Poincaré algebra $ISO(1,2)$ ⁶ [12]. The generators of these algebras are the ones of the Lorentz transformations J_{ab} together with those of local translations P_a , where $a, b = 1, 2, 3$. These generators satisfy the following commutation relations:

$$[J_{ab}, J_{cd}] = 4\eta_{[a[c}J_{d]b]} , \quad [P_a, J_{bc}] = 2\eta_{[a[b}P_{c]} , \quad [P_a, P_b] = \Lambda J_{ab} , \quad (1)$$

where $\eta_{ab} = \text{diag}(-1, 1, 1)$ is the Minkowski metric and Λ is the cosmological constant. Specifically in the three-dimensional case, using the definition $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$, one is allowed to rewrite the above commutation relations in the following convenient way:

$$[J_a, J_b] = \epsilon_{abc}J^c , \quad [P_a, J_b] = \epsilon_{abc}P^c , \quad [P_a, P_b] = \Lambda\epsilon_{abc}J^c . \quad (2)$$

A gauge field is then introduced for each generator of the algebra. More specifically the dreibein⁷ $e_\mu{}^a$ is introduced as the gauge field corresponding to translations and the spin

⁶ In the case that a cosmological constant is present, the corresponding algebra would be the de Sitter $SO(1,3)$ or the Anti de Sitter $SO(2,2)$, depending on the sign of the constant.

⁷ The dreibein is the three-dimensional expression for the vielbein.

connection $\omega_\mu^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}$ as the one corresponding to the Lorentz transformations. Consequently the gauge connection will be given as

$$A_\mu(x) = e_\mu^a(x)P_a + \omega_\mu^a(x)J_a. \quad (3)$$

Its transformation in the adjoint representation will follow the standard rule

$$\delta A_\mu = \partial_\mu \varepsilon + [A_\mu, \varepsilon], \quad (4)$$

where

$$\varepsilon(x) = \xi^a(x)P_a + \lambda^a(x)J_a \quad (5)$$

is an infinitesimal gauge transformation parameter. Therefore, the transformations of the dreibein and the spin connection are found to be

$$\delta e_\mu^a = \partial_\mu \xi^a - \epsilon^{abc}(\xi_b \omega_{\mu c} + \lambda_b e_{\mu c}), \quad (6)$$

$$\delta \omega_\mu^a = \partial_\mu \lambda^a - \epsilon^{abc}(\lambda_b \omega_{\mu c} + \Lambda \xi_b e_{\mu c}). \quad (7)$$

Furthermore, the field strength tensor can be found using the standard formula

$$R_{\mu\nu}(A) = 2\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]. \quad (8)$$

Given the above expressions, together with the decomposition of the field strength tensor on the generators

$$R_{\mu\nu}(A) = T_{\mu\nu}^a P_a + R_{\mu\nu}^a J_a, \quad (9)$$

the corresponding curvatures of the dreibein (torsion tensor) and the spin connection (curvature 2-form) are found

$$T_{\mu\nu}^a = 2\partial_{[\mu}e_{\nu]}^a + 2\epsilon^{abc}\omega_{[\mu b}e_{\nu]c}, \quad (10)$$

$$R_{\mu\nu}^a = 2\partial_{[\mu}\omega_{\nu]}^a + \epsilon^{abc}(\omega_{\mu b}\omega_{\nu c} + \Lambda e_{\mu b}e_{\nu c}). \quad (11)$$

Concluding, the Einstein–Hilbert action in the three dimensional case

$$S_{EH3} = \frac{1}{16\pi G} \int_M \epsilon^{\mu\nu\rho} \left(e_\mu^a (\partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a}) + \epsilon_{abc} e_\mu^a \omega_\nu^b \omega_\rho^c + \frac{1}{3} \Lambda \epsilon_{abc} e_\mu^a e_\nu^b e_\rho^c \right) \quad (12)$$

is identical to the action functional of a Chern–Simons gauge theory of the Poincaré algebra, upon the choice of an appropriate quadratic form in the algebra [12]. The standard choice for the above is

$$tr(J^a P^b) = \delta^{ab}, \quad tr(J^a J^b) = tr(P^a P^b) = 0. \quad (13)$$

At this point, it is noted that, specifically in three dimensions, for a non-zero value of the cosmological constant, there is an alternative non-degenerate, invariant quadratic form given by

$$tr(J^a P^b) = 0, \quad \frac{1}{\Lambda} tr(P^a P^b) = tr(J^a J^b) = \delta^{ab}. \quad (14)$$

The above, alternative consideration yields a different action, that is classically equivalent to the one in (12) [12].

Given the above, it is shown that gravity in three dimensions is formulated with success as a Chern–Simons type gauge theory, in equal parts for the transformations of its gauge fields and its dynamics.

2.2. Four-dimensional case

Like in the three-dimensional case described in the previous subsection, we have to employ the vierbein formalism in order to construct four-dimensional gauge theory of gravity. In absence of cosmological constant, the isometry group of the Minkowski spacetime is $ISO(1, 3)$ (the Poincaré group) and it is the one that will be considered as the gauge group, in accordance with the three-dimensional case, where isometry groups of the Minkowski, dS and AdS were considered as the gauge groups. The Poincaré algebra comprises of ten generators, four local translations, P_a and six Lorentz transformations, M_{ab} , satisfying the following commutation relations:

$$[M_{ab}, M_{cd}] = 4 \eta_{[a[c} M_{d]b]}, \quad [P_a, M_{bc}] = 2 \eta_{a[b} P_{c]}, \quad [P_a, P_b] = 0, \quad (15)$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the four-dimensional Minkowski metric. Following the standard procedure, the gauge covariant derivative is defined as:

$$D_\mu = \partial_\mu + [\mathcal{A}_\mu, \cdot], \quad (16)$$

where $\mathcal{A}_\mu(X)$ is the gauge connection. Expansion of the connection on the generators of $ISO(1, 3)$ gives the expression:

$$\mathcal{A}_\mu(X) = e_\mu^a(X) P_a + \frac{1}{2} \omega_\mu^{ab}(X) M_{ab}, \quad (17)$$

where e_μ^a and ω_μ^{ab} are identified as the component gauge fields for the translations and Lorentz transformations, respectively. By definition, transformation of D_μ is covariant, therefore, the transformation law for the gauge connection is given by:

$$\delta \mathcal{A}_\mu = D_\mu \varepsilon = \partial_\mu \varepsilon + [\mathcal{A}_\mu, \varepsilon], \quad (18)$$

where $\varepsilon = \varepsilon(X)$ is a gauge transformation parameter, which, as an element of the $ISO(1, 3)$ algebra, it may be written as an expansion on the generators:

$$\varepsilon(X) = \xi^a(X) P_a + \frac{1}{2} \lambda^{ab}(X) M_{ab}, \quad (19)$$

with $\xi^a(X)$ and $\lambda^{ab}(X)$ being infinitesimal parameters. Combination of (17)–(19) leads to the expression for the transformation of the component gauge fields:

$$\begin{aligned} \delta e_\mu^a &= \partial_\mu \xi^a + \omega_\mu^{ab} \xi_b - \lambda_b^a e_\mu^b \\ \delta \omega_\mu^{ab} &= \partial_\mu \lambda^{ab} + \lambda_c^a \omega_\mu^{bc} - \lambda_c^b \omega_\mu^{ac}. \end{aligned} \quad (20)$$

The corresponding field strength tensors, $T_{\mu\nu}^a$ and $R_{\mu\nu}^{ab}$, of the component fields, e and ω , are obtained by the definition of the field strength tensor $R_{\mu\nu}$, of A_μ :

$$R_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (21)$$

after its expansion on the generators:

$$R_{\mu\nu} = T_{\mu\nu}^a P_a + \frac{1}{2} R_{\mu\nu}^{ab} M_{ab}. \quad (22)$$

Therefore, combining (17), (21) and (22), the expressions of the component tensors are:

$$\begin{aligned} T_{\mu\nu}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a - \omega_\mu^{ab} e_{\nu b} + \omega_\nu^{ab} e_{\mu b} \\ R_{\mu\nu}^{ab} &= \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} - \omega_\mu^{ac} \omega_{\nu c}^b + \omega_\nu^{ac} \omega_{\mu c}^b. \end{aligned} \quad (23)$$

Until this point, the construction of the gauge-theoretic version of four-dimensional gravity has been unfolding in a straightforward way. Moving on to the dynamical part of the theory, the

obvious choice would be an action of Yang-Mills type of the Poincaré group. Nevertheless, in order to claim a successful relation of four-dimensional gravity to a gauge theory, it is necessary to result with the Einstein–Hilbert action, which is, of course, not of Yang-Mills type.

First, it has to be noted that the desired action has to be invariant under the Lorentz transformations and not under the total Poincaré symmetry. Therefore, in order to reduce the symmetry of the action, a spontaneous symmetry breaking mechanism can be employed by the inclusion of a scalar field [3, 6]. For the present purpose, in order to achieve the incorporation of the spontaneous symmetry breaking mechanism, the gauge group, i.e. the gauge symmetry of the action of Yang-Mills type, has to be the de Sitter, $SO(1, 4)$, group, instead of the Poincaré. The choice of the de Sitter group is strategic, in the sense that it comprises of the same number of generators as the Poincaré, but carries an extra and useful virtue, that is all generators can be considered on equal footing, denoting them all with a single gauge field, since it is a semisimple group. Spontaneous symmetry breaking can be induced by assigning the scalar field to the fundamental representation of $SO(1, 4)$ [3, 5]. Thus, the symmetry is reduced to the Lorentz with four out of ten generators, the translations, having been broken.

The above described procedure leads to the desired Einstein–Hilbert action and, therefore, Einstein four-dimensional gravity with cosmological constant is retrieved as a gauge theory of the de Sitter group.

3. Gauge theories and noncommutativity

In order to proceed to the noncommutative framework of the theory, first we have to recall how gauge theories are formulated in noncommutative spaces, as described in [104].

Let us consider a scalar field, $\phi(X)$, where X are the coordinates of the noncommutative space. The gauge transformation of the scalar field will be non-trivial and infinitesimally it will be:

$$\delta\phi(X) = \varepsilon(X)\phi(X), \quad (24)$$

where $\varepsilon(X)$ is the gauge parameter. Now, because of the trivial gauge transformation of the coordinates, the quantity $X_\mu\phi(X)$ transforms as following:

$$\delta(X_\mu\phi(X)) = X_\mu\varepsilon(X)\phi(X). \quad (25)$$

Due to noncommutativity of the coordinates, it can be easily proven that the above transformation is not covariant. In order to resolve that problem, the covariant coordinate is introduced, which is defined through its covariant transformation,

$$\delta(\mathcal{X}_\mu\phi(X)) \equiv \varepsilon(X)\mathcal{X}_\mu\phi(X), \quad (26)$$

and restores the analogy with the ordinary gauge theories. The above covariant transformation holds if $\delta\mathcal{X}_\mu = [\varepsilon(X), \mathcal{X}_\mu]$ or, in other words, if the covariant coordinate transforms covariantly, which is something true by definition. In order to configure the covariant coordinate, a field $\mathcal{A}_\mu(X)$ has to be introduced with the following transformation rule:

$$\delta\mathcal{A}_\mu(X) = -[X_\mu, \varepsilon(X)] + [\varepsilon(X), \mathcal{A}_\mu(X)]. \quad (27)$$

Thus, the ordinary coordinate is being replaced by the covariant one, which is equal to $\mathcal{X}_\mu = X_\mu + \mathcal{A}_\mu(X)$. It is now clear that the \mathcal{A}_μ field is interpreted as gauge connection and thus its corresponding strength tensor has to be defined. The field strength tensor, besides the commutator of the covariant coordinates, will also include an extra term in order to transform covariantly.

Formulating the theory on noncommutative spaces, we have to treat properly the anticommutators of the various operators. The commutator of two elements of an arbitrary gauge algebra, $\varepsilon(X) = \varepsilon^a(X)T_a$ and $\phi(X) = \phi^a(X)T_a$, where T_a are the generators of the algebra, is:

$$[\varepsilon, \phi] = \frac{1}{2} \{ \varepsilon^a, \phi^b \} [T_a, T_b] + \frac{1}{2} [\varepsilon^a, \phi^b] \{ T_a, T_b \}. \quad (28)$$

In the commutative case, ε^a and ϕ^b are ordinary functions that depend on the coordinates, thus their commutator vanishes and so does the last term of the above relation. However, since in the noncommutative setting the coordinates do not commute, in turn, these functions also do not and therefore the second term is now not vanishing. This naturally means that in the noncommutative case the gauge theory contains anticommutators of the generators, contrary to the ordinary gauge theories. Generally, the products of such anticommutators do not belong to the algebra, so their presence causes the problem that the algebra does not close. One way to address that problem would be to enlarge the algebra and include as generators all the possible operators that could come from these anticommutators. This solution would require more and more extensions of the algebra, as the new anticommutators would also not close, and, eventually, one would end up with an infinite-dimensional algebra, which, although useful in other contexts (e.g. in [17, 18, 51]), it is not optimal for our purposes here. A very meaningful way out of the above drawback is to consider that the products of these anticommutators also depend on the representation. This means that in a specific representation, the anticommutators of the generators will produce a limited number of different operators and, therefore, including them in the initial algebra eventually extends it only to a larger but finite one.

4. Noncommutative gravity: a matrix model

In this section we present the gauge-theoretic construction of the three and four-dimensional matrix models of noncommutative gravity. Naturally, in the process of building a noncommutative gauge theory, a noncommutative background space will be needed to accommodate it. Thus, we begin by specifying the appropriate three and four-dimensional covariant, noncommutative spaces that will act as background spaces for each case respectively. Following that, we will present the three and four-dimensional gravity models, constructed as noncommutative gauge theories on the above spaces.

4.1. Three-dimensional case

4.1.1. The \mathbb{R}_λ^3 space. The most well-known covariant, noncommutative space is the fuzzy sphere [76, 105]. It is defined in terms of three rescaled angular momentum operators $X_i = \lambda J_i$, the Lie algebra generators of a unitary, irreducible representation of $SU(2)$, that satisfy the following relations:

$$[X_i, X_j] = i\lambda\epsilon_{ijk}X_k, \quad \sum_{i=1}^3 X_i X_i = \lambda^2 s(s+1) := r^2, \quad (29)$$

where $i, j, k = 1, \dots, 3$, $\lambda \in \mathbb{R}$ and $2s \in \mathbb{N}$. The second of the above relations comes from the quadratic Casimir operator. If one relaxes this condition, allowing the coordinates X_i to live in unitary, reducible representations of $SU(2)$, and at the same time keeps λ fixed, one is lead to the three-dimensional noncommutative space known as \mathbb{R}_λ^3 [92], which is expressed as a direct sum of fuzzy spheres with every possible radius determined by $2s \in \mathbb{N}$ [92, 93, 106, 107]

$$\mathbb{R}_{\lambda}^3 = \sum_{2s \in \mathbb{N}} S_{\lambda,s}^2. \quad (30)$$

Consequently, \mathbb{R}_{λ}^3 can be seen as a discrete foliation of the three-dimensional Euclidean space by several fuzzy 2-spheres, with each of them being a ‘leaf’ of the foliation [108].

4.1.2. Gauge theory of three-dimensional gravity on \mathbb{R}_{λ}^3 . In the following, we will review a description for a noncommutative version of the three-dimensional gravity on the fuzzy space that was mentioned in the previous paragraph. This description is achieved by following the same steps that were followed in the commutative case, albeit this time using the tools of non-commutative gauge theories that were mentioned in section 3. Thus, the covariant coordinate should now contain information about the noncommutative counterparts of the dreibein and spin connection⁸.

The relevant group that describes the symmetry of the fuzzy space \mathbb{R}_{λ}^3 , which will be used, is $SO(4)$ ⁹. This will lead to a non-abelian noncommutative gauge theory, which in turn will introduce the unwelcome feature of the non-closure of the anticommutators of the generators that was described in section 3. Following the procedure that was explained in that section, motivated by the approach that was followed in the Moyal-Weyl case in [16], the algebra shall be extended appropriately, so that the products of the anticommutators are also included in it.

According to the above, the first to be considered as the symmetry group is the spin group. In this case, it is the $Spin(4)$ group, which is isomorphic to $SU(2) \times SU(2)$. Choosing a specific representation, and given the elements that are yielded from the anticommutators of the algebra generators in that representation, when the latter are included in the algebra as generators, one is lead to the extension of the $SU(2) \times SU(2)$ symmetry to the $U(2) \times U(2)$, which will be considered as the gauge group of the theory¹⁰. Since each $U(2)$ is represented by the Pauli matrices and the identity, the $U(2) \times U(2)$ gauge group will consist of the following 4×4 matrices

$$J_a^L = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, J_0^L = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, J_a^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}, J_0^R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad (31)$$

where $a = 1, 2, 3$. Nevertheless, the identification of the noncommutative dreibein and spin connection should be treated carefully. In order to interpret the above gauge fields correctly the following linear combination of the above matrices are considered as generators instead:

$$P_a = \frac{1}{2} (J_a^L - J_a^R) = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, M_a = \frac{1}{2} (J_a^L + J_a^R) = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \quad (32)$$

as well as

$$\mathbb{I} = J_0^L + J_0^R, \gamma_5 = J_0^L - J_0^R. \quad (33)$$

Given the known commutation and anticommutation relations of the Pauli matrices, the above generators will satisfy the following

$$\begin{aligned} [P_a, P_b] &= i\epsilon_{abc}M_c, \quad [P_a, M_b] = i\epsilon_{abc}P_c, \quad [M_a, M_b] = i\epsilon_{abc}M_c, \\ \{P_a, P_b\} &= \frac{1}{2}\delta_{ab}\mathbb{I}, \quad \{P_a, M_b\} = \frac{1}{2}\delta_{ab}\gamma_5, \quad \{M_a, M_b\} = \frac{1}{2}\delta_{ab}\mathbb{I}, \\ [\gamma_5, P_a] &= [\gamma_5, M_a] = 0, \quad \{\gamma_5, P_a\} = 2M_a, \quad \{\gamma_5, M_a\} = 2P_a. \end{aligned} \quad (34)$$

⁸ Similar approaches can be found in [31, 32, 34].

⁹ That is in the Euclidean case. In the Lorentzian case, the relevant group would have been the $SO(1,3)$.

¹⁰ Similarly, in the Lorentzian case the initial $SL(2; \mathbb{C})$ symmetry would be enlarged to $GL(2; \mathbb{C})$.

As was mentioned in the previous paragraph, the noncommutative coordinates, X_a , will be given in terms of the three operators of the fuzzy space that was discussed. Thus, in the same way as described in section 3, the covariant coordinate will include information about the deformation of space, through the gauge connection \mathcal{A}_μ , since

$$\mathcal{X}_\mu = \delta_\mu^a X_a + \mathcal{A}_\mu, \quad (35)$$

where \mathcal{A}_μ can be expanded on the generators of the algebra as $\mathcal{A}_\mu = \mathcal{A}_\mu^I(X_a) \otimes T^I$, T^I being the generators with $I = 1, \dots, 8$, and \mathcal{A}_μ^I the $U(2) \times U(2)$ -valued gauge fields. The reason behind the tensor product in the previous relation is that the gauge fields are no longer functions of coordinates in a classical manifold, but are now operator valued (since the coordinates got promoted), while the generators are represented by 4×4 matrices. Having in mind the generators we chose for the $U(2) \times U(2)$ algebra, the explicit expansion of the gauge connection over them will be

$$\mathcal{A}_\mu(X) = e_\mu^a(X) \otimes P_a + \omega_\mu^a(X) \otimes M_a + A_\mu(X) \otimes i\mathbb{I} + \tilde{A}_\mu(X) \otimes \gamma_5, \quad (36)$$

consequently forming the covariant coordinate

$$\mathcal{X}_\mu = X_\mu \otimes i\mathbb{I} + e_\mu^a(X) \otimes P_a + \omega_\mu^a(X) \otimes M_a + A_\mu(X) \otimes i\mathbb{I} + \tilde{A}_\mu(X) \otimes \gamma_5, \quad (37)$$

where $e_\mu^a, \omega_\mu^a, A_\mu, \tilde{A}_\mu$ are the components of the connection \mathcal{A}_μ (gauge fields). In the same spirit, the gauge parameter $\varepsilon(X)$ is also an element of the algebra and will be expanded on its generators as

$$\varepsilon(X) = \xi^a(X) \otimes P_a + \lambda^a(X) \otimes M_a + \varepsilon_0(X) \otimes i\mathbb{I} + \tilde{\varepsilon}_0(X) \otimes \gamma_5, \quad (38)$$

where $\xi^a, \lambda^a, \varepsilon_0, \tilde{\varepsilon}_0$ denote its components. Given the above expansions, by making use of relations (27) and (28), the transformations of the component gauge fields are calculated, in a similar way to the commutative case. The explicit formulae for these transformations are shown below:

$$\begin{aligned} \delta e_\mu^a &= -i[X_\mu + A_\mu, \xi^a] + \frac{i}{2}\{\xi_b, \omega_{\mu c}\}\epsilon^{abc} + \frac{i}{2}\{\lambda_b, e_{\mu c}\}\epsilon^{abc} \\ &\quad + i[\varepsilon_0, e_\mu^a] + [\lambda^a, \tilde{A}_\mu] + [\tilde{\varepsilon}_0, \omega_\mu^a], \\ \delta \omega_\mu^a &= -i[X_\mu + A_\mu, \lambda^a] + \frac{i}{2}\{\xi_b, e_{\mu c}\}\epsilon^{abc} + \frac{i}{2}\{\lambda_b, \omega_{\mu c}\}\epsilon^{abc} \\ &\quad + i[\varepsilon_0, \omega_\mu^a] + [\xi^a, \tilde{A}_\mu] + [\tilde{\varepsilon}_0, e_\mu^a], \\ \delta A_\mu &= -i[X_\mu + A_\mu, \varepsilon_0] - \frac{i}{4}[\xi^a, e_{\mu a}] - \frac{i}{4}[\lambda^a, \omega_{\mu a}] - i[\tilde{\varepsilon}_0, \tilde{A}_\mu], \\ \delta \tilde{A}_\mu &= -i[X_\mu + A_\mu, \tilde{\varepsilon}_0] + \frac{1}{4}[\xi^a, \omega_{\mu a}] + \frac{1}{4}[\lambda^a, e_{\mu a}] + i[\varepsilon_0, \tilde{A}_\mu]. \end{aligned} \quad (39)$$

Following the determination of the transformation rules of the component gauge fields above, their behavior in two limits is going to be considered.

First, the Abelian limit is considered, or, in other words, the case in which the gauge group that was used were an Abelian $U(1)$ group. This would have, of course, led to an Abelian gauge theory on the three-dimensional, fuzzy space that was used. It effectively means setting $e_\mu^a, \omega_\mu^a, \tilde{A}_\mu$, as well as their corresponding parameters $\xi^a, \lambda^a, \tilde{\varepsilon}_0$ equal to zero, leaving A_μ as

the only non-vanishing gauge field and ε_0 as the only non-vanishing gauge parameter. Thus, the only non-trivial transformation out of the above is

$$\delta A_\mu = -i[X_\mu, \varepsilon_0] + i[\varepsilon_0, A_\mu], \quad (40)$$

which is the anticipated transformation of a noncommutative Maxwell gauge field. Therefore, this limit emphasizes on the Maxwellian nature of the ‘new’ fields, A, \tilde{A} . Also, it is understood that the Maxwellian sector is present in the theory, independently of whether the dreibein is trivial or not, with the covariant coordinate being $X_\mu + A_\mu$.

On the other hand, the second limit that will be considered is the commutative one ($\lambda \rightarrow 0$), at which gravity and the Yang-Mills fields disentangle, consequently making the gauge fields that were introduced due to the noncommutativity, A_μ and \tilde{A}_μ , vanish in this limit. That causes the inner derivation to reduce to the commutative one, that is $[X_\mu, f] \rightarrow -i\partial_\mu f$, therefore leading to the following transformation rules of the dreibein and spin connection:

$$\delta e_\mu^a = -\partial_\mu \xi^a - \epsilon^{abc}(-i\xi_b \omega_{\mu c} - i\lambda_b e_{\mu c}) \quad (41)$$

$$\delta \omega_\mu^a = -\partial_\mu \lambda^a - \epsilon^{abc}(-i\lambda_b \omega_{\mu c} - i\xi_b e_{\mu c}). \quad (42)$$

At this point, it is observed that the above transformations closely resemble the corresponding commutative ones, in relations (6) and (7). More specifically, after performing the following re-definitions of the generators, gauge fields and parameters:

$$P_a \rightarrow -\frac{i}{\sqrt{\Lambda}} P_a, \quad M_a \rightarrow iM_a,$$

as well as

$$e_\mu^a \rightarrow i\sqrt{\Lambda} e_\mu^a, \quad \xi^a \rightarrow -i\sqrt{\Lambda} e_\mu^a, \quad \omega_\mu^a \rightarrow -i\omega_\mu^a, \quad \lambda_\mu^a \rightarrow i\lambda_\mu^a,$$

the aforementioned transformations, exactly coincide with their commutative counterparts (6) and (7). Thus, it is evident that in the commutative limit, the transformations of the gauge fields of the three-dimensional gravity, presented in [12], are recovered.

Following the above, the curvature tensors of the theory will be obtained, through the calculation of the commutator of the covariant derivatives. It should be noted that since the right hand side of the commutator of the coordinates is linear with respect to the coordinates—as shown in the first equation of (29)—an additional linear term should be included in the definition of the curvature as indicated below:

$$\mathcal{R}_{\mu\nu}(X) = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\lambda \epsilon_{\mu\nu\rho} \mathcal{X}^\rho. \quad (43)$$

The curvature tensor $\mathcal{R}_{\mu\nu}$ is, too, an element of the $U(2) \times U(2)$ and as such, it can be expanded on the algebra’s generators according to

$$\mathcal{R}_{\mu\nu}(X) = T_{\mu\nu}^a(X) \otimes P_a + R_{\mu\nu}^a(X) \otimes M_a + F_{\mu\nu}(X) \otimes i\mathbb{I} + \tilde{F}_{\mu\nu}(X) \otimes \gamma_5. \quad (44)$$

Following a similar procedure as when calculating the transformation laws of the gauge fields, using the definition of the curvature (43), together with the expansions of the curvature tensor and the covariant coordinate (44) and (37) respectively, the component curvature tensors are calculated:

$$\begin{aligned}
T_{\mu\nu}^a &= i[X_\mu + A_\mu, e_\nu^a] - i[X_\nu + A_\nu, e_\mu^a] + \frac{i}{2} \{e_{\mu b}, \omega_{\nu c}\} \epsilon^{abc} + \frac{i}{2} \{\omega_{\mu b}, e_{\nu c}\} \epsilon^{abc} \\
&\quad + [\omega_\mu^a, \tilde{A}_\nu] - [\omega_\nu^a, \tilde{A}_\mu] - i\lambda \epsilon_{\mu\nu\rho} e^{\rho a}, \\
R_{\mu\nu}^a &= i[X_\mu + A_\mu, \omega_\nu^a] - i[X_\nu + A_\nu, \omega_\mu^a] + \frac{i}{2} \{\omega_{\mu b}, \omega_{\nu c}\} \epsilon^{abc} + \frac{i}{2} \{e_{\mu b}, e_{\nu c}\} \epsilon^{abc} \\
&\quad + [e_\mu^a, \tilde{A}_\nu] - [e_\nu^a, \tilde{A}_\mu] - i\lambda \epsilon_{\mu\nu\rho} \omega^{\rho a}, \\
F_{\mu\nu} &= i[X_\mu + A_\mu, X_\nu + A_\nu] - \frac{i}{4} [e_\mu^a, e_{\nu a}] - \frac{i}{4} [\omega_\mu^a, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] \\
&\quad - i\lambda \epsilon_{\mu\nu\rho} (X^\rho + A^\rho), \\
\tilde{F}_{\mu\nu} &= i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + \frac{1}{4} [e_\mu^a, \omega_{\nu a}] + \frac{1}{4} [\omega_\mu^a, e_{\nu a}] \\
&\quad + -i\lambda \epsilon_{\mu\nu\rho} \tilde{A}^\rho. \tag{45}
\end{aligned}$$

Once again, when the commutative limit as well as the re-scalings that were mentioned before are considered, the tensors of equations (10) and (11) are recovered, exactly as expected.

4.1.3. Action of three-dimensional fuzzy gravity. Concluding with the three-dimensional case, an action for the aforementioned theory should be found. Considering that the space on which the theory has been formulated is a three-dimensional space, inspiration is gained by the gauge-theoretic approach of three-dimensional gravity that was presented back in section 2.1. Therefore, the obvious choice would be to take an action of Chern–Simons type. For the Euclidean case, which we have discussed so far, the action that was considered is [91, 107]

$$S_0 = \frac{1}{g^2} \text{Tr} \left(\frac{i}{3} \epsilon^{\mu\nu\rho} X_\mu X_\nu X_\rho - m^2 X_\mu X^\mu \right), \tag{46}$$

which, following its variation, leads to the field equation

$$[X_\mu, X_\nu] + 2im^2 \epsilon_{\mu\nu\rho} X^\rho = 0. \tag{47}$$

The above field equation admits the space \mathbb{R}_λ^3 that we have used as a solution, for $2m^2 = -\lambda$.

Next, in order for the gauge fields to be introduced in the aforementioned action, there are two possible paths one could follow. The first would be to consider fluctuations of the above equation of motion by replacing the coordinates with their covariant counterparts. The other, less straightforward path would be to replace the coordinates in the action with the covariant coordinates and then complete the variation of the action, in order to obtain the field equations. Furthermore, since, eventually, the action shall be written in terms of the gauge fields, an additional trace, tr , over the gauge indices should be involved. Consequently, the proposed action is

$$S = \frac{1}{g^2} \text{Tr} \text{tr} \left(\frac{i}{3} \epsilon^{\mu\nu\rho} \mathcal{X}_\mu \mathcal{X}_\nu \mathcal{X}_\rho + \frac{\lambda}{2} \mathcal{X}_\mu \mathcal{X}^\mu \right), \tag{48}$$

where the first trace is over the matrices X and the second over the gauge indices. The above action can be rewritten as

$$\begin{aligned}
S &= \frac{1}{6g^2} \text{Tr} \text{tr} (i \epsilon^{\mu\nu\rho} \mathcal{X}_\mu \mathcal{R}_{\nu\rho}) + \frac{\lambda}{6g^2} \text{Tr} \text{tr} (\mathcal{X}_\mu \mathcal{X}^\mu) \\
&= \mathcal{S} + S_\lambda, \tag{49}
\end{aligned}$$

where all the λ -related terms have been isolated in the S_λ term, which vanishes for $\lambda \rightarrow 0$.

Now, making use of the non-vanishing traces of the generators of the algebra

$$tr(P_a P_b) = \delta_{ab}, \quad tr(M_a M_b) = \delta_{ab}, \quad (50)$$

which are obtained starting from the expressions for the anticommutators in (34), following the calculation of the traces over the gauge indices, the first term \mathcal{S} of the above action turns out equal to

$$\mathcal{S} = \frac{i}{6g^2} \text{Tr} \epsilon^{\mu\nu\rho} (e_{\mu a} T_{\nu\rho}^a + \omega_{\mu a} R_{\nu\rho}^a - 4(X_\mu + A_\mu) F_{\nu\rho} + 4\tilde{A}_\mu \tilde{F}_{\nu\rho}). \quad (51)$$

This action is similar to the one presented in [12]; when the commutative limit is considered and the re-definitions that were mentioned before are applied, the first two terms of the above action are identical to the one presented in [12]. Nevertheless, in this case, an additional sector is unavoidably obtained. This sector is evidently associated with the additional gauge fields, which cannot decouple in the present, noncommutative case.

Finally, variation of the action (49) with respect to the covariant coordinate yields the following field equations:

$$T_{\mu\nu}^a = 0, \quad R_{\mu\nu}^a = 0, \quad F_{\mu\nu} = 0, \quad \tilde{F}_{\mu\nu} = 0. \quad (52)$$

At this point, it is noted that the same equations of motion are obtained, following the variation of (49) with respect to the gauge fields, after using the algebra trace and replacing the tensors with their expansions on the generators of the algebra (45).

4.2. Four-dimensional case

4.2.1. A Fuzzy Version of the Four-Sphere. The noncommutative space that we are going to use is the fuzzy four-sphere, S_F^4 , that is the four-dimensional analogue of the fuzzy sphere, S_F^2 , the discrete (matrix) approximation of the regular sphere¹¹. The group that naturally should be considered is the $SO(5)$ group, since it amounts to the corresponding isometry group, and thus one should be able to identify the coordinates with a subset of its generators. However, the subalgebra is not closing, which leads to the covariance not being preserved [78]. This fact forces us to use a larger symmetry group, in which we should be able to incorporate all generators and the noncommutativity in it, with an appropriate identification. We should end up with a construction in which the coordinates will transform as vectors under the action of the rotational transformations. The minimal extension of the symmetry leads to the $SO(6)$ group [96, 100]. The 15 generators of $SO(6)$, J_{AB} , with $A, B = 1, \dots, 6$, obey the following algebra:

$$[J_{AB}, J_{CD}] = i(\delta_{AC} J_{BD} + \delta_{BD} J_{AC} - \delta_{BC} J_{AD} - \delta_{AD} J_{BC}). \quad (53)$$

In $SO(4)$ notation, the above generators are decomposed as following:

$$J_{\mu\nu} = \frac{1}{\hbar} \Theta_{\mu\nu}, \quad J_{\mu 5} = \frac{1}{\lambda} X_5, \quad J_{\mu 6} = \frac{\lambda}{2\hbar} P_\mu, \quad J_{56} = \frac{1}{2} H, \quad (54)$$

where $\mu, \nu = 1, \dots, 4$, λ is a dimensionful parameter and h is a radius constraint related operator. The X_μ, P_μ and $\Theta_{\mu\nu}$ are identified as coordinates, momenta and noncommutativity tensor respectively. The commutation relations which they obey are the following:

¹¹ Here we are presenting the whole construction in the Euclidean signature, although in the introduction we discussed in terms of the Lorentzian one. This choice is motivated by the disadvantage of working with representations of non-compact groups, which, if unitary, are infinite-dimensional (see [96] for the construction in Lorentzian language).

$$[X_\mu, X_\nu] = i \frac{\lambda^2}{\hbar} \Theta_{\mu\nu}, \quad [P_\mu, P_\nu] = 4i \frac{\hbar}{\lambda^2} \Theta_{\mu\nu}, \quad (55)$$

$$[X_\mu, P_\nu] = i\hbar \delta_{\mu\nu} H, \quad [X_\mu, H] = i \frac{\lambda^2}{\hbar} P_\mu, \quad (56)$$

$$[P_\mu, H] = 4i \frac{\hbar}{\lambda^2} X_\mu. \quad (57)$$

The above commutation relations imply that coordinates and momenta separately close into an $SO(4)$ subalgebra of the $SO(6)$ symmetry. Also, the commutation relations between the coordinates and momenta, show the quantum structure of the noncommutative space. The algebra of spacetime transformations is:

$$[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] = i\hbar(\delta_{\mu\rho} \Theta_{\nu\sigma} + \delta_{\nu\sigma} \Theta_{\mu\rho} - \delta_{\nu\rho} \Theta_{\mu\sigma} - \delta_{\mu\sigma} \Theta_{\nu\rho}), \quad (58)$$

$$[X_\mu, \Theta_{\nu\rho}] = i\hbar(\delta_{\mu\rho} X_\nu - \delta_{\mu\nu} X_\rho), \quad (59)$$

$$[P_\mu, \Theta_{\nu\rho}] = i\hbar(\delta_{\mu\rho} P_\nu - \delta_{\mu\nu} P_\rho) \quad (60)$$

$$[H, \Theta_{\mu\nu}] = 0. \quad (61)$$

The first commutation relation defines the $SO(4)$ subalgebra. The second and the third show the vector-like transformation of the coordinates and the momenta under rotations.

Finally, we should emphasize that the above algebra admits finite-dimensional representations of its generators, and, for that reason, the noncommutative spacetime constructed is actually a finite quantum system. On this space we are going to construct the four-dimensional gravity model as a noncommutative gauge theory.

4.2.2. Gauge group and representation. As mentioned earlier, when working on the noncommutative framework, one has to use also the anticommutators of the generators. Employing these anticommutators of generators without having their representation fixed, one generally does not end up with closed algebras, but instead the anticommutators keep producing new operators which do not belong to the algebra. This is also the case when it comes to $SO(5)$. This problem is resolved by abandoning the idea of the arbitrary representation of the generators and actually pick one specific representation in which the generators of $SO(5)$ belong and, afterwards, include the operators that the anticommutators produce into the algebra, by extending it to a larger one in which the new operators are also generators. In our specific case, the extension of $SO(5)$ leads to the $SO(6) \times U(1)$ gauge group, whose generators are represented by 4×4 matrices¹² (see [96, 100]):

$$M_{ab} = -\frac{i}{4} [\Gamma_a, \Gamma_b], \quad K_a = \frac{1}{2} \Gamma_a, \quad P_a = -\frac{i}{2} \Gamma_a \Gamma_5, \quad D = -\frac{1}{2} \Gamma_5, \quad \mathbb{I}_4. \quad (62)$$

The above Γ matrices are the 4×4 gamma matrices in the Euclidean signature, which satisfy the relation $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbb{I}_4$, where $a, b = 1, \dots, 4$ and $\Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$.

The algebra and the anticommutation relations the above generators follow is:

¹² The choice of the representation is motivated by the idea of minimal extension but larger representations could be also employed. Such a choice could be possibly made in case matter fields accommodated in a preferred representation were involved. Moreover, inclusion of matter fields would imply that anomaly cancellation should be checked.

$$\begin{aligned}
[K_a, K_b] &= iM_{ab}, \quad [P_a, P_b] = iM_{ab}, \\
[P_a, D] &= iK_a, \quad [K_a, P_b] = i\delta_{ab}D, \quad [K_a, D] = -iP_a, \\
[K_a, M_{bc}] &= i(\delta_{ac}K_b - \delta_{ab}K_c), \\
[P_a, M_{bc}] &= i(\delta_{ac}P_b - \delta_{ab}P_c), \\
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}), \\
[D, M_{ab}] &= 0, \\
\{M_{ab}, M_{cd}\} &= \frac{1}{8}(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})\mathbb{I}_4 - \frac{\sqrt{2}}{4}\epsilon_{abcd}D, \\
\{M_{ab}, K_c\} &= \sqrt{2}\epsilon_{abcd}P_d, \quad \{M_{ab}, P_c\} = -\frac{\sqrt{2}}{4}\epsilon_{abcd}K_d, \\
\{K_a, K_b\} &= \frac{1}{2}\delta_{ab}\mathbb{I}_4, \quad \{P_a, P_b\} = \frac{1}{8}\delta_{ab}\mathbb{I}_4, \quad \{K_a, D\} = \{P_a, D\} = 0, \\
\{P_a, K_b\} &= \{M_{ab}, D\} = -\frac{\sqrt{2}}{2}\epsilon_{abcd}M_{cd}. \tag{63}
\end{aligned}$$

4.2.3. Action and equations of motion. We will now try to find the action of the $SO(6) \times U(1)$ gauge theory of gravity (for a more detailed presentation see the original works [96, 100]). Starting from the noncommutativity of the background space, we consider the following topological action:

$$\mathcal{S} = \text{Tr}([X_\mu, X_\nu] - \kappa^2\Theta_{\mu\nu})([X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma}. \tag{64}$$

Variation of the above action with respect to X and Θ fields, gives the following field equations, respectively:

$$\epsilon^{\mu\nu\rho\sigma}[X_\nu, [X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma}] = 0, \quad \epsilon^{\mu\nu\rho\sigma}([X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma}) = 0. \tag{65}$$

The second relation, when $\kappa^2 = \frac{i\lambda^2}{\hbar}$, recovers the noncommutativity of the space and, in that case, the first relation is trivially satisfied. In the case that we had chosen the fields X and Θ as coordinate-dependent, we would still end up with the first of the above field equations following the same procedure.

Next, we continue to the dynamical part of the above action. First, we need to rewrite the action in terms of the curvature field strength tensor in order to be able to do comparisons to the commutative analogue at any step of the process. For that reason, we have to include the gauge fields in the action (64), perceiving them as fluctuations of the X and Θ fields:

$$\begin{aligned}
\mathcal{S} = \text{Tr} &\text{tr} \epsilon^{\mu\nu\rho\sigma} ([X_\mu + A_\mu, X_\nu + A_\nu] - \kappa^2(\Theta_{\mu\nu} + \mathcal{B}_{\mu\nu})) \\
&\cdot ([X_\rho + A_\rho, X_\sigma + A_\sigma] - \kappa^2(\Theta_{\rho\sigma} + \mathcal{B}_{\rho\sigma})). \tag{66}
\end{aligned}$$

In the last expression for the action we have included a trace over the gauge algebra, as well¹³. At this point, we have to make some new definitions:

- $\mathcal{X}_\mu = X_\mu + A_\mu$ is defined as the covariant coordinate of the noncommutative gauge theory, where A_μ is the gauge connection,

¹³ The traces on the various products of generators that emerge are obtained by their anticommutation relations, equation (63), after tracing both sides.

b. $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, is defined as the covariant noncommutative tensor, where $\mathcal{B}_{\mu\nu}$ is the 2-form field,
c. $\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - \kappa^2 \hat{\Theta}_{\mu\nu}$ is defined as the field strength tensor of the theory.

In addition to the above, we also replace $\kappa^2 = \frac{i\lambda^2}{\hbar}$ and, finally, we get the following expression for the action:

$$\mathcal{S} = \text{Trtr} \left([\mathcal{X}_\mu, \mathcal{X}_\nu] - \frac{i\lambda^2}{\hbar} \hat{\Theta}_{\mu\nu} \right) \left([\mathcal{X}_\rho, \mathcal{X}_\sigma] - \frac{i\lambda^2}{\hbar} \hat{\Theta}_{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} := \text{Trtr} \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}. \quad (67)$$

We can immediately notice that the above expression is none other than a noncommutative analogue of the four-dimensional Chern–Simons action. Variations of this action with respect to \mathcal{X} and \mathcal{B} , lead to the field equations:

$$\epsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\rho\sigma} = 0, \quad \epsilon^{\mu\nu\rho\sigma} [\mathcal{X}_\nu, \mathcal{R}_{\rho\sigma}] = 0. \quad (68)$$

The interpretation of the first one is of course trivial, which is the vanishing of the curvature tensor. As far as the second one is concerned, it can be considered as the noncommutative counterpart of the Bianchi identity. For later convenience, we express the curvature field strength tensor decomposed in terms of the generators of the gauge algebra:

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}^a \otimes P_a + R_{\mu\nu}^{ab} \otimes M_{ab} + R_{\mu\nu}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbb{I}_4, \quad (69)$$

where the quantities attached to the generators are the component curvature tensor associated to the corresponding gauge fields, as given from the decomposition of the gauge connection A_μ :

$$A_\mu = e_\mu^a \otimes P_a + \omega_\mu^{ab} \otimes M_{ab} + b_\mu^a \otimes K_a + \tilde{a}_\mu \otimes D + a_\mu \otimes \mathbb{I}_4. \quad (70)$$

For completion, we write the explicit expressions of the component tensors:

$$\begin{aligned} \tilde{R}_{\mu\nu}^a &= [X_\mu + a_\mu, e_\nu^a] - [X_\nu + a_\nu, e_\mu^a] - \frac{i}{2} \{b_\mu^a, \tilde{a}_\nu\} + \frac{i}{2} \{b_\nu^a, \tilde{a}_\mu\} \\ &\quad - \frac{\sqrt{2}}{2} ([b_\mu^b, \omega_\nu^{cd}] - [b_\nu^b, \omega_\mu^{cd}]) \epsilon_{abcd} - i \{\omega_\mu^{ab}, e_{\nu b}\} + i \{\omega_\nu^{ab}, e_{\mu b}\} - \frac{i\lambda^2}{\hbar} \tilde{B}_{\mu\nu}^a, \\ R_{\mu\nu}^{ab} &= [X_\mu + a_\mu, \omega_\nu^{ab}] - [X_\nu + a_\nu, \omega_\mu^{ab}] + \frac{i}{2} \{b_\mu^a, b_\nu^b\} + \frac{\sqrt{2}}{4} ([b_\mu^c, e_\nu^d] - [b_\nu^c, e_\mu^d]) \epsilon_{abcd} \\ &\quad - \frac{\sqrt{2}}{4} ([\tilde{a}_\mu, \omega_\nu^{cd}] - [\tilde{a}_\nu, \omega_\mu^{cd}]) \epsilon_{abcd} + 2i \{\omega_\mu^{ac}, \omega_\nu^{cb}\} + \frac{i}{2} \{e_\mu^a, e_\nu^b\} - \frac{i\lambda^2}{\hbar} B_{\mu\nu}^{ab}, \\ R_{\mu\nu}^a &= [X_\mu + a_\mu, b_\nu^a] - [X_\nu + a_\nu, b_\mu^a] + i \{b_{\mu b}, \omega_\mu^{ab}\} - i \{b_{\nu b}, \omega_\mu^{ab}\} \\ &\quad - \frac{i}{2} \{\tilde{a}_\mu, e_\nu^a\} + \frac{i}{2} \{\tilde{a}_\nu, e_\mu^a\} + \frac{\sqrt{2}}{8} \epsilon_{abcd} ([e_\mu^b, \omega_\nu^{cd}] - [e_\nu^b, \omega_\mu^{cd}]) - \frac{i\lambda^2}{\hbar} B_{\mu\nu}^a, \\ \tilde{R}_{\mu\nu} &= [X_\mu + a_\mu, \tilde{a}_\nu] - [X_\nu + a_\nu, \tilde{a}_\mu] + \frac{i}{2} \{b_{\mu a}, e_\nu^a\} - \frac{i}{2} \{b_{\nu a}, e_\mu^a\} \\ &\quad - \frac{\sqrt{2}}{8} \epsilon_{abcd} [\omega_\mu^{ab}, m\omega_\nu^{cd}] - \frac{i\lambda^2}{\hbar} \tilde{B}_{\mu\nu}, \\ R_{\mu\nu} &= [X_\mu, a_\nu] - [X_\nu, a_\mu] + [a_\mu, a_\nu] + \frac{1}{4} [b_\mu^a, b_{\nu a}] + \frac{1}{4} [\tilde{a}_\mu, \tilde{a}_\nu] + \frac{1}{8} [\omega_\mu^{ab}, \omega_{\nu ab}] \\ &\quad + \frac{1}{16} [e_{\mu a}, e_\nu^a] - \frac{i\lambda^2}{\hbar} B_{\mu\nu}, \end{aligned}$$

where the last terms of all the above expressions include components of the decomposition of the 2-form field $\mathcal{B}_{\mu\nu}$ on the various generators.

4.2.4. Spontaneous symmetry breaking of the noncommutative action. In order to proceed to the spontaneous symmetry breaking of the action (67), we have to introduce a scalar field, Φ , along with a dimensionful parameter λ , which will set the length scale of the theory. Then, the action is modified as [96, 100]:

$$\mathcal{S} = \text{Tr} \text{tr}_G \lambda \Phi(X) \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + \eta (\Phi(X)^2 - \lambda^{-2} \mathbf{I}_N \otimes \mathbf{I}_4), \quad (71)$$

where η is a Lagrange multiplier with dimension $[M^{-2}]$. Obviously, in the on-shell case the above action coincides with the previously introduced dynamic one, (67). By the term ‘on-shell’ it is meant, of course, that the following condition holds:

$$\Phi^2(X) = \lambda^{-2} \mathbf{I}_N \otimes \mathbf{I}_4.$$

Variation of the action (71) with respect to η yields the above constraint equation as a field equation. If we consider that the scalar field consists only of the symmetric part of the decomposition on the generators, it can be explicitly written as¹⁴:

$$\Phi(X) = \tilde{\phi}^a(X) \otimes P_a + \phi^a(X) \otimes K_a + \phi(X) \otimes \mathbf{I}_4 + \tilde{\phi}(X) \otimes D.$$

Finally, we have to gauge fix the scalar field, Φ . The gauge fixing that we choose is in the direction of the generator D . Specifically we choose the value of $\tilde{\phi}(x)$ such that:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi}=-2\lambda^{-1}} = -2\lambda^{-1} \mathbf{I}_N \otimes D.$$

Now, we continue by calculating the traces over the algebra in the modified action, (71), and we perform the substitution of the gauge-fixed scalar field. The action, finally, takes the following form:

$$\mathcal{S}_{\text{br}} = \text{Tr} \left(\frac{\sqrt{2}}{4} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} - 4 R_{\mu\nu} \tilde{R}_{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma}. \quad (72)$$

Because of the fact that we kept only the symmetric part of the scalar field’s decomposition under the $SO(6)$ generators that was not charged under the initial $U(1)$, the remaining symmetry of the spontaneously broken action is $SO(4) \times U(1)$. On the generators’ level, that means that out of the 16 total generators, only seven remain unbroken while the rest break: (i) the generators of the translations, P_a , which imply the torsionless condition, $\tilde{R}_{\mu\nu}^a = 0$, which in turn leads to a relation between ω and e, \tilde{a} , (ii) the K_a generators, which implies $R_{\mu\nu}^a = 0$ leading to a proportionality relation between e, b gauge fields and (iii) the D generator which accommodates the gauge fixing of $\tilde{a}_\mu = 0$ [99]. In a few words, the gauge group after the spontaneous symmetry breaking is the $SO(4) \times U(1)$ and the only independent fields of the theory are the e and a .

Replacing $\tilde{a}_\mu = 0$ and $b_\mu^a = \frac{i}{2} e_\mu^a$ in the expression for the component tensor $R_{\mu\nu}^{ab}$ we get:

$$\begin{aligned} R_{\mu\nu}^{ab} = & [X_\mu + a_\mu, \omega_\nu^{ab}] - [X_\nu + a_\nu, \omega_\mu^{ab}] + i \{ \omega_\mu^{ac}, \omega_\nu^b \} - i \{ \omega_\mu^{bc}, \omega_\nu^a \} \\ & + \frac{3i}{8} \{ e_\mu^a, e_\nu^b \} - \frac{i\lambda^2}{\hbar} B_{\mu\nu}^{ab}. \end{aligned}$$

The above, is a relation that holds also to the commutative limit of the theory as it will be shown in the next subsection.

¹⁴ Had we taken the antisymmetric part as well the symmetry breaking would lead to the same gauge symmetry enhanced by a $U(1)$.

4.2.5. The commutative limit. It is crucial, for the consistency of the theory, that its predictions should coincide with the ones of GR in the below-Planck scale energy regime, in case we turn off the noncommutativity at once and consider that the effects related to it are completely ignored. Although this is a simplistic assumption, since we expect that the effects of noncommutativity will eventually have a quantitative imprint on the low-energy regime, we insist on it because it will consist a solid argument that noncommutativity will give low-energy predictions, as small modifications, around the existing and valid theory of GR. Therefore, in order to examine this and establish GR as the guide of our results, we move on with examining the theory at the level of the vanishing of all its noncommutative-related features. To begin with, for signature compatibility, we consider the fuzzy space to have Lorentzian signature in this limit¹⁵, which is fuzzy dS_4 . Then, we have to take into account the following considerations:

- The 2-form field $\mathcal{B}_{\mu\nu}$ and the a_μ decouple as, the first one was related to the preservation of covariance of the fuzzy space and the latter was used to extend the gauge group in order for the anticommutators of the generators to be closing;
- As functions become commutative, their commutators vanish, $[f(x), g(x)] \rightarrow 0$ and their anticommutators become products, $\{f(x), g(x)\} \rightarrow 2f(x)g(x)$;
- The inner derivation reduces to the simple derivative: $[X_\mu, f] \rightarrow \partial_\mu f$ and the traces reduce to integrations, $\frac{\sqrt{2}}{4} \text{Tr} \rightarrow \int d^4x$;
- In the chosen gauge in which the spontaneous symmetry breaking is induced, the expression for the D -related component tensor $\tilde{R}_{\mu\nu}$ of the field strength tensor reduces to:

$$\tilde{R}_{\mu\nu} = -\frac{\sqrt{2}}{8} \epsilon_{abcd} [\omega_\mu{}^{ab}, \omega_\nu{}^{cd}] - \frac{i\lambda^2}{\hbar} \tilde{B}_{\mu\nu}.$$

Because of this, the second term of the corresponding action, (72), will vanish in the commutative limit, since the commutator of the spin connection will be zero. Also the $\tilde{B}_{\mu\nu}$ and the a_μ will decouple, as mentioned above, so the latter will not be included in the first term of the aforementioned action.

- In order to achieve an exact matching with the results of the commutative case, we also need to make following reparametrizations:

$$\begin{aligned} e_\mu{}^a &\rightarrow im e_\mu{}^a, & P_a &\rightarrow -\frac{i}{m} P_a, & \tilde{R}_{\mu\nu}{}^a &\rightarrow im T_{\mu\nu}{}^a \\ \omega_\mu{}^{ab} &\rightarrow -\frac{i}{2} \omega_\mu{}^{ab}, & M_{ab} &\rightarrow 2iM_{ab}, & R_{\mu\nu}{}^{ab} &\rightarrow -\frac{i}{2} R_{\mu\nu}{}^{ab}, \end{aligned}$$

where m is an arbitrary, complex constant of dimensions $[L]^{-1}$, which is introduced in order for the $e_\mu{}^a$ to remain dimensionless in the commutative limit, so that it can admit the interpretation of the actual vielbein field.

After the inclusion of all of the above, the torsion tensor, $\tilde{R}_{\mu\nu}{}^a$, takes the following form:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu{}^{ab} e_{\nu b} + \omega_\nu{}^{ab} e_{\mu b} = 0$$

¹⁵ As mentioned in the previous footnote, the choice of the Euclidean signature addresses the problem of the infinite-dimensional representations in the noncommutative framework but it is obvious that it cannot produce a realistic commutative limit. Therefore, for the discussion of the commutative limit to be meaningful, we continue the discussion in the Lorentzian signature.

which is exactly the torsionless condition of the first-order formulation of GR. Furthermore, because of the latter, the relation between ω and e will also be exactly the same as in the first-order formulation of GR, as shown in [97].

As far as the curvature 2-form, $R_{\mu\nu}^{ab}$, is concerned, its expression will become the following:

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^b{}_c - \omega_\mu^{bc} \omega_\nu^a{}_c + \frac{3}{2} m^2 e_\mu^a e_\nu^b = R_{\mu\nu}^{(0)ab} + \frac{3}{2} m^2 e_\mu^a e_\nu^b.$$

It is clear that the curvature 2-form is exactly the same as that of the first order formulation of GR, plus an extra term that involves only the vielbein fields.

Finally, concerning the action, as it is already mentioned, the second term of (72) will vanish, and it will now consist only of the first term. It is of utmost importance to be understood that, in the commutative limit, the action is only Lorentz-invariant since all the rest of the symmetry is broken. The final expression for the action in the commutative limit will take a form originally proposed by MacDowell–Mansouri, which eventually leads to the so-called Palatini action, the gauge-theoretic equivalent of the Einstein–Hilbert action.

5. Conclusions—future plans

In the present article first we have reviewed in some detail the gauge theoretic approach to the three- and four-dimensional gravity. Then after a short reminder of the formulation of gauge theories on noncommutative spaces we have presented our approach of constructing gauge fuzzy gravities, as matrix models, in the corresponding dimensions.

In particular for the four-dimensional fuzzy Euclidean case we would like to emphasize that the constructed matrix gravity model describes finitely many degrees of freedom giving promises for improved UV properties as compared to ordinary gravity. A main future project is to explore further this possibility and examine to which extent the hopefully positive results can be realized in the spaces with Minkowskian signature too. Obvious extensions of our studies concern the inclusion of matter fields—fermions and scalars (e.g. the scalar fields used to break the gauge symmetry will be upgraded to dynamical ones).

After the completion of the theoretical part of our construction our plan is to study systematically the cosmological consequences of our gravitational model. Finding the solutions of the fuzzy gravity that has been developed will allow us to proceed with the construction of cosmological models and other implications, such as consideration of spherically symmetric solutions that could result in the description of Black Holes from a noncommutative perspective. In the cosmological studies, we expect that the presence of the U(1), due to the anticommutators, as well as and maybe more importantly the Chern–Simons form of the action could provide the characteristic signature of noncommutativity. Of current interest is to examine the possibility that the Planck scale quantum structure of the spacetime we have constructed induces a modification of the gravitational wave dispersion relation and compare with the data received by recent experiments in order to obtain bounds on the scale of noncommutativity. Last, we plan to examine whether the above four-dimensional fuzzy gravity model could be unified with internal interactions of particle physics extending the rationale of Chamseddine and Mukhanov [109].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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