SLAC-PUB-356
November 1967

ASYMPTOTIC ESTIMATES OF FEYNMAN INTEGRALS*

James P. Fink[†,††]
Department of Mathematics and Stanford Linear Accelerator Center
Stanford University, Stanford, California

ABSTRACT

In this paper, we consider the problem of determining logarithmic as well as polynomial, asymptotic estimates for certain convergent integrals containing parameters. We state and prove an asymptotic theorem which gives the logarithmic asymptotic behavior of such a convergent integral as any subset of the parameters becomes large while the remaining parameters remain bounded. This theorem is then applied to the photon and electron self-energy graphs of quantum electrodynamics.

(To be submitted to Journal of Math. Phys.)

---

*The preparation of this manuscript was supported in part by U. S. Atomic Energy Commission.

†National Science Foundation pre-doctoral fellow in Mathematics at Stanford University during the years 1963-67.

††Present address is Department of Mathematics and Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania.

Print-67-2341

I. INTRODUCTION

The renormalization procedures of F. Dyson[1, 2] and A. Salam[3, 4] depend upon a certain convergence criterion for integrals which was proved in a paper by S. Weinberg.[5] In addition to stating and proving conditions under which a Feynman integral converges, Weinberg develops a method for determining a polynomial bound on the value of the integral as subsets of the external momenta become large, provided the usual rotations of energy contours can be performed. The value of his technique is that one need not evaluate the integrals under consideration. The bound on the integral is determined simply from the asymptotic properties of the integrand alone.

Weinberg's analysis, however, does not determine the logarithmic asymptotic behavior of convergent integrals. A method which provides some clue to the logarithmic asymptotic behavior of the photon and electron self-energy graphs of quantum electrodynamics is the renormalization group approach (cf., Bjorken and Drell,[6] Bogoliubov and Shirkov,[7] and Landau[8]). The renormalization group is, by definition, the group of transformations which when applied to the propagators, charges, and masses of a theory yield new propagators, charges, and masses which do not change the expressions for observable quantities. The arguments of the renormalization group approach rely upon several fundamental assumptions which lead to anomalous results which in turn make one suspect the original assumptions.

In this paper, we develop a technique for determining the logarithmic asymptotic behavior of a certain class of convergent integrals and apply it to various Feynman integrals of quantum electrodynamics. We use Weinberg's results[5] as a basis, although we are required to modify and extend them.

II. ASYMPTOTIC THEOREMS FOR INTEGRALS

A. Introduction

In this section we will be concerned with extending the results of Weinberg.[5] Before undertaking this task, however, we will briefly summarize his results and, in doing so, we will use essentially the notation used by Weinberg.

B. Summary of Weinberg's Results

Let $f(p_1, \ldots, p_n)$ be a complex-valued function of the n real variables $p_1, \ldots, p_n$. We will consider the variables $p_1, \ldots, p_n$ as the components of a vector $\overline{P}$ in R^n, and we will be concerned only with those functions $f(\overline{P})$ which belong to a certain class A_n defined as follows:

Definition: A function $f(\overline{P})$ is an element of the class A_n if and only if, for each subspace $S \subset R^n$, there exist coefficients $\alpha(S)$, $\beta(S)$ such that, for any choice of $m \leq n$ independent vectors $\overline{L}_1, \ldots, \overline{L}_m$ and bounded region $W \subset R^n$, we have

$$f(\overline{L}_1 \eta_1 \cdots \eta_m + \overline{L}_2 \eta_2 \cdots \eta_m + \ldots + \overline{L}_m \eta_m + \overline{C})$$
$$= O\left\{ \eta_1^{\alpha(\{\overline{L}_1\})} (\log \eta_1)^{\beta(\{\overline{L}_1\})} \cdots \eta_m^{\alpha(\{\overline{L}_1, \ldots, \overline{L}_m\})} (\log \eta_m)^{\beta(\{\overline{L}_1, \ldots, \overline{L}_m\})} \right\}$$

when $\eta_1, \ldots, \eta_m$ tend independently to infinity and $\overline{C} \in W$. The notation $\{\overline{L}_1, \ldots, \overline{L}_r\}$ denotes the subspace spanned by the vectors $\overline{L}_1, \ldots, \overline{L}_r$.

Let I be a subspace of R^n spanned by some set of orthonormal vectors $\overline{L}'_1, \ldots, \overline{L}'_k$, and consider the integral

$$f_I(\overline{P}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dy_1 \cdots dy_k \, f(\overline{P} + \overline{L}'_1 y_1 + \ldots + \overline{L}'_k y_k)$$

$$= \int_{\overline{P}' \in I} d^k \overline{P}' \, f(\overline{P} + \overline{P}') \; .$$

Provided this integral exists, $f_I(\overline{P})$ is a function which depends only on the projection of $\overline{P}$ along the subspace I; that is, $f_I(\overline{P})$ depends only upon the component of $\overline{P}$ in the subspace complementary to I .

The following theorem was proved by Weinberg.[5]

Theorem 1 : Suppose $f(\overline{P}) \epsilon A_n$ with asymptotic coefficients $\alpha(S)$ and $\beta(S)$ for any non-zero subspace S of R^n. Let $f(\overline{P})$ be integrable over any bounded region in R^n (local integrability), and let

$$D_I = \max_{S' \subset I} \left\{ \alpha(S') + \dim S' \right\} ,$$

where dim S' is the dimension of S'. If $D_I < 0$, then

(a) $f_I(\overline{P})$ exists ;

(b) $f_I(\overline{P}) \epsilon A_{n-k}$ with asymptotic coefficient $\alpha_I(S)$ for $S \subset E$, where

$$R^n = I \oplus E, \text{ given by } \alpha_I(S) = \max_{\Lambda(I) S' = S} \left\{ \alpha(S') + \dim S' - \dim S \right\} .$$

$\Lambda(I)$ is the operation of projection along the subspace I and $\max\limits_{\Lambda(I)S'=S}$ means that the maximum is taken over all those subspaces S' which project onto S.

C. Definition of the Subclass B_n

Let $f(\overline{P}) \epsilon A_n$ with asymptotic coefficients $\alpha(S)$ and $\beta(S)$. Let $\overline{L}_1, \ldots, \overline{L}_m$ be $m \leq n$ independent vectors and W a finite region in R^n. We arrange the logarithmic asymptotic coefficients $\beta(\{\overline{L}_1\}), \ldots, \beta(\{\overline{L}_1, \ldots, \overline{L}_m\})$ in increasing order, and suppose that

$$\beta\left(\{\overline{L}_1, \ldots, \overline{L}_{\pi_1}\}\right) \leq \beta\left(\{\overline{L}_1, \ldots, \overline{L}_{\pi_2}\}\right) \leq \cdots \leq \beta\left(\{\overline{L}_1, \ldots, \overline{L}_{\pi_m}\}\right) ,$$

where $\pi_1, \ldots, \pi_m$ is a permutation of the integers $1, \ldots, m$.

Definition: A function $f(\overline{P})$ is an element of the subclass B_n if and only if $f(\overline{P}) \in A_n$ with asymptotic coefficients $\alpha(S)$ and $\beta(S)$ such that $\beta(S)$ is a non-negative integer for all $S \subset R^n$ and

$$f(\overline{L}_1 \eta_1 \ldots \eta_m + \overline{L}_2 \eta_2 \ldots \eta_m + \ldots + \overline{L}_m \eta_m + \overline{C})$$

$$= O\left\{\eta_1^{\alpha(\{\overline{L}_1\})} \ldots \eta_m^{\alpha(\{\overline{L}_1, \ldots, \overline{L}_m\})} \sum_{\gamma_1, \ldots, \gamma_m} \left(\log \eta_{\pi_1}\right)^{\gamma_1} \left(\log \eta_{\pi_2}\right)^{\gamma_2} \ldots \left(\log \eta_{\pi_m}\right)^{\gamma_m}\right\}$$

when $\eta_1, \ldots, \eta_m$ tend independently to infinity and $\overline{C} \in W$, where the sum ranges over all non-negative integers $\gamma_1, \ldots, \gamma_m$ satisfying

$$\gamma_1 \leq \beta\left(\left\{\overline{L}_1, \ldots, \overline{L}_{\pi_1}\right\}\right),$$

$$\gamma_1 + \gamma_2 \leq \beta\left(\left\{\overline{L}_1, \ldots, \overline{L}_{\pi_2}\right\}\right),$$

$$\vdots$$

$$\gamma_1 + \ldots + \gamma_m \leq \beta\left(\left\{\overline{L}_1, \ldots, \overline{L}_{\pi_m}\right\}\right).$$

Since $B_n \subset A_n$, Theorem 1 applies to the subclass B_n.

D. Generalization for One-Dimensional Integrals

Our goal is to obtain a formula for $\beta_I(S)$ for integrable functions in the subclass B_n similar to the formula for $\alpha_I(S)$ given in Theorem 1 for integrable functions in A_n. We begin with a definition based on this theorem.

Definition: A subspace S' is said to be a maximizing subspace for the I-integration (relative to a given subspace $S \subset E$) if

$$\Lambda(I) S' = S \quad \text{and} \quad \alpha_I(S) = \alpha(S') + \dim S' - \dim S.$$

The proof of Theorem 1 given in Ref. 5 shows that maximizing subspaces always exist.

Let us first consider the case when $\dim I = 1$. For this case, the maximizing subspaces fall into two categories - those for which $\dim S' = \dim S$ and those for which $\dim S' = \dim S + 1$. Let p be the number of non-empty categories of maximizing subspaces; that is, $p = 1$ if all maximizing subspaces have the same dimension and $p = 2$ otherwise. By repeating the proof of Theorem 1 for the subclass B_n, we arrive at the following theorem:

Theorem 2: Let $f(\overline{P}) \in B_n$ satisfy all the conditions of Theorem 1 and suppose that $\dim I = 1$. Then $f_I(\overline{P}) \in B_{n-1}$ with asymptotic coefficients $\alpha_I(S)$ given by Theorem 1 and $\beta_I(S)$ given by

$$\beta_I(S) = \max_{S' \in M} \beta(S') + p - 1 ,$$

where M is the set of all maximizing subspaces.

E. Generalization for Two-Dimensional Integrals

In order to generalize this result when $\dim I > 1$, let us examine next the case $\dim I = 2$. We write $I = I_1 \oplus I_2$ where $\dim I_1 = \dim I_2 = 1$ and integrate first with respect to the I_1 variable and then with respect to the variable in I_2 and vice versa. Since we will be dealing only with integrable functions in B_n in the following, Fubini's theorem applies and we conclude that the integral is independent of the order of integration and of the particular choice of I_1 and I_2.

Let us perform the I_2 - integration first and then the I_1 - integration.

We have

$$\alpha_{I_2}(S') = \max_{\Lambda(I_2)S''=S'} \left\{\alpha(S'') + \dim S'' - \dim S'\right\} ,$$

$$\alpha_I(S) = \max_{\Lambda(I_1)S'=S} \left\{\alpha_{I_2}(S') + \dim S' - \dim S\right\}$$

$$= \max_{\Lambda(I)S''=S} \left\{\alpha(S'') + \dim S'' - \dim S\right\} ,$$

where

$$S \subset E \quad \text{with } R^n = I \oplus E ,$$

$$S' \subset E_2 \text{ with } R^n = I_2 \oplus E_2 ,$$

$$S'' \subset R^n .$$

Let $S'_\mu \subset E_2$ be the maximizing subspaces for the I_1-integration relative to S after performing the I_2-integration. For each S'_μ, let $S''_{\mu\nu} \subset R^n$ be the maximizing subspaces for the I_2-integration relative to S'_μ. We have the relations

$$\Lambda(I_2)S''_{\mu\nu} = S'_\mu , \qquad \alpha_{I_2}(S'_\mu) = \alpha(S''_{\mu\nu}) + \dim S''_{\mu\nu} - \dim S'_\mu ,$$

$$\Lambda(I_1)S'_\mu = S , \qquad \alpha_I(S) = \alpha_{I_2}(S'_\mu) + \dim S'_\mu - \dim S .$$

We now want to determine the maximizing subspaces for the full I-integration relative to S; that is, we want to determine the subspaces $S'' \subset R^n$ for which $\Lambda(I)S'' = S$ and $\alpha_I(S) = \alpha(S'') + \dim S'' - \dim S$.

Lemma 1: The $S''_{\mu\nu}$ are precisely the maximizing subspaces for the I-integration relative to S; that is, each $S''_{\mu\nu}$ is a maximizing subspace for the I-integration relative to S and any such maximizing subspace for the I-integration relative to S must be one of the $S''_{\mu\nu}$.

Proof: To show that each $S''_{\mu\nu}$ is a maximizing subspace for the I-integration relative to S, we note that $\Lambda(I)S''_{\mu\nu} = S$ and

$$\alpha_I(S) = \alpha_{I_2}(S'_\mu) + \dim S'_\mu - \dim S$$

$$= \alpha(S''_{\mu\nu}) + \dim S''_{\mu\nu} - \dim S'_\mu + \dim S'_\mu - \dim S$$

$$= \alpha(S''_{\mu\nu}) + \dim S''_{\mu\nu} - \dim S \ .$$

Conversely, suppose that S''_o is a maximizing subspace for the I-integration relative to S and let $S'_o = \Lambda(I_2)S''_o$. We have

$$S = \Lambda(I_1)S'_o = \Lambda(I)S''_o \quad ,$$

$$\alpha_I(S) = \alpha(S''_o) + \dim S''_o - \dim S \ .$$

Now

$$\alpha_I(S) = \max_{\Lambda(I_1)S' = S} \left\{\alpha_{I_2}(S') + \dim S' - \dim S\right\}$$

$$\geq \alpha_{I_2}(S'_o) + \dim S'_o - \dim S$$

$$= \max_{\Lambda(I_2)S'' = S'_o} \left\{\alpha(S'') + \dim S'' - \dim S'_o\right\} + \dim S'_o - \dim S$$

$$\geq \alpha(S''_o) + \dim S''_o - \dim S$$

$$= \alpha_I(S) \quad ,$$

where the last equality follows from the assumptions on S''_o . Since the first and last terms in this chain are the same quantity, all inequalities must be equalities, and hence

$$\alpha_I(S) = \alpha_{I_2}(S'_o) + \dim S'_o - \dim S \ , \quad \Lambda(I_1)S'_o = S \quad ,$$

$$\alpha_{I_2}(S'_o) = \alpha(S''_o) + \dim S''_o - \dim S'_o \ , \quad \Lambda(I_2)S''_o = S'_o \ .$$

Thus, S'_o is a mazimizing subspace for the I_1-integration relative to S after performing the I_2-integration and so must be one of the S'_μ . S''_o is a maximizing subspace for the I_2-integration relative to S'_o (which is one of the S'_μ) and consequently must be one of the $S''_{\mu\nu}$. ∎

We observe that Lemma 1 does not depend upon the fact that we are assuming $\dim I_1 = \dim I_2 = 1$.

Let p_1 be the number of different dimensions among the maximizing subspaces for the I_1-integration relative to S after performing the I_2-integration, and let $p_{2\mu}$ be the number of different dimensions among the maximizing subspaces for the I_2-integration relative to S'_μ .

Lemma 2: $p_{2\mu}$ is independent of μ .

Proof: Suppose not. Then there exist two maximizing subspaces for the I_1-integration relative to S after performing the I_2-integration, say S'_1 and S'_2 , such that $p_{21} = 1$ and $p_{22} = 2$.

There are several cases to be considered. We will work out the details for one case only because the others are all similar.

Let S''_{11} be maximizing for the I_2-integration relative to S'_1 and let S''_{21} and S''_{22} be maximizing for the I_2-integration relative to S'_2 . Suppose $\dim S'_1 = \dim S'_2 = \dim S''_{11} = \dim S''_{21} = \dim S''_{22} - 1$. Performing the I_2-integration first and then the I_1-integration, we obtain

$$\beta_I(S) = \max\left\{\beta_{I_2}(S'_1),\ \beta_{I_2}(S'_2)\right\}$$

$$= \max\left\{\beta(S''_{11}),\ \max\left\{\beta(S''_{21}),\ \beta(S''_{22})\right\} + 1\right\} \quad .$$

On the other hand, reversing the order of integration gives

$$\beta_I(S) = \max\left\{\beta(S''_{11}),\ \beta(S''_{21}),\ \beta(S''_{22})\right\} + 1 \quad .$$

These two expressions are not equal for all non-negative integral values of $\beta(S'')$ and hence we have a contradiction. ▌

Since $p_{2\mu}$ is independent of μ, we will denote it simply by p_2.

Now let $\widetilde{I}_1$ and $\widetilde{I}_2$ be two one-dimensional subspaces of I different from I_1 and I_2, respectively, such that $I = \widetilde{I}_1 \oplus \widetilde{I}_2$. Just as will I_1 and I_2, we let T'_ρ be the maximizing subspaces for the $\widetilde{I}_1$-integration relative to S after performing the $\widetilde{I}_2$-integration, and, for each T'_ρ, we let $T''_{\rho\sigma}$ be the maximizing subspaces for the $\widetilde{I}_2$-integration relative to T'_ρ. Let $\widetilde{p}_1$ be the number of different dimensions among the maximizing subspaces for the $\widetilde{I}_1$-integration relative to S after performing the $\widetilde{I}_2$-integration, and let $\widetilde{p}_2$ be the number of different dimensions among the maximizing subspaces for the $\widetilde{I}_2$-integration relative to T'_ρ. By Lemma 2, $\widetilde{p}_2$ is independent of ρ and we have the following lemma:

Lemma 3: Let $I = I_1 \oplus I_2$ and $I = \widetilde{I}_1 \oplus \widetilde{I}_2$ be two decompositions of the two-dimensional space of integration I into one-dimensional components. Let $p_1, p_2, \widetilde{p}_1$, and $\widetilde{p}_2$ be defined as above. Then

$$p_1 + p_2 = \widetilde{p}_1 + \widetilde{p}_2 \quad .$$

Proof: Consider the decomposition $I = I_1 \oplus I_2$. Performing first the I_2-integration and then the I_1-integration, we obtain by Theorem 2,

$$\beta_{I_2}(S'_\mu) = \max_\nu \beta(S''_{\mu\nu}) + p_2 - 1 \quad ,$$

$$\beta_I(S) = \max_\mu \beta_{I_2}(S'_\mu) + p_1 - 1 \quad .$$

Combining these two expressions,

$$\beta_I(S) = \max_\mu \left\{ \max_\nu \beta(S''_{\mu\nu}) + p_2 - 1 \right\} + p_1 - 1$$

$$= \max_{\mu,\nu} \beta(S''_{\mu\nu}) + p_1 + p_2 - 2$$

since p_2 does not depend upon μ.

Similarly, for the decomposition $I = \tilde{I}_1 \oplus \tilde{I}_2$, we have

$$\beta_I(S) = \max_{\rho,\sigma} \beta(T''_{\rho\sigma}) + \tilde{p}_1 + \tilde{p}_2 - 2 \quad .$$

By Lemma 1, however, the $S''_{\mu\nu}$ are precisely the maximizing subspaces for the I-integration relative to S and so are the $T''_{\rho\sigma}$. Consequently, the $T''_{\rho\sigma}$ are merely the $S''_{\mu\nu}$ relabeled. Thus, $\max_{\mu,\nu} \beta(S''_{\mu\nu}) = \max_{\rho,\sigma} \beta(T''_{\rho\sigma})$ and we obtain the desired result. ▌

We see that the proof of Lemma 3 provides us with a formula for $\beta_I(S)$ when $\dim I = 2$.

Theorem 3: Let $f(\overline{P}) \in B_n$ satisfy all the conditions of Theorem 1 and suppose that $\dim I = 2$. Then

$$\beta_I(S) = \max_{S' \in M} \beta(S') + p_1 + p_2 - 2 \; ,$$

where M is the set of all maximizing subspaces for the I-integration relative to S.

F. The General Asymptotic Theorem

The generalization to the case $\dim I = k$ is now reasonably straightforward. We write $I = I_1 \oplus \ldots \oplus I_k$ where each component subspace I_j has dimension one.

Definition: Let $I = I_1 \oplus \ldots \oplus I_k$ with $\dim I_j = 1$. The dimension numbers $p_1, \ldots, p_k$ are defined inductively as follows: p_1 is the number of dimensions among the maximizing subspaces for the I_1-integration relative to S after performing the $I_2 \oplus \ldots \oplus I_k$-integration. $p_j, j = 2, \ldots, k,$ is the number of dimensions among the maximizing subspaces for the I_j-integration after performing the $I_{j+1} \oplus \ldots \oplus I_k$-integration relative to any one of the maximizing subspaces for the I_{j-1}-integration after performing the $I_j \oplus \ldots \oplus I_k$-integration.

By definition, the dimension numbers p_j can take on only the values 1 and 2. The definition of $p_j, j = 2, \ldots, k,$ appears to be ambiguous, however, because it does not specify the maximizing subspace for the I_{j-1}-integration relative to which p_j is computed. The next lemma shows that this ambiguity actually does not exist.

Lemma 4: The dimension numbers $p_j, j = 2, \ldots, k,$ are independent of the maximizing subspaces for the I_{j-1}-integration relative to which they are computed.

Proof: The result for $\dim I = k = 2$ was proved already as Lemma 2 in Section II E. Therefore, if $k > 2$, we assume that the $p_j, j = 2, \ldots, k - 1,$ are independent of the maximizing subspaces for the I_{j-1}-integration relative to which they are computed.

Suppose that p_k does not enjoy this property. Then there exist two maximizing subspaces S_1' and S_2' for the I_{k-1}-integration after performing the

I_k-integration for which $p_k = p_{k1} = 1$ and $p_k = p_{k2} = 2$. Using Lemma 2.1, S'_1 and S'_2 are maximizing subspaces for the $I_1 \oplus \ldots \oplus I_{k-1}$-integration relative to S after performing the I_k-integration. Let $S''_{1\nu}$ and $S''_{2\nu}$ be the maximizing subspaces for the I_k-integration relative to S'_1 and S'_2, respectively.

Since we are assuming that S'_1 and S'_2 are two different maximizing subspaces for the $I_1 \oplus \ldots \oplus I_{k-1}$-integration relative to S after performing the I_k-integration, there exists a one-dimensional subspace J_1 of $I_1 \oplus \ldots \oplus I_{k-1}$ and its orthogonal complement J_2 in $I_1 \oplus \ldots \oplus I_{k-1}$ $(I_1 \oplus \ldots \oplus I_{k-1} = J_1 \oplus J_2)$ such that the subspaces $\Lambda(J_2)\, S'_1$ and $\Lambda(J_2)\, S'_2$ are different. (See Proposition A.1 of the Appendix.) We now integrate out the J_2 subspace leaving the J_1 subspace. Let

$$\Lambda(J_2)S'_1 = T'_1 \ , \qquad \Lambda(J_2)S''_{1\nu} = T''_{1\nu} \ ,$$

$$\Lambda(J_2)S'_2 = T'_2 \ , \qquad \Lambda(J_2)S''_{2\nu} = T''_{2\nu} \ .$$

By Lemma 1, T'_1 and T'_2 are maximizing subspaces for the J_1-integration relative to S after performing the $I_k \oplus J_2$-integration, and $T''_{1\nu}$ and $T''_{2\nu}$ are maximizing subspaces for the I_k-integration relative to T'_1 and T'_2, respectively, after performing the J_2-integration.

Let p'_{k1} and p'_{k2} be the numbers of different dimensions among the subspaces $T''_{1\nu}$ and $T''_{2\nu}$, respectively. Then $p'_{k1} = p_{k1}$ and $p'_{k2} = p_{k2}$. Therefore, $p_{k1} \neq p_{k2}$ implies that $p'_{k1} \neq p'_{k2}$ which contradicts Lemma 2 because $\dim J_1 = \dim I_k = 1$. ▌

<u>Lemma 5:</u> Let $I = I_1 \oplus I_2 \oplus \ldots \oplus I_k$ and $I = \tilde{I}_1 \oplus \tilde{I}_2 \oplus \ldots \oplus \tilde{I}_k$ be two decompositions of the k-dimensional space of integration I into one-dimensional

components. Let $p_1, p_2, \ldots, p_k$ and $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k$ be the corresponding dimension numbers as defined above. Then

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \tilde{p}_i \quad .$$

The proof of this lemma is almost identical to that of Lemma 3 where it is assumed that dim I = 2 .

The proof of Lemma 5 now gives us the general asymptotic theorem.

Theorem 4: Let $f(\overline{P}) \in B_n$ satisfy all the conditions of Theorem 1 and suppose that dim I = k. Let $p_1, p_2, \ldots, p_k$ be the dimension numbers corresponding to any decomposition of I into one-dimensional components. Then $f_I(\overline{P}) \in B_{n-k}$ with asymptotic coefficients $\alpha_I(S)$ given by Theorem 1 and $\beta_I(S)$ given by

$$\beta_I(S) = \max_{S' \in M} \beta(S') + \sum_{i=1}^{k} p_i - k \ ,$$

where M is the set of all maximizing subspaces for the I-integration relative to S.

III. ASYMPTOTIC ESTIMATES FOR SELF-ENERGY GRAPHS

A. Introduction

We now apply Theorems 1 and 4 to photon and electron self-energy graphs in order to obtain asymptotic bounds for the corresponding renormalized Feynman integrals. We remark that, although our discussion centers around photon self-energy graphs, the same results apply to electron self-energy graphs with the obvious modifications.

B. Degree of Divergence of a Subgraph

Weinberg shows in his article[5] that the integrand of any Feynman integral corresponding to a certain Feynman diagram is an element of the class A_{4N}, where N is the number of independent four-momenta in the diagram, provided the energy contour can be rotated from the real to the imaginary axis. Thus, if q is a four-momentum, the hyperbolic metric

$$q^2 = q_o^2 - q_1^2 - q_2^2 - q_3^2$$

becomes negative definite for q_o purely imaginary. We will therefore assume that this well-known energy contour rotation[2,6] has always been carried out. Furthermore, since the logarithmic asymptotic coefficients of any Feynman integrand are zero, the integrands belong to the subclass B_{4N} defined in Section II C.

For a detailed discussion of the connection between subgraphs of a Feynman graph and the corresponding subspaces of R^{4N}, where N is the number of independent four-momenta in the Feynman graph, we again refer to Weinberg[5] and also to Bjorken and Drell.[6] In the following, the subspace S of Theorems 1 and 4 is always the subspace associated with the external momenta of the Feynman diagram, which for the case of a photon self-energy graph is simply the photon four-momentum q. The maximizing subspaces $S' \in M$ correspond to those subgraphs of the original Feynman graph with maximum degree of divergence. For a subgraph g' corresponding to a subspace S', the degree of divergence $D_I(g')$ is defined as

$$D_I(g') = \alpha(S') + \dim S' - \dim S \,, \tag{1}$$

where $\alpha(S')$ is the asymptotic coefficient for the integrand corresponding to the original graph. In renormalizable field theories, it turns out that

$$D_I(g') = 4 - \frac{3}{2}\, F(g') - B(g') \quad ,$$

where $F(g')$ and $B(g')$ are the numbers of fermion and boson lines, respectively, attached to the subgraph g', including external lines belonging to g'. (See, for example, Dyson[2] and Bjorken and Drell.[6])

Rules for determining the degree of divergence of a subgraph in which there are subtraction terms are given in Bjorken and Drell.[6] By a simple counting technique, we can determine the degree of divergence $D_I(g')$ of a subgraph g' which, according to Equation (1) and Theorems 1 and 4, is the quantity we need to know in order to calculate the asymptotic coefficients $\alpha_I(S)$ and $\beta_I(S)$ of the integral.

In order to calculate the logarithmic asymptotic coefficient $\beta_I(S)$, we must first determine the dimension numbers p_j defined in Section II F. Before we can do this, however, we need some facts concerning maximizing subspaces of convergent Feynman integrals.

C. Irreducible Subspaces of the Space of Integration

Suppose that the space of integration I of a convergent integral has dimension $4k$ as is the case for Feynman integrands. Let $I = I_1 \oplus \ldots \oplus I_{4k}$ be a decomposition of I into one-dimensional components I_j, and let $I'_1, \ldots, I'_k$ be the four-dimensional subspaces of I defined as

$$I'_1 = I_1 \oplus I_2 \oplus I_3 \oplus I_4 \,,$$

$$I'_2 = I_5 \oplus I_6 \oplus I_7 \oplus I_8 \,,$$

$$\vdots$$

$$I'_k = I_{4k-3} \oplus I_{4k-2} \oplus I_{4k-1} \oplus I_{4k} \,.$$

Furthermore, suppose that the maximizing subspaces for the I-integration relative to S are of the form

$$S,$$

$$S \oplus I'_{\ell_1} \,,$$

$$S + I'_{\ell_1} \oplus I'_{\ell_2} \,,$$

$$\vdots$$

$$S \oplus \bigoplus_{i=1}^{j} I'_{\ell_i} \,,$$

where

$$j = 1, \ldots, k, \qquad \ell_i = 1, \ldots, k,$$

$$\ell_{i_1} < \ell_{i_2} \quad \text{if} \quad i_1 < i_2 \,.$$

Definition: Suppose that the maximizing subspaces for the I-integration relative to S are of the form just given. A direct sum $J = \bigoplus_{i=1}^{j} I_{\ell_i}$, where $j = 1, \ldots, k$, $\ell_i = 1, \ldots, k$, and $\ell_{i_1} < \ell_{i_2}$ if $i_1 < i_2$, is called an irreducible subspace of I if every maximizing subspace for the I-integration relative to S which contains one or more of the components I_{ℓ_i} of J actually contains the entire sum J.

For example, if $I = I_1' \oplus I_2' \oplus I_3'$ with I_1' and $I_2' \oplus I_3'$ irreducible, then the possible maximizing subspaces for the I-integration relative to S are

$$S, \; S \oplus I_1', \quad S \oplus I_2' \oplus I_3' , \quad S \oplus I_1' \oplus I_2' \oplus I_3' = S \oplus I \; .$$

The subspace $S \oplus I_2'$, in particular, could not be maximizing because, by the irreducibility of $I_2' \oplus I_3'$, the subspaces I_2' and I_3' cannot be split up.

D. Maximizing Subspaces and Dimension Numbers of Convergent Feynman Integrals

We begin with a lemma which applies to any photon self-energy graph.

Lemma 6: For any renormalized photon self-energy graph, the subgraph shown in Fig. 1 has degree of divergence equal to 2. In other words, the subspace S is itself maximizing for the I-integration relative to S.

Proof: Clearly, $\Lambda(I)\, S = S$.

For a given photon self-energy graph, let the corresponding Feynman integral be denoted by

$$\Pi_{\mu\nu}(q) = \int d\, P_I \, R_{\mu\nu}(P_I, q) \quad ,$$

where P_I denotes the integration variables in the space of integration I. Suppose that $R_{\mu\nu}$ is the integrand which results after all subtractions have been performed with the exception of the over-all subtractions. Performing the over-all subtractions, we obtain

$$\Pi^c_{\mu\nu}(q) = \int dP_I \left\{ R_{\mu\nu}(P_I, q) - R_{\mu\nu}(P_I, 0) \right.$$
$$\left. - q_\rho \frac{\partial R_{\mu\nu}(P_I, 0)}{\partial q_\rho} - \frac{q_\rho q_\sigma}{2} \frac{\partial^2 R_{\mu\nu}(P_I, 0)}{\partial q_\rho \, \partial q_\sigma} \right\} \qquad .$$

For this new integrand, we have that

$$\alpha(S) + \dim S - \dim S = \alpha(S)$$
$$= 2$$

which is equal to $\alpha_I(S)$ for a photon self-energy graph. We recall that S is the subspace associated with the external momentum q. In S, all the variables denoted by P_I are zero.

Thus, S is maximizing for the I-integration relative to S. ∎

The counting technique for determining the degree of divergence of a subgraph gives the value of the expression

$$D_I(g') = \alpha(S') + \dim S' - \dim S ,$$

and hence we can determine the maximizing subspaces for the I-integration relative to S. In order to calculate the dimension numbers p_j, however, we must be able to determine which subspaces maximize subintegrations of the full I-integration.

We again write $I = I_1 \oplus \ldots \oplus I_{4k}$, where $\dim I = 4k$, and define the four-dimensional subspaces $I_1', \ldots, I_k'$, as in Section III C. Let p_j, $j = 1, \ldots, 4k$, be the corresponding dimension numbers defined in Section II F.

Theorem 5: Suppose there exists a decomposition of the space of integration I such that the irreducible subspaces of I can be written as

$$I_1'' = I_1' \oplus \ldots \oplus I_{k_1}' ,$$

$$I_2'' = I_{k_1+1}' \oplus \ldots \oplus I_{k_2}' ,$$

$$\vdots$$

$$I_m'' = I_{k_{m-1}+1}' \oplus \ldots \oplus I_{k_m}' , \quad k_1 < k_2 < \ldots < k_m = k ,$$

for some integers $m, k_1, k_2, \ldots, k_m = k$. In other words, we assume that the maximizing subspaces for the I-integration relative to S are of the form

$$S ,$$

$$S \oplus I_{\ell_1}'' ,$$

$$S \oplus I_{\ell_1}'' \oplus I_{\ell_2}'' ,$$

$$\vdots$$

$$S \oplus \bigoplus_{i=1}^{j} I_{\ell_i}'' ,$$

where

$$j = 1, \ldots, m, \qquad \ell_i = 1, \ldots, m,$$

$$\ell_{i_1} < \ell_{i_2} \quad \text{if} \quad i_1 < i_2 \quad .$$

Furthermore, suppose that the subspaces

$$S,\ S \oplus I''_1,\ S \oplus I''_1 \oplus I''_2,\ \ldots,\ S \oplus I''_1 \oplus \ldots \oplus I''_m = S \oplus I$$

are included in the set of all maximizing subspaces. Then

$$\sum_{j=1}^{4k} p_j = 4k + m,$$

where m, defined implicitly above, is the number of irreducible subspaces of I.

<u>Proof</u>: The proof of this theorem, although somewhat long, is not difficult. It amounts to calculating each of the dimension numbers p_j, and this is done by determining the maximizing subspaces for the subintegrations of the I-integration.

Consider the sequence of maximizing subspaces

$$S,\ S \oplus I''_1,\ S \oplus I''_1 \oplus I''_2,\ \ldots,\ S \oplus I''_1 \oplus \ldots \oplus I''_m = S \oplus I,$$

and take any two adjacent subspaces from this sequence

$$S \oplus I''_1 \oplus \ldots \oplus I''_r, \qquad S \oplus I''_1 \oplus \ldots \oplus I''_{r+1}, \qquad 0 \leq r \leq m - 1.$$

In terms of the I'_j, these two subspaces are

$$S \oplus I'_1 \oplus \ldots \oplus I'_{k_r} \quad \text{and} \quad S \oplus I'_1 \oplus \ldots \oplus I'_{k_{r+1}},$$

and in terms of the I_j, they are

$$S \oplus I_1 \oplus \ldots \oplus I_{4k_r} \quad \text{and} \quad S \oplus I_1 \oplus \ldots \oplus I_{4k_{r+1}} \quad .$$

(For $r = 0$, we define $k_r = 0$. Then $S \oplus I''_1 \oplus \ldots \oplus I''_r$ and $S \oplus I'_1 \oplus \ldots \oplus I'_{k_r}$ refer to the subspace S.) Set

$$S_o = S,$$

$$S_j = S \oplus I_1 \oplus \ldots \oplus I_j \quad , \quad j = 1, \ldots, 4k .$$

The two maximizing subspaces we are considering then are denoted S_{4k_r} and $S_{4k_{r+1}}$. Now

$$\Lambda\left(I_{4k_r+1}\right) S_{4k_r} = S_{4k_r} ,$$

$$\Lambda\left(I_{4k_r+1}\right) S_{4k_{r+1}} = S_{4k_r} . \qquad (2)$$

Also,

$$\alpha_I(S) = \max_{\substack{\Lambda\left(I_1 \oplus \cdots \oplus I_{4k_r}\right) S' = S \\ S' \subset S_{4k_r}}} \left\{ \alpha_{I_{4k_r+1} \oplus \cdots \oplus I_{4k}}(S') + \dim S' - \dim S \right\}$$

$$\geq \alpha_{I_{4k_r+1} \oplus \cdots \oplus I_{4k}}\left(S_{4k_r}\right) + \dim S_{4k_r} - \dim S$$

$$= \max_{\substack{\Lambda\left(I_{4k_r+1}\right) S'' = S_{4k_r} \\ S'' \subset S_{4k_r+1}}} \left\{ \alpha_{I_{4k_r+2} \oplus \cdots \oplus I_{4k}}(S'') + \dim S'' - \dim S_{4k_r} \right\} + \dim S_{4k_r} - \dim S$$

$$\geq \alpha_{I_{4k_r+2} \oplus \cdots \oplus I_{4k}}\left(S_{4k_r}\right) + \dim S_{4k_r} - \dim S$$

$$= \max_{\Lambda\left(I_{4k_r+2} \oplus \cdots \oplus I_{4k}\right) S''' = S_{4k_r}} \left\{ \alpha(S''') + \dim S''' - \dim S_{4k_r} \right\} + \dim S_{4k_r} - \dim S$$

$$\geq \alpha\left(S_{4k_r}\right) + \dim S_{4k_r} - \dim S$$

$$= \alpha_I(S) ,$$

where this last step follows because $S_{4k_r} = S \oplus I_1'' \oplus \cdots \oplus I_r''$ is a maximizing subspace for the I-integration relative to S.

Thus,

$$\alpha_{I_{4k_r+1}\oplus\ldots\oplus I_{4k}}\left(S_{4k_r}\right) = \alpha_{I_{4k_r+2}\oplus\ldots\oplus I_{4k}}\left(S_{4k_r}\right) + \dim S_{4k_r} - \dim S_{4k_r}. \quad (3)$$

Similarly,

$$\alpha_I(S) \geq \alpha_{I_{4k_r+1}\oplus\ldots\oplus I_{4k}}\left(S_{4k_r}\right) + \dim S_{4k_r} - \dim S$$

$$\geq \alpha_{I_{4k_r+2}\oplus\ldots\oplus I_{4k}}\left(S_{4k_r+1}\right) + \dim S_{4k_r+1} - \dim S$$

$$= \Lambda\left(I_{4k_r+2}\oplus\ldots\oplus I_{4k}\right)\max_{S''' = S_{4k_r+1}}\left\{\alpha(S''') + \dim S''' - \dim S_{4k_r+1}\right\}$$
$$+ \dim S_{4k_r+1} - \dim S$$

$$\geq \alpha\left(S_{4k}\right) + \dim S_{4k} - \dim S$$

$$= \alpha_I(S)$$

by the maximizing property of $S_{4k} = S \oplus I$. Hence,

$$\alpha_{I_{4k_r+1}\oplus\ldots\oplus I_{4k}}\left(S_{4k_r}\right) = \alpha_{I_{4k_r+2}\oplus\ldots\oplus I_{4k}}\left(S_{4k_r+1}\right) + \dim S_{4k_r+1} - \dim S_{4k_r}. \quad (4)$$

Relations (2), (3), and (4) together imply that both S_{4k_r} and S_{4k_r+1} are maximizing subspaces for the I_{4k_r+1}-integration relative to S_{4k_r} after performing the $I_{4k_r+2}\oplus\ldots\oplus I_{4k}$-integration. Since $\dim S_{4k_r+1} = \dim S_{4k_r} + 1$, the corresponding dimension number has the value 2; that is,

$$p_{4k_r+1} = 2, \quad r = 0, 1, \ldots, m-1.$$

We next consider the dimension numbers $p_{4k_r+\ell}$ for $r = 0, \ldots, m-1$ and $\ell = 2, \ldots, 4k_{r+1} - 4k_r$. Our task is to determine the maximizing subspaces for the

$I_{4k_r+\ell}$-integration relative to $S_{4k_r+\ell-1}$ after performing the $I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}$-integration. $S_{4k_r+\ell}$ is such a subspace because

$$\Lambda\left(I_{4k_r+\ell}\right) S_{4k_r+\ell} = S_{4k_r+\ell-1}$$

and

$$\alpha_I(S) \geq \alpha_{I_{4k_r+\ell} \oplus \ldots \oplus I_{4k}}\left(S_{4k_r+\ell-1}\right) + \dim S_{4k_r+\ell-1} - \dim S$$

$$\geq \alpha_{I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}}\left(S_{4k_r+\ell}\right) + \dim S_{4k_r+\ell} - \dim S$$

$$\geq \alpha\left(S_{4k}\right) + \dim S_{4k} - \dim S$$

$$= \alpha_I(S) .$$

Since $\dim S_{4k_r+\ell} = \dim S_{4k_r+\ell-1} + 1$, any other maximizing subspace for the $I_{4k_r+\ell}$-integration relative to $S_{4k_r+\ell-1}$ after performing the $I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}$-integration must have the same dimension as $S_{4k_r+\ell-1}$. Let T be such a subspace; that is, assume that T is a maximizing subspace for the $I_{4k_r+\ell}$-integration relative to $S_{4k_r+\ell-1}$ after performing the $I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k_r+\ell}$ - integration with

$$T \subsetneq S_{4k_r+\ell} \quad \text{and} \quad \dim T = \dim S_{4k_r+\ell} - 1.$$

Then T is also maximizing for the $I_1 \oplus \ldots \oplus I_{4k_r+\ell}$-integration relative to S after performing the $I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}$-integration because

$$\Lambda\left(I_1 \oplus \ldots \oplus I_{4k_r+\ell}\right) T = S$$

$$\alpha_I(S) = \alpha_{I_{4k_r+\ell} \oplus \ldots \oplus I_{4k}}\left(S_{4k_r+\ell-1}\right) + \dim S_{4k_r+\ell-1} - \dim S$$

$$= \alpha_{I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}}(T) + \dim T - \dim S.$$

Now take any maximizing subspace T' for the $I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}$-integration relative to T. By Lemma 1, T' is a maximizing subspace for the full I-integration relative to S.

By assumption T does not contain the subspace $I_{4k_r+\ell}$ and, consequently, neither does T'. Since $I_{4k_r+\ell} \subset I''_{r+1}$ for $\ell = 2, \ldots, 4k_{r+1} - 4k_r$, the hypotheses of the theorem imply that we can write T' in the form

$$T' = S \quad \text{or} \quad T' = S \oplus \bigoplus_{i=1}^{j} I''_{\ell_i} \tag{5}$$

for some $j = 1, \ldots, m$, where $\ell_i = 1, \ldots, m$, $\ell_{i_1} < \ell_{i_2}$ if $i_1 < i_2$, and $\ell_i \neq r + 1$.

Now we also have that

$$\begin{aligned} \Lambda\left(I_{4k_r+\ell} \oplus \ldots \oplus I_{4k}\right) T' &= \Lambda\left(I_{4k_r+\ell}\right) T \\ &= S_{4k_r+\ell-1} \quad , \end{aligned} \tag{6}$$

and we recall that

$$S_{4k_r+\ell-1} = S \oplus I''_1 \oplus \ldots \oplus I''_r \oplus I_{4k_r+1} \oplus \ldots \oplus I_{4k_r+\ell-1} .$$

Thus, if $2 \leq \ell \leq 4k_{r+1} - 4k_r$, $S_{4k_r+\ell-1}$ contains a nontrivial part of I''_{r+1}. The statements (5) and (6) are, therefore, not compatible for $2 \leq \ell \leq 4k_{r+1} - 4k_r$ because no ℓ_i in the direct sum in (5) can take the value $r+1$, and we have a contradiction.

Hence, there are no maximizing subspaces for the $I_{4k_r+\ell}$ integration relative to $S_{4k_r+\ell-1}$ after performing the $I_{4k_r+\ell+1} \oplus \ldots \oplus I_{4k}$-integration other than $S_{4k_r+\ell}$. Thus,

$$p_{4k_r+\ell} = 1 \quad \text{for} \quad r = 0, 1, \ldots, m-1, \quad \ell = 2, \ldots, 4k_{r+1} - 4k_r .$$

Therefore, we have that

$$\sum_{j=1}^{4k} p_j = \sum_{r=0}^{m-1}\left(p_{4k_r+1} + \sum_{\ell=2}^{4k_{r+1}-4k_r} p_{4k_r+\ell}\right)$$

$$= \sum_{r=0}^{m-1}\left(4k_{r+1} - 4k_r + 1\right)$$

$$= 4k + m . \blacksquare$$

E. An Asymptotic Theorem for Self-Energy Graphs

We now turn to the case of an arbitrary photon or electron self-energy graph of electrodynamics.

Definition: The order of a self-energy graph is defined as the number of vertex points in the graph. With this definition, the order of a photon or electron self-energy graph is always an even number.

For photon self-energy graphs, we have the following theorem:

Theorem 6: Any n-th order photon self-energy graph with m irreducible insertions ($m \leq n/2$) has asymptotic coefficients

$$\alpha_I\left(\{q\}\right) = 2 \quad , \quad \beta_I\left(\{q\}\right) = m \ ,$$

where q is the momentum of the photon.

Proof: The fact that $\alpha_I\left(\{q\}\right) = 2$ follows directly from Theorem 1.

Consider a photon self-energy graph with m irreducible insertions. Using Lemma 6, the counting technique for determining the degree of divergence of a subgraph, Theorem 4, and Theorem 5, we obtain

$$\beta_I\left(\{q\}\right) = \max_{S' \in M} \beta(S') + \sum_{j=1}^{4k} p_j - 4k$$

$$= 0 + 4k + m - 4k$$

$$= m . \blacksquare$$

Thus, the logarithmic asymptotic coefficient of the graph shown in Fig. 2(a) is $\beta_I(\{q\}) = 2$, and the logarithmic asymptotic coefficient of the graph shown in Fig. 2(b) is $\beta_I(\{q\}) = 3$. However, the graph shown in Fig. 2(c) has the logarithmic asymptotic coefficient $\beta_I(\{q\}) = 2$ due to the irreducibility of the vertex insertion shown in Fig. 3.

An analogous theorem for electron self-energy graphs is the following:

Theorem 7: Any n-th order electron self-energy graph with m irreducible insertions $(m \leq n/2)$ has asymptotic coefficients

$$\alpha_I(\{q\}) = 1 \quad , \qquad \beta_I(\{q\}) = m \ ,$$

where q is the momentum of the electron.

Theorem 7 for electron self-energy graphs is proved in exactly the same way as Theorem 6 for photon self-energy graphs. The only difference is that the maximum degree of divergence of a subgraph is 1 instead of 2 .

IV. SUMMARY AND CONCLUSIONS

A. Conclusions About the Perturbation Expansion Parameter

In Section III we show that if

(a) the energy contours of the Feynman integral corresponding to a photon or electron self-energy graph are rotated from the real axis to the imaginary axis, and

(b) the momentum q of the photon or electron is replaced by tq, where t is a real scalar,

then the asymptotic behavior of the photon or electron self-energy graph as $t \rightarrow \infty$ is given by

$$ct^{\alpha} (\log t)^{\beta} ,$$

where c is a constant, $\alpha=1$ for electron self-energy graphs and 2 for photon self-energy graphs, and $\beta = m$, the number of irreducible insertions in the graph. For a given order n, the maximum value of the logarithmic asymptotic coefficient β is $n/2$. Consequently, in a perturbation expansion of the total photon propagator or electron propagator, we would expect the expansion parameter to involve, not only the square of the charge e^2, but the quantity

$$e^2 \log \frac{q^2}{\lambda^2} ,$$

where renormalization is carried out by subtracting at the point $q^2 = \lambda^2 < 0$. In perturbation expansions and renormalization group arguments, one usually assumes that the expansion paramter is $e^2 \log (q^2/\lambda^2)$. (See, for example, Bjorken and Drell,[6] Bogoliubov and Shirkov,[7] and Landau.[8]) That this assumption is the correct one is supported by our results.

B. Summing Different Graphs

Although the maximum value of the logarithmic asymptotic coefficient β for n-th order self-energy graphs is $n/2$, it may be that the sum of all the n-th order graphs has a logarithmic asymptotic coefficient less than $n/2$. For example, consider the three fourth order photon self energy graphs shown in Fig. 4. Each one of these graphs has logarithmic asymptotic coefficient $\beta = 2$, but when the three graphs are summed together, the $\log^2 (q^2/\lambda^2)$ terms cancel (see Bjorken and Drell[6]). Thus, the total fourth order photon propagator has logarithmic asymptotic coefficient $\beta = 1$.

The arguments of the renormalization group predict this cancellation at least for the fourth-order and sixth-order graphs in the perturbation expansion of the photon propagator (cf., Bjorken and Drell,[6] Bogoliubov and Shirkov[7]), and perhaps a similar cancellation occurs for the graphs of other orders. (There is, of course, no cancellation for the single second-order self-energy graph.) This question is unanswered by our results as they stand. We are able to give the asymptotic behavior of any self-energy graph of arbitrary order, but we do so without regard for multiplicative constants.

The problem of summing and determining asymptotic estimates for the entire perturbation expansion remains open. In the first place, it is not even clear that the perturbation series of quantum electrodynamics actually converge. Assuming they do converge, it may turn out that the individual terms have an asymptotic behavior quite unlike that of their sum.

C. Graphs Other Than Self-Energy Graphs; The Problem of Unphysical Momenta

We point out that the general theorems of Section II and the theorems about maximizing subspaces in Section III are applicable to any convergent Feynman integral. Although we have concentrated on self-energy graphs, one could just as well determine the asymptotic behavior of a graph like that shown in Fig. 5, a contribution to eighth-order electron-proton scattering. Three of the four external momenta are independent, say p_1, p_2, and p_1'. Therefore, the asymptotic behavior of this graph will depend upon which subset of p_1, p_2, and p_1' becomes large

In all of these results there remains one undesirable feature, the necessity of performing energy contour rotations in order to avoid the singularities associated with the hyperbolic metric. Quantities of the type $p^2 - m^2 \equiv p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2$, where m is a constant, appear in the denominators of Feynman integrals; however, if the energy contours can be rotated from the real up to the imaginary axis ($p_0 \rightarrow i p_0$), then the expression $p^2 - m^2$ never vanishes.

By performing this rotation, we are restricting ourselves to unphysical momenta. It would be useful to determine the asymptotic behavior of graphs like that shown in Fig. 5 when a certain subset of the external momenta remain on the mass shell while others become large. In other words, one would like to apply asymptotic estimates to real physical experiments. This more difficult problem is not yet solved.

ACKNOWLEDGMENTS

The author wishes to thank Professor Sidney Drell for his valuable suggestions and encouragement throughout the course of this work which forms part of a Ph. D. dissertation submitted to Stanford University (1967).

V. APPENDIX

In this appendix, we prove a statement which was used in the proof of Lemma 4. Let S and I be two subspaces of R^n whose direct sum $S \oplus I$ is R^n.

Proposition A.1: Let S_1 and S_2 be two different subspaces of R^n satisfying $\Lambda(I) S_i = S$, $i = 1,2$, where $\Lambda(I)$ is the operation of projection along the subspace I, and suppose that $\dim I > 1$. Then there exists a one-dimensional subspace J_1 of I such that $\Lambda(J_2) S_1 \neq \Lambda(J_2) S_2$, where J_2 is the orthogonal complement of J_1 in $I (I = J_1 \oplus J_2)$.

The proof of this proposition will follow from the next three lemmas.

If σ is a point in S, we let

$$\Lambda(I)^{-1} \sigma = \left\{x : x \in R^n , \Lambda(I)x = \sigma\right\}.$$

Lemma A.1: Suppose $\Lambda(I) S_i = S$, $i = 1,2$. Then $S_1 = S_2$ if and only if $\Lambda(I)^{-1} \sigma \cap S_1 = \Lambda(I)^{-1} \sigma \cap S_2$ for all $\sigma \in S$.

Proof: Since $\Lambda(I) S_i = S$, then $S_i = \bigcup_{\sigma \in S} \left\{\Lambda(I)^{-1} \sigma \cap S_i\right\}$, $i = 1,2$. If $S_1 = S_2$, then clearly

$$\Lambda(I)^{-1} \sigma \cap S_1 = \Lambda(I)^{-1} \sigma \cap S_2 \qquad \text{for all } \sigma \in S.$$

Conversely, if $\Lambda(I)^{-1}\sigma \cap S_1 = \Lambda(I)^{-1}\sigma \cap S_2$ for all $\sigma \in S$, then we obviously have $S_1 = S_2$ because $S_i = \bigcup_{\sigma \in S} \left\{\Lambda(I)^{-1}\sigma \cap S_i\right\}$, $i = 1,2$. ∎

Lemma A.2: Suppose $\sigma \in S$ and $\Lambda(I) S_i = S$, $i = 1,2$. Then

$$\Lambda(I)^{-1}\sigma \cap S_1 = \Lambda(I)^{-1}\sigma \cap S_2$$

if, and only if,

$$\Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} = \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}.$$

Proof: If $\Lambda(I)^{-1}\sigma \cap S_1 = \Lambda(I)^{-1}\sigma \cap S_2$, then clearly

$$\Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} = \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}.$$

Conversely, suppose that $\Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} = \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}$. Let $x \in \Lambda(I)^{-1}\sigma \cap S_1$. Then x has the same S-coordinates as σ. Now

$$\Lambda(S)x \in \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} = \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}.$$

Thus, there exists a point $y \in \Lambda(I)^{-1}\sigma \cap S_2$ such that $\Lambda(S)y = \Lambda(S)x$. Since $y \in \Lambda(I)^{-1}\sigma \cap S_2$, y has the same S-coordinates as σ and hence as x. Since $\Lambda(S)y = \Lambda(S)x$, y has the same I-coordinates as x. Consequently, $x = y$ and so $x \in \Lambda(I)^{-1}\sigma \cap S_2$. Thus, we have that $\Lambda(I)^{-1}\sigma \cap S_1 \subset \Lambda(I)^{-1}\sigma \cap S_2$. A similar argument gives containment the other way. ▌

Now suppose that $\dim I = k$ and write

$$I = I_1 \oplus \dots \oplus I_k,$$

$$S = I_{k+1} \oplus \dots \oplus I_n,$$

where each of the component subspaces I_j, $j = 1, \dots, n$, is one-dimensional.

<u>Lemma A.3</u>: Let I_j be one of the component subspaces of I; that is take $j = 1, \dots, k$. Let S_o be a subspace of R^n for which $\Lambda(I)S_o = S$. Then for any $\sigma \in S$,

$$\Lambda(I_j)\left\{\Lambda(I)^{-1}\sigma \cap S_o\right\} = \left\{\Lambda(I_j)\,\Lambda(I)^{-1}\sigma\right\} \cap \Lambda(I_j)S_o .$$

We remark that, in general, if f maps X into Y, and A and B are two subsets of X, then

$$f(A \cap B) \subset f(A) \cap f(B).$$

Lemma A.3 says that, in our special case, we actually have equality.

<u>Proof</u>: Let $x \in \Lambda(I_j)\left\{\Lambda(I)^{-1}\sigma \cap S_o\right\}$. Then there exists a point $y \in \Lambda(I)^{-1}\sigma \cap S_o$ such that $\Lambda(I_j)y = x$. We have

(i) $y \in \Lambda(I)^{-1}\sigma$,

(ii) $y \in S_o$,

(iii) $\Lambda(I_j)y = x$.

Now (i) and (ii) imply that $x \in \Lambda(I_j)\Lambda(I)^{-1}\sigma$, and (ii) and (iii) imply that $x \in \Lambda(I_j)S_o$. Thus $x \in \left\{\Lambda(I_j)\Lambda(I)^{-1}\sigma\right\} \cap \Lambda(I_j)S_o$.

Conversely, suppose that $x \in \left\{\Lambda(I_j)\Lambda(I)^{-1}\sigma\right\} \cap \Lambda(I_j)S_o$. Then $x \in \Lambda(I_j)\Lambda(I)^{-1}\sigma$ and $x \in \Lambda(I_j)S_o$. Since $x \in \Lambda(I_j)S_o$, there exists a point $z \in S_o$ such that $\Lambda(I_j)z = x$. Then

$$\begin{aligned}\Lambda(I)z &= \Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)\Lambda(I_j)z \\ &= \Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)x,\end{aligned}$$

where the hat over $\Lambda(I_j)$ means that $\Lambda(I_j)$ does not appear in the product. But since $x \in \Lambda(I_j)\Lambda(I)^{-1}\sigma$, we have that $\Lambda(I)z = \sigma$, and so $z \in \Lambda(I)^{-1}\sigma$. Thus,

$$z \in \Lambda(I)^{-1}\sigma \cap S_o \quad \text{and} \quad x = \Lambda(I_j)z \in \Lambda(I_j)\left\{\Lambda(I)^{-1}\sigma \cap S_o\right\}. \ \blacksquare$$

We can now prove the proposition.

<u>Proof of Proposition A.1</u>: Since $S_1 \neq S_2$ and $\Lambda(I)S_i = S$, $i = 1,2$, Lemmas A.1 and A.2 imply that there exists a point $\sigma \in S$ such that

$$\Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} \neq \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}.$$

Thus, there exists a point

$$x \in \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} \text{ but } x \notin \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\} \text{ (or vice versa).}$$

We can, therefore, find a point $y \in \Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}$ such that the I_j-component of x is not equal to the I_j-component of y for some $j = 1, \ldots, k$.

Now consider $S \oplus I_j$. Since the I_j-components of x and y are unequal.

$$\begin{aligned}&\Lambda(I)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)\Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} \\ &\qquad \neq \Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)\Lambda(S)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}\end{aligned}$$

or

$$\begin{aligned}&\Lambda(S)\Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)\left\{\Lambda(I)^{-1}\sigma \cap S_1\right\} \\ &\qquad \neq \Lambda(S)\Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)\left\{\Lambda(I)^{-1}\sigma \cap S_2\right\}.\end{aligned}$$

Using Lemma A.3,

$$\Lambda(S)\left\{\Lambda(I_j)^{-1}\sigma \cap \Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)S_1\right\}$$

$$\neq \Lambda(S)\left\{\Lambda(I_j)^{-1}\sigma \cap \Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)S_2\right\} .$$

By Lemmas A.1 and A.2 again, this last statement implies that

$$\Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)S_1 \neq \Lambda(I_1)\cdots\widehat{\Lambda(I_j)}\cdots\Lambda(I_k)S_2$$

or

$$\Lambda\left(I_1\oplus\cdots\oplus\widehat{I_j}\oplus\cdots\oplus I_k\right)S_1 \neq \Lambda\left(I_1\oplus\cdots\oplus\widehat{I_j}\oplus\cdots\oplus I_k\right)S_2 .$$

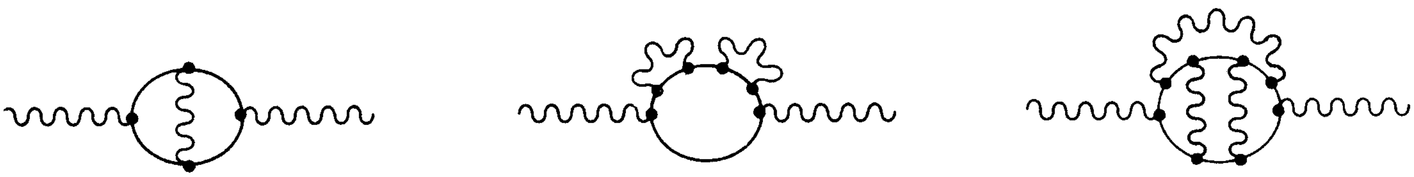
Therefore, we can take $J_1 = I_j$ and $J_2 = I_1\oplus\cdots\oplus\widehat{I_j}\oplus\cdots\oplus I_k$. ▌

REFERENCES

1. F. Dyson, "The Radiation Theories of Tomonaga, Schwinger, and Feynman," Phys. Rev. 75, 486 (1949).
2. F. J. Dyson, "The S-Matrix in Quantum Electrodynamics," Phys. Rev. 75, 1736 (1949).
3. A. Salam, "Overlapping Divergences and the S-Matrix," Phys. Rev. 82, 217 (1951).
4. A. Salam, "Divergent Integrals in Renormalizable Field Theories," Phys. Rev. 84, 426 (1951).
5. S. Weinberg, "High Energy Behavior in Quantum Field Theory," Phys. Rev. 118, 838 (1960).
6. J. Bjorken and S. Drell, Relativistic Quantum Fields (McGraw-Hill, New York, 1965).
7. N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, New York, 1959).
8. L. D. Landau, "On the Quantum Theory of Fields," in Niels Bohr and the Development of Physics W. Pauli (ed.) (McGraw-Hill, New York, 1955).
9. J. Bjorken and S. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).

920A1

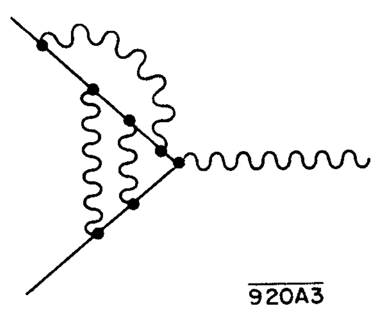
FIG. 1



(a) (b) (c)

920A2

FIG. 2



920A3

FIG. 3

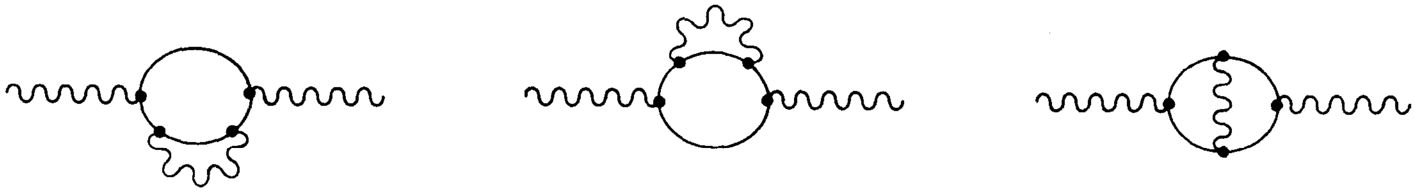


FIG. 4

920A4

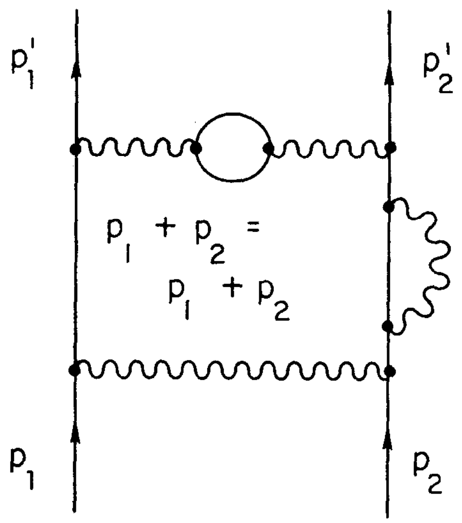


FIG. 5

920A5