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Article

A Probe into the Evolution of Primordial Perturbations in the $f(T)$ Gravity Framework with Chaplygin Gas

Sanjeeda Sultana [1,†], Surajit Chattopadhyay [1,*,†] and Antonio Pasqua [2,†]

[1] Department of Mathematics, Amity University, Major Arterial Road, Action Area II, Rajarhat, Newtown, Kolkata 700135, India; sanjeeda.sultana1@s.amity.edu
[2] Department of Physics, University of Trieste, Via Valerio 2, 34127 Trieste, Italy; toto.pasqua@gmail.com
* Correspondence: schattopadhyay1@kol.amity.edu
† These authors contributed equally to this work.

Abstract: This work is focused on studying the cosmology of variable modified Chaplygin gas (VMCG) in the framework of exponential and logarithmic $f(T)$ theory. The equation of state (EoS) for VMCG in exponential and logarithmic $f(T)$ gravity shows quintom behavior. Primordial perturbations were studied for VMCG in both exponential and logarithmic $f(T)$ gravity, and it was observed that the potential increases with cosmic time t, and the scalar field decreases toward the minimum value of the potential. The squared speed of sound was positive, meaning that VMCG in both exponential and logarithmic $f(T)$ gravity shows stability against small gravitational perturbations.

Keywords: variable modified Chaplygin gas; exponential $f(T)$ gravity; logarithmic $f(T)$ gravity; primordial perturbations

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1. Introduction

Observations from cosmic microwave background (CMB) radiation [1,2], Supernovae Ia (SNe Ia) [3,4], baryon acoustic oscillations (BAOs) [5], large-scale structures (LSSs) [6], and weak lensing [7] have implied that the universe is currently expanding with acceleration. In modern physics, this is one of the most important issues. There are two main types of methods used to explain the late-time cosmic acceleration: One is to add "dark energy" to the right-hand side of the Einstein equation within the context of general relativity (see [8] for a discussion on dark energy). The other method is to alter the Einstein equation's left-hand side, known as a modified gravitational theory for $f(R)$ gravity [9–11].

Alternatively, the Weitzenböck connection, which contains torsion instead of curvature as defined by the Levi–Civita connection, could be used to investigate gravity beyond the framework of general relativity. This method, known as "teleparallelism" (see, for example, [12–15]), was also adopted by Einstein [16]. The teleparallel Lagrangian density given by the torsion scalar T has been expanded to a function of T to explain the universe's late-time acceleration [17,18]. In [19], inflation models based on modified teleparallel gravity were studied. This notion is comparable to the theory of $f(R)$ gravity, which promotes the Ricci scalar R in the Einstein–Hilbert action to a function of R. In the recent literature [20–27], $f(T)$ gravity has been thoroughly examined.

In this work, we used the analysis method in [28] to explicitly investigate the cosmic evolution in the exponential $f(T)$ hypothesis [18,24] in greater depth. We specifically studied the energy density (ρ_{DE}) and EoS (w_{DE}) for dark energy. The latest observational evidence on cosmology [29,30] appears to suggest dynamic dark energy in the EoS, in which there is a crossing of the phantom division line $w_{DE} = 1$ from the non-phantom phase to the phantom phase as redshift z decreases in the near past. Nevertheless, we show that the universe described by the exponential $f(T)$ theory is always in the phantom or non-phantom (quintessence) phase. As a result, the crossing of the phantom divide cannot be perceived [24]. It is noteworthy that Refs. [31–34] investigated gravity models of an exponential form, such as the $f(R)$ model. We have also provided a logarithmic $f(T)$ theory and demonstrated that it has characteristics similar to the exponential one. In this study, we aimed to develop a realistic $f(T)$ theory that can produce the same behavior of the phantom divide crossing as shown by the data. A combined $f(T)$ theory with both exponential and logarithmic terms is constructed to achieve this. Moreover, the most recent observational evidence from SNe Ia, BAOs, and CMB is used to investigate the observational limitations of the combined $f(T)$ theory. It should be noted that Ref. [23] has also provided two $f(T)$ models with the phantom divide crossing.

Recent findings of type Ia supernova luminosity [3,35] have led to the search for a new kind of matter that defies the strong energy condition $\rho + 3p < 0$. At a particular stage of the evolution of the universe, dark energy is referred to as the matter constant, which is responsible for satisfying this condition. There are several contenders for the role of dark energy. Quintessence is a type of dark energy represented by a scalar field. Specifically, one may experiment with a different kind of dark energy, known as pure Chaplygin gas, which follows an EoS similar to [36]

$$p = -\frac{B}{\rho}, \tag{1}$$

where ρ and p are the energy density and pressure, respectively, and B is a positive constant. A cosmological model with Chaplygin gas was suggested for the first time in [37]. In [37], an FRW cosmological model with Chaplygin gas was considered. The authors showed that the resulting evolution of the universe is in agreement with the current accelerated expansion of the universe. An increasing value for the effective cosmological constant was predicted using their model [37]. Frolov et al. [38] reported a linear perturbation of tachyon matter rolling toward its potential minimum by considering a tachyon coupled to gravity, where the tachyon potential had its quadratic minimum at a finite value of the tachyon field. The authors of [38] concluded that the rolling tachyon condensate in an expanding universe behaves like a pressure-like fluid, and its linear fluctuations coupled with small metric perturbations have evolving patterns similar to those in pressureless fluid. The equation, as mentioned earlier, was then modified to what is known as the generalized Chaplygin gas (GCG) equation, which is of the following form:

$$p = -\frac{B}{\rho^{\alpha}}, \tag{2}$$

where $0 \le \alpha \le 1$. This generalized model had already been studied in [39–41]. Some works on modified Chaplygin gas (MCG) are [42,43] and have an EoS of the following form:

$$p = A\rho - \frac{B}{\rho^{\alpha}}, \tag{3}$$

where B is a positive constant. In this EoS, the radiation period is represented by $A = \frac{1}{3}$ at one extreme (i.e., when the scale factor $a(t)$ is vanishingly small) and a ΛCDM model at the other extreme (i.e., when the scale factor $a(t)$ is infinitely large). With the EoS (1), where B is a positive function of the scale factor a, i.e., $B = B(a)$, Guo and Jhang [44] proposed the variable Chaplygin gas model for the first time. When assuming the Chaplygin gas to be a Born–Infeld scalar field [45], this assumption makes sense because $B(a)$ is connected to the scalar potential. A few studies on the variable Chaplygin gas model were published later [46]. Further, a variable modified Chaplygin gas (VMCG) was introduced by Debnath [47] for the acceleration of the universe. The EoS for VMCG [47] is

$$p = A\rho - \frac{B(a)}{\rho^{\alpha}}. \tag{4}$$

Several authors [48–51] have shown several physical interpretations and interesting features of VMCG.

The paper is organized as follows: In Section 2, we discuss $f(T)$ gravity. In Section 3, we demonstrate the cosmology of VMCG in both exponential and logarithmic $f(T)$ theory. In Section 4, primordial perturbations are studied in the cosmological settings of VMCG in exponential and logarithmic $f(T)$ theory. Section 5 concludes the paper.

2. $f(T)$ Gravity

Orthonormal tetrad components $e_A(x^\mu)$, where an index A runs over 0, 1, 2, 3 for the tangent space at each point x^μ of the manifold, are employed in teleparallelism. Their relation to the metric $g^{\mu\nu}$ is

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu, \tag{5}$$

where e^μ_A is the tangent vector of the manifold and μ and ν are coordinate indices on the manifold and also run over 0, 1, 2, 3.

The torsion $T^\rho{}_{\mu\nu}$ and contorsion $K^{\mu\nu}{}_\rho$ tensors are defined as

$$T^\rho{}_{\mu\nu} \equiv e^\rho_A(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu), \tag{6}$$

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}). \tag{7}$$

The torsion scalar T in general relativity describes the teleparallel Lagrangian density rather than the Ricci scalar R for the Lagrangian density, and it is defined as

$$T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \tag{8}$$

where

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha). \tag{9}$$

As a result, for $f(T)$ theory, the modified teleparallel action is given by [18]

$$I = \frac{1}{16\pi G}\int d^4x|e|(T + f(T)), \tag{10}$$

where $|e| = det(e^A_\mu) = \sqrt{-g}$.

The flat Friedmann–Lematre–Robertson–Walker (FLRW) space-time is assumed with the metric

$$ds^2 = dt^2 - a^2(t)dx^2, \tag{11}$$

where $a(t)$ is the scale factor. $g_{\mu\nu} = diag(1, -a^2, -a^2, -a^2)$ in this space-time and hence the exact value of the torsion scalar $T = -6H^2$ are yielded by the tetrad components $e^A_\mu = (1, a, a, a)$, where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The unit used is the gravitational constant $G = M_{pl}^{-2}$ where Planck mass $M_{pl} = 1.2 \times 10^{19}$GeV and $\kappa_B = c = \hbar = 1$.

From the variation principle, the modified Friedmann equations in the flat FLRW background are given by [17,18]

$$H^2 = \frac{8\pi G}{3}\rho_M - \frac{f}{6} - 2H^2 f_T, \tag{12}$$

$$(H^2)' = \frac{16\pi G P_M + 6H^2 + f + 12H^2 f_T}{24H^2 f_{TT} - 2 - 2f_T}, \tag{13}$$

where $'$ denotes the derivative with respect to $ln\ a$, and P_M and ρ_M are the pressure and energy density of all perfect fluids of generic matter, respectively, $f_T \equiv \frac{df(T)}{dT}$ and $f_{TT} \equiv \frac{d^2 f(T)}{dT^2}$.

When comparing the above-mentioned modified Friedmann equations Equations (12) and (13) with the ones in general relativity

$$H^2 = \frac{8\pi G}{3}(\rho_M + \rho_{DE}), \tag{14}$$

$$(H^2)' = -8\pi G(\rho_M + P_M + \rho_{DE} + P_{DE}), \tag{15}$$

we have the pressure and energy density of the effective dark energy,

$$P_{DE} = \frac{1}{16\pi G}\left(\frac{f - Tf_T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}\right), \tag{16}$$

$$\rho_{DE} = \frac{1}{16\pi G}(-f + 2Tf_T). \tag{17}$$

The EoS for dark energy is defined as

$$w_{DE} = \frac{P_{DE}}{\rho_{DE}} = -1 + \frac{T'}{3T}\frac{f_T + 2Tf_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2Tf_{TT}}{(1 + f_T + 2Tf_{TT})(f/T - 2f_T)}. \tag{18}$$

The squared speed of sound is defined as

$$c_s^2 = \frac{1 + f_T}{1 + f_T - 12H^2 f_{TT}}. \tag{19}$$

We will only take into account non-relativistic matter (cold dark matter and baryon) with $\rho_M = \rho_m$ and $P_M = P_m = 0$, where P_m and ρ_m are the pressure and energy density of non-relativistic matter, respectively, since we are interested in the late-time universe. Moreover, it can be shown from Equations (12) and (13) that the effective dark energy satisfies the continuity equation

$$\frac{d\rho_{DE}}{dN} \equiv \rho'_{DE} = -3(1 + w_{DE})\rho_{DE}, \tag{20}$$

where $N \equiv ln\ a$.

In the subsequent section, we investigate VMCG in the framework of exponential and logarithmic $f(T)$ theory.

3. $f(T)$ Theory Cosmology: Exponential and Logarithmic

The cosmological evolution of the EoS for dark energy w_{DE} in $f(T)$ theory is analyzed by defining a dimensionless variable [28]

$$y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{DE}}{\rho_m^{(0)}}, \tag{21}$$

where

$$\bar{m}^2 \equiv \frac{8\pi G\rho_m^{(0)}}{3}, \tag{22}$$

and $\rho_m^{(0)} = \rho_m$ at $z = 0$, which is the current density parameter of non-relativistic matter with the redshift $z = \frac{1-a}{a}$. The evolution equation of the universe is obtained from Equation (20) as

$$y'_H = -3(1 + w_{DE})y_H. \tag{23}$$

It is to be noted that w_{DE} is a function of T, whereas T is a function of H^2. Moreover, from Equation (21), it follows that $H^2 = \bar{m}^2(y_H + a^{-3})$.

3.1. Exponential $f(T)$ Theory

In this subsection, we will study the cosmological evolution of the VMCG under the purview of $f(T)$ gravity. For that purpose, we will consider some viable models of $f(T)$ gravity. Such models are already established as viable models in the literature. First, we will consider exponential $f(T)$ gravity discussed in [52]. With this model [52], the authors demonstrated some dependencies of the exponential $f(T)$ theory for its deviation from the ΛCDM model on some model parameters. It was also established in [52] that the EoS parameter can realize both phantom and non-phantom phases for this exponential $f(T)$ gravity model. Our study is motivated by the choice made in [52]. Here, we intend to study the cosmological consequences of $f(T)$ gravity with VMCG as the background fluid. The remaining part of the section is framed accordingly. The exponential $f(T)$ theory in [18,52] is considered, and it is given by

$$f(T) = \alpha T(1 - e^{\frac{pT_0}{T}}) \tag{24}$$

where

$$\alpha = -\frac{1-\Omega_m^{(0)}}{1-(1-2p)e^p}, \tag{25}$$

where p is a constant with $p = 0$ corresponding to the ΛCDM model and $T_0 = T$ at $z = 0$, which is the current torsion. Here, $\Omega_m^{(0)} \equiv \frac{\rho_m^{(0)}}{\rho_{crit}^{(0)}}$ where $\rho_{crit}^{(0)} = \frac{3H_0^2}{8\pi G}$ is the critical density (H_0 is the current Hubble parameter) and $\rho_m^{(0)}$ is the current energy density of non-relativistic matter. It is to be noted that by using Equation (17) with $T = T_0$ and $\rho_{DE}^{(0)} = \rho_{DE}$ at $z = 0$, α in Equation (25) is derived from $\Omega_{DE}^{(0)} \equiv \frac{\rho_{DE}^{(0)}}{\rho_{crit}^{(0)}} = 1 - \Omega_m^{(0)} = -\alpha[1-(1-2p)e^p]$. We have also noted that if the value of $\Omega_m^{(0)}$ is known, then the theory in Equation (24) has only one parameter p. By using $\Omega_m^{(0)}$ and p, the values of other dimensionless quantities can be calculated at $z = 0$, like $y_H(z=0) = \frac{H_0^2}{\bar{m}^2} - 1 = \frac{1}{\Omega_m^{(0)}} - 1$ and $\frac{\bar{m}^2}{T_0} = \frac{8\pi G\rho_m^{(0)}/3}{-6H_0^2} = \frac{-\Omega_m^{(0)}}{6}$.

VMCG in Exponential $f(T)$ Theory

In this subsection, we will study the cosmological evolution of the VMCG model in exponential $f(T)$ theory. Since the discovery of the late-time acceleration of the universe, the Chaplygin gas in its original form as well as in various generalized versions has extensively been considered as a candidate of dark energy that has the potential of unifying dark energy and dark matter [36,39,40,42,44,45]. The current study involves the modified theory of gravity, and we consider the background curvature energy density as a manifestation of VMCG. This approach of considering curvature fluid in the form of Chaplygin gas for modified theories is not new in the literature. Several studies [53–56] have considered this approach to study the modified theory of gravity and its cosmological consequences. In this context, the conservation equation is defined as

$$\dot{\rho}_{total} + 3H(\rho_{total} + p_{total}) = 0, \tag{26}$$

where $\rho_{total} = \rho_{DE} + \rho_m$ and $p_{total} = P_{DE}$ (as $p_m = 0$). Considering $B(a) = B_0 a^{-n}$ in the EoS of VMCG Equation (4) where $n > 0$ and $B_0 > 0$, we have the following form of the EoS parameter,

$$w = A - \frac{B_0 a^{-n}}{\rho^{\alpha+1}}. \tag{27}$$

From Equations (26) and (27), we have the reconstructed density for VMCG,

$$\rho_{VMCG} = \left(\frac{3a^{-n}B_0(1+\alpha)}{3-n+3\alpha+3A(1+\alpha)} + a^{-3(1+A)(1+\alpha)}C_1\right)^{\frac{1}{1+\alpha}}. \tag{28}$$

Using Equations (22) and (28) in Equation (21), we have the expression for Hubble parameter,

$$H = \frac{\sqrt{\frac{\rho_m^{(0)}}{a^3} + \left(a^{-3(1+A)(1+\alpha)}C_1 + \frac{3a^{-n}B_0(1+\alpha)}{3+3A-n+3(1+A)\alpha}\right)^{\frac{1}{1+\alpha}}}}{\sqrt{3}}. \tag{29}$$

The expression for the torsion scalar is

$$T = -2\left(\frac{\rho_m^{(0)}}{a^3} + \left(a^{-3(1+A)(1+\alpha)}C_1 + \frac{3a^{-n}B_0(1+\alpha)}{3+3A-n+3(1+A)\alpha}\right)^{\frac{1}{1+\alpha}}\right). \tag{30}$$

Hence, by incorporating Equations (25) and (30) in Equation (24), we obtain the reconstructed exponential $f(T)$ as

$$f(T)_{EXP} = \frac{1}{a^3(1+e^p(-1+2p))} \times 2\left(1 - e^{-\frac{a^3 p T_0}{2\left(a^3\left(a^{-\frac{3(1+A)\left(\Omega_m^{(0)}+e^p(-1+2p)\right)}{1+e^p(-1+2p)}}C_1 - \frac{3a^{-n}B_0\left(\Omega_m^{(0)}+e^p(-1+2p)\right)}{n-3\Omega_m^{(0)}-3A\Omega_m^{(0)}+e^p(-3-3A+n)(-1+2p)}\right)^{1+\frac{1-\Omega_m^{(0)}}{\Omega_m^{(0)}+e^p(-1+2p)}} + \rho_m^{(0)}\right)}}\right) \left(1-\Omega_m^{(0)}\right)\left(a^3\left(a^{-\frac{3(1+A)\left(\Omega_m^{(0)}+e^p(-1+2p)\right)}{1+e^p(-1+2p)}}C_1 - \frac{3a^{-n}B_0\left(\Omega_m^{(0)}+e^p(-1+2p)\right)}{n-3\Omega_m^{(0)}-3A\Omega_m^{(0)}+e^p(-3-3A+n)(-1+2p)}\right)^{1+\frac{1-\Omega_m^{(0)}}{\Omega_m^{(0)}+e^p(-1+2p)}} + \rho_m^{(0)}\right). \tag{31}$$

From Equation (31), we can obtain the derivative of reconstructed exponential $f(T)$ with respect to T as $f_{T,EXP}$ and its second derivative as $f_{TT,EXP}$ (expressions are not provided in the paper due to their complexity). By incorporating Equations (30) and (31) and the expressions of $f_{T,EXP}$ and $f_{TT,EXP}$ in Equation (18), we have the reconstructed EoS parameter $w_{EXP,VMCG}$ for VMCG in exponential $f(T)$ gravity, and the same is plotted in Figure 1. From Figure 1, we can conclude that it is showing a quintom behavior, i.e., a transition from quintessence to a phantom phase at the early stage of the universe, and it crosses the phantom boundary again in the later stage, which implies avoidance of the Big-Rip singularity (at $z \approx 0.8$).

By implementing Equation (29) and the expressions of $f_{T,EXP}$ and $f_{TT,EXP}$ in Equation (19), we have the reconstructed squared speed of sound $c^2_{EXP,VMCG}$ for VMCG in exponential $f(T)$ gravity, and the same is plotted in Figure 2. From Figure 2, we observed that it lies in the positive region and it is also increasing from early to later stage of the universe. Thus, we conclude that the VMCG in exponential $f(T)$ gravity is classically stable against small gravitational perturbations.

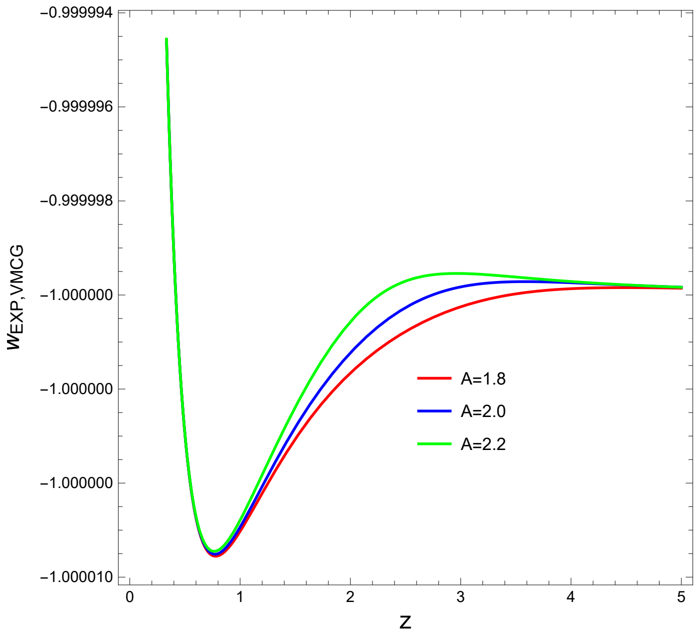

Figure 1. Evolution of reconstructed EoS parameter $w_{EXP,VMCG}$ with respect to redshift z for VMCG in exponential $f(T)$ gravity. The parameters chosen are $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $p = 0.0001$, $\alpha = 3$, $n = 0.01$, $B_0 = 0.9$, $C_1 = 0.006$, and $T_0 = -29{,}400$.

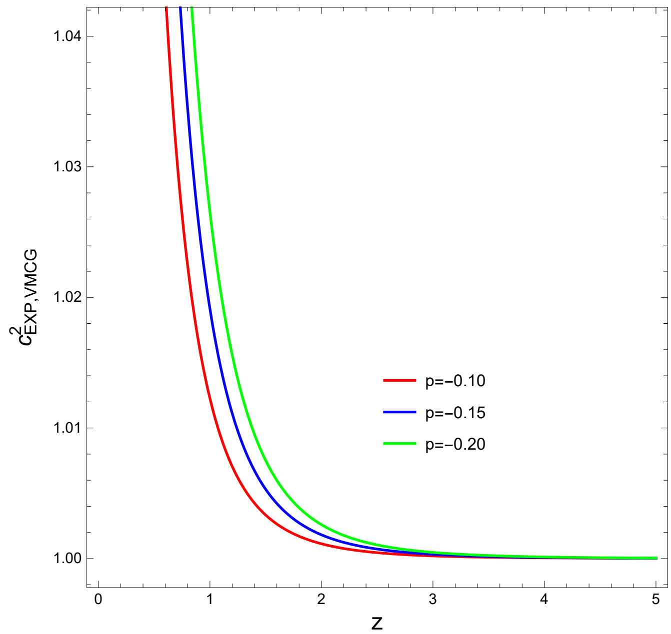

Figure 2. Evolution of reconstructed squared speed of sound $c^2_{EXP,VMCG}$ with respect to redshift z for VMCG in exponential $f(T)$ gravity. The parameters chosen are $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $A = 1$, $\alpha = 3$, $n = 0.01$, $B_0 = 0.9$, $C_1 = 0.006$, and $T_0 = -29{,}400$.

3.2. Logarithmic $f(T)$ Theory

Unlike exponential $f(T)$ gravity, only the non-phantom scenario was realized in the cosmological setting of logarithmic $f(T)$ gravity in [52], which is a viable form of $f(T)$ gravity. Given the study of [52], we incorporated VMCG as the background fluid and accordingly reconstructed $f(T)$ gravity to determine whether, with this reconstruction, we can realize both phantom and non-phantom phases. The logarithmic $f(T)$ theory is given by [52]

$$f(T) = \beta T_0 \left(\frac{qT_0}{T}\right)^{-\frac{1}{2}} ln\left(\frac{qT_0}{T}\right) \tag{32}$$

where

$$\beta \equiv \frac{1-\Omega_m^{(0)}}{2q^{-\frac{1}{2}}}, \tag{33}$$

where q is a positive constant. It is observed that if the value of $\Omega_m^{(0)}$ is acquired using the same method as the exponential $f(T)$ theory, then we can say that the logarithmic $f(T)$ theory in Equation (32) consists of only one parameter q.

VMCG in Logarithmic $f(T)$ Theory

In this subsection, we will study the cosmological consequences of VMCG model in logarithmic $f(T)$ theory. By using Equations (30) and (33) in Equation (32), we obtain the reconstructed logarithmic $f(T)$ as

$$f(T)_{LOG} = \frac{(1-\Omega_m^{(0)})\sqrt{q}T_0 Log\left[-\frac{qT_0}{2\left(\frac{\rho_m^{(0)}}{a^3}+\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3+3A-n+3(1+A)\alpha}\right)^{\frac{1}{1+\alpha}}\right)}\right]}{\sqrt{2}\sqrt{-\frac{qT_0}{\frac{\rho_m^{(0)}}{a^3}+\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3+3A-n+3(1+A)\alpha}\right)^{\frac{1}{1+\alpha}}}}Log[2]}. \tag{34}$$

The derivative of reconstructed logarithmic $f(T)$ with respect to T is

$$\begin{aligned} f_{T,LOG} = \Bigg(& 3(-1+\Omega_m^0)\sqrt{-\frac{a^3 qT_0}{\rho_m^0+a^3\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3-n+3\alpha+3A(1+\alpha)}\right)^{\frac{1}{1+\alpha}}}} \\ & \Big(3a^{3(1+A)(1+\alpha)}B_0\rho_m^0(1+\alpha) - a^n C_1\rho_m^0(n-3(1+\alpha)-3A(1+\alpha)) + \\ & a^{3(2+A+\alpha+A\alpha)}B_0 n\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3-n+3\alpha+3A(1+\alpha)}\right)^{\frac{1}{1+\alpha}} + \\ & a^{3+n}(1+A)C_1(3-n+3\alpha+3A(1+\alpha))\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3-n+3\alpha+3A(1+\alpha)}\right)^{\frac{1}{1+\alpha}}\Big) \\ & \left(-2+Log\left[-\frac{qT_0}{2\left(\frac{\rho_m^0}{a^3}+\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3+3A-n+3(1+A)\alpha}\right)^{\frac{1}{1+\alpha}}\right)}\right]\right)\Bigg) \\ & \Big(4\sqrt{2}a^4\sqrt{q}\left(3a^{3(1+A)(1+\alpha)}B_0(1+\alpha)-a^n C_1(-3+n-3\alpha-3A(1+\alpha))\right) \\ & \left(-\frac{3\rho_m^0}{a^4}+\frac{3\left(-a^{-3(1+A)(1+\alpha)}(1+A)C_1+\frac{a^{-n}B_0 n}{-3+n-3\alpha-3A(1+\alpha)}\right)\left(a^{-3(1+A)(1+\alpha)}C_1+\frac{3a^{-n}B_0(1+\alpha)}{3-n+3\alpha+3A(1+\alpha)}\right)^{-1+\frac{1}{1+\alpha}}}{a}\right)Log[2]\Big)^{-1}. \end{aligned} \tag{35}$$

From Equation (35), we can obtain the second derivative of reconstructed logarithmic $f(T)$ with respect to T as $f_{TT,LOG}$ (Expression is not provided in the paper due to its complexity). By incorporating Equations (30), (34), and (35) and the expression of $f_{TT,LOG}$ in Equation (18), we have the reconstructed EoS parameter $w_{LOG,VMCG}$ for VMCG in logarithmic $f(T)$ gravity and the same is plotted in Figure 3. From Figure 3, we can conclude that it is showing a similar kind of behavior to the case of Figure 1, i.e., a quintom behavior at the early stage of the universe, and again crosses the phantom boundary, which signifies avoidance of the Big-Rip singularity (at $z \approx 0.45$), and in the later stage it becomes asymptotic to -0.8.

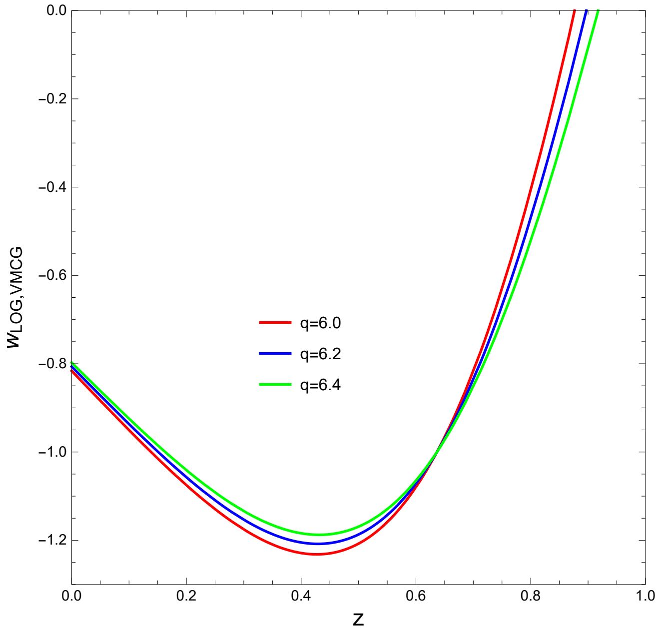

Figure 3. Evolution of reconstructed EoS parameter $w_{LOG,VMCG}$ with respect to redshift z for VMCG in logarithmic $f(T)$ gravity. The parameters chosen are $A = 0.005$ $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $\alpha = 0.001$, $n = 0.002$, $B_0 = 0.003$, $C_1 = 0.004$, and $T_0 = -29{,}400$.

By substantial substitutions of Equations (29) and (35) and the expression of $f_{TT,LOG}$ in Equation (19), we have the reconstructed squared speed of sound $c^2_{LOG,VMCG}$ for VMCG in logarithmic $f(T)$ gravity, and the same is plotted in Figure 4. From Figure 4, we observed that it lies in the positive region, and it shows an increasing pattern at the early stage of the universe and a decreasing pattern at the later stage. Thus, we conclude that the VMCG in logarithmic $f(T)$ gravity is classically stable against small gravitational perturbations.

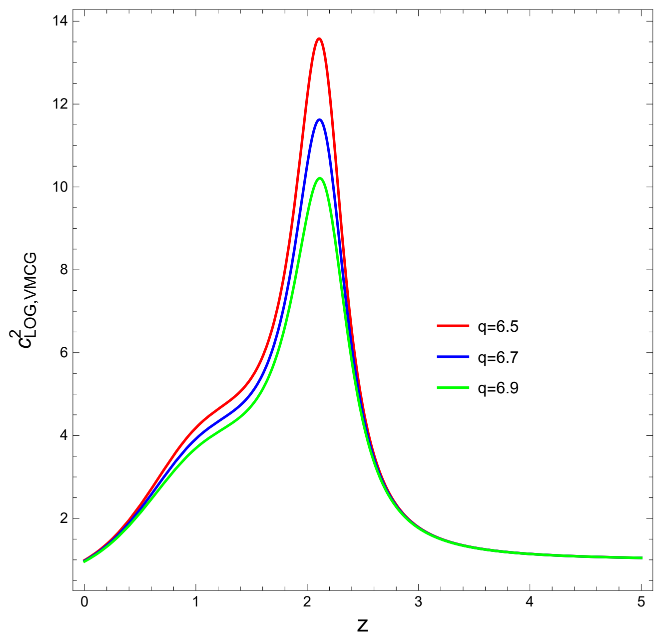

Figure 4. Evolution of reconstructed squared speed of sound $c^2_{LOG,VMCG}$ with respect to redshift z for VMCG in logarithmic $f(T)$ gravity. The parameters chosen are $A = 3.4$ $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $\alpha = 0.01$, $n = 0.02$, $B_0 = 0.03$, $C_1 = 0.004$, and $T_0 = -29{,}400$.

In the next section, we explore the evolution of primordial perturbations in the cosmological settings of VMCG in exponential and logarithmic $f(T)$ gravity.

4. Primordial Perturbations in the Cosmological Settings of VMCG in Exponential and Logarithmic $f(T)$ Theory

In the work of [57], the technique of cosmological reconstruction, which builds a model for an arbitrary evolution of the scale factor, was studied for the matter density contrast's evolution behavior. A model comprising a tachyon and scalar phantom with conformal quantum matter has been presented in a notable work [58]. Their model, perturbed by quantum effects, realized two de Sitter phases: the phantom/tachyon was responsible for the late-time accelerating universe, and quantum effects produced the early universe inflation. The authors of the study [59] explicitly demonstrated that the perturbation modes exit the horizon during the pre-bounce contraction era at a sizable negative time. In recent work, [59] proposed a nonsingular bounce cosmology in the context of $f(R)$ generalized by the Lagrange multiplier. A fluid model incorporating the EoS for a fluid with bulk viscosity was studied by [60] by examining the tensor-to-scalar ratio of the density perturbations and the spectral index of the curvature perturbations. In [61], it has been demonstrated that the power spectrum of primordial scalar curvature perturbations in the early-time bounce is almost scale-invariant.

A density perturbation analysis of the reconstructed $f(T)$ gravity within the framework of VMCG is presented in the present work. Numerous studies on the density perturbation of a universe dominated by other forms of Chaplygin gas were conducted in [62,63]. By establishing a free parameter in the EoS of Chaplygin gas, [64] demonstrated that it is feasible to obtain the value for the density contrast found in the universe's large-scale structure. We reconstructed the exponential and logarithmic $f(T)$ theory in Equations (31) and (34), respectively, by considering the background evolution of VMCG, and, subsequently, the derivatives of different orders were obtained. We investigated primordial perturbation using this reconstructed $f(T)$. Refs. [65–67] previously used this strategy in various settings. The details are demonstrated in the following paragraphs.

We consider the matter component to be a canonical scalar field ϕ with the Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{36}$$

By following [68], we applied the perturbed equation of motion for the gravitational potential Φ to comprehend the evolution of scalar-sector metric perturbations. As explained in [68], the final form of the equation of motion of one Fourier mode Φ_κ is

$$\ddot{\Phi}_\kappa + \alpha\dot{\Phi}_\kappa + \mu^2\Phi_\kappa + c_s^2\frac{\kappa^2}{a^2}\Phi_\kappa = 0 \tag{37}$$

with

$$\alpha = 7H + \frac{2V_\phi}{\dot{\phi}} - \frac{36H\dot{H}(f_{TT} - 4H^2 f_{TTT})}{1 + f_T - 12H^2 f_{TT}} \tag{38}$$

and

$$\mu^2 = 6H^2 + 2\dot{H} + \frac{2HV_\phi}{\dot{\phi}} - \frac{36H^2\dot{H}(f_{TT} - 4H^2 f_{TTT})}{1 + f_T - 12H^2 f_{TT}}. \tag{39}$$

where α is the frictional term and μ_2 is the effective mass for the gravitational potential Φ. The background equation for the scalar field is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0. \tag{40}$$

In the present scenario, the equation $\dot{H} = -\frac{4\pi G(\rho_m + p_m)}{1 + f_T - 12H^2 f_{TT})}$ can be written as

$$(1 + f_T - 12H^2 f_{TT})\dot{H} = -4\pi G\dot{\phi}^2, \tag{41}$$

where $4\pi G = \frac{1}{2}$.

We have the time derivative of H as

$$\dot{H} = \frac{1}{6}a\left(-\frac{3\rho_m^{(0)}}{a^4} + \left(3\left(-a^{-3(1+A)(1+\alpha)}(1+A)C_1 + \frac{a^{-n}B_0 n}{-3+n-3\alpha-3A(1+\alpha)}\right)\left(a^{-3(1+A)(1+\alpha)}C_1 + \frac{3a^{-n}B_0(1+\alpha)}{3-n+3\alpha+3A(1+\alpha)}\right)^{-1+\frac{1}{1+\alpha}}\right)a^{-1}\right). \tag{42}$$

Using Equations (29) and (42) and the expressions for $f_{T,EXP}$, $f_{TT,EXP}$ in Equation (41), we have $\dot{\phi}_{EXP,VMCG}$ for VMCG in exponential $f(T)$ theory and by incorporating the same in Equation (40) we have the time derivative of the self-interacting potential $\dot{V}_{EXP,VMCG}$ for VMCG in exponential $f(T)$ theory. Similarly, by using Equations (29), (35) and (42) and the expression for $f_{TT,LOG}$ in Equation (41), we have $\dot{\phi}_{LOG,VMCG}$ for VMCG in logarithmic $f(T)$ theory, and by implementing the same in Equation (40), we have the time derivative of the self-interacting potential $\dot{V}_{LOG,VMCG}$ for VMCG in logarithmic $f(T)$ theory. $\dot{V}_{EXP,VMCG}$ and $\dot{V}_{LOG,VMCG}$ are plotted in Figures 5 and 6, respectively. In both Figures 5 and 6, we can see that the time derivative of the self-interacting potential lies in the positive region, and it decreases with the evolution of the universe, which means the potential increases with cosmic time t.

We also depict the derivative of potential V with respect to the scalar field ϕ versus z for VMCG in both exponential and logarithmic $f(T)$ theory in Figures 7 and 8, respectively. From both the Figures 7 and 8, we can conclude that the scalar field ϕ decreases with the universe's evolution and becomes asymptotic towards 0, i.e., towards the minimum value of the potential V.

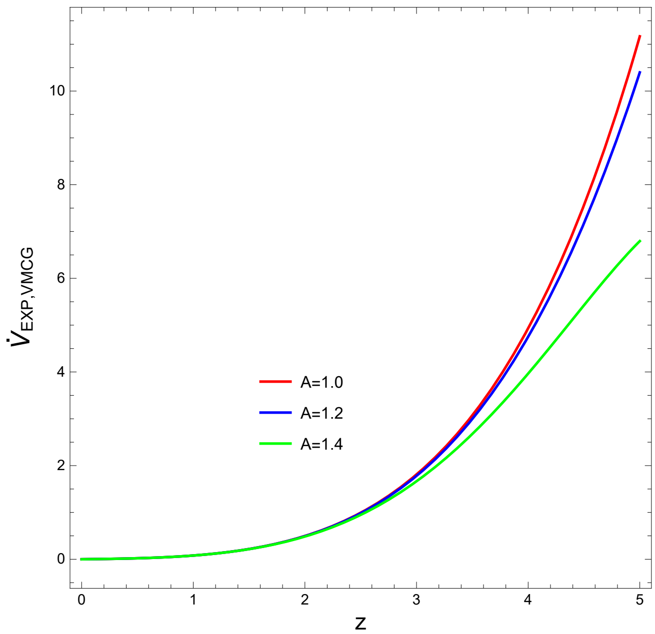

Figure 5. Evolution of the time derivative of the self interacting potential $\dot{V}_{EXP,VMCG}$ with respect to redshift z for VMCG in exponential $f(T)$ gravity. The parameters chosen are $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $p = 0.06$, $\alpha = 3$, $n = 0.01$, $B_0 = 0.9$, $C_1 = 0.006$, and $T_0 = -29{,}400$.

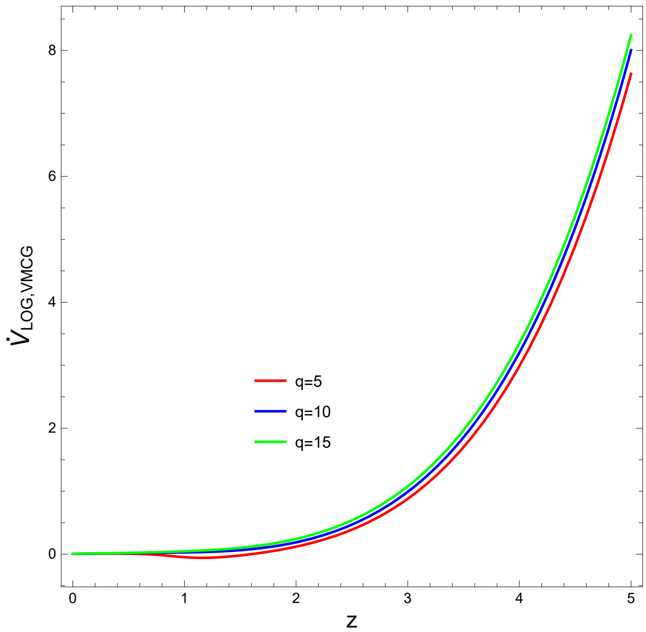

Figure 6. Evolution of the time derivative of the self interacting potential $\dot{V}_{LOG,VMCG}$ with respect to redshift z for VMCG in logarithmic $f(T)$ gravity. The parameters chosen are $A = 0.005$, $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $\alpha = 0.001$, $n = 0.002$, $B_0 = 0.003$, $C_1 = 0.004$, and $T_0 = -29{,}400$.

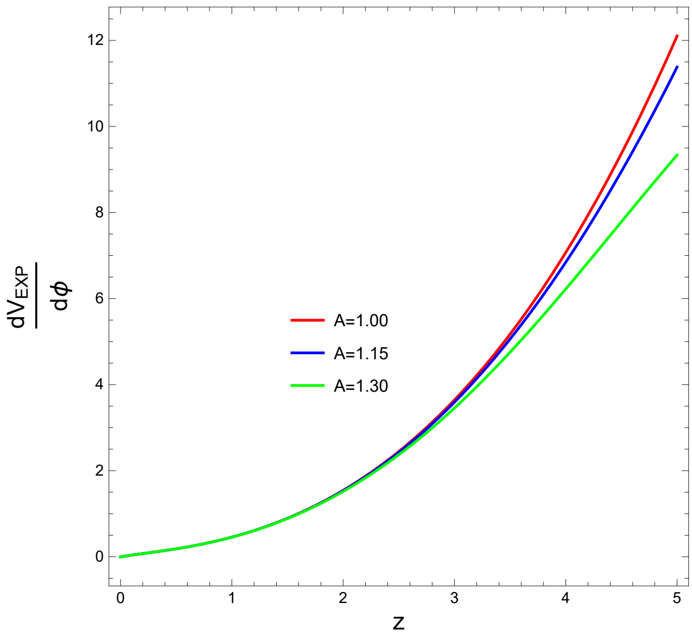

Figure 7. Plot of $\frac{dV_{EXP}}{d\phi}$ versus z for VMCG in exponential $f(T)$ gravity. The parameters chosen are $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $p = 0.06$, $\alpha = 3$, $n = 0.01$, $B_0 = 0.9$, $C_1 = 0.006$, and $T_0 = -29{,}400$.

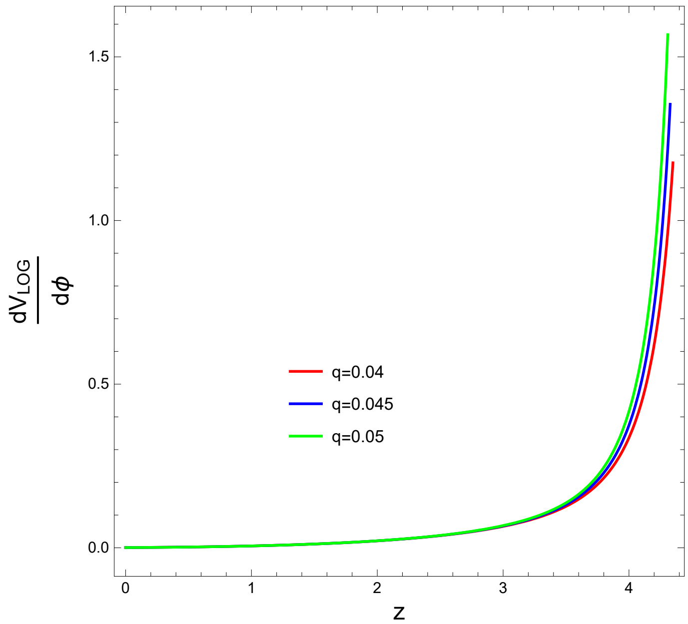

Figure 8. Plot of $\frac{dV_{LOG}}{d\phi}$ versus z for VMCG in logarithmic $f(T)$ gravity. The parameters chosen are $A = 0.005$, $\Omega_m^{(0)} = 0.26$, $\rho_m^{(0)} = 3822$, $\alpha = 0.001$, $n = 0.002$, $B_0 = 0.003$, $C_1 = 0.004$, and $T_0 = -29{,}400$.

5. Conclusions

In this work, we studied the VMCG model in the cosmological settings of exponential and logarithmic $f(T)$ theory. We observed the behavior of the reconstructed EoS parameter $w_{EXP,VMCG}$ for VMCG in exponential $f(T)$ gravity, and it shows a quintom behavior, i.e., a transition from quintessence to a phantom phase at the early stage of the universe, and it crosses the phantom boundary again in the later stage, which implies avoidance of the Big-Rip singularity (at $z \approx 0.8$). We observed that the reconstructed squared speed of sound $c^2_{EXP,VMCG}$ for VMCG in exponential $f(T)$ gravity lies in the positive region, and it is also increasing from the early to later stages of the universe. Thus, we understand that the VMCG in exponential $f(T)$ gravity is classically stable against small gravitational perturbations.

We also observed the reconstructed EoS parameter $w_{LOG,VMCG}$ for VMCG in logarithmic $f(T)$ gravity. It shows a similar kind of behavior to the case of $w_{EXP,VMCG}$, i.e., a quintom behavior at the early stage of the universe, and again crosses the phantom boundary, which signifies avoidance of the Big-Rip singularity (at $z \approx 0.45$), and in the later stage it becomes asymptotic to -0.8. The reconstructed squared speed of sound $c^2_{LOG,VMCG}$ for VMCG in logarithmic $f(T)$ gravity is also observed, and it lies in the positive region. An increasing pattern at the early stage of the universe is seen, while there is a decreasing pattern in the later stage. Thus, we conclude that the VMCG in logarithmic $f(T)$ gravity is classically stable against small gravitational perturbations.

We studied the evolution of the time derivative of the self-interacting potential $\dot{V}_{EXP,VMCG}$ and $\dot{V}_{LOG,VMCG}$ for VMCG in exponential and logarithmic $f(T)$ gravity, respectively, with respect to redshift z and observed that it lies in the positive region, and it decreases with the evolution of the universe which means the potential increases with cosmic time t. We also observed the evolution of $\frac{dV_{EXP}}{d\phi}$ and $\frac{dV_{LOG}}{d\phi}$ with respect to z for VMCG in both exponential and logarithmic $f(T)$ theory, respectively, and inferred that the scalar field ϕ decreases with the evolution of the universe and becomes asymptotic towards 0, i.e., towards the minimum value of the potential V.

While concluding, let us comment on the outcomes of the current work with respect to the existing literature. A recent study [69] demonstrated the cosmology of some generalized versions of Chaplygin gas in $f(T)$ gravity framework. The study reported here is in line with that work, and here, in addition to the late-time cosmology, we carried out primordial perturbation analyses in this framework. In another recent study [70], different cosmological CG equations of state were used and compared with the EoS for the modified teleparallel gravity, and its corresponding Lagrangian densities were reconstructed. In our study, we focused more on the reconstruction of $f(T)$ gravity itself and on seeing the consequences for primordial perturbations, and the results are demonstrated accordingly. In a notable work [71], various $f(T)$ gravity models were reconstructed that correspond to a range of dark energy scenarios, which include the polytropic, the standard Chaplygin, the generalized Chaplygin, and the modified Chaplygin gas models. In our study, apart from reconstructions, we also examined the reconstructed model's stability. In the future, we intend to find the results obtained here in torsion-free gravity theories.

Author Contributions: The work was formulated jointly by S.S. and S.C., and they jointly carried out the computation. The first draft was prepared by S.S. under the supervision of S.C. The third author, A.P., participated in generating the plots along with S.S. The final draft was prepared and checked by all the authors. All authors have read and agreed to the published version of the manuscript.

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