"CLASSICAL" IN TERMS OF GENERAL STATISTICAL MODELS

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From the statistical point of view a physical system is called classical if there are no uncertainly relations. In the language of ordinary quantum mechanics where observables correspond to self-adjoint operators, this means that for a classical system there is a fundamental self-adjoint operator such that any observable can be represented by a function of it. Hence for a classical physical system ordinary quantum mechanics reduces to classical probability theory, where the measure space is given by the spectrum of the fundamental operator and the Borel sets on it.

Both, classical probability theory and ordinary quantum mechanics are special cases in the category of general statistical models. For introduction and overview see e.g. G. Ludwig [4], A.S. Holevo [3], S. Gudder [2]. A general statistical model consists of two sets $\mathcal{S}$ and $\mathcal{R}$, with some structure: $\mathcal{S}$, interpreted to be the set of all preparation procedures, is endowed with a convex structure which expresses the obvious possibility to form statistical mixtures. Combining a preparation procedure with a registration procedure one gets a setup to produce statistically arising events. Such setup is called statistical experiment. The set $\mathcal{R}$ is interpreted to be the set of all registration procedures. The propositional calculus for the resulting events in a statistical experiment obviously forms a Boolean algebra and the probabilities by which the various events arise in a statistical experiment are the values of a probability content on this algebra. Here, by a probability content is understood a real, non negative, finitely additive function on the Boolean algebra which takes the value one on the identity. For a Boolean algebra $\mathcal{L}$ let $\mathcal{W}\ell(\mathcal{L})$ denote the set of probability contents on $\mathcal{L}$. Obviously, $\mathcal{W}\ell(\mathcal{L})$ has a natural convex structure. It is assumed that any fixed registration procedure can be combined with each preparation procedure to give statistical experiments. Hence, any registration procedure not only assigns a Boolean algebra but also a function $M: \mathcal{S} \to \mathcal{W}\ell(\mathcal{L})$. This function M, evidently, has to be affine. In other words, an element $R \in \mathcal{R}$ is formally characterized by two ingredients: a Boolean algebra and an affine vector valued function M.

Consider an element $R \in \mathcal{R}$ with Boolean algebra $\mathcal{L}$. One may look at the corresponding M as at a real, [0.1] valued function with two arguments, one varying in $\mathcal{L}$ and the other in $\mathcal{S}$. It is common language to call R a $\mathcal{L}$-measurement if it is represented by the affine map

$$\mathcal{S} \ni s \to M(s,.) \in \mathcal{W}\ell(\mathcal{L}),$$

and to call R a $\mathcal{L}$ -observable if it is represented by the vector valued measure

$$\mathcal{L} \ni E \to M(., E).$$

In the following we assume $\mathcal{S}$ to be separated by $\mathcal{R}$, i.e. if for all $R \in \mathcal{R}$ and two preparation procedures s, $s' \in \mathcal{S}$ there holds $M(s,.) = M(s',.)$, then $s = s'$. This means that knowledge of all distributions in statistical experiments allows to identify the preparation procedure. It may happen that the distribution of only one $R_o \in \mathcal{R}$ suffices: A registration procedure is called informationally complete if $s \to M(s,.)$ is injective. Let Γ denote the phase space of classical Mechanics and let $\mathcal{B}(\Gamma)$ denote its Borel sets. By

$$(\rho, E) \to M(\rho, E) := \int_E \rho \, d\Gamma,$$

where $E \in \mathcal{B}(\Gamma)$ and ρ is a density function, an informationally complete registration procedure (Laplacian demon) is defined. The fundamental observable mentioned at the beginning of this note is informationally complete as well. In Hilbert space quantum mechanics observables which can be represented by self-adjoint operators or, equivalently, by projection valued measures, are not informationally complete if there are uncertainty relations. Hence one is tempted to assume that the existence of an informationally complete registration procedure implies the physical system to be a classical one. But this assumption is wrong in general!

In Hilbert space quantum mechanics considered as a general statitical model, positive operator valued measures do represent registration procedures as well. Among them there are informationally complete ones as has been shown by Ali and Prugovecki [1]. In [6], Stulpe recently gave a sufficient condition for the existence of such observables in general statistical models: The state space, which can be introduced canonically, has to be separable.

In general, existence of an informationally complete registration procedure seems to be necessary but not sufficient to distinct classical physical systems. What requirement is lacking?

The commutation criterion for commensurability in quantum mechanics works only for self-adjoint operators. In general statistical models one has to introduce commensurability as follows: Let be given registration procedures $R_i \in \mathcal{R}$ (i = 1 ... n) with Boolean algebras $\mathcal{L}_i$ and vector valued measures $\mathcal{L}_i \ni E_i \to M_i(., E_i)$. If there is a Boolean algebra $\mathcal{L}$ and injective homomorphisms $j_i : \mathcal{L}_i \to \mathcal{L}$ and a vector measure $\mathcal{L} \ni E \to M(., E)$, $M(., E)$ being an affine function on $\mathcal{S}$ such that $M_i(., E_i) = M(., j(E_i))$, then $(R_i)_{i=1 \ldots n}$ forms a commensurable set. For projection valued measures this clearly reduces to the commutation criterion. One may call a general statistical model classical if $\mathcal{R}$ or at least any finite subset of $\mathcal{R}$ forms a commensurable set.

The lacking requirement concerns conditional probabilities or, more precisely, whether there are respective preparation procedures in $\mathcal{S}$. Let $R \in \mathcal{R}$ with respective $\mathcal{L}$-measurement M. Let $M(S, E) \neq 0$ for some $s \in \mathcal{S}$ and $E \in \mathcal{L}$. If there is some $S_E \in \mathcal{S}$ fulfilling

$$M(S_E, .) = \frac{M(S_E ; E \wedge (.))}{M(s, E)} ,$$

then S_E is called the conditional preparation under hypothesis E.

Now a criterion for a general statistical model $(\mathcal{S}, \mathcal{R})$ to be a classical one may be stated as follows: If there is a preparation procedure $R \in \mathcal{R}$ with the respective $\mathcal{L}$-measurement M being informationally complete and if for each $(S, E) \in \mathcal{S} \times \mathcal{L}$ with $M(S, E) \neq 0$ there is a conditional preparation S_E under hypothesis E in $\mathcal{S}$ then $(\mathcal{S}, \mathcal{R})$ is classical. - It has been shown by M. Singer [5], that under this assumption any finite subset of $\mathcal{R}$ forms a commensurable set.

References:

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