



Rigorous Lower Bound of the Dynamical Critical Exponent of the Ising Model

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Abstract

We study the kinetic Ising model under Glauber dynamics and establish an upper bound on the spectral gap for finite systems. This bound implies the critical exponent inequality $z \geq 2$, thereby rigorously improving the previously known estimate $z \geq 2 - \eta$. Our proof relies on the mapping from stochastic processes to frustration-free quantum systems and leverages the Simon–Lieb and Gosset–Huang inequalities.

Keywords Ising model · Glauber dynamics · Markov process · Critical systems · Dynamical critical exponent · Frustration-free Hamiltonian

1 Introduction

The Ising model is a paradigmatic model of phase transition. As the namesake model for the Ising universality class, its critical exponents in particular have been studied extensively with numerical methods [1, 2]. Rigorous results, including the exact solution in two dimensions, have also been explored [3, 4]. An important version of the Ising model is its *stochastic* variant, which captures how spins equilibrate when there are no conservation laws. The kinetic Ising model, which implements the Glauber dynamics of the Ising model, belongs to this universality class.

When the Ising model is near or right at criticality, the relaxation time τ is believed to undergo critical slowing down. The degree of the critical slowing down can be captured by the dynamical critical exponent. Near the critical point in an infinite system, τ is expected to behave as

$$\tau \simeq \xi^z, \quad (1)$$

where ξ is the correlation length, which also is expected to grow as $\xi \simeq |T - T_c|^{-\nu}$. For the kinetic Ising model, Halperin [5, 6] showed $z\nu \geq \gamma$, where γ is the critical exponent of the

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susceptibility i.e. $\chi \simeq |T - T_c|^{-\gamma}$.¹ Using (unproven) Fisher’s identity $\gamma = (2 - \eta)v$, this bound is converted to

$$z \geq 2 - \eta, \tag{2}$$

where η is the anomalous dimension characterizing the decay of correlation functions at the critical point, i.e., $\langle \sigma_i \sigma_j \rangle \simeq |i - j|^{-d+2-\eta}$ as $|i - j| \rightarrow \infty$. The values of η are known exactly in $d = 2$, and $d \geq 4$, and very accurately in $d = 3$; $\eta = 1/4$ for $d = 2$, $\eta = 0.0362978(20)$ for $d = 3$ [2], and $\eta = 0$ for $d \geq 4$ [4]. Numerical results performed with models in the stochastic Ising model give $z = 2.1667(5)$ for $d = 2$ [7] and $z = 2.0245(15)$ for $d = 3$ [8], which are consistent with this bound. A recent five-loop ϵ -expansion analysis yields compatible estimates, $z = 2.14(2)$ for $d = 2$ and $z = 2.0236(8)$ for $d = 3$ [9].

Equivalently, right at the critical point in a finite system, the system size L replaces the correlation length:

$$\tau \simeq L^z. \tag{3}$$

The best rigorous bound so far, based on this definition, for the two-dimensional kinetic Ising model is $z \geq 7/4$ [10], which coincides with the bound (2).

In this work, we present a rigorous improvement of this bound for the kinetic Ising model in any dimension $d \geq 2$:

$$z \geq 2 \tag{4}$$

using Eq. (3) as the definition of z . This bound for the two-dimensional Ising model was conjectured before in Ref. [11]. The key step is our recent work, which proved $z \geq 2$ quite generally for continuous-time Markov chains corresponding to a statistical mechanics model, assuming only the detailed balance condition and the locality of the update rule [12]. This work presents a shorter and mathematically more careful version of the proof focusing on the Ising model.

Our proof is based on a lower-bound on correlation functions that follows from the correlation inequality due to Simon and Lieb [13–15] and an upper-bound on correlation function by Gosset and Huang [16]. Our proof not only applies to the critical point, but also to any equilibrium state in which the two-point correlation function does not exhibit exponential decay.

2 Definitions

2.1 Ising Model in Equilibrium

In this work, we consider the standard ferromagnetic Ising model with the nearest neighbor interaction on the d -dimensional hypercubic lattice $\Lambda_L = \{ -\frac{L-1}{2}, -\frac{L-3}{2}, \dots, \frac{L-1}{2} \}^d \subset \mathbb{Z}^d$, where L is odd. We denote the a -th component of i by $i^{(a)}$ ($a = 1, 2, \dots, d$). For each site $i \in \Lambda_L$, we associate a spin variable $\sigma_i = \pm 1$. We denote by $\sigma = (\sigma_i)_{i \in \Lambda_L}$ a spin configuration, and by \mathcal{S}_L the set of all spin configurations.

The energy of a spin configuration $\sigma \in \mathcal{S}_L$ is given by

$$E(\sigma) = - \sum_{(i,j)} \sigma_i \sigma_j = -\frac{1}{2} \sum_{i \in \Lambda_L} \sigma_i \sum_{j \in B_i} \sigma_j, \tag{5}$$

¹ We review the discussion in Ref. [5] in Appendix A.

where the summation in the first expression is over nearest neighbor pairs, and B_i in the second expression is the set of sites adjacent to the site i . We use periodic boundary conditions to regard Λ_L as a torus. For any function O of spin configurations, we denote its expectation value in the equilibrium state at inverse temperature $\beta > 0$ as

$$\langle O \rangle := \frac{1}{Z} \sum_{\sigma \in \mathcal{S}_L} O(\sigma) w(\sigma), \tag{6}$$

where

$$w(\sigma) := e^{-\beta E(\sigma)} > 0 \tag{7}$$

is the Boltzmann weight and

$$Z := \sum_{\sigma \in \mathcal{S}_L} w(\sigma) \tag{8}$$

is the partition function. We note that the spin flipping symmetry $w(\sigma) = w(-\sigma)$ implies $\langle \sigma_i \rangle = 0$. Also, the first Griffiths inequality [17] states

$$0 \leq \langle \sigma_i \sigma_j \rangle \leq 1. \tag{9}$$

It is well known that the model undergoes a ferromagnetic phase transition if $d \geq 2$ [18, 19]. It has been proved that

$$\limsup_{L \rightarrow \infty} \sum_{i \in \Lambda_L} \langle \sigma_o \sigma_i \rangle_L \begin{cases} < \infty & \text{if } \beta < \beta_c; \\ = \infty & \text{if } \beta \geq \beta_c, \end{cases} \tag{10}$$

where $o \in \Lambda_L$ is the origin $(0, 0, \dots, 0)$ and $\beta_c < \infty$ is the critical inverse temperature.

2.2 Kinetic Ising Model

Now we introduce the kinetic Ising model, also called the stochastic Ising model, which implements the Glauber dynamics. Let $p(t, \sigma) \geq 0$ be the probability for the spin configuration σ to be realized at time t . We consider a continuous-time Markov process described by the master equation²

$$\frac{d}{dt} p(t, \sigma) = \sum_{\sigma' \in \mathcal{S}_L} W_{\sigma, \sigma'} p(t, \sigma'). \tag{11}$$

Let $P_i(\sigma) \geq 0$ be the rate at which the spin at i is flipped. Let us denote by $\tau_i(\sigma)$ the spin configuration obtained by flipping the spin at site i in $\sigma \in \mathcal{S}_L$. Then the transition rate matrix W is given by $\sum_{i \in \Lambda_L} W_i$, where

$$(W_i)_{\sigma, \sigma'} = \begin{cases} P_i(\sigma') & (\sigma = \tau_i(\sigma')) \\ -P_i(\sigma') & (\sigma = \sigma') \\ 0 & (\text{otherwise}) \end{cases}, \tag{12}$$

which satisfies

$$\sum_{\sigma \in \mathcal{S}_L} (W_i)_{\sigma, \sigma'} = 0. \tag{13}$$

² See, for example, Ref. [20] for a general introduction to stochastic processes.

This relation is necessary for the conservation of probability $\sum_{\sigma \in S_L} p(t, \sigma) = 1$.

For concreteness, in this work we assume the heat-bath algorithm [21]

$$P_i(\sigma) := \frac{1}{2} - \frac{1}{2} \sigma_i \tanh \left(\beta \sum_{j \in B_i} \sigma_j \right) > 0, \tag{14}$$

which satisfies the detailed balance condition

$$(W_i)_{\sigma, \sigma'} w(\sigma') = (W_i)_{\sigma', \sigma} w(\sigma) \tag{15}$$

for any $\sigma, \sigma' \in S_L$. Other choices of $P_i(\sigma)$ can be treated in the same way with minor modifications as discussed in Appendix C. The Boltzmann weight $w(\sigma) > 0$ is an eigenvector of W with the eigenvalue 0:

$$\sum_{\sigma' \in S_L} (W_i)_{\sigma, \sigma'} w(\sigma') = \left(\sum_{\sigma' \in S_L} (W_i)_{\sigma', \sigma} \right) w(\sigma) = 0, \tag{16}$$

As we will see in Sect. 2.3, $w(\sigma)/Z$ is the unique stationary distribution of our Markov process.

2.3 Real Symmetric Matrices

In order to relate the Markov process to a quantum mechanical problem, we use the standard similarity transformation to define a matrix H_i for each $i \in \Lambda_L$ by

$$(H_i)_{\sigma, \sigma'} := -\sqrt{\frac{w(\sigma')}{w(\sigma)}} (W_i)_{\sigma, \sigma'}. \tag{17}$$

The matrix will be used in (26) to define the (generalized) Rokhsar-Kivelson Hamiltonian [11, 22–24]. Note that the detailed balance condition (15) implies $(H_i)_{\sigma, \sigma'} = (H_i)_{\sigma', \sigma}$ and hence H_i is a real symmetric matrix. Since H_i and $-W_i$ are related to each other by the similarity transformation in (17), their eigenvalues are common. Also, the non-positive nature of off-diagonal components of H_i , combined with Eq. (16), implies that H_i is positive semi-definite.³ We also find from (16) that the normalized vector Φ_0 defined by

$$(\Phi_0)_\sigma := \sqrt{\frac{w(\sigma)}{Z}} \tag{18}$$

is an eigenvector of H_i with eigenvalue 0. It should be noted that the vector Φ_0 does not depend on i . As long as $0 \leq \beta < \infty$, the Perron–Frobenius theorem [25] implies that the eigenvalue 0 of $H = \sum_{i \in \Lambda_L} H_i$ is nondegenerate. Therefore, we have $\lim_{t \rightarrow \infty} (e^{-Ht})_{\sigma, \sigma'} = \sqrt{w(\sigma)w(\sigma')}/Z$.

Let us denote by ϵ_L the smallest eigenvalue of $-W$ other than 0. The quantity ϵ_L , which is called the spectral gap in the mathematical literature, essentially determines the decay

³ Let v be an eigenvector of H_i with the lowest eigenvalue μ_i and u be a vector whose components are given by $u_\sigma = |v_\sigma|$. Since off-diagonal components of H_i are nonpositive, $\mu_i = \sum_{\sigma, \sigma'} v_\sigma (H_i)_{\sigma, \sigma'} v_{\sigma'} \geq \sum_{\sigma, \sigma'} u_\sigma (H_i)_{\sigma, \sigma'} u_{\sigma'}$. Then the variational principle implies that u_σ is also an eigenvector of H_i with the eigenvalue μ_i . On the other hand, $w_\sigma = \sqrt{w(\sigma)} > 0$ is an eigenvector of H_i with the eigenvalue 0 and $\sum_{\sigma \in S_L} w_\sigma u_\sigma = \sum_{\sigma \in S_L} w_\sigma |v_\sigma| \neq 0$. Therefore, $\mu_i = 0$.

property of autocorrelation functions. For example, the equilibrium autocorrelation function of an operator O is defined by

$$\langle Oe^{Wt}O \rangle = \frac{1}{Z} \sum_{\sigma, \sigma' \in \mathcal{S}_L} O(\sigma)(e^{Wt})_{\sigma, \sigma'} O(\sigma')w(\sigma'). \tag{19}$$

For any σ' , $\lim_{t \rightarrow \infty} (e^{Wt})_{\sigma, \sigma'} = w(\sigma)/Z$, which implies

$$\lim_{t \rightarrow \infty} \langle Oe^{Wt}O \rangle = \sum_{\sigma \in \mathcal{S}_L} O(\sigma) \frac{w(\sigma)}{Z} \sum_{\sigma' \in \mathcal{S}_L} O(\sigma') \frac{w(\sigma')}{Z} = \langle O \rangle^2. \tag{20}$$

Then the autocorrelation function behaves as

$$|\langle Oe^{Wt}O \rangle - \langle O \rangle^2| \simeq Ce^{-\epsilon_L t}. \tag{21}$$

Suppose that the relaxation time $\tau = 1/\epsilon_L$ grows with L as L^z . This scaling defines the dynamical critical exponent z .

2.4 Theorem

Our main theorem is the following upper bound for the spectral gap ϵ_L .

Theorem 1 *Let $d \geq 2$. Then there exist constants C and L_0 that depend only on d such that*

$$0 < \epsilon_L \leq C \left(\frac{\log L}{L} \right)^2 \tag{22}$$

for any $\beta \geq \beta_c$ and $L \geq L_0$.

Comparing the bound (22) at $\beta = \beta_c$ with the expected finite size scaling behavior $\epsilon_L \simeq L^{-z}$, we see that the theorem implies $z \geq 2$ (assuming the existence of the exponent z). We also note that the bound (22) is most meaningful for $\beta = \beta_c$ since it is expected that ϵ_L decays much faster as L grows in the low-temperature phase with $\beta > \beta_c$. See Ref. [26] for known rigorous results for sufficiently large β .

3 Proof

We examine the behavior of the quantity

$$c(L) := 2d \sum_{j \in \partial \Lambda_L} \langle \sigma_o \sigma_j \rangle \tag{23}$$

as a function of L , where $\partial \Lambda_L$ is the set of lattice sites in which at least one of its components satisfies $|i^{(a)}| = \frac{L-1}{2}$. Note that $\partial \Lambda_L$ does not represent the boundary since we are imposing periodic boundary conditions.

We shall prove a lower bound (24) and an upper bound (34) for the same quantity $c(L)$. The combination of the two bounds leads to the desired upper bound for the spectral gap.

3.1 Lower Bound

A lower bound of $c(L)$ can be derived based on the Simon-Lieb inequality [13–15]. If the correlation function $\langle \sigma_i \sigma_j \rangle_{L'}$ does not decay exponentially, then

$$c(L) \geq 1 \tag{24}$$

for any $L = 1, 3, 5, \dots$ [13–15]. For readers' convenience, we review the proof in the Appendix B.

3.2 Upper Bound

In order to make use of results from quantum information, we shall identify the matrices discussed in the previous section with quantum mechanical operators. For each $i \in \Lambda_L$, let the local Hilbert space \mathcal{H}_i be the two-dimensional Hilbert space with the basis $\{|+1\rangle_i, |-1\rangle_i\}$. We consider a quantum system with the Hilbert space $\otimes_{i \in \Lambda_L} \mathcal{H}_i$ and denote its basis states as

$$|\sigma\rangle = \otimes_{i \in \Lambda_L} |\sigma_i\rangle_i, \tag{25}$$

where $\sigma \in \mathcal{S}_L$. Given $(H_i)_{\sigma, \sigma'}$ in (17), we define the Rokhsar-Kivelson Hamiltonian $\hat{H} = \sum_{i \in \Lambda_L} \hat{H}_i$ by

$$\hat{H}_i := \sum_{\sigma, \sigma' \in \mathcal{S}_L} (H_i)_{\sigma, \sigma'} |\sigma\rangle \langle \sigma'|. \tag{26}$$

As detailed in the previous section, the spectrum of this Hamiltonian coincides with that of the transition rate matrix.

For the heat-bath algorithm (14), we find

$$\hat{H}_i = \frac{1}{2 \cosh(\beta \sum_{j \in B_i} \hat{\sigma}_j^z)} \left(e^{-\beta \hat{\sigma}_i^z \sum_{j \in B_i} \hat{\sigma}_j^z} - \hat{\sigma}_i^x \right), \tag{27}$$

which includes up to $(2d + 1)$ -spin interactions. Here $\hat{\sigma}_i^x$ and $\hat{\sigma}_i^z$ are operators defined by

$$\hat{\sigma}_i^x = \sum_{\sigma \in \mathcal{S}_L} |\tau_i(\sigma)\rangle \langle \sigma|, \quad \hat{\sigma}_i^z = \sum_{\sigma \in \mathcal{S}_L} \sigma_i |\sigma\rangle \langle \sigma|, \tag{28}$$

which can be represented by Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{29}$$

on \mathcal{H}_i . It is easily checked that $[\hat{H}_i, \hat{H}_j] = 0$ unless $|i - j| = 1$. As one can easily show, $\hat{H}_i^2 = \hat{H}_i$, implying that \hat{H}_i is a projector. Furthermore,

$$|\Phi_0\rangle := \sum_{\sigma \in \mathcal{S}_L} (\Phi_0)_\sigma |\sigma\rangle \tag{30}$$

is a simultaneous ground state of all \hat{H}_i 's with the eigenvalue 0:

$$\hat{H}_i |\Phi_0\rangle = - \sum_{\sigma \in \mathcal{S}_L} |\sigma\rangle (H_i \Phi_0)_\sigma = 0. \tag{31}$$

In general, $\hat{H} = \sum_i \hat{H}_i$ is said to be frustration-free if a ground state $|\Phi\rangle$ of \hat{H} is a simultaneous ground state of all \hat{H}_i 's. (31) implies that \hat{H} is frustration-free.

For a general frustration-free Hamiltonian $\hat{H} = \sum_i \hat{H}_i$ with $\hat{H}_i^2 = \hat{H}_i$ and for arbitrary operators \hat{O} and \hat{O}' , the following inequality holds [16]⁴:

$$\frac{\langle \Phi_0 | \hat{O}(1 - \hat{G})\hat{O}' | \Phi_0 \rangle}{\|\hat{O}^\dagger | \Phi_0 \rangle\| \|\hat{O}' | \Phi_0 \rangle\|} \leq 2e^2 \exp\left(-\frac{D(\hat{O}, \hat{O}') - 1}{c - 1} \sqrt{\frac{\epsilon_L}{g^2 + \epsilon_L}}\right), \tag{32}$$

where \hat{G} is the projector onto the ground states, ϵ_L is the smallest eigenvalue of \hat{H} apart from 0, and $D(\hat{O}, \hat{O}')$ is a ‘‘distance’’ between \hat{O} and \hat{O}' with respect to the Hamiltonian \hat{H} . See Appendix F for the definition. For our Rokhsar-Kivelson Hamiltonian, $D(\hat{\sigma}_o^z, \hat{\sigma}_i^z)$ coincides with the L^1 -norm $\|i\|_1$ under periodic boundary condition defined by $\|i\|_1 := \sum_{a=1}^d \min\{|i^{(a)}|, L - |i^{(a)}|\}$. We also have $c = 2, g = 2d, \hat{G} = |\Phi_0\rangle\langle\Phi_0|, \langle\Phi_0|\hat{\sigma}_o^z|\Phi_0\rangle = \langle\Phi_0|\hat{\sigma}_i^z|\Phi_0\rangle = 0$, and $\|\hat{\sigma}_o^z|\Phi_0\rangle\| = \|\hat{\sigma}_i^z|\Phi_0\rangle\| \leq 1$. Plugging these in, we find⁵

$$\langle \Phi_0 | \hat{\sigma}_o^z \hat{\sigma}_i^z | \Phi_0 \rangle \leq 2e^2 \exp\left(-(\|i\|_1 - 1) \sqrt{\frac{\epsilon_L}{(2d)^2 + \epsilon_L}}\right). \tag{33}$$

Since $\langle \Phi_0 | \hat{\sigma}_o^z \hat{\sigma}_i^z | \Phi_0 \rangle = \frac{1}{Z} \sum_{\sigma \in \mathcal{S}_L} \sigma_o \sigma_i w(\sigma) = \langle \sigma_o \sigma_i \rangle$, the inequality (32) gives an upper bound of $c(L)$:

$$c(L) \leq 2d|\partial\Lambda_L| \cdot 2e^2 \exp\left(-\frac{L - 3}{2} \sqrt{\frac{\epsilon_L}{(2d)^2 + \epsilon_L}}\right), \tag{34}$$

where $|\partial\Lambda_L| \leq 2dL^{d-1}$.

3.3 Proof of (22)

Inequalities (24) and (34) imply

$$\frac{\epsilon_L}{\epsilon_L + (2d)^2} \leq f(L)^2, \tag{35}$$

where

$$f(L) := \frac{2}{L - 3} \ln(8e^2 d^2 L^{d-1}). \tag{36}$$

When $f(L)^2 < 1$, this can be written as

$$0 < \epsilon_L \leq (2d)^2 \frac{f(L)^2}{1 - f(L)^2}. \tag{37}$$

We find that

$$0 < \epsilon_L \leq (2d)^4 \left(\frac{\log L}{L}\right)^2 \tag{38}$$

for any $L \geq 10e^2 d^2$.

⁴ This expression improves the original inequality in Ref. [16] in several ways: (i) the operator norm $\|\hat{O}\|$ is replaced by $\|\hat{O}|\Phi_0\rangle\| \leq \|\hat{O}\|$, (ii) $2c - 1$ in the exponent is replaced by $2(c - 1)$, and (iii) the definition of the distance between \hat{O} and \hat{O}' is refined. We include the proof of this version in the Appendix F.

⁵ The inequality (32) for $L \rightarrow \infty$ in the disordered phase directly implies that the correlation length ξ is smaller than or equal to $\sqrt{\frac{(2d)^2 + \epsilon_L}{\epsilon_L}}$. This suggests that $\tau = \frac{1}{\epsilon_L} \geq \frac{\xi^2 - 1}{(2d)^2}$. Hence, if one uses Eq. (1) as the definition of z , then this relation already implies that $z \geq 2$ for the critical point.

4 Discussions

Our results are applicable not only to the critical point but also to the entire ordered phase. In the ordered phase, ϵ_L measures the finite-size gap of quasi-degenerate ground states, which is typically exponentially small. A conjecture in Ref. [27] suggests that excitations are gapless throughout the ordered phase. Furthermore, our methods naturally extend to broader classes of Markov chains; see Ref. [12] for a comprehensive discussion.

Before closing, we note that Refs. [28–31] establish upper bounds on the spectral gap for frustration-free Hamiltonians under *open* boundary conditions without boundary terms⁶. The bound $\epsilon_L \leq C/L^2$ derived in the first three works is not directly applicable to our model (27), as it assumes strictly nearest-neighbor interactions. The result in Ref. [31] implies a similar bound, but there are important differences: it considers anisotropic shapes in addition to open boundary conditions and focuses on the asymptotic behavior in the large L limit, rather than providing a bound for each finite system. Hence, the conclusion is not directly comparable to ours.

Appendix A Halperin’s bound

Here we review the argument for the previous best bound $z \geq 2 - \eta$ obtained by Halperin [5] in terms of the Rokhsar–Kivelson Hamiltonian.

Let us consider the autocorrelation function of the total magnetization $\hat{M} := \sum_{i \in \Lambda_L} \hat{s}_i^z$.

$$C(t) := \langle \Phi_0 | \hat{M} e^{-\hat{H}t} (1 - \hat{G}) \hat{M} | \Phi_0 \rangle, \tag{A1}$$

where $\hat{G} = |\Phi_0\rangle\langle\Phi_0|$ is the projector onto the ground state of $\hat{H} = \sum_{i \in \Lambda_L} \hat{H}_i$ in Eq. (27). Let $|\Phi_n\rangle$ ($n = 1, 2, \dots$) be excited states and E_n be their eigenenergies. It follows that

$$C(t) = \sum_n |\langle \Phi_n | \hat{M} | \Phi_0 \rangle|^2 e^{-E_n t}. \tag{A2}$$

We define the characteristic time $\bar{\tau}$ by

$$\bar{\tau} := \frac{\int_0^\infty dt C(t)}{C(0)} = \frac{\langle \Phi_0 | \hat{M} \frac{1-\hat{G}}{\hat{H}} \hat{M} | \Phi_0 \rangle}{\langle \Phi_0 | \hat{M} (1 - \hat{G}) \hat{M} | \Phi_0 \rangle} = \frac{\sum_n \frac{|\langle \Phi_n | \hat{M} | \Phi_0 \rangle|^2}{E_n}}{\sum_n |\langle \Phi_n | \hat{M} | \Phi_0 \rangle|^2} \leq \frac{1}{\epsilon} = \tau. \tag{A3}$$

To derive a lower bound of this quantity, let us define the uniform susceptibility (T is the temperature)

$$\chi := \frac{1}{TL^d} \langle \Phi_0 | \hat{M} (1 - \hat{G}) \hat{M} | \Phi_0 \rangle \tag{A4}$$

and a quantity

$$\begin{aligned} R &:= \frac{1}{L^d} \langle \Phi_0 | \hat{M} \hat{H} (1 - \hat{G}) \hat{M} | \Phi_0 \rangle \\ &= \frac{1}{2L^d} \langle \Phi_0 | [\hat{M}, [\hat{H}, \hat{M}]] | \Phi_0 \rangle \end{aligned}$$

⁶ Under the open boundary condition without boundary terms, the Hamiltonian decomposes into a direct sum of sectors, each defined by fixing the boundary spins in every possible configuration. This situation differs from the familiar plus/minus boundaries, where all boundary spins are fixed to +1 or -1, and from free boundaries, where the boundary spins remain completely unconstrained.

$$= \frac{1}{L^d} \sum_i \left\langle \Phi_0 \left| \frac{1}{\cosh(\beta \sum_{j \in B_i} \hat{\sigma}_j^z)} \hat{\sigma}_i^x \right| \Phi_0 \right\rangle. \tag{A5}$$

The Schwartz inequality

$$\left\langle \Phi_0 \left| \hat{M} \frac{1 - \hat{G}}{\hat{H}} \hat{M} \right| \Phi_0 \right\rangle \langle \Phi_0 | \hat{M} \hat{H} (1 - \hat{G}) \hat{M} | \Phi_0 \rangle \geq \langle \Phi_0 | \hat{M} (1 - \hat{G}) \hat{M} | \Phi_0 \rangle^2 \tag{A6}$$

leads to

$$\tau \geq \bar{\tau} \geq \frac{T}{R} \chi. \tag{A7}$$

Now, we take the $L \rightarrow \infty$ limit. The susceptibility behaves as $\chi \simeq |T - T_c|^{-\nu}$, while R does not show singularity around the critical point $T = T_c$. Hence, if we assume $\bar{\tau} \simeq |T - T_c|^{-z\nu}$, one gets

$$z\nu \geq \gamma. \tag{A8}$$

Appendix B Derivation of the lower bound

Here we review Simon and Lieb’s results [13–15] on the correlation function of Ising model (7). In this section, we specify the system size in the expectation value (6) and write $\langle O \rangle_L$. We recall that we always use periodic boundary conditions.

B.1 Statement

Suppose that $c(L)$ defined in (23) satisfies $0 < c(L) < 1$ for some $L = 1, 3, 5, \dots$. Then the correlation function $\langle \sigma_o \sigma_i \rangle_{L'}$ decays exponentially for any $L' > L$, i.e.,

$$\langle \sigma_o \sigma_i \rangle_{L'} \leq C e^{-\|i\|_\infty / \xi}, \tag{B9}$$

where $\xi := -\frac{L+1}{2 \log c(L)} > 0$, $C := c(L)^{-1}$, and $\|i\|_\infty$ is the L^∞ -norm defined by

$$\|i\|_\infty := \max_a \left\{ \min \left\{ |i^{(a)}|, L - |i^{(a)}| \right\} \right\}. \tag{B10}$$

Note that B9 implies

$$\limsup_{L' \rightarrow \infty} \sum_{i \in \Lambda_{L'}} \langle \sigma_o \sigma_i \rangle_{L'} < \infty \tag{B11}$$

and hence $\beta < \beta_c$. See [18, 19]. We thus find that $c(L) > 1$ for any $L = 1, 3, 5, \dots$ if $\beta \geq \beta_c$.

B.2 Proof

Let us define an integer n_i by

$$n_i := \left\lfloor \frac{2}{L+1} \|i\|_\infty \right\rfloor, \tag{B12}$$

where $\lfloor r \rfloor$ represents the integer part of a real number r . If $n_i \geq 1$, the Simon-Lieb inequality [13–15] implies

$$\langle \sigma_o \sigma_i \rangle_{L'} \leq \sum_{j \in \partial \Lambda_L} \sum_{k_1 \in B_j \cap \partial \Lambda_{L+2}} \langle \sigma_o \sigma_j \rangle_L^{\text{free}} \langle \sigma_{k_1} \sigma_i \rangle_{L'}, \tag{B13}$$

where B_i denotes the set of lattice sites adjacent to i and $\langle \sigma_o \sigma_j \rangle_L^{\text{free}}$ is the correlation function with free boundary conditions, which can be bounded as $\langle \sigma_o \sigma_j \rangle_L^{\text{free}} \leq \langle \sigma_o \sigma_j \rangle_L$ by the Griffiths second inequality. Thus the right-hand side of B13 can be bounded from above to give

$$\langle \sigma_o \sigma_i \rangle_{L'} \leq \sum_{j \in \partial \Lambda_L} |B_j| \langle \sigma_o \sigma_j \rangle_L \max_{k_1 \in \partial \Lambda_{L+2}} \langle \sigma_{k_1} \sigma_i \rangle_{L'}. \tag{B14}$$

Plugging $|B_j| = 2d$ and the definition of $c(L)$ in (23), we find

$$\langle \sigma_o \sigma_i \rangle_{L'} \leq c(L) \max_{k_1 \in \partial \Lambda_{L+2}} \langle \sigma_o \sigma_{i-k_1} \rangle_{L'}. \tag{B15}$$

Repeating this process n_i times and using (9), we find

$$\langle \sigma_o \sigma_i \rangle_{L'} \leq c(L)^{n_i} \max_{k_1, k_2, \dots, k_{n_i} \in \partial \Lambda_{L+2}} \left\langle \sigma_o \sigma_{i - \sum_{j=1}^{n_i} k_j} \right\rangle_{L'} \leq c(L)^{n_i} \leq C e^{-\|i\|_\infty / \xi}, \tag{B16}$$

where we used $\lfloor r \rfloor \geq r - 1$ in the last step.

Appendix C Generalization

Our discussion presented in the main text can be readily extended to other algorithms in which $P_i(\sigma)$ (i) satisfies the detailed balance condition (15), (ii) depends only on spins in a finite distance from the site i , (iii) gives the transition matrix W with ergodicity.

To see this, note that the corresponding Rokhsar-Kivelson Hamiltonian

$$\hat{H}^{(\text{RK})} = \sum_{i \in \Lambda_L} \hat{H}_i^{(\text{RK})}, \tag{C17}$$

defined by (12) and (17), is frustration free and its unique ground state is given by $|\Phi_0\rangle$ in (30).

Let $|i, n\rangle$ ($n = 1, 2, \dots$) be the eigenvectors of $\hat{H}_i^{(\text{RK})}$ with nonzero eigenvalues. Then $\hat{P}_i^{(\text{RK})} := \sum_n |i, n\rangle \langle i, n|$ is the projector version of $\hat{H}_i^{(\text{RK})}$. Let $\epsilon_L^{(\text{RK})}$ and $\tilde{\epsilon}_L^{(\text{RK})}$ be the second smallest eigenvalue of $\hat{H}^{(\text{RK})}$ and $\sum_{i \in \Lambda_L} \hat{P}_i^{(\text{RK})}$, respectively. Then we have

$$\hat{H}^{(\text{RK})} \leq \max_{i \in \Lambda_L} \|\hat{H}_i^{(\text{RK})}\| \sum_{i \in \Lambda_L} \hat{P}_i^{(\text{RK})}, \tag{C18}$$

which implies

$$\epsilon_L^{(\text{RK})} \leq \max_{i \in \Lambda_L} \|\hat{H}_i^{(\text{RK})}\| \tilde{\epsilon}_L^{(\text{RK})}. \tag{C19}$$

Therefore, our discussion for projector Hamiltonians is sufficient.

For example, in the Metropolis algorithm, the flipping rate $P_i(\sigma)$ is given by

$$P_i(\sigma) := \min(1, e^{-2\beta \sigma_i \sum_{j \in B_i} \sigma_j}) \tag{C20}$$

and the corresponding Rokhsar-Kivelson Hamiltonian is

$$\hat{H}_i^{(RK)} = \min \left(e^{\beta \sum_{j \in B_i} \hat{\sigma}_j^z}, e^{-\beta \sum_{j \in B_i} \hat{\sigma}_j^z} \right) \left(e^{-\beta \hat{\sigma}_i^z \sum_{j \in B_i} \hat{\sigma}_j^z} - \hat{\sigma}_i^x \right). \tag{C21}$$

The largest eigenvalue of $\hat{H}_i^{(RK)}$ gives $\|\hat{H}_i^{(RK)}\| = 2$.

Another simple choice will be

$$P_i(\sigma) := e^{-\beta \sigma_i \sum_{j \in B_i} \sigma_j}, \tag{C22}$$

which corresponds to

$$\hat{H}_i^{(RK)} = e^{-\beta \hat{\sigma}_i^z \sum_{j \in B_i} \hat{\sigma}_j^z} - \hat{\sigma}_i^x \tag{C23}$$

with $\|\hat{H}_i^{(RK)}\| = 2 \cosh(2d\beta)$.

Appendix D Interaction graph

Let us define some notions that will play fundamental roles in Appendices E and F.

Consider a quantum system with Hilbert space \mathcal{H} and let \hat{H}_i with $i \in V$ be an arbitrary collection of operators on \mathcal{H} . Here, V is an arbitrary finite set, which we identify as the set of vertices. We then define the set of edges E as a collection of unordered pairs $\{i, j\}$ such that $[\hat{H}_i, \hat{H}_j] \neq 0$. The graph (V, E) is called the interaction graph associated with the collection of \hat{H}_i

Let $V_i := \{j \in V \mid [\hat{H}_i, \hat{H}_j] \neq 0\}$ be the set of vertices adjacent to $i \in V$. Let $g := \max_{i \in V} |V_i|$ be the maximum degree and c be the chromatic number of the interaction graph. We assume $c \geq 2$. Also, let $d(i, j)$ be the graph theoretic distance (the shortest-path distance) between $i \in V$ and $j \in V$ on the interaction graph.

Appendix E Detectability Lemma

The proof of the Gosset–Huang inequality is based on the detectability lemma [32, 33].

E.1 Statement

Suppose that the Hamiltonian

$$\hat{H} = \sum_{i=1}^N \hat{H}_i \tag{E24}$$

is a sum of projectors $\hat{H}_i^2 = \hat{H}_i$ and is frustration free, i.e., all \hat{H}_i 's can be simultaneously minimized. We assume that the lowest eigenvalue of \hat{H} is 0.

Let \hat{G} be the projector onto the space of all the ground states of \hat{H} . Note that $\hat{H}_i \hat{G} = 0$ for any i . Let ϵ be the smallest eigenvalue of \hat{H} other than 0. Let us write $\hat{P}_i := 1 - \hat{H}_i$ and define

$$\hat{P} := \hat{P}_{\sigma(1)} \hat{P}_{\sigma(2)} \cdots \hat{P}_{\sigma(N)}, \tag{E25}$$

where σ is an arbitrary permutation of $1, 2, \dots, N$. Furthermore, we consider the interaction graph (V, E) associated with \hat{H}_i and $V = \{1, 2, \dots, N\}$ and define the maximal degree g (see Sec. D). Then,

$$\|\hat{P} - \hat{G}\| \leq \sqrt{\frac{g^2}{g^2 + \epsilon}}. \tag{E26}$$

E.2 Proof

Take a state $|\psi\rangle$ and we perform the following operations to the quantity $\|\hat{H}_i \hat{P}_{\sigma(j)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|$. If \hat{H}_i commutes with $\hat{H}_{\sigma(j)}$,

$$\begin{aligned} \|\hat{H}_i \hat{P}_{\sigma(j)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| &= \|\hat{P}_{\sigma(j)} \hat{H}_i \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| \\ &\leq \|\hat{H}_i \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|. \end{aligned} \tag{E27}$$

Otherwise,

$$\begin{aligned} \|\hat{H}_i \hat{P}_{\sigma(j)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| &= \|\hat{H}_i (1 - \hat{H}_{\sigma(j)}) \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| \\ &\leq \|\hat{H}_i \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| + \|\hat{H}_i \hat{H}_{\sigma(j)} \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| \\ &\leq \|\hat{H}_i \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| + \|\hat{H}_{\sigma(j)} \hat{P}_{\sigma(j+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|. \end{aligned} \tag{E28}$$

Note that the first term in the last expression is the original quantity but j is shifted to $j + 1$. If we start from $\|\hat{H}_i \hat{P} |\psi\rangle\|$ and repeat this procedure, \hat{H}_i never reaches $|\psi\rangle$, because $\hat{H}_i \hat{P}_i = 0$. At the end, we obtain the following terms:

$$\|\hat{H}_i \hat{P} |\psi\rangle\| \leq \sum_{\sigma(l) \in V_i} \|\hat{H}_{\sigma(l)} \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|. \tag{E29}$$

Since the square of an average is less than or equal to the average of the squares,

$$\begin{aligned} \|\hat{H}_i \hat{P} |\psi\rangle\|^2 &\leq |V_i| \left(\frac{1}{|V_i|} \sum_{\sigma(l) \in V_i} \|\hat{H}_{\sigma(l)} \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\| \right)^2 \\ &\leq |V_i| \left(\frac{1}{|V_i|} \sum_{\sigma(l) \in V_i} \|\hat{H}_{\sigma(l)} \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|^2 \right) \\ &\leq g \sum_{\sigma(l) \in V_i} \|\hat{H}_{\sigma(l)} \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|^2. \end{aligned} \tag{E30}$$

Therefore,

$$\begin{aligned} \langle \psi | \hat{P}^\dagger \hat{H} \hat{P} | \psi \rangle &= \sum_{i=1}^N \langle \psi | \hat{P}^\dagger \hat{H}_i \hat{P} | \psi \rangle = \sum_{i=1}^N \langle \psi | \hat{P}^\dagger \hat{H}_i^2 \hat{P} | \psi \rangle = \sum_{i=1}^N \|\hat{H}_i \hat{P} |\psi\rangle\|^2 \\ &\leq g \sum_{i=1}^N \sum_{\sigma(l) \in V_i} \|\hat{H}_{\sigma(l)} \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} |\psi\rangle\|^2. \end{aligned} \tag{E31}$$

Exchanging the order of the summations, we find

$$\begin{aligned}
 \langle \psi | \hat{P}^\dagger \hat{H} \hat{P} | \psi \rangle &\leq g^2 \sum_{l=1}^N \|\hat{H}_{\sigma(l)} \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} | \psi \rangle\|^2 \\
 &= g^2 \sum_{l=1}^N \|(1 - \hat{P}_{\sigma(l)}) \hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} | \psi \rangle\|^2 \\
 &= g^2 \sum_{l=1}^N (\|\hat{P}_{\sigma(l+1)} \cdots \hat{P}_{\sigma(N)} | \psi \rangle\|^2 - \|\hat{P}_{\sigma(l)} \cdots \hat{P}_{\sigma(N)} | \psi \rangle\|^2) \\
 &= g^2 (\|\psi\rangle\|^2 - \|\hat{P} | \psi \rangle\|^2).
 \end{aligned}
 \tag{E32}$$

Applying this result to $|\psi^\perp\rangle := (1 - \hat{G})|\psi\rangle$ and using the definition of ϵ ,

$$\epsilon \|\hat{P} |\psi^\perp\rangle\|^2 \leq \langle \psi^\perp | \hat{P}^\dagger \hat{H} \hat{P} | \psi^\perp \rangle \leq g^2 (\|\psi^\perp\rangle\|^2 - \|\hat{P} |\psi^\perp\rangle\|^2),
 \tag{E33}$$

which can be rewritten as

$$\|\hat{P} |\psi^\perp\rangle\| \leq \sqrt{\frac{g^2}{g^2 + \epsilon}} \|\psi^\perp\rangle\|.
 \tag{E34}$$

Therefore, using $\hat{P}\hat{G} = \hat{G}$,

$$\begin{aligned}
 \|(\hat{P} - \hat{G})|\psi\rangle\| &= \|\hat{P}(1 - \hat{G})|\psi\rangle\| = \|\hat{P} |\psi^\perp\rangle\| \leq \sqrt{\frac{g^2}{g^2 + \epsilon}} \|\psi^\perp\rangle\| \leq \sqrt{\frac{g^2}{g^2 + \epsilon}} \|\psi\rangle\|
 \end{aligned}
 \tag{E35}$$

for any $|\psi\rangle$, which implies (E26).

Appendix F Gosset–Huang Inequality

F.1 Statement

Let V be a finite set and suppose that the Hamiltonian

$$\hat{H} = \sum_{i \in V} \hat{H}_i
 \tag{F36}$$

is a sum of projectors $\hat{H}_i^2 = \hat{H}_i$ and is frustration free. We assume that the lowest eigenvalue of \hat{H} is 0. Let \hat{G} be the projector onto the space of all the ground states of \hat{H} . Let ϵ be the smallest eigenvalue of \hat{H} other than 0.

We consider the interaction graph (V, E) associated with \hat{H}_i (see Sect. D). Let g be the maximum degree and c be the chromatic number of the interaction graph. We assume $c \geq 2$. Also, let $d(i, j)$ be the graph theoretic distance between $i, j \in V$ on the interaction graph.

Then, for any operators \hat{O} and \hat{O}' , the following inequality holds [16]:

$$\frac{\langle \Phi_0 | \hat{O}(1 - \hat{G})\hat{O}' | \Phi_0 \rangle}{\|\hat{O}^\dagger | \Phi_0 \rangle\| \|\hat{O}' | \Phi_0 \rangle\|} \leq 2e^2 \exp\left(-\frac{D(\hat{O}, \hat{O}') - 1}{c - 1} \sqrt{\frac{\epsilon}{g^2 + \epsilon}}\right),
 \tag{F37}$$

where ϵ is the smallest eigenvalue of \hat{H} apart from 0 and

$$D(\hat{O}, \hat{O}') := \min_{i,j} \{d(i, j) \mid [\hat{H}_i, \hat{O}] \neq 0, [\hat{H}_j, \hat{O}'] \neq 0\}.
 \tag{F38}$$

F.2 Proof

We assume a vertex coloring of the interaction graph and decompose the set V according to the colors of vertices as $V = \bigcup_{j=1}^c V^{(j)}$ so that $\{i, i'\} \notin E$ for any $i, i' \in V^{(j)}$. We shall relabel the operators \hat{H}_i as $\hat{H}_i^{(j)}$ where $j \in \{1, \dots, c\}$ and $i \in V^{(j)}$. By definition, $\hat{H}_i^{(j)}$ with the same color j commutes with each other. Set $\hat{P}_i^{(j)} := 1 - \hat{H}_i^{(j)}$ and $\hat{P}^{(j)} := \prod_{i \in V^{(j)}} \hat{P}_i^{(j)}$ for each $j \in \{1, \dots, c\}$. Furthermore, define

$$\hat{P} := \hat{P}^{(c)} \dots \hat{P}^{(2)} \hat{P}^{(1)}. \tag{F39}$$

Let n be a nonnegative integer. Note that $(\hat{P}^\dagger \hat{P})^n$ contains $2n(c - 1) + 1$ layers. The definition of $D(\hat{O}, \hat{O}')$ in (F38) implies that any path that connects \hat{O} and \hat{O}' on the interaction graph contains $D(\hat{O}, \hat{O}') + 1$ non-commuting operators. However, as far as

$$n \leq m := \left\lfloor \frac{D(\hat{O}, \hat{O}') - 1}{2(c - 1)} \right\rfloor, \tag{F40}$$

$2n(c - 1) + 1 \leq D(\hat{O}, \hat{O}')$. Hence, using $\hat{P}_i^{(j)}|\Phi_0\rangle = |\Phi_0\rangle$ for any i, j , we obtain

$$\langle \Phi_0 | \hat{O} (\hat{P}^\dagger \hat{P})^n \hat{O}' | \Phi_0 \rangle = \langle \Phi_0 | \hat{O} \hat{O}' | \Phi_0 \rangle, \tag{F41}$$

For any m -th order polynomial $Q_m(x)$ with $Q_m(1) = 1$, we have

$$\langle \Phi_0 | \hat{O} \hat{O}' | \Phi_0 \rangle = \langle \Phi_0 | \hat{O} Q_m(\hat{P}^\dagger \hat{P}) \hat{O}' | \Phi_0 \rangle. \tag{F42}$$

Writing $\hat{G}^\perp := 1 - \hat{G}$ and using $\hat{P} \hat{G} = \hat{G}$,

$$(\hat{P}^\dagger \hat{P})^n - \hat{G} = (\hat{P}^\dagger \hat{P} - \hat{G})^n = (\hat{P}^\dagger \hat{P} - \hat{G})^n \hat{G}^\perp \tag{F43}$$

for $n \geq 1$. For $n = 0$,

$$(\hat{P}^\dagger \hat{P})^0 - \hat{G} = \hat{G}^\perp = (\hat{P}^\dagger \hat{P} - \hat{G})^0 \hat{G}^\perp \tag{F44}$$

Therefore,

$$\begin{aligned} \langle \Phi_0 | \hat{O} (1 - \hat{G}) \hat{O}' | \Phi_0 \rangle &= \langle \Phi_0 | \hat{O} (Q_m(\hat{P}^\dagger \hat{P}) - \hat{G}) \hat{O}' | \Phi_0 \rangle \\ &= \langle \Phi_0 | \hat{O} Q_m(\hat{P}^\dagger \hat{P} - \hat{G}) \hat{G}^\perp \hat{O}' | \Phi_0 \rangle \\ &\leq \| \hat{O}^\dagger | \Phi_0 \rangle \| \| \hat{O}' | \Phi_0 \rangle \| \| Q_m(\hat{P}^\dagger \hat{P} - \hat{G}) \|. \end{aligned} \tag{F45}$$

Now we set $\delta := \epsilon / (g^2 + \epsilon)$. Since the detectability lemma (E26) implies $\| \hat{P}^\dagger \hat{P} - \hat{G} \| \leq 1 - \delta$, we obtain

$$\| Q_m(\hat{P}^\dagger \hat{P} - \hat{G}) \| \leq \max_{0 \leq x \leq 1 - \delta} |Q_m(x)|. \tag{F46}$$

We set

$$Q_m(x) = \frac{T_m\left(\frac{2x}{1-\delta} - 1\right)}{T_m\left(\frac{2}{1-\delta} - 1\right)}. \tag{F47}$$

Here, $T_m(x)$ is the m -th order Chebyshev polynomial of the first kind defined by $T_m(x) = \cos(m \arccos x)$ ($|x| \leq 1$) and $T_m(x) = \cosh(m \operatorname{arccosh} x)$ ($x > 1$), which satisfies

$$T_m(x) > \frac{1}{2} e^{2m\sqrt{(x-1)/(x+1)}} \tag{F48}$$

for $x > 1^7$ and $|T_m(x)| \leq 1$ for $|x| \leq 1$. Hence, $Q_m(x)$ satisfies

$$|Q_m(x)| \leq \frac{1}{T_m\left(\frac{1+\delta}{1-\delta}\right)} \leq 2e^{-2m\sqrt{\delta}} = 2 \exp\left(-2m\sqrt{\frac{\epsilon}{g^2 + \epsilon}}\right) \tag{F49}$$

for $0 \leq x \leq 1 - \delta$. Therefore, using $\lfloor r \rfloor \geq r - 1$,

$$\begin{aligned} \frac{\langle \Phi_0 | \hat{O}(1 - \hat{G}) \hat{O}' | \Phi_0 \rangle}{\|\hat{O}^\dagger | \Phi_0 \rangle\| \|\hat{O}' | \Phi_0 \rangle\|} &\leq 2 \exp\left(-2m\sqrt{\frac{\epsilon}{g^2 + \epsilon}}\right) \\ &\leq 2 \exp\left(2\sqrt{\frac{\epsilon}{g^2 + \epsilon}}\right) \exp\left(-\frac{D(\hat{O}, \hat{O}') - 1}{c - 1} \sqrt{\frac{\epsilon}{g^2 + \epsilon}}\right). \end{aligned} \tag{F50}$$

Finally using $\sqrt{\epsilon/(g^2 + \epsilon)} \leq 1$, we arrive at (F37).

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References

1. Hasenbusch, M., Pinn, K., Vinti, S.: Critical exponents of the three-dimensional Ising universality class from finite-size scaling with standard and improved actions. *Phys. Rev. B* **59**(17), 11471–11483 (1999). <https://doi.org/10.1103/PhysRevB.59.11471>
2. Poland, D., Simmons-Duffin, D.: The conformal bootstrap. *Nat. Phys.* **12**(6), 535–539 (2016). <https://doi.org/10.1038/nphys3761>
3. Onsager, L.: Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev.* **65**(3–4), 117–149 (1944). <https://doi.org/10.1103/PhysRev.65.117>
4. Fernández, R., Fröhlich, J., Sokal, A.D.: *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*. Springer, Berlin (1992)
5. Halperin, B.I.: Rigorous inequalities for the spin-relaxation function in the kinetic Ising model. *Phys. Rev. B* **8**, 4437–4440 (1973). <https://doi.org/10.1103/PhysRevB.8.4437>
6. Abe, R., Hatano, A.: Dynamics of the Ising model near the transition point. ii. *Prog. Theor. Phys.* **41**(4), 941–948 (1969). <https://doi.org/10.1143/PTP.41.941>
7. Nightingale, M.P., Blöte, H.W.J.: Monte Carlo computation of correlation times of independent relaxation modes at criticality. *Phys. Rev. B* **62**, 1089–1101 (2000). <https://doi.org/10.1103/PhysRevB.62.1089>
8. Hasenbusch, M.: Dynamic critical exponent z of the three-dimensional Ising universality class: Monte Carlo simulations of the improved Blume-Capel model. *Phys. Rev. E* **101**, 022126 (2020). <https://doi.org/10.1103/PhysRevE.101.022126>

⁷ We used $T_m(x) > \frac{1}{2}e^{m \operatorname{arccosh} x} > \frac{1}{2}e^{2m \tanh(\frac{1}{2} \operatorname{arccosh} x)} = \frac{1}{2}e^{2m\sqrt{(x-1)/(x+1)}}$ for $x > 1$.

9. Adzhemyan, L.Ts., Evdokimov, D.A., Hnatič, M., Ivanova, E.V., Kompaniets, M.V., Kudlis, A., Zakharov, D.V.: The dynamic critical exponent z for 2D and 3D Ising models from five-loop ε expansion. *Phys. Lett. A* **425**, 127870 (2022) <https://doi.org/10.1016/j.physleta.2021.127870>
10. Lubetzky, E., Sly, A.: Critical Ising on the square lattice mixes in polynomial time. *Commun. Math. Phys.* **313**(3), 815–836 (2012). <https://doi.org/10.1007/s00220-012-1460-9>
11. Isakov, S.V., Fendley, P., Ludwig, A.W.W., Trebst, S., Troyer, M.: Dynamics at and near conformal quantum critical points. *Phys. Rev. B* **83**, 125114 (2011). <https://doi.org/10.1103/PhysRevB.83.125114>
12. Masaoka, R., Soejima, T., Watanabe, H.: Rigorous lower bound of dynamical critical exponents in critical frustration-free systems [arXiv:2406.06415](https://arxiv.org/abs/2406.06415) (2024). <https://doi.org/10.48550/arXiv.2406.06415>
13. Boel, R.J., Kasteleyn, P.W.: Correlation-function identities and inequalities for Ising models with pair interactions. *Commun. Math. Phys.* **61**(3), 191–208 (1978). <https://doi.org/10.1007/BF01940764>
14. Simon, B.: Correlation inequalities and the decay of correlations in ferromagnets. *Commun. Math. Phys.* **77**(2), 111–126 (1980). <https://doi.org/10.1007/BF01982711>
15. Lieb, E.H.: A refinement of Simon’s correlation inequality. *Commun. Math. Phys.* **77**(2), 127–135 (1980). <https://doi.org/10.1007/BF01982712>
16. Gosset, D., Huang, Y.: Correlation length versus gap in frustration-free systems. *Phys. Rev. Lett.* **116**, 097202 (2016). <https://doi.org/10.1103/PhysRevLett.116.097202>
17. Griffiths, R.B.: Correlations in Ising ferromagnets. i. *J. Math. Phys.* **8**(3), 478–483 (1967). <https://doi.org/10.1063/1.1705219>
18. Friedli, S., Velenik, Y.: *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, Cambridge (2017)
19. Simon, B.: *The Statistical Mechanics of Lattice Gases*, vol. I. Princeton University Press, Princeton (1993)
20. Shiraishi, N.: *An Introduction to Stochastic Thermodynamics*. Springer, Singapore (2023)
21. Abe, R.: Dynamics of the Ising model near the transition point. *Prog. Theor. Phys.* **39**(4), 947–956 (1968). <https://doi.org/10.1143/PTP.39.947>
22. Henley, C.L.: From classical to quantum dynamics at Rokhsar–Kivelson points. *J. Phys.: Condens. Matter* **16**(11), 891 (2004). <https://doi.org/10.1088/0953-8984/16/11/045>
23. Castelnuovo, C., Chamon, C., Mudry, C., Pujol, P.: From quantum mechanics to classical statistical physics: generalized Rokhsar–Kivelson Hamiltonians and the “stochastic matrix form” decomposition. *Ann. Phys.* **318**(2), 316–344 (2005). <https://doi.org/10.1016/j.aop.2005.01.006>
24. Verstraete, F., Wolf, M.M., Perez-Garcia, D., Cirac, J.I.: Criticality, the area law, and the computational power of projected entangled pair states. *Phys. Rev. Lett.* **96**, 220601 (2006). <https://doi.org/10.1103/PhysRevLett.96.220601>
25. Tasaki, H.: *Physics and Mathematics of Quantum Many-Body Systems*, vol. 66. Springer, Singapore (2020)
26. Thomas, L.E.: Bound on the mass gap for finite volume stochastic Ising models at low temperature. *Commun. Math. Phys.* **126**(1), 1–11 (1989). <https://doi.org/10.1007/BF02124328>
27. Masaoka, R., Soejima, T., Watanabe, H.: Quadratic dispersion relations in gapless frustration-free systems. *Phys. Rev. B* **110**, 195140 (2024). <https://doi.org/10.1103/PhysRevB.110.195140>
28. Gosset, D., Mozgunov, E.: Local gap threshold for frustration-free spin systems. *J. Math. Phys.* **57**(9), 091901 (2016). <https://doi.org/10.1063/1.4962337>
29. Anshu, A.: Improved local spectral gap thresholds for lattices of finite size. *Phys. Rev. B* **101**(16), 165104 (2020). <https://doi.org/10.1103/PhysRevB.101.165104>
30. Lemm, M., Xiang, D.: Quantitatively improved finite-size criteria for spectral gaps. *J. Phys. A* **55**(29), 295203 (2022). <https://doi.org/10.1088/1751-8121/ac7989>
31. Lemm, M., Lucia, A.: On the critical finite-size gap scaling for frustration-free Hamiltonians. [arXiv:2409.09685](https://arxiv.org/abs/2409.09685) (2024). <https://doi.org/10.48550/arXiv.2409.09685>
32. Aharonov, D., Arad, I., Vazirani, U., Landau, Z.: The detectability lemma and its applications to quantum Hamiltonian complexity. *New J. Phys.* **13**(11), 113043 (2011). <https://doi.org/10.1088/1367-2630/13/11/113043>
33. Anshu, A., Arad, I., Vidick, T.: A simple proof of the detectability lemma and spectral gap amplification. *Phys. Rev. B* **93**(20), 205142 (2016). <https://doi.org/10.1103/PhysRevB.93.205142>