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Emergent Geometry from D-Branes

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Avant-propos

Cette thèse est le fruit du travail fourni durant ces quatre dernières années passées à l'Université Libre de Bruxelles en tant qu'étudiant en doctorat. Elle est destinée aux membres de mon jury de thèse afin qu'ils puissent en évaluer la qualité. Cependant, de nombreux néophytes sont ou seront curieux de savoir ce qu'elle contient et, *a fortiori*, ce qui m'a occupé et passionné pendant ces années d'études. C'est pourquoi j'ai décidé d'écrire, en français, ces quelques lignes d'explications très générales et non-techniques à propos du contexte dans lequel mes travaux s'inscrivent ainsi que l'objet de ceux-ci.

À l'heure actuelle, il est possible de décrire avec une redoudable précision un nombre impressionnant de phénomènes naturels, allant du mouvement des planètes jusqu'aux interactions entre les particules dites élémentaires. Dans la conception actuelle de la physique moderne, la quasi-totalité des phénomènes observés dans la nature découle, au moins en principe, de quatre forces (aussi appelées "interactions") regroupées comme suit:

- la force gravitationnelle. La meilleure description que nous en avons est donnée par la théorie de la Relativité Générale d'Einstein. Elle décrit par exemple le mouvement des astres et des satellites, et fut élaborée au début du XX^{e}.

- Les forces non-gravitationnelles: la force éléctromagnétique, la force faible et la force forte. Alors que la première est bien connue et décrit par exemple le courant électrique ou la lumière, les deux dernières sont moins familières. La raison est que ces deux forces ne sont observables qu'aux échelles sub-atomiques, et ne font donc pas partie de notre expérience quotidienne directe. Elles sont néanmoins indispensables pour comprendre par exemple la cohésion des noyaux des atomes ou la radioactivité. La théorie décrivant ces trois interactions est appelée le Modèle Standard de la Physique des Particules et fut élaborée au cours du XX^{e} siècle.

Cette situation n'est cependant pas totalement satisfaisante. En effet, nous n'avons par exemple pas encore de description complète du Big Bang, qui est souvent présenté comme étant "le début de l'univers". En fait, les théories mentionnées ci-dessus ne permettent pas de décrire la physique dans ce régime extrême. L'origine du problème est essentiellement la suivante: la Relativié Générale n'est qu'une approximation. Plus précisément, cette description de l'interaction gravitationnelle n'est plus valable si l'on considère des systèmes dont la taille caractéristique est inférieure à 10^{-35}m, qui est l'ordre de grandeur d'une constante appelée la constante de Planck. En d'autres termes, la description microscopique de la gravitation est inconnue. Les physiciens cherchent donc une théorie valable aux échelles microscopiques et telle que, si l'on se restreint aux systèmes macroscopiques, elle coïncide avec la Relativité Générale.

Afin de bien comprendre la problématique, il est utile de considérer l'analogie suivante. Jusqu'au XIX^{e} siècle, les scientifiques décrivaient la matière en considérant qu'elle était un milieu continu. Cette description était tout à fait satisfaisante, jusqu'au jour où certaines observations faites au microscope suggérèrent que l'hypothèse du continu n'était en fait pas valable aux échelles de l'ordre de 10^{-6}m. La raison est qu'en réalité, la matière est composée d'atomes et de molécules, et n'est donc *pas* un milieu continu. L'hypothèse du continu est donc une approximation, qui permet de décrire *effectivement* la matière lorsque l'on considère de grands systèmes: on dit qu'il s'agit d'une *théorie effective*. Dans cet exemple, la théorie microscopique est donc la théorie atomique, qui fut largement acceptée par la communauté scientifique durant le XX^{e} siècle.

Remarquons que dans cet exemple, la théorie microscopique et la théorie effective sont très différentes. D'une part, les atomes et les molécules sont décrits par leurs positions et leurs vitesses. D'autre part, si l'on considère par exemple un milieu continu comme un gaz, celui-ci est décrit par des propriétés telles que sa pression, sa température et son volume. Ces propriétés sont donc propres aux milieux continus, et n'ont aucun sens à l'échelle microscopique. Pour cette raison, on les appelle des *propriétés émergentes*: elles ne sont pas fondamentales, mais sont nécessaires à la description effective du système.

La Relativité Générale est donc une théorie effective, dont la description microscopique est encore inconnue. À l'instar de la physique des milieux continus, il se pourrait donc qu'il y ait des ingrédients de la Relativité Générale qui ne soient pas fondamentaux mais emergents. Si cela s'avère correct, lesquels sont-ils?

Sans entrer dans trop de détails, de nombreuses considérations théoriques suggèrent fortement que l'une des propriétés émergentes de la Relativité Générale soit l'existence-même des *dimensions spatiales* dans lesquelles nous vivons. En d'autres termes, la théorie microscopique que nous cherchons est une théorie décrivant des phénomènes physiques ayant lieu dans un univers ayant *moins* de trois dimensions spatiales. L'existence de ces dernières ne serait donc pas un ingrédient fondamental, mais plutôt la conséquence de *nouvelles* interactions encore à déterminer. Hors du régime microscopique, ces interactions se comportent *comme si* il y avait des dimensions spatiales supplémentaires, c'est-à-dire que les dimensions *émergent* de la description microscopique.

Afin de mieux comprendre ces concepts relativement abstraits, nous avons dans cette thèse construit et étudié des modèles simples les illustrant. En particulier, nous décrivons explicitement comment les dimensions spatiales émergent à partir d'interactions plus fondamentales. Il est intéressant de remarquer que dans ces modèles, ces nouvelles interactions sont en fait du même type que celles du Modèle Standard de la Physique des Particules mentionné plus haut, ce qui suggère qu'une théorie unifiant toutes les interactions de la nature est de ce type.

Par l'analyse de ces modèles, nous essayons donc de répondre aux questions suivantes : quelle est la nature profonde des dimensions spatiales ? Pourquoi vivons-nous dans un espace à trois dimensions ? Grâce aux modèles de gravitation émergente dont nous disposons aujourd'hui, ces questions ne sont plus uniquement des questions philosophiques, mais aussi des questions scientifiques précises et bien posées. Dans cette thèse, nous avons donc modestement essayé d'apporter des éléments de réponse à ces problématiques, qui m'ont toujours fasciné.

Emergent Geometry form D-Branes

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In this thesis, we explain and illustrate on several examples how to derive supergravity solutions by computing observables in the corresponding dual, lower-dimensional field theory. In particular, no *a priori* knowledge on the gravitational dual is assumed, including its dimensionality. The basic idea to construct the pre-geometric models is to consider the world-volume theory of probe D-branes in the presence of a large number N of higher-dimensional background branes. In the standard decoupling limit, the probes are moving only in the flat directions parallel to the background D-branes. We show however that the quantum effective action of the probe world-volume theory, obtained at large N using standard vector model techniques, has the required field content to be interpreted as the action describing the probes in a higher-dimensional, curved and classical spacetime. The properties of the emerging supergravity solution are easily found by comparing the quantum effective action of the pre-geometric model with the non-abelian D-brane action. In all the examples we consider, this allows us to derive the metric, the dilaton and various form fields, overall performing exclusively field theoretic computations.

The first part of the thesis consists of introductory chapters, where we review vector models at large N, aspects of brane physics in supergravity and string theory and the gauge/gravity correspondence. The second part contains the original contributions of this thesis, consisting of various explicit emergent geometry examples.

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[1]FNRS Research Fellow 2009–2013

The original contributions of this thesis are based on the two following published papers:

- F. Ferrari, M. Moskovic, and A. R., "Examples of Emergent Type IIB Backgrounds from Matrices," *Nucl.Phys.* **B872** (2013) 184–212, `arXiv:1301.3738 [hep-th]`,
- F. Ferrari and A. R.,"Emergent D5-brane Background from D-strings, *Physics Letters B* (2013), `arXiv:1303.7254 [hep-th]`.

In the main text, they are referred to as [1] and [2] respectively.

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[1] By the way, it was my pleasure to save your life using instantons.

[2] Most of the paper used during this thesis was scratch paper from the printer bins.

Contents

Part I

Introduction and Background

Chapter 1

Introduction

The quest for the theory of quantum gravity is far from being over. There is indeed up to now no completely satisfactory description of the gravitational field at the quantum level, despite many interesting developments for instance in string theory. In this thesis, we explore in detail scenarios in which the gravitational interaction is basically not a fundamental interaction but is rather an emergent concept. This is done in the context of the gauge/gravity correspondence, which offers a very natural framework for such emergent geometry scenarios. After recalling the general context in which these developments take place, we present in the second section of this chapter what has been done more precisely during the thesis, leaving the technical developments for the next chapters and part 2.

1.1 General context

The theory of General Relativity is a very successful theory that allows us to describe with remarkable accuracy the gravitational force at the classical level. The gravitational interaction is elegantly described in terms of a metric on the spacetime manifold, and the dynamics of the metric is given by Einstein's equations. In this context, the very existence of spacetime is assumed from the start. This description is expected to be reliable at least as long as the spacetime curvature is much smaller than the Planck length, which is of the order of magnitude of 10^{-35} meters. At smaller scales, the gravitational field is not expected to be classical anymore, and a new description of the gravitational interaction is required. Typically, we suspect new degrees of freedom to appear at these very small scales. New degrees of freedom are also required to account for the entropy associated to realistic black holes.

Superstring theory (see [3–7] for standard references) is at present the most developed proposal for a quantum theory of gravity. It is a UV complete theory and its spectrum automatically contains a graviton, which describes fluctuations of the gravitational field. Moreover, superstring theory reduces in the low-energy limit to supergravity. It also describes the non-gravitational interactions of the type of those present in the Standard Model, and thus yields interesting proposals for a unified theory of all the fundamental interactions of Nature. It is also possible to construct black holes in string theory and in specific cases it is even possible to reproduce the associated entropy formula [8]. However, superstring theory is consistent only in ten spacetime dimensions, and to reproduce a four-dimensional world like ours requires appropriate compactifications of six dimensions. Such modifications of the original theory bring a lot of problems, starting with the fact that there seems to be a *huge* number of compactification possibilities [9]. This is the so-called "landscape problem" and suggests that string theory is actually losing its predictive power [10], as the number of parameters that must be tuned to specify a "good" compactification seems to be very large. Moreover, it is difficult to answer questions about the nature of the Big Bang singularity or the fate of an evaporating black hole and the associated information paradox directly in the context of string theory.

Despite these various drawbacks, string theory provides us with a very interesting and powerful conjecture, stating that certain ordinary gauge theories are equivalent to superstring theories [11–14]. This actually brings us back to the original motivation of string theory as a theory for the strong interaction in the end of the sixties. It is only later on, when it was realised that the spin 2 massless particle present in the string spectrum could be interpreted as the graviton, that string theory became a candidate for a theory of quantum gravity unifying all known fundamental interactions. In fact, the presence of a graviton in the strongly coupled regime of gauge theories is already a hint that gravity may be an emergent phenomenon. This is more precisely formulated in the aforementioned conjectured equivalence (or duality), which in the best understood example relates a conformal gauge field theory in four dimensions to a superstring theory on a ten-dimensional curved background containing an AdS factor: this is the AdS/CFT correspondence, which is an example of a broader gauge/gravity duality. The correspondence is a realisation of the holographic principle [15,16], which states that a theory of gravity in a D-dimensional spacetime is equivalent to a non-gravitational theory in $d < D$ dimensions. It is difficult to emphasize enough why the gauge/gravity correspondence is so important to, at least, theoretical physicists. Not only does it provide very nice examples of emergent spacetime scenarios, but it may also be used as an effective tool to compute strong coupling effects in gauge theories: in fact, the strong coupling regime of the gauge theory is dual to the classical limit of superstring theory. One can then address issues about the strong coupling regime of the gauge theory (that are in general very difficult to answer from first principles) by much simpler questions in classical supergravity; in other words, the less understood regime on the field theory side is related to the best understood regime of superstring theory! This opened up the possibility of understanding real-world strong coupling effects in terms of some gravitational theory. While we are not (yet?) in a position to study directly concrete problems like quark confinement, a lot of progress has been made to extend the original AdS/CFT correspondence to non-conformal and/or less-or even non-supersymmetric set-ups. From this viewpoint, the dual gravitational theory is simply an auxiliary theory that is cooked up to allow for interesting effects on the field theory side.

More in the spirit of the holographic principle, which was first suggested by properties of classical gravity like the black hole area law, the viewpoint in this thesis will be different: we would like to understand the fundamental nature of gravity by directly studying gauge theories at strong coupling. In fact, we shall consider that *gravity and spacetime are emergent concepts*: at the fundamental level we typically have a Yang-Mills SU(N) gauge theory with coupling $g_{\rm YM}$, in which there are no ingredients associated to gravity: in this sense, the models are "pre-geometric." However, at large N and strong 't Hooft coupling $g_{\rm YM}^2 N$ where the theory naively gets extremely complicated, a very simple, classical description is available *at the cost of introducing new dimensions and a new interaction corresponding to gravity*.

One should not be surprised if this way of understanding the fundamental nature of gravity turns out to be correct. In fact, most of the phenomena that surrounds us are of this type, that is to say, emergent. For instance, hydrodynamics is very well approximated by the Navier-Stokes equations, which provides an excellent description of the immensely complicated system composed of a macroscopic number of molecules of water. It would be rather remarkable that gravity, which from a historical point of view was the first interaction to be precisely described, would actually turn out to be fundamental!

Let us mention that the idea of emergent gravity is actually older than the gauge/gravity correspondence or even than the holographic principle. In 1967, Sakharov [17] proposed a theory of "induced gravity," which is inspired by condensed matter systems having an effective description in terms of curved geometries. However, these "world crystal" models are not good candidates for our universe, because they typically predict huge values for the cosmological constant. Moreover, it turned out that models having the graviton as a composite particle are severely constrained by the Weinberg-Witten theorem [18]. Using S-matrix techniques, they studied the existence of interacting massless spin two particles in unitary theories with a conserved energy-momentum tensor in flat space. The conclusions include a no-go theorem excluding most of the naive emergent gravity scenarios. Interestingly, the basic idea of holography provides an obvious way out of the theorem, as in holographic scenarios, the composite graviton propagates in a spacetime containing (at least) one extra dimension than in the original theory.

Let us add a word of caution. In emergent spacetime models that we consider in this thesis, we always deal with emerging *spacelike* dimensions. The problem of emergent time is very subtle and there is no concrete, well-understood example where the field theory is Euclidean while the dual gravitational theory is Lorentzian. We refer to [19] for a general presentation of ideas related to emergent spacetime, including a discussion on time emergence and associated conceptual puzzles.

1.2 About this thesis

In this thesis, we illustrate on several examples how we can derive non-trivial supergravity solutions dual to certain gauge field theories from the computation of particular observables in the gauge theory. The fact that we are able to derive full spacetime geometries (including non-trivial Ramond-Ramond and Neveu-Schwarz forms, if any) from purely field theoretic computations provides good evidence for the validity of the rather general emergent geometry framework that we use.

The examples that we present are built in string theory, where we consider the field theory describing the low-energy limit of probe D-branes in the presence of a large number N of background D-branes. Taking the usual decoupling limit of this system, the resulting theory describes the probes living in the spacetime parallel to the background D-branes. We then show that the quantum effective action for the probe D-branes, obtained by integrating out the fields associated to the presence of the background branes, has the field content required to interpret the probes as moving in a higher-dimensional, curved spacetime.

By comparing the probe effective action to the D-brane action for arbitrary background, the properties of the emergent spacetime are straightforwardly extracted. We are able to determine in this way the metric, the dilaton, the Kalb-Ramond two-form, as well as the non-trivial Ramond-Ramond form field sourced by the background D-branes. We thus derive a solution of the supergravity equations in ten dimensions by exclusively performing computations in a lower-dimensional, ordinary gauge theory.

In the course of the derivation, we identify the mechanism by which the emerging classical coordinates appear in the original gauge theory. They correspond to composite scalar operators which are classical in the large N limit. From a technical point of view, this is closely related to the well-known trick of introducing auxiliary variables to solve vector models in a very elegant way. In this thesis however, we show on several examples that this simple trick has a very interesting physical interpretation in terms of emergent, classical coordinates.

Let us explain the logic of the presentation of this work while presenting its content. In part 1, we review the relevant background for the developments presented in part 2, which is based on the two published papers [1, 2].

Part 1: As a starter, we review in chapter 2 how vector models may be solved at large N using auxiliary variables. We focus on the simplest $\mathrm{O}(N)$-invariant model with quartic interaction, which already features all the properties necessary to understand how a classical dimension can emerge from a field theory at large N. When we consider more involved pre-geometric models (see part 2), it is essentially thanks to these properties that an interpretation in terms of a higher-dimensional, classical spacetime is possible. For this reason, we close the chapter by a clear summary of the key ingredients.

In chapter 3, we review some basic facts about branes. We first review in section

3.1 what are branes in supergravity, from a purely classical point of view. We derive the extremal brane solutions, compute their tension and charges and discuss their regularity. We decided to include quite a few details in the presentation. The reason is that we will find some of these brane solutions in part 2 of the thesis by a radically different approach, involving exclusively ordinary gauge field theory computations. Having at hand the two kinds of derivation in detail, we hope to be able to appreciate how non-trivial and miraculous is the fact that gauge theories contain gravity.

Our next task will be to set the stage on which our pre-geometric models are constructed. This requires to understand various properties of D-branes in string theory that we review in section 3.2. The presentation is not self-contained, and many results will be directly taken out of the literature. We will nevertheless briefly review the origin of many D-brane properties, sometimes simply by sketching the argument (when not included in standard textbooks, references containing the proofs are provided). We will in particular discuss the action describing the world-volume dynamics of a D-brane and present its non-abelian generalisation. We close this section by the presentation of several low-energy non-abelian D-brane actions in flat space and various dimensions, including the fermionic fields and the associated supersymmetry transformations. We construct these actions using supersymmetry and the technique of trivial dimensional reduction, that we describe in detail; in particular, we include the full technical treatment of fermions for the reduction from six to two dimensions in subsection 3.3.4.

In chapter 4, we review the original argument leading to the AdS/CFT conjecture. We also review how the decoupling limit is defined on Dp-branes, with a special emphasis on the D5-brane case. This is the last ingredient we need to construct our pre-geometric models.

Part 2: The original contributions of this thesis are presented. In chapter 5 we review the general emergent geometry framework, and explain the general ideas underlying our derivations. We explain how to construct the pre-geometric models by taking the corresponding decoupling limit of the probe D-brane action in flat space built in chapter 3. We also describe how the dual geometry can be extracted from the models. Finally, we move to the concrete examples that we have studied during this thesis. In section 6, we derive the supergravity dual of three deformations of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions: the Coulomb branch deformation, the non-commutative deformation and the β-deformation. In section 7 we derive the near-horizon geometry of the supergravity solution sourced by a large number of D5-branes.

We close this thesis with some conclusions and possible future directions.

All our notations and conventions are summarized in appendix A. Appendix B presents general considerations relevant for the dimensional reduction of section 3.3. In appendix C, we briefly review the supergravity solutions (derived in our emergent geometry framework in sections 6.5 and 6.6) that are proposed to be dual to the non-commutative and β-deformations of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions. Finally, explicit formulas for the terms up to order five of the non-abelian D-instanton action are provided in appendix D.

Chapter 2

Elements of large N vector models

Quantum field theories are usually studied in the realm of perturbation theory, where the coupling constant g is assumed to be small. The reason is technical: at least in principle, it is easy to compute observables like scattering amplitudes as an asymptotic power series in g, and one reaches the desired accuracy by computing up to a sufficiently high order in g. There are however many interesting strong coupling effects in Nature, and it is thus necessary to go beyond perturbation theory to understand these phenomena. We know only few examples of quantum field theories that are solvable at strong coupling. Some of these models have enough supersymmetries to constrain the dynamics so strongly that we can solve it at any values of the coupling constant. Others enjoy the property of being solvable, or at least they simplify, when the rank N of their gauge group goes to infinity. In particular, when the fields transform as vectors under the gauge group, an explicit resolution can be possible in a power series in $1/N$.

In this chapter we review how we can solve the simplest $\mathrm{O}(N)$ vector model, with quartic interaction, when $N \to \infty$ (standard references on vector models are [20–22]). The resolution relies on the existence of a scalar composite operator whose quantum fluctuations tend to zero as $N \to \infty$. Integrating out the original variables, one then obtain an effective field theory for the classical scalar operator, from which the dynamics of the original fields is easily recovered. Our approach will be pedestrian and we shall not try to give a complete treatment of the subject. In fact, we are mostly interested in the trick itself, that actually lies at the heart of the mechanism responsible for the emergence of classical coordinates in emergent geometry models, as we will explain in detail in part 2 of this thesis.

We start by defining the model and discuss the issue of defining a non-trivial large N limit. We then introduce an auxiliary scalar field that reduces on-shell to a scalar composite operator that is classical at large N. We explain how to compute its effective action and illustrate the procedure in some detail by showing that it has a kinetic term, as in chapter 7 we will consider very similar computations. We close this chapter by summarizing the main steps of the reasoning, highlighting the properties that will turn out to be crucial in the emergent geometry framework explained in part 2 of this thesis.

2.1 A simple model and a useful trick

We consider N scalar fields $(Q^1, \ldots, Q^N) = \vec{Q}$. Their dynamics in Euclidean d-dimensional spacetime is defined by the O(N)-invariant Lagrangian

$$L(\vec{Q}) = \frac{1}{2}\partial_\mu Q^f \partial_\mu Q^f + \frac{m^2}{2}\vec{Q}^2 + \frac{g}{8}(\vec{Q}^2)^2 \tag{2.1.1}$$

where $\vec{Q}^2 = Q^f Q^f$, $1 \leq f \leq N$ and $1 \leq \mu \leq d$. A generic diagram generated by (2.1.1) scales as $g^a N^b$ for some numbers a and b; see figure 2.1 for several examples of diagrams contributing to the process $(f, f) \to (f', f')$ for some f and f' fixed, with $1 \leq f, f' \leq N$. Note that a loop does not necessarily brings a power of N. On the other hand, only loop diagrams can have an explicit dependence on N; in particular, b is always non-negative. In order to have a well-defined limit for $N \to \infty$, we therefore need to scale g as some negative power of N in such a way that the total power of N in $g^a N^b$ is non-positive. It turns out that this requirement is fulfilled when $g \sim N^{-1}$, that is, we take the large N limit while keeping the combination $\lambda \equiv gN$ fixed. In principle, one should then analyse all diagrams of the theory, to any order in perturbation theory, and verify that indeed N^{b-a} do not diverge when $N \to \infty$. This will typically require some topological and combinatoric analysis, which is very difficult to do directly on a generic diagram generated by the Lagrangian (2.1.1). There is fortunately a more clever way to proceed.

The trick is to define a new theory by the Lagrangian

$$L(\vec{Q}, \phi) = L(\vec{Q}) - \frac{1}{2g}\left(\phi - \frac{g}{2}\vec{Q}^2\right)^2 . \tag{2.1.2}$$

The new field ϕ is not a dynamical field, as there is no kinetic term for it in the Lagrangian (2.1.2). We can thus eliminate ϕ in (2.1.2) by replacing it by the solution $\phi_\star$ of its equation of motion, which reads

$$\phi_\star = \frac{g}{2}\vec{Q}^2 = \frac{\lambda}{2N}\vec{Q}^2 , \tag{2.1.3}$$

where we inserted the finite combination $\lambda = gN$. Since $L(\vec{Q}, \phi_\star) = L(\vec{Q})$, we conclude that *the two Lagrangians* (2.1.1) *and* (2.1.2) *define the same theory.* Let us take a closer look at the interaction vertices of the theory (2.1.2). Expanding the square, we find

$$L(\vec{Q}, \phi) = \frac{1}{2}\partial_\mu Q^f \partial_\mu Q^f + \frac{m^2}{2}\vec{Q}^2 - \frac{N}{2\lambda}\phi^2 + \frac{1}{2}\phi\vec{Q}^2 . \tag{2.1.4}$$

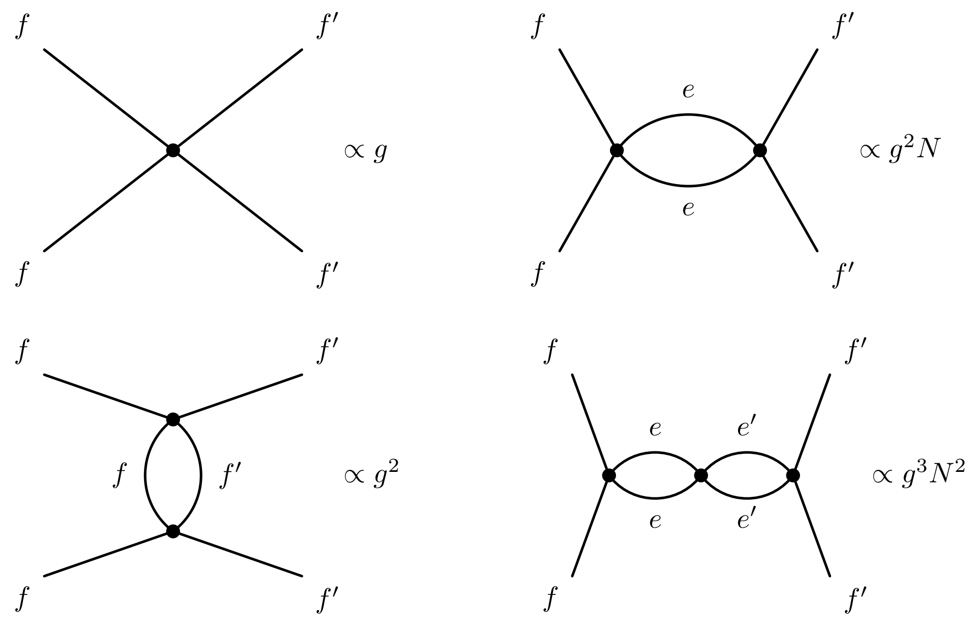


Figure 2.1: Some diagrams contributing to the scattering $(f, f) \to (f', f')$ with their scaling in the coupling g and the parameter N in the theory defined by the Lagrangian (2.1.1). Observe that the presence of a loop in a diagram does not necessarily bring a factor on N.

The crucial point here is that there are no self-interaction terms for the original vector variables $\vec{Q}$ in (2.1.4). Moreover, the only interaction term is of the form $\phi\vec{Q}^2$. The corresponding Feynman diagram is shown on figure 2.2, diagram (a), with the dashed line representing the field ϕ.

Let us remark that although the saddle point $\phi_\star$ given in (2.1.3) is real, the field ϕ must be imaginary off-shell in order to have a positive quadratic term in (2.1.4).

Since the field ϕ does not carry any $\mathrm{O}(N)$ indices, a loop for the fields $\vec{Q}$ automatically brings a factor of N. Moreover, any "propagator" for the auxiliary field ϕ will bring a factor of $1/N$ to the diagram. At fixed λ, the dependence on N of any diagram in the theory (2.1.4) is thus trivial to determine: for L loops of the original fields $\vec{Q}$ and P "propagators" of the auxiliary field ϕ, the diagram will behave as N^{L-P}. Diagrams similar to those of the original theory shown in figure 2.1 are presented in figure 2.3.

Let us now consider all diagrams with only dashed external lines. The knowledge of these diagrams allows us to recover the diagrams of the original theory (2.1.1) easily, since the only interactions between the auxiliary field ϕ and the vector variables $\vec{Q}$ is the diagram (a) of figure 2.2. The diagrams with only dashed external lines are contained in the *effective action* $S_{\text{eff}}(\phi)$ for the auxiliary field ϕ, obtained by

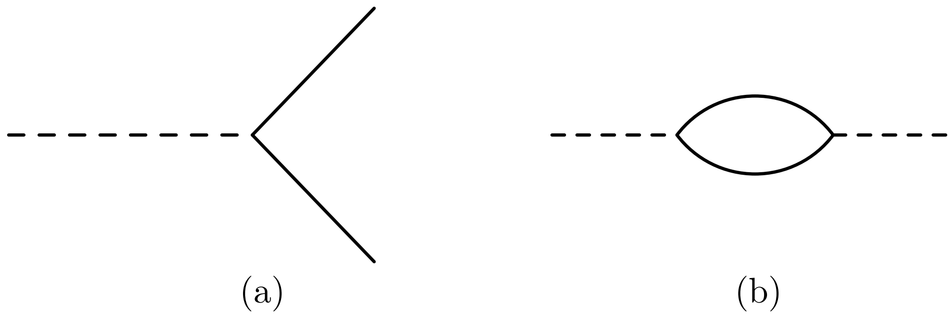


Figure 2.2: With a dashed line representing the new field ϕ, (a) is the only interaction vertex in Lagrangian (2.1.4). (b) represents a typical loop for the fields $\vec{Q}$.

integrating out the original vector variables $\vec{Q}$:

$$e^{-S_{\text{eff}}(\phi)} \equiv \int \mathcal{D}\vec{Q}\; e^{-S(\vec{Q},\phi)}\,, \tag{2.1.5}$$

where $\mathcal{D}\vec{Q}$ stands for the formal integration measure over the fields $\vec{Q}$ and $S(\vec{Q},\phi)$ is the action associated with Lagrangian (2.1.4). Since the variables $\vec{Q}$ appear only quadratically in the Lagrangian (2.1.4), the integral in (2.1.5) is trivial to perform. The result may be formally written as

$$S_{\text{eff}}(\phi) = \frac{N}{2}\,\mathrm{tr}\log\bigl(-\partial_\mu\partial_\mu + (m^2+\phi)\bigr) - \frac{N}{2\lambda}\int \mathrm{d}^d x\,\phi(x)^2\,. \tag{2.1.6}$$

Let us already observe one of the key features of this effective action: being proportional to N, *it is classical when N is large.* We shall have more to say about this in section 2.2. For now, let us study the quantum effective action (2.1.6) for any values of N. We write $\phi = \phi_0 + \varphi$ where ϕ_0 is a solution of the equation of motion derived from (2.1.6) and we expand $S_{\text{eff}}(\phi_0+\varphi)$ in powers of φ. An infinite number of vertices are generated in this way; we represent these vertices by blob-vertices, see figure 2.4. Moreover, a generic term in the expansion in φ will typically contain an infinite number of derivatives, reflecting the fact that the theory (2.1.6) is non-local. If we further expand in derivatives of φ, it is easy to show that the theory (2.1.6) contains a kinetic term for the fluctuation φ. In other words, *the auxiliary field ϕ acquired a non-trivial dynamics thanks to the quantum effects of the original vector variables $\vec{Q}$.* The computation of the aforementioned kinetic term is very similar to the computations that we will perform in more sophisticated models in part 2 of this thesis, so let us work out in detail the case of $d=4$ to see how it goes. For convenience, we absorb ϕ_0 in a redefinition of m^2 from now on. We expand the log in (2.1.6) using the formula

$$\log(1+x) = -\sum_{k=1}^{\infty}\frac{(-x)^k}{k}\,\cdot \tag{2.1.7}$$

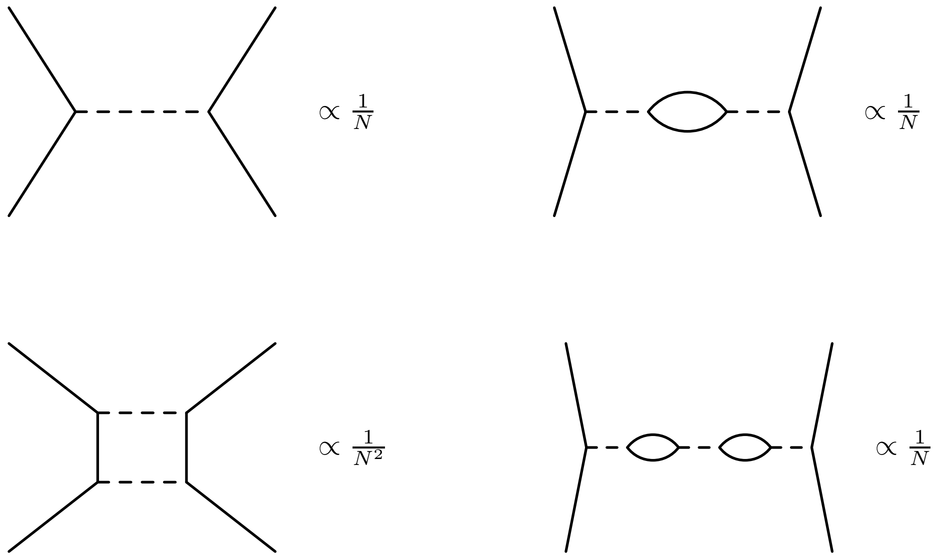


Figure 2.3: Some diagrams of the theory (2.1.4) with their scaling in N, with $\lambda = gN$ kept fixed. Note that the power of N is very easy to find, because the field ϕ does not carry $\mathrm{O}(N)$ indices.

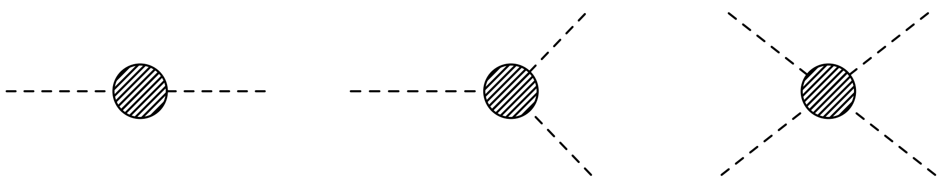


Figure 2.4: Some vertices of the effective theory (2.1.6) for the auxiliary field ϕ. Notice that the tadpole diagram is zero because we expand ϕ around the saddle point ϕ_0.

Setting $\Delta \equiv (-\partial_\mu \partial_\mu + m^2)^{-1}$, the quadratic term in ϕ is given by

$$S_{\text{eff}}^{(2)}(\phi) = -\frac{1}{2}\,\mathrm{tr}\,(\Delta\phi)^2 \ . \tag{2.1.8}$$

The operator Δ admits the following integral representation:

$$\Delta(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2 + m^2} \,. \tag{2.1.9}$$

Inserting formula (2.1.9) into (2.1.8) and performing some trivial integrations, we find

$$\mathrm{tr}\,(\Delta\phi)^2 = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\tilde{\phi}(k)\tilde{\phi}(-k)}{(p^2+m^2)((p-k)^2+m^2)} \,, \tag{2.1.10}$$

where $\tilde{\phi}$ denotes the Fourier transform of ϕ,

$$\tilde{\phi}(k) \equiv \int \mathrm{d}^4 x \ \phi(x) e^{-ik\cdot x} \,. \tag{2.1.11}$$

The derivative expansion of the field ϕ is equivalent to a Taylor expansion in powers of the four-momentum k in the integrand of (2.1.10). Using

$$\frac{1}{(p+k)^2+m^2} = \frac{1}{p^2+m^2} - \frac{2p\cdot k}{(p^2+m^2)^2} + \frac{4(p\cdot k)^2}{(p^2+m^2)^3} - \frac{k^2}{(p^2+m^2)^2} + O(k^3)\,, \quad (2.1.12)$$

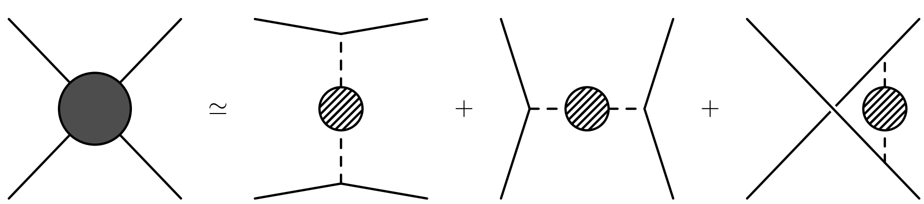
and focusing on the term with two derivatives (namely the term with $(p\cdot k)^2$ and k^2 in the expansion (2.1.12)), we find the following kinetic term:

$$S^2_{\text{eff}}(\phi)\big|_{\partial^2} = -\frac{\pi^2 N}{18m^2}\int \mathrm{d}^4x\ \partial_\mu\phi\partial_\mu\phi\,. \quad (2.1.13)$$

As promised, the auxiliary field ϕ introduced to make the original vector variables $\vec{Q}$ appear quadratically in the Lagrangian (2.1.2) becomes dynamical once the original variables $\vec{Q}$ have been integrated out. This is an important ingredient of our mechanism for emergent coordinates, as we will see in part 2 of this work. Notice that the unusual minus sign in (2.1.13) should not worry us because as explained earlier, the field ϕ is imaginary off-shell.

2.2 The solution at large N

Let us now consider the theory defined by the effective action (2.1.6) in the regime where $N\to\infty$ and λ is fixed. As we already noted, N appears only as a global factor in (2.1.6) and therefore the quantum fluctuations of the field ϕ are suppressed when N is large: *the field ϕ is thus classical when $N\to\infty$*. The dominant diagrams are therefore the tree diagrams built out of the vertices contained in the effective action (2.1.6) and shown on figure 2.4, with the dashed lines connected to the external $\vec{Q}$-lines by the vertex (a) of figure 2.2. For instance, the scattering $(f,f)\to(f',f')$ considered above is determined in the large N limit by the sum of three tree diagrams:



, (2.2.1)

where the diagram with the dark blob on the left-hand side represents the (full) four-point function. The diagrams on the right-hand side of (2.2.1) can be straightforwardly computed using the action (2.1.6). Let us stress that in terms of the original fields $\vec{Q}$, the diagrams on the right-hand side of (2.2.1) include *the whole series of the dominant bubble diagrams for the fields $\vec{Q}$, which contain diagrams with arbitrary high number of loops* (a typical diagram is shown on figure 2.5).

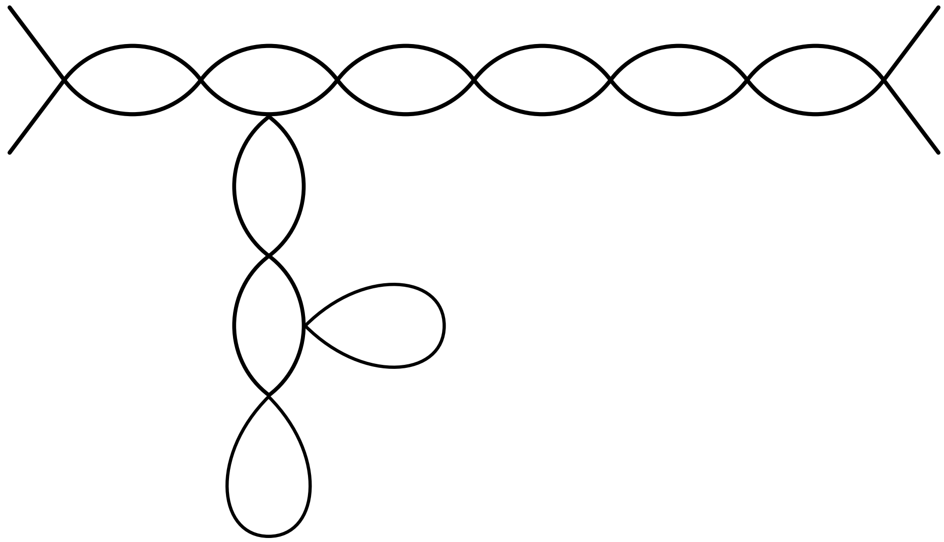


Figure 2.5: A typical leading multi-loop diagram at large N in the theory (2.1.1).

The present study of the O(N) vector model at large N is far from complete. We could for instance compute critical exponents, compare with the usual perturbative treatment, discuss symmetry breaking and determine the first $1/N$ corrections. As we will not need these developments explicitly, we refer the interested reader to the literature [20–22] for more on these topics.

2.3 Summary of the key ingredients

The mechanism presented in the two previous sections plays a crucial role in the emergent geometry framework used in part 2 of this thesis. Let us therefore list the steps that we have followed and emphasize the crucial elements.

- Our starting point was a vector model for the interacting fields $\vec{Q}$ with the Lagrangian (2.1.1). We then considered an equivalent theory, defined by the Lagrangian (2.1.2), that contains *an auxiliary field ϕ such that the original vector variables only appear quadratically*, see (2.1.4). Let us insist on the fact that this extra field ϕ may be eliminated by solving its (algebraic) equation of motion (2.1.3), thus reproducing the original action (2.1.1).

- The original fields $\vec{Q}$ are trivially integrated out using the Lagrangian (2.1.4) containing the auxiliary scalar field. We thus obtain the effective action $S_{\text{eff}}(\phi)$ for ϕ (see equation (2.1.5)), from which one can reconstruct very easily all the scattering amplitudes of the original theory (2.1.1) for the vector variables $\vec{Q}$. In the effective theory (2.1.4), *the field ϕ is dynamical thanks to the quantum* *effects of the original variables.*

- Since N appears only as a global factor in the effective action for ϕ, *the field ϕ is classical in the large N limit.*

When we will consider models of emergent space, we will apply step by step the above reasoning to particular vector-like models. Moreover, it is exactly the properties emphasized in the last two steps that will allow us to conclude that a classical geometry has emerged.

Chapter 3

Aspects of brane physics

In this chapter we review some elementary aspects of brane physics, first from the supergravity point of view and then from the string perspective. Anticipating our needs for the second part of this thesis, we shall mainly focus on type IIB superstring theory.

3.1 Brane solutions in supergravity

We present the derivation of the well-known extremal p-brane solutions in supergravity (see e.g. [23] for a review). We decided to include this derivation in this work in order to be able to appreciate how curious and intriguing is the fact that (at least some of) these solutions may be recovered from the radically different approach presented in part 2 of this thesis.

3.1.1 p-brane solutions

Needless to say, finding exact and non-trivial solutions to supergravity equations is very challenging. These equations are in general highly non-linear coupled partial differential equations, and it is only under suitable simplifying assumptions that an analytic treatment is possible. The first step that we take on the road to simplification is to consider a truncation of the original theory by setting various fields to zero already at the level of the action. When the solutions of the resulting simplified equations of motion are still solutions of the original theory, we say that we have a *consistent truncation* of the original supergravity theory. The next step is to make Ansätze for the solution. This is a delicate step, as not enough constraints might not lead to an exact analytical treatment, while too many might not allow for interesting solutions.

In this section, we define an action generalizing several consistent truncations of type IIA and type IIB supergravity in ten dimensions, present its equations of motion and derive the extremal p-brane solutions. We will actually be slightly more general and keep the spacetime dimension $D > 2$ as a free parameter. The precise relation between the action we consider and the type IIB supergravity action is presented in subsection 3.1.6.

Without loss of generality, we work in the Euclidean D-dimensional spacetime $\mathbf{R}^D$. The spacetime indices are labelled by $M, N, \dots$ with $1 \leq M, N, \dots \leq D$. The dynamical fields remaining after the consistent truncation are the spacetime metric g_{MN}, a scalar field[1] ϕ and an $(n-1)$-form gauge potential $A_{[n-1]}$ (we denote by a subscript in square brackets the degree of the forms that we shall encounter). The action that we consider reads[2]

$$S = -\frac{1}{2\kappa^2} \int \mathrm{d}^D x \, \sqrt{g} \left[R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{2n!} e^{a\phi} \mathcal{F}_{[n]}^2 \right] , \tag{3.1.1}$$

where $\mathcal{F}_{[n]} \equiv \mathrm{d}A_{[n-1]}$, $\mathcal{F}_{[n]}^2 \equiv \mathcal{F}_{[n]M_1\cdots M_n} \mathcal{F}_{[n]}^{M_1\cdots M_n}$, a is a free parameter and we introduced the constant κ with dimensions of $(\text{length})^{\frac{D-2}{2}}$ in order to have a dimensionless action. Newton's constant G_{N} in D dimensions is related to κ by

$$2\kappa^2 = 16\pi G_{\mathrm{N}} \, . \tag{3.1.2}$$

The equations of motion derived from action (3.1.1) read

$$R_{MN} = \frac{1}{2} \partial_M \phi \, \partial_N \phi + S_{MN} \, , \tag{3.1.3a}$$

$$\Box \, \phi = \frac{a}{2n!} e^{a\phi} \mathcal{F}_{[n]}^2 \, , \tag{3.1.3b}$$

$$\nabla_{M_1} \left(e^{a\phi} \mathcal{F}_{[n]}^{M_1 M_2 \cdots M_n} \right) = 0 \, , \tag{3.1.3c}$$

where the box-operator $\Box$ is defined as usual by $\Box \equiv g^{MN} \nabla_M \nabla_N$ and where the source term S_{MN} is defined by

$$S_{MN} \equiv \frac{e^{a\phi}}{2(n-1)!} \left(\mathcal{F}_{[n]MM_2\cdots M_n} \mathcal{F}_{[n]N}{}^{M_2\cdots M_n} - \frac{n-1}{n(D-2)} g_{MN} \mathcal{F}_{[n]}^2 \right) . \tag{3.1.4}$$

The Ansatz that we are going to consider is motivated by invariance properties. If we look for a solution that preserves some supersymmetries of the untruncated action, this solution must necessarily also be invariant under the translations given by the anti-commutator of the conserved supercharges. Motivated by this, we consider solutions to equations (3.1.3) that are invariant under the d-dimensional Euclidean group $\mathrm{ISO}(d)$, where $d < D$ is a fixed number. Let us denote $(x^1, \dots, x^d) = (x^\mu)$ with $1 \leq \mu \leq d \equiv p + 1$ the coordinates corresponding to the directions along which the

[1]This scalar field should not be confused with the auxiliary field introduced in chapter 2.

[2]The unusual global minus sign comes from the fact that we are working in the Euclidean.

solution is translation invariant, and $(x^{d+1}, \dots, x^D) = (y^m)$ with $1 \leq m \leq D - d$ the coordinates corresponding to the remaining directions. Assuming that the Euclidean time coordinate is included in the coordinates x^μ, we can see the d-dimensional region covered by the coordinates x^μ as the d-dimensional volume swiped by a spatial p-dimensional hypersurface called the p-brane. Thus a 0-brane is a particle, a 1-brane is a string, a 2-brane is a membrane and so on. The x^μs thus correspond to the *parallel* directions while the y^ms correspond to the *transverse* directions of the p-brane.

We further simplify the situation by asking that the solution is isotropic in the transverse space, that is, the solution should be invariant under the rotations acting on the y^m. The metric g_{MN} in our coordinates (x^μ, y^m) must then take the form

$$\mathrm{d}s^2 = g_{MN}\,\mathrm{d}x^M \mathrm{d}x^N = e^{2A(r)}\,\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2B(r)}\,\delta_{mn}\mathrm{d}y^m \mathrm{d}y^n \tag{3.1.5}$$

for some functions A and B of $r \equiv \sqrt{y^m y^m}$. The scalar field ϕ can depend only on r,

$$\phi = \phi(r)\,. \tag{3.1.6}$$

Concerning the gauge potential $A_{[n-1]}$, we first have to understand which values the degree $(n-1)$ can take. In principle, one could start by keeping n arbitrary and study the equations of motion (3.1.3) with the conditions (3.1.5) and (3.1.6). It will then turn out that it is only for some specific values of n depending on D and d that solutions with a non-trivial gauge field exist. This can be understood more simply by considering the physical interpretation of our Ansatz: a p-brane with a non-trivial gauge field can be seen as a p-dimensional extended object carrying a non-zero charge density. We are thus looking for gauge potentials that can be coupled to a p-dimensional object. There are two possibilities. The simpler one is inspired by the charged point-particle coupling which looks like

$$\int_{\mathcal{C}} A_{[1]\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau = \int_{\mathcal{C}} A_{[1]}\,, \tag{3.1.7}$$

where $\mathcal{C}$ is the world-line of the particle and τ is a world-line parameter. The obvious generalization to p-dimensional sources is

$$\int_{\mathcal{V}} A_{[p+1]}\,, \tag{3.1.8}$$

where $\mathcal{V}$ is the p-brane world-volume. The coupling (3.1.8) is called the *elementary* (or *electric*) coupling, and thus a p-brane solution sourcing a non-trivial $(p+1)$-form gauge potential is called an *electric p-brane*. Similarly to the familiar point-particle case, the electric charge q_e carried by the electric p-brane is obtained by integrating the Hodge dual $\star\mathcal{F}_{[d+1]}$ of the gauge field-strength over a $(D-d-1)$-dimensional hypersurface Σ_{D-d-1} of constant Euclidean time,

$$q_e = \frac{1}{2\kappa^2} \int_{\Sigma_{D-d-1}} \star(e^{a\phi}\mathcal{F}_{[d+1]})\,. \tag{3.1.9}$$

The factor of $e^{a\phi}$ is necessary to ensure that q_e is conserved when the equation of motion (3.1.3c) is satisfied. Our definition of the Hodge $\star$-operator is the same as in [24] and is given in the appendix, see (A.2.1). As we will see when we compute the charges of our solutions in subsection 3.1.2, it is only for a particular choice of the hypersurface Σ_{D-d-1} that the charge is non-zero.

There is another way extended objects can couple to gauge fields. This comes from the observation that one can also define a charge for a p-brane by integrating directly the field-strength $\mathcal{F}_{[D-d-1]}$ of the gauge potential $A_{[D-d-2]}$ over Σ_{D-d-1}. This charge is conserved independently of the equations of motion thanks to the Bianchi identity $\mathrm{d}\mathcal{F}_{[D-d-1]} = 0$ and is thus dubbed "topological." Notice that, in the language of differential forms, the equation of motion (3.1.3c) for $\mathcal{F}_{[D-d-1]}$ reads

$$\mathrm{d} \star (e^{a\phi}\mathcal{F}_{[D-d-1]}) = 0\,. \tag{3.1.10}$$

We thus see that it is locally possible to define a dual gauge potential for the combination $\star(e^{a\phi}\mathcal{F}_{[D-d-1]})$, that would directly couple to the p-brane by a term similar to (3.1.8). This coupling is called the *solitonic* (or *magnetic*) coupling, and thus a p-brane solution sourcing a non-trivial $(D-d-2)$-form gauge potential is called a *magnetic p-brane*. Including convenient numerical coefficients, the magnetic charge q_m is defined by

$$q_m = \frac{1}{2\kappa^2}\int_{\Sigma_{D-d-1}} \mathcal{F}_{[D-d-1]}\,. \tag{3.1.11}$$

The degrees d and $D-d-2$ of the gauge potentials for the electric and magnetic Ansätze respectively are related by the duality transformation defined by $\tilde{d} = D-d-2$. This transformation is idem potent: $\tilde{\tilde{d}} = d$. The case $p=3$ in $D=10$ dimensions, that will be important for us in chapters 4 and 6, is such that $\tilde{d} = d = 4$ and as a consequence its electric and magnetic field-strengths coincide: it is a *self-dual* solution of the equations of motion (3.1.3). Section 3.1.3 is devoted to the study of this particular solution. For the rest of this subsection, we assume $d \neq \tilde{d}$.

We now turn to the constraints imposed by the invariance conditions defining the p-brane solution on the two possible field-strengths that can couple to the p-brane. The reason why we consider the field-strengths rather than the gauge potentials is because the field-strengths are gauge-covariant quantities, and the consequence of invariance properties of the solution are thus easily written down. In both the electric and the magnetic cases, there is not much freedom. In the case of the electric p-brane, where $n = d+1$, it is easy to see that the only possible field-strength is

$$\mathcal{F}_{[d+1]\mu_1\cdots\mu_d m} = \partial_m e^{C(r)}\epsilon_{\mu_1\cdots\mu_d}\,, \tag{3.1.12}$$

with all other components (not related to those shown in (3.1.12) by permutations of the indices) vanishing, and where C is some function of r to be determined later. The completely antisymmetric tensor in d dimensions $\epsilon_{\mu_1\cdots\mu_d}$ is defined in appendix A. The field-strength (3.1.12) identically satisfies the Bianchi identity $\mathrm{d}\mathcal{F}_{[d+1]} = 0$.

For the magnetic p-brane, the symmetry properties defining our Ansatz are easily shown to impose

$$\mathcal{F}_{[\tilde{d}+1]m_1\cdots m_{\tilde{d}+1}} = h(r)\, y^m \epsilon_{mm_1\cdots m_{\tilde{d}+1}} \tag{3.1.13}$$

with all other components vanishing, where h is some function of r and we used the completely antisymmetric tensor in $\tilde{d}+2$ dimensions $\epsilon_{m_1\cdots m_{\tilde{d}+2}}$ defined in appendix A. The function h turns out to be completely fixed (up to a multiplicative constant) by the Bianchi identity $\mathrm{d}\mathcal{F}_{[\tilde{d}+1]} = 0$, which reduces to a differential equation for h. The solution reads

$$h(r) = \frac{\lambda}{r^{\tilde{d}+2}}\,, \tag{3.1.14}$$

where λ is some constant. This yields the following expression for $\mathcal{F}_{[\tilde{d}+1]}$:

$$\mathcal{F}_{[\tilde{d}+1]m_1\cdots m_{\tilde{d}+1}} = \lambda\, \frac{y^m}{r^{\tilde{d}+2}}\, \epsilon_{mm_1\cdots m_{\tilde{d}+1}}\,. \tag{3.1.15}$$

Let us now plug our two Ansätze (3.1.5), (3.1.6) and (3.1.12) or (3.1.15) into the equations of motion for the metric and the scalar field given in (3.1.3). The resulting systems of equations are conveniently packaged by defining a parameter ς, with $\varsigma = +1$ for the electric Ansatz and $\varsigma = -1$ for the magnetic Ansatz, and read

$$A'' + (\tilde{d}+1)\frac{A'}{r} + dA'^2 + \tilde{d}A'B' = \frac{\tilde{d}}{2(D-2)}S_\varsigma^2\,, \tag{3.1.16a}$$

$$B'' + dA'B' + \tilde{d}B'^2 + (2\tilde{d}+1)\frac{B'}{r} + d\frac{A'}{r} = \frac{-d}{2(D-2)}S_\varsigma^2\,, \tag{3.1.16b}$$

$$\tilde{d}B'' + dA'' - 2dA'B' + dA'^2 - \tilde{d}B'^2 - \tilde{d}\frac{B'}{r} - d\frac{A'}{r} + \frac{1}{2}\phi'^2 = \frac{1}{2}S_\varsigma^2\,, \tag{3.1.16c}$$

$$\phi'' + (\tilde{d}+1)\frac{\phi'}{r} + \tilde{d}B'\phi' + dA'\phi' = -\frac{a\varsigma}{2}S_\varsigma^2\,, \tag{3.1.16d}$$

where a prime denotes the differentiation with respect to r and we set

$$S_{+1} \equiv C' e^{\frac{a}{2}\phi - dA + C}\,, \tag{3.1.17}$$

$$S_{-1} \equiv \frac{\lambda e^{\frac{a}{2}\phi - \tilde{d}B}}{r^{\tilde{d}+1}}\,. \tag{3.1.18}$$

We shall consider the equation of motion for the gauge fields (3.1.3c) shortly. Our task now is to look for a solution of the equations (3.1.16). These equations are however still rather complicated, and an easy analytic solution is still out of reach. We thus add an extra constraint on the derivatives of the functions A and B, reading

$$dA' + \tilde{d}B' = 0\,. \tag{3.1.19}$$

Although we will not present this viewpoint here, the condition (3.1.19) actually comes from the requirement that the p-brane solution is invariant under some supersymmetry transformations. Combining (3.1.19) with the equations (3.1.16) we get the following equations

$$\nabla^2 A = \frac{\tilde{d}}{2(D-2)} S_\varsigma^2 \,, \tag{3.1.20a}$$

$$\nabla^2 \phi = -\frac{a\varsigma}{2} S_\varsigma^2 \,, \tag{3.1.20b}$$

$$d(D-2)A'^2 + \frac{\tilde{d}}{2}\phi'^2 = \frac{\tilde{d}}{2} S_\varsigma^2 \,, \tag{3.1.20c}$$

where $\nabla^2 \equiv g^{mn}\nabla_m \nabla_n$. When acting on a function of r only, as in (3.1.20), ∇^2 reduces to the operator $\partial_r^2 + (\tilde{d}+1)/r\ \partial_r$. We now arrive at the last constraint of the p-brane solution. Consistently with the first two equations of (3.1.20), we impose the following linear relation between A' and ϕ',

$$A' = -\frac{\varsigma \tilde{d}}{a(D-2)} \phi' \,. \tag{3.1.21}$$

Thanks to this last simplifying condition, we are now in a position to solve exactly the remaining differential equations (3.1.20). Equation (3.1.20c) boils down to

$$\nabla^2 e^{\frac{\varsigma\Delta}{2a}\phi} = 0 \,, \tag{3.1.22}$$

where for convenience we set

$$\Delta \equiv a^2 + \frac{2d\tilde{d}}{D-2} \cdot \tag{3.1.23}$$

Equation (3.1.22) implies that $e^{\frac{\varsigma\Delta}{2a}\phi}$ is a harmonic function H in the transverse space and is thus parameterised in general by two integration constants. One of the two integration constants is fixed by the asymptotic behaviour of ϕ that we choose to be simply $\phi(r) \to 0$ when $r \to \infty$.[3] The solution of (3.1.22) therefore reads

$$e^{\frac{\varsigma\Delta}{2a}\phi(r)} = H(r) = 1 + \frac{k}{r^{\tilde{d}}} \,, \tag{3.1.24}$$

where k is the remaining integration constant. Since k is of dimension $(\text{length})^{\tilde{d}}$, it sets the mass scale of the solution. The precise physical interpretation of k will be given when we compute the ADM mass of the solution in section 3.1.2. Similarly to the case of the Schwarzschild black hole, naked singularities are avoided if we restrict our attention to $k > 0$. Using asymptotic flatness in the transverse direction to fix

[3]This is not restricting the asymptotic value of the dilaton field Φ that appears in string theory, as we will see on the precise relation between ϕ and Φ, see formula (3.1.60).

the integration constants in the solution of (3.1.19) and (3.1.21), we finally find the metric

$$\mathrm{d}s^2 = H^{-\frac{4\tilde{d}}{(D-2)\Delta}}\ \delta_{\mu\nu}\ \mathrm{d}x^\mu \mathrm{d}x^\nu + H^{\frac{4d}{(D-2)\Delta}}\ \delta_{mn}\ \mathrm{d}y^m \mathrm{d}y^n\,. \tag{3.1.25}$$

We now specialize to the electric Ansatz (3.1.12), as in this case we still have to find the function C. The equation of motion (3.1.3c) with $\varsigma = +1$ reduces to

$$C'' + (\tilde{d}+1)\frac{C'}{r} + C'^2 + \frac{\Delta}{a}\phi' C' = 0\,. \tag{3.1.26}$$

Equation (3.1.26) is actually a consequence of equations (3.1.20). Eliminating the second derivative of ϕ using its own equation of motion in (3.1.20), we find the following first order differential equation for C:

$$\left(e^C\right)' = -\frac{\sqrt{\Delta}}{a}\phi' e^{-\frac{a}{2}\phi + dA}\,. \tag{3.1.27}$$

We fix the integration constant in the solution of (3.1.27) to zero. The function C is finally found to be given by

$$e^C = \frac{2}{\sqrt{\Delta}} e^{-\frac{\Delta}{2a}\phi}\,. \tag{3.1.28}$$

In the case of the magnetic Ansatz (3.1.15), the equation of motion (3.1.3) for the gauge field is trivially satisfied. The remaining equations in (3.1.20) are satisfied when the constant λ is given by

$$\lambda = \frac{2\tilde{d}k}{\sqrt{\Delta}}\,\cdot \tag{3.1.29}$$

3.1.2 Tension, charges and scalar curvature

We now wish to compute the mass associated to the p-brane metric (3.1.25). In fact, the total mass of the solution is trivially divergent as the volume of the p-brane is infinite. In the transverse space however, the p-brane is represented as a point and the metric is asymptotically flat, and thus we will obtain a finite result if we compute the ADM "mass" of this point-like object by integrating over the hypersphere of infinite radius in transverse space $S^\perp_\infty \equiv \{x^\mu \text{ fixed},\ r \to \infty\}$. This corresponds to computing the density of mass per unit of p-volume, that is, the *tension* τ_p of the p-brane; the dimensions of τ_p are thus (length)$^{-d}$. The tension τ_p is thus given by the standard ADM formula:

$$\tau_p \equiv \frac{1}{2\kappa^2}\int_{S^\perp_\infty} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \mathrm{d}^{\tilde{d}+1} S_i\,, \tag{3.1.30}$$

where $2 \le i, j, \ldots \le D$ if x^1 is taken to be the Euclidean time. The volume elements $\mathrm{d}^{\tilde{d}+1} S_i$ are non-zero only for $i = m$ and read

$$\mathrm{d}^{\tilde{d}+1} S_m = y_m r^{\tilde{d}} \mathrm{d}\Omega_{\tilde{d}+1}\,, \tag{3.1.31}$$

where $\mathrm{d}\Omega_{\tilde{d}+1}$ is the volume element of the unit $(\tilde{d}+1)$-dimensional round sphere. Plugging the metric (3.1.25) into (3.1.30) we find

$$\tau_p = \frac{2\mathrm{Vol}(S^{\tilde{d}+1})\tilde{d}k}{\kappa^2 \Delta}\,, \tag{3.1.32}$$

where $\mathrm{Vol}(S^{\tilde{d}+1})$ is the volume of the unit $(\tilde{d}+1)$-dimensional round sphere. As anticipated, the constant k sets the mass scale of the solution (3.1.25).

We now compute the electric and the magnetic charges defined by equations (3.1.9) and (3.1.11) for the electric and the magnetic solutions (3.1.12) and (3.1.15) respectively. To get a non-zero result for q_e and q_m, one must choose the hypersurface $\Sigma_{\tilde{d}+1}$ to be the transverse hypersphere $S_r^{\perp} \equiv \{x^\mu \text{ and } r \text{ fixed}\}$. Of course the result must be independent of r, which is a good thing to check on our explicit solutions. Plugging (3.1.12) with the function C given by (3.1.28) into (3.1.9) one gets

$$q_e = \frac{\mathrm{Vol}(S^{\tilde{d}+1})\tilde{d}k}{\kappa^2 \sqrt{\Delta}}\,, \tag{3.1.33}$$

while plugging (3.1.15) with λ given by (3.1.29) into (3.1.11) yields

$$q_m = \frac{\mathrm{Vol}(S^{\tilde{d}+1})\tilde{d}k}{\kappa^2 \sqrt{\Delta}}\,. \tag{3.1.34}$$

Note that if we change the sign in the electric and magnetic field-strengths Ansätze (3.1.12) and (3.1.15), the equations of motion (3.1.3) are unchanged and the signs of the associated charges (3.1.33) and (3.1.34) are simply flipped. The p-brane tension (3.1.32) and charges (3.1.33), (3.1.34) are such that

$$\tau_p = \frac{2q_e}{\sqrt{\Delta}} \quad (\text{for } \varsigma = +1) \quad \text{and} \quad \tau_p = \frac{2q_m}{\sqrt{\Delta}} \quad (\text{for } \varsigma = -1). \tag{3.1.35}$$

Although we will not present the details of this fact, equalities (3.1.35) are actually a particular case of more general statements known as the *BPS inequalities*

$$\tau_p \geq \frac{2|q_e|}{\sqrt{\Delta}} \quad (\text{for } \varsigma = +1) \quad \text{and} \quad \tau_p \geq \frac{2|q_m|}{\sqrt{\Delta}} \quad (\text{for } \varsigma = -1), \tag{3.1.36}$$

that must be satisfied by any solutions and are consequences of the superalgebra. The p-brane solutions that we have constructed thus *saturate* the BPS inequalities, which can in turn be shown to be equivalent to the fact that the p-brane solution preserves some supercharges of the original (untruncated) theory. In other words, the p-brane that we consider is a BPS solution.

Let us now address the question of the regularity of the metric (3.1.25). The only candidate for a singularity is the point $r = 0$, where the p-brane itself is located and where the harmonic function H given in (3.1.24) diverges. Let us show that there is indeed a curvature singularity at $r = 0$. For a metric of the form

$$ds^2 = e^{2A(r)}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2B(r)}\mathrm{d}y^m \mathrm{d}y^m\,, \tag{3.1.37}$$

with arbitrary functions $A(r)$ and $B(r)$, the scalar curvature R reads

$$\begin{aligned} R = -e^{-2B}\Big(2dA'' + 2(\tilde{d}+1)B'' + \tilde{d}(d+1)A'B' + \tilde{d}(\tilde{d}+1)B'^2 \\ + d^2A'^2 + 2d(\tilde{d}+1)\frac{A'}{r} + 2(\tilde{d}+1)^2\frac{B'}{r}\Big)\,, \end{aligned} \tag{3.1.38}$$

where we remind the reader that a prime denotes the differentiation with respect to r. Focusing on the metric (3.1.25), we find

$$R \propto r^{-\frac{2a^2}{\Delta}} \quad \text{for } r \simeq 0\,. \tag{3.1.39}$$

Hence for $a \neq 0$, the p-brane solution has a curvature singularity at $r = 0$. Moreover, $r = 0$ is an event horizon of the metric (3.1.25). When $a = 0$, the geometry is regular and the scalar field ϕ is decoupled from the metric and the gauge potential. A particular case of this situation is analysed in subsection 3.1.3.

Summary of the electric and magnetic p-brane solution

Let us summarize what we have done and found in this section. We started from action (3.1.1) that reduces, for particular values of the parameter a, to consistent truncations of supergravity actions. We then solved the equations of motion (3.1.3) deriving from action (3.1.1) by assuming that the solution is invariant under the Euclidean group in d dimensions and isotropic in the remaining $D-d$ dimensions. We argued that the solution admits non-trivial gauge fields of degree n only for $n = d+1$ (electric solution) and $n = D-d-1$ (magnetic solution). Assuming that some supercharges are preserved (see equation (3.1.19) and the discussion below) and adding an extra simplifying condition (3.1.21), we were able to fully solve the equations of motion for both types of p-brane. The solutions read

$$ds^2 = H^{-\frac{4\tilde{d}}{(D-2)\Delta}}\ \delta_{\mu\nu}\ \mathrm{d}x^\mu \mathrm{d}x^\nu + H^{\frac{4d}{(D-2)\Delta}}\ \delta_{mn}\ \mathrm{d}y^m \mathrm{d}y^n\,, \tag{3.1.40a}$$

$$e^\phi = H^{\frac{2a\varsigma}{\Delta}}\,, \qquad H(r) = 1 + \frac{k}{r^{\tilde{d}}}\,, \tag{3.1.40b}$$

$$\mathcal{F}_{[d+1]} = \frac{2\tilde{d}k\, y_m \epsilon_{\mu_1\cdots\mu_d}}{d!\sqrt{\Delta}H^2 r^{\tilde{d}+2}}\ \mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_d}\wedge \mathrm{d}y^m \quad (\text{electric, } \varsigma = +1)\,, \tag{3.1.40c}$$

$$\mathcal{F}_{[\tilde{d}+1]} = \frac{2\tilde{d}k\, y^m \epsilon_{mm_1\cdots m_{\tilde{d}+1}}}{(\tilde{d}+1)!\sqrt{\Delta}r^{\tilde{d}+2}}\ \mathrm{d}y^{m_1}\wedge\cdots\wedge \mathrm{d}y^{m_{\tilde{d}+1}} \quad (\text{magnetic, } \varsigma = -1)\,. \tag{3.1.40d}$$

The dual dimension $\tilde{d}$ is defined by $\tilde{d} \equiv D - d - 2$ and $\Delta \equiv a^2 + 2d\tilde{d}/(D-2)$. The tension τ_p, the electric charge q_e and the magnetic charge q_m are related to the integration constant k by

$$\tau_p = \frac{2q_{e,m}}{\sqrt{\Delta}} = \frac{2\mathrm{Vol}(S^{\tilde{d}+1})\tilde{d}k}{\kappa^2\Delta} \tag{3.1.41}$$

for the electric and the magnetic solution respectively.

As a consistency check, it is easy to show using the explicit formula for the electric solution $\mathcal{F}_{[d+1]}$ in (3.1.40c) that the combination

$$\tilde{\mathcal{F}}_{[\tilde{d}+1]} \equiv \star(e^{a\phi}\mathcal{F}_{[d+1]}) \tag{3.1.42}$$

is equal (up to an irrelevant sign) to the magnetic solution (3.1.40d) and satisfies the equations of motion (3.1.3) with $\mathcal{F}$ replaced by $\tilde{\mathcal{F}}$ and a replaced by $-a$, as it should.

3.1.3 The self-dual 3-brane in $D = 10$

Let us now consider the particular case of $D = 10$, $n = 5$ and $a = 0$. As we will see in subsection 3.1.6, this corresponds to the Ramond-Ramond five-form of type IIB supergravity theory. The special property of this situation can be traced back to the fact that the dimension $d = 4$ of the brane carrying the $\mathcal{F}_{[5]}$-charge is equal to the dual dimension $\tilde{d} = 8 - d = 4$. As a consequence, it is impossible to have a non-trivial pure electric solution: if we impose the electric Ansatz (3.1.12), then we have $\star\mathcal{F}_{[5]} = 0$, that is, $\mathcal{F}_{[5]} = 0$ since $\star^2$ is always proportional to the identity (see (A.2.2)). Similarly, it is impossible to have a pure magnetic solution. We are thus naturally led to consider a *dyonic* solution, that is, a solution having both non-zero electric and magnetic charges. The field-strength $\mathcal{F}_{[5]}$ then satisfies the self-duality[4] condition:

$$\star\,\mathcal{F}_{[5]} = -i\mathcal{F}_{[5]}\,. \tag{3.1.43}$$

An immediate consequence of the self-duality condition (3.1.43) is that it is impossible to write a kinetic term for $\mathcal{F}_{[5]}$ as we did for the other n-form fields until now. More precisely, we have the identity $\mathcal{F}_{[5]} \wedge \star\mathcal{F}_{[5]} = 0$, as can be seen by the following manipulation:

$$\mathcal{F}_{[5]} \wedge \star\mathcal{F}_{[5]} = \star\mathcal{F}_{[5]} \wedge \mathcal{F}_{[5]} = (-1)^{25}\mathcal{F}_{[5]} \wedge \star\mathcal{F}_{[5]} = 0\,, \tag{3.1.44}$$

where in the first equality we used the self-duality condition (3.1.43). The self-dual field-strength $\mathcal{F}_{[5]}$ is thus absent from the action (3.1.1), and is therefore also absent from the equations of motion obtained by varying it. To circumvent this technical

[4] Formula $\star^2 = -1$ (valid when acting on five-forms in the Euclidean) also allows for the condition $\star\mathcal{F}_{[5]} = i\mathcal{F}_{[5]}$.

obstruction, we simply impose *by hand* the self-duality condition (3.1.43) at the level of the equations of motion. This procedure yields the following equations of motion:

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{2.4!}\mathcal{F}_{[5]MM_2\cdots M_5}\mathcal{F}_{[5]N}{}^{M_2\cdots M_5}\,, \qquad \nabla^2\phi = 0\,, \tag{3.1.45a}$$

$$-i\mathcal{F}_{[5]} = \star\mathcal{F}_{[5]}\,, \qquad \nabla_{M_1}\mathcal{F}_{[5]}{}^{M_1M_2\cdots M_5} = 0\,. \tag{3.1.45b}$$

Notice that there is no source term in the equation of motion for the scalar field ϕ: it is a free field, and we take ϕ = constant. Since ϕ must go to zero as $r\to\infty$, we must have $\phi = 0$ everywhere. For the metric, we consider the usual Ansatz (3.1.5) with the extra constraint (3.1.19),

$$\mathrm{d}s^2 = e^{2A(r)}\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{-2A(r)}\mathrm{d}y^m\mathrm{d}y^m\,, \tag{3.1.46}$$

where $A(r)$ is some function to be determined and we set the integration constant in the solution of (3.1.19) to zero to ensure that the metric is asymptotically flat as $r\to\infty$. For the form $\mathcal{F}_{[5]}$ we impose *simultaneously* the conditions (3.1.12) and (3.1.13):

$$\mathcal{F}_{[5]\mu_1\cdots\mu_4 m} = \partial_m e^{C(r)}\epsilon_{\mu_1\cdots\mu_4}\,, \qquad \mathcal{F}_{[5]m_1\cdots m_5} = h(r)y^m\epsilon_{mm_1\cdots m_5}\,, \tag{3.1.47}$$

where $C(r)$ and $h(r)$ are two functions of $r=\sqrt{y^my^m}$. The condition $\mathrm{d}\mathcal{F}_{[5]} = 0$ yields $h(r) = \lambda r^{-6}$ for some constant λ and C is fixed in term of A and λ by the self-duality condition (3.1.43). Moreover, the equation of motion $\mathrm{d}\star\mathcal{F}_{[5]} = 0$ is automatically satisfied. The equations (3.1.45) reduce to

$$A'' + \frac{5A'}{r} = \frac{\lambda^2 e^{8A}}{2r^{10}}\,, \qquad 8A'^2 = \frac{\lambda^2e^{8A}}{r^{10}}\,\cdot \tag{3.1.48}$$

These equations imply that e^{-4A} is a harmonic function, $e^{-4A}\equiv H = 1 + k\,r^{-4}$, while the constant λ must be related to k by the relation

$$\lambda = 2\sqrt{2}k\,. \tag{3.1.49}$$

The full solution for the self-dual 3-brane thus reads

$$\mathrm{d}s^2 = H^{-1/2}\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + H^{1/2}\mathrm{d}y^m\mathrm{d}y^m\,, \tag{3.1.50}$$

$$\mathcal{F}_{[5]m_1\cdots m_5} = 2\sqrt{2}k\;\epsilon_{m_1\cdots m_5 m}\frac{y^m}{r^6}\,, \qquad \mathcal{F}_{[5]m\mu_1\cdots\mu_4} = \frac{2i\sqrt{2}k\;y_m}{H^2r^6}\epsilon_{\mu_1\cdots\mu_4}\,, \tag{3.1.51}$$

$$H = 1 + \frac{k}{r^4}\,, \qquad \phi = 0\,. \tag{3.1.52}$$

The tension (3.1.30) and the charges (3.1.9), (3.1.11) for this solution read

$$\tau_p = \frac{2\mathrm{Vol}(S^5)k}{\kappa^2}\,, \qquad q_e = -iq_m = -\frac{i\sqrt{2}\mathrm{Vol}(S^5)k}{\kappa^2}\,\cdot \tag{3.1.53}$$

In particular we have the following relation between the tension, the electric and magnetic charges:

$$\tau_p^2 = |q_e|^2 + |q_m|^2 \,, \tag{3.1.54}$$

which is (the Euclidean version of) the BPS equality for dyonic states in supergravity [25, 26]. To close this discussion on the special case of the self-dual 3-brane, let us remark that the scalar curvature (3.1.38) for the metric in (3.1.50) is regular:

$$R \simeq -\frac{8}{\sqrt{k}} \qquad \text{for } r \simeq 0. \tag{3.1.55}$$

In fact, one can show that the solution has no curvature singularity. This is in contrast with the cases where $a \neq 0$, as discussed previously in subsection 3.1.2.

3.1.4 Multi-centered solutions and p-brane stack

Let us now discuss a very simple generalisation of the p-brane solutions that we described (valid also for the self-dual 3-brane in $D = 10$). In our general analysis of subsection 3.1.1, we encountered two linear differential equations: one for the function $h(r)$ appearing in the magnetic Ansatz (3.1.15) and one for the combination $e^{\frac{\varsigma\Delta}{2a}\phi}$ in (3.1.22). The obvious generalisation of the solutions (3.1.14) and (3.1.24) for $h(r)$ and $H(r)$ respectively is to consider a linear combination of such solutions. With the asymptotic condition $H \to 1$ as $r \to \infty$, this yields the following new solutions:

$$h(\vec{y}) = \sum_{\alpha=1}^{N} \frac{\lambda_\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}+2}} \,, \qquad H(\vec{y}) = 1 + \sum_{\alpha=1}^{N} \frac{k_\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}} \,, \tag{3.1.56}$$

where N is a fixed number. In (3.1.56), we use the convenient vector notation for $(y^m) = \vec{y}$ and λ_α, k_α and y^m_α are some constants with $1 \leq \alpha \leq N$. The equations of motion require the constants λ_α to be related to the k_αs by

$$\lambda_\alpha = \frac{2\tilde{d}k_\alpha}{\sqrt{\Delta}} \,. \tag{3.1.57}$$

Of course the $\mathrm{SO}(\tilde{d}+2)$ invariance in the transverse space is broken by this solution, which describes N parallel p-branes located at different positions in transverse space, $\vec{y}_\alpha$ being the position of the brane α. The tension and the total charges turn out to be simply given by the sum of the tension and the charges of the individual branes. In particular, the BPS condition (3.1.35) is still satisfied. When all the $\vec{y}_\alpha$ coincide, we have a stack of N p-branes on top of each other.

The self-dual case $n = 5$ with $D = 10$ studied in subsection 3.1.3 allows for a similar generalisation, the only difference being that the constants λ_α and k_α are now related by

$$\lambda_\alpha = 2\sqrt{2}k_\alpha \qquad \text{(Self-dual solution } n = 5,\ D = 10,\ a = 0). \tag{3.1.58}$$

To close this subsection, let us mention that it is also possible to find non-BPS solutions to the equations of motion (3.1.3) that are still invariant under the Euclidean group in d dimensions. The constraint (3.1.19) is not satisfied for these solutions, and they have two horizons in transverse space, much like the non-extremal Reissner-Nortström solution in four dimensions. We will not present these solutions (known as *black p-branes*) and we refer the interested reader to the original literature [27] for more details (for a review, see e.g. [23]). The solutions that we derived here are thus called *extremal p-branes*.

3.1.5 Einstein metric versus string metric

The metric g_{MN} in action (3.1.1) is conventionally called the *Einstein metric*, and differs from the so-called *string metric* $g_{(\mathrm{s})MN}$, arising in string theory from particular excitations of closed strings. The closed string spectrum also contains a scalar field Φ called the dilaton, and we would like to establish the link between $g_{(\mathrm{s})MN}$, g_{MN}, Φ and the scalar field ϕ that we considered so far. The difference between the Einstein metric and the string metric comes from the fact that the low-energy effective action for the string metric $g_{(\mathrm{s})MN}$ and for the dilaton Φ typically has the form

$$\int \mathrm{d}^D x \sqrt{g_{(\mathrm{s})}}\, e^{-2\Phi} \left(R(g_{(\mathrm{s})}) + 4 {g_{(\mathrm{s})}}^{MN} \partial_M \Phi \partial_N \Phi + \cdots \right) , \tag{3.1.59}$$

where $\cdots$ represents terms which depend on the considered string theory (see subsection 3.1.6 for an example in $D = 10$), $R(g_{(\mathrm{s})})$ is the Ricci scalar for the metric $g_{(\mathrm{s})MN}$ and Φ is allowed to have a non-vanishing vacuum expectation value Φ_0. The scalar field ϕ, which was assumed to be zero at infinity, is related to Φ through the relation

$$\tilde{\Phi} \equiv \Phi - \Phi_0 = \sqrt{\frac{D-2}{8}}\phi \,, \tag{3.1.60}$$

the proportionality factor being fixed by matching the kinetic terms. To bring the action (3.1.1) into the form of (3.1.59), we simply need to set

$$g_{MN} = e^{-4\tilde{\Phi}/(D-2)} g_{(\mathrm{s})MN} \,. \tag{3.1.61}$$

The action (3.1.1) with g_{MN} and ϕ expressed in terms of $g_{(\mathrm{s})MN}$ and Φ using (3.1.60) and (3.1.61) then reads

$$S = -\frac{1}{2\kappa_0^2} \int \mathrm{d}^D x \sqrt{g_{(\mathrm{s})}}\, e^{-2\Phi} \Big[R(g_{(\mathrm{s})}) + 4 {g_{(\mathrm{s})}}^{MN} \partial_M \Phi \partial_N \Phi \\ - \frac{1}{2n!} e^{\left(\sqrt{\frac{8}{D-2}}a + \frac{4(n-1)}{D-2}\right)(\Phi - \Phi_0)} \, \mathcal{F}^2_{[n](\mathrm{s})} \Big] \,, \tag{3.1.62}$$

where we introduced the constant $\kappa_0 = \kappa e^{-\Phi_0}$ and where the subscript (s) on $\mathcal{F}^2_{[n](\mathrm{s})}$ means that we contracted the indices using the string frame metric $g_{(\mathrm{s})MN}$; explicitly,

$$\mathcal{F}^2_{[n](\mathrm{s})} \equiv {g_{(\mathrm{s})}}^{M_1 N_1} \cdots {g_{(\mathrm{s})}}^{M_n N_n} \mathcal{F}_{[n]M_1 \cdots M_n} \mathcal{F}_{[n]N_1 \cdots N_n} \,. \tag{3.1.63}$$

For future reference, let us note that using the formula (A.2.3) relating the Hodge $\star$-operators associated to metrics differing by a conformal factor, we have

$$\star\left(e^{a\phi}\mathcal{F}_{[n]}\right) = \star_{(\mathrm{s})}\left(e^{\frac{1}{2}(a+\frac{n-5}{2})\phi}\mathcal{F}_{[n]}\right), \tag{3.1.64}$$

where $\star_{(\mathrm{s})}$ is the Hodge $\star$-operator for the string frame metric $g_{(\mathrm{s})MN}$.

3.1.6 The case of type IIB supergravity

In this section we focus on the case $D = 10$, and consider the low-energy limit of type IIB superstring theory. The action can be found for example in [3] and reads

$$S_{\mathrm{IIB}} = S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}}, \tag{3.1.65}$$

with the various pieces given by[5]

$$S_{\mathrm{NS}} = -\frac{1}{2\kappa_0^2}\int \mathrm{d}^{10}x\,\sqrt{g_{(\mathrm{s})}}e^{-2\Phi}\Big(R(g_{(\mathrm{s})}) + 4g_{(\mathrm{s})}{}^{MN}\partial_M\Phi\partial_N\Phi - \frac{1}{12}H_{[3]}^2\Big), \tag{3.1.66a}$$

$$S_{\mathrm{R}} = \frac{1}{4\kappa_0^2}\int \mathrm{d}^{10}x\,\sqrt{g_{(\mathrm{s})}}\Big(F_{[1]}^2 + \frac{1}{3!}\tilde{F}_{[3]}^2 + \frac{1}{2.5!}\tilde{F}_{[5]}^2\Big), \tag{3.1.66b}$$

$$S_{\mathrm{CS}} = -\frac{i}{4\kappa_0^2}\int C_{[4]}\wedge H_{[3]}\wedge F_{[3]}, \tag{3.1.66c}$$

where $H_{[3]} = \mathrm{d}B_{[2]}$ is the field-strength of the Kalb-Ramond field $B_{[2]MN}$, $F_{[5]} = \mathrm{d}C_{[4]}$, $F_{[1]} = \mathrm{d}C_{[0]}$ where $C_{[0]}$ is a scalar field called the *axion* and we set

$$\tilde{F}_{[3]} \equiv F_{[3]} - C_{[0]}H_{[3]}, \qquad \tilde{F}_{[5]} \equiv F_{[5]} - \frac{1}{2}C_{[2]}\wedge H_{[3]} + \frac{1}{2}B_{[2]}\wedge F_{[3]}. \tag{3.1.67}$$

The form fields $C_{[q]}$ are called the *Ramond-Ramond* form fields (RR for short), while $B_{[2]}$ is called the *Neveu-Schwarz* form field (NS for short). The equations of motion of type IIB supergravity are obtained by adding the self-duality condition $\star\tilde{F}_{[5]} = -i\tilde{F}_{[5]}$ to the equations of motion derived from the action (3.1.65). This rather *ad hoc* procedure is necessary because the kinetic term of a self-dual 5-form is identically zero, as we explained in subsection 3.1.1.

Consistent truncations of (3.1.65) are obtained by setting all but one of the field-strengths $H_{[3]}, F_{[1]}, F_{[3]}$ and $F_{[5]}$ to zero. If we denote by $F_{[n]}$ the field (either RR of NS) that we do not set to zero, the action (3.1.65) reduces to

$$S = -\frac{1}{2\kappa_0^2}\int \mathrm{d}^{10}x\,\sqrt{g_{(\mathrm{s})}}\left[e^{-2\Phi}\big(R(g_{(\mathrm{s})})+4g_{(\mathrm{s})}{}^{MN}\partial_M\Phi\partial_N\Phi\big)-\frac{1}{2n!}e^{2\alpha\Phi}\,F_{[n](\mathrm{s})}^2\right], \tag{3.1.68}$$

[5] We remind the reader that we are working in the Euclidean, hence the different signs and factors of i with [3].

where the parameter α is 0 for a RR field and -1 for the NS field. Matching (3.1.68) with (3.1.62) requires the two following identifications:

$$\alpha = \frac{1}{2}\Big(a + \frac{n-5}{2}\Big) , \qquad \mathcal{F}_{[n]} = e^{(\alpha+1)\Phi_0} F_{[n]} \,. \tag{3.1.69}$$

Both for the RR and NS fields we find $\Delta = 4$, where Δ was defined in (3.1.23). The relation (3.1.41) between the tension and the charges thus reduces to

$$\tau_p = q_{e,m} = \frac{(7-p)\mathrm{Vol}(S^{8-p})k}{2\kappa^2} , \tag{3.1.70}$$

for the electric and the magnetic solution respectively. Let us observe that an electric source for $F_{[1]}$ corresponds to an object whose world-volume has zero dimension: it is simply a point in spacetime.

Let us now determine the behaviour of the brane tension (3.1.32) in ten dimensions when the asymptotic value of the dilaton Φ_0 varies. Of course in expression (3.1.32) for τ_p there is no explicit dependence on Φ_0 because we started from action (3.1.1) in which there is no explicit reference to Φ_0. But now that we have carefully identified the action (3.1.1) with the type IIB action (3.1.65), we see that there are two different places where Φ_0 is hidden: first, the parameter κ is related to κ_0, which is Φ_0 independent, by $\kappa = e^{\Phi_0}\kappa_0$. Second, we learned that $\mathcal{F}_{[n]}$ is actually Φ_0-dependent, see (3.1.69). As a consequence, for the electric Ansatz the charge q_e defined in (3.1.9) and associated to $\mathcal{F}_{[n]}$ depends on Φ_0. Defining the electric charge Q_e associated to the field $F_{[n]}$ by

$$Q_e \equiv \frac{1}{2\kappa_0^2} \int_{\Sigma_{D-d-1}} \star(e^{a\phi} F_{[d+1]}) \,, \tag{3.1.71}$$

we find that the Φ_0 dependence of q_e is given by

$$q_e = e^{(\alpha-1)\Phi_0} Q_e \,. \tag{3.1.72}$$

Since the physical tension τ_p of the solution is equal to the charge q_e (see formula (3.1.70)), we conclude that τ_p has the following dependence on Φ_0:

$$\tau_p \propto \begin{cases} e^{-\Phi_0} & \text{for RR fields } (\alpha = 0) \,, \\ e^{-2\Phi_0} & \text{for NS fields } (\alpha = -1) \,. \end{cases} \tag{3.1.73}$$

Note in particular that the behaviour of the tension is independent of the dimensionality of the brane. Similar considerations apply for the magnetic solution and the self-dual 3-brane.

For future reference, let us write explicitly the RR field sourced by a 5-brane in $D = 10$ dimensions. In this case we have $p = 5$ and $a = -1$ and thus the harmonic function H and the scalar field ϕ are given by

$$H(r) = e^{-2\phi} = 1 + \frac{k}{r^2} \,. \tag{3.1.74}$$

The Einstein and string metrics are respectively given by

$$\mathrm{d}s^2 = H^{-1/4}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + H^{3/4}\mathrm{d}y^m \mathrm{d}y^m \,, \tag{3.1.75a}$$

$$\mathrm{d}s^2_{(\mathrm{s})} = H^{-1/2}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + H^{1/2}\mathrm{d}y^m \mathrm{d}y^m \,, \tag{3.1.75b}$$

while the electric field $F_{[7]}$ and its dual $F_{[3]} = \star_{(\mathrm{s})}F_{[7]}$ read[6]

$$F_{[7]} = \frac{1}{g_\mathrm{s}6!}\partial_m H^{-1}\epsilon_{\mu_1\cdots\mu_6}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_6}\wedge\mathrm{d}y^m \,, \tag{3.1.76a}$$

$$F_{[3]} = -\frac{1}{g_\mathrm{s}3!}\partial_m H \;\epsilon_{mm_1m_2m_3}\mathrm{d}y^{m_1}\wedge\mathrm{d}y^{m_2}\wedge\mathrm{d}y^{m_3} \,. \tag{3.1.76b}$$

3.2 Branes in string theory

The extended objects that we described as solutions of supergravity theories in the previous section are generically singular. Since superstring theory is a UV-complete theory for the supergravity fields, it must contain regular extended objects that reduce, in the supergravity limit, to brane solutions. The fact that the supergravity brane solutions are singular means that close to the singularity, the supergravity approximation breaks down and finite α' effects cannot be neglected anymore.

In this section we review the concept of Dirichlet-branes (D-branes for short), a particular type of branes of string theory. We start by considering the bosonic oriented closed string theory and recall the basics of T-duality. Then we move to open strings and show how D-branes naturally arise by T-duality. We explain why D-branes have to be considered as dynamical objects and present their effective action in a general string background. The discussion is extended to superstring theory and we discuss the coupling between D-branes and RR fields. We also discuss the low-energy non-abelian D-brane action that describes the collective dynamics of a stack of D-branes, as well as its limitations. Preparing the ground for the applications presented in part 2 of this thesis, we then write the actions for the bosonic and fermionic fields describing the low-energy limit of a stack of D9-branes in a trivial closed string background. We then deduce by T-duality the action for a stack of D5-branes. Next, we explain how the D5-brane theory is modified if we add D9-branes into the system, and we write explicitly the terms that will later be important to us. We further reduce the theory down to two dimensions, yielding (part of) the action of the D1/D5 system. This latter step is explained in detail, including the treatment of the spinor fields in the process of dimensional reduction. Finally, by continuing to reduce the theory down to zero dimension, we describe the D(−1)/D3 system. The actions for the D(−1)/D3 and the D1/D5 systems, in the decoupling limit reviewed in chapter 4, will be our starting points for the computations presented in chapters 6 and 7 respectively.

[6]Remember that $\alpha = 0$ for RR fields and thus using formula (3.1.64) we have $\star(e^{a\phi}F_{[n]}) = \star_{(\mathrm{s})}F_{[n]}$.

Standard references on string theory are [3–7] (see also e.g. [28] for a useful set of lecture notes) and for references on D-branes see e.g. [29, 30]. Since these references contain all technical details, we will not derive all the relevant formulas in this section, but simply state the results, briefly recall their origin and, more importantly, stress their physical interpretation.

3.2.1 T-duality and D-Branes

A natural way to introduce the concept of D-branes is through T-duality, a remarkable property of string theory that we now describe. We consider the theory of bosonic oriented closed strings propagating on flat Euclidean spacetime. The string spectrum generically contains a tachyon and at the massless level there is a scalar particle, a graviton and an antisymmetric two-form excitation. The fields describing coherent states of these massless particles are the dilaton Φ, the metric $g_{(\mathrm{s})MN}$ and the Kalb-Ramond two-form $B_{[2]MN}$, where the spacetime indices are $1 \leq M, N \leq D$. For definiteness, we assume that X^D is the Euclidean time. Except in particular situations (like for example the linear dilaton CFT), consistency of the quantum world-sheet theory requires $D = 26$.

The presence of the tachyon in the spectrum suggests that the bosonic string theory is ill-defined. Despite this, there are a lot of interesting things to learn by considering the bosonic string, most of which will essentially remain true when we consider superstrings in subsection 3.2.5. Of course there can also be tachyons in superstring spectra, but this will simply mean that we are considering an unstable state, which will eventually decay into a tachyon-free, stable one [31].

Let us get started and consider a target space with topology $\mathbf{R}^{25} \times S^1$ where the circle S^1 is of radius R and is taken in a direction different from X^{26}, say along the coordinate X^{25} for definiteness: X^{25} is thus identified with $X^{25} + 2\pi R$. As a consequence of the periodicity of X^{25}, the momentum component p^{25} must be quantized according to

$$p^{25} = \frac{n}{R}\,, \qquad n \in \mathbf{Z}\,. \tag{3.2.1}$$

Moreover, in deriving the spectrum of this theory, we must take into account new configurations that are *winding* around the circle S^1, that is, such that

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2w\pi R\,, \qquad w \in \mathbf{Z}\,. \tag{3.2.2}$$

The *winding number* w counts the number of times the closed string winds around the compact dimension. It is a new conserved quantum number, and will thus label the states of the theory.

We now consider the physics from the viewpoint of the non-compact dimensions X^μ, with μ ranging on all values from 1 to 26, except 25. Let us take a look at the mass spectrum $M^2 = p^\mu p_\mu$ of the quantized string. We introduce the complex coordinates $(z, \bar{z})$ on the world-sheet, related to the usual coordinates (τ, σ) on the cylinder by

$$z = e^{\tau + i\sigma}\,, \qquad \bar{z} = e^{\tau - i\sigma}\,. \tag{3.2.3}$$

Using the coodinates $(z, \bar{z})$, X^{25} decomposes as

$$X^{25}(z, \bar{z}) = X^{25}_L(z) + X^{25}_R(\bar{z})\,, \tag{3.2.4}$$

with the left-moving and right-moving pieces given in terms of the oscillators α^{25}_n and $\tilde{\alpha}^{25}_n$ respectively by

$$X^{25}_L(z) = \frac{x^{25}}{2} - i\sqrt{\frac{\alpha'}{2}}\alpha^{25}_0 \log z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}\alpha^{25}_n z^{-n}\,, \tag{3.2.5a}$$

$$X^{25}_R(\bar{z}) = \frac{\tilde{x}^{25}}{2} - i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}^{25}_0 \log \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}^{25}_n \bar{z}^{-n}\,, \tag{3.2.5b}$$

where $(x^{25} + \tilde{x}^{25})/2$ is the position of the string center of mass. The condition (3.2.2) implies that w is related to α^{25}_0 and $\tilde{\alpha}^{25}_0$ through

$$wR = \sqrt{\frac{\alpha'}{2}}(\alpha^{25}_0 - \tilde{\alpha}^{25}_0)\,, \tag{3.2.6}$$

while the 25th component p^{25} of the momentum is as usual given by

$$p^{25} = \frac{n}{R} = \frac{1}{\sqrt{2\alpha'}}(\alpha^{25}_0 + \tilde{\alpha}^{25}_0)\,. \tag{3.2.7}$$

Defining the standard number operators N and $\tilde{N}$ for the left- and right-moving sectors by

$$N = \sum_{n=1}^{\infty} \alpha^M_{-n}\alpha^M_n\,, \qquad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}^M_{-n}\tilde{\alpha}^M_n\,, \tag{3.2.8}$$

the mass-squared operator M^2 for the states in the uncompactified world reads

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right)\,. \tag{3.2.9}$$

Consistency of the quantum theory also require to have the following relation between N and $\tilde{N}$:

$$nw + N - \tilde{N} = 0\,, \tag{3.2.10}$$

which reduces for zero winding number $w = 0$ to the usual level matching condition.

Let us now determine the massless states in the spectrum (3.2.9) satisfying the constraint (3.2.10) for a generic compactification radius R. In this case, the condition $M^2 = 0$ implies $n = w = 0$, which combined with (3.2.10) yields $N = \tilde{N} = 1$. This is basically the same spectrum as in the uncompactified theory, the only difference being that the states are now arranged into representations of the Euclidean group in 25 dimensions instead of 26: nothing really interesting happened.

We now consider also the massive states of the spectrum defined by (3.2.9) and (3.2.10). To get some intuition, let us see what happens as we vary the radius R. For larger and larger R, the winding energies w^2R^2/α'^2 increase, as we might expect since it takes more energy to wind a string around a larger loop. Meanwhile, the momentum quanta n^2/R^2 decrease as the compactified dimension is bigger and bigger. In the limiting case where $R \to \infty$, *the winding modes are excluded from the spectrum while the momentum quanta go to a continuum*, and we recover our original, uncompactified theory. The take-home message of this paragraph is the following: the fact that one sector of the energy spectrum goes to a continuum (here the momentum modes) is the typical signature of the appearance of an uncompactified direction (here X^{25}).

On the other hand, when R gets smaller and smaller, the momentum energies n^2/R^2 increase while the winding energies w^2R^2/α'^2 decrease. In the limit $R \to 0$, the momentum modes are completely excluded from the spectrum, reflecting the fact that the compactified direction X^{25} has disappeared: geometrically, the compactified dimension is shrunk to zero. The crucial point is that the winding energies, however, go to a continuum and we essentially end up with the *same* spectrum as in the $R \to \infty$ limit, *describing 26 dimensions*. Since the direction X^{25} disappeared, *there must be a new dimension X'^{25} appearing in the theory.* Momentum quanta along this new dimension correspond, in the original theory, to winding quanta. The roles of momentum and winding are thus exchanged as we consider $R \to 0$ instead of $R \to \infty$, despite their very different nature as quantum numbers.

Although remarkable, this result should not be a complete surprise to us, since the theory is consistent only in 26 dimensions. Loosely speaking, if we try to dimensionally reduce the theory on a circle, consistency of the theory prevents us from reaching our goal by making a new dimension appear. Note also that this is really a stringy effect, as for point-like particles there is no notion of winding number. The conclusion is that from the viewpoint of the uncompactified dimensions, the theories $R \to \infty$ and $R \to 0$ are *equivalent*: it is impossible to make the difference between the two limits; we say that the two theories are *dual* to each other. The precise duality transformation, exchanging momentum and winding, is given by

$$R \mapsto R' = \frac{\alpha'}{R}\,, \qquad n \leftrightarrow w\,, \tag{3.2.11}$$

where R' sets the periodicity of the dual coordinate X'^{25}, defined by

$$X'^{25}(z,\bar{z}) \equiv X_L^{25}(z) - X_R^{25}(\bar{z})\,. \tag{3.2.12}$$

As expected, the constraint (3.2.10) and the spectrum (3.2.9) take the same form before and after the transformation (3.2.11), provided we interpret R' as the radius of the new compactified dimension. The duality transformation (3.2.11) is called a *T-duality transformation*, the letter "T" standing for "Target-space."

What happens to the vibration modes in the direction X^{25} when $R \to 0$? Of course, since the dimension X^{25} disappears as $R \to 0$, it is meaningless to say that the string vibrates in this direction. According to (3.2.12) the effect of the transformation (3.2.11) on the left-moving and right-moving oscillators α^{25}_{-n} and $\tilde{\alpha}^{25}_{-n}$ respectively reads

$$\alpha^{25}_{-n} \mapsto \alpha^{25}_{-n}\,, \qquad \tilde{\alpha}^{25}_{-n} \mapsto -\tilde{\alpha}^{25}_{-n}\,. \tag{3.2.13}$$

The oscillator modes in the shrunk direction X^{25} are thus re-interpreted as oscillator modes in the new dimension X'^{25} (up to a sign flip for the right-moving sector). We thus conclude that *the string vibrates in the new dimension X'^{25} as a consequence of its vibrations in the original dimension X^{25}*.

Let us remark that to show the equivalence between two theories, we should not content ourselves by showing that the spectra of both theories coincide, as interactions might be different. In fact, using (3.2.12), one can show that interactions between the excitations of the two dual spectra are essentially unchanged and thus *T-duality is an exact duality of perturbative bosonic closed string theory.*

Before we add open strings in the game, let us present the spacetime effective action S_{closed} for the closed bosonic string fields and determine how the dilaton field Φ transforms under T-duality. The action S_{closed} is defined by the requirement that its equations of motion are equivalent to the condition that the world-sheet theory has no conformal anomaly. When the spacetime curvature is small in string units, the action S_{closed} reads

$$S_{\text{closed}} = -\frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\,\sqrt{g_{(\text{s})}}\,e^{-2\Phi}\Big(R - \frac{1}{12}H_{[3]MNP}H_{[3]}^{MNP} + 4\partial_M\Phi\partial^M\Phi\Big)\,, \tag{3.2.14}$$

where $H_{[3]} = \mathrm{d}B_{[2]}$ and the indices are raised using the inverse metric ${g_{(\text{s})}}^{MN}$. The constant κ_0 has dimensions of $(\text{length})^{24}$ and is related to Newton's constant in 26 dimensions using $\kappa = \kappa_0 e^{\Phi_0}$ and (3.1.2), where Φ_0 is the vacuum expectation value of the dilaton Φ. If we compactify one direction, say X^{25} as above, on a circle of radius R, then the effective Newton's constant $\kappa_{25}^2/8\pi$ in the uncompactified 25 dimensions is related to κ by

$$\frac{1}{\kappa_{25}^2} = \frac{2\pi R}{\kappa^2}\,. \tag{3.2.15}$$

On the other hand, the constant κ_{25} must be invariant under T-duality, as it measures the strength of the gravitational interactions in the uncompactified spacetime. Using the relation between the radius R and its T-dual R' in (3.2.11), we deduce that the constant κ in the original theory must transforms under T-duality as

$$\kappa \mapsto \frac{\sqrt{\alpha'}}{R}\kappa\,. \tag{3.2.16}$$

The dilaton Φ must transform to accommodate (3.2.16), namely

$$e^{\Phi} \mapsto \frac{R}{\sqrt{\alpha'}} e^{\Phi} \,. \tag{3.2.17}$$

Although we will not present it here, it is also possible to find the transformation laws of the other closed string fields G_{MN} and B_{MN} under T-duality. The resulting rules have been derived in [32–34] (see also [35]) and are known as the *Buscher's rules*.

We now consider the addition of oriented open strings in the theory. Since the open string endpoints are loose, there is no notion of conserved winding number as a winding string around the compactified dimension can be un-winded continuously. In the limit $R \to 0$, the direction X^{25} disappears and the string endpoints are thus restricted to move in the remaining 24 uncompactified dimensions. To see this explicitly, observe that while the original coordinate X^{25} satisfies the Neumann boundary condition

$$\partial_n X^{25} = 0 \qquad \text{at endpoints } (\sigma = 0 \text{ and } \pi), \tag{3.2.18}$$

where ∂_n denotes the derivative in the direction normal to the world-sheet boundary, the new coordinate X'^{25} automatically satisfies Dirichlet boundary condition, that is,

$$\partial_t X'^{25} = 0 \qquad \text{at endpoints } (\sigma = 0 \text{ and } \pi), \tag{3.2.19}$$

where ∂_t denotes the derivative in the direction tangent to the world-sheet boundary. On the other hand, the oscillators in the direction X^{25} of the open string transform into oscillators vibrating in the new direction X'^{25}, similarly to the closed string modes: despite the original dimension has disappeared, the open strings still vibrate in 26 directions. We thus conclude that in the T-dual picture, we have a bosonic string theory in 26 dimensions with open string endpoints restricted to move on a 24-dimensional hyperplane called a *D24-brane*, where "D" stands for "Dirichlet."

In the original uncompactified theory, the spacetime fields describing coherent states of massless excitations of the strings include an abelian gauge field A_M. This gauge field is truly a field on the whole spacetime, $A_M(X^N)$, since the photon state arising from the quantization of the open string can have momentum in any of the 26 directions. In the T-dual picture, however, the Dirichlet conditions prevent the open string from having momentum in the new direction X'^{25}. As a consequence, the fields associated to the open string massless states are functions only of the 25 coordinates that are parallel to the D24-brane: we say that the D24-brane has a field "living" on it. It is thus natural to split $A_M(X^\mu)$ into fields transforming under irreducible representation of the world-volume symmetry group: we have a gauge field $A_\mu(X^\nu)$ and a scalar field

$$Z(X^\mu) = A_{25}(X^\mu) \,. \tag{3.2.20}$$

We will shortly give a nice geometric interpretation to the scalar field Z.

To conclude this subsection, let us examine what happens when we perform a T-duality transformation along the dual coordinate X'^{25} itself. For the closed string sector, there is nothing really new as $R' \to 0$: the momentum modes along X'^{25} decouple while the winding modes go to a continuum, reproducing the original direction X^{25}. For the open strings, however, we have a novelty: *winding number is well-defined*. Indeed, as the endpoints are confined on the D24-brane, an un-winding process would require the endpoint to move along the compactified direction X'^{25} which is transverse to the D24-brane: this is by definition impossible. The spectrum thus contains winding modes as well and, analogously to the closed string case, these form a continuum when $R' \to 0$. The net effect is to *reintroduce* the dependence on the original coordinate X^{25} in the open string fields: they are again fields on the 26-dimensional spacetime. As a conclusion, *two T-duality transformations along the same direction leave the theory unchanged*. Moreover, if we apply several T-duality transformations in different directions, we will decrease the dimension of the hyperplane where the open string endpoints move, thus producing Dp-branes for various values of $p \leq 25$, the D25-brane corresponding to the case where the open string endpoints are free to move in any direction, which is nothing but the original theory with open strings satisfying Neumann boundary conditions in every directions. For a Dp-brane, the open string satisfies Dirichlet boundary conditions in $D - p - 1$ directions (geometrically corresponding to the *transverse* directions of the Dp-brane) and Neumann boundary conditions in the remaining p directions (geometrically corresponding to the *parallel* directions of the Dp-brane). Generalising the discussion of the previous paragraph, we see that a Dp-brane has one gauge field $A_\mu(X^\nu)$ with $1 \leq \mu, \nu \leq p + 1$ living on it, as well as one scalar Z_m for each transverse direction X^{p+m} with $1 \leq m \leq D - p - 1$. A T-duality along a parallel direction transforms a Dp-brane into a D$(p - 1)$-brane, while a T-duality transformation in a transverse direction maps it to a D$(p + 1)$-brane.

3.2.2 Chan-Patton factors and Wilson lines

Let us now extend the discussion to oriented open strings with Chan-Patton factors labelled by (i, j) with $1 \leq i, j \leq n$. The gauge field arising from the massless states of the open string is now an $n \times n$ matrix $A_M = ({A_M}^i{}_j)$ transforming under the adjoint representation of the gauge group U(n). Moreover, as the endpoints are now labelled by the indices (i, j), we may interpret n as the *number* of D-branes: in the T-dual picture and when the gauge field is trivial, the open string is attached to the D-brane i at one end and to the D-brane j at the other end.

We now consider the non-trivial background gauge field A_M given by

$$A_\mu = 0 \,, \qquad A_{25} = \frac{1}{2\pi R} \,\mathrm{diag}(\theta_1, \ldots, \theta_n) \,, \tag{3.2.21}$$

where the $\theta_1, \ldots, \theta_n = (\theta_i)$ are some real numbers. In a non-compact space, a constant gauge field is always pure gauge. Although the gauge field component A_{25} can be written as

$$A_{25} = -i\Lambda\,\partial_{25}\Lambda \quad \text{with} \quad \Lambda = \mathrm{diag}(e^{\frac{i\theta_1 X^{25}}{2\pi R}}, \ldots, e^{\frac{i\theta_n X^{25}}{2\pi R}})\,, \tag{3.2.22}$$

A_M is not pure gauge for generic (θ_i) because Λ is not a well-defined function on the circle of radius R. To find out what is the effect of the background field (3.2.21) on the open string spectrum, recall that an open string couples to a non-trivial background gauge field A_M through its boundary. At the world-sheet level, we simply have to add a term of the form

$$\int_{\partial\mathcal{M}} A_M \,\mathrm{d}X^M\,, \tag{3.2.23}$$

where $\partial\mathcal{M}$ is the world-sheet boundary. The new spectrum, including the effect of the coupling (3.2.23) and with A_M given by (3.2.21), reads

$$M^2 = \left(\frac{2\pi l - \theta_i + \theta_j}{2\pi R}\right)^2 + \frac{N-1}{\alpha'}\,, \tag{3.2.24}$$

where the quantized internal momentum is $p^{25} = l/R$ with $l \in \mathbf{Z}$ and (i,j) are the Chan-Patton labels of the state. As a consequence of (3.2.24), the scalar field Z is massive when $\theta_i \neq \theta_j$. This has a very natural interpretation in the T-dual picture. Let us compute the length of the open string in the dual direction X'^{25}: it is given by the difference $\Delta X'^{25}$ of X'^{25} evaluated at one end minus X'^{25} evaluated at the other end and reads

$$\Delta X'^{25} = (2\pi l - \theta_i + \theta_j)\,R'\,, \tag{3.2.25}$$

where $R' = \alpha'/R$ is the dual radius. Formula (3.2.25) indicates that the endpoints are not lying on the same hyperplane, but rather lie in parallel hyperplanes separated by $|\theta_i - \theta_j|\,R'$: *the effect of the non-trivial expectation value* (3.2.21) *is to separate the D-branes.* The contribution proportional to $\theta_i - \theta_j$ in the mass formula (3.2.24) corresponds to the energy that is necessary to stretch a string between two separated D-branes. In particular, there are no massless states for the string stretched between two fixed D-branes.[7] Moreover, according to (3.2.20), the scalar state of the open string corresponds to fluctuations of the distance between the D-branes. We thus conclude that for a Dp-brane, *the $D-p-1$ scalar fields Z_m determine the shape of the D-brane.* In other words, the Z_m are the embedding functions of the D-brane into spacetime.

If $\theta_i \neq \theta_j$ for all $1 \leq i \neq j \leq n$, we have one U(1) gauge potential for each D-branes. The total gauge group is then simply $\mathrm{U}(1)^n$. When D-brane i is brought on top of D-brane j, that is, when we set $\theta_i = \theta_j$, the (i,j) string is not stretched anymore and formula (3.2.24) allows for new massless states. These correspond to extra gauge bosons, indicating a *gauge symmetry enhancement.* For $r \leq n$ D-branes brought on top of each other, the gauge symmetry group gets enhanced according to

$$\mathrm{U}(1)^n \to \mathrm{U}(1)^{n-r} \times \mathrm{U}(r)\,. \tag{3.2.26}$$

[7] Except for exceptional values of θ_i, θ_j and l, of course.

Conversely, a non-zero vacuum expectation value for the scalar fields Z_m breaks part of the gauge symmetry group by giving masses to the gauge bosons. This picture thus provides us with a nice geometrical representation of the Higgs symmetry breaking mechanism occuring on the D-brane world-volume theory.

Now that we have seen that a gauge field background like (3.2.21) describes separated D-branes, we might wonder "where" the open string fields of this system are living. As the n parallel Dp-branes have the same dimension, the open string fields will depend on $p+1$ coordinates, and may thus be seen to live in an auxiliary space-time of dimension $d = p+1$. Alternatively, we could imagine that the fields live on *all* the n Dp-branes at the same time.

Let us now consider another interesting set-up: the parallel Dp/Dq system, with $p > q$ for definiteness. It is composed of a Dq-brane lying in directions parallel to the Dp-branes, and stretched open strings have thus three different possible boundary conditions: either both endpoints satisfy Neumann boundary conditions (abbreviated "NN boundary conditions"), or both satisfy Dirichlet boundary conditions (abbreviated "DD boundary conditions"), or one endpoint satisfies Neumann boundary conditions while the other satisfies Dirichlet boundary conditions (abbreviated "mixed" or "ND boundary conditions"). The quantization of the strings in this set-up turn out to be quite interesting, as the fields associated to the states arising from the stretched strings depend only on the $q+1$ coordinates parallel to the Dq-brane. The basic reason is that there is no momentum in the directions involving at least one Dirichlet condition. Since there are $q+1$ NN boundary conditions, *the fields corresponding to the stretched strings live on the Dq-brane world-volume.*

3.2.3 D-brane dynamics: the Dirac-Born-Infeld action

We now wish to address the question of D-brane dynamics, focusing for now on a single D-brane (the case of several D-branes will be considered in subsection 3.2.6). As we have seen in section 3.2.2, the most general set-up with one D-brane includes a world-volume gauge field A_μ and one scalar Z_m for each direction transverse to the D-brane X^m, $1 \leq m \leq D-p-1$. These, together with the usual closed string background fields Φ, G_{MN} and B_{MN}, are constrained by the requirement that the world-sheet theory has no conformal anomaly.

We should thus in principle compute the β-functions of the world-sheet theory in a general background to obtain the D-brane dynamics. Next, we should find an action such that its equations of motion reproduce the conditions obtained by setting to zero the β-functions. Instead of carrying this lengthy program, it is much easier to guess the resulting action and motivate it by showing that it is consistent with various requirement like T-duality and gauge invariance. In the case of the bosonic string, this yields the following action known as the *Dirac-Born-Infeld action*:

$$S_{\mathrm{DBI}} = T_p \int_\Sigma \mathrm{d}^d\xi \, e^{-\Phi} \sqrt{\det\left[\mathrm{P}(G + B_{[2]}) + 2\pi\alpha' F\right]}\,, \qquad (3.2.27)$$

where T_p is a parameter that we will shortly relate to the D-brane tension, Σ is the world-volume of the D-brane, ξ^a with $1 \leq a \leq d$ are some coordinates on Σ, P denotes the pullback of tensors from spacetime to Σ and $F = \mathrm{d}A$ is the field-strength for the world-volume gauge potential $A = A_a \mathrm{d}\xi^a$. The action (3.2.27) depends on the D world-sheet scalars X^M through the closed string background fields $\Phi(X), G_{MN}(X)$ and $B_{MN}(X)$ on one hand and on the other hand trough the pullback, since by definition

$$P(G + B_{[2]})_{ab} = \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^b} (G + B_{[2]})_{MN} \,. \tag{3.2.28}$$

Finally, the determinant in (3.2.27) is taken over $d \times d$ matrices. The Dirac-Born-Infeld action is valid only for constant gauge field F and is exact to all orders in α'.

The action (3.2.27) is manifestly invariant under world-volume reparametrizations $\xi^a \mapsto \xi'^a$, reflecting the fact that not all the D scalar fields X^M are dynamical. In the so-called *static gauge*, we set

$$\xi^a = X^a \,, \qquad 1 \leq a \leq d \,. \tag{3.2.29}$$

This choice completely fixes the reparametrization ambiguity. The remaining dynamical fields are, in addition to the gauge potential A_μ, the $D - d$ scalar fields $Z^m = X^{d+m}$, with $1 \leq m \leq D - d$, describing the transverse embedding coordinates of the D-brane and introduced in the paragraph below (3.2.20).

Let us now turn to the motivations for the D-brane action (3.2.27). First of all, the interactions of the D-brane action should be weighted by the open string coupling $e^{\Phi/2}$, because they are given by the low-energy limit of disk amplitudes. We thus need to have a global factor $e^{-\Phi}$ multiplying the Lagrangian.[8]

Next, consider the D-brane with vanishing B-field and world-volume gauge field F. Then the simplest candidate for a coordinate invariant Lagrangian, focusing on low-derivative terms, is simply the volume density $\sqrt{\det \mathrm{P}(G)}$ induced by the spacetime metric G. The coefficient $T_p e^{-\Phi}$ thus corresponds to the physical tension of the D-brane.

To find out how the D-brane Lagrangian depends on the world-volume gauge field, let us consider a flat D2-brane along directions X^1 and X^2 with a constant field-strength F_{12}. An admissible world-volume gauge potential consists of A^μ with A^1 independent of X^2 and $A^2 = X^1 F_{12}$. If we now perform a T-duality transformation in the direction X^2, the D2-brane is mapped onto a D1-brane lying along the X^1 direction. Moreover, according to our discussion of subsection 3.2.2, the scalar field living on the D1-brane is determined by the component A_2 of the gauge potential. In other words, the D1-brane is not located at the origin in the new direction X'^2, but

[8] This is similar to what happens for an ordinary Yang-Mills gauge theory with coupling constant g_{YM}, where it appears as a global factor g_{YM}^{-2} in front of the action (possibibly after a field redefinition).

rather at the position given by

$$X'^2 = 2\pi\alpha' A_2 = 2\pi\alpha' X^1 F_{12}\,. \tag{3.2.30}$$

Up to proportionality factors that we will discuss shortly, the D2-brane action should match the D1-brane action to which it is dual to. We already know the latter: in the static gauge $\xi^0 = X^0, \xi^1 = X^1$, it is proportional to

$$\begin{aligned}\int \mathrm{d}X^0\mathrm{d}X^1\sqrt{\det \mathrm{P}(G)} &= \int \mathrm{d}X^0\mathrm{d}X^1\sqrt{1+(\partial_1 X'^2)^2}\\ &= \int \mathrm{d}X^0\mathrm{d}X^1\sqrt{1+(2\pi\alpha' F_{12})^2}\,.\end{aligned} \tag{3.2.31}$$

As expected, (3.2.31) reproduces (up to a constant factor) the postulated D-brane action (3.2.27) for flat space and vanishing B-field. To generalise to a D-brane of any dimensionality $p+1$ and for general (constant) world-volume field-strength F, we can simply boost and rotate the world-volume coordinates to make F block-diagonal, each bloc being a two-by-two matrix. Repeating the steps for the case $p = 2$ above for each bloc, we end up with an action containing products of terms similar to $(1+(2\pi\alpha' F_{12})^2)$ but for each two-dimensional plane along the D-brane world-volume, building up for us the determinant $\det(\mathbf{1} + 2\pi\alpha' F)$ under the square-root in (3.2.27).

Finally, let us understand where the B-field dependence in the D-brane action (3.2.27) comes from. Remember that the B-field and the gauge potential form $A_M \mathrm{d}X^M$ couple to the word-sheet through the terms

$$\frac{i}{2\pi\alpha'}\int_{\mathcal{M}} B_{[2]} + i\int_{\partial\mathcal{M}} A_M \mathrm{d}X^M\,, \tag{3.2.32}$$

where $\mathcal{M}$ is the world-sheet and $\partial\mathcal{M}$ its boundary. Recall also that there are gauge invariances associated to the fields A and $B_{[2]}$: the spacetime physics is invariant under the two different gauge transformations $A_M \mapsto A_M + \partial_M\lambda$ and $B_{[2]} \mapsto B_{[2]} + \mathrm{d}\zeta$, where λ is a function and $\zeta = \zeta_M \mathrm{d}X^M$ is a one-form. The world-sheet theory must therefore be invariant under these transformations. Under the former, the world-sheet terms (3.2.32) are obviously invariant. Under the latter, the world-sheet action (3.2.32) is invariant provided we also transform A_M according to

$$A_M \mapsto A_M - \frac{1}{2\pi\alpha'}\zeta_M\,, \tag{3.2.33}$$

which is not an ordinary gauge transformation for A_M since ζ_M is arbitrary. As a consequence, the world-volume field-strength F transforms according to

$$F \mapsto F - \frac{1}{2\pi\alpha'}\mathrm{P}(\mathrm{d}\zeta)\,. \tag{3.2.34}$$

The D-brane action must also be invariant under this gauge transformation. Since we already know how the world-volume field-strength F enters in the Lagrangian, we simply have to replace F by the invariant combination

$$F + \frac{1}{2\pi\alpha'}\mathrm{P}(B_{[2]})\,. \tag{3.2.35}$$

We have thus fully motivated the form of the general D-brane action (3.2.27).

3.2.4 The D-brane tension

The arguments of section (3.2.3) proved that the D-brane action (3.2.27) was consistent with T-duality and gauge invariance. The parameter T_p however remains unknown at this stage. Before we explain how it can be determined explicitly, let us derive a recursion relation for T_p using T-duality. Consider a static Dp-brane wrapped on the p-torus $T^p = S^1 \times \cdots \times S^1$ (with p factors of S^1), each circle S^1 having an arbitrary radius R_i, $1 \leq i \leq p$, and with trivial world-volume and spacetime fields. From the viewpoint of the remaining transverse directions, the D-brane is point-like and its energy is given by its effective tension $T_p e^{-\Phi_0}$ times its internal volume. The energy of the D-brane thus reads

$$T_p e^{-\Phi_0}\mathrm{Vol}(T^p) = T_p e^{-\Phi_0}\prod_{i=1}^{p}(2\pi R_i)\,. \tag{3.2.36}$$

By T-duality along, say, the X^p direction, the energy (3.2.36) must match the energy of a D$(p-1)$-brane wrapped around the $(p-1)$-torus T^{p-1} with circles of radius R_i, with $1 \leq i \leq p-1$, which reads

$$T_{p-1} e^{-\tilde{\Phi}_0}\prod_{i=1}^{p-1}(2\pi R_i)\,, \tag{3.2.37}$$

where $\tilde{\Phi}_0$ is the dilaton expectation value in the T-dual theory. It is related to the original dilaton expectation value Φ_0 using formula (3.2.17). Matching (3.2.36) with (3.2.37) yields the following recursion formula for the D-brane tensions:

$$T_p = \frac{T_{p-1}}{2\pi\sqrt{\alpha'}}\,. \tag{3.2.38}$$

Let us now explain how one can actually find the value of the D-brane tension T_p. The idea is to compute the scattering amplitude measuring the coupling between a closed string mode and the D-brane in two different ways: on one hand, it is computed using the total action $S_{\text{closed}} + S_{\text{DBI}}$ in the semi-classical approximation, where S_{closed} gives the dynamics of the closed background fields Φ, G_{MN}, B_{MN} and is given in (3.2.14). On the other hand, we can compute the scattering amplitude directly in string theory; the value of T_p is then fixed by asking that the two results match in the small α' limit and at weak string coupling. As this procedure is explained in detail in the literature, we will not reproduce it here. The result reads

$$T_p = \frac{\sqrt{\pi}}{2^4 \kappa_0} (4\pi^2\alpha')^{\frac{11-p}{2}} \qquad \text{(bosonic string)} \,. \tag{3.2.39}$$

Of course, the result (3.2.39) satisfies the T-duality consistency requirement (3.2.38). Note that because we are using a semi-classical approximation, the global coefficient of the action $S_{\text{closed}}+S_{\text{DBI}}$ *does* matter, and indeed the result (3.2.39) fixes the value of the combination $T_p\kappa_0$ (in terms of the microscopic parameter α') instead of fixing the relative coefficient $T_p\kappa_0^2$. This is due to the fact that for a typical path integral weight such as $e^{-S/\hbar}$, the leading term in the semi-classical approximation is proportional to $\hbar^{-1}$.

3.2.5 D-branes in superstring theories

We now move to superstring theories, which are consistent in ten-dimensional spacetimes. The spectrum of the theory is richer than in the bosonic case: in particular, the closed string massless spectrum now contains excitations of spacetime gauge potentials $C_{[q]}$ called the *Ramond-Ramond fields* (RR fields for short) as well as a number of fermionic superpartners. The possible values of q depend on the considered superstring theory. For example, type IIB superstring theory reduces in the low-energy approximation to type IIB supergravity, which contains RR forms $C_{[q]}$ for $q = 0, 2, 4$, as we have reviewed in subsection 3.1.6. The properties of electric and magnetic RR sources in the supergravity approximation have been studied in section 3.1.

There is an important difference between the RR gauge fields and the gauge field arising from open string excitations: for the latter, the associated charges are from the start identified with the open string endpoints, while non-trivial RR fields are not straightforwardly related to any explicit source in the theory, and in fact fundamental strings turn out to be *neutral* under the RR fields. Still, the RR sources in superstring theory are intimately related to the open string endpoints: they turn out to be the D-branes. The first evidence for this is the fact that D-branes have their tension behaving like g_{s}^{-1}, exactly like RR sources in supergravity as we have shown in subsection 3.1.6, see in particular equation (3.1.73). Moreover, D-branes in superstring theory turn out to be BPS configurations, and must then carry conserved charges. Since a p-dimensional object naturally sources a $(p+1)$-form gauge potential through the electric coupling, D-branes are perfect candidates for RR sources. These arguments can be strengthened by superstring amplitude computations showing explicitly that D-branes have the correct coupling to RR fields.

Under a T-duality transformation, the chiral type IIB superstring theory is mapped to the non-chiral type IIA superstring theory. The reason is essentially because as we have seen in the bosonic case, T-duality amounts to a world-sheet parity transformation on the right-movers only, see (3.2.12). World-sheet supersymmetry then requires that one also changes the sign of the world-sheet right-moving fermions, hence the chirality change in the theory. By performing two T-duality transformations, we recover the original type IIB theory. Since we have RR field-strengths $F_{[q+1]}$ with $q = 0, 2, 4$, we must have Dp-branes with $p = -1, 1, 3$. Moreover, by T-duality, we must also have Dp-branes with $p = 5, 7$, which are nothing but the magnetic sources for the original field-strengths $F_{[q+1]}$ for $q = 0$ and 2. Finally, the case $p = 9$ corresponds to the space-filling brane.

Since D-branes couple to RR fields, it follows that the D-brane action S_{DBI} should be extended to describe the interactions between the D-brane and non-trivial RR fields. The precise coupling is given by the Wess-Zumino (or Chern-Simons) action S_{CS}, reading

$$S_{\mathrm{CS}} = i\mu_p \int_\Sigma \Big[\sum_q \mathrm{P}\big(C_{[q]} \wedge e^{B_{[2]}}\big) \wedge e^{2\pi\alpha' F}\Big]_{\mathrm{top}} , \tag{3.2.40}$$

where the sum over q runs over the values allowed in a given theory and the effect of the subscript "top" is to select the form of degree $p + 1$ in the square brackets. The prefactor μ_p corresponds to the charge of the Dp-brane, and the factor i comes from the fact that we are working in the Euclidean. The general form of (3.2.40) can be understood by arguments based on the consistency with T-duality and gauge invariance, similarly to the discussion we did for the Dirac-Born-Infeld action, see below (3.2.27). Basically, the pull-back $\mathrm{P}(C_{[q]})$ will require, in the T-dual picture, the presence of the world-volume field-strength in a way very similarly to what appears in (3.2.40). The B-field dependence comes from the requirement of gauge invariance under $B_{[2]} \mapsto B_{[2]} + \mathrm{d}\zeta$, $F \mapsto F - \mathrm{P}(\mathrm{d}\zeta)/(2\pi\alpha')$.

The complete Dp-brane action is thus obtained by adding the Dirac-Born-Infeld action (3.2.27) to the Chern-Simons action (3.2.40). Moreover, the Dp-brane charge μ_p can be determined by comparing field theory amplitudes to the actual superstring result in the semi-classical approximation. The result yields $\mu_p = T_p$, as required by supersymmetry. One can also determine the value of T_p. The result reads

$$T_p = \frac{\sqrt{\pi}}{\kappa_0}(4\pi^2\alpha')^{\frac{3-p}{2}} \qquad \text{(Superstrings)} . \tag{3.2.41}$$

Let us compare the fundamental string tension $\tau_{\mathrm{F}} \equiv (2\pi\alpha')^{-1} \equiv \ell_{\mathrm{s}}^{-2}$ to the D-string tension $\tau_1 = T_1/g_{\mathrm{s}}$. Using (3.2.41), we have

$$\frac{\tau_{\mathrm{F}}}{\tau_1} = \frac{g_{\mathrm{s}}\kappa_0}{8\pi^{7/2}\alpha'^2} . \tag{3.2.42}$$

It turns out to be very convenient for various formulas to define the string coupling as the ratio (3.2.42). In other words, we choose the additive normalisation of Φ such that

$$\frac{\tau_{\mathrm{F}}}{\tau_1} = e^{\Phi} = g_{\mathrm{s}} . \tag{3.2.43}$$

With this choice, the constant κ is given by

$$\kappa = 8\pi^{7/2}\alpha'^2 g_{\rm s}\,, \tag{3.2.44}$$

while T_p reads

$$T_p = \tau_p g_{\rm s} = \frac{1}{(2\pi)^p \alpha'^{(p+1)/2}}\,. \tag{3.2.45}$$

For future reference, let us also express the constant k introduced in section 3.1 and related to the tension τ_p by (3.1.70) in terms of the microscopic parameters α' and $g_{\rm s}$:

$$k = 2^{5-p}\pi^{\frac{5-p}{2}} g_{\rm s}\alpha'^{\frac{7-p}{2}}\Gamma\Big(\frac{7-p}{2}\Big)\,. \tag{3.2.46}$$

3.2.6 Non-abelian extension of the D-brane action

So far, we have only considered the low-energy dynamics of a single Dp-brane moving in a general background. The next natural step is to study the dynamics of a stack of N Dp-branes on top of each other which, according to the discussion of subsection 3.2.2, have an U(N) world-volume gauge invariance. In principle, one could compute tree-level string amplitudes in the low-energy limit to find the D-brane action. This is however a very tedious and complicated procedure in general. Fortunately, the result can be determined by considering consistency of the total action with T-duality [36]. We now briefly review the steps of the rationale.

Let us first give some details on the non-abelian D-brane theory in flat space. The bosonic sector of the theory contains two types of fields: an U(N) gauge potential A_μ and $9-p$ scalar fields X^m describing the fluctuations of the D-brane shape in transverse space. The scalar fields are now $N \times N$ matrices transforming in the adjoint representation of the gauge group U(N). Since the configuration preserves half of the original supercharges, the low-energy action for the world-volume theory must be super-Yang-Mills theory in $p+1$ dimensions with sixteen real supercharges, the bosonic fields A_μ and X^m being supplemented with their fermionic superpartners. In our conventions, the corresponding Yang-Mills coupling $g_{\rm YM}$ appears as $g_{\rm YM}^{-2}$ in front of the gauge field kinetic term $\frac{1}{4}\,{\rm tr}\,F_{\mu\nu}F_{\mu\nu}$. By analogy with the abelian case, we deduce by expanding the square-root in (3.2.27) that $g_{\rm YM}$ is related to the Dp-brane effective tension τ_p through

$$g_{\rm YM}^2 = \frac{1}{(2\pi\alpha')^2\tau_p} = (2\pi)^{p-2} g_{\rm s}\alpha'^{\frac{p-3}{2}}\,. \tag{3.2.47}$$

In particular, $g_{\rm YM}^2$ has the dimensions of (length)$^{p-3}$ and thus for $p \le 3$, the theory is renormalizable while for $p > 3$ it is not.[9] In the limiting case $p = 3$ where $g_{\rm YM}$ is

[9] This is not in itself a problem, because remember that these are *low-energy* actions. In general, the UV completion of this system involves new degrees of freedom corresponding to massive excitations of the strings.

dimensionless, the world-volume is four-dimensional and the theory is $\mathcal{N} = 4$ super-Yang-Mills on which we will have more to say in chapter 4 and in part 2 of this thesis. The action for super-Yang-Mills in $p+1$ dimensions typically contains potential terms involving commutators of the scalar fields X^m, as we will shortly see for the special cases $p = 5, 1$ and -1 in subsections 3.3.2, 3.3.4 and 3.3.5 respectively.

When the closed string background is arbitrary, things are more complicated. The most naive modification of (3.2.27) and (3.2.40) would be to simply put the integrands in a trace on the gauge group $\mathrm{U}(N)$ and replace the scalar field derivatives appearing in the pull-backs by gauge-covariant derivatives. The resulting candidate for the non-abelian D-brane action is however clearly incorrect, as for example it does not reproduce the potential terms involving commutators of the scalar fields in flat space. Moreover, the action is not consistent with T-duality, and should thus be further modified. Note, however, that for N D9-branes there are no scalar fields because the brane is space-filling, and the above naive procedure turns out to yields the correct action (up to the symmetrized trace prescription that we will shortly define). One can then infer the lower-dimensional non-abelian D-brane actions by acting with T-duality transformations and using the Buscher's rules for the transformations of the background fields. This procedure typically yields the various commutators of the scalar fields X^m that were missing in the first naive guess.

The above reasoning fixes the explicit dependence on the scalar fields X^m and the gauge potential A_μ. There is however a place where the scalar fields appear implicitly: that is through the background fields, which in the abelian case are evaluated at the D-brane position. Now that the "position" is given by $N \times N$ matrices, we must define precisely how the dependence on X^m is implemented. Let us illustrate how this is done for instance on the dilaton Φ. In the abelian case and in the static gauge, it is evaluated at X^m: $\Phi(X^m)$, where X^m are the scalar fields. In the non-abelian case, we write $X^m = x^m \mathbf{1}_{N\times N} + 2\pi\alpha'\phi^m$ where x^m are some numbers and ϕ^m are some $N \times N$ matrices, and we expand $\Phi(X^m)$ about $x^m\mathbf{1}$ as

$$\Phi(X^m = x^m\mathbf{1} + 2\pi\alpha'\phi^m) = \sum_{n=0}^{\infty} \frac{(2\pi\alpha')^n}{n!} \phi^{m_1} \cdots \phi^{m_n} (\partial_{x^{m_1}} \cdots \partial_{x^{m_n}}) \Phi(x^m) \,. \qquad (3.2.48)$$

Since each term of the sum in (3.2.48) is completely symmetrized in the indices $m_1, \ldots, m_n$, the order in which the fields ϕ^{m_i} appear does not matter. For the other background fields G_{MN}, B_{MN} and the various RR forms $C_{[q]}$, we proceed in a similar way.

There is one last complication. In fact, the above procedure is ambiguous because we did not specified in which order we should write the different matrices appearing in the action. The last step thus consists of defining the *symmetrized trace prescription*, that we denote by Str and define as the $\mathrm{U}(N)$ trace of the completely symmetric combination of the three quantities F_{ab}, $D_a\phi^m$ and $[\phi^m, \phi^n]$, where we write

$$X^m = x^m\mathbf{1} + 2\pi\alpha'\phi^m \qquad (3.2.49)$$

as above. This *ad hoc* procedure yields the correct result for the low-energy non-abelian D-brane action at least up to fifth order in F_{ab} and ϕ^m. At sixth order however, discrepancies with direct superstring amplitude computations have been noticed [37, 38], so one should not trust this procedure beyond fifth order.

We now present the result for the bosonic sector of the non-abelian D-brane action (known as *Myers' action*) in a general closed string background. In the Euclidean, the action is $S_{\text{DBI}} + S_{\text{CS}}$ with

$$S_{\text{DBI}} = T_p \int \mathrm{d}^{p+1}\xi \, \mathrm{Str}\left(e^{\Phi} \sqrt{\det\left[\mathrm{P}(T)_{ab} + 2\pi\alpha' F_{ab}\right] \det(Q^m_n)} \right), \tag{3.2.50a}$$

$$S_{\text{CS}} = i\mu_p \int \mathrm{Str}\left(\mathrm{P}\left[e^{2\pi i\alpha' \iota_\phi \iota_\phi} \left(\sum_q C_{[q]} \wedge e^{B_{[2]}} \right) \right] \wedge e^{2\pi\alpha' F} \right)_{\text{top}} . \tag{3.2.50b}$$

The parameters T_p and μ_p are simply given by their values in the abelian case multiplied by N. Explicitly,

$$T_p = \mu_p = \tau_p g_{\text{s}} = \frac{N}{(2\pi)^p \alpha'^{(p+1)/2}} \,. \tag{3.2.51}$$

Some definitions are in order to understand the actions (3.2.50). The spacetime tensor T is defined by

$$T_{MN} \equiv E_{MN} + E_{Mm}(Q^{-1} - \delta)^m_k E^{kn} E_{nN} \,, \tag{3.2.52}$$

where $E_{MN} \equiv G_{MN} + B_{[2]MN}$, $Q^m_n \equiv \delta^m_n + 2\pi i\alpha' [\phi^m, \phi^k] E_{kn}$ and E^{mn} is defined as the inverse of E_{mn}, i.e. $E_{mn}E^{nk} = \delta^k_m$. We remind the reader that the indices $1 \leq M, N \leq 10$ are ten-dimensional spacetime indices, $1 \leq m, n, k \leq 9-p$ are indices corresponding to the directions transverse to the Dp-brane and $1 \leq a, b \leq p+1$ label the parallel directions. The first determinant under the square-root in (3.2.50a) is taken over $(p+1) \times (p+1)$ matrices carrying the indices a, b, while the second determinant is taken over $(9-p) \times (9-p)$ matrices carrying the indices m, n. We also remind the reader that the pull-back operator P is, in the static gauge, given by

$$\mathrm{P}(T)_{ab} \equiv T_{ab} + D_b X^n T_{an} + D_a X^m T_{mb} + D_a X^m D_b X^n T_{mn} \,, \tag{3.2.53}$$

where $X^m = x^m \mathbf{1} + 2\pi\alpha' \phi^m$ are the matrix coordinates in the transverse space and D_a denotes the gauge-covariant derivative, and similarly for any q-form.

In (3.2.50b), we used the interior product ι_ϕ, which is defined on any spacetime n-form $\omega = \frac{1}{n!}\omega_{M_1 \cdots M_n} \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_n}$ by

$$\iota_\phi(\omega) \equiv \frac{1}{(n-1)!} \phi^m \omega_{m M_2 \cdots M_n} \mathrm{d}x^{M_2} \wedge \cdots \wedge \mathrm{d}x^{M_n} \,. \tag{3.2.54}$$

In particular, acting twice with ι_ϕ produces a commutator of the scalar fields ϕ^m:

$$\begin{aligned} \iota_\phi \iota_\phi(\omega) &= \frac{1}{(n-2)!} \phi^m \phi^n \omega_{n m M_3 \cdots M_n} \mathrm{d}x^{M_3} \wedge \cdots \wedge \mathrm{d}x^{M_n} \\ &= -\frac{1}{2(n-2)!} [\phi^m, \phi^n] \omega_{m n M_3 \cdots M_n} \mathrm{d}x^{M_3} \wedge \cdots \wedge \mathrm{d}x^{M_n} \,. \end{aligned} \tag{3.2.55}$$

A particular feature of the non-abelian Chern-Simons action (3.2.50b) is that thanks to the presence of commutators of ϕ^m, it allows for couplings involving gauge potentials of degrees *higher* than the Dp-brane dimension $p+1$. For example, in the type IIB theory, a stack of K D-instantons is coupled to a five-form $F_{[5]}$ through terms of the form

$$\phi^{m_1}[\phi^{m_2},\phi^{m_3}][\phi^{m_4},\phi^{m_5}]\,, \tag{3.2.56}$$

that appear when we expand (3.2.50b). Note also that this term is of fifth order in ϕ^m, and is thus in the range of validity of Myers' action. Finally, one should not forget that all background fields present in (3.2.50) are evaluated at X^m and are expanded as explained above formula (3.2.48).

The actions (3.2.50) describe only the bosonic sector of the low-energy spacetime theory; however, a complete description should also include the various fermions required by supersymmetry. Unfortunately the full supersymmetric completion of the non-abelian theory $S_{\rm DBI}+S_{\rm CS}$ is not known (for more on this topic, see e.g. [39, 40] and references therein). In chapter 5 we will sketch a procedure from which the full supersymmetric D-brane action can be determined, at least in some particular (but still non-trivial) backgrounds.

3.3 Explicit low-energy actions of some D-brane systems

To close this chapter, we provide explicit formulas for low-energy non-abelian D-brane actions in flat spacetime, including the fermionic fields and the corresponding supersymmetry transformations for systems that will be of interest in part 2. The strategy that we follow to find these actions is by using T-duality, the starting point being the action for N space-filling D9-branes. In this case, the low-energy action is pure super-Yang-Mills in ten dimensions with sixteen supercharges. Since we are in the field theory limit, T-duality transformation amounts to trivial dimensional reduction. Anticipating our needs for the computations in part 2, the presentation will be rather explicit.

We start by writing the low-energy action for N D9-brane in flat space. We then present the result of dimensional reduction from ten to six dimensions, yielding the N D5-brane world-volume theory. We then move to a different set-up: we consider a stack of K D5-branes parallel to N D9-branes. Let us introduce some terminology, that will be meaningful in part 2: in any system that we consider, the lower-dimensional D-branes are called *probe D-branes*, while the higher-dimensional ones are called *background D-branes.* Moreover, we will always consider in this thesis the case where the background D-branes have four dimensions more than the probe D-branes. The reason is because these systems still preserve one quarter of the original thirty-two supercharges. The total low-energy action is still quite complicated and tedious to determine from first principles, as it typically requires to compute various low-energy string scattering amplitudes.[10] Fortunately, as we will explain in section 5.2, we do not need to know all the terms in the action to find the emergent geometry. The only part of the action that we will actually need and will consider here is the one containing only fields associated to strings having at least one endpoint on the probe D-brane stack. In particular, we shall not consider the couplings between these fields and those coming from strings with both endpoints attached to the background D-branes. On the other hand, we need to know the terms in the action describing the dynamics of the fields associated to the open strings stretched between the two stacks of D-branes, as well as their couplings to the probe D-brane fields. The result is strongly constrained by supersymmetry: we simply need to add hypermultiplets transforming under the bi-fundamental of the gauge group $\mathrm{U}(N)\times\mathrm{U}(K)$, as it should for fields coming from open strings attached to the two stacks. Once these hypermultiplets are added to the action for K D5-branes, we continue to dimensionally reduce the theory down to two dimensions, yielding the terms of the action of the D1/D5 system that will be relevant for us in chapter 7. For completeness, we present the details of the dimensional reduction from six to two dimensions, including the treatment of the fermions. We finish by presenting the relevant terms of the zero-dimensional theory of the D(−1)-branes in the D(−1)/D3 system, that will be relevant for us in chapter 6.

We remind the reader that our conventions are summarized in appendix A. See also appendix B for general considerations about dimensional reduction.

3.3.1 D9-brane action in flat space

We consider the low-energy limit of a stack of N space-filling D9-branes in the trivial closed string background. Working in the Euclidean, the spacetime symmetry group is SO(10) and the gauge group is $\mathrm{U}(N)$. The field content is as follows: there is a gauge potential A_M with $1 \le M \le 10$ in ten dimensions together with a fermionic superpartner λ sitting in the Majorana-Weyl spinor representation of SO(10) and in the adjoint representation of the gauge group $\mathrm{U}(N)$. Without loss of generality, we choose the chirality of λ to be positive, i.e. it is a left-handed Weyl spinor. Since there are no transverse directions, there are no scalar fields living on the D9-brane worldvolume. Moreover, the system must be invariant under sixteen real supercharges, fixing completely the Lagrangian that we denote by L_{D9}. Explicitly,

$$L_{\mathrm{D9}} = \frac{1}{(2\pi)^4 g_{\mathrm{s}}\ell_{\mathrm{s}}^6}\,\mathrm{tr}_{\mathrm{U}(N)}\Big(\ell_{\mathrm{s}}^{-4} + \frac{1}{4}F_{MN}F_{MN} + \frac{i}{2}\bar\lambda\,\Gamma_M\nabla_M\lambda\Big)\,, \tag{3.3.1}$$

where $g_{\mathrm{s}} = e^{\Phi_0}$ is the string coupling set by the constant dilaton $\Phi=\Phi_0$ and the ten 32×32 matrices Γ_M satisfy the Euclidean Clifford algebra

$$\{\Gamma_M,\Gamma_N\} = 2\delta_{MN}\mathbf{1}_{32\times 32}\,. \tag{3.3.2}$$

[10]Note that these computations have been done for the D(−1)/D3 system in [41, 42].

The field-strength F_{MN} and the covariant derivative $\nabla_M\lambda$ are defined by

$$F_{MN} \equiv \partial_M A_N - \partial_N A_M + i[A_M, A_N]\,, \qquad \nabla_M\lambda \equiv \partial_M\lambda + i[A_M, \lambda]\,. \tag{3.3.3}$$

The global factor in (3.3.1) is fixed by the D9-brane tension τ_9, see formula (3.2.51) with $p = 9$, while the kinetic term for the gauge potential A_M is found by expanding the non-abelian Dirac-Born-Infeld action (3.2.50a) in the limit of small α' and evaluated in the trivial closed string background. Since we set all RR fields to zero, the non-abelian Chern-Simons action (3.2.50b) vanishes. The gauge-invariant kinetic term for the fermionic field λ is normalized such that the Lagrangian (3.3.1) is invariant under the following supersymmetry transformations:

$$\delta A_M = -i\bar{\varepsilon}\,\Gamma_M\lambda\,, \qquad \delta\lambda = \frac{1}{2}F_{MN}\Gamma_{MN}\,\varepsilon\,, \tag{3.3.4}$$

where $\Gamma_{MN} \equiv \frac{1}{2}[\Gamma_M, \Gamma_N]$ and ε is any constant left-handed Majorana-Weyl spinor in ten dimensions. Finally, the Dirac conjugate $\bar{\lambda}$ of λ is defined by

$$\bar{\lambda} \equiv \lambda^T C\,, \tag{3.3.5}$$

where λ^T is the transposed of λ while C is the charge conjugation matrix in ten dimensions such that

$$C\Gamma_M C^{-1} = -\Gamma_M^T\,, \qquad C^\dagger = C^{-1} = -C^T = C\,. \tag{3.3.6}$$

3.3.2 D5-brane action from T-duality: dimensional reduction

Starting from the D9-brane Lagrangian (3.3.1), it is trivial to obtain the low-energy world-volume theory of N D5-branes by performing four T-duality transformations in different directions parallel to the D9-brane. In practice, this simply amounts to perform trivial dimensional reductions.

The procedure of trivial dimensional reduction, including the precise treatment of fermions, will be explained in detail when we obtain the low-energy action of the D1/D5 system. For the case at hand, let us simply present the result.

The six-dimensional theory describing a stack of N D5-branes is invariant under the spacetime symmetry group SO(6), R-symmetry group SO(4) and gauge group U(N). The field content is as follows: there is a gauge potential A_r with $1 \le r \le 6$, together with its superpartner Λ_α with $1 \le \alpha \le 2$ transforming as a left-handed Weyl spinor of SO(4), as well as four adjoint scalars a_μ with $1 \le \mu \le 4$ transforming under the fundamental of SO(4) and their superpartner $\bar{\Lambda}^{\dot{\alpha}}$ with $1 \le \dot{\alpha} \le 2$ transforming as right-handed Weyl spinors of SO(4).

For later convenience, we also introduce an auxiliary field $D_{\mu\nu}$ in the adjoint of U(N) and satisfying the self-duality condition $D_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D_{\rho\sigma}$. The field $D_{\mu\nu}$ has the dimensions of (length)$^{-2}$ and has no kinetic term. It could thus be integrated out trivially, by simply replacing it by the solution of its algebraic equation of motion.

The Lagrangian L_{D5} for the N D5-brane world-volume theory, obtained by dimensional reduction from (3.3.1) and after the introduction of the auxiliary field $D_{\mu\nu}$ reads

$$L_{\mathrm{D5}} = \frac{1}{2(2\pi)^4 g_{\mathrm{s}}\ell_{\mathrm{s}}^2}\,\mathrm{tr}_{\mathrm{U}(N)}\Big(2\ell_{\mathrm{s}}^{-4} + \frac{1}{2}F_{rs}F_{rs} + \nabla_r a_\mu \nabla_r a_\mu + 2i\,[a_\mu, a_\nu]\,D_{\mu\nu} \\ + i\Lambda_a^\alpha \bar{\Sigma}_r^{ab}\nabla_r\Lambda_{\alpha b} - i\bar{\Lambda}_{\dot{\alpha}}^a \Sigma_{rab}\nabla_r\bar{\Lambda}^{\dot{\alpha}b} + 2i\sigma_{\mu\alpha\dot{\alpha}}\Lambda_a^\alpha[a_\mu, \bar{\Lambda}^{\dot{\alpha}a}] - D_{\mu\nu}D_{\mu\nu}\Big)\,, \quad (3.3.7)$$

with $1 \leq a, b \leq 4$ being the indices for the spinor representation of SO(6). The various matrices $\Sigma_r, \bar{\Sigma}_r, \sigma_\mu$ and $\bar{\sigma}_\mu$ are defined in appendix A. The Lagrangian (3.3.7) is invariant (up to total derivative terms) under the following supersymmetry transformations

$$\delta A_r = -i\varepsilon_a^\alpha \bar{\Sigma}_r^{ab}\Lambda_{\alpha b}\,, \quad (3.3.8a)$$

$$\delta\Lambda_{\alpha a} = F_{rs}\Sigma_{rsa}{}^b\varepsilon_{\alpha b} - \vec{D}\cdot\vec{\sigma}_\alpha{}^\beta\varepsilon_{\beta a}\,, \quad (3.3.8b)$$

$$\delta a_\mu = -\varepsilon_a^\alpha \sigma_{\mu\alpha\dot{\alpha}}\bar{\Lambda}^{\dot{\alpha}a}\,, \quad (3.3.8c)$$

$$\delta\bar{\Lambda}^{\dot{\alpha}a} = i\nabla_r a_\mu \bar{\Sigma}_r^{ab}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\varepsilon_{\alpha b}\,, \quad (3.3.8d)$$

$$\delta D_{\mu\nu} = -i\varepsilon_a^\alpha \bar{\Sigma}_r^{ab}\nabla_r\Lambda_{\beta b}\sigma_{\mu\nu\alpha}{}^\beta\,, \quad (3.3.8e)$$

where ε_a^α are the supersymmetry parameters. Note that we have only written half of the supersymmetry transformations. We do not need the other half explicitly because in the next subsection, when we add hypermultiplets to describe the presence of background D9-branes, half of the supercharges will be broken. Of course the complete set of supersymmetry transformations can be trivially deduced from the supersymmetry transformations (3.3.4) of the original ten-dimensional theory.

3.3.3 The D5/D9 system: addition of hypermultiplets

We now consider the system composed of a stack of K D5-branes parallel to N D9-branes on top of each other, and focus on the D5-brane world-volume theory. This system breaks half of the supercharges of the original D5-brane theory, and is thus invariant under eight real supercharges that can be chosen to be those written explicitly in (3.3.8). Moreover, the field content must now include the fields associated to the strings having one endpoint on a D9-brane and the other endpoint on a D5-brane. From the D5-brane world-volume viewpoint, these are fields in the bi-fundamental of $\mathrm{U}(N)\times\mathrm{U}(K)$. We thus need to add hypermultiplets to the Lagrangian (3.3.7) containing the bosonic scalar fields $q_{\alpha fi}$ and $q_\alpha^{\dagger fi}$ where $1 \leq f \leq N$ and $1 \leq i \leq K$ transforming as right-handed Weyl spinors of the internal SO(4) symmetry group, together with their fermionic superpartners χ_{afi} and $\tilde{\chi}^{afi}$. The total six-dimensional Lagrangian is obtained by adding the hypermultiplet Lagrangian L_{HM} to L_{D5} given in (3.3.7), with $L_{\rm HM}$ given by

$$L_{\rm HM} = \frac{1}{2}\nabla_r q^{\dagger\alpha f}\nabla_r q_{\alpha f} + \frac{i}{4}\tilde{\chi}^{af}\Sigma_{rab}\nabla_r\chi^b_f - \frac{i}{4}\nabla_r\tilde{\chi}^{af}\Sigma_{rab}\chi^b_f + \frac{1}{\sqrt{2}}q^{\dagger\alpha f}\Lambda_{\alpha a}\chi^a_f + \frac{1}{\sqrt{2}}\tilde{\chi}^{af}\Lambda^\alpha_a q_{\alpha f} + \frac{i}{2}q^{\dagger\alpha f}D_{\mu\nu}\sigma_{\mu\nu\alpha}{}^\beta q_{\beta f}\,. \quad (3.3.9)$$

The covariant derivatives in (3.3.9) are defined by

$$\nabla_r q_{\alpha fi} = \partial_r q_{\alpha fi} + iA_{ri}{}^{j}q_{\alpha fj}\,, \qquad \nabla_r q^{\dagger fi}_\alpha = \partial_r q^{\dagger fi}_\alpha + iA_r{}^i{}_j q^{\dagger fj}_\alpha\,, \qquad (3.3.10)$$

with $A_r{}^i{}_j \equiv -A_{rj}{}^i$, and similarly for $\chi_{afi}, \tilde{\chi}^{afi}$. The supersymmetry transformations leaving the total Lagrangian $L_{\rm D5}+L_{\rm HM}$ unchanged (up to a total derivative) are given by

$$\begin{aligned}
\delta A_r &= -i\varepsilon^\alpha_a\bar{\Sigma}^{ab}_A\Lambda_{\alpha b}\,, & \delta\Lambda_{\alpha a} &= F_{rs}\Sigma_{rsa}{}^b\varepsilon_{\alpha b} - D_{\mu\nu}\sigma_{\mu\nu\alpha}{}^\beta\varepsilon_{\beta a}\,,\\
\delta a_\mu &= -\varepsilon^\alpha_a\sigma_{\mu\alpha\dot{\alpha}}\bar{\Lambda}^{\dot{\alpha}a}\,, & \delta\bar{\Lambda}^{\dot{\alpha}a} &= i\nabla_r a_\mu\bar{\Sigma}^{ab}_r\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\varepsilon_{\alpha b}\,,\\
\delta q_\alpha &= i\sqrt{2}\chi^a\varepsilon_{a\alpha}\,, & \delta\chi^a &= \sqrt{2}\nabla_r q_\alpha\bar{\Sigma}^{ab}_r\varepsilon^\alpha_b\,,\\
\delta q^{\dagger\alpha} &= i\sqrt{2}\tilde{\chi}^a\varepsilon^\alpha_a\,, & \delta\tilde{\chi}^a &= -\sqrt{2}\nabla_r q^{\dagger\alpha}\bar{\Sigma}^{ab}_r\varepsilon_{\alpha b}\,,\\
\delta D_{\mu\nu} &= -i\varepsilon^\alpha_a\bar{\Sigma}^{ab}_r\nabla_r\Lambda_{\beta b}\vec{\sigma}_{\mu\nu\alpha}{}^\beta\,.
\end{aligned}$$

The theory $L_{\rm D5} + L_{\rm HM}$ is still invariant under the internal group SO(4) $\sim$ SU(2)$_+$ $\times$ SU(2)$_-$, but the R-symmetry group consists only of the SU(2)$_+$ factor, as the supercharges of the original D5-brane world-volume theory forming a doublet of SU(2)$_-$ are broken by the presence of the hypermultiplets.

Note that since in the total Lagrangian $L_{\rm D5} + L_{\rm HM}$ there is no gauge potential for the U(N) gauge group, the latter is a *flavour* group from the six-dimensional viewpoint. Of course the full low-energy action for the D5/D9 system includes a kinetic term for the U(N) gauge field, but as will explain in chapter 5, we do not need to consider it for our purposes.

3.3.4 The D1/D5 system

We now explain how to obtain the Lagrangian describing K D1-branes (also called D-strings) in the presence of N D5-branes from the D5/D9 Lagrangian (given by the sum of the Lagrangian (3.3.7) for the adjoint sector and the Lagrangian (3.3.9) for the hypermultiplets) by trivial dimensional reduction. Although this is a straightforward procedure, we present here the machinery of dimensional reduction in full detail, including the treatment of the spinor fields. In appendix B we review some general considerations on dimensional reduction.

After the dimensional reduction down to two dimensions, the six-dimensional spacetime symmetry group SO(6) yields the two-dimensional spacetime symmetry group SO(2) and a new global $SO(4)' \sim SU(2)'_+ \times SU(2)'_-$, where we put primes in order to avoid any confusion with the SO(4) symmetry group corresponding to rotation in the direction transverse to both the D1- and the D5-branes. The fields of the six-dimensional theory are then reorganized into representations of SO(2) and $SO(4)'$, while their transformation laws under $U(K)$, $U(N)$, $SO(4) \sim SU(2)_+ \times SU(2)_-$ are left unchanged.

Let λ_a and ψ^a be any left- and right-handed six-dimensional Weyl spinors respectively. Since a two-dimensional Weyl spinor has only one component, each six-dimensional Weyl spinor yields four two-dimensional Weyl spinors, that will mix with each other under a transformation of the internal symmetry group $SO(4)'$. Performing the reduction along the directions x^3, x^4, x^5 and x^6, it is very convenient to look for a basis in which the generators $S^{(6)}_{(2+m)(2+n)}$ have a simple expression in terms of the four-dimensional generators of the spinor representation of $SO(4)'$ $S^{(4)}_{mn}$, with $1 \leq m, n \leq 4$. We then consider a new basis for the left- and right-handed six-dimensional spinors, defined using two four-by-four unitary matrices U and V in terms of which the spinors λ'_a and ψ'^a in the new basis read

$$\lambda'_a = U_a{}^b \lambda_b \,, \qquad \psi'^a = V^a{}_b \psi^b \,. \tag{3.3.12}$$

Under the change of basis (3.3.12), the matrices Σ_r are transformed into Σ'_r with

$$\Sigma'_r = U \Sigma_r V^{-1} \,. \tag{3.3.13}$$

We *choose* the matrices U and V such that

$$\Sigma'_{(2+m)(2+n)} = -\begin{pmatrix} \sigma_{mn} & 0 \\ 0 & \bar{\sigma}_{mn} \end{pmatrix} \,. \tag{3.3.14}$$

Using the explicit value for the matrices Σ_A and $\bar{\Sigma}_A$ in appendix A.5, we find that the matrices U and V can be taken to be

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \,, \qquad V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \,. \tag{3.3.15a}$$

In this basis, the six-dimensional generators along the reduced directions $S'^{(6)}_{(2+m)(2+n)}$ read

$$S'^{(6)}_{(2+m)(2+n)} = -i \begin{pmatrix} \sigma_{mn} & 0 & 0 & 0 \\ 0 & \bar{\sigma}_{mn} & 0 & 0 \\ 0 & 0 & \sigma_{mn} & 0 \\ 0 & 0 & 0 & \bar{\sigma}_{mn} \end{pmatrix} = S^{(4)}_{mn} \otimes \mathbf{1}_2 \,, \tag{3.3.16}$$

where $S^{(4)}_{mn}$ are the four-dimensional generators given explicitly in (A.4.11). Formula (3.3.16) shows that λ'_1 and λ'_2 are the components of a left-handed Weyl spinor in four dimensions, while λ'_3 and λ'_4 are the components of a right-handed Weyl spinor in four dimensions. We thus write

$$(\lambda'_\zeta) \equiv \begin{pmatrix} \lambda'_1 \\ \lambda'_2 \end{pmatrix} , \qquad (\lambda'^{\dot\zeta}) \equiv \begin{pmatrix} \lambda'_3 \\ \lambda'_4 \end{pmatrix} , \tag{3.3.17}$$

where $1 \leq \zeta, \dot\zeta \leq 2$ are the indices for the left- and right-handed Weyl spinors of $\mathrm{SO}(4)'$ respectively. Similarly, the right-handed six-dimensional spinor ψ'^a is rewritten in terms of left- and right-handed four-dimensional spinors according to

$$(\psi'_\zeta) \equiv \begin{pmatrix} \psi'^1 \\ \psi'^2 \end{pmatrix} , \qquad (\psi'^{\dot\zeta}) \equiv \begin{pmatrix} \psi'^3 \\ \psi'^4 \end{pmatrix} . \tag{3.3.18}$$

Let us now look at how the components of the four-dimensional Weyl spinors in (3.3.17) and (3.3.18) transform under a two-dimensional spacetime transformation. In the basis defined by (3.3.14), the generator $S'^{(6)}_{12}$ reads

$$S'^{(6)}_{12} = \frac{1}{2} \begin{pmatrix} \mathbf{1}_2 \otimes \sigma_3 & 0 \\ 0 & -\mathbf{1}_2 \otimes \sigma_3 \end{pmatrix} . \tag{3.3.19}$$

As a consequence, λ'_1 and λ'_2 are left-handed two-dimensional Weyl spinor and λ'_3 and λ'_4 are right-handed two-dimensional Weyl spinor, while ψ'^1 and ψ'^2 are right-handed two-dimensional Weyl spinor and ψ'^3 and ψ'^4 are left-handed two-dimensional Weyl spinors.

Let us now look at the six-dimensional scalar $\lambda_a\psi^a$. Using the definitions (3.3.12), we have

$$\lambda_a\psi^a = \lambda'_a (UV^T)^a{}_b \chi'^b = \lambda'^\zeta \chi'_\zeta + \lambda'^{\dot\zeta} \chi'_{\dot\zeta} , \tag{3.3.20}$$

where we used the explicit expressions (3.3.15) for the matrices U and V and the conventions (A.4.13) for rising and lowering four-dimensional Weyl spinor indices. Finally, for the kinetic term, we must work out the combinations $\Sigma'_I v_I$ and $\bar\Sigma'_I v_I$, where v_I is a two-dimensional vector. Using formula (3.3.13) together with (A.5.1) and (3.3.15), we find

$$\Sigma'_I v_I = -2 \begin{pmatrix} \bar v \mathbf{1}_2 & 0 \\ 0 & v\mathbf{1}_2 \end{pmatrix} , \qquad \bar\Sigma'_I v_I = -2 \begin{pmatrix} v \mathbf{1}_2 & 0 \\ 0 & \bar v\mathbf{1}_2 \end{pmatrix} , \tag{3.3.21}$$

where we have defined the numbers v and $\bar v$ by

$$v \equiv \frac{1}{2}(v_1 - iv_2) , \qquad \bar v \equiv \frac{1}{2}(v_1 + iv_2) . \tag{3.3.22}$$

We are now ready to apply the dimensional reduction to the fermions of the D5/D9 system. As we have seen in subsection 3.3.3, the six-dimensional theory contains the following Weyl spinors: $\Lambda^\alpha_a, \bar\Lambda^a_{\dot\alpha}, \chi^a$ and $\tilde\chi^a$. Using our change of basis defined in (3.3.12), we apply the decompositions (3.3.17) and (3.3.18) to the left- and right-handed six-dimensional Weyl spinors respectively. This yields the following new fermions: Λ'_ζ,$\Lambda'^{\dot\zeta}$, $\bar\Lambda'_\zeta$,$\bar\Lambda'^{\dot\zeta}$,χ'_ζ,$\chi'^{\dot\zeta}$,$\tilde\chi'_\zeta$ and $\tilde\chi'^{\dot\zeta}$. From now on, we delete all the primes in order to avoid the clutter.

Before we write explicitly the Lagrangian resulting from the dimensional reduction, let us summarize its field content and make explicit the various symmetry properties. All these symmetry properties of the fields are also summarized in table A.2 in appendix A.

The six dimensional vector multiplet yields the two-dimensional gauge field A_I, the four scalars $\phi_m = A_{2+m}$ transforming in the vector representation of $\mathrm{SO}(4)'$ and the spinors $\Lambda_{\alpha\zeta}$, $\Lambda_{\alpha\dot\zeta}$ transforming in the representations $(1/2,0)_{1/2}$ and $(0,1/2)_{-1/2}$ of $\mathrm{SO}(4)'\times\mathrm{SO}(2)$. The adjoint hypermultiplet yields four scalars a_μ and spinors $\bar\Lambda_{\dot\alpha\zeta}$, $\bar\Lambda_{\dot\alpha\dot\zeta}$ transforming in the representations $(1/2,0)_{-1/2}$ and $(0,1/2)_{1/2}$ of $\mathrm{SO}(4)'\times\mathrm{SO}(2)$. Finally, the fundamental hypermultiplets yield the scalars q^α_f and $\tilde q^{\alpha f}$ together with fermions $(\chi_{\zeta f},\tilde\chi_\zeta{}^f)$ in the $(1/2,0)_{-1/2}$ and $(\chi_{\dot\zeta f},\tilde\chi_{\dot\zeta}{}^f)$ in the $(0,1/2)_{1/2}$ representations of $\mathrm{SO}(4)'\times\mathrm{SO}(2)$ respectively. Let us also note that the theory is invariant under worldsheet parity transformations which act by exchanging the $\mathrm{SU}(2)'_+$ and $\mathrm{SU}(2)'_-$ factors of $\mathrm{SO}(4)'$.

We are now ready to write down the Lagrangian. Using formulas (3.3.20) and (3.3.21), the result of the dimensional reduction from six to two dimensions yields:

$$\begin{aligned}
L_{\mathrm{D1/D5}} = \frac{\ell_{\mathrm s}^2}{2g_{\mathrm s}}\,\mathrm{tr}_{\mathrm{U}(K)}\biggl(&\frac{2}{\ell_{\mathrm s}^4}\mathbf{1}_{K\times K}+\frac12 F_{IJ}F_{IJ}+\nabla_I\phi_m\nabla_I\phi_m-\frac12\bigl[\phi_m,\phi_n\bigr]\bigl[\phi_m,\phi_n\bigr]\\
&+\nabla_I a_\mu\nabla_I a_\nu-\bigl[\phi_m,a_\mu\bigr]\bigl[\phi_m,a_\mu\bigr]+2i\bigl[a_\mu,a_\nu\bigr]D_{\mu\nu}-D_{\mu\nu}D_{\mu\nu}-2\Lambda^{\alpha\zeta}\sigma_{m\zeta\dot\zeta}\bigl[\phi_m,\Lambda^{\dot\zeta}_\alpha\bigr]\\
&-2i\Lambda^{\alpha\zeta}\nabla\Lambda_{\alpha\zeta}+2i\Lambda^\alpha_{\dot\zeta}\bar\nabla\Lambda^{\dot\zeta}_\alpha+2\bar\Lambda^\zeta_{\dot\alpha}\sigma_{m\zeta\dot\zeta}\bigl[\phi_m,\bar\Lambda^{\dot\alpha\dot\zeta}\bigr]-2i\bar\Lambda^\zeta_{\dot\alpha}\bar\nabla\bar\Lambda^{\dot\alpha}_\zeta+2i\bar\Lambda_{\dot\alpha\dot\zeta}\nabla\bar\Lambda^{\dot\alpha\dot\zeta}\\
&-2i\sigma_{\mu\alpha\dot\alpha}\Lambda^\alpha_\zeta\bigl[a_\mu,\bar\Lambda^{\dot\alpha\zeta}\bigr]+2i\sigma_{\mu\alpha\dot\alpha}\Lambda^{\alpha\dot\zeta}\bigl[a_\mu,\bar\Lambda^{\dot\alpha}_{\dot\zeta}\bigr]\biggr)\\
&+\frac12\nabla_I\tilde q^{\alpha f}\nabla_I q_{\alpha f}-\frac12\phi_m\tilde q^{\alpha f}\phi_m q_{\alpha f}-\frac12\tilde\chi^{f\zeta}\sigma_{m\zeta\dot\zeta}\phi_m\chi^{\dot\zeta}_f+\frac12\tilde\chi^f_{\dot\zeta}\bar\sigma^{\dot\zeta\zeta}_m\phi_m\chi_{f\zeta}\\
&+i\tilde\chi^{f\zeta}\bar\nabla\chi_{f\zeta}-i\tilde\chi^f_{\dot\zeta}\nabla\chi^{\dot\zeta}_f-\frac{1}{\sqrt2}\tilde q^{\alpha f}\Lambda_{\alpha\zeta}\chi^\zeta_f+\frac{1}{\sqrt2}\tilde q^{\alpha f}\Lambda^{\dot\zeta}_\alpha\chi_{f\dot\zeta}\\
&-\frac{1}{\sqrt2}\tilde\chi^{\zeta f}\Lambda^\alpha_\zeta q_{\alpha f}+\frac{1}{\sqrt2}\tilde\chi^f_{\dot\zeta}\Lambda^{\alpha\dot\zeta}q_{\alpha f}+\frac i2\tilde q^{\alpha f}D_{\mu\nu}\sigma_{\mu\nu\alpha}{}^\beta q_{\beta f}\,,
\end{aligned}\qquad(3.3.23)$$

where $\nabla\equiv(\nabla_1-i\nabla_2)/2$ and $\bar\nabla\equiv(\nabla_1+i\nabla_2)/2$ and we used the expression for the D-string tension τ_2 in terms of $\ell_{\mathrm s}$ and $g_{\mathrm s}$, see (3.2.51).

We can also deduce the supersymmetry transformations leaving the Lagrangian (3.3.23) unchanged (up to a total derivative) from (3.3.11). The supersymmetry parameters ε^α_a split according to (3.3.17) into two four-dimensional Weyl spinors ε^α_ζ and $\varepsilon^{\alpha\dot\zeta}$ (where we deleted the primes as above). The supersymmetry transformations leaving (3.3.23) invariant (up to total derivatives) are given by

$$
\begin{aligned}
\delta\phi_m &= -i\varepsilon^{\alpha\zeta}\sigma_{m\zeta\dot\zeta}\Lambda^{\dot\zeta}_{\alpha} + i\varepsilon^{\alpha}_{\dot\zeta}\bar\sigma^{\dot\zeta\zeta}_{m}\Lambda_{\alpha\zeta}\,,\\
\delta A &= -i\varepsilon^{\alpha}_{\dot\zeta}\Lambda^{\dot\zeta}_{\alpha}\,,\\
\delta\bar A &= i\varepsilon^{\alpha\zeta}\Lambda_{\alpha\zeta}\\
\delta\Lambda_{\alpha\zeta} &= -i[\phi_m,\phi_n]\sigma_{mn\zeta}{}^{\xi}\varepsilon_{\alpha\xi} + s_{IJ}F_{IJ}\varepsilon_{\alpha\zeta} + 2\bar\nabla\phi_m\sigma_{m\zeta\dot\zeta}\varepsilon^{\dot\zeta}_{\alpha} - D_{\mu\nu}\sigma_{\mu\nu\alpha}{}^{\beta}\varepsilon_{\beta\zeta}\,,\\
\delta a_\mu &= -\varepsilon^{\alpha}_{\zeta}\sigma_{\mu\alpha\dot\alpha}\bar\Lambda^{\dot\alpha\zeta} - \varepsilon^{\alpha\dot\zeta}\sigma_{\mu\alpha\dot\alpha}\bar\Lambda^{\dot\alpha}_{\dot\zeta}\,,\\
\delta\bar\Lambda^{\dot\alpha}_{\zeta} &= [\phi_m,a_\mu]\sigma_{m\zeta\dot\zeta}(\bar\sigma_\mu)^{\dot\alpha\alpha}\varepsilon^{\dot\zeta}_{\alpha} + 2\nabla a_\mu\bar\sigma^{\dot\alpha\alpha}_{\mu}\varepsilon_{\alpha\zeta}\,,\\
\delta q_\alpha &= i\sqrt2\chi^{\zeta}\varepsilon_{\zeta\alpha} + i\sqrt2\chi_{\dot\zeta}\varepsilon^{\dot\zeta}_{\alpha}\,,\\
\delta q^{\dagger\alpha} &= i\sqrt2\tilde\chi^{\zeta}\varepsilon^{\alpha}_{\zeta} + i\sqrt2\tilde\chi_{\dot\zeta}\varepsilon^{\alpha\dot\zeta}\,,\\
\delta\chi^{\dot\zeta} &= i\sqrt2\left([\phi_m,q_\alpha]\bar\sigma^{\dot\zeta\zeta}_{m}\varepsilon^{\alpha}_{\zeta} + 2i\bar\nabla q_\alpha\varepsilon^{\alpha\dot\zeta}\right)\,,\\
\delta\chi_{\zeta} &= -i\sqrt2\left([\phi_m,q^{\dagger\alpha}]\sigma_{m\zeta\dot\zeta}\varepsilon^{\dot\zeta}_{\alpha} + 2i\nabla q_\alpha\varepsilon^{\alpha\dot\zeta}\right)\,,\\
\delta\tilde\chi_{\zeta} &= i\sqrt2\left([\phi_m,q^{\dagger\alpha}]\sigma_{m\zeta\dot\zeta}\varepsilon^{\dot\zeta}_{\alpha} + 2i\nabla q^{\dagger\alpha}\varepsilon_{\alpha\zeta}\right)\,,\\
\delta\tilde\chi^{\dot\zeta} &= -i\sqrt2\left([\phi_m,q^{\dagger\alpha}]\bar\sigma^{\dot\zeta\zeta}_{m}\varepsilon_{\alpha\zeta} + 2i\bar\nabla q^{\dagger\alpha}\varepsilon^{\dot\zeta}_{\alpha}\right)\,,\\
\delta D_{\mu\nu} &= \varepsilon^{\alpha\zeta}\sigma_{\mu\nu\alpha}{}^{\beta}\left(\sigma_{m\zeta\dot\zeta}[\phi_m,\Lambda^{\dot\zeta}_{\beta}] + 2i\nabla\Lambda_{\alpha\zeta}\right) - \varepsilon^{\alpha}_{\dot\zeta}\sigma_{\mu\nu\alpha}{}^{\beta}\left(\bar\sigma^{\dot\zeta\zeta}_{m}[\phi_m,\Lambda_{\beta\zeta}] + 2i\bar\nabla\Lambda^{\dot\zeta}_{\beta}\right)\,,
\end{aligned}
$$

where we set $A \equiv (A_1 - iA_2)/2$ and $\bar A \equiv (A_1 + iA_2)/2$

Let us insist on the fact that the Lagrangian (3.3.23) is *not* the Lagrangian for the *full* D1/D5 system since we did not consider the couplings to the fields living on the background D5-branes.

3.3.5 The D(–1)/D3 system

We can further reduce the theory (3.3.23) down to zero dimension. This yields the terms in the Lagrangian for the zero-dimensional theory living on the K D(–1)-branes in the presence of N D3-branes that will be relevant for us in chapter 6. It is actually much easier to reduce directly the theory in six dimensions obtained by adding (3.3.7) and (3.3.9) down to zero dimensions, since then the whole spacetime symmetry group SO(6) becomes altogether an internal, R-symmetry group. The decomposition of the original fields in terms of representations of the symmetry group of the lower-dimensional theory is trivial. In particular, the gauge field A_r decomposes into six scalar ϕ_A, with $1 \le A \le 6$. Setting $X_\mu = \ell_{\mathrm s}^2 a_\mu$ and $\bar\psi^{\dot\alpha a} \equiv -i\ell_{\mathrm s}^2\bar\Lambda^{\dot\alpha a}$ for later convenience, the result reads

$$
\begin{aligned}
L_{\mathrm{D}(-1)/\mathrm{D}3} = \frac{\pi}{g_{\mathrm{s}}}\,\mathrm{tr}_{\mathrm{U}(K)}\Big\{ 2\,\mathbf{1}_{K\times K} + 2iD_{\mu\nu}\big[X_\mu, X_\nu\big] \\
- \frac{\ell_{\mathrm{s}}^4}{2}[\phi_A,\phi_B][\phi_A,\phi_B] - \big[X_\mu,\phi_A\big]\big[X_\mu,\phi_A\big] - \ell_{\mathrm{s}}^4\Lambda^\alpha{}_a\bar{\Sigma}^{ab}_A[\phi_A,\Lambda_{\alpha b}] \\
- 2\Lambda^\alpha{}_a\sigma_{\mu\alpha\dot{\alpha}}\big[X_\mu,\bar{\psi}^{\dot{\alpha}a}\big] - \bar{\psi}_{\dot{\alpha}}{}^a\Sigma_{Aab}\big[\phi_A,\bar{\psi}^{\dot{\alpha}b}\big] - \ell_{\mathrm{s}}^4 D_{\mu\nu}D_{\mu\nu}\Big\} \\
+ \frac{i}{2}\tilde{q}^\alpha D_{\mu\nu}\sigma_{\mu\nu\alpha}{}^\beta q_\beta + \frac{1}{2}\tilde{q}^\alpha\phi_A\phi_A q_\alpha - \frac{1}{2}\tilde{\chi}^a\Sigma_{Aab}\phi_A\chi^b \\
+ \frac{1}{\sqrt{2}}\tilde{q}^\alpha\Lambda_{\alpha a}\chi^a + \frac{1}{\sqrt{2}}\tilde{\chi}^a\Lambda^\alpha{}_a q_\alpha\,.
\end{aligned}
\tag{3.3.24}
$$

The supersymmetry transformations leaving (3.3.24) invariant are straightforwardly obtained from (3.3.11) and read

$$\delta X_\mu = i\varepsilon^\alpha{}_a\sigma_{\mu\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}a}\,, \tag{3.3.25a}$$

$$\delta\phi_A = -i\varepsilon^\alpha{}_a\bar{\Sigma}^{ab}_A\Lambda_{\alpha b}\,, \tag{3.3.25b}$$

$$\delta D_{\mu\nu} = -\varepsilon^\alpha{}_a\sigma_{\mu\nu\alpha}{}^\beta\bar{\Sigma}^{ab}_A[\phi_A,\Lambda_{\beta b}]\,, \tag{3.3.25c}$$

$$\delta\bar{\psi}^{\dot{\alpha}a} = i\varepsilon_{\beta b}\bar{\sigma}^{\dot{\alpha}\beta}_\mu\bar{\Sigma}^{ba}_A[X_\mu,\phi_A]\,, \tag{3.3.25d}$$

$$\delta\Lambda_{\alpha a} = -\varepsilon_{\beta a}\sigma_{\mu\nu\alpha}{}^\beta D_{\mu\nu} - i\varepsilon_{\alpha b}\bar{\Sigma}_{AB}{}^b{}_a[\phi_A,\phi_B]\,. \tag{3.3.25e}$$

Notice that the theory (3.3.24) being a theory in zero dimension, the "fields" are more properly called "moduli." They organize themselves into a vector multiplet $(\phi_A, \Lambda_{\alpha a}, D_{\mu\nu})$ of six-dimensional $\mathcal{N} = 1$ supersymmetry and an adjoint $(X_\mu, \bar{\psi}^{\dot{\alpha}a})$ and fundamentals $(q_\alpha, \chi^a, \tilde{q}^\alpha, \tilde{\chi}^a)$ hypermultiplets. Their symmetry properties are summarized in appendix A, table A.1.

Chapter 4

The gauge/gravity correspondence

The gauge/gravity correspondence [11–13] (for reviews see e.g. [14, 43]) is one of the most impressive advances in theoretical physics of the last two decades. Simply put, it states that a gauge theory on flat space is equivalent to a higher-dimensional string theory on a curved background. It is remarkable that information about spacetime geometry and gravity are somehow encoded into a lower-dimensional, ordinary field theory. Moreover, the strong coupling regime of the gauge theory is related to the regime where the gravity approximation of string theory is reliable: questions about strong coupling effects, that are usually very difficult to address from first principles, are then related to much easier classical gravity problems. Conversely, gravity and extra dimensions are seen as resulting from strong coupling effects of the gauge theory; in other words, *spacetime and gravity are emerging from the strongly coupled quantum effects of the lower-dimensional, gauge theory.* This is a central idea exploited in this thesis, on which we shall have more to say in part 2.

To be precise, we should talk about the gauge/gravity *conjecture*, as there is no general proof of the famous correspondence. Historically, the first precise example of a gauge/gravity duality involved on the gauge theory side the conformal $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions and on the string theory side type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. This is the celebrated AdS/CFT correspondence. Right after its formulation, various consistency checks were completed and by now nobody doubts the validity of the correspondence. The duality was also rapidly extended and tested in more general contexts. The majority of the tests consisted of computing quantities on both sides of the duality and verify that they agree (see however [44–49]). In part 2, we illustrate how it is possible to go one step further by completely reconstructing the emergent dimensions as well as all properties of the dual supergravity solution directly from the gauge theory.

The goal of this chapter is to review the basic ingredients leading to the conjecture. Although we will not make use of the conjecture, this will allows us to understand how to construct specific pre-geometric models in part 2. We start this chapter by presenting the original argument leading to the AdS/CFT conjecture and introduce the decoupling limit for D3-branes. We then explain how the decoupling limit is generalised in other set-ups, with a particular emphasis on the case of the D5-branes that will be important for us in chapter 7.

4.1 The original argument: the AdS/CFT correspondence

Let us consider type IIB superstring theory in the presence of N D3-branes on top of each other in flat space. At fixed energy, the coupling of a single D-brane to the closed strings in the bulk is proportional to the string coupling g_s. For N D-branes, the coupling between the stack and the closed strings is thus $g_\mathrm{s}N$. We consider two regimes, depending on the value of $g_\mathrm{s}N$. Firstly, when $g_\mathrm{s}N \ll 1$, the backreaction of the D-branes on the ambient spacetime is negligible. The original description of D-branes in flat space as hyperplanes where open strings can end is thus reliable, and the open strings interact with the excitations around flat space of the closed strings. Secondly, when $g_\mathrm{s}N \gg 1$, the backreaction of the D-branes cannot be neglected anymore. The closed strings therefore propagate on the curved geometry sourced by the D-branes.

The basic idea leading to the conjecture is to compare these two regimes in a particular low-energy limit, such that while $\alpha' \to 0$, the strings attached to the D3-branes decouples from the closed strings in the bulk. For this reason, it is called the *decoupling limit*. We now describe and discuss the effect of this limit on the two regimes.

4.1.1 First regime: $g_\mathrm{s}N \ll 1$. The open string picture

Since the backreaction of the N D3-branes can be neglected, the low-energy dynamics of the massless excitations of the open and closed strings in this system is simply given by the total action $S_\mathrm{IIB} + S_\mathrm{DBI} + S_\mathrm{CS}$ with the type IIB supergravity action S_IIB given in (3.1.65), while S_DBI and S_CS are given in (3.2.50) and where the fields are expanded around the trivial, flat spacetime. For instance, for the graviton h_{MN}, we write $G_{MN} = \delta_{MN} + \kappa\, h_{MN}$ and expand the action in powers of h_{MN}, the constant κ given in (3.2.44) being introduced in order to have a canonically normalised kinetic term for h_{MN}.

As $\alpha' \to 0$, the gravitational constant $\kappa \propto \alpha'^2$ goes to zero. In other words, the gravitational interaction becomes weaker and weaker at low-energies. We thus end up with open strings that are completely decoupled from the closed strings, which propagates freely in the bulk. The important point is that unlike the gravitational sector, *the gauge theory living on D3-branes remains non-trivial in the limit*; it is

$\mathcal{N} = 4$ super-Yang-Mills in four dimensions with a *finite* coupling constant

$$g_{\rm YM} = \sqrt{2\pi g_{\rm s}} = \sqrt{2\pi e^{\Phi_0}}\,, \tag{4.1.1}$$

see (3.2.47) with $p = 3$ and where $\Phi = \Phi_0$ is the constant value of the dilaton.

In conclusion, the low-energy limit of the N D3-brane system in this picture consist of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions plus free closed strings excitations around flat space.

Let us mention that if we consider a constant RR scalar $C_{[0]}$, the D3-brane world-volume theory has a non-zero ϑ-angle given by

$$\vartheta = -2\pi C_{[0]}\,. \tag{4.1.2}$$

4.1.2 Second regime: $g_{\rm s}N \gg 1$. The closed string picture

In this regime, the backreaction of the D3-branes on spacetime cannot be neglected anymore. At low energies, the strings propagate on the non-trivial geometry sourced by the N D3-branes which we will shortly show to be well approximated by the solution of type IIB supergravity derived in section 3.1. Choosing the origin in transverse space where the D3-branes are located, the string frame metric $\mathrm{d}s^2_{\rm (s)}$ and the dilaton Φ of this solution read

$$\mathrm{d}s^2_{\rm (s)} = H^{-1/2}\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + H^{1/2}\delta_{mn}\mathrm{d}y^m\mathrm{d}y^n\,, \qquad e^\Phi = e^{\Phi_0} = g_{\rm s}\,, \tag{4.1.3}$$

with $1 \le \mu,\nu \le 4$ and $1 \le m,n \le 6$ labelling the parallel and transverse directions respectively and where H is a harmonic function in transverse space,

$$H = 1 + \frac{k}{r^4} \qquad \text{with } r^2 = y^m y^m\,. \tag{4.1.4}$$

The constant k is expressed in terms of $g_{\rm s}$ and α' using (3.1.32), (3.2.51) and (3.2.44) as

$$k = 4\pi N g_{\rm s}\alpha'^2 = \lambda\alpha^2\,, \tag{4.1.5}$$

where we set

$$\lambda = 4\pi g_{\rm s} N\,. \tag{4.1.6}$$

The flux of the self-dual 5-form $F_{[5]}$ across the 5-sphere in transverse space is independent of the radius of the sphere and is given by

$$\int_{\rm S^5} F_{[5]} = 2\pi N\alpha'\,. \tag{4.1.7}$$

The supergravity approximation for this background is valid when the curvature is small in string units. Using the expression (3.1.55) for the scalar curvature and (4.1.5) to express k in terms of $g_{\rm s}$, N and α', we have

$$\alpha' R \propto \frac{1}{\sqrt{g_{\rm s}N}}\,\cdot \tag{4.1.8}$$

We thus conclude that the supergravity approximation is reliable precisely in the regime that we consider, where $g_{\rm s}N \gg 1$.

Let us now consider the physics perceived by an observer sitting at infinity in transverse space, that is, where $r \to \infty$. There are no open strings in this region, as they are attached to the D-branes that are located far away at $r = 0$: for this observer, the only effect of the D-branes is to source the non-trivial geometry (4.1.3). There are two sectors of closed strings with which the observer can interact: there are closed strings at infinity that propagate on flat space (because the metric in (4.1.3) is asymptotically flat in the transverse directions), and there are closed strings that propagate in the spacetime bulk. Let us imagine that the observer performs scattering experiments by sending wave packets with fixed typical proper energy $E_\infty = E(r \to \infty)$ towards the horizon, where $E(r)$ is the proper energy of the wave packets at a distance r of the D-brane stack. If a wave packet gets sufficiently close to the horizon, say at $r = r_0$, its energy $E(r_0)$ will be arbitrary large due to the blue-shift effect predicted by the formula

$$\frac{E(r)}{E_\infty} = (1 + \frac{k}{r^4})^{1/4}\,, \tag{4.1.9}$$

valid for any $r > 0$. This suggests an interesting limit for this system. Consider that as we take $\alpha' \to 0$, we choose to scale down to zero *all* distances r as well, in such a way that *all* incoming wave packets are infinitely blue-shifted. In this case, *the low-energy approximation for the observer sitting at infinity is automatically reliable*, since $E(r)/E_\infty \to \infty$ even for fixed E_∞. Moreover, if we also impose that the energy $E(r)$ of the incoming wave packets scales in such a way that it remains finite in string units, the product $\sqrt{\alpha'}E(r)$ is fixed and *all massive string states are kept in the spectrum*. Using the formula (4.1.9) relating E_∞ to $E(r)$ as well as the relation (4.1.5) between the constant k and α', we find that the combination r/α' must be kept fixed as we take the limit $\alpha' \to 0$.

Let us work out the geometry in the limit $\alpha' \to 0$ while keeping the new coordinate $U \equiv r/\alpha'$ fixed. Using (4.1.5), the harmonic function H given by (4.1.4) is approximated by

$$H \simeq \frac{\lambda}{\alpha'^2 U^4}\,. \tag{4.1.10}$$

The metric in (4.1.3) in the near-horizon region is thus approximated by

$$\mathrm{d}s^2_{(\mathrm{s})} \simeq \alpha'\Big(\frac{U^2}{\sqrt{\lambda}}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{\sqrt{\lambda}}{U^2}\left(\mathrm{d}U^2 + U^2\mathrm{d}\Omega_5^2\right)\Big)\,, \tag{4.1.11}$$

where $\mathrm{d}\Omega_5^2$ is the round metric on the unit 5-sphere in transverse space. The near-horizon geometry (4.1.11) correspond to $\mathrm{AdS}_5 \times \mathrm{S}^5$, with the radius of AdS_5 and S^5 both equal to $k^{1/4}$.

In conclusion, the low-energy limit in this second regime consists of type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the supergravity approximation, plus free closed strings propagating on flat space.

4.1.3 The correspondence

The two discussions above suggest a very interesting property. Let us consider $\mathcal{N} = 4$ super-Yang-Mills in four dimensions with gauge group $\mathrm{U}(N)$. Although we will not present these details here, it is known that this theory admits a non-trivial large N limit if we keep the 't Hooft coupling $g_{\mathrm{YM}}^2 N$ fixed. Notice that this requires that $g_{\mathrm{YM}} \to 0$. Moreover, in the limiting theory, $g_{\mathrm{YM}}^2 N$ is the new coupling constant; in particular, for large $g_{\mathrm{YM}}^2 N$, the theory is strongly coupled and thus becomes highly non-trivial.

On the other hand, we have seen in subsection 4.1.1 that for $g_{\mathrm{s}} N \ll 1$, $\mathcal{N} = 4$ super-Yang-Mills in four dimensions can be seen as the low-energy D3-brane world-volume theory (plus free closed strings on flat space), where in this case the Yang-Mills coupling g_{YM} depends on g_{s} as $g_{\mathrm{YM}}^2 \propto g_{\mathrm{s}}$. This implies that the 't Hooft coupling $g_{\mathrm{YM}}^2 N \propto g_{\mathrm{s}} N$ is small. When $g_{\mathrm{YM}}^2 N$ is large howerver, we have seen in subsection 4.1.2 that the *same* system is described by classical closed superstrings propagating on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Moreover, the strings are weakly coupled since $g_{\mathrm{s}} \propto g_{\mathrm{YM}}^2 \to 0$.

We are thus very naturally lead to conjecture that *the very non-trivial limit of $\mathcal{N} = 4$ super-Yang-Mills at large N and strong 't Hooft coupling is equivalent to the low-energy limit of perturbative type IIB superstring theory on* $\mathrm{AdS}_5 \times \mathrm{S}^5$.

In other words, the non-trivial strong 't Hooft coupling regime of $\mathcal{N} = 4$ super-Yang-Mills at large N becomes very simple, *at the cost of introducing extra dimensions and a new interaction corresponding to classical gravity.* The extra dimensions and the gravitational interaction are thus strong coupling effects. From this point of view, they are not fundamental properties: they are *emergent concepts.*

The above form of the conjecture is called the *weak form.* Stronger forms are obtained by waiving the restrictions on the parameters of the system. In the *medium form*, we still assume that N is large, but $g_{\mathrm{YM}}^2 N$ is allowed to take any value. In this case, the supergravity approximation is in general not valid, but the strings are still weakly interacting and string perturbation theory is reliable. Finally, in the *strong form*, we assume that N takes any values. The string coupling g_{s} is thus arbitrary, and a non-perturbative treatment of type IIB superstring theory is necessary.

Although there is no complete proof of the conjecture, the correspondence can be checked in many ways. The first obvious check is to compare the symmetry groups of the two sides of the duality. $\mathcal{N} = 4$ super-Yang-Mills in four dimensions is invariant under the conformal group,[1] which in four dimensions is $\mathrm{SO}(5,1)$ (or $\mathrm{SO}(4,2)$ in Minkowski spacetime). Moreover, the R-symmetry group acting on the sixteen supercharges is $\mathrm{SU}(4) \sim \mathrm{SO}(6)$. On the superstring side, the isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is precisely $\mathrm{SO}(5,1) \times \mathrm{SO}(6)$, the first and second factors being the isometry groups of AdS_5 and S^5 respectively, in perfect agreement with the global symmetries of the gauge theory.

To close this section, let us see what happens if we modify slightly our original

[1] Actually it is invariant under the full superconformal group.

set-up by pulling one of the D3-branes away from the stack and bring it to a distance r in transverse space. As explained in subsection 3.2.2, the gauge group $\mathrm{U}(N)$ is broken down to $\mathrm{U}(N-1) \times \mathrm{U}(1)$, as a consequence of the Higgs mechanism. The scalar field arising from the string stretched between the separated brane and the stack has therefore a non-zero vacuum expectation value given by $U = r/\alpha'$, which is precisely the combination that is kept fixed in the decoupling limit. The fixed energy U then sets the mass scale of the system. This suggests that *the rescaled AdS radial coordinate* $U = r/\alpha'$ *is dual to the energy scale of the gauge theory*; moreover, in the decoupling limit, the energies on the field theory side are *fixed*. In the conformal case where the N branes are on top of each other, the Yang-Mills coupling $g_{\mathrm{YM}}^2 \propto g_{\mathrm{s}}$ is energy-independent, consistently with the fact that for a stack of D3-branes the dilaton $e^{\Phi} = e^{\Phi_0} = g_{\mathrm{s}}$ is constant, see (4.1.3).

4.2 Generalization: the decoupling limit for Dp-branes

It is also possible to define an interesting decoupling limit for Dp-branes for $p \neq 3$ [50]. The basic idea is essentially as above: we take $\alpha' \to 0$ while keeping the Yang-Mills coupling g_{YM} of the world-volume gauge theory fixed:

$$g_{\mathrm{YM}}^2 = (2\pi)^{p-2} g_{\mathrm{s}} \alpha'^{\frac{p-3}{2}} \quad \text{fixed as } \alpha' \to 0, \tag{4.2.1}$$

where we used the formula (3.2.47) relating g_{YM} to the stringy parameters g_{s} and α' for any p. Condition (4.2.1) requires that the string coupling g_{s} must scale as $\alpha'^{(3-p)/2}$, and we are not free to tune the value of the combination $g_{\mathrm{s}}N$. Let us take a closer look at the effective couplings in the open and closed string pictures in order to understand under which conditions they are valid. Of course, we still have to scale the radial coordinate r in transverse space such that $U = r/\alpha'$ is kept fixed as $\alpha' \to 0$, and hence the energies on the field theory side are fixed as well.

For $p \neq 3$, g_{YM} is dimensionful. Since the coordinate U is interpreted as the energy scale of the word-volume gauge theory, the effective coupling g_{eff} of the field theory living on the Dp-branes is given by

$$g_{\mathrm{eff}}^2 = g_{\mathrm{YM}}^2 N U^{p-3}\,, \tag{4.2.2}$$

which is indeed a dimensionless quantity. Similarly to the situation described in subsection 4.1.1, the D-brane backreaction is negligible when $g_{\mathrm{eff}} \ll 1$, which corresponds to the regime where the perturbative description of the fields living on the branes is reliable. Since $p \neq 3$, this condition restricts the possible values of U. Explicitly we have

$$g_{\mathrm{eff}} \ll 1 \Leftrightarrow \begin{cases} (g_{\mathrm{YM}}^2 N)^{\frac{1}{3-p}} \ll U & \text{for } p < 3\,, \\ (g_{\mathrm{YM}}^2 N)^{\frac{1}{3-p}} \gg U & \text{for } p > 3\,. \end{cases} \tag{4.2.3}$$

Conditions (4.2.3) imply that the field theory description of the D-brane world-volume dynamics is defined in the UV, where $U \to \infty$, only for $p < 3$; in other words, the theory is asymptotically free: the effective coupling decreases at high energies. On the other hand, for $p > 3$ the theory is ill-defined in the UV: as $U \to \infty$, the coupling grows indefinitely. As already explained in subsection 3.2.6, this simply means that new degrees of freedom, corresponding to excitations that we neglected in the field theory limit of the full string theory, cannot be neglected at high energies.

Let us now consider the dual picture where the open strings are absent and the closed strings propagate on the curved geometry produced by the N Dp-branes. The corresponding supergravity solution has been derived in section 3.1. Choosing the origin in transverse space where the Dp-branes are located, the string frame metric $\mathrm{d}s^2_{(\mathrm{s})}$ and the dilaton Φ read

$$\mathrm{d}s^2_{(s)} = H^{-\frac{1}{2}}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + H^{\frac{1}{2}}\mathrm{d}y^m \mathrm{d}y^m \,, \qquad e^\Phi = g_\mathrm{s} H^{\frac{3-p}{4}} \tag{4.2.4}$$

with $1 \leq \mu, \nu \leq p+1$ and $1 \leq m, n \leq 9-p$ labelling the parallel and transverse directions respectively, $g_\mathrm{s} = e^{\Phi_0}$ and the harmonic function in transverse space H is given by

$$H = 1 + \frac{k}{r^{7-p}} = 1 + \frac{2^{5-p}\pi^{\frac{5-p}{2}} g_\mathrm{s} \alpha'^{\frac{7-p}{2}} \Gamma(\frac{7-p}{2})}{r^{7-p}} \,\cdot \tag{4.2.5}$$

In the last equality we used the explicit expression (3.2.46) for the constant k in terms of the string parameters α' and g_s. In the decoupling limit (4.2.1), we find

$$r^{7-p} \ll k \,. \tag{4.2.6}$$

As a consequence, the geometry (4.2.4) is approximated by the near-horizon geometry, where $H \simeq k\, r^{p-7}$. The effective string coupling e^Φ in this region reads:

$$e^\Phi \simeq \frac{g_\mathrm{eff}^{\frac{7-p}{2}}}{N} \,, \tag{4.2.7}$$

and is thus *fixed* as $\alpha' \to 0$ for any values of p. Moreover, for $p < 7$, string perturbation theory is reliable when $g_\mathrm{eff} \ll 1$, which corresponds to the regime where the perturbation theory on the D-brane world-volume is also reliable.

In order for the supergravity approximation to be valid, we have to make sure that the curvature in string units remains small. The scalar curvature for the metric (4.2.4) is such that

$$\alpha' R \propto \frac{1}{g_\mathrm{eff}} \,\cdot \tag{4.2.8}$$

We thus conclude that, similarly to the conformal case $p = 3$, the supergravity approximation is reliable when the perturbative description of the world-volume gauge theory is not valid, and vice-versa. In the light of (4.2.3) and using the rescaled radius $U = r/\alpha'$, we conclude that supergravity is not reliable when we look close to the

D-branes when $p < 3$, while for $p > 3$ supergravity is not reliable only far away from the D-branes.

Let us now specialize to the case $p = 5$ that will be of interest in chapter 7. From the above discussion, we know that in the decoupling limit (4.2.1) with r/α' fixed, the supergravity approximation is reliable close to the branes, where the system is described in terms of the D5-brane geometry. The D5-brane solution in the near-horizon region is deduced from the formulas (3.1.75) and (3.1.76). Explicitly, it reads

$$\mathrm{d}s^2_{(s)} \simeq \sqrt{\frac{2\pi r^2}{g_\mathrm{s} N \ell_\mathrm{s}^2}} \delta_{\mu\nu} \mathrm{d}^\mu \mathrm{d}x^\nu + \sqrt{\frac{g_\mathrm{s} N \ell_\mathrm{s}^2}{2\pi r^2}} (\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2) \,, \tag{4.2.9a}$$

$$F_{[7]} = \frac{1}{g_\mathrm{s} 6!} \partial_m e^{2\phi} \epsilon_{\mu_1 \cdots \mu_6} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_6} \wedge \mathrm{d}y^m \,, \tag{4.2.9b}$$

$$e^\phi \simeq \sqrt{\frac{2\pi r^2}{g_\mathrm{s} N \ell_\mathrm{s}^2}} \,, \tag{4.2.9c}$$

where $\mathrm{d}\Omega_3^2$ is the round metric on the unit radius three-sphere of constant r in transverse space. For future reference, let us also write the three-from field-strength $F_{[3]} \equiv \star_{(\mathrm{s})} F_{[7]}$ that is magnetically sourced by the D5-branes:

$$F_{[3]} = \frac{1}{3! g_\mathrm{s}} \epsilon_{m_1 m_2 m_3 m} \partial_m e^{-2\phi} \mathrm{d}y^{m_1} \wedge \mathrm{d}y^{m_2} \wedge \mathrm{d}y^{m_3} \,. \tag{4.2.10}$$

As a final remark, let us mention that far way from the D5-branes in transverse space, the effective string coupling e^Φ is large and we have to change the description of the system by going to the S-dual picture. The system is then described by the classical geometry sourced by the type IIB NS5-branes, obtained by taking the magnetic solution of section 3.1 with the values $D = 10, n = 3$ and $a = -1$.

Part II

Examples of Emergent Geometries

Chapter 5

The general framework

We open the second part of this thesis by presenting the general set-up in which our emergent geometry models are constructed. We explain the basic idea as well as the general strategy that we will apply in chapters 6 and 7 to derive various supergravity solutions from pure field theoretic computations.

5.1 The basic idea

The basic idea is to consider the scattering of K probe branes, K being fixed, off a large number N of background branes, as depicted in figure 5.1. This system contains three types of open strings, depending on their boundary conditions. The effective action S_{eff} for the probe branes can be obtained by integrating out the background/background and background/probe open strings. In the usual decoupling limit reviewed in chapter 4, this amounts to compute a standard gauge-theoretic path integral. As we will see in chapters 6 and 7, S_{eff} has the correct field content to match with the non-abelian D-brane action for the probe branes moving in the non-trivial supergravity solution created by the background branes. Using the formulas for this non-abelian action reviewed in subsection 3.2.6, the supergravity background can then be read off straightforwardly from S_{eff}, as we will explain in detail.

5.2 The general strategy

We consider the path integral for a system of $N \gg 1$ background D-branes and K probe D-branes with K fixed. In the decoupling limit, where we take $\alpha' \to 0$ while keeping the Yang-Mills coupling g_{YM} fixed, the path integral reads

$$\int \mathrm{d}\mu_{\text{b}} \mathrm{d}\mu_{\text{p}}\, e^{-S_{\text{b}}-S_{\text{p}}}\,, \tag{5.2.1}$$

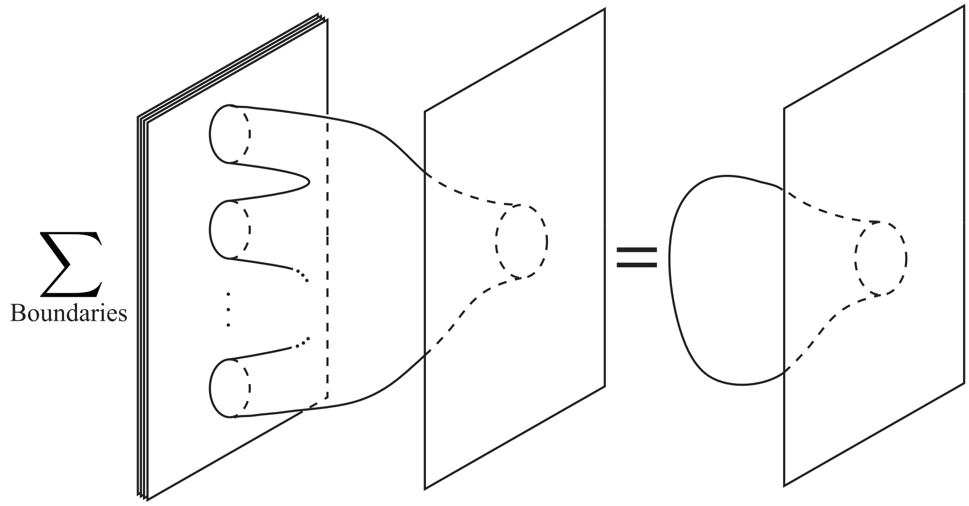


Figure 5.1: On the left, the world-sheets describing the leading large N interaction between K probe branes and a stack of N background branes. The number of boundaries on the background branes world volume can be arbitrary, corresponding to a sum over loops in the microscopic gauge theoretic path integral (5.2.1). This sum is replaced on the right by a unique open string disk diagram in a non-trivial curved background.

where $S_{\rm b}$ is the low energy world-volume action on the background D-branes and $S_{\rm p}$ is the action for the probe D-brane fields, taking into account their coupling to the background D-brane local fields.

In the simplest case already studied in [51], the background D-branes are taken to be D3-branes while the probe D-branes are D-instantons. The action $S_{\rm b}$ is thus the $\mathcal{N}=4$ super-Yang-Mills action and the action $S_{\rm p}$ is derived in [41, 42, 52]. In chapter 6, we consider deformations of this set-up: we take $\mathcal{N}=4$ super-Yang-Mills on the Coulomb branch in section 6.4; we consider the non-commutative deformation in 6.5 and the β-deformation in 6.6. In chapter 7, the background D-branes are D5-branes while the probes are D-strings.

From the gauge/gravity duality, we expect (5.2.1) to be equivalent to the path integral for K D-branes in the non-trivial near-horizon closed string background generated by the background D-branes,

$$\int \mathrm{d}Z\mathrm{d}\Psi\, e^{-S_{\rm eff}(Z,\Psi)}\,. \tag{5.2.2}$$

In (5.2.2), Z and Ψ are the probe bosonic and fermionic world-volume fields and the effective action $S_{\rm eff}$ is the non-abelian action describing the probe moving in ten dimensions (whose bosonic sector is reviewed in subsection 3.2.6). It depends non-trivially on the supergravity background and can thus be used to obtain the supergravity fields.

In practice, to derive the emerging geometry from S_{eff}, we need to cast (5.2.1) in the form (5.2.2). In order to do this, we first integrate over the background-probe string degrees of freedom in (5.2.1). Since these fields are transforming under the fundamental representation of $\mathrm{U}(N)$, they are vector-like variables from the probe world-volume point of view. As explained in chapter 2, this integration can always be done exactly at large N, by introducing suitable auxiliary variables. Some of these variables turn out to correspond to emerging space coordinates, providing precisely the required fields to write the result in the form of (5.2.2). The factor $e^{-S_{\text{eff}}}$ is related to a local operator $\mathscr{D}$ in the background D-branes world-volume gauge theory. Integrating over the background-background strings in (5.2.1) then amounts to computing the expectation value of this operator,

$$\big\langle \mathscr{D} \big\rangle(Z,\Psi) = e^{-S_{\text{eff}}(Z,\Psi)}\,. \tag{5.2.3}$$

This equality provides a precise mapping between any state in the background D-branes world-volume theory, in which we take the expectation value on the left-hand side, and a spacetime geometry, which is encoded in S_{eff}. When we discussed vector models in chapter 2, we insisted on the crucial following property: the action S_{eff} obtained in this way will always be proportional to N. As a result, S_{eff} *yields a classical, non-fluctuating emergent geometry at large* N.

In general, the computation of the expectation value of the operator $\mathscr{D}$ involves an intractable sum over planar diagrams of the background D-brane world-volume theory. However, in some interesting cases, drastic simplifications can occur. In particular, for the deformed D(–1)/D3 systems that we study in chapter 6, it corresponds to a one-point function which cannot be quantum corrected if conformal invariance is unbroken. This is the case, for example, in the planar β-deformed theory studied in section 6.6. More generally, we shall assume that when eight or more supercharges are preserved, including when conformal invariance is broken, the expectation value $\langle\mathscr{D}\rangle$ is not quantum corrected or, more mildly, that the terms in the effective action S_{eff} that we use to derive the supergravity background are insensitive to the possible quantum corrections in $\langle\mathscr{D}\rangle$. This is a very plausible assumption, which is strongly supported by the consistency of the results obtained in chapters 6 and 7 (see also [53]).

It is important to realize that, even when $\langle\mathscr{D}\rangle$ is not quantum corrected, the effective action derived from (5.2.3) has an explicit non-trivial dependence on the 't Hooft coupling constant

$$\lambda = 4\pi g_{\mathrm{s}} N\,, \tag{5.2.4}$$

coming from the *exact* integration over the probe/background degrees of freedom. As we have explained in section 2.2, this integration amounts to summing an infinite class of diagrams, with an arbitrary number of loops. We shall see explicit examples in the following chapters.

Chapter 6

Emergent geometry from Matrices

The aim of the present chapter is to study the emergence of type IIB geometries from field theory, by considering three deformations of the conformal $\mathcal{N} = 4$ super-Yang-Mills theory with broken conformal invariance or supersymmetry.

The simplest deformation we consider is the Coulomb branch deformation, which corresponds to turning on the vacuum expectation values of the scalar fields of the $\mathcal{N} = 4$ theory. This breaks both conformal invariance and R-symmetry but preserves sixteen supersymmetries. The resulting dual geometry asymptotically coincides with the usual $\mathrm{AdS}_5 \times \mathrm{S}^5$ background in the UV but the metric and the Ramond-Ramond five-form field-strength are modified in the IR at the scales set by the scalar expectation values. As we will see, our field theory calculations yield a perfect match with the near-horizon limit of the general multi-centered D3-brane solution reviewed in subsection 3.1.4, for both the metric and the Ramond-Ramond five-form.

The second case we consider is the non-commutative deformation [54, 55]. It breaks conformal invariance but preserves both supersymmetry and R-symmetry. This model does not seem to have a UV fixed point and, accordingly, the known supergravity dual [56, 57] does not have a boundary in the UV and it is likely that a purely field theoretic description does not exist. However, at sufficiently large distance scales, the model approaches the undeformed $\mathcal{N} = 4$ theory and the physical interpretation of both the field theory and its dual supergravity background become clear. The emergent geometry we find is then fully consistent with the background proposed in [56, 57].

Finally, we investigate the so-called β-deformation [58]. In its most general form [59], it breaks supersymmetry completely but preserves conformal invariance in the planar limit [60,61]. The supergravity solution [59,62] is known when the deformation parameters are small, which ensures that the α' corrections can be neglected. Again, our solution is fully consistent with supergravity, including for the Neveu-Schwarz and Ramond-Ramond three-form field-strengths. Let us note that the form of the dilaton was already derived from instanton calculus in previous papers (see [63, 64] and related research in [44–49, 65]).

The plan of the chapter is as follows. In section 6.1, we give some details on the Myers non-abelian action for D-instantons. In section 6.2, we emphasize the subtleties associated with the use of D-instantons in backgrounds that have a non-constant dilaton [66]. The undeformed system, which is studied in detail in [51], is reviewed in section 6.3 and will serve as the starting point for the deformations. We then start the study of the emergent geometry examples associated to the deformations: section 6.4 is devoted to the Coulomb branch set-up, the non-commutative deformation is presented in section 6.5 and we finally close this chapter with the analysis of the β-deformation in section 6.6.

Our notations and conventions as well as useful algebraic identities are presented in appendix A. For completeness, we have also included in appendix C a brief review of the supergravity solutions dual to the non-commutative and β-deformations. Finally, appendix D contain explicit formulas for the non-abelian D-instanton action up to order five.

6.1 On Myers' D-instanton action

To analyse the action S_{eff}, we limit ourselves to the bosonic part, setting $\Psi = 0$ in (5.2.3). As explained in subsection 3.2.6, we write the ten $K \times K$ matrices Z_M, $1 \leq M \leq 10$, as

$$Z_M = z_M \mathbf{1} + \ell_{\mathrm{s}}^2 \epsilon_M \tag{6.1.1}$$

and expand S_{eff} in powers of ϵ,

$$S_{\text{eff}} = \sum_{n\geq 0} S_{\text{eff}}^{(n)} = \sum_{n\geq 0} \frac{1}{n!} \ell_{\mathrm{s}}^{2n} c_{M_1\cdots M_n}(z) \operatorname{tr} \epsilon_{M_1} \cdots \epsilon_{M_n} \,. \tag{6.1.2}$$

The coordinates z_M correspond to a given ten-dimensional spacetime point and we have introduced powers of the string length $\ell_{\mathrm{s}}^2 = 2\pi\alpha'$ for convenience. Myers' prescription for the non-abelian D-instanton action yields the coefficients $c_{M_1\cdots M_n}$ in terms of the supergravity fields, see formula (D.3) in appendix D. Many terms in (D.3) are actually redundant, being fixed by general consistency conditions [66]. In order to derive the full set of supergravity fields, it is enough to consider the following combinations,

$$c = -2i\pi\tau = 2i\pi\big(C_{[0]} - ie^{-\Phi}\big) \tag{6.1.3}$$

$$c_{[MNP]} = -\frac{12\pi}{\ell_{\mathrm{s}}^2} \partial_{[M}\big(\tau B_{[2]} - C_{[2]}\big)_{NP]} \tag{6.1.4}$$

$$c_{[MN][PQ]} = -\frac{18\pi}{\ell_{\mathrm{s}}^4} e^{-\Phi}\big(G_{MP}G_{NQ} - G_{MQ}G_{NP}\big) \tag{6.1.5}$$

$$c_{[MNPQR]} = -\frac{120 i\pi}{\ell_{\mathrm{s}}^4} \partial_{[M}\Big(C_{[4]} + C_{[2]} \wedge B_{[2]} - \frac{1}{2}\tau B_{[2]} \wedge B_{[2]}\Big)_{NPQR]} \,. \tag{6.1.6}$$

Myers' action has two basic limitations. The first comes from the symmetrized trace prescription [39,67] used to fix the ordering ambiguities due to the non-commuting nature of the variables Z and reviewed in section 3.2.6. As we explained, this prescription is valid up to order five in the expansion (6.1.2) but is known to fail at higher orders. This caveat will be of no concern to us, since equations (6.1.3)–(6.1.6) show that the expansion up to order five is sufficient to fix unambiguously the non-trivial supergravity fields.

The second limitation comes from the fact that the formulas (6.1.3)–(6.1.6) are valid only to leading order in the small ℓ_{s}^2, or supergravity, approximation. This implies that our microscopic calculations of S_{eff}, which do not rely on a small ℓ_{s}^2 approximation, can be compared with Myers' only when $\ell_{\mathrm{s}}^2 \to 0$. When comparing our results with the known supergravity solutions, this restriction is harmless, since the solutions are themselves known at small ℓ_{s}^2 only.

Let us point out, however, that some of the basic structural properties of the action, which are visible in the formulas (6.1.3)–(6.1.6), must be valid to all orders in ℓ_{s}^2 because they are consequences of the general consistency conditions discussed in [66]. One of the most interesting properties is that the coefficients $c_{[MNP]}$ and $c_{[MNPQR]}$, viewed as the components of differential forms

$$\hat{F}_{[3]} \equiv \frac{1}{3!} c_{[MNP]}\, \mathrm{d}z^M \wedge \mathrm{d}z^N \wedge \mathrm{d}z^P \,, \tag{6.1.7}$$

$$\hat{F}_{[5]} \equiv \frac{1}{5!} c_{[MNPQR]}\, \mathrm{d}z^M \wedge \mathrm{d}z^N \wedge \mathrm{d}z^P \wedge \mathrm{d}z^Q \wedge \mathrm{d}z^R \,, \tag{6.1.8}$$

must always be closed,

$$\mathrm{d}\hat{F}_{[3]} = 0\,, \quad \mathrm{d}\hat{F}_{[5]} = 0\,. \tag{6.1.9}$$

Locally, we can thus write

$$\hat{F}_{[3]} = -\frac{4\pi}{\ell_{\mathrm{s}}^2}\, \mathrm{d}\hat{C}_{[2]}\,, \quad \hat{F}_{[5]} = -\frac{24 i\pi}{\ell_{\mathrm{s}}^4}\, \mathrm{d}\hat{C}_{[4]}\,. \tag{6.1.10}$$

Since the two- and four-form potentials $\hat{C}_{[2]}$ and $\hat{C}_{[4]}$ are well-defined to all order in ℓ_{s}^2, formulas (6.1.4) and (6.1.6) can actually be used to *define* the Ramond-Ramond and Neveu-Schwarz form fields to all order in ℓ_{s}^2,

$$\hat{C}_{[2]} = \tau B_{[2]} - C_{[2]}\,, \quad \hat{C}_{[4]} = C_{[4]} + C_{[2]} \wedge B_{[2]} - \frac{1}{2}\tau B_{[2]} \wedge B_{[2]}\,, \tag{6.1.11}$$

modulo the general gauge transformations that are discussed in [66]. One of our main goal is to compute the forms (6.1.7) and (6.1.8) for the Coulomb branch, non-commutative and β-deformations of the conformal $\mathcal{N} = 4$ gauge theory. As explained in the next section we can then use (6.1.11) to compare with supergravity in appropriate limits.

Other properties of the Myers action will not, however, be preserved by the ℓ_{s}^2 corrections. For example, the only general constraint on the fourth order coefficient

$c_{[MN][PQ]}$ is that it should have the same tensorial symmetries as the Riemann tensor. This does not imply a factorization in terms of a second rank symmetric tensor as in (6.1.5) and thus such a factorization property is generically lost when ℓ_s^2 corrections are included.

6.2 On the use of the non-abelian D-instanton action

There is one last limitation associated with the use of D-instantons to derive the supergravity background [66]. Intuitively, this limitation is related to the fact that a D-instanton, sitting at a particular point, cannot be expected in general to probe the geometry of the full spacetime manifold. This restriction is waived if the effective action, evaluated at $Z_M = z_M\mathbf{1}$, $S_{\text{eff}}(z\mathbf{1}) = Kc(z)$, does not depend on z, or, equivalently, if the axion-dilaton τ is constant. This is the case for the $\mathcal{N}=4$ gauge theory at any point on its Coulomb branch. However, for a generic background with non-constant axion-dilaton, the instantons are forced to sit at the critical points of $c(z) = -2i\pi\tau(z)$. This condition becomes strict when $N\to\infty$, being equivalent to the saddle-point approximation of the integral (5.2.2).

An alternative way to understand the same limitation is to study the effect of general matrix coordinate redefinitions on the effective action. It is explained in [66] that, when $\mathrm{d}c$ is generic, one can actually gauge away the coefficients $c_{M_1\cdots M_n}$ for $n\geq 2$ in the expansion (6.1.2) by an allowed matrix transformation $Z\mapsto Z'$.

For our purposes, we shall deal with this difficulty by using a perturbative approach around the $\mathrm{AdS}_5\times\mathrm{S}^5$ background on which the instantons can freely move. This is possible because the non-commutative and β-deformed models are continuous deformations of the $\mathcal{N}=4$ gauge theory and thus the associated dual backgrounds will be themselves continuous deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ background.

Let us denote by η the deformation parameter; η is the dimensionless ratio θ/ℓ_s^2 for the non-commutative theory discussed in section 6.5 or the combination $\lambda\gamma^2$ for the β-deformed theory studied in section 6.6. Let us also denote by $c^*_{M_1\cdots M_n}$ the coefficients in the expansion (6.1.2) for the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ background. In our models, the gradient of the axion-dilaton and the corrections to the metric and five-form field-strength turn out to be of order η^2. Hence,

$$c(z) = c^* + O(\eta^2)\,, \tag{6.2.1}$$

$$c_{[MN][PQ]}(z) = c^*_{[MN][PQ]}(z) + O(\eta^2)\,, \tag{6.2.2}$$

$$c_{[MNPQR]}(z) = c^*_{[MNPQR]}(z) + O(\eta^2)\,, \tag{6.2.3}$$

whereas the three-form field-strengths are turned on at leading order in η,

$$c_{[MNP]}(z) = O(\eta)\,. \tag{6.2.4}$$

The general variation of $c_{[MNP]}$ under an arbitrary redefinition of the matrix coordinates corresponds to a standard tensorial transformation under diffeomorphisms plus terms proportional to the gradient of c [66] which, by (6.2.1), are $O(\eta^2)$. *This means that the Neveu-Schwarz and Ramond-Ramond forms $B_{[2]}$ and $C_{[2]}$ are unambiguously fixed in terms of the microscopic calculation of the coefficient $c_{[MNP]}$ of the D-instanton effective action to leading order in the deformation parameter η.*

Moreover, since the background derived from S_{eff} unambiguously matches with the $\text{AdS}_5 \times \text{S}^5$ supergravity background in the undeformed theory, we can always choose the same coordinate systems in both points of view at $\eta = 0$. In the deformed $\eta \neq 0$ models, the coordinate systems z_{mic} and z_{SUGRA} used in the effective action S_{eff} and in the supergravity solution respectively no longer necessarily agree, but the discrepancy must be of order η,

$$z_{\text{mic}} = z_{\text{SUGRA}} + O(\eta)\,. \tag{6.2.5}$$

The associated ambiguity in the axion-dilaton field $c(z)$ is then of order

$$\delta c = \delta z_M \partial_M c = O(\eta \partial c) = O(\eta^3)\,. \tag{6.2.6}$$

This means that the leading $O(\eta^2)$ non-constant term in the axion-dilaton field, see (6.2.1)*, is unambiguously fixed in terms of the microscopic calculation of $c(z)$.*

The conclusion is that, by using D-instantons, we have only access to the leading deformations of the $\text{AdS}_5 \times \text{S}^5$ background, through the $O(\eta)$ terms in $B_{[2]}$ and $C_{[2]}$ and the $O(\eta^2)$ term in τ. Beyond this order, the instantons can no longer probe the full spacetime geometry due to the non-trivial dilaton profile. In particular, the backreaction on the metric and five-form cannot be obtained.

Of course, the above restrictions do not apply if we use particles or higher-dimensional branes, which can probe the geometry with their kinetic term. A concrete example of this situation is the subject of chapter 7, where we use D-strings to probe the geometry sourced by N D5-branes (see also [53] where the geometry sourced by N D4-branes is derived using D-particles as probes).

6.3 The undeformed set-up

As usual, we separate the ten spacetime coordinates z^M into four coordinates x_μ parallel to the background branes and six emergent transverse coordinates $(y_A) = \vec{y}$, with $1 \leq \mu \leq 4$ and $1 \leq A \leq 6$. The radial coordinate r is defined by

$$r^2 \equiv y_A y_A \equiv \vec{y}^{\,2}\,. \tag{6.3.1}$$

Our starting point is the microscopic probe action S_{p} for K D-instantons in the undeformed conformal $\mathcal{N} = 4$ model. The relevant terms can be obtained by taking the $\ell_{\rm s} \to 0$ limit of the action (3.3.24). Using notations explained in appendix A and allowing for a non-zero ϑ-angle, the total action reads

$$
\begin{aligned}
S_{\rm p} = K\Big(\frac{8\pi^2 N}{\lambda} + i\vartheta\Big) + \frac{4\pi^2 N}{\lambda}\, \mathrm{tr}_{\mathrm{U}(K)}\Big\{ & 2iD_{\mu\nu}\big[X_\mu, X_\nu\big] - \big[X_\mu, \phi_A\big]\big[X_\mu, \phi_A\big] \\
& - 2\Lambda^\alpha{}_a \sigma_{\mu\alpha\dot\alpha}\big[X_\mu, \bar\psi^{\dot\alpha a}\big] - \bar\psi_{\dot\alpha}{}^a \Sigma_{Aab}\big[\phi_A, \bar\psi^{\dot\alpha b}\big]\Big\} \\
& + \frac{i}{2}\tilde q^\alpha D_{\mu\nu}\sigma_{\mu\nu\alpha}{}^\beta q_\beta + \frac{1}{2}\tilde q^\alpha \phi_A\phi_A q_\alpha - \frac{1}{2}\tilde\chi^a \Sigma_{Aab}\phi_A \chi^b \\
& + \frac{1}{\sqrt2}\tilde q^\alpha \Lambda_{\alpha a}\chi^a + \frac{1}{\sqrt2}\tilde\chi^a \Lambda^\alpha{}_a q_\alpha + \cdots
\end{aligned}
\tag{6.3.2}
$$

The $\cdots$ represent couplings with the local fields of the $\mathcal{N}=4$ gauge theory living on the background D3-brane world-volume. These terms are described in [41, 42, 51] and enter into the computation of the expectation value (5.2.3) of the operator $\mathscr{D}$, but play no role when this determinant is not quantum corrected. As discussed in chapter 5, we can thus discard them for our present purposes.

Notice that the action (6.3.2) is the standard sigma model action for the ADHM instanton moduli. The fields in the vector multiplet $(\phi_A, \Lambda_{\alpha a}, D_{\mu\nu})$ are auxiliary fields that can be easily integrated out from (6.3.2) to yield the usual ADHM constraints and measure on the instanton moduli space. However, keeping these variables is crucial to solve the model at large N. In particular, the action (6.3.2) is quadratic in the hypermultiplet fields, a property that would be lost if we integrate out the six scalars ϕ_A. Instead, we can integrate exactly over the moduli $q, \tilde q, \chi, \tilde\chi$ which belong to the fundamental of $\mathrm{U}(N)$. This yields an effective action which is automatically proportional to N and can thus be treated classically when $N \to \infty$.

The microscopic actions for the deformed theories that we will study are simple modifications of (6.3.2) and their large N limit can be studied along the same lines. Since our goal is to obtain the bosonic effective action, we shall always set $\Lambda_{\alpha a}$ and $\bar\psi^{\dot\alpha a}$ to zero in the following. We also introduce the notation

$$
Y_A = \ell_{\rm s}^2 \phi_A\,, \tag{6.3.3}
$$

since the auxiliary fields Y_A will turn out to play the role of the six emerging transverse coordinates.

6.4 $\mathcal{N}=4$ on the Coulomb branch

Our first example is the Coulomb branch deformation of the conformal $\mathrm{U}(N)$, $\mathcal{N}=4$ gauge theory. This deformation is parameterized by the scalar expectation values as

$$
\langle \varphi_A \rangle = \ell_{\rm s}^{-2}\, \mathrm{diag}(y_{1A}, \ldots, y_{NA})\,, \quad 1 \le A \le 6\,. \tag{6.4.1}
$$

6.4.1 The microscopic action

The microscopic action is modified by making the replacement

$$\phi_{Ai}{}^{j}\delta_{f}^{f'} \mapsto \phi_{Ai}{}^{j}\delta_{f}^{f'} - \langle\varphi_{Af}{}^{f'}\rangle\delta_{i}^{j} = \phi_{Ai}{}^{j}\delta_{f}^{f'} - \ell_{\mathrm{s}}^{-2}y_{fA}\delta_{f}^{f'}\delta_{i}^{j} \tag{6.4.2}$$

in the third line of (6.3.2). We have indicated all the $\mathrm{U}(N)$ and $\mathrm{U}(K)$ indices explicitly for clarity. This modification is actually best understood as coming from the coupling of the scalar fields φ_A to the moduli in the $\cdots$ part of the action (6.3.2) that we have not written down explicitly.

6.4.2 The effective action

Integrating out $q, \tilde{q}, \chi, \tilde{\chi}$ yields the following effective action:

$$\begin{aligned} S_{\text{eff}}(X,Y,D) &= K\Big(\frac{8\pi^2 N}{\lambda} + i\vartheta\Big) \\ &+ \frac{4\pi^2 N}{\ell_{\mathrm{s}}^4\lambda}\operatorname{tr}_{\mathrm{U}(K)}\Big\{2i\ell_{\mathrm{s}}^4 D_{\mu\nu}\big[X_\mu, X_\nu\big] - \big[X_\mu, Y_A\big]\big[X_\mu, Y_A\big]\Big\} + \ln\Delta_{q,\tilde{q}} - \ln\Delta_{\chi,\tilde{\chi}}\,. \end{aligned} \tag{6.4.3}$$

The logarithm $\ln(\Delta_{q,\tilde{q}}/\Delta_{\chi,\tilde{\chi}})$ is the sum of the term obtained by integrating over the bosonic variables $q, \tilde{q}$,

$$\ln\Delta_{q,\tilde{q}} = \sum_{f=1}^{N}\ln\det\Big(\big(Y_A - y_{fA}\big)^2\otimes\mathbf{1}_{2\times2} + i\ell_{\mathrm{s}}^4 D_{\mu\nu}\otimes\sigma_{\mu\nu}\Big) \tag{6.4.4}$$

and the term obtained by integrating over the fermionic variables $\chi, \tilde{\chi}$,

$$-\ln\Delta_{\chi,\tilde{\chi}} = -\sum_{f=1}^{N}\ln\det\big(\Sigma_A\otimes\big(Y_A - y_{fA}\big)\big)\,. \tag{6.4.5}$$

As expected, this action is proportional to N and thus can be treated classically at large N. In particular, the fluctuations of X, Y and D are suppressed. Since the moduli Y_A are scalars in the $\mathcal{N}=4$ theory, they are interpreted as the six coordinates for the emerging space transverse to the background D3-branes. Together with the four X_μs, they correspond to the ten matrix coordinates Z_M in the non-abelian D-instanton action (6.1.2). Consequently, to compare (6.4.3) with (6.1.2), we simply need to integrate out the additional variables $D_{\mu\nu}$ by solving the saddle-point equation

$$\frac{\partial S_{\text{eff}}}{\partial D_{\mu\nu i}{}^{j}} = 0 \tag{6.4.6}$$

and plugging the solution $D_{\mu\nu} \equiv \langle D_{\mu\nu}\rangle$ back into (6.4.3),

$$S_{\text{eff}}(X,Y) = S_{\text{eff}}\big(X,Y,\langle D\rangle\big)\,. \tag{6.4.7}$$

Our goal is to expand $S_{\text{eff}}(X,Y)$ as in (6.1.2), up to the fifth order and then use (6.1.3)–(6.1.6) to read off the supergravity background. We set

$$X_\mu = x_\mu \mathbf{1} + \ell_{\text{s}}^2 \epsilon_\mu \,, \quad Y_A = y_A \mathbf{1} + \ell_{\text{s}}^2 \epsilon_A \tag{6.4.8}$$

and solve (6.4.6) perturbatively in ϵ. Using the standard notation $[\epsilon_\mu, \epsilon_\nu]^+$ for the self-dual part of the commutator (see (A.4.4)) and defining the harmonic function

$$H(\vec{y}) = \frac{1}{N} \sum_{f=1}^{N} \frac{R^4}{\left|\vec{y} - \vec{y}_f\right|^4} \,, \tag{6.4.9}$$

where R is given by

$$R^4 = \alpha'^2 \lambda = \frac{\ell_{\text{s}}^4 \lambda}{4\pi^2} \,, \tag{6.4.10}$$

we obtain

$$\langle D_{\mu\nu} \rangle = i H^{-1} \left[\epsilon_\mu, \epsilon_\nu\right]^+ + \frac{i \ell_{\text{s}}^2}{2} \partial_A H^{-1} \left(\epsilon_A [\epsilon_\mu, \epsilon_\nu]^+ + [\epsilon_\mu, \epsilon_\nu]^+ \, \epsilon_A \right) + O(\epsilon)^4 \,. \tag{6.4.11}$$

Let us note that since $\langle D \rangle$ solves the equation of motion (6.4.6), it enters into (6.4.3) at order $\langle D \rangle^2$ and thus the expansion (6.4.11) to third order in ϵ is sufficient to get the expansion of (6.4.3) to fifth order.

Plugging (6.4.11) into (6.4.3), expanding the determinants by using the relation

$$\ln \det(M + \delta M) = \ln \det M + \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \operatorname{tr}(M^{-1} \delta M)^n \tag{6.4.12}$$

and computing the resulting traces by using the identities (A.4.3) and (A.5.11)–(A.5.15) in appendix A, we find that the first, second and third order action in (6.1.2) vanish, due to many cancellations between the bosonic and fermionic contributions (6.4.4) and (6.4.5),

$$S_{\text{eff}}^{(1)} = S_{\text{eff}}^{(2)} = S_{\text{eff}}^{(3)} = 0 \,. \tag{6.4.13}$$

On the other hand, the action is non-trivial at the fourth and fifth orders,

$$\begin{aligned} S_{\text{eff}}^{(4)} &= -\frac{\ell_{\text{s}}^8}{2R^4} \operatorname{tr}_{\mathrm{U}(K)} \Big\{ 2[\epsilon_A, \epsilon_\mu][\epsilon_A, \epsilon_\mu] + H^{-1} [\epsilon_\mu, \epsilon_\nu][\epsilon_\mu, \epsilon_\nu] \\ &\qquad\qquad + \frac{1}{2} H [\epsilon_A, \epsilon_B][\epsilon_A, \epsilon_B] \Big\} \,, \end{aligned} \tag{6.4.14}$$

$$\begin{aligned} S_{\text{eff}}^{(5)} &= -\frac{\ell_{\text{s}}^{10}}{2R^4} \partial_A H^{-1} \operatorname{tr}_{\mathrm{U}(K)} \Big\{ \epsilon_A [\epsilon_\mu, \epsilon_\nu][\epsilon_\mu, \epsilon_\nu] + 2 \epsilon_{\mu\nu\rho\lambda} \epsilon_A \epsilon_\mu \epsilon_\nu \epsilon_\rho \epsilon_\lambda \\ &\qquad - H^2 \epsilon_A [\epsilon_B, \epsilon_C][\epsilon_B, \epsilon_C] - \frac{2iH^2}{5} \epsilon_{ABCDEF} \epsilon_B \epsilon_C \epsilon_D \epsilon_E \epsilon_F \Big\} \,. \end{aligned} \tag{6.4.15}$$

6.4.3 The emergent geometry

The results of the previous subsection are consistent with the general ideas explained in chapter 5. The effective action that we have obtained can be matched with the non-abelian action for D-instantons embedded in a non-trivial ten-dimensional emergent geometry, with background supergravity fields fixed by comparing (6.4.13), (6.4.14) and (6.4.15) with (D.3) or, equivalently, (6.1.3)–(6.1.6).

The conditions $S^{(1)}_{\text{eff}} = S^{(2)}_{\text{eff}} = 0$ imply that the axion-dilaton is a constant,

$$\tau = \frac{4i\pi N}{\lambda} - \frac{\vartheta}{2\pi}\,, \tag{6.4.16}$$

whereas $S^{(3)}_{\text{eff}} = 0$ yields

$$B_{[2]} = C_{[2]} = 0\,. \tag{6.4.17}$$

On the other hand, the fourth order term (6.4.14) allows to identify the coefficient $c_{[MN][PQ]}$ which turns out to be precisely of the required form (6.1.5), with a metric

$$G_{\mu\nu} = H^{-1/2}\delta_{\mu\nu}\,,\quad G_{AB} = H^{1/2}\delta_{AB}\,,\quad G_{A\mu} = 0\,. \tag{6.4.18}$$

Finally, we get the completely antisymmetric coefficient $c_{[MNPQR]}$ from (6.4.15), which yields the five-form field-strength by comparing with (6.1.6) and using (6.4.17),

$$(F_{[5]})_{ABCDE} = -\frac{N\ell_{\text{s}}^4}{\pi R^4}\partial_F H \epsilon_{ABCDEF}\,,\quad (F_{[5]})_{A\mu_1\cdots\mu_4} = -\frac{iN\ell_{\text{s}}^4}{\pi R^4}\partial_A H^{-1}\epsilon_{\mu_1\cdots\mu_4}\,, \tag{6.4.19}$$

and all the other independent components (not related to those in (6.4.19) by anti-symmetry) vanishing.

6.4.4 Summary and discussion

To summarise, the supergravity fields derived from the expansion of the D-instanton effective action by comparing with (6.1.3)–(6.1.6), read

$$\tau = \frac{4i\pi N}{\lambda} - \frac{\vartheta}{2\pi}, \tag{6.4.20}$$

$$\mathrm{d}s^2 = H^{-1/2}\mathrm{d}x_\mu \mathrm{d}x_\mu + H^{1/2}\Big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\Big)\,, \tag{6.4.21}$$

$$F_{[5]} = -\frac{N\ell_{\text{s}}^4}{\pi R^5}\Big(\frac{r^4}{R^4}\,y_A\frac{\partial H}{\partial y_A}\,\omega_{\text{S}^5} + i\frac{R^4}{r^4}\,y_A\frac{\partial H^{-1}}{\partial y_A}\omega_{\text{AdS}_5}\Big)\,. \tag{6.4.22}$$

We have denoted the metric on the unit round five-sphere by $\mathrm{d}\Omega_5^2$ and used the definitions

$$H(\vec{y}) = \frac{1}{N}\sum_{f=1}^{N}\frac{R^4}{\left(\vec{y}-\vec{y}_f\right)^4}\,, \tag{6.4.23}$$

$$\omega_{\mathrm{AdS}_5} = \frac{\vec{y}^{\,2} y_A}{R^3}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_4\wedge\mathrm{d}y_A\,, \tag{6.4.24}$$

$$\omega_{\mathrm{S}^5} = \frac{1}{5!}\frac{R^5 y_F}{\vec{y}^{\,6}}\epsilon_{ABCDEF}\,\mathrm{d}y_A\wedge\cdots\wedge\mathrm{d}y_E\,. \tag{6.4.25}$$

The radius R is related to the string scale ℓ_{s} and the 't Hooft coupling λ by (6.4.10). The solution given by (6.4.20), (6.4.21) and (6.4.22) matches the supergravity solution for the multi-centered D3-brane background derived in subsection 3.1.3 taken in the decoupling limit explained in chapter 4.

Let us note that the axion-dilaton τ given by (6.4.20) is constant for the present solution. The D-instantons can thus move freely on the entire spacetime geometry and the restriction discussed in section 6.2 does not apply. Moreover, the match between the microscopic calculation and the supergravity solution is found at finite ℓ_{s}^2 or, equivalently, for any value of the 't Hooft coupling λ. This suggests that, similarly to the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ background [68–70], the near-horizon multi-centered D3-brane background could be exact, with vanishing ℓ_{s}^2 corrections to both Myers' action and to the supergravity equations of motion.

Beyond the details of the solution, let us emphasize that general properties like the self-duality of the five-form field-strength with respect to the metric (6.4.21),

$$\star\, F_5 = -iF_5\,, \tag{6.4.26}$$

or the quantization of the five-form flux in units of the D3-brane charge,

$$\int_{\vec{y}^{\,2}=r^2} F_5 = 4\pi^2\ell_{\mathrm{s}}^4 N(r)\,, \tag{6.4.27}$$

where $N(r)$ counts the number of D3-branes with $\vec{y}_f^{\,2} < r^2$, which are consistency requirements from the point of view of the closed string theory, are highly non-trivial and rather mysterious consequences of the microscopic, field theoretic calculation of the effective action.

6.5 The non-commutative deformation

Our second example is the non-commutative deformation of the $\mathcal{N}=4$ gauge theory. This deformation amounts to imposing non-trivial commutation relations among the spacetime coordinates [54, 55]. The most general deformation is parameterized by a real antisymmetric matrix $\theta_{\mu\nu}$, with

$$[x_\mu, x_\nu] = -i\theta_{\mu\nu}\,. \tag{6.5.1}$$

Up to an SO(4) rotation, we may assume that the only non-vanishing components are $\theta_{12} = -\theta_{21}$ and $\theta_{34} = -\theta_{43}$, with corresponding self-dual and anti self-dual parts

$$\theta^\pm_{12} = \theta^\pm_{34} = \frac{1}{2}\bigl(\theta_{12} \pm \theta_{34}\bigr)\,, \quad \theta^2_\pm = \theta^\pm_{\mu\nu}\theta^\pm_{\mu\nu} = (\theta_{12} \pm \theta_{34})^2\,. \tag{6.5.2}$$

It is convenient for some purposes discussed later in this section to make the rotation to imaginary Euclidean time $x^4 \to ix^4$, in which case θ_{34} is imaginary and $(\theta_\pm)^* = \theta_\mp$.

6.5.1 The microscopic action

The non-commutative deformation can be elegantly implemented by replacing all ordinary products fg of some functions f and g appearing in the microscopic action by the so-called Moyal $*$-product defined by

$$f * g \equiv e^{-\frac{i}{2}\theta_{\mu\nu}P^f_\mu P^g_\nu} \cdot (fg)\,, \tag{6.5.3}$$

where P^f_μ and P^g_μ are the translation operators acting on f and g respectively [54, 55]. The only moduli in (6.3.2) transforming non-trivially under translations are the matrices X_μ, with $P_\mu \cdot X_\nu = -i\delta_{\mu\nu}$. It is then easy to check that the only term affected by the use of the $*$-product is the commutator term,

$$\operatorname{tr} D_{\mu\nu}[X_\mu, X_\nu] \mapsto \operatorname{tr} D_{\mu\nu}\bigl(X_\mu * X_\nu - X_\nu * X_\mu\bigr) = \operatorname{tr} D_{\mu\nu}\bigl([X_\mu, X_\nu] + i\theta_{\mu\nu}\bigr)\,. \tag{6.5.4}$$

This simple reasoning reproduces the well-known modification of the ADHM construction in non-commutative gauge theories [71]. Note that, in particular, the action only depends on the self-dual part $\theta^+_{\mu\nu}$ of the non-commutative parameters because the modulus $D_{\mu\nu}$ is itself self-dual.

6.5.2 The effective action

Integrating out $q, \tilde{q}, \chi, \tilde{\chi}$ from the microscopic action yields

$$\begin{aligned} S_{\text{eff}}(X, Y, D) = K\Bigl(\frac{8\pi^2 N}{\lambda} + i\vartheta\Bigr) - \frac{4\pi^2 N}{\ell_{\text{s}}^4 \lambda} \operatorname{tr}\bigl[X_\mu, Y_A\bigr]\bigl[X_\mu, Y_A\bigr] \\ - N \ln\det\bigl(\Sigma_A \otimes Y_A\bigr) + \mathcal{S}\bigl([X_\mu, X_\nu]^+, \vec{Y}^2, D; \theta_+\bigr)\,. \end{aligned} \tag{6.5.5}$$

We have singled out the D-dependent piece in the action,

$$\begin{aligned} \mathcal{S}\bigl([X_\mu, X_\nu]^+, \vec{Y}^2, D\bigr) = \frac{8i\pi^2 N}{\lambda} \operatorname{tr} D_{\mu\nu}\bigl(\bigl[X_\mu, X_\nu\bigr] + i\theta_{\mu\nu}\bigr)^+ \\ + N \ln\det\bigl(\vec{Y}^2 \otimes \mathbf{1}_2 + i\ell_{\text{s}}^4 D_{\mu\nu} \otimes \sigma_{\mu\nu}\bigr)\,. \end{aligned} \tag{6.5.6}$$

Let us note that the determinants appearing in (6.5.5) and (6.5.6) are special cases of the determinants (6.4.4) and (6.4.5) studied in the previous subsection. The crucial difference comes from the saddle-point equation (6.4.6), which now picks a new term in $\theta_{\mu\nu}$,

$$\frac{\partial \mathcal{S}}{\partial D_{\mu\nu}{}^{i}_{j}} = \frac{8\pi^2}{\lambda}\Big([X_\mu, X_\nu]^{+}{}_{i}^{j} + i\theta^{+}_{\mu\nu}\delta_i^j\Big) + \ell_{\mathrm{s}}^4\Big(\vec{Y}^2\otimes \mathbf{1}_2 + i\ell_{\mathrm{s}}^4 D_{\rho\kappa}\otimes\sigma_{\rho\kappa}\Big)^{-1}{}_{i\alpha}^{\ j\beta}\sigma_{\mu\nu\beta}{}^{\alpha} = 0\,. \tag{6.5.7}$$

This equation must be solved for $D_{\mu\nu} = \langle D_{\mu\nu}\rangle$, order by order in the expansion (6.4.8).

By using (A.4.7), we find a quadratic equation for the zeroth order solution. Picking the root that behaves smoothly when $\theta_{\mu\nu}\to 0$ yields

$$\langle D_{\mu\nu}\rangle = \frac{\lambda}{8\pi^2\theta_+^2}\Big(1-\sqrt{1+4\theta_+^2 r^4/R^8}\Big)\theta^{+}_{\mu\nu} + O(\epsilon) \tag{6.5.8}$$

in terms of the transverse radial coordinate (6.3.1) and the parameter θ_+ defined in (6.5.2). Plugging this result into (6.5.6) and (6.5.5) and computing the determinants using (A.4.6) and (A.5.8), we get the zeroth order coefficient (6.5.18) for the effective action.

The first, second and completely symmetric third order coefficients in the expansion (6.1.2) of the effective action are fixed in terms of the derivatives of c by consistency conditions [66]. To get further information, we thus need to compute the completely antisymmetric third order coefficient or equivalently the three-form $\hat{F}_{[3]}$ defined in (6.1.7). From (A.5.13), we see that the determinant in (6.5.5) cannot contribute to the completely antisymmetric coefficient. *A priori*, we thus simply need to plug the solution of (6.5.7) to the third order in ϵ into (6.5.6). However, the algebra to do this calculation explicitly is very tedious. Fortunately, the discussion can be greatly simplified by using the following argument.

The basic idea is to note that the D-dependent piece (6.5.6) of the effective action and thus the saddle-point equation (6.5.7) as well depend only on the combinations $\vec{Y}^2$ and $[X_\mu, X_\nu]^+ = \ell_{\mathrm{s}}^4[\epsilon_\mu,\epsilon_\nu]^+$ of the matrices Y_As and X_μs. The same must be true after plugging $D_{\mu\nu}=\langle D_{\mu\nu}\rangle$ into $\mathcal{S}$. If we define

$$\vec{Y}^2 \equiv r^2 + \ell_{\mathrm{s}}^2\epsilon_r = r^2 + 2\ell_{\mathrm{s}}^2\vec{y}\cdot\vec{\epsilon} + \ell_{\mathrm{s}}^4\vec{\epsilon}^{\,2}\,, \tag{6.5.9}$$

the expansion of $\mathcal{S}$ in powers of ϵ is then most conveniently written in terms of $[\epsilon_\mu,\epsilon_\nu]^+$ and ϵ_r. It will actually be useful to replace $[\epsilon_\mu,\epsilon_\nu]^+$ by a completely general self-dual matrix $M^+_{\mu\nu}$ in (6.5.6) and (6.5.7), which is not necessarily a commutator, and solve the equations in term of this more general matrix. We simply have to keep in mind that $M^+_{\mu\nu}$ will be identified with $[\epsilon_\mu,\epsilon_\nu]^+$ at the end of the calculation and is thus of order ϵ^2. The most general single-trace expansion up to order three then reads

$$\mathcal{S}\big(M^+_{\mu\nu}, r^2 + \ell_s^2\epsilon_r\big) = Ks(r^2) + \ell_s^2 s'(r^2)\,\mathrm{tr}\,\epsilon_r + \frac{\ell_s^4}{2}s''(r^2)\,\mathrm{tr}\,\epsilon_r^2 + \frac{\ell_s^6}{6}s'''(r^2)\,\mathrm{tr}\,\epsilon_r^3 \\ + \ell_s^4 s_{\mu\nu}(r^2)\,\mathrm{tr}\,M^+_{\mu\nu} + \ell_s^6 s'_{\mu\nu}(r^2)\,\mathrm{tr}\,\epsilon_r M^+_{\mu\nu} + O\big(\epsilon^4\big)\,, \quad (6.5.10)$$

where the primes denote the derivatives with respect to r^2. The zeroth order coefficient $s(r^2)$ is determined by the zeroth order solution (6.5.8) or equivalently (6.5.18),

$$s(r^2) = c - i\vartheta - \frac{8\pi^2 N}{\lambda} + N\ln r^4 \quad (6.5.11)$$

$$= N\Big(\sqrt{1+4\theta_+^2 r^4/R^8} - 1\Big) + N\ln\bigg(\frac{\sqrt{1+4\theta_+^2 r^4/R^8}-1}{2\theta_+^2/R^8}\bigg)\,. \quad (6.5.12)$$

Since $\mathcal{S}$ does not depend on r^2 and ϵ_r independently but only through the combination $r^2 + \ell_s^2\epsilon_r$, the expansion (6.5.10) must be invariant under the simultaneous shifts

$$r^2 \mapsto r^2 + \ell_s^2 a\,, \quad \epsilon_r \mapsto \epsilon_r - a\mathbf{1}\,, \quad (6.5.13)$$

for any real number a. This fixes the terms in $\mathrm{tr}\,\epsilon_r$, $\mathrm{tr}\,\epsilon_r^2$ and $\mathrm{tr}\,\epsilon_r^3$ in terms of the derivatives of s and the term in $\mathrm{tr}\,\epsilon_r M^+_{\mu\nu}$ in terms of $s'_{\mu\nu}$ as indicated. To fix $s_{\mu\nu}(r^2)$, we can then use another shift symmetry, under

$$M^+_{\mu\nu} \mapsto M^+_{\mu\nu} + i\xi^+_{\mu\nu}\,, \quad \theta^+_{\mu\nu} \mapsto \theta^+_{\mu\nu} - \ell_s^4\xi^+_{\mu\nu}\,, \quad (6.5.14)$$

for any self-dual $\xi^+_{\mu\nu}$. This symmetry comes from the fact that only the combination $\ell_s^4 M^+_{\mu\nu} + i\theta^+_{\mu\nu}$ enters in the generalized versions of the equations (6.5.6) and (6.5.7), in which $[X_\mu, X_\nu]^+$ has been replaced by $\ell_s^4 M^+_{\mu\nu}$. This replacement is useful precisely because it allows to consider the symmetry (6.5.14), by waiving the tracelessness condition that any commutator must satisfy. The invariance of (6.5.10) under (6.5.14) then yields

$$s_{\mu\nu} = -i\frac{\partial s}{\partial\theta^+_{\mu\nu}} = \frac{iN}{\theta_+^2}\Big[1 - \sqrt{1+4\theta_+^2 r^4/R^8}\Big]\theta^+_{\mu\nu}\,. \quad (6.5.15)$$

Plugging this result into (6.5.10) for $M^+_{\mu\nu} = [\epsilon_\mu, \epsilon_\nu]^+$ and using (6.5.9) immediately yields the piece

$$2\ell_s^6 s'_{\mu\nu} y_A\,\mathrm{tr}\,\epsilon_A[\epsilon_\mu, \epsilon_\nu] \quad (6.5.16)$$

of the effective action contributing to the three-form $\hat{F}_{[3]}$ in (6.1.7), from which we obtain

$$\hat{F}_{[3]} = 4s'_{\mu\nu}y_A \mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}y^A = \mathrm{d}\big(2s_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\big)\,. \quad (6.5.17)$$

6.5.3 The emergent geometry

In this example, there is a non-trivial contribution to the action at order ϵ^0:

$$c = i\vartheta + \frac{8\pi^2 N}{\lambda} + N\left(\sqrt{1+4\theta_+^2 r^4/R^8} - 1\right) + N\ln\left(\frac{\sqrt{1+4\theta_+^2 r^4/R^8}-1}{2\theta_+^2 r^4/R^8}\right). \quad (6.5.18)$$

As we have discussed in section 6.2 (see also below for the details on the case at hand), the physical content of this formula is obtained by expanding up to quadratic order in the deformation parameter θ_+ and comparing with (6.1.3). This yields

$$\tau = ie^{-\Phi} - C_{[0]} = -\frac{\vartheta}{2\pi} + \frac{4i\pi N}{\lambda}\left(1 + \frac{\theta_+^2 r^4}{2\ell_{\mathrm{s}}^4 R^4}\right) + O\big(\ell_{\mathrm{s}}^{-2}\theta\big)^3 . \quad (6.5.19)$$

To disentangle the dilaton and the axion fields from (6.5.19), one has to be careful because the fields do not need to be real-valued in the Euclidean. It is thus convenient to rotate the x_4 coordinate to Minkowskian time which, from (6.5.1), implies that θ_{34} is purely imaginary. After this rotation, the dilaton Φ and the axion $C_{[0]}$ are real and we can then take the real and imaginary parts of (6.5.19) to find them.

Similarly, the action at third order yields $\hat{C}_{[2]}$ as we have shown. The physical content of this contribution is found by expanding to linear order in θ_+. From (6.1.10) and (6.1.4), this yields

$$\tau B_{[2]} - C_{[2]} = \frac{4i\pi N}{\lambda}\frac{r^4}{R^4}\frac{\theta^+_{\mu\nu}}{\ell_{\mathrm{s}}^2}\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu + O\big(\ell_{\mathrm{s}}^{-2}\theta\big). \quad (6.5.20)$$

To disentangle the Neveu-Schwarz and Ramond-Ramond fields $B_{[2]}$ and $C_{[2]}$ from (6.5.20), we again rotate to Minkowskian signature in which x_4 and θ_{34} are purely imaginary and the fields $B_{[2]}$ and $C_{[2]}$ are real.

As a final remark, let us note that we have also computed the effective action to fourth order. As mentioned in section 6.2, only the term linear in the deformation parameter θ is physical. Consistently with the supergravity solution reviewed in appendix C.1, this linear term is found to vanish. At quadradic order in θ, we find a coefficient $c_{[MN][PQ]}$ which does not factorize as in (6.1.5), as expected.

6.5.4 Summary and discussion

The large N solution of the microscopic model yields an effective action (6.1.2) with c given in (6.5.18). Since c depends non-trivially on the transverse coordinates $\vec{y}$, the discussion of section 6.2 implies that the physical information contained in the effective action is obtained by expanding in $\eta_\pm = \theta_\pm/\ell_{\mathrm{s}}^2$ around the undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Precisely, (6.5.18) can be used to find the axion-dilaton

$\tau = ic/(2\pi)$ up to terms of order η^3, giving the predictions

$$C_{[0]} = \frac{\vartheta}{2\pi} - \frac{4i\pi N}{\lambda}\frac{\theta_{12}\theta_{34}}{\ell_s^4}\frac{r^4}{R^4} + O(\ell_s^{-2}\theta)^3\,, \tag{6.5.21}$$

$$e^{-\Phi} = \frac{4\pi N}{\lambda}\left[1 + \left(\left(\frac{\theta_{12}}{\ell_s^2}\right)^2 + \left(\frac{\theta_{34}}{\ell_s^2}\right)^2\right)\frac{r^4}{R^4}\right] + O\big(\ell_s^{-2}\theta\big)^3\,. \tag{6.5.22}$$

Moreover, our microscopic calculation yields a third order coefficient $c_{[MNP]}$ and thus a three-form $\hat{F}_{[3]}$ of the form (6.1.10), with a two-form potential $\hat{C}_{[2]}$ given by

$$\hat{C}_{[2]} = \frac{N\ell_s^2}{2i\pi\theta_+^2}\left[1 - \sqrt{1 + 4\theta_+^2 r^4/R^8}\right]\theta^+_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,. \tag{6.5.23}$$

From the discussion of section 6.2, we know that only the term linear in the deformation parameter is physical. By using (6.1.11), we have shown that this yields the predictions

$$\begin{aligned} C_{[2]} = -\frac{r^4}{R^4}\bigg[&\Big(\frac{4i\pi N}{\lambda}\frac{\theta_{34}}{\ell_s^2} + \frac{\vartheta}{2\pi}\frac{\theta_{12}}{\ell_s^2}\Big)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &+ \Big(\frac{4i\pi N}{\lambda}\frac{\theta_{12}}{\ell_s^2} + \frac{\vartheta}{2\pi}\frac{\theta_{34}}{\ell_s^2}\Big)\mathrm{d}x_3 \wedge \mathrm{d}x_4\bigg] + O\big(\ell_s^{-2}\theta\big)^2\,, \end{aligned} \tag{6.5.24}$$

$$B_{[2]} = \frac{r^4}{R^4}\left[\frac{\theta_{12}}{\ell_s^2}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{\theta_{34}}{\ell_s^2}\mathrm{d}x_3 \wedge \mathrm{d}x_4\right] + O\big(\ell_s^{-2}\theta\big)^2\,. \tag{6.5.25}$$

We can now compare the above results with the supergravity solution reviewed in appendix C.1. This solution was derived in [56] and [57] independently. As explained previously, to compare the supergravity and microscopic solutions, we must expand in the deformation parameters θ_{12}/ℓ_s^2 and θ_{34}/ℓ_s^2, which enter into the functions Δ_{12} and Δ_{34} defined in (C.1.7). For the axion $C_{[0]}$, this expansion plays no role and indeed equations (6.5.21) and (C.1.4) match. For the dilaton field, we find a match between (6.5.22) and (C.1.2) to quadratic order, consistently with our discussion in section 6.2. For the $B_{[2]}$ and $C_{[2]}$ fields, to compare supergravity with (6.5.25) and (6.5.21), we must use the approximation $\Delta_{12} \simeq \Delta_{34} \simeq 1$ to keep the leading contribution in the deformation parameter only. We again find a perfect match with the microscopic calculation, in the regime where both can a priori be compared.

As a final remark, let us note that the dimensionless expansion parameter governing the deformation with respect to the conformal $\mathcal{N} = 4$ model is not really $\eta \sim \theta/\ell_s^2$ but rather the combination

$$\eta_{\text{mic}} = \frac{\theta\, r^2}{R^4} \sim \frac{\theta}{\ell_s^2}\frac{r^2}{\ell_s^2\lambda} \tag{6.5.26}$$

in the microscopic formulas (6.5.18), (6.5.23) and

$$\eta_{\text{SUGRA}} = \frac{\theta}{\ell_s^2}\frac{r^2}{R^2} \sim \frac{\theta}{\ell_s^2}\frac{r^2}{\ell_s^2\sqrt{\lambda}} \tag{6.5.27}$$

in the supergravity solution. In the microscopic formulas, λ is a priori arbitrary, but the supergravity solution can be trusted only at large λ. The condition $\eta_{\text{SUGRA}} \ll 1$ thus automatically implies $\eta_{\text{mic}} \ll 1$ in the supergravity limit. However, the condition $\eta_{\text{SUGRA}} \ll 1$ cannot be satisfied for all r, even if we choose the deformation parameter θ/ℓ_s^2 to be arbitrarily small; we have to restrict ourselves to the region $r \ll \ell_s^2\lambda^{1/4}/\theta^{1/2}$, where the solution is indeed a small deformation of the $\text{AdS}_5 \times \text{S}^5$ background. This means that, even for infinitesimal θ, the theory is completely changed in the UV, a well-known difficulty associated with non-commutative field theories.

6.6 The beta-deformation

Our last example of this chapter is the β-deformed $\mathcal{N} = 4$ gauge theory. The most general deformation that we study is parameterised by three real numbers γ_1, γ_2 and γ_3 and breaks all supersymmetries.

To describe the solution of the model it is convenient to introduce the polar coordinates (ρ_i, θ_i), $1 \le i \le 3$, defined in terms of the transverse coordinates $\vec{y}$ by

$$\begin{aligned} y_1 &= \rho_1 \cos\theta_1 \,, \quad y_3 = \rho_2 \cos\theta_2 \,, \quad y_5 = \rho_3 \cos\theta_3 \,, \\ y_2 &= \rho_1 \sin\theta_1 \,, \quad y_4 = \rho_2 \sin\theta_2 \,, \quad y_6 = \rho_3 \sin\theta_3 \,, \end{aligned} \tag{6.6.1}$$

together with

$$r_i = \frac{\rho_i}{\sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2}} = \frac{\rho_i}{|\vec{y}|} \,, \tag{6.6.2}$$

which satisfy the constraint

$$r_1^2 + r_2^2 + r_3^2 = 1 \,. \tag{6.6.3}$$

We shall also use the spherical angles (θ, ϕ) defined by

$$r_1 = \sin\theta\cos\phi \,, \quad r_2 = \sin\theta\sin\phi \,, \quad r_3 = \cos\theta \,. \tag{6.6.4}$$

6.6.1 The microscopic action

In parallel with the case of the non-commutative theory, the β-deformation can be implemented by replacing the ordinary products fg appearing in the microscopic action by a $*$-product [62]. Let us denote by Q_i, $1 \le i \le 3$, the charges associated with the $\text{U}(1)_1 \times \text{U}(1)_2 \times \text{U}(1)_3$ subgroup of SO(6) corresponding to the rotations in the 1-2, 3-4 and 5-6 planes in $\vec{y}$-space respectively. The charge assignments according to the SU(4) quantum numbers is indicated in table A.3. The $*$-product is then defined by

$$f * g = e^{i\pi\epsilon_{ijk}\gamma_i Q_j^f Q_k^g} fg \,, \tag{6.6.5}$$

where ϵ_{ijk} is the totally antisymmetric symbol, the charges Q_i^f and Q_i^g act on f and g respectively. When $\gamma_1 = \gamma_2 = \gamma_3$, $\mathcal{N} = 1$ supersymmetry is preserved, but supersymmetry is completely broken otherwise. In all cases, the model is conformal in the planar limit [60, 61].

The only terms in (6.3.2) that are affected when we use the $*$-product are the Yukawa couplings $\bar{\psi}[\phi, \bar{\psi}]$ and $\tilde{\chi}\phi\chi$. To compute the bosonic part of the effective action, we only need $\tilde{\chi}\phi\chi$. According to (A.5.18), the effect of the $*$-product on this term is equivalent to replacing the matrices Σ_A by deformed versions $\tilde{\Sigma}_A$,

$$\tilde{\chi}^a * \Sigma_{Aab}\phi_A * \chi^b = \tilde{\chi}^a \tilde{\Sigma}_{Aab}\phi_A \chi^b \,. \tag{6.6.6}$$

The explicit formulas for the matrices $\tilde{\Sigma}_A$ are given in (A.5.19).

6.6.2 The effective action

Integrating out q, $\tilde{q}$, χ and $\tilde{\chi}$ from the deformed microscopic action, we get

$$S_{\text{eff}}(X, Y, D) = K\Big(\frac{8\pi^2 N}{\lambda} + i\vartheta\Big) + \frac{4\pi^2 N}{\ell_{\text{s}}^4 \lambda}\,\text{tr}\Big\{2i\ell_{\text{s}}^4 D_{\mu\nu}\big[X_\mu, X_\nu\big] - \big[X_\mu, Y_A\big]\big[X_\mu, Y_A\big]\Big\} + \ln\Delta_{q,\tilde{q}} - \ln\tilde{\Delta}_{\chi,\tilde{\chi}}\,, \tag{6.6.7}$$

where

$$\ln\Delta_{q,\tilde{q}} = N\ln\det\big(\vec{Y}^2\otimes \mathbf{1}_2 + i\ell_{\text{s}}^4 D_{\mu\nu}\otimes\sigma_{\mu\nu}\big)\,, \tag{6.6.8}$$

$$\ln\tilde{\Delta}_{\chi,\tilde{\chi}} = N\ln\det\big(\tilde{\Sigma}_A\otimes Y_A\big)\,. \tag{6.6.9}$$

The dependence of $S_{\text{eff}}(X, Y, D)$ on $D_{\mu\nu}$ is exactly the same as in the undeformed model. The solution of the saddle-point equation (6.4.6) is thus given by (6.4.11) for $\vec{y}_f = \vec{0}$. In particular, when we write (6.4.8), $\langle D_{\mu\nu}\rangle$ is of order ϵ^2 and will contribute to S_{eff} only at order four or higher in ϵ.

To leading order, (6.6.7) yields

$$c = \frac{8\pi^2 N}{\lambda} + i\vartheta + 2N\ln\vec{y}^{\,2} - N\ln\det U\,, \tag{6.6.10}$$

where the matrix U is defined by

$$U = y_A\tilde{\Sigma}_A\,. \tag{6.6.11}$$

The determinant of U can be computed straightforwardly in terms of the polar coordinates introduced in (6.6.1),

$$\det U = \rho_1^4 + \rho_2^4 + \rho_3^4 + 2\cos(2\pi\gamma_1)\rho_2^2\rho_3^2 + 2\cos(2\pi\gamma_2)\rho_1^2\rho_3^2 + 2\cos(2\pi\gamma_3)\rho_1^2\rho_2^2\,. \tag{6.6.12}$$

Plugging this result in (6.6.10) and using the coordinates r_i defined in (6.6.2) yields

$$c = \frac{8\pi^2 N}{\lambda} + i\vartheta - N\ln\Big[1 - 4\big(r_2^2 r_3^2\sin^2(\pi\gamma_1) + r_1^2 r_3^2\sin^2(\pi\gamma_2) + r_1^2 r_2^2\sin^2(\pi\gamma_3)\big)\Big]\,. \tag{6.6.13}$$

Let us note that this result was also obtained in the context of standard instanton calculus in [63, 64].

The effective action at first and second order is fixed in terms of the derivatives of c. New information is found in the completely antisymmetric coefficient at order three, which yields the three-form $\hat{F}_{[3]}$ defined in (6.1.7). Expanding in ϵ using (6.4.12), we see that both determinants (6.6.8) and (6.6.9) contribute to the third order action, but only (6.6.9) yields a completely antisymmetric term. Explicitly, we get the following compact result:

$$\hat{F}_{[3]} = -\frac{N}{3}\,\mathrm{tr}\big(U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U\big)\,. \tag{6.6.14}$$

In particular, this formula makes manifest the fact that $\mathrm{d}\hat{F}_{[3]} = 0$. However, the evaluation of the trace on the right-hand side is extremely tedious to perform by hand, because the explicit expressions for the matrix U and its inverse U^{-1} are quite involved. We have thus implemented the calculation in Mathematica. The resulting formulas greatly simplify when using the coordinates defined in (6.6.1), (6.6.2) and (6.6.4). To linear order in the deformation parameters, which is all we need to compare with supergravity, we find, for the two-form potential defined in (6.1.10),

$$\begin{aligned}\hat{C}_{[2]} = 8N\ell_{\mathrm{s}}^2\Big[&\omega_1 \wedge \big(\gamma_1 \mathrm{d}\theta_1 + \gamma_2 \mathrm{d}\theta_2 + \gamma_3 \mathrm{d}\theta_3\big)\\ &- \frac{i}{4}\big(\gamma_1 r_2^2 r_3^2\,\mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 + \gamma_2 r_3^2 r_1^2\,\mathrm{d}\theta_3 \wedge \mathrm{d}\theta_1 + \gamma_3 r_1^2 r_2^2\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\big)\Big] + O\big(\gamma^2\big)\,,\end{aligned} \tag{6.6.15}$$

where the one-form ω_1 is defined by the condition

$$\mathrm{d}\omega_1 = r_1 r_2 r_3 \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{6.6.16}$$

The general formula for arbitrary finite γ_is is quite involved and we shall refrain from writing it down explicitly.

6.6.3 The emergent geometry

Expanding (6.6.13) to quadratic order in the deformation parameters and using (6.1.3) yields

$$e^{-\Phi} = \frac{4\pi N}{\lambda}\Big(1 + \frac{1}{2}\lambda\big(\gamma_1 r_2^2 r_3^2 + \gamma_2 r_3^2 r_1^2 + \gamma_3 r_1^2 r_2^2\big) + O\big(\lambda\gamma^4\big)\Big)\,. \tag{6.6.17}$$

When the background is a small deformation of the undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution, i.e. when $\lambda\gamma_i^2 \ll 1$, this is a perfect match with the supergravity solution (C.2.2) and (C.2.8), consistently with the discussion in section 6.2. Similarly, (6.6.15) and

(6.1.11) yield

$$\begin{aligned} B_{[2]} &= \frac{\lambda}{4\pi N}\operatorname{Im}\hat{C}_{[2]} \\ &= -\frac{\ell_{\mathrm{s}}^2\lambda}{2\pi}\bigl(\gamma_1 r_2^2 r_3^2\,\mathrm{d}\theta_2\wedge\mathrm{d}\theta_3 + \gamma_2 r_3^2 r_1^2\,\mathrm{d}\theta_3\wedge\mathrm{d}\theta_1 + \gamma_3 r_1^2 r_2^2\,\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\bigr) + O\bigl(\gamma^2\bigr)\,, \end{aligned} \tag{6.6.18}$$

$$\begin{aligned} C_{[2]} &= -\operatorname{Re}\hat{C}_{[2]} - \frac{\vartheta}{2\pi}B_{[2]} \\ &= -8N\ell_{\mathrm{s}}^2\omega_1\wedge\bigl(\gamma_1\mathrm{d}\theta_1+\gamma_2\mathrm{d}\theta_2+\gamma_3\mathrm{d}\theta_3\bigr) - \frac{\vartheta}{2\pi}B_{[2]} + O\bigl(\gamma^2\bigr)\,. \end{aligned} \tag{6.6.19}$$

After making the $\mathrm{SL}(2,\mathbb{R})$ transformation $C_{[0]}\mapsto C_{[0]}+\frac{\vartheta}{2\pi}$, $C_{[2]}\mapsto C_{[2]}-\frac{\vartheta}{2\pi}B_{[2]}$ to generalize the solution to an arbitrary ϑ-angle, we find again a match with the supergravity background (C.2.3) and (C.2.4) in the appropriate limit.

Actually, in the present case, it seems that the discussion of section 6.2 can be slightly refined. Indeed, because the imaginary part of c given by (6.6.10) is a constant, it turns out that the general matrix coordinates redefinitions do not act on $\operatorname{Re}\hat{F}_{[3]}$ [66]. This three-form is thus unambiguously fixed by our microscopic calculations, even when the perturbation with respect to the undeformed conformal $\mathcal{N}=4$ gauge theory is large. As a consequence, to compare with supergravity, we do not have to impose $\lambda\gamma_i^2$ to be small. The only relevant constraint is of course the validity of the supergravity solution itself, which is the weaker condition $\lambda\gamma_i^4\ll 1$ together with $\lambda\gg 1$. In this limit, we are allowed to expand the microscopic results as in (6.6.15), since $\gamma_i\ll 1$. However, we are not allowed to simplify the function G defined by (C.2.8) in the supergravity solution, because $\lambda\gamma_i^2$ may be large. Remarquably, we do find agreement with the microscopic prediction, because the real part of $\hat{C}_{[2]}$ is related to the right-hand side of (C.2.4) which does not depend on G!

6.6.4 Summary and discussion

Let us discuss here the slightly simpler $\mathcal{N}=1$ preserving case $\gamma=\gamma_1=\gamma_2=\gamma_3$. In $\mathcal{N}=1$ language, the $\mathcal{N}=4$ multiplet decomposes into one vector multiplet and three chiral multiplets Φ_1, Φ_2 and Φ_3. The β-deformation then simply amounts to replacing the $\mathcal{N}=4$ preserving superpotential term $\operatorname{tr}[\Phi_1,\Phi_2]\Phi_3$ by $\operatorname{tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2)$.

As we have shown, the large N solution of the microscopic theory yields

$$c = \frac{8\pi^2 N}{\lambda} + i\vartheta - N\ln\bigl(1-4(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)\sin^2(\pi\gamma)\bigr)\,. \tag{6.6.20}$$

Expanding to second order in the deformation parameter γ as required by the discussion in section 6.2, we obtain the prediction

$$e^{-\Phi} = \frac{4\pi N}{\lambda}\Bigl(1+\frac{1}{2}\lambda\gamma^2\bigl(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2\bigr) + O\bigl(\lambda\gamma^4\bigr)\Bigr)\,. \tag{6.6.21}$$

Using (6.6.15), the two-form $\hat{C}_{[2]}$ defined in (6.1.10) is found to be

$$\hat{C}_{[2]} = \frac{4N\ell_s^2}{\pi}\sin\big(2\pi\gamma\big)\bigg[G_1\wedge \mathrm{d}\theta_1 + G_2\wedge \mathrm{d}\theta_2 + G_3\wedge \mathrm{d}\theta_3 - \frac{i}{4}\frac{r_1^2r_2^2\mathrm{d}\theta_1\wedge \mathrm{d}\theta_2 + r_1^2r_3^2\mathrm{d}\theta_1\wedge \mathrm{d}\theta_3 + r_2^2r_3^2\mathrm{d}\theta_2\wedge \mathrm{d}\theta_3}{1-4(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)\sin^2(\pi\gamma)}\bigg]\,, \quad (6.6.22)$$

with

$$\mathrm{d}G_1 = \frac{r_1r_2r_3\big(r_1^2+(r_2^2+r_3^2)\cos(2\pi\gamma)\big)}{\big(1-4(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)\sin^2(\pi\gamma)\big)^2}\,\sin\theta\,\mathrm{d}\theta\wedge \mathrm{d}\phi\,, \quad (6.6.23)$$

$$\mathrm{d}G_2 = \frac{r_1r_2r_3\big(r_2^2+(r_1^2+r_3^2)\cos(2\pi\gamma)\big)}{\big(1-4(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)\sin^2(\pi\gamma)\big)^2}\,\sin\theta\,\mathrm{d}\theta\wedge \mathrm{d}\phi\,, \quad (6.6.24)$$

$$\mathrm{d}G_3 = \frac{r_1r_2r_3\big(r_3^2+(r_1^2+r_2^2)\cos(2\pi\gamma)\big)}{\big(1-4(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)\sin^2(\pi\gamma)\big)^2}\,\sin\theta\,\mathrm{d}\theta\wedge \mathrm{d}\phi\,. \quad (6.6.25)$$

To obtain a prediction for $B_{[2]}$ and $C_{[2]}$, we are instructed by the discussion in section 6.2 to expand to linear order in the deformation parameter γ. In this limit,

$$\mathrm{d}G_1 \simeq \mathrm{d}G_2 \simeq \mathrm{d}G_3 \simeq r_1r_2r_3\,\sin\theta\,\mathrm{d}\theta\wedge \mathrm{d}\phi = \mathrm{d}\omega_1 \quad (6.6.26)$$

and (6.1.11) then yields

$$C_{[2]} = -8N\ell_s^2\gamma\omega_1\wedge\big(\mathrm{d}\theta_1+\mathrm{d}\theta_2+\mathrm{d}\theta_3\big) + O\big(\gamma^2\big)\,, \quad (6.6.27)$$

$$B_{[2]} = -\frac{\ell_s^2\lambda}{2\pi}\,\gamma\big(r_1^2r_2^2\mathrm{d}\theta_1\wedge \mathrm{d}\theta_2 + r_1^2r_3^2\mathrm{d}\theta_1\wedge \mathrm{d}\theta_3 + r_2^2r_3^2\mathrm{d}\theta_2\wedge \mathrm{d}\theta_3\big) + O\big(\gamma^2\big)\,. \quad (6.6.28)$$

The supergravity solution is reviewed in appendix C.2 and can be trusted as long as the two conditions

$$\lambda \gg 1\,, \quad \lambda\gamma^4 \ll 1\,, \quad (6.6.29)$$

are satisfied. The discussion in section 6.2 implies that supergravity can be compared with the above microscopic solution only when the background is a small perturbation of the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ solution. This occurs when $\lambda\gamma^2 \ll 1$, in which case the functions $1/\sqrt{G}$ and $\sqrt{G}$ in equations (C.2.2) and (C.2.3) can be simplified. This yields a perfect match with (6.6.21), (6.6.27) and (6.6.28).

Chapter 7

Emergent geometry from D-Strings

In this chapter, we apply the general framework of chapter 5 to derive the emergent near-horizon geometry of a large number of D5-branes by studying the corresponding probe D-string world-sheet model. We start in section 7.1 by presenting the relevant D-string microscopic pre-geometric world-sheet theory. We solve the model at large N in section 7.3 and find that the solution is expressed as a classical action which contains the right dynamical fields to describe the motion of the D-strings in a ten-dimensional background. In section 7.4, we compare the expansion of this action around a flat world-sheet with the corresponding terms derived from the D-string action in arbitrary supergravity background. This allows us to identify the string-frame metric, dilaton and Ramond-Ramond three-form field-strength. The result matches perfectly the near-horizon supergravity solution sourced by the background D5-branes.

The branes are located in $\mathbf{R}^{10}$ according to table 7.1. We denote by w_I the two coordinates parallel to the D-strings and by $(z^i) = (x_\mu, y_m)$ the coordinates transverse to the D-strings. As usual, we also define the radial coordinate r by

$$r^2 = y_m y_m \,. \tag{7.0.1}$$

	1	2	3	4	5	6	7	8	9	10
D5	×	×	×	×	×	×				
D1	×	×								
	w_1	w_2	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4

(7.0.2)

Table 7.1: Location of the D1- and D5-branes in $\mathbf{R}^{10}$. The third row indicates the notation we use for the various types of coordinates.

7.1 The D-string microscopic action

In principle, we need the action describing the dynamics of the open string modes of a system composed of a fixed number K of D-strings and a large number N of D5-branes, in the appropriate decoupling limit explained in section 4.2. As we reviewed in subsection 3.3.4, the model preserves eight supercharges and there is an $\mathrm{SO}(2)\times\mathrm{SO}(4)\times\mathrm{SO}(4)'$ global symmetry group corresponding to rotations in spacetime preserving the brane configuration: SO(2) is associated with rotations on the D-string world-sheet, $\mathrm{SO}(4)'$ with rotations on the D5-brane world-volume transverse to the D-strings and SO(4) with rotations transverse to both the D-strings and the D5-branes.

As explained in section 3.3, this action could be studied by evaluating appropriate low-energy limits of open string disk diagrams with various boundary conditions. The result is a sum of a world-sheet action for the D-strings and a world-volume action for the D5-branes, with couplings between the D-string and D5-brane degrees of freedom. The naive action obtained in this way for the degrees of freedom living on the D5-branes would not be renormalizable and thus could be used only in the infrared. In other words, the full description of the D5-brane is not field theoretic. Fortunately, an explicit description of the D5-brane degrees of freedom and their couplings to the D-strings will not be required for our purposes, as we have explained in section 5.2: thanks to supersymmetry, these couplings are not expected to contribute to the terms in the effective action on which we shall focus. A similar non-renormalization theorem was discussed in [72] in the case of D-particles. We thus focus on the D-string world-sheet Lagrangian, without referring any longer to possible couplings to the D5-brane fields.

7.2 The decoupling limit

The relevant terms for the D-string world-sheet theory were obtained in subsection 3.3.4 by performing the dimensional reduction of the $\mathrm{U}(K)$ $\mathcal{N}=1$ gauge theory in six dimensions down to two dimensions. The theory contain one hypermultiplet in the adjoint, corresponding to D1/D1 string degrees of freedom, and N hypermultiplets in the fundamental corresponding to the D1/D5 strings. In this section, we implement the decoupling explained in section 4.2 on this Lagrangian.

The scalar fields a_μ and ϕ_m in the world-sheet Lagrangian (3.3.23) are associated with the motion of the D-strings parallel and transverse to the D5-branes respectively. The corresponding coordinates are

$$X_\mu = \ell_{\mathrm{s}}^2 a_\mu\,,\quad Y_m = \ell_{\mathrm{s}}^2 \phi_m\,. \tag{7.2.1}$$

To implement the decoupling of chapter 4, we take $\ell_{\mathrm{s}}\to 0$ while keeping X_μ, $Y_m/\ell_{\mathrm{s}}^2 = \phi_m$ and the six-dimensional 't Hooft coupling $N\ell_{\mathrm{s}}^2 g_{\mathrm{s}}$ fixed. Compatibility with the supersymmetry transformations presented in subsection 3.3.4 then require that the fermionic superpartners Λ_α and $\bar{\psi}^{\dot{\alpha}} = \ell_{\rm s}^2 \bar{\Lambda}^{\dot{\alpha}}$ of ϕ_m and X_μ respectively must also be kept fixed. Introducing $\psi_\alpha = \ell_{\rm s}^2 \Lambda_\alpha$, the Lagrangian (3.3.23) then simplifies in the scaling to

$$\begin{aligned} L_{\rm p} = \frac{1}{2\ell_{\rm s}^2 g_{\rm s}} \operatorname{tr}_{\mathrm{U}(K)} \Big(& 2\mathbf{1}_K + \nabla_I X_\mu \nabla_I X_\nu + 2i \big[X_\mu, X_\nu\big] D_{\mu\nu} - \ell_{\rm s}^{-4} \big[Y_m, X_\mu\big]\big[Y_m, X_\mu\big] \\ & - 2i \bar{\psi}^{\zeta}_{\dot{\alpha}} \nabla_{\bar{w}} \bar{\psi}^{\dot{\alpha}}_{\zeta} + 2i \bar{\psi}_{\dot{\alpha}\dot{\zeta}} \nabla_w \bar{\psi}^{\dot{\alpha}\dot{\zeta}} + 2\ell_{\rm s}^{-2} \bar{\psi}^{\zeta}_{\dot{\alpha}} \sigma_{m\zeta\dot{\zeta}} [Y_m, \bar{\psi}^{\dot{\alpha}\dot{\zeta}}] \\ & - 2i\ell_{\rm s}^{-2} \psi^{\alpha}_{\zeta} \sigma_{\mu\alpha\dot{\alpha}} \big[X_\mu, \bar{\psi}^{\dot{\alpha}\zeta}\big] + 2i\ell_{\rm s}^{-2} \psi^{\alpha\dot{\zeta}} \sigma_{\mu\alpha\dot{\alpha}} \big[X_\mu, \bar{\psi}^{\dot{\alpha}}_{\dot{\zeta}}\big] \Big) \\ & + \frac{1}{2} \nabla_I \tilde{q}^{\alpha f} \nabla_I q_{\alpha f} - \frac{1}{2\ell_{\rm s}^4} Y_m \tilde{q}^{\alpha f} Y_m q_{\alpha f} - \frac{1}{2\ell_{\rm s}^2} \tilde{\chi}^{f\zeta} \sigma_{m\zeta\dot{\zeta}} Y_m \chi^{\dot{\zeta}}_f + \frac{1}{2\ell_{\rm s}^2} \tilde{\chi}^{f}_{\dot{\zeta}} \bar{\sigma}^{\dot{\zeta}\zeta}_m Y_m \chi_{f\zeta} \\ & + i\tilde{\chi}^{f\zeta} \nabla_{\bar{w}} \chi_{f\zeta} - i\tilde{\chi}^{f}_{\dot{\zeta}} \nabla_w \chi^{\dot{\zeta}}_f - \frac{1}{\sqrt{2}\ell_{\rm s}^2} \tilde{q}^{\alpha f} \psi_{\alpha\zeta} \chi^{\zeta}_f + \frac{1}{\sqrt{2}\ell_{\rm s}^2} \tilde{q}^{\alpha f} \psi^{\dot{\zeta}}_{\alpha} \chi_{f\dot{\zeta}} \\ & - \frac{1}{\sqrt{2}\ell_{\rm s}^2} \tilde{\chi}^{\zeta f} \psi^{\alpha}_{\zeta} q_{\alpha f} + \frac{1}{\sqrt{2}\ell_{\rm s}^2} \tilde{\chi}^{f}_{\dot{\zeta}} \psi^{\alpha\dot{\zeta}} q_{\alpha f} + \frac{i}{2} \tilde{q}^{\alpha f} D_{\mu\nu} \sigma_{\mu\nu\alpha}{}^{\beta} q_{\beta f}\,. \end{aligned} \tag{7.2.2}$$

This Lagrangian will be our starting point for the probe D-string world-sheet theory.

It is important to emphasize that the scalar fields Y_m are non-dynamical auxiliary variables in (7.2.2). Indeed, their kinetic term is subleading in the decoupling limit, and thus the Y_ms could be trivially integrated out by solving their algebraic equation of motion. However, as explained in chapter 2 and in the next section, these fields become dynamical due to the quantum corrections and play a central role both in the mathematics and the physical interpretation of the solution of the model at large N.

7.3 The solution of the model at large N

We now solve the model defined by the Lagrangian (7.2.2). The crucial property is that the fields $(q, \tilde{q}, \chi, \tilde{\chi})$ carry only one $\mathrm{U}(N)$ index and are thus vector-like variables. The large N path integral over these fields can then always be performed exactly, using the standard techniques for large N vector models reviewed in chapter 2. In our case, the relevant auxiliary fields making the vector fields appear quadratically are precisely the variables $(Y_m, \psi_{\alpha\zeta}, \psi_{\alpha\dot{\zeta}}, D_{\mu\nu})$ which we have already included when writing (7.2.2). The path integral over the vector variables is then Gaussian. The result is an effective action for the auxiliary fields. Moreover, this effective action is automatically proportional to N because the vector fields have N components.

The resulting structure is thus perfectly consistent with the D-string seen as moving in a higher dimensional classical non-trivial background. Indeed, the fields Y_m can be interpreted as the emerging coordinates which behave classically at large N and the metric on the emerging space will be related to the kinetic term for the Y_m.

Let us now carry out this procedure explicitly for our model, mainly focusing on the case of a single D-string probe.

7.3.1 Integrating out

The effective action NS_{eff} is given by

$$e^{-NS_{\text{eff}}} = \int \mathrm{d}q\mathrm{d}\tilde{q}\mathrm{d}\chi\mathrm{d}\tilde{\chi}\; e^{-S_{\text{p}}}\,, \tag{7.3.1}$$

where S_{p} is the action for the Lagrangian (7.2.2). In order to derive the emergent geometry, we can focus on the bosonic part of the effective action and thus set the fermionic fields ψ and $\bar{\psi}$ to zero. Note, however, that computing the fermionic terms in the effective action could also be done.

The integral (7.3.1) yields

$$\begin{aligned} S_{\text{eff}}(A,X,Y,D) = \frac{1}{2N\ell_{\text{s}}^2 g_{\text{s}}}\int \mathrm{d}^2 w\; \mathrm{tr}_{\mathrm{U}(K)}\Big(2\mathbf{1}_K + \nabla_I X_\mu \nabla_I X_\mu + 2i\big[X_\mu, X_\nu\big]D_{\mu\nu} \\ - \ell_{\text{s}}^{-4}\big[Y_m, X_\mu\big]\big[Y_m, X_\mu\big]\Big) + \ln\Delta_{q,\tilde{q}} - \ln\Delta_{\chi,\tilde{\chi}}\,, \end{aligned} \tag{7.3.2}$$

where the determinants $\Delta_{q,\tilde{q}}$ and $\Delta_{\chi,\tilde{\chi}}$ are given by

$$\Delta_{q,\tilde{q}} = \det\big(-\mathbf{1}_K\otimes\mathbf{1}_2\,\nabla^2 + \ell_{\text{s}}^{-4}Y_mY_m\otimes\mathbf{1}_2 + iD_{\mu\nu}\otimes\sigma_{\mu\nu}\big)\,, \tag{7.3.3}$$

$$\Delta_{\chi,\tilde{\chi}} = \det\begin{pmatrix} -2i\mathbf{1}_K\otimes\mathbf{1}_2\,\nabla_{\bar{w}} & \ell_{\text{s}}^{-2}Y_m\otimes\sigma_m \\ -\ell_{\text{s}}^{-2}Y_m\otimes\bar{\sigma}_m & 2i\mathbf{1}_K\otimes\mathbf{1}_2\,\nabla_w \end{pmatrix}\,. \tag{7.3.4}$$

At large N, the field $D_{\mu\nu}$ is fixed in terms of the other variables by the saddle point equation

$$\frac{\delta S_{\text{eff}}}{\delta D_{\mu\nu}} = 0\,. \tag{7.3.5}$$

If we specialize to the case $K = 1$ of a single D-string probe, then the solution is simply $D_{\mu\nu} = 0$. This follows from the vanishing of the linear term in D in the expansion of (7.3.3) around $D = 0$ or, equivalently, from the commuting nature of the X_μ and SO(4) invariance. We thus get

$$\begin{aligned} S_{\text{eff}}(A,X,Y) = \frac{1}{N\ell_{\text{s}}^2 g_{\text{s}}}\int \mathrm{d}^2 w\;\Big(1 + \frac{1}{2}\partial_I X_\mu \partial_I X_\mu\Big) \\ + 2\ln\det\big(-\nabla^2 + \ell_{\text{s}}^{-4}Y_mY_m\big) - \ln\det\begin{pmatrix} -2i\mathbf{1}_2\nabla_{\bar{w}} & \ell_{\text{s}}^{-2}Y_m\sigma_m \\ -\ell_{\text{s}}^{-2}Y_m\bar{\sigma}_m & 2i\mathbf{1}_2\nabla_w \end{pmatrix}\,. \end{aligned} \tag{7.3.6}$$

7.3.2 The effective action up to cubic order

We are going to use (7.3.6) up to order three in an expansion in the constant field-strength F_{IJ} and around constant values of the coordinate world-sheet fields,

$$X_\mu = x_\mu + \ell_{\rm s}^2 \epsilon_\mu \,, \quad Y_m = y_m + \ell_{\rm s}^2 \epsilon_m \,. \tag{7.3.7}$$

Eventually, we shall match $NS_{\rm eff}$ up to this order in the next section with the D-string action in a general type IIB background.

The explicit computation of the expansion is straightforward. We write

$$\ln \Delta_{q,\tilde q} = 2 \ln\det K_B + 2\,{\rm tr}\ln(\mathbf{1} + K_B^{-1}\varphi)\,, \tag{7.3.8}$$

$$\ln \Delta_{\chi,\tilde\chi} = \ln\det K_F + {\rm tr}\ln(\mathbf{1} + K_F^{-1}\xi)\,, \tag{7.3.9}$$

in terms of the bosonic and fermionic propagators

$$K_B^{-1}(w,w') = \int \frac{{\rm d}^2 p}{(2\pi)^2} \frac{e^{ip\cdot(w-w')}}{p^2 + \ell_{\rm s}^{-4} r^2}\,, \tag{7.3.10}$$

$$K_F^{-1}(w,w') = \int \frac{{\rm d}^2 p}{(2\pi)^2} \frac{e^{ip\cdot(w-w')}}{p^2 + \ell_{\rm s}^{-4} r^2} \begin{pmatrix} 2\mathbf{1}_2 p_w & \ell_{\rm s}^{-2} y_m \sigma_m \\ -\ell_{\rm s}^{-2} y_m \bar\sigma_m & -2\mathbf{1}_2 p_{\bar w}\end{pmatrix} \tag{7.3.11}$$

where we use the radial coordinate r defined in (7.0.1) and with

$$\varphi \equiv 2\ell_{\rm s}^{-2} y_m \epsilon_m + \epsilon_m\epsilon_m - i\partial_I A_I - 2iA_I\partial_I + A_I A_I\,, \tag{7.3.12}$$

$$\xi \equiv \begin{pmatrix} 2\mathbf{1}_2 A_{\bar w} & \epsilon_m\sigma_m \\ -\epsilon_m\bar\sigma_m & -2\mathbf{1}_2 A_w \end{pmatrix}\,. \tag{7.3.13}$$

We then expand the traces in (7.3.8) and (7.3.9) using (6.4.12), and compute the resulting one-loop Feynman integrals. The computation are very similar to those presented in section 2.1, and we simply present the results.

At zeroth and first order, the contributions from the bosonic and fermionic determinants cancel each other and only the constant D-string tension term in (7.3.6) remains. At second and third order, we obtain a non-local effective action, which we write as a power series in derivatives by expanding the associated Feynman integrals for small external momenta. Overall, we get

$$\begin{aligned} NS_{\rm eff}(A,X,Y) = \int {\rm d}^2 w \Big(&\frac{1}{\ell_{\rm s}^2 g_{\rm s}} + \frac{\ell_{\rm s}^2}{2g_{\rm s}}\partial_I\epsilon_\mu\partial_I\epsilon_\mu + \frac{N\ell_{\rm s}^4}{4\pi r^2}\partial_I\epsilon_m\partial_I\epsilon_m + \frac{N\ell_{\rm s}^4}{8\pi r^2} F_{IJ}F_{IJ} \\ &- \frac{N\ell_{\rm s}^6}{2\pi r^4} y_m\epsilon_m\partial_I\epsilon_n\partial_I\epsilon_n - \frac{N\ell_{\rm s}^6}{4\pi r^4}\epsilon_m y_m F_{IJ}F_{IJ} - \frac{iN\ell_{\rm s}^6}{6\pi r^4}\epsilon_{IJ} y_m \epsilon_{mnlp}\epsilon_n\partial_I\epsilon_l\partial_J\epsilon_p\Big) + \cdots \end{aligned} \tag{7.3.14}$$

where the $\cdots$ stand for terms of quartic or higher order and terms with more than two derivatives. Of course, the result is consistent with the symmetries of the microscopic theory discussed in section 7.2, including the world-sheet parity which must come accompanied by a parity transformation in the directions y_m.

7.4 Extraction of the D5 background

As explained in section 3.2, the action for a D-string moving in a general type IIB supergravity background is the sum of a Dirac-Born-Infeld term and a Chern-Simons term,

$$S = \frac{1}{\ell_{\rm s}^2}\int \mathrm{d}^2\xi\, e^{-\Phi}\sqrt{\det\left[\mathrm{P}(G+B_{[2]})+\ell_{\rm s}^2 F\right]} + \frac{i}{\ell_{\rm s}^2}\int \Big[\mathrm{P}(C_{[0]}B_{[2]}+C_{[2]})+\ell_{\rm s}^2 C_{[0]}F\Big]\,, \tag{7.4.1}$$

where the fields Φ, G, $B_{[2]}$, $C_{[0]}$ and $C_{[2]}$ are the dilaton, string-frame metric, Kalb-Ramond two-form and Ramond-Ramond potentials respectively, F is the world-sheet field-strength and P denotes as usual the pull-back of the spacetime fields to the world-sheet. Working in the static gauge and writing the fields Z_i, $1\leq i\leq 8$, corresponding to the coordinates transverse to the D-string world-sheet as

$$Z_i = z_i + \ell_{\rm s}^2\epsilon_i\,, \tag{7.4.2}$$

we can expand (7.4.1) in powers of ϵ_i and F. Following the basic idea presented in chapter 5, this expansion should match with the similar expansion (7.3.14) of the effective action describing the solution of the large N microscopic model of the D-strings in the presence of the N D5-branes. Morevover, we should be able to derive the supergravity background sourced by the D5-branes from the coefficients in the expansion (7.3.14). Let us check that this is indeed the case.

The zeroth order Lagrangian derived in this way from (7.4.1) reads

$$L^{(0)} = \frac{1}{\ell_{\rm s}^2}\Big[e^{-\Phi}\sqrt{\det(G_{IJ}+B_{[2]IJ})}+\frac{i}{2}\epsilon^{IJ}\big(C_{[0]}B_{[2]IJ}+C_{[2]IJ}\big)\Big]\,, \tag{7.4.3}$$

where the capital Latin indices $1\leq I,J,\ldots\leq 2$ correspond as usual to the directions parallel to the D-string world-sheet. Matching with the microscopic result (7.3.14) and taking into account world-sheet parity invariance yields the conditions

$$e^{-\Phi}\sqrt{\det(G_{IJ}+B_{[2]IJ})} = \frac{1}{g_{\rm s}}\,,\quad C_{[0]}B_{[2]IJ}+C_{[2]IJ}=0\,, \tag{7.4.4}$$

where $g_{\rm s}=e^{\Phi_0}$. Taking these constraints into account, the Lagrangian derived from (7.4.1) at first order in ϵ is, up to an irrelevant total derivative,

$$L^{(1)} = \frac{1}{2}F_{IJ}\big(-E^{IJ}+iC_{[0]}\epsilon^{IJ}\big)\,, \tag{7.4.5}$$

where the matrix E^{IJ} is the inverse of $G_{IJ}+B_{[2]IJ}$. Using the fact that $L^{(1)}=0$ in (7.3.14) and world-sheet parity, we get

$$B_{[2]IJ}=0\,,\quad C_{[0]}=0\,. \tag{7.4.6}$$

The conditions (7.4.4) thus reduce to

$$e^{-\Phi}\sqrt{\det G_{IJ}} = \frac{1}{g_\mathrm{s}}\,, \quad C_{[2]IJ} = 0\,. \tag{7.4.7}$$

Taking these results into account as well as the fact that $G_{iI} = 0$ from the ISO(2) invariance of the world-sheet, the second order Lagrangian derived from (7.4.1) then reads

$$L^{(2)} = \ell_\mathrm{s}^2\Big[\big(\frac{1}{2}\tilde{G}^{IJ}G_{ij} + \tilde{G}^{I[J}\tilde{G}^{K]L}B_{[2]iL}B_{[2]jK}\big)\,\partial_I\varepsilon_i\partial_J\varepsilon_j \\ + \frac{1}{4}\tilde{G}^{IJ}\tilde{G}^{KL}\,F_{IK}F_{JL} - i\epsilon^{IJ}\partial_{[i}(C_{[2]})_{j]I}\,\varepsilon_i\partial_J\varepsilon_j\Big]\,, \tag{7.4.8}$$

where the matrix $\tilde{G}^{IJ}$ is the inverse of the two-by-two matrix G_{IJ}. Comparing with (7.3.14) and using again parity invariance yields

$$G_{IJ} = \sqrt{\frac{2\pi r^2}{\ell_\mathrm{s}^2 g_\mathrm{s} N}}\,\delta_{IJ}\,, \quad \partial_{[i}(C_{[2]})_{j]I} = 0 \tag{7.4.9}$$

and then

$$e^{\phi} \equiv e^{\Phi-\Phi_0} = \sqrt{\frac{2\pi r^2}{\ell_\mathrm{s}^2 g_\mathrm{s} N}} \tag{7.4.10}$$

by using (7.4.7). Comparing the terms $\partial_I\varepsilon_i\partial_J\varepsilon_j$ in (7.3.14) and (7.4.8), using (7.4.9) and the ISO(2) symmetry to fix $B_{[2]Ii} = 0$, we find

$$G_{\mu\nu} = \sqrt{\frac{2\pi r^2}{\ell_\mathrm{s}^2 g_\mathrm{s} N}}\delta_{\mu\nu}\,, \quad G_{mn} = \sqrt{\frac{\ell_\mathrm{s}^2 g_\mathrm{s} N}{2\pi r^2}}\delta_{mn}\,. \tag{7.4.11}$$

We thus find that the components G_{IJ} and $G_{\mu\nu}$ match, which shows that the metric has the expected SO(6) isometry of the background sourced by D5-branes. We can continue the same analysis at third order. Using the constraints on the background that we have already derived, (7.4.1) yields the third order Lagrangian up to two derivative terms,

$$L^{(3)} = \frac{\ell_\mathrm{s}^4}{2}\Big[\partial_i\big(\tilde{G}^{IJ}G_{jk}\big)\,\varepsilon_i\partial_I\varepsilon_j\partial_J\varepsilon_k + i\epsilon^{IJ}\partial_{[i}(C_{[2]})_{jk]}\,\varepsilon_i\partial_I\varepsilon_j\partial_J\varepsilon_k \\ + \frac{1}{2}\partial_i(\tilde{G}^{IJ}\tilde{G}^{KL})\,\varepsilon_i F_{IL}F_{JK} + \tilde{G}^{IK}\tilde{G}^{JL}B_{[2]ij}\,F_{KL}\partial_I\varepsilon_i\partial_J\varepsilon_j\Big]\,. \tag{7.4.12}$$

Matching with (7.3.14) and using the ISO(2) invariance and SO(6) isometry of the background, we then obtain

$$B_{[2]ij} = 0\,, \quad F_{[3]} = \mathrm{d}C_{[2]} = \frac{1}{6g_\mathrm{s}}\epsilon_{mnlp}\partial_p e^{-2\phi}\mathrm{d}y_m\wedge\mathrm{d}y_n\wedge\mathrm{d}y_l\,. \tag{7.4.13}$$

Overall, we have derived the following type IIB supergravity background

$$e^{\phi} = \sqrt{\frac{2\pi r^2}{\ell_{\rm s}^2 g_s N}}\,, \quad {\rm d}s^2 = e^{\phi}({\rm d}w_I{\rm d}w_I + {\rm d}x_\mu{\rm d}x_\mu) + e^{-\phi}{\rm d}y_m{\rm d}y_m\,,$$
$$F_{[3]} = {\rm d}C_{[2]} = \frac{1}{3!g_{\rm s}}\epsilon_{mnlp}\partial_p e^{-2\phi}{\rm d}y_m \wedge {\rm d}y_n \wedge {\rm d}y_l\,, \quad B_{[2]} = C_{[0]} = 0\,, \tag{7.4.14}$$

which perfectly matches with the near-horizon geometry of N D5-branes, see (4.2.9a), (4.2.9c) and (4.2.10).

Conclusions and Outlook

In the first part of this thesis, we reviewed various topics that are necessary to understand the developments of part 2, which is based on the two published papers [1, 2]. Let us quickly summarize the key steps of what we have done. We explained in chapter 2 how to solve a simple vector model at large N by introducing an auxiliary scalar field, that makes the original vector field appears only quadratically in the Lagrangian. The effective action for the auxiliary field obtained by trivially integrating out the original fields has two very important features: first, it comes with a global factor of N, and thus *the auxiliary scalar field is classical at large N*. Second, *the scalar field becomes dynamical*. In fact, since the integration over the original fields is *exact*, the dynamics of the scalar field in the effective action automatically includes the effects of diagrams of any loop order in the original theory; in other words, *the auxiliary field acquire a non-trivial dynamics by taking into account non-perturbative quantum effects of the original vector fields*.

It is essentially these crucial properties of vector models at large N that allows us to understand how classical, dynamical dimensions may emerge in a strongly coupled field theory. For the concrete examples of part 2, the vector-like models that we considered are built from the low-energy world-volume actions describing probe D-branes in the presence of large number N of higher-dimensional background D-branes. The vector variables are here represented by fields arising from the strings stretched between the probes and the background D-branes. In the decoupling limit, where $\alpha' \to 0$ while the distance $r \sim \alpha'$ between the probe branes and the background branes goes to zero, the fields describing fluctuations of the probes in transverse space become auxiliary because their kinetic terms are subleading in the small α' limit, as we have explicitly seen in chapter 7 in the case of D-string probes. The probes then effectively live in a flat, lower-dimensional spacetime corresponding to the background brane world-volume. Moreover, all the vector-like fields appear quadratically in the low-energy action thanks to the auxiliary fields. According to our previous discussion about vector models, one should thus keep the auxiliary *scalar* fields, as they are the perfect, natural candidates for the emergent dimensions. To see if this works, we solve the low-energy world-volume theory by integrating out the vector-like fields, producing the effective action describing the probe branes. Interestingly, this effective action precisely match with the non-abelian D-brane action describing the probes moving in a *higher-dimensional, curved geometry*. By reading off the coefficient of various terms in the effective action, we are able to determine the full emergent geometry, including the metric, the dilaton, as well as various non-trivial form fields. In each and every examples that we consider, the spacetime geometry that we extract in this way perfectly match with the expected supergravity solutions.[1]

In conclusion, the results of this thesis strongly support the validity of the emergent geometry framework described in [51] and reviewed in chapter 5. One of the main lesson is the following: *space and gravity are intrinsically quantum effects*. As we reviewed in chapter 4, this remarkable property was already present in the gauge/gravity duality, but in the light of the examples studied in this thesis (as well as those in [51,53]), this property is now manifest. As many emergent phenomenon in physics, space and gravity correspond to an effective description of a more fundamental and highly non-trivial theory, which in this case is a strongly coupled quantum gauge theory. In particular, it is meaningless to consider the "quantization of the gravitational field."

Let us now turn to possible future directions. At the technical level, it would be interesting to understand presicely why the vacuum expectation value of the operator $\mathscr{D}$, defined in section 5.2 (see in particular equation (5.2.3)) and which allows us to compute the probe effective action, is not quantum corrected in all the examples we considered. As we explained, this property is strongly supported by our ability to derive the correct supergravity backgrounds. Let us also repeat that in the case of D-instantons probing the D3-brane geometry, conformal invariance is enough to prove this property, as the expectation value $\langle \mathscr{D} \rangle$ is expressed in terms of one-point functions of local operators on the D3-branes [51]. On the other hand, in the non-conformal cases that we considered, there were eight conserved supercharges and it is very likely that this is enough to prove the non-renormalization of $\mathscr{D}$.

It would also be interesting to find other models that could be studied along the same lines as those presented in part 2. In particular, one could consider to use D-instantons to probe other deformations of $\mathcal{N} = 4$ super-Yang-Mills theory like for instance the dipole deformation[2] [73–76]. This field theory can be obtained by introducing a deformed product for the fields charged under an $\mathrm{SO}(2) \sim \mathrm{U}(1)$ subgroup of the spacetime symmetry group and a U(1) factor of the total R-symmetry group. The simpler modification of the probe is obtained by applying this deformation directly on the undeformed probe theory, similarly to what we have done in the non-commutative and β-deformations studied in sections 6.5 and 6.6 respectively. The resulting effective action for the scalar auxiliary fields should thus allows us to read-off the dual supergravity solution (see [62, 74, 75, 77] for related works).

We could aslo study how the D3-brane geometry can be derived using D3-brane probes. The action to study can be obtained by splitting each adjoint field of $\mathcal{N} = 4$ super-Yang-Mills with $\mathrm{U}(N+K)$ gauge group into $N \times N$ and $K \times K$ square matrices

[1] Up to limitations that we state clearly when the probes are D-instantons.

[2] Thanks to Wei Song for suggesting me this possibility.

together with $N \times K$ rectangular matrices, which corresponds to the fields associated to the probe/background strings. These $N \times K$ matrices are transforming under the fundamental (or anti-fundamental) representation of $\mathrm{U}(N)$ and are thus the vector variables which should be intergrated out by introducing suitable auxiliary fields. Sending N to infinity and assuming that the operator $\mathscr{D}$ associated with this system is not quantum corrected, which amounts to set the $\mathrm{U}(N)$-adjoint fields to zero in the vacuum expectation value $\langle\mathscr{D}\rangle$, one should then be able to read-off the self-dual D3-brane supergravity solution, providing a new check of the framework.

A more ambitious goal would be to address questions related to black hole physics in the emergent geometry picture. According to the state/geometry relation (5.2.3), this could be achieved in principle by computing the effective action for the probes when the operator $\mathscr{D}$ is evaluated in a thermal state of the background brane world-volume theory. Since there are no conserved supercharges in this state, the expectation value $\langle\mathscr{D}\rangle$ is not simply given by its classical value, and more work is necessary to compute the terms in the effective action required to read-off the dual geometry.

One could also probe the geometry sourced by the D1/D5 system, where the D1-branes are parallel to the D5-branes and wraped on a circle S^1 while the other directions parallel to the D5-branes are wraped on a four-torus T^4. When the quantized momentum along the circle S^1 is non-zero, the classical geometry produced by this system is a three-charge black hole with non-zero horizon area. Moreover, the system preserves one quarter of the original supercharges and the near-horizon region contains a BTZ black hole with non-zero entropy [78].[3] This system could be probed using a fixed number of D-strings wraped on the S^1 (see [79] for a similar set-up). Since the full system is supersymmetric, we expect that the expectation value $\langle\mathscr{D}\rangle$ is not quantum corrected.

[3]Thanks to Geoffrey Compère for various discussions about this set-up.

Appendices

Appendix A

Notations, conventions and useful formulas

In this appendix, we specify all our notations and conventions. We also provide various identities that are used in the main text.

A.1 General conventions

We work in the Euclidean throughout all chapters. The coordinates in $\mathbf{R}^D$ are denoted by x^M or z^M with $1 \leq M \leq D$. When we consider a flat p-brane (that can be a Dp-brane or not), we choose the coordinates x^M such that the brane extends in the first $d \equiv p+1$ directions. We can therefore denote by x^μ with $1 \leq \mu \leq d$ and $y^m = x^{d+m}$ with $1 \leq m \leq D-d$ the coordinates parallel and transverse to the brane respectively. The radial coordinate r in transverse space is defined by $r^2 = y^m y^m$.

We use the standard Pauli matrices $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.1.1}$$

They satisfy

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i\epsilon_{ijk}\, \sigma_k \, , \tag{A.1.2}$$

where $i, j, k = 1, 2, 3$ and ϵ_{ijk} is the completely antisymmetric tensor such that $\epsilon_{123} = +1$.

Our conventions in string theory are the same as in the String Theory books by Polchinski, see [3, 4]. In particular, the dilaton Φ is such that the constant κ, related to the ten-dimensional Newton's constant G_N by

$$2\kappa^2 = 16\pi G_N \, , \tag{A.1.3}$$

is given by (see (3.2.44))

$$\kappa = 8\pi^{7/2} \alpha'^2 g_{\mathrm{s}} \, . \tag{A.1.4}$$

The parameter T_p appearing in the D-brane action (3.2.50a) is related to the D-brane tension τ_p by

$$T_p = \tau_p g_\mathrm{s}\,, \tag{A.1.5}$$

where g_s is the string coupling constant defined as the ratio τ_F/τ_1, where $\tau_F = (2\pi\alpha')^{-1}$ is the fundamental string tension and τ_1 is the D-string tension. The explicit value of T_p reads (see (3.2.45))

$$T_p = \tau_p g_\mathrm{s} = \frac{1}{(2\pi)^p \alpha'^{(p+1)/2}} = \frac{1}{(2\pi)^{\frac{p-1}{2}} \ell_\mathrm{s}^{1+p}}\,, \tag{A.1.6}$$

where ℓ_s is the fundamental string length defined by

$$\ell_\mathrm{s}^2 \equiv 2\pi\alpha'\,. \tag{A.1.7}$$

In various expressions of chapters 3 and 4 we use the constant k, which is given in term of the p-brane tension τ_p by (3.1.32). In the case of Dp-branes in superstring theory, it reads (see (3.2.46))

$$k = 2^{5-p}\pi^{\frac{5-p}{2}} g_\mathrm{s} \alpha'^{\frac{7-p}{2}} \Gamma(\frac{7-p}{2}) = 2^{\frac{3-p}{2}}\pi^{-1} g_\mathrm{s} \ell_\mathrm{s}^{7-p} \Gamma(\frac{7-p}{2})\,. \tag{A.1.8}$$

The Yang-Mills coupling g_YM for the low-energy world-volume action of a Dp-brane is given by (see (3.2.47))

$$g_\mathrm{YM}^2 = \frac{1}{(2\pi\alpha')^2 \tau_p} = (2\pi)^{p-2} g_\mathrm{s} \alpha'^{\frac{p-3}{2}} = (2\pi)^{\frac{p-1}{2}} g_\mathrm{s} \ell_\mathrm{s}^{p-3}\,. \tag{A.1.9}$$

A.2 Diffential Geometry

Hodge dual

The Hodge $\star$-operator is defined as the linear operator such that

$$\star\,(\mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_r}) \equiv \frac{\sqrt{\det g}}{(D-r)!} \epsilon^{M_1 \dots M_r}{}_{M_{r+1} \dots M_D}\, \mathrm{d}x^{M_{r+1}} \wedge \cdots \wedge \mathrm{d}x^{M_D}\,, \tag{A.2.1}$$

where $\det g$ is the determinant of the metric g_{MN}. When acting on a p-form, its square is proportional to the identity:

$$\star^2 = (-1)^{p(D-p)}\,. \tag{A.2.2}$$

Let $g'_{MN} = \Lambda g_{MN}$ for some function Λ. Then the Hodge $\star$-operator associated to the metric g' is denoted by $\star'$ and is proportional to $\star$:

$$\star = \Lambda^{p-D/2} \star'\,, \tag{A.2.3}$$

where p is the degree of the form on which the $\star$-operator acts.

Integration

Integration of an n-form ω on a n-dimensional manifold $\mathcal{M}$ is defined with the following convention. For simplicity, we assume that $x^{M_1}, \ldots, x^{M_n}$ are coordinates on the whole manifold $\mathcal{M}$. Then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{M}} \frac{1}{n!} \omega_{M_1 \cdots M_n} \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_n} \equiv \int_{\mathbb{R}^n} \omega_{1 \cdots n} \; \mathrm{d}^n x \,, \tag{A.2.4}$$

where $\mathrm{d}^n x$ is the Lebesgue measure on $\mathbf{R}^n$.

Levi-Civita tensor

For any dimension D, we define the completely antisymmetric Levi-Civita tensor $\epsilon_{M_1 \ldots M_D}$ by

$$\epsilon_{1 \ldots D} = +1 \tag{A.2.5}$$

Indices are raised using the inverse metric g^{MN} in a standard way: for instance,

$$\epsilon^{M_1}{}_{M_2 \cdots M_n} \equiv g^{M_1 N_1} \epsilon_{N_1 M_2 \ldots M_D} \,. \tag{A.2.6}$$

As a consequence, we have

$$\epsilon^{1 \ldots D} = \frac{1}{\det g} \,. \tag{A.2.7}$$

A.3 Indices, fields and representations

For maximum clarity, the notations for the fields, indices and the associated transformation laws for the chapter 6 and 7 are presented on two separated tables, see table A.1 and table A.2 respectively.

A.4 $D = 4$ algebra

There are several places where SO(4)s appear. In the D(–1)/D3 system, there is one SO(4) corresponding to rotation in the directions parallel to the D3. In the D1/D5 system, there are *two* internal SO(4) symmetry groups. One of these corresponds to the rotations in directions transverse to both the D-strings and the D5-branes: we denote it simply by SO(4). The second, that we write SO(4)′ to avoid any ambiguity, corresponds to rotations in the directions transverse to the D-strings and parallel to the D5-branes. For maximum clarity, we use different indices for the representations of SO(4) and SO(4)′:

	Spin(4)	SU(4)	U(N)	U(K)
$\alpha, \beta, ...$ (upper or lower)	$(1/2, 0)$	**1**	**1**	**1**
$\dot{\alpha}, \dot{\beta}, ...$ (upper or lower)	$(0, 1/2)$	**1**	**1**	**1**
$\mu, \nu, ...$	$(1/2, 1/2)$	**1**	**1**	**1**
$a, b, ...$ (lower)	$(0, 0)$	**4**	**1**	**1**
$a, b, ...$ (upper)	$(0, 0)$	$\bar{\mathbf{4}}$	**1**	**1**
$A, B, ...$	$(0, 0)$	**6**	**1**	**1**
$f, f', ...$ (lower)	$(0, 0)$	**1**	**N**	**1**
$f, f', ...$ (upper)	$(0, 0)$	**1**	$\bar{\mathbf{N}}$	**1**
$i, j, ...$ (lower)	$(0, 0)$	**1**	**1**	**K**
$i, j, ...$ (upper)	$(0, 0)$	**1**	**1**	$\bar{\mathbf{K}}$
$X_{\mu i}{}^{j} = \ell_{\mathrm{s}}^2 A_{\mu i}{}^{j}$	$(1/2, 1/2)$	**1**	**1**	**Adj**
$Y_{Ai}{}^{j} = \ell_{\mathrm{s}}^2 \phi_{Ai}{}^{j}$	$(0, 0)$	**6**	**1**	**Adj**
$\psi_{\alpha a i}{}^{j} = \ell_{\mathrm{s}}^2 \Lambda_{\alpha a i}{}^{j}$	$(1/2, 0)$	**4**	**1**	**Adj**
$\bar{\psi}^{\dot{\alpha} a j}{}_{i} = \ell_{\mathrm{s}}^2 \bar{\Lambda}^{\dot{\alpha} a j}{}_{i}$	$(0, 1/2)$	$\bar{\mathbf{4}}$	**1**	**Adj**
$D_{\mu\nu i}{}^{j}$	$(1, 0)$	**1**	**1**	**Adj**
$q_{\alpha f i}$	$(1/2, 0)$	**1**	**N**	**K**
$\tilde{q}^{\alpha f i}$	$(1/2, 0)$	**1**	$\bar{\mathbf{N}}$	$\bar{\mathbf{K}}$
$\chi^{a}{}_{fi}$	$(0, 0)$	$\bar{\mathbf{4}}$	**N**	**K**
$\tilde{\chi}^{afi}$	$(0, 0)$	$\bar{\mathbf{4}}$	$\bar{\mathbf{N}}$	$\bar{\mathbf{K}}$

Table A.1: Conventions for the transformation laws of indices and moduli relevant for **chapter 6**. For maximum clarity, we have indicated all the indices associated to each modulus, whereas in the main text the gauge U(N) and U(K) indices are usually suppressed. The representations of $\mathrm{Spin}(4) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ are indicated according to the spin in each SU(2) factor. The $(1/2, 1/2)$ of $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ and the **6** of $\mathrm{SU}(4) = \mathrm{Spin}(6)$ correspond to the fundamental representations of SO(4) and SO(6) respectively.

	Spin(4)	Spin(4)′	U(1)	U(N)	U(K)
$I, J, \ldots$	$(0,0)$	$(0,0)$	1	**1**	**1**
$\mu, \nu, \ldots$	$(1/2,1/2)$	$(0,0)$	0	**1**	**1**
$m, n, \ldots$	$(0,0)$	$(1/2,1/2)$	0	**1**	**1**
$\alpha, \beta, \ldots$ (upper or lower)	$(1/2,0)$	$(0,0)$	0	**1**	**1**
$\dot\alpha, \dot\beta, \ldots$ (upper or lower)	$(0,1/2)$	$(0,0)$	0	**1**	**1**
$\zeta, \xi, \ldots$ (upper or lower)	$(0,0)$	$(1/2,0)$	0	**1**	**1**
$\dot\zeta, \dot\xi, \ldots$ (upper or lower)	$(0,0)$	$(0,1/2)$	0	**1**	**1**
$f, f', \ldots$ (lower)	$(0,0)$	$(0,0)$	0	**N**	**1**
$f, f', \ldots$ (upper)	$(0,0)$	$(0,0)$	0	$\bar{\mathbf{N}}$	**1**
$i, j, \ldots$ (lower)	$(0,0)$	$(0,0)$	0	**1**	**K**
$i, j, \ldots$ (upper)	$(0,0)$	$(0,0)$	0	**1**	$\bar{\mathbf{K}}$
A_I	$(0,0)$	$(0,0)$	1	**1**	**Adj**
${X_{\mu i}}^j = \ell_s^2 {A_{\mu i}}^j$	$(1/2,1/2)$	$(0,0)$	0	**1**	**Adj**
${Y_{mi}}^j = \ell_s^2 {\phi_{mi}}^j$	$(0,0)$	$(1/2,1/2)$	0	**1**	**Adj**
${\psi_{\alpha\zeta i}}^j = \ell_s^2 {\Lambda_{\alpha\zeta i}}^j$	$(1/2,0)$	$(1/2,0)$	$1/2$	**1**	**Adj**
${\psi_{\alpha\dot\zeta i}}^j = \ell_s^2 {\Lambda_{\alpha\dot\zeta i}}^j$	$(1/2,0)$	$(0,1/2)$	$-1/2$	**1**	**Adj**
${\bar\psi^{\dot\alpha}{}_{\zeta i}}^j = \ell_s^2 {\bar\Lambda^{\dot\alpha}{}_{\zeta i}}^j$	$(0,1/2)$	$(1/2,0)$	$-1/2$	**1**	**Adj**
${\bar\psi^{\dot\alpha}{}_{\dot\zeta i}}^j = \ell_s^2 {\bar\Lambda^{\dot\alpha}{}_{\dot\zeta i}}^j$	$(0,1/2)$	$(0,1/2)$	$1/2$	**1**	**Adj**
${D_{\mu\nu i}}^j$	$(1,0)$	$(0,0)$	0	**1**	**Adj**
$q_{\alpha f i}$	$(1/2,0)$	$(0,0)$	0	**N**	**K**
$\tilde q^{\alpha f i}$	$(1/2,0)$	$(0,0)$	0	$\bar{\mathbf{N}}$	$\bar{\mathbf{K}}$
$\chi_{\zeta f i}$	$(0,0)$	$(1/2,0)$	$-1/2$	**N**	**K**
$\chi_{\dot\zeta f i}$	$(0,0)$	$(0,1/2)$	$1/2$	**N**	**K**
$\tilde\chi_\zeta{}^{fi}$	$(0,0)$	$(1/2,0)$	$-1/2$	$\bar{\mathbf{N}}$	$\bar{\mathbf{K}}$
$\tilde\chi_{\dot\zeta}{}^{fi}$	$(0,0)$	$(0,1/2)$	$1/2$	$\bar{\mathbf{N}}$	$\bar{\mathbf{K}}$

Table A.2: Conventions for the transformation laws of indices and fields relevant for **chapter 7**. For maximum clarity, we have indicated all the indices associated to each field, whereas in the main text the gauge U(N) and U(K) indices are usually suppressed. The representations of $\text{Spin}(4) = \text{SU}(2)_+ \times \text{SU}(2)_-$ and $\text{Spin}(4)' = \text{SU}(2)'_+ \times \text{SU}(2)'_-$ are indicated according to the spin in each SU(2) factor. The $(1/2, 1/2)$ of $\text{SU}(2)_+ \times \text{SU}(2)_-$ corresponds to the fundamental representations of SO(4). The U(1) group corresponds to the world-sheet rotations under which positive and negative chirality spinors have charge $1/2$ and $-1/2$ respectively.

- SO(4): for the vectors, we denote the components using the indices $1 \leq \mu, \nu \ldots \leq 4$, while we use the indices $1 \leq \alpha, \beta, \ldots \leq 2$ and $1 \leq \dot{\alpha}, \dot{\beta}, \ldots \leq 2$ for the left- and right-handed Weyl spinors respectively.
- SO(4)′: for the vectors, we denote the components using the indices $1 \leq m, n \ldots \leq 4$, while we use the indices $1 \leq \zeta, \xi, \ldots \leq 2$ and $1 \leq \dot{\zeta}, \dot{\xi}, \ldots \leq 2$ for the left- and right-handed Weyl spinors respectively.

We define

$$\sigma_{\mu\alpha\dot{\alpha}} = (\vec{\sigma}, -i\mathbf{1}_{2\times 2})_{\alpha\dot{\alpha}}\,, \quad \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha} = (-\vec{\sigma}, -i\mathbf{1}_{2\times 2})^{\dot{\alpha}\alpha} \tag{A.4.1}$$

where $\vec{\sigma} = (\sigma_i)$ are the three Pauli matrices (A.1.1) and

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)\,, \quad \bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)\,. \tag{A.4.2}$$

The following identity is very useful (in particular for the computations in chapter 6):

$$\sigma_{\mu\nu}\sigma_{\rho\kappa} = \frac{1}{4}(-\epsilon_{\mu\nu\rho\kappa} + \delta_{\nu\rho}\delta_{\mu\kappa} - \delta_{\mu\rho}\delta_{\nu\kappa})\mathbf{1}_2 + \left(\delta_{\kappa[\nu}\sigma_{\mu]\rho} - \delta_{\rho[\nu}\sigma_{\mu]\kappa}\right)\,, \tag{A.4.3}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor with $\epsilon_{1234} = +1$.

We denote by an upper "+" the projection of an antisymmetric tensor on its self-dual part,

$$a^+_{\mu\nu} \equiv \frac{1}{2}(a_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\kappa}a_{\rho\kappa})\,. \tag{A.4.4}$$

With these definitions $\sigma_{\mu\nu}$ is self-dual,

$$\sigma_{\mu\nu} = \sigma^+_{\mu\nu}\,. \tag{A.4.5}$$

We have the following useful identities,

$$\det(\mathbf{1}_2 + a_{\mu\nu}\sigma_{\mu\nu}) = 1 + a_+^2\,, \tag{A.4.6}$$

$$\left(\mathbf{1}_2 + a_{\mu\nu}\sigma_{\mu\nu}\right)^{-1} = \frac{\mathbf{1}_2 - a_{\rho\sigma}\sigma_{\rho\sigma}}{1 + a_+^2}\,, \tag{A.4.7}$$

where

$$a_+^2 \equiv a^+_{\mu\nu}a^+_{\mu\nu}\,. \tag{A.4.8}$$

For the computation of the effective action at second and third order in chapter 7, we use the following identities:

$$\mathrm{tr}(\sigma_\mu\bar{\sigma}_\nu\sigma_\rho\bar{\sigma}_\lambda) = 2(\delta_{\mu\nu}\delta_{\rho\lambda} - \delta_{\mu\rho}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\rho\nu} - \epsilon_{\mu\nu\rho\lambda})\,, \tag{A.4.9}$$

$$a_{\mu_1}a_{\mu_2}a_{\mu_3}\,\mathrm{tr}(\sigma_{\mu_1}\bar{\sigma}_{\nu_1}\sigma_{\mu_2}\bar{\sigma}_{\nu_2}\sigma_{\mu_3}\bar{\sigma}_{\nu_3} + \bar{\sigma}_{\mu_1}\sigma_{\nu_1}\bar{\sigma}_{\mu_2}\sigma_{\nu_2}\bar{\sigma}_{\mu_3}\sigma_{\nu_3}) = 4\left[a^2(a_{\nu_1}\delta_{\nu_2\nu_3} + a_{\nu_2}\delta_{\nu_1\nu_3} + a_{\nu_3}\delta_{\nu_1\nu_2}) - 4a_{\nu_1}a_{\nu_2}a_{\nu_3}\right]\,, \tag{A.4.10}$$

for any numbers a_μ and where we set $a^2 \equiv a_\mu a_\mu$.

In the Weyl basis, the generators $S^{(4)}_{\mu\nu}$ of the spinor representation in four dimensions satisfying the SO(4) algebra are

$$S^{(4)}_{\mu\nu} = -i \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma}_{\mu\nu} \end{pmatrix} . \tag{A.4.11}$$

We also define the Levi-Civita symbol $(\epsilon_{\dot\zeta\dot\xi})$ and its inverse $(\epsilon^{\zeta\xi})$ by

$$(\epsilon^{\zeta\xi}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad (\epsilon_{\dot\zeta\dot\xi}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{A.4.12}$$

Note that this is *not* the same object than the Levi-Civita *tensor* in two dimensions, that we decided to define with the opposite sign, see (A.2.5). The matrices (A.4.12) are used to raise and lower Weyl spinor indices according to the rules

$$\lambda'^{\zeta} \equiv \epsilon^{\zeta\xi}\lambda'_{\xi} , \qquad \chi'_{\dot\zeta} \equiv \epsilon_{\dot\zeta\dot\xi}\chi'^{\dot\xi} . \tag{A.4.13}$$

A.5 $D = 6$ algebra

In subsections 3.3.2, 3.3.3 and 3.3.4, the *spacetime* symmetry group is SO(6). Its vectors are labelled by $1 \le r, s, \ldots \le 6$ while the Weyl spinors are labelle by $1 \le a, b, \ldots \le 4$.
In subsection 3.3.5 and in chapter 6, there is an *internal* SO(6) symmetry group. Its vectors are labelled by $1 \le A, B, \ldots \le 6$ while the Weyl spinors are labelle by $1 \le a, b, \ldots \le 4$.

To construct a Weyl representation of the six-dimensional Clifford algebra, we use the following matrices Σ_A:

$$\Sigma_1 = -i\sigma_2 \otimes \sigma_3 , \qquad \Sigma_2 = \sigma_2 \otimes \mathbf{1}_2 , \qquad \Sigma_3 = -i\mathbf{1}_2 \otimes \sigma_2 , \tag{A.5.1a}$$

$$\Sigma_4 = \sigma_3 \otimes \sigma_2 , \qquad \Sigma_5 = -i\sigma_2 \otimes \sigma_1 , \quad \Sigma_6 = \sigma_1 \otimes \sigma_2 , \tag{A.5.1b}$$

and

$$\bar\Sigma_A = \Sigma_A^\dagger . \tag{A.5.2}$$

Explictly,

$$\Sigma_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \ \Sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \ \Sigma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ,$$
$$\Sigma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} , \ \Sigma_5 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \ \Sigma_6 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} . \tag{A.5.3}$$

These matrices satisfy the algebra

$$\Sigma_A\bar{\Sigma}_B + \Sigma_B\bar{\Sigma}_A = 2\delta_{AB}\mathbf{1}_4 \tag{A.5.4}$$

as well as the relations

$$\bar{\Sigma}_A^{ab} = \frac{1}{2}\epsilon^{abcd}\Sigma_{Acd}\,, \quad \Sigma_{Aab} = \frac{1}{2}\epsilon_{abcd}\bar{\Sigma}_A^{cd} \tag{A.5.5}$$

where the ϵs are as usual the completely antisymmetric symbols with $\epsilon_{1234} = \epsilon^{1234} = +1$. Euclidean six-dimensional Dirac matrices, satisfying

$$\big\{\Gamma_A, \Gamma_B\big\} = 2\delta_{AB}\,, \tag{A.5.6}$$

can then be defined by

$$\Gamma_A = \begin{pmatrix} 0 & \Sigma_A \\ \bar{\Sigma}_A & 0 \end{pmatrix}\,. \tag{A.5.7}$$

If $\vec{v} = (v_A)_{1\leq A\leq 6}$ is a six-dimensional vector, one can check that

$$\det(v_A\Sigma_A) = \vec{v}^{\,4}\,, \tag{A.5.8}$$

$$(v_A\Sigma_A)^{-1} = \frac{v_A\bar{\Sigma}_A}{\vec{v}^{\,2}}\,\cdot \tag{A.5.9}$$

In sections 6.4 and 6.5, we have to compute the expansion of some determinants of the form

$$\ln\det\big(\Sigma_A\otimes(v_A + \ell_{\mathrm{s}}^2\epsilon_A)\big) = \ln\vec{v}^{\,4} + \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\,\mathrm{tr}\,\big((v_A\Sigma_A)^{-1}\,\Sigma_B\otimes\epsilon_B\big)^k = \sum_{k=0}^{\infty} t^{(k)}\,. \tag{A.5.10}$$

Up to order five, this is done by using the trace formulas in [51], which yield

$$t^{(1)} = -\frac{4}{v^2}\,\mathrm{tr}_{\mathrm{U}(K)}(\vec{v}\cdot\vec{\epsilon}\,)\,, \tag{A.5.11}$$

$$t^{(2)} = \frac{2}{\vec{v}^{\,4}}\,\mathrm{tr}_{\mathrm{U}(K)}\left[2(\vec{v}\cdot\vec{\epsilon}\,)^2 - \vec{v}^{\,2}\vec{\epsilon}^{\,2}\right]\,, \tag{A.5.12}$$

$$t^{(3)} = -\frac{4}{3\vec{v}^{\,6}}\,\mathrm{tr}_{\mathrm{U}(K)}\left[4(\vec{v}\cdot\vec{\epsilon}\,)^3 - 3\vec{v}^{\,2}(\vec{v}\cdot\vec{\epsilon}\,)\vec{\epsilon}^{\,2}\right]\,, \tag{A.5.13}$$

$$t^{(4)} = \frac{8}{\vec{v}^{\,8}}\,\mathrm{tr}_{\mathrm{U}(K)}\left[(\vec{v}\cdot\vec{\epsilon}\,)^4 - \vec{v}^{\,2}(\vec{v}\cdot\vec{\epsilon}\,)^2\vec{\epsilon}^{\,2} + \frac{1}{4}\vec{v}^{\,4}\vec{\epsilon}^{\,4} - \frac{1}{8}\vec{v}^{\,4}\epsilon_A\epsilon_B\epsilon_A\epsilon_B\right]\,, \tag{A.5.14}$$

$$t^{(5)} = -\frac{4}{\vec{v}^{\,10}}\,\mathrm{tr}_{\mathrm{U}(K)}\bigg[\frac{16}{5}(\vec{v}\cdot\vec{\epsilon}\,)^5 - 4\vec{v}^{\,2}(\vec{v}\cdot\vec{\epsilon}\,)^3\vec{\epsilon}^{\,2} + \vec{v}^{\,4}\left(\vec{v}\cdot\vec{\epsilon}\,\vec{\epsilon}^{\,4} - \vec{v}\cdot\vec{\epsilon}\,\epsilon_B\epsilon_C\epsilon_B\epsilon_C + \vec{v}\cdot\vec{\epsilon}\,\epsilon_B\vec{\epsilon}^{\,2}\epsilon_B\right) + \frac{i}{5}\vec{v}^{\,4}v_A\epsilon_{A_1\cdots A_5A}\epsilon_{A_1}\cdots\epsilon_{A_5}\bigg]\,. \tag{A.5.15}$$

Weyl spinors λ_a and ψ^a in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of the rotation group Spin(6) = SU(4) transform under a six-dimensional rotation parametrized by the antisymmetric matrix Ω, $\delta x_A = -\Omega_{AB} x_B$, as

$$\delta\lambda_a = -\frac{1}{2}\Omega_{AB}\Sigma_{ABa}{}^{b}\lambda_b\,, \quad \delta\psi^a = -\frac{1}{2}\Omega_{AB}\bar{\Sigma}_{AB}{}^{a}{}_{b}\psi^b\,, \tag{A.5.16}$$

where the generators of the rotation group are defined by

$$\Sigma_{AB} = \frac{1}{4}\big(\Sigma_A\bar{\Sigma}_B - \Sigma_B\bar{\Sigma}_A\big)\,, \quad \bar{\Sigma}_{AB} = \frac{1}{4}\big(\bar{\Sigma}_A\Sigma_B - \bar{\Sigma}_B\Sigma_A\big)\,. \tag{A.5.17}$$

This yields in particular the charges under the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{U}(1)_3$ subgroup of SO(6) corresponding to rotations in the 1-2, 3-4 and 5-6 planes respectively, see Table A.3.

β-deformed case

The $\mathrm{U}(1)_i$ charges in Table A.3 are used to compute the $*$-product in section 6.6. In particular, deformed Σ_A matrices can be defined by the identity

$$\psi_1^a * \phi_A * \psi_2^b\,\Sigma_{Aab} = \psi_1^a\phi_A\psi_2^b\,\tilde{\Sigma}_{Aab}\,. \tag{A.5.18}$$

Explicitly, we have

$$\tilde{\Sigma}_1 = \begin{pmatrix} 0 & -i^{\gamma_1-\gamma_2} & 0 & 0 \\ i^{-\gamma_1+\gamma_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & i^{-\gamma_1-\gamma_2} \\ 0 & 0 & -i^{\gamma_1+\gamma_2} & 0 \end{pmatrix},$$
$$\tilde{\Sigma}_2 = \begin{pmatrix} 0 & i^{\gamma_1-\gamma_2-1} & 0 & 0 \\ i^{-\gamma_1+\gamma_2+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & i^{-\gamma_1-\gamma_2-1} \\ 0 & 0 & i^{\gamma_1+\gamma_2+1} & 0 \end{pmatrix},$$
$$\tilde{\Sigma}_3 = \begin{pmatrix} 0 & 0 & -i^{-\gamma_1+\gamma_3} & 0 \\ 0 & 0 & 0 & -i^{\gamma_1+\gamma_3} \\ i^{\gamma_1-\gamma_3} & 0 & 0 & 0 \\ 0 & i^{-\gamma_1-\gamma_3} & 0 & 0 \end{pmatrix},$$
$$\tilde{\Sigma}_4 = \begin{pmatrix} 0 & 0 & i^{-\gamma_1+\gamma_3-1} & 0 \\ 0 & 0 & 0 & i^{\gamma_1+\gamma_3+1} \\ i^{\gamma_1-\gamma_3+1} & 0 & 0 & 0 \\ 0 & i^{-\gamma_1-\gamma_3-1} & 0 & 0 \end{pmatrix}, \qquad \text{(A.5.19)}$$
$$\tilde{\Sigma}_5 = \begin{pmatrix} 0 & 0 & 0 & -i^{\gamma_2-\gamma_3} \\ 0 & 0 & i^{-\gamma_2-\gamma_3} & 0 \\ 0 & -i^{\gamma_2+\gamma_3} & 0 & 0 \\ i^{-\gamma_2+\gamma_3} & 0 & 0 & 0 \end{pmatrix},$$
$$\tilde{\Sigma}_6 = \begin{pmatrix} 0 & 0 & 0 & i^{\gamma_2-\gamma_3-1} \\ 0 & 0 & i^{-\gamma_2-\gamma_3-1} & 0 \\ 0 & i^{\gamma_2+\gamma_3+1} & 0 & 0 \\ i^{-\gamma_2+\gamma_3+1} & 0 & 0 & 0 \end{pmatrix}.$$

	y_1+iy_2	y_3+iy_4	y_5+iy_6	λ_1	λ_2	λ_3	λ_4	ψ^1	ψ^2	ψ^3	ψ^4
$\mathrm{U}(1)_1$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathrm{U}(1)_2$	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\mathrm{U}(1)_3$	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Table A.3: Charges under $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{U}(1)_3 \subset \mathrm{SO}(6)$. The spinors λ_a and ψ^a are arbitrary spinors in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of Spin(6) respectively.

Appendix B

On trivial dimensional reduction

In this appendix we present some basic considerations about trivial dimensional reduction. The goal is to provide the necessary tools to understand the machinery of the computations underlying section 3.3. For a presentation on spinors in various dimensions, see e.g. [80].

Trivial dimensional reduction is a recipe to construct a d dimensional theory from a $D > p$ dimensional theory, such that the lower-dimensional theory possesses some of the symmetries and invariances of the original, higher-dimensional theory. The recipe starts with the *trivial dimensional reduction Ansatz*, stating that all fields of the original theory are independent of some coordinates. Compatibility of this Ansatz with the spacetime symmetries requires to restrict the full spacetime transformations to a subset of *allowed* transformations. The fields of the original theory must then be reorganised into representations of the group of allowed transformations. This last technical step is straightforward as far as bosonic fields are concerned, while for fermions some non-trivial work might be required.

Consider a theory in D spacetime dimensions invariant under SO(D). A generic SO(D) transformation acts on the coordinates x^M with $1 \leq M \leq D$ as

$$x^M \mapsto x'^M = R^M{}_N x^N \,, \tag{B.1}$$

where the matrix $R = (R^M{}_N) \in \mathrm{SO}(D)$ is such that $R^T R = \mathbf{1}$ and $\det R = 1$. Let us perform the trivial dimensional reduction on the last $D - d - 1$ coordinates $(x^{d+1}, \ldots, x^D) = (x^{d+m})$ with $1 \leq m \leq D - d$. Restricting to orientation preserving transformations in the remaining d directions $(x^1, \ldots, x^d) = (x^I)$ with $1 \leq I \leq d$, the set of allowed transformation then forms the group

$$\mathrm{SO}(d) \times \mathrm{SO}(D - d) \subset \mathrm{SO}(D) \tag{B.2}$$

formed by the $D \times D$ matrices R such that $R^I{}_{d+m} = 0$ and $R^{d+m}{}_I = 0$ while the submatrices $(R^I{}_J) \in \mathrm{SO}(d)$ and $(R^m{}_n) \in \mathrm{SO}(D - d)$. The first factor in the decomposition (B.2) is the new spacetime symmetry group, while the second factor in (B.2) is a new internal symmetry group of the lower dimensional theory. Let us parameterise the matrix $R \in \mathrm{SO}(D)$ as

$$R = e^{\frac{i}{2}\omega_{MN}J_{MN}}\,, \tag{B.3}$$

where $\omega_{MN} = -\omega_{MN}$ and the hermitian matrices $J_{MN} = J^\dagger_{MN}$ form a basis of the algebra $\mathfrak{so}(D)$. The allowed transformations are then obtained by restricting to parameters ω_{MN} such that $\omega_{MI} = 0$.

Let $V^M(x^N)$ be a vector field in the original, D-dimensional theory. Under the spacetime transformation $R \in \mathrm{SO}(D)$, V^M transforms as $V^M \mapsto V'^M$ with V'^M given by the rule

$$V'^M(R^N{}_P x^P) = R^M{}_P V^P(x^N)\,. \tag{B.4}$$

Under an allowed transformation $R^I{}_J$ corresponding to a spacetime symmetry transformation in the lower-dimensional theory, we see that V^I transforms as a vector of $\mathrm{SO}(d)$, while the remaining $D-d$ components $V^{d+m} = \phi^m$ are scalars of $\mathrm{SO}(d)$. On the other hand, under the internal transformations $(R^m{}_n) \in \mathrm{SO}(D-d)$ the vector in d dimensions V^I is invariant while the $D-d$ scalars ϕ^m transform as the fundamental of $\mathrm{SO}(D-d)$.

For a spinor Ψ of $\mathrm{SO}(D)$, the situation is slightly more complicated. Let us denote by Ψ^a with $1 \leq a \leq 2^{[D/2]}$ the components of Ψ, where $[D/2]$ is the integral part of $D/2$, and $S^{(D)}_{MN}$ the generators of the spinor representation of $\mathrm{SO}(D)$. For $R \in \mathrm{SO}(D)$, the spinor Ψ transforms as $\Psi \mapsto \Psi'$ where

$$\Psi'^a(R^M{}_N x^N) = \left(e^{\frac{i}{2}\omega_{MN}S^{(D)}_{MN}}\right)^a{}_b \Psi^b(x^M)\,. \tag{B.5}$$

Our goal is to identify the linear combinations of the components Ψ^a that transform as a spinor of $\mathrm{SO}(d)$, or, equivalently, we need to find a basis such that $S^{(D)}_{IJ}$ has a simple expression in terms of $S^{(d)}_{IJ}$. Typically, we find a basis such that the former is written as a tensor product of the latter with the identity matrix, allowing for a trivial identification of the $\mathrm{SO}(d)$ spinors.

Let us give some details on the concrete example of subsection 3.3.4. In this case, we have $D = 6$ and $d = 2$. Since a generic spinor in six dimensions has eight components, it must contain four spinors in two dimensions. It is in this case more convenient to rewrite the six-dimensional spinors in terms of spinors of the internal symmetry group, that we denoted $\mathrm{SO}(4)'$ in the main text. We thus have to decompose a spinor in six dimensions into two spinors of $\mathrm{SO}(4)'$, because a generic spinor in four dimensions has four components; this is precisely what is done explicitly in subsection 3.3.4. In particular, we define in (3.3.12) the change of basis which allows us to trivially identify the spinors of $\mathrm{SO}(4)'$, see (3.3.17) and (3.3.18). It was necessary in this case to define a new basis for the left- and right-handed spinors of $\mathrm{SO}(6)$ independently. The defining property of the new basis is that the matrices Σ'_μ have simple expressions, see (3.3.14), which yield the relation (3.3.16) between the six-dimensional generators $S'^{(6)}_{(2+m)(2+m)}$ and the four dimensional ones $S^{(4)}_{mn}$ defined in appendix A, see (A.4.11).

Appendix C

Some type IIB supergravity backgrounds

In this appendix we review the known supergravity backgrounds dual to the non-commutative and β-deformed Euclidean $\mathcal{N} = 4$ super-Yang-Mills theories studied in chapter 6. We use the standard relation between the radius R and the 't Hooft coupling λ,

$$R^4 = \alpha'^2 \lambda = \frac{\ell_{\rm s}^4 \lambda}{4\pi^2} \cdot$$

The backgrounds are written at zero bare ϑ angle. The solutions at non-zero ϑ can be obtained by performing the SL$(2, \mathbb{R})$ transformation $C_{[0]} \mapsto C_{[0]} + \frac{\vartheta}{2\pi}$, $C_{[2]} \mapsto C_{[2]} - \frac{\vartheta}{2\pi} B_{[2]}$ and $C_{[4]} \mapsto C_{[4]} + \frac{\vartheta}{4\pi} B_{[2]} \wedge B_{[2]}$, which automatically yields a new solution to the supergravity equations of motion.

C.1 The dual to the non-commutative gauge theory

The gravitational dual of the non-commutative deformation of the $\mathcal{N} = 4$ super-Yang-Mills theory was derived in [56, 57].[1] With non-vanishing non-commutative parameters $\theta_{12} = -\theta_{21}$ and $\theta_{34} = -\theta_{43}$, the solution for the string-frame metric and

[1] Our formulas can be matched with those in [57] by making the replacements $R^2 \mapsto \alpha' R^2$, $\theta_{12} \mapsto \tilde{b}'/(2\pi)$, $\theta_{34} \mapsto \tilde{b}/(2\pi)$, $r \mapsto \alpha' R^2 u$, $\lambda/(4\pi N) \mapsto \hat{g}$ and $C_{[0]} \mapsto -\chi$, $C_{[2]} \mapsto -A$, $F_{[5]} \mapsto -F$.

the other supergravity fields reads

$$ds^2 = \frac{r^2}{R^2}\left[\frac{dx_1^2 + dx_2^2}{\Delta_{12}} + \frac{dx_3^2 + dx_4^2}{\Delta_{34}}\right] + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2\,, \tag{C.1.1}$$

$$e^{-\Phi} = \frac{4\pi N}{\lambda}\sqrt{\Delta_{12}\Delta_{34}}\,, \tag{C.1.2}$$

$$B_{[2]} = \frac{r^4}{R^4}\left(\frac{\theta_{12}}{\ell_s^2}\frac{dx_1\wedge dx_2}{\Delta_{12}} + \frac{\theta_{34}}{\ell_s^2}\frac{dx_3\wedge dx_4}{\Delta_{34}}\right)\,, \tag{C.1.3}$$

$$C_{[0]} = -\frac{4i\pi N}{\lambda}\frac{\theta_{12}\theta_{34}}{\ell_s^4}\frac{r^4}{R^4}\,, \tag{C.1.4}$$

$$C_{[2]} = -\frac{4i\pi N}{\lambda}\frac{r^4}{R^4}\left(\frac{\theta_{34}}{\ell_s^2}\frac{dx_1\wedge dx_2}{\Delta_{12}} + \frac{\theta_{12}}{\ell_s^2}\frac{dx_3\wedge dx_4}{\Delta_{34}}\right)\,, \tag{C.1.5}$$

$$C_{[4]} = \frac{16\pi r^2}{R^3}\omega_4 - 4i\pi\frac{r^6}{R^6}\frac{dx_1\wedge dx_2\wedge dx_3\wedge dx_4}{\Delta_{12}\Delta_{34}}\,, \tag{C.1.6}$$

where the functions Δ_{12} and Δ_{34} are defined by

$$\Delta_{12} = 1 + \left(\frac{\theta_{12}}{\ell_s^2}\right)^2\frac{r^4}{R^4}\,,\quad \Delta_{34} = 1 + \left(\frac{\theta_{34}}{\ell_s^2}\right)^2\frac{r^4}{R^4}\,. \tag{C.1.7}$$

The x_1, x_2, x_3 and x_4 are the world-volume coordinates on which the gauge theory live, r is the transverse radial coordinate, expressed in terms of the six transverse coordinates $\vec{y} = (y_A)_{1\leq A\leq 6}$ as $r^2 = |\vec{y}|^2$, $d\Omega_5^2$ is the metric on the five-dimensional round sphere of radius one and ω_4 is a four-form defined in terms of the volume form

$$\omega_{S^5} = \frac{1}{5!}\frac{R^5 y_F}{r^6}\epsilon_{ABCDEF}\, dy_A\wedge\cdots\wedge dy_E \tag{C.1.8}$$

on S^5 of radius R by

$$d\omega_4 = \omega_{S^5}\,. \tag{C.1.9}$$

The consistency of the supergravity approximation for the above solution requires as usual $\lambda \gg 1$. In the far infrared region $r \ll R\ell_s/\sqrt{\theta} \sim \ell_s^2\lambda^{1/4}/\sqrt{\theta}$, the solution is a small deformation of the usual $AdS_5\times S^5$ background and can be compared with the microscopic calculations presented in the main text. On the other hand, in the far ultraviolet region $r \gg R\ell_s/\sqrt{\theta}$, the metric (C.1.1) approximates another $AdS_5\times S^5$ space, with a new radial coordinate $\tilde{r} = 1/r$. Thus there is no conformal boundary at infinity, which signals that the non-commutative theory is not a standard UV-complete quantum field theory.

C.2 The dual to the β-deformed theory

The gravitational dual of the β-deformed $\mathcal{N} = 4$ super-Yang-Mills theory was derived in [62] in the $\mathcal{N} = 1$ supersymmetry preserving case $\gamma_1 = \gamma_2 = \gamma_3$ and generalized in [59] to arbitrary deformation parameters γ_1, γ_2 and γ_3. The solution for the string-frame metric and the other non-trivial supergravity fields reads

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{r^2}{R^2}\mathrm{d}x_\mu \mathrm{d}x_\mu + \frac{R^2}{r^2}\mathrm{d}r^2 + R^2 \mathrm{d}\tilde{\Omega}_5^2\,, && \text{(C.2.1)}\\
e^{-\Phi} &= \frac{4\pi N}{\lambda\sqrt{G}}\,, && \text{(C.2.2)}\\
B_{[2]} &= -\frac{\ell_\mathrm{s}^2\lambda}{2\pi}\,G\left(\gamma_3 r_1^2 r_2^2 \mathrm{d}\theta_1\wedge\mathrm{d}\theta_2 + \gamma_2 r_1^2 r_3^2 \mathrm{d}\theta_3\wedge\mathrm{d}\theta_1 + \gamma_1 r_2^2 r_3^2 \mathrm{d}\theta_2\wedge\mathrm{d}\theta_3\right), && \text{(C.2.3)}\\
C_{[2]} &= -8N\ell_\mathrm{s}^2\,\omega_1\wedge\left(\gamma_1\mathrm{d}\theta_1 + \gamma_2\mathrm{d}\theta_2 + \gamma_3\mathrm{d}\theta_3\right), && \text{(C.2.4)}\\
C_{[4]} &= \frac{4N\ell_\mathrm{s}^4}{\pi}\left(G\,\omega_1\wedge\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\theta_3 - i\omega_4\right). && \text{(C.2.5)}
\end{aligned}
$$

The coordinates x_μ, $1 \le \mu \le 4$, can be viewed as the world-volume coordinates of the background D3-branes. The coordinate r is the usual transverse radial coordinate, expressed in terms of the six transverse coordinates $\vec{y} = (y_A)_{1\le A\le 6}$ as $r^2 = \vec{y}^2$. The coordinates $(r_i, \theta_i)_{1\le i\le 3}$ are defined by the relations

$$
\begin{aligned}
&y_1 = \rho_1\cos\theta_1\,, \quad y_3 = \rho_2\cos\theta_2\,, \quad y_5 = \rho_3\cos\theta_3\,,\\
&y_2 = \rho_1\sin\theta_1\,, \quad y_4 = \rho_2\sin\theta_2\,, \quad y_6 = \rho_3\sin\theta_3
\end{aligned} \tag{C.2.6}
$$

and

$$
r_i = \frac{\rho_i}{\sqrt{\rho_1^2+\rho_2^2+\rho_3^2}} = \frac{\rho_i}{|\vec{y}|}\,, \quad r_1^2 + r_2^2 + r_3^2 = 1\,. \tag{C.2.7}
$$

The function G is given by

$$
\frac{1}{G} = 1 + \lambda\left(\gamma_1^2 r_2^2 r_3^2 + \gamma_2^2 r_1^2 r_3^2 + \gamma_3^2 r_1^2 r_2^2\right). \tag{C.2.8}
$$

The metric (C.2.1) describes an $\mathrm{AdS}_5\times\tilde{\mathrm{S}}^5$ geometry for a deformed five-sphere $\tilde{\mathrm{S}}^5$ endowed with the metric

$$
\mathrm{d}\tilde{\Omega}_5^2 = \sum_{i=1}^{3}\left(\mathrm{d}r_i^2 + G\,r_i^2\mathrm{d}\theta_i^2\right) + \lambda G\, r_1^2 r_2^2 r_3^2\left(\sum_{i=1}^{3}\gamma_i\mathrm{d}\theta_i\right)^2. \tag{C.2.9}
$$

Defining the angles θ and ϕ by

$$
r_1 = \sin\theta\cos\phi\,, \quad r_2 = \sin\theta\sin\phi\,, \quad r_3 = \cos\theta\,, \tag{C.2.10}
$$

the one-form ω_1 in (C.2.4) and (C.2.5) satisfies

$$
\mathrm{d}\omega_1 = r_1 r_2 r_3\,\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{C.2.11}
$$

and can be chosen to be

$$
\omega_1 = \frac{1}{4}\sin^4\theta\cos\phi\sin\phi\,\mathrm{d}\phi\,. \tag{C.2.12}
$$

The four-form ω_4 in (C.2.5) satisfies

$$\mathrm{d}\omega_4 = \omega_{\mathrm{AdS}_5}\,, \tag{C.2.13}$$

where

$$\omega_{\mathrm{AdS}_5} = \frac{1}{R^8}\,r^3 \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_4 \wedge \mathrm{d}r \tag{C.2.14}$$

is the volume form on the unit radius AdS_5 space. Explicitly, one can choose

$$\omega_4 = \frac{1}{4R^8} r^4 \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_4\,. \tag{C.2.15}$$

Changes of ω_1 and ω_4 by exact forms correspond to a supergravity gauge transformation.

The β-deformed theory is conformal in the planar limit, which explains the fact that the AdS_5 factor in the metric (C.1.1) is undeformed. The consistency of the supergravity approximation requires, on top of the usual condition $\lambda \gg 1$, that $\gamma_i^4 \lambda \ll 1$, as can be checked by evaluating the curvature of the deformed sphere (C.2.9). In particular, the γ_is must be very small. This explains why the periodicity in the deformation parameters, $(\gamma_1, \gamma_2, \gamma_3) \equiv (\gamma_1 + n_1, \gamma_2 + n_2, \gamma_3 + n_3)$ for any integers n_1, n_2, n_3, which is manifest in the microscopic theory and in particular in the effective action computed in section 6.6, cannot be seen in the supergravity solution. Finally, let us note that the background is a small deformation of the usual $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution when $\gamma_i^2 \lambda \ll 1$, a condition often used in the main text.

Appendix D

Myers' non-abelian D-instanton action

The Myers' non-abelian D-instanton action is obtained by adding the non-abelian Dirac-Born-Infeld action (3.2.50a) to the non-abelian Chern-Simons action (3.2.50b), with $p = -1$. Since for a D-instanton there are no parallel directions, the transverse matrix coordinates X^m correspond to the ten-dimensional matrix coordinates X^M. Following the discussion around (3.2.49) the matrices X^M are written as

$$X^M = x^M \mathbf{1} + \ell_{\mathrm{s}}^2 \epsilon^M \,, \tag{D.1}$$

where we decided to introduce a factor of ℓ_{s}^2 for convenience. The action $S_{\mathrm{DBI}} + S_{\mathrm{CS}}$ is then expanded as

$$S_{\text{eff}} = \sum_{n\geq 0} S_{\text{eff}}^{(n)} = \sum_{n\geq 0} \frac{1}{n!} \ell_{\mathrm{s}}^{2n} c_{M_1\cdots M_n}(z)\, \mathrm{tr}(\epsilon_{M_1} \cdots \epsilon_{M_n})\,. \tag{D.2}$$

Up to order five, the action is then given in terms of the type IIB supergravity fields by the following formulas:

$$
\begin{aligned}
S^{(0)}_{\text{eff}} &= -2i\pi K\tau\,,\\
S^{(1)}_{\text{eff}} &= -2i\pi\ell_{\text{s}}^2\partial_M\tau\,\operatorname{tr}\epsilon_M\,,\\
S^{(2)}_{\text{eff}} &= -i\pi\ell_{\text{s}}^4\partial_M\partial_N\tau\,\operatorname{tr}\epsilon_M\epsilon_N\,,\\
S^{(3)}_{\text{eff}} &= \bigl(-\frac{i\pi}{3}\ell_{\text{s}}^6\partial_M\partial_N\partial_P\tau - 2\pi\ell_{\text{s}}^4\partial_{[M}(\tau B_{[2]}-C_{[2]})_{NP]}\bigr)\operatorname{tr}\epsilon_M\epsilon_N\epsilon_P\,,\\
S^{(4)}_{\text{eff}} &= \bigl(-\frac{i\pi}{12}\ell_{\text{s}}^8\partial_M\partial_N\partial_P\partial_Q\tau - \frac{3\pi}{2}\ell_{\text{s}}^6\partial_M\partial_{[N}(\tau B_{[2]}-C_{[2]})_{PQ]}\\
&\qquad\qquad - \pi\ell_{\text{s}}^4 e^{-\Phi}(G_{MP}G_{NQ}-G_{MQ}G_{NP})\bigr)\operatorname{tr}\epsilon_M\epsilon_N\epsilon_P\epsilon_Q\,,\\
S^{(5)}_{\text{eff}} &= \Bigl(-\frac{i\pi}{60}\ell_{\text{s}}^{10}\partial_M\partial_N\partial_P\partial_Q\partial_R\tau - \frac{\pi}{3}\ell_{\text{s}}^8\partial_P\partial_Q\partial_R(\tau B_{[2]}-C_{[2]})_{MN}\\
&\qquad\qquad - \pi\ell_{\text{s}}^6\partial_R\bigl(e^{-\Phi}(G_{MP}G_{NQ}-G_{MQ}G_{NP})\bigr)\\
&\qquad - i\pi\ell_{\text{s}}^6\partial_{[M}(C_{[4]}+C_{[2]}\wedge B_{[2]}-\frac{\tau}{2}B_{[2]}\wedge B_{[2]})_{NPQR]}\Bigr)\operatorname{tr}\epsilon_M\epsilon_N\epsilon_P\epsilon_Q\epsilon_R\,.
\end{aligned}
\tag{D.3}
$$

The complex field τ is defined by

$$\tau = -C_{[0]} + ie^{-\Phi}\,. \tag{D.4}$$

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