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Approximated analytical solution of the Landau–Lifshitz equation in tightly focused laser beams in the ultrarelativistic limit

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Abstract. The Landau–Lifshitz equation is widely considered as the equation, which correctly includes the effects of radiation reaction in the classical motion of an electric charge. Here, we provide an approximated analytical solution of the Landau–Lifshitz equation in the presence of a virtually arbitrary electromagnetic field by making the assumptions that the electron's initial energy is the largest dynamical energy in the problem. We show that in a regime where radiation–reaction effects are a small correction of the Lorentz dynamics, the four-momentum of the electron can be determined perturbatively in the ultrarelativistic limit. We explicitly compute the electron four-momentum up to the first order and in the experimentally relevant case of an ultrashort, tightly focused laser beam as an external field.

1 Introduction

An accelerated electric charge emits electromagnetic radiation. In the case of the charge (an electron, for definiteness) being accelerated by a background electromagnetic field, the system is described by the combined Lorentz equation of the electron and Maxwell's equations of the electromagnetic field (the background electromagnetic field plus the electromagnetic field produced by the charge) [1]. By expressing the electromagnetic field in terms of the electron's trajectory (Liénard–Wiechert field) and by replacing it into the Lorentz equation, one faces a divergence in the case of a point-like charge related to the Coulomb structure of the field close to its source. However, by first considering a charge of finite size, it can be shown that the divergence appears only in a term proportional to the four-acceleration of the charge, such that it can be absorbed in the renormalization of the mass of the charge. The resulting equation is known as Lorentz–Abraham–Dirac (LAD) equation [2–4], it contains, apart from the Lorentz force, an additional "radiation–reaction" force, and it has the unique characteristics that the radiation–reaction force features the time derivative of the acceleration of the electron. This causes serious and well-known difficulties related to the appearance of so-called runaway solutions as well as to violation of causality [5–10].

Landau and Lifshitz have shown in Ref. [1] that within classical electrodynamics, i.e., under the assumption that quantum effects are negligible, the radiation–reaction force in the instantaneous rest frame of the electron is much smaller than the Lorentz force, such that a perturbative "reduction of order" can be performed on the LAD equation. This amounts in replacing the acceleration appearing in the radiation–reaction force with the Lorentz force divided by the electron mass. The resulting equation, known as the Landau–Lifshitz (LL) equation, does not contain the time derivative of the electron acceleration by construction and it is therefore not plagued by the existence of runaway solutions. Now, in the present context quantum effects can be negligible if in the instantaneous rest frame of the electron 1) the amplitude of the background field is much smaller than the critical electromagnetic field of QED $E_{\mathrm{cr}} = B_{\mathrm{cr}} = m^2/|e|$ (m and $e < 0$ are the mass and the charge of the electron, respectively and units with $\epsilon_0 = \hbar = c = 1$ are used throughout, such that $\alpha = e^2/(4\pi) \approx 1/137$ is the fine-structure constant) and 2) its typical frequency is much smaller than the electron rest energy (or equivalently, its typical reduced wavelength is much larger than the Compton wavelength $\lambda_C = 1/m$). The crucial observation in Ref. [1] is that the reduction of order is valid for background field strengths much smaller than the critical fields of classical electrodynamics, which are $1/\alpha$ larger than E_{cr} and B_{cr}, and for background wavelengths much larger than the classical electron radius,

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which is α times smaller than the Compton wavelength. Thus, in the realm of classical electrodynamics, the reduction of order is always self-consistently permitted.

In Ref. [11] a rigorous derivation of the LL equation is presented within classical electrodynamics by considering a particle whose size, mass and charge tend to zero in such a way that the charge/mass ratio tends to a finite limit. In Ref. [12] terms of order e^3 in the LL equation have been explicitly derived from QED in the case of an arbitrary background field, while the authors state that the terms of the order e^4 can be derived analogously (assuming that quantum effects are small according to the above discussion). Apparently, the approach of Ref. [12] is only valid in the perturbative regime, where the radiation–reaction force is much smaller than the Lorentz force in the laboratory frame. However, as pointed out by the authors, the results are derived in the average rest frame of the electron wave packet where indeed the radiation–reaction force can be assumed to be much smaller than the Lorentz force in the classical limit. Then, covariance considerations, analogous to those presented in Ref. [1], allow to conclude that the derivation is valid in general. Finally, the LL equation has been solved analytically in the case of an arbitrary plane wave [13] and in Ref. [14] the classical solution has been derived from QED.

Classical as well as quantum radiation reaction is an ongoing and active field of research. The specific case of radiation reaction of ultrarelativistic electrons in high-intensity lasers has been reviewed in Refs. [10,15]. Additional theoretical studies of radiation reaction using the LL equation have been carried out in Refs. [16–20]. Alternative descriptions of classical radiation reaction are also being investigated, see, e.g., Refs. [21,22] for recent works (see also the reviews [10,23,24] for a comprehensive list of related references).

Intense laser beams represent a unique experimental tool to investigate classical and quantum electrodynamics in the presence of strong fields. Laser intensities of the order of 10^{23} W/cm^2 have already been achieved experimentally [25] and several multipetawatt facilities are under construction or planned [26–30], which can overcome the present record by one–two orders of magnitude. In addition, ultrarelativistic electron beams can nowadays be produced not only in conventional accelerators but also via laser wakefield acceleration [31]. In the presence of such intense electromagnetic fields, the dynamics of an ultrarelativistic electron is strongly influenced by its own radiation and testing the validity of the LL equation experimentally will soon become possible. Indeed, experiments about radiation reaction at the edge between the classical and the quantum regime have been already carried out by employing laser accelerated electron beams [32,33]. In Ref. [34] the LL equation has been tested experimentally using aligned crystals, under the consideration of quantum effects. It would also be valuable to design experiments where one can "cleanly" test the LL equation without the interference of quantum effects.

Since realistic ultrastrong laser beams are obtained by tightly focusing the laser energy both in space and time, it is important to investigate the solutions of the LL equation beyond the plane-wave approximation. Here, we present an approximated solution of the LL equation valid in principle for an arbitrarily tightly focused laser beam under the approximation that the energy of the electron is the largest dynamical energy of the problem. This regime has been already investigated in Ref. [35], where a perturbative next-to-leading-order solution of the Lorentz equation has been presented. That solution has been employed in a series of papers to construct quasiclassical electron states in the presence of tightly focused laser beams [36–38]. Here, we apply the same technique to obtain an approximated, analytical solution of the LL equation, where corrections due to radiation reaction are incorporated to leading order. Unlike in the case of the Lorentz equation, the fact to neglect quantum effects from the onset will introduce constraints, which have to be taken into account in the analysis. We also impose certain conditions on the external field, which we consider to be achievable in a real experiment. To be more specific, we have in mind the case where the background electromagnetic field represents an intense, few-cycle, and tightly focused laser beam. These conditions and their implications will be specified below. From the derivation in Sect. 3, it is clear that our assumptions are in line with the regime where radiation–reaction effects are small corrections as compared to the Lorentz dynamics, which ultimately validates the perturbative approach to radiation reaction.

Throughout the paper, the Minkowski metric tensor $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is assumed and the four-dimensional product of two generic four-vectors a^μ and b^μ is indicated as (ab), i.e., $(ab) \equiv a^\mu b_\mu$ (in particular, it is $a^2 \equiv (aa)$).

2 The LAD and the LL equation

The relativistic covariant form of the LAD equation for an electron in a general electromagnetic field $F^{\mu\nu}(x)$ reads [1]

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = eF^{\mu\nu}(x)u_\nu + \frac{2}{3}\frac{e^2}{4\pi}\left(\frac{\mathrm{d}^2u^\mu}{\mathrm{d}s^2} + \frac{\mathrm{d}u^\nu}{\mathrm{d}s}\frac{\mathrm{d}u_\nu}{\mathrm{d}s}u^\mu\right). \quad (1)$$

In this equation, x^μ and $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}s$ are the space–time coordinates and the four-velocity of the electron, respectively, and s is its proper time. The terms proportional to $e^2/(4\pi)$ in Eq. (1) represent the radiation–reaction force. As we have mentioned in the introduction, the reduction of order employed in Ref. [1] consists in replacing the first derivatives of the four-velocity in the radiation–reaction force with their zero-order expression $eF^{\mu\nu}(x)u_\nu/m$. The resulting LL equation is [1]

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = eF^{\mu\nu}u_\nu + \frac{2}{3}\frac{e^2}{4\pi}\left[\frac{e}{m}\left(\partial_\alpha F^{\mu\nu}\right)u^\alpha u_\nu - \frac{e^2}{m^2}F^{\mu\nu}F_{\alpha\nu}u^\alpha + \frac{e^2}{m^2}\left(F^{\alpha\nu}u_\nu\right)\left(F_{\alpha\lambda}u^\lambda\right)u^\mu\right]. \quad (2)$$

In the following, we are going to derive an approximated, analytical expression of the classical four-momentum of the electron according to the LL equation by assuming that the initial energy of the electron is the largest dynamical energy of the problem.

3 Results

We consider a background electromagnetic field, described by the four-vector potential $A^\mu(x)$ in the Lorenz gauge $\partial_\mu A^\mu = 0$. We work in the laboratory frame with space–time coordinates $x^\mu = (t, \boldsymbol{x})$ where the electron initial four-momentum is $p_0^\mu = (\varepsilon_0, \boldsymbol{p}_0) = \left(\sqrt{m^2+\boldsymbol{p}_0^2}, \boldsymbol{p}_0\right)$, and we have in mind the case where $A^\mu(x)$ represents an intense, few-cycle, and tightly focused laser beam. Thus, the field tensor $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$, which can be expressed in terms of the electric field $\boldsymbol{E}(x)$ and of the magnetic field $\boldsymbol{B}(x)$, is localized in space and time. The fact that the initial electron's energy is the largest dynamical energy in the problem means that, if the background field has a maximum amplitude F_0 and if it is characterized by a typical angular frequency ω_0, such that the classical nonlinearity parameter is $\xi_0 = |e|F_0/m\omega_0$, the strong inequalities $m \ll m\xi_0 \ll \varepsilon_0$ are satisfied. In this work we employ a similar technique as in Ref. [35]. The approach in Ref. [35] is also valid for $\xi_0 \sim 1$ but here, where the aim is to investigate radiation–reaction effects, we consider the strong-field regime where $\xi_0 \gg 1$. The above assumptions well fit present and near-future experimental conditions envisaged to test the LL equation with intense lasers. In fact, even for a laser beam of peak intensity $I_0 \approx 10^{21}$ W/cm^2, as that considered in Ref. [33], it is $\xi_0 \approx 15$ (assuming a Ti:Sa laser with central wavelength $\lambda_0 = 0.8$ μm), and $m\xi_0 \approx 8$ MeV, which is much smaller than the energy of already operating accelerators. Moreover, electron beams with energies of about 8 GeV have been already demonstrated experimentally also with laser wakefield accelerators [31] (below, we discuss also the limits on the electron energy in such a way that quantum effects are negligible).

We have in mind the experimentally relevant situation where the electron is initially almost counterpropagating with respect to the laser field. Since, under the above conditions $m \ll m\xi_0 \ll \varepsilon_0$, the electron is expected to be only slightly deflected from its initial direction by the background field [35], we also assume that the incoming transverse momentum of the electron is at most of the order of $m\xi_0$. In order to clearly define the meaning of "longitudinal" and "transverse", it is convenient to introduce light-cone coordinates, defined by

$$T = \frac{t + \boldsymbol{n}\cdot\boldsymbol{x}}{2}, \quad \boldsymbol{x}_\perp = \boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}, \quad \phi = t - \boldsymbol{n}\cdot\boldsymbol{x}, \quad (3)$$

where $\boldsymbol{n}$ is a unit vector. We also define the quantities $n^\mu = (1, \boldsymbol{n}), \tilde{n}^\mu = (1/2)(1, -\boldsymbol{n})$, and $a_j^\mu = (0, \boldsymbol{a}_j)$, with $j = 1, 2$. The quantities $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ introduced above are two unit vectors perpendicular to $\boldsymbol{n}$ and to each other, and such that $\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \boldsymbol{n}$. An arbitrary four-vector $v^\mu = \left(v^0, \boldsymbol{v}\right)$ can be expressed as: $v^\mu = v_+ n^\mu + v_- \tilde{n}^\mu + v_1 a_1^\mu + v_2 a_2^\mu$, where $v_+ = (\tilde{n}v) = \left(v^0 + \boldsymbol{n}\cdot\boldsymbol{v}\right)/2, v_- = (nv) = v^0 - \boldsymbol{n}\cdot\boldsymbol{v}$, and $v_j = -\left(a_j v\right) = \boldsymbol{a}_j\cdot\boldsymbol{v}$ (note that $a_j^2 = -\boldsymbol{a}_j^2 = -1$). While the unit vector $\boldsymbol{n}$ is arbitrary in principle, it is of course convenient to choose it to approximately coincide with the electron's initial momentum, such that the conditions $p_z > 0, |\boldsymbol{p}_\perp| \lesssim m\xi_0$, and $p_z \sim \varepsilon$ are satisfied, where $p^\mu = (\varepsilon, \boldsymbol{p}) = \left(\sqrt{m^2+\boldsymbol{p}^2}, \boldsymbol{p}\right)$ is the electron's four-momentum at a generic time.

In the above light-cone coordinates, the variable T plays the role of the time and the LL equation can be written as

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}T} = \frac{e}{p_+}F^{\mu\nu}p_\nu + \frac{2}{3}\frac{e^3}{4\pi m^2}\left[\frac{1}{p_+}\left(\partial_\alpha F^{\mu\nu}\right)p^\alpha p_\nu - \frac{e}{p_+}F^{\mu\nu}F_{\alpha\nu}p^\alpha + \frac{e}{m^2 p_+}\left(F^{\alpha\nu}p_\nu\right)\left(F_{\alpha\lambda}p^\lambda\right)p^\mu\right]. \quad (4)$$

The light-cone components of the field tensor can be expressed in terms of the electromagnetic field as $F_{\tilde{n},n} = \tilde{n}_\mu F^{\mu\nu}n_\nu = \boldsymbol{n}\cdot\boldsymbol{E} = E_n, F_{\tilde{n},j} = \tilde{n}_\mu F^{\mu\nu}a_{j,\nu} = \boldsymbol{a}_j\cdot\boldsymbol{F}_m/2$, $F_{n,j} = n_\mu F^{\mu\nu}a_{j,\nu} = \boldsymbol{a}_j\cdot\boldsymbol{F}_p$, and $F_{1,2} = a_{1,\mu}F^{\mu\nu}a_{2,\nu} = -\boldsymbol{n}\cdot\boldsymbol{B} = -B_n$, with $\boldsymbol{F}_{p/m} = \boldsymbol{E}_\perp \pm \boldsymbol{n}\times\boldsymbol{B}_\perp$. The LL equation in light-cone coordinates can be written as

$$\begin{aligned}\frac{\mathrm{d}p_+}{\mathrm{d}T} = eE_n + \frac{e}{2}\frac{\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp}{p_+} + \frac{2}{3}\frac{e^3}{4\pi m^2}\Bigg[&p_+\partial_T E_n \\ &+ \frac{m^2+\boldsymbol{p}_\perp^2}{2p_+}\partial_\phi E_n + \boldsymbol{p}_\perp\cdot\boldsymbol{\nabla}_\perp E_n \\ &+ \frac{1}{2}(\boldsymbol{p}_\perp\cdot\partial_T\boldsymbol{F}_m) + \frac{m^2+\boldsymbol{p}_\perp^2}{4p_+^2}(\boldsymbol{p}_\perp\cdot\partial_\phi\boldsymbol{F}_m) \\ &+ \frac{1}{2p_+}(((\boldsymbol{p}_\perp\cdot\boldsymbol{\nabla}_\perp)\boldsymbol{F}_m)\cdot\boldsymbol{p}_\perp) \\ &+ eE_n^2 + \frac{eE_n}{2p_+}\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp + \frac{e}{8}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+^2}\boldsymbol{F}_m^2 \\ &+ \frac{e}{2}\boldsymbol{F}_m\cdot\boldsymbol{F}_p - \frac{eB_n}{2p_+}\boldsymbol{n}\cdot(\boldsymbol{p}_\perp\times\boldsymbol{F}_m) \\ &+ \left(\frac{2eE_n p_+}{m^2} + e\frac{\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp}{m^2}\right)\left(\frac{-E_n}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+} + \boldsymbol{F}_p\cdot\boldsymbol{p}_\perp\right) \\ &- \frac{e}{m^2}\left(\frac{m^2+\boldsymbol{p}_\perp^2}{4p_+}\boldsymbol{F}_m + p_+\boldsymbol{F}_p - B_n(\boldsymbol{n}\times\boldsymbol{p}_\perp)\right)^2\Bigg],\end{aligned} \quad (5)$$

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}T} &= e\boldsymbol{F}_p - eB_n\frac{\boldsymbol{n}\times\boldsymbol{p}_\perp}{p_+} + \frac{e}{4}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+^2}\boldsymbol{F}_m \\ &+ \frac{2}{3}\frac{e^3}{4\pi m^2}\Bigg[\frac{m^2+\boldsymbol{p}_\perp^2}{4p_+}\partial_T\boldsymbol{F}_m + \frac{(m^2+\boldsymbol{p}_\perp^2)^2}{8p_+^3}\partial_\phi\boldsymbol{F}_m \\ &+ \frac{m^2+\boldsymbol{p}_\perp^2}{4p_+^2}(\boldsymbol{p}_\perp\cdot\boldsymbol{\nabla}_\perp)\boldsymbol{F}_m \\ &+ p_+\partial_T\boldsymbol{F}_p + \frac{m^2+\boldsymbol{p}_\perp^2}{2p_+}\partial_\phi\boldsymbol{F}_p \\ &+ (\boldsymbol{p}_\perp\cdot\boldsymbol{\nabla}_\perp)\boldsymbol{F}_p - \Bigg(\partial_T B_n + \frac{m^2+\boldsymbol{p}_\perp^2}{2p_+^2}\partial_\phi B_n \\ &+ \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{\nabla}_\perp B_n}{p_+}\Bigg)(\boldsymbol{n}\times\boldsymbol{p}_\perp) \\ &+ \left(\frac{e}{2}\frac{\boldsymbol{F}_p\cdot\boldsymbol{p}_\perp}{p_+} - eE_n\frac{m^2+\boldsymbol{p}_\perp^2}{4p_+^2}\right)\boldsymbol{F}_m \\ &+ \left(eE_n + \frac{e}{2}\frac{\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp}{p_+}\right)\boldsymbol{F}_p \\ &- \frac{eB_n}{4}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+^2}(\boldsymbol{n}\times\boldsymbol{F}_m) - eB_n(\boldsymbol{n}\times\boldsymbol{F}_p) \\ &- \frac{eB_n^2}{p_+}\boldsymbol{p}_\perp + \left(\frac{2eE_n}{m^2} + e\frac{\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp}{m^2p_+}\right)\Bigg(\frac{-E_n}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+} \\ &+ \boldsymbol{F}_p\cdot\boldsymbol{p}_\perp\Bigg)\boldsymbol{p}_\perp \\ &- \frac{e}{m^2p_+}\left(\frac{m^2+\boldsymbol{p}_\perp^2}{4p_+}\boldsymbol{F}_m + p_+\boldsymbol{F}_p - B_n(\boldsymbol{n}\times\boldsymbol{p}_\perp)\right)^2\boldsymbol{p}_\perp\Bigg],\end{aligned} \tag{6}$$

whereas the on-shell condition $p^2(T) = m^2$ implies that the remaining minus light-cone component of the four-momentum of the electron at a generic time T is given by

$$p_-(T) = \frac{m^2 + \boldsymbol{p}_\perp^2(T)}{2p_+(T)}. \tag{7}$$

3.1 Approximations and scale analysis

In our physical situation of interest, we require that

$$m \ll m\xi_0 \ll p_{0,+} \quad \text{and} \quad |\boldsymbol{p}_{0,\perp}| \lesssim m\xi_0. \tag{8}$$

Under these conditions it is $p_{0,+} \approx \varepsilon_0$ and, as we have already mentioned, this is what “the electron energy is the largest dynamical energy scale in the problem” means. As we will also see, under the conditions in Eq. (8), the energy of the electron is almost constant such that these conditions are verified during the whole dynamics (in a few-cycle laser pulse).

Now, since the LL equation is a classical equation of motion, we require quantum effects to be small. As we have mentioned in the introduction, this implies that the typical strength of the electromagnetic field measured in the instantaneous rest system of the electron is much smaller than the critical field scale $F_{cr} = E_{cr} = B_{cr}$. This condition can be formalized by introducing the quantum nonlinearity parameter χ_0 which is defined as [9,10,24]

$$\chi_0 := \frac{2p_{0,+}}{m}\frac{F_0}{F_{cr}} = \frac{2p_{0,+}|e|F_0}{m^3} \tag{9}$$

and by requiring $\chi_0 \ll 1$. We notice that in the presence of a generic electromagnetic field $F^{\mu\nu}$ the more general time-dependent definition $\sqrt{|F_{\mu\nu}p^\nu|^2}/mF_{cr}$ along the electron trajectory should be used, but under the conditions in Eq. (8), the two definitions are approximately equivalent. As it results from the derivation of the LL equation in Ref. [1], another condition, which is essential for the LL equation to hold, is that the characteristic angular frequency ω_0 of the external electromagnetic field measured in the instantaneous rest system of the electron is much smaller than the electron mass m. This condition can be formalized by introducing another parameter η_0 defined as

$$\eta_0 := \frac{2p_{0,+}}{m}\frac{\omega_0}{m}, \tag{10}$$

for which we also require that $\eta_0 \ll 1$. With the above definitions, we have $\xi_0 = \chi_0/\eta_0$.

By recalling the numerical example mentioned above of a Ti:Sa laser with $\xi_0 \approx 16$, we see that electron energies up to about 0.5 GeV can be considered, which would correspond to $\chi_0 \approx 0.1$ and then $\eta_0 \approx 6 \times 10^{-3}$.

3.1.1 Scale analysis

In order to evaluate the importance of each term in Eqs. (5) and (6), we have in mind an experimental setup where a Gaussian beam of central angular frequency ω_0 is focused to a spot radius σ, such that the field reaches the maximal field strength F_0 within an area of the order of $\pi\sigma^2$. We assume that σ is of the same order of $\lambda_0 = 2\pi/\omega_0$. The longitudinal extension of such a tightly focused laser pulse is described by the Rayleigh length $l_R = \pi\sigma^2/\lambda_0 \sim \pi\lambda_0$. Here, one should observe that along the longitudinal direction the field decreases only linearly, unlike on the transverse plane where it features an exponential decay. Although presently laser pulses of 40–50 fs are employed for strong-field experiments, the experimental aim is to produce shorter and shorter pulses such that, at a given energy, the power is higher and higher. Indeed lasers with pulse lengths of down to 10 fs are already being tested [39]. Assuming a Ti:Sa laser with central wavelength $\lambda_0 = 0.8$ μm this would correspond to about 3.7 cycles. Thus, we first make the assumption that the field contains only one or a few cycles and then we briefly comment on the case of longer pulses.

The mentioned conditions on $p_{0,+}/m$, χ_0, and η_0 can now be used to perform a scale analysis of the terms in the LL equation. For this analysis all field components are estimated to be of the same order as F_0 and the derivatives $\partial_T, \partial_\phi$, and $\boldsymbol{\nabla}_\perp$ are considered to be of the same order of $1/\lambda_0$. For example, for the first two terms in Eq. (5), resulting from the Lorentz equation, we have

$$eE_n \sim eF_0 \sim \eta_0 \left(\frac{m\xi_0}{p_{0,+}}\right) m^2, \tag{11}$$

$$\frac{e}{2}\frac{\boldsymbol{F}_m \cdot \boldsymbol{p}_\perp}{p_+} \sim e\frac{F_0|\boldsymbol{p}_\perp|}{p_+} \sim \eta_0 \left(\frac{m\xi_0}{p_{0,+}}\right)^2 m^2. \tag{12}$$

Therefore, the ratio of the two terms is of first order in $m\xi_0/p_{0,+} \ll 1$, which means that the second term (12) is much smaller than the first one. Since we seek for a first-order solution, we can neglect all terms which are in this sense much smaller than the term in Eq. (12) either by having higher orders of $\xi_0 m/p_{0,+}$ or also of χ_0 and η_0. In this way, by using orders-of-magnitude estimations such as the following ones:

$$e\boldsymbol{F}_p \sim eE_n \sim \eta_0 \left(\frac{m\xi_0}{p_{0,+}}\right) m^2, \tag{13}$$

$$\frac{e}{2}\frac{\boldsymbol{F}_m \cdot \boldsymbol{p}_\perp}{p_+} \sim eB_n\frac{\boldsymbol{n} \times \boldsymbol{p}_\perp}{p_+} \sim \eta_0 \left(\frac{m\xi_0}{p_{0,+}}\right)^2 m^2, \tag{14}$$

$$\begin{aligned}\frac{2}{3}\frac{e^3}{4\pi m^2}\frac{2e}{m^2}E_n p_+ (\boldsymbol{F}_p \cdot \boldsymbol{p}_\perp) &\\ \sim \frac{2}{3}\frac{e^3}{4\pi m^2}\frac{2e}{m^2}p_+ B_n (\boldsymbol{n}\cdot(\boldsymbol{p}_\perp \times \boldsymbol{F}_p)) &\\ \sim \frac{2}{3}\frac{e^4}{4\pi m^4}p_+\boldsymbol{F}_p^2\boldsymbol{p}_\perp \sim \chi_0^2\alpha\left(\frac{m\xi_0}{p_{0,+}}\right) m^2, &\end{aligned} \tag{15}$$

$$\frac{2}{3}\frac{e^3}{4\pi m^2}\frac{e}{m^2}p_+^2\boldsymbol{F}_p^2 \sim \chi_0^2\alpha m^2, \tag{16}$$

$$\begin{aligned}\frac{2}{3}\frac{e^3}{4\pi m^2}p_+\partial_T E_n \sim \frac{2}{3}\frac{e^3}{m^2}p_+\partial_T \boldsymbol{F}_p &\\ \sim \eta_0^2\alpha\left(\frac{m\xi_0}{p_{0,+}}\right) m^2, &\end{aligned} \tag{17}$$

the LL equations can be simplified as:

$$\left(\frac{\mathrm{d}p_+}{\mathrm{d}T}\right)^{(1)} = eE_n + \frac{e}{2}\frac{\boldsymbol{F}_m \cdot \boldsymbol{p}_\perp}{p_+} - \frac{2}{3}\frac{e^4}{4\pi m^4}p_+^2\boldsymbol{F}_p^2, \tag{18}$$

$$\left(\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}T}\right)^{(1)} = e\boldsymbol{F}_p - eB_n\frac{\boldsymbol{n} \times \boldsymbol{p}_\perp}{p_+} - \frac{2}{3}\frac{e^4}{4\pi m^4}p_+\boldsymbol{F}_p^2\boldsymbol{p}_\perp. \tag{19}$$

The upper index (1) in the derivatives of the four-momentum components has to be interpreted as we keep up to the next-to-leading order terms in the Lorentz force and the leading-order term in the radiation–reaction force. The relative importance of the remaining terms depends on the precise values of the parameters. In the case of a plane wave, it would be $E_n = 0$ and $\boldsymbol{F}_m = \boldsymbol{0}$ such that the component p_+ of the four-momentum would significantly change in the so-called "classical radiation-dominated" regime where $\alpha\chi_0\xi_0 \sim 1$ [13]. This, however, would occur for $\xi_0 \gtrsim 1/\alpha\chi_0 \gtrsim 10^3$, which, together with the condition $p_{0,+} \gg m\xi_0$ would imply that quantum effects are dominating the dynamics. In this respect, the present approach allows to investigate regimes where radiation–reaction effects are corrections of the Lorentz dynamics. However, it is interesting to notice that the radiation–reaction term in Eq. (18) can be of the same order of the leading-order term corresponding to the Lorentz force under the condition that $\alpha\chi_0 p_{0,+}/m \sim 1$, which is feasible, as discussed in the numerical example at the end of the previous paragraph.

3.2 Calculation of the momentum to first order

In the following we will present an iterative method to solve Eqs. (18) and (19) by exploiting the appearance of different powers of the small quantities according to the conditions (8)–(10).

First, by using the identity $p^\mu = p_+\mathrm{d}x^\mu/\mathrm{d}T$, we obtain an approximated expression of the "spatial" coordinates $\boldsymbol{r}(T) = (\phi(T), \boldsymbol{x}_\perp(T))$, which we are going to replace in the fields. We have that

$$\frac{\mathrm{d}\phi}{\mathrm{d}T} = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p_+^2}, \tag{20}$$

$$\frac{\mathrm{d}^2\boldsymbol{x}_\perp}{\mathrm{d}T^2} = \frac{1}{p_+}\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}T} - \frac{\mathrm{d}p_+}{\mathrm{d}T}\frac{\boldsymbol{p}_\perp}{p_+^2}. \tag{21}$$

By inserting Eqs. (18) and (19) here, we have

$$\begin{aligned}\left(\frac{\mathrm{d}^2\boldsymbol{x}_\perp}{\mathrm{d}T^2}\right)^{(1)} = \frac{e\boldsymbol{F}_p}{p_+} - eB_n\frac{\boldsymbol{n} \times \boldsymbol{p}_\perp}{p_+^2} - \frac{eE_n}{p_+^2}\boldsymbol{p}_\perp &\\ +\frac{2}{3}\frac{e^3}{4\pi m^2}\partial_T\boldsymbol{F}_p. &\end{aligned} \tag{22}$$

It has to be noticed that the leading-order terms in the radiation–reaction force cancel out exactly in Eq. (21). Thus, consistently with the meaning of the upper index (1), we had to keep the next-to-leading order contribution to the radiation–reaction force in Eq. (22).

Now, on the one hand, since the expression (20) is of second order in $m\xi_0/p_{0,+}$ we assume that the coordinate $\phi(T) = \phi_0$ is constant for all times. On the other hand, in order to employ the transverse coordinates inside the fields and replace them in terms of the initial coordinates, it is sufficient to only consider the leading-order term, i.e.,

$$\left(\frac{\mathrm{d}^2\boldsymbol{x}_\perp}{\mathrm{d}T^2}\right)^{(1)} \approx \frac{e}{p_+}\boldsymbol{F}_p. \tag{23}$$

Note, in fact, that the radiation–reaction term is much smaller than the leading-order term by a factor of the order of $\alpha\eta_0$.

Thus, to first order the spatial light-cone coordinates of the electron are given by the same expression without radiation reaction, i.e.,

$$\boldsymbol{r}^{(1)}(T) = \left(\phi_0, \boldsymbol{x}_{0,\perp} + \frac{e}{p_{0,+}} \int_{T_0}^{T} \mathrm{d}T' \boldsymbol{G}_p\left(T', \boldsymbol{r}_0\right) + \frac{\boldsymbol{p}_{0,\perp}}{p_{0,+}}(T - T_0)\right), \tag{24}$$

where $\boldsymbol{G}_p\left(T, \boldsymbol{r}_0\right) = \int_{T_0}^{T} \mathrm{d}T' \boldsymbol{F}_p\left(T', \boldsymbol{r}_0\right)$, with $\boldsymbol{r}_0 = (\phi_0, \boldsymbol{x}_{0,\perp})$. Equation (24) justifies the insertion of $\boldsymbol{p}_\perp(T) = \boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p(T, \boldsymbol{r}_0)$ into Eqs. (18) and (19). In order to perform the integration of Eqs. (18) and (19), we need to expand the field components in their transverse spatial dependence such that there remains no explicit dependence on $\boldsymbol{r}(T)$. Such an expansion is based on the argument that due to the electron's high energy we expect that its transverse position $\boldsymbol{x}_\perp$ only changes by a small amount. Indeed, by considering Eq. (24) we see that for an ultrashort laser pulse $(\boldsymbol{x}_\perp - \boldsymbol{x}_{0,\perp})/\lambda_0 \sim m\xi_0/p_{0,+} \ll 1$. For example, we may perform the following expansion

$$E_n\left(T', \boldsymbol{r}\left(T'\right)\right) = E_n\left(T', \boldsymbol{r}_0\right) + \boldsymbol{\nabla}_\perp E_n\left(T', \boldsymbol{r}_0\right) \cdot \left(\boldsymbol{x}_\perp(T') - \boldsymbol{x}_{0,\perp}\right). \tag{25}$$

By using $(\boldsymbol{x}_\perp(T') - \boldsymbol{x}_{0,\perp}) \approx \frac{1}{p_{0,+}} \int_{T_0}^{T'} (\boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p(T'', \boldsymbol{r}_0))\mathrm{d}T''$, we obtain

$$E_n\left(T', \boldsymbol{r}\left(T'\right)\right) \approx E_n\left(T', \boldsymbol{r}_0\right) + \frac{1}{p_{0,+}} \int_{T_0}^{T'} (\boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p(T'', \boldsymbol{r}_0))\mathrm{d}T'' \cdot \boldsymbol{\nabla}_\perp E_n\left(T', \boldsymbol{r}_0\right). \tag{26}$$

By writing $\boldsymbol{\nabla}_\perp E_n = \frac{\mathrm{d}}{\mathrm{d}T} \boldsymbol{\nabla}_\perp \int \mathrm{d}T E_n$ and by employing partial integration, we can write

$$\int_{T_0}^{T} \mathrm{d}T' E_n(T', \boldsymbol{r}\left(T'\right)) \approx \int_{T_0}^{T} \mathrm{d}T' \Big[E_n\left(T', \boldsymbol{r}_0\right) + \frac{1}{p_{0,+}}\left(\boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p\left(T', \boldsymbol{r}_0\right)\right) \cdot \boldsymbol{\nabla}_\perp \int_{T'}^{T} E_n\left(T'', \boldsymbol{r}_0\right) \mathrm{d}T''\Big]. \tag{27}$$

This expansion can be carried out for any other expressions of field components. The result for $p_+(T)$ to first order is then obtained by integrating Eq. (18) and is given by

$$p_+^{(1)}(T) = p_{0,+} + \int_{T_0}^{T} \mathrm{d}T' \Big[eE_n\left(T', \boldsymbol{r}_0\right) - \frac{2}{3}\frac{e^4}{4\pi m^4} p_{0,+}^2 \boldsymbol{F}_p^2\left(T', \boldsymbol{r}_0\right)\Big]. \tag{28}$$

The transverse components are obtained in a similar way by integrating Eq. (19) and are given by

$$\begin{aligned}\boldsymbol{p}_\perp^{(1)}(T) = \boldsymbol{p}_{0,\perp} + \int_{T_0}^{T} \mathrm{d}T' \Big[& e\boldsymbol{F}_p\left(T', \boldsymbol{r}_0\right) \\ & + \frac{e}{p_{0,+}}\left[\left(\boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p\left(T', \boldsymbol{r}_0\right)\right) \cdot \boldsymbol{\nabla}_\perp\right] \\ & \int_{T'}^{T} \boldsymbol{F}_p\left(T'', \boldsymbol{r}_0\right) \mathrm{d}T'' - \frac{e}{p_{0,+}} B_n\left(T', \boldsymbol{r}_0\right) \\ & \left[\boldsymbol{n} \times \left(\boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p\left(T', \boldsymbol{r}_0\right)\right)\right] \\ & - \frac{2}{3}\frac{e^4}{4\pi m^4} p_{0,+} \boldsymbol{F}_p^2(T', \boldsymbol{r}_0) \\ & \left(\boldsymbol{p}_{0,\perp} + e\boldsymbol{G}_p\left(T', \boldsymbol{r}_0\right)\right)\Big].\end{aligned} \tag{29}$$

In order to keep the electron exactly on shell, the remaining light-cone component $p_-(T)$ can be computed from the identity $p_-(T) = \left[m^2 + \boldsymbol{p}_\perp^2(T)\right]/2p_+(T)$.

Finally, the corresponding dependence of the spatial coordinates on T can be obtained by using the identity $\mathrm{d}x^\mu/\mathrm{d}T = p^\mu/p_+$, which would provide a more accurate solution of the original one in Eq. (24) in line with the iterative approach.

3.2.1 Qualitative considerations on the effects of the laser pulse duration

So far we have assumed small integration time scales, which means that the integration interval $T - T_0$ can be estimated to be effectively of the order of λ_0. This corresponds to the physical situation where the electron is exposed to a single-cycle or few-cycle laser pulse. If one wants to account for longer pulse durations, i.e., for pulses with $N \gg 1$ cycles, one has to include a factor N for each integration of functions which accumulate while the electron propagates inside the laser field. Such terms which increase with time, thus limiting the time interval over which the perturbative approach is valid, are commonly referred to as secular terms. On the contrary, when oscillating functions are integrated, the factor N will be absent. Indeed, we expect that any single field component does not accumulate when being integrated. As an example, we conclude that the term $\boldsymbol{G}_p\left(T, \boldsymbol{r}_0\right)$ is of the order of $\lambda_0 F_0$.

For pulses with multiple laser cycles, i.e., large N, the accumulation effects when performing the integrations may invalidate the perturbative approach which we have followed. It could happen, for example, that further terms in Eq. (22) contribute after the integration, because the leading-order term does not accumulate whereas the other terms may do. More precisely the second and the third term in Eq. (22) are next-to-leading order but can in principle accumulate because they include products of fields with $\boldsymbol{p}_\perp(T)$, which also depends on the fields. If we require these terms to stay negligible after integrating twice, we need to require that $N^2 m\xi_0 \ll p_{0,+}$. Otherwise, the approximated expression (24) may not be sufficient anymore.

We conclude that including long pulse durations would additionally complicate the scale analysis and, as a first investigation of this problem, we limit here to the simpler case of a single-cycle or of a few-cycle laser pulse.

3.3 The case of a plane-wave background field

As a sanity check of the calculations above, we can specialize to the plane-wave case and to compare the results with those obtained from the exact analytical solution given in Ref. [13]. In the case of a plane wave, the field can be expressed as a function of the coordinate $T=(\tilde{n}x)=(t+\boldsymbol{n}\cdot\boldsymbol{x})/2$ only. A general four-potential in Lorenz gauge, with the additional condition $A^0(T)=0$, for such a field can be written as

$$A^\mu(T)=a_1^\mu\psi_1(T)+a_2^\mu\psi_2(T), \tag{30}$$

where $\psi_1(T)$ and $\psi_2(T)$ are arbitrary functions and the four-vectors a_j^μ are defined as in Sect. 3. The field tensor $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ is then given by

$$F^{\mu\nu}(T)=\tilde{f}_1^{\mu\nu}\psi_1'(T)+\tilde{f}_2^{\mu\nu}\psi_2'(T), \tag{31}$$

where $\tilde{f}_j^{\mu\nu}\equiv\tilde{n}^\mu a_j^\nu-\tilde{n}^\nu a_j^\mu$ and the prime denotes a derivative with respect to T. From this expression it follows that

$$E_n=B_n=0, \tag{32}$$

$$\boldsymbol{F}_m=\boldsymbol{0}, \tag{33}$$

and

$$\boldsymbol{F}_p(T)=-\psi_1'(T)\boldsymbol{a}_1-\psi_2'(T)\boldsymbol{a}_2. \tag{34}$$

Therefore, for plane waves the expressions (28) and (29) simplify to

$$p_+^{(1)}(T)=p_{0,+}-\frac{2}{3}\frac{e^4}{4\pi m^4}p_{0,+}^2\int_{T_0}^{T}\mathrm{d}T'\boldsymbol{F}_p^2(T'), \tag{35}$$

$$\boldsymbol{p}_\perp^{(1)}(T)=\boldsymbol{p}_{0,\perp}+e\boldsymbol{G}_p(T)-\frac{2}{3}\frac{e^4}{4\pi m^4}p_{0,+}\int_{T_0}^{T}\mathrm{d}T'\left(\boldsymbol{p}_{0,\perp}+e\boldsymbol{G}_p(T')\right)\boldsymbol{F}_p^2(T'). \tag{36}$$

Now, it can be easily shown by expanding the solution in Ref. [13] according to the analysis above, that indeed the two expressions of the four-momenta coincide. One sees from Eq. (35) that, as we have already mentioned, in our approach radiation–reaction effects are ultimately treated perturbatively.

4 Conclusion

In conclusion, we have found an approximated expression of the classical four-momentum of an ultrarelativistic electron according to the Landau–Lifshitz (LL) equation. The main expansion parameter is inversely proportional to the initial energy of the electron, which is assumed to be the largest dynamical energy in the problem. Although we have not made specific assumptions about the space–time structure of the background field, we had in mind the case of an ultrashort, tightly focused laser beam. The results are applicable under the general assumptions made in the derivation of the LL equation and under the assumption that quantum effects are negligible. These conditions can be realized in experiments using an intense, short, and tightly focused laser beam as a background field. Therefore, our results can also be used for testing the validity of the LL equation experimentally. As a limitation of our approach, we point out that, since in order to treat a complicated background electromagnetic field we need the condition $\varepsilon\gg m\xi_0$, we are not able within our approach to describe the classical radiation-dominated regime but only the regime where radiation–reaction effects are small corrections as compared to the Lorentz dynamics.

Author contributions

Both authors have performed the calculations. AH has written the paper under the guidance of ADP, who has then revised it before submission.

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