# Slow-roll Dark Energy

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Abstract

We derive slow-roll conditions for thawing quintessence. We solve the equation of motion of $\phi$ for a Taylor expanded potential (up to the quadratic order) in the limit where the equation of state $w$ is close to $-1$ to derive the equation of state as a function of the scale factor. We find that the evolution of $\phi$ and hence $w$ are described by only two parameters, $w_0$ (the present-day value of $w$), and the $K$, which parametrizes the curvature of the potential, and $w(a)$ is model-independent. We derive observational constraints on these two parameters. We also derive the slow-roll conditions for a non-minimally coupled scalar field (extended quintessence) during the radiation/matter dominated era extending our previous results for thawing quintessence. We find that the ratio $\ddot{\phi}/3H\dot{\phi}$ becomes constant but negative, in sharp contrast to the ratio for the minimally coupled scalar field. We also find that $w(a)$ asymptotically approaches that of the minimally coupled thawing quintessence.

## 1 Introduction

It is my great pleasure to give my talk at this special occasion celebrating Prof. Maeda and Prof. Nakamura's 60th birthday. I have several joint papers with both of them about black holes and cosmic no hair conjecture, which I am very proud of. The topic I discuss is dark energy, about which I wrote several papers (including X-matter paper [1]) with Takashi Nakamura more than ten years ago when I was a postdoc at Yukawa Institute. More specifically, I consider slow-roll dark energy. Firstly I explain the motivation of my study and then I derive the slow-roll conditions of a certain type of scalar field dark energy models (called thawing model) and derive the equation of state of such a dark energy model. I also fit the equation of state to observational data and put constraints on the parameters involved in the equation of state. Next, I extend the dark energy model to include non-minimal coupling with gravity. I derive the slow-roll conditions of such a non-minimally coupled quintessence and find such a dark energy behaves very differently from a minimally coupled quintessence. I also find however that the equation of state approaches to that of minimally coupled quintessence. Finally I summarize my talk.

## 2 Slow-roll Dark Energy

There is strong evidence that the Universe is dominated by dark energy, and the current cosmological observations seem to be consistent with ΛCDM. The equations of state of dark energy, $w$, is close to $-1$ within 10% or less. However, how much is a dark energy model close to the cosmological constant? In order to quantify such "distance from the cosmological constant" in the dark energy theory space, we need to introduce a parametrization of the equation of state, $w(a)$, which parametrizes the deviation from the cosmological constant, $w = -1$. Moreover, since $w$ is close to $-1$ This implies that even if a scalar field (dubbed "quintessence" [2]) plays the role of dark energy, it should roll down its potential slowly because its kinetic energy density should be much smaller than its potential. In this situation, as in the case of inflation, it is useful to derive the slow-roll conditions for quintessence because the dynamics of the scalar field can be discussed only by simple conditions without having to solve its equation of motion directly. Quintessence models are classified according to their motion [3]: In "thawing" models the scalar fields hardly move in the past and begin to roll down the potential recently, while in "freezing" models the scalar fields move in the opposite ways and gradually slow down the motion. We will mostly consider the slow-roll conditions for thawing models since the observational data already favor them and there are several particle physics models for them. Note however that our consideration will not be limited to quintessence but will be applied to the case when the scalar fields which are subdominant components in the universe move slowly. Axions, curvatons, and moduli before the oscillation can be such fields.

## 2.1 Slow-roll Conditions

We first derive the slow-roll conditions for thawing quintessence models [4]. Working in units of $8\pi G = 1$, the basic equations in a flat universe are

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{1}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}(\rho_B + \rho_\phi), \tag{2}$$

$$\dot{H} = -\frac{1}{2}\left((\rho_B + p_B) + (\rho_\phi + p_\phi)\right) = -\frac{1}{2}\left((1 + w_B)\rho_B + \dot{\phi}^2\right), \tag{3}$$

where $V' = dV/d\phi$, $H = \dot{a}/a$ is the Hubble parameter with $a$ being the scale factor, $\rho_B(p_B)$ is the energy density (pressure) of matter/radiation, $\rho_\phi = \dot{\phi}^2/2 + V(\phi)(p_\phi = \dot{\phi}^2/2 - V(\phi))$ is the scalar field energy density (pressure), and $w_B$ is the equation of state of matter/radiation.

By slow-roll quintessence we mean a model of quintessence whose kinetic energy density is much smaller than its potential,

$$\frac{1}{2}\dot{\phi}^2 \ll V. \tag{4}$$

Unlike the case of inflation, we do not require that $\ddot{\phi}$ is smaller than the friction term $3H\dot{\phi}$ in Eq. (1) since $H$ is not determined by the potential alone, but by the matter/radiation along with the scalar field energy density.

With fixed $w_0$, slowly rolling thawing models correspond to the equation of state $w = p_\phi/\rho_\phi$ very close to $-1$, so that the Hubble friction is not effective and hence $\ddot{\phi}$ is not necessarily small compared with $3H\dot{\phi}$ in Eq. (1). Slowly rolling freezing models correspond to models whose $w$ is not so close to $-1$ compared with thawing models so that the Hubble friction is effective and $\ddot{\phi}$ is smaller than $3H\dot{\phi}$ in Eq. (1).

We derive the slow-roll conditions for thawing quintessence during the matter/radiation dominated epoch. We first introduce the following function :

$$\beta = \frac{\ddot{\phi}}{3H\dot{\phi}}. \tag{5}$$

As stated above, for thawing models, $\beta$ is a quantity of $\mathcal{O}(1)$. We assume $\beta$ is an approximately constant in the sense $|\dot{\beta}| \ll H|\beta|$, and the consistency of the assumption will be checked later. In terms of $\beta$, using Eq. (1), $\dot{\phi}$ is written as

$$\dot{\phi} = -\frac{V'}{3(1+\beta)H}, \tag{6}$$

and the slow-roll condition Eq. (4) becomes

$$\epsilon := \frac{V'^2}{6H^2V} \ll 1, \tag{7}$$

where we have omitted $1 + \beta$ since it is an $\mathcal{O}(1)$ quantity and introduced the factor of $1/6$ so that $\epsilon$ coincides with the inflationary slow-roll parameter, $\epsilon = \frac{1}{2}(V'/V)^2$, if the scalar field dominates the expansion: $H^2 \simeq V/3$. Eq. (7) is a quintessence counterpart of the inflationary slow-roll condition $(V'/V)^2 \ll 1$.

Similar to the case of inflation, the consistency of Eq. (5) and Eq. (1) should give the second slow-roll condition. In fact, from the time derivative of Eq. (6)

$$\ddot{\phi} = \frac{V''V'}{9(1+\beta)^2H^2} - \frac{1 + w_B}{2(1+\beta)}V' + \frac{\dot{\beta}V'}{3(1+\beta)^2H}, \tag{8}$$

where we have used $\dot{H}/H^2 \simeq -3(1+w_B)/2$ from Eq. (2) and Eq. (3). On the other hand, from Eq. (5) and Eq. (6), $\ddot{\phi} = 3\beta H\dot{\phi} = -\beta V'/(1+\beta)$, and so we obtain

$$\beta = -\frac{V''}{9(1+\beta)H^2} + \frac{(1+w_B)}{2} - \frac{\dot{\beta}}{3(1+\beta)H} \simeq -\frac{V''}{9(1+\beta)H^2} + \frac{(1+w_B)}{2}, \tag{9}$$

where we have used $\dot{\beta} \ll H\beta$. While the left-hand-side of Eq. (9) is an almost time-independent quantity by assumption, the first term in the right-hand-side is a time-dependent quantity in general. Therefore the equality holds if the first term is negligible:

$$\eta := \frac{V''}{3H^2}; \qquad |\eta| \ll 1, \tag{10}$$

so that $\beta$ becomes

$$\beta = \frac{1+w_B}{2}, \tag{11}$$

*or* the left-hand-side is negligible:

$$|\beta| \ll 1, \tag{12}$$

so that

$$\eta = \frac{3}{2}(1+w_B). \tag{13}$$

The former condition corresponds to the slow-roll thawing models, while the latter corresponds to the slow-roll freezing models. $\beta$ given by Eq. (11) is an approximately constant, which is consistent with our assumption. Here the factor 1/3 is introduced in Eq. (10) so that $\eta$ coincides with the inflationary slow-roll parameter, $\eta = V''/V$, if $H^2 \simeq V/3$. Eq. (10) is a quintessence counterpart of the inflationary slow-roll condition $|V''|/V \ll 1$.

Eq. (7) and Eq. (10) constitute the slow-roll conditions for thawing quintessence during the matter/radiation epoch. Moreover once the universe becomes dominated by the scalar field, the two conditions reduce to the usual inflationary slow-roll conditions from $H^2 \simeq V/3$. Therefore, these conditions (Eq. (7) and Eq. (10)) are the slow-roll conditions for thawing quintessence at all times, both during the matter/radiation era and during the scalar field dominated era. Note that since $H^2 \gtrsim V/3$, the inflationary slow-roll conditions are sufficient conditions for slow-roll thawing quintessence during the matter/radiation era, not necessary conditions. In Fig. 1, the evolution of $\beta$ is shown for a thawing quintessence model ($V = M^4(1-\cos\phi)$). The evolution of $\beta$ agrees nicely with Eq. (11).

## 2.2 Equation of State

Next we derive general solutions of $\phi$ in the limit of $|1+w| \ll 1$ and derive $w$ as a function of $a$. To do so, we first note that the Hubble friction term in Eq. (1) can be eliminated by the following change of variable

$$u = (\phi - \phi_i)a^{3/2}, \tag{14}$$

where $\phi_i$ is an arbitrary constant, which is introduced for later use, and then Eq. (1) becomes

$$\ddot{u} + \frac{3}{4}(p_B + p_\phi)u + a^{3/2}V' = 0. \tag{15}$$

We assume a universe consisting of matter and quintessence with $w \simeq -1$. Then the pressure is well approximated by a constant: $p_B + p_\phi \simeq p_\phi \simeq -\rho_{\phi 0}$, where $\rho_{\phi 0}$ is the nearly constant density contributed by the quintessence in the limit $w \simeq -1$. Eq. (15) then becomes

$$\ddot{u} - \frac{3}{4}\rho_{\phi 0}u + a^{3/2}V' = 0. \tag{16}$$

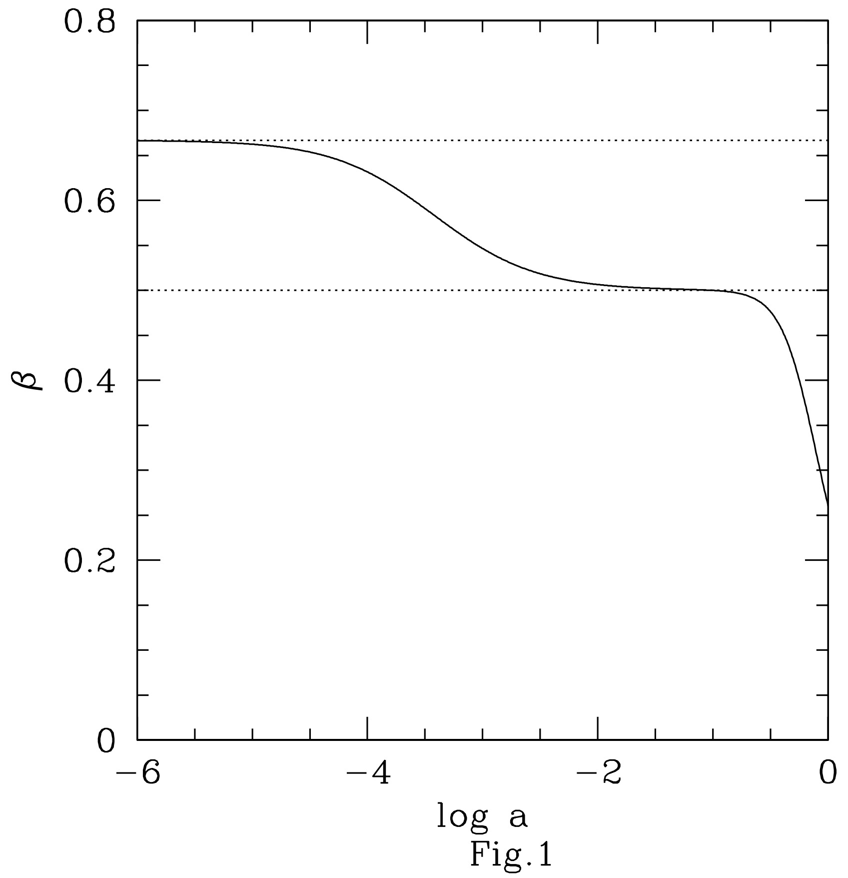


Figure 1: $\beta$ as a function of $a$ for thawing quintessence model with the axion-like potential $V = M^4(1 - \cos\phi)$. The dotted lines are $\beta = 2/3, 1/2$, respectively.

Since we consider a slow-roll scalar field, the potential may be generally expanded around some value $\phi_i$, which we identify with the initial value, in the form (up to the quadratic order)

$$V(\phi) = V(\phi_i) + V'(\phi_i)(\phi - \phi_i) + \frac{1}{2}V''(\phi_i)(\phi - \phi_i)^2. \tag{17}$$

Substituting the expansion Eq. (17) into Eq. (16) and taking $\rho_{\phi 0} = V(\phi_i)$ gives

$$\ddot{u} + \left(V''(\phi_i) - \frac{3}{4}V(\phi_i)\right)u = -V'(\phi_i)a^{3/2}. \tag{18}$$

Being consistent with $|w+1| \ll 1$, we assume $a(t)$ is well approximated by its value in the ΛCDM model which is given by

$$a(t) = \left(\frac{1-\Omega_{\phi 0}}{\Omega_{\phi 0}}\right)^{1/3} \sinh^{2/3}(t/t_\Lambda), \tag{19}$$

where $\Omega_{\phi 0}$ is the present-day value of density parameter of quintessence, $a = 1$ at present, and $t_\Lambda$ is defined as $t_\Lambda = 2/\sqrt{3\rho_{\phi 0}} = 2/\sqrt{3V(\phi_i)}$. Since Eq. (18) is a second order linear ordinary differential equation, it can be solved analytically. The solution, which is valid if the initial time is much earlier than the present time, is [4]

$$\phi(t) = \phi_i + \frac{V'(\phi_i)}{V''(\phi_i)}\left(\frac{\sinh(kt)}{kt_\Lambda \sinh(t/t_\Lambda)} - 1\right), \tag{20}$$

where $k = \sqrt{(3/4)V(\phi_i) - V''(\phi_i)}$. Then the equation of state becomes [4]

$$1 + w(a) = (1+w_0)a^{3(K-1)}\left(\frac{(K - F(a))(F(a)+1)^K + (K + F(a))(F(a)-1)^K}{(K - \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2}+1)^K + (K + \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2}-1)^K}\right)^2, \tag{21}$$

where

$$K = kt_\Lambda = \sqrt{1 - \frac{4}{3}\frac{V''(\phi_i)}{V(\phi_i)}}, \tag{22}$$

$$F(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}}. \tag{23}$$

We also studied the slow-roll conditions for k-essence [5] and find that the equation of state obeys the same functional form as Eq. (23) since the k-essence Lagrangian can be Taylor-expanded for small kinetic energy, $X = \dot{\phi}^2/2$, if it is analytical

$$p(X,\phi) = p(0,\phi) + (\partial p/\partial X)X + \dots, \tag{24}$$

and it reduces to that of canonical scalar field by field redefinition [6].

## 2.3 Observational Constraints

The results of this paper indicate that Eq. (21) applies both to quintessence models and to a subset of k-essence models with $w \simeq -1$. Hence Eq. (21) is a useful and physically well-motivated parametrization for $w(a)$ that can be compared with the observations. So, in this section, we present the observational constraints on the equation of state parameters $w_0$ and on $K$ [6].

First, we note that the cosmological constant corresponds to a *line* in the $(w_0, K)$ plane: $w_0 = -1$ *irrespective of* $K$. This can be understood for a canonical scalar field by noting that $w_0 = -1$ corresponds to the case where the scalar field sits at the minimum ($K < 1$) or the maximum ($K > 1$) of the potential.

As observational data we consider the recent compilation of 397 Type Ia supernovae (SNIa), called the Constitution set with the light curve fitter SALT, by Hicken et al. [7] and the measurements of BAO from the recent SDSS data [8].

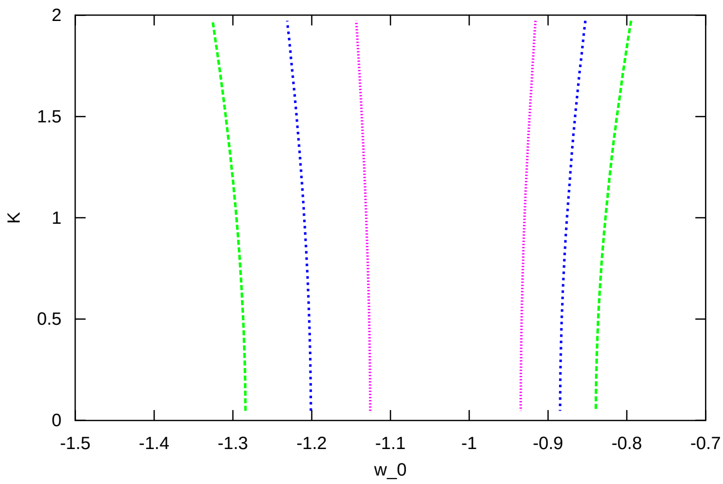


Figure 2: Contours at 68.3% (red, inner), 95.4% (blue, middle), 99.7% (green, outer) confidence level on $w_0$ and $K$. The Constitution SN set was combined with BAO constraint.

BAO measurements from the SDSS data provide a constraint on the distance parameter $A$ defined by

$$A(z) = (\Omega_m H_0^2)^{1/2} \left( \frac{1}{H(z) z^2} \right)^{1/3} \left( \int_0^z \frac{dz'}{H(z')} \right)^{2/3} \tag{25}$$

to be $A(z = 0.35) = 0.493 \pm 0.017$ [8].

The joint constraints from SNIa and BAO are shown in Fig. 2. We marginalize over $\Omega_m$ to derive the constraints. The allowed range of $w_0$ is narrow: $-1.14 \lesssim w_0 \lesssim -0.92 (1\sigma)$. We find that the cosmological constant $w_0 = -1$ is fully consistent with the current data. Note that $K$, which parametrizes the curvature of $V(\phi_i)$, is not well-constrained by current SNIa and BAO data.

# 3 Slow-roll Extended Quintessence

In this section, we further study the slow-roll conditions for a scalar field *non-minimally coupled* to gravity (called extended quintessence) and examine to what extent the results for minimally coupled quintessence are universal [9].

We consider the cosmological dynamics described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - F(\phi) R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] + S_m. \tag{26}$$

Here $\kappa^2 \equiv 8\pi G_{bare}$ is the bare gravitational constant, $F(\phi)$ is the non-minimal coupling and $S_m$ denotes the action of matter (radiation and nonrelativistic particle).

The equations of motion in a flat FRW universe model are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + 6F'(\phi)\left(\dot{H} + 2H^2\right) = 0, \tag{27}$$

$$3H^2 = \kappa^2\left(\rho_B + \frac{1}{2}\dot{\phi}^2 + U\right) =: \kappa^2(\rho_B + \rho_\phi) =: \kappa^2\rho_{\rm tot}, \tag{28}$$

$$\begin{aligned} 2\dot{H} &= -\kappa^2\left(\rho_B + p_B + \rho_\phi + \dot{\phi}^2/2 - V - 2\ddot{F} - 4H\dot{F} - 2F(2\dot{H} + 3H^2)\right) \\ &=: -\kappa^2\left((1 + w_B)\rho_B + \rho_\phi + p_\phi\right), \end{aligned} \tag{29}$$

$$U := V + 6H\left(\dot{F} + HF\right), \tag{30}$$

where $' = d/d\phi$, $\rho_B$ and $p_B$ denote the background (radiation and matter) energy density and pressure, respectively, and $w_B = p_B/\rho_B$ is the equation of state of radiation and matter.

### 3.1 Slow-roll Conditions

We derive the slow-roll conditions for extended (thawing) quintessence during the matter/radiation dominated epoch [9]. Eq. (27) then becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'_{\rm eff} = 0, \tag{31}$$

$$V'_{\rm eff} \equiv V' + 3F'H^2(1 - 3w_B). \tag{32}$$

By "slow-roll", we mean that the movement of $\phi$ during one Hubble time is much smaller than $\phi$. On the other hand, the condition that the kinetic energy density of the scalar field is much smaller than the potential $U$ (Eq. (30)) in the energy density of the scalar field $\rho_\phi$ (Eq. (28))

$$\frac{1}{2}\dot{\phi}^2 \ll U, \tag{33}$$

implies that

$$\dot{\phi}^2 H^{-2} \ll \kappa^{-2} \lesssim \phi^2, \tag{34}$$

from $U \ll \rho_{\rm tot} \simeq \kappa^{-2}H^2$ if $\kappa\phi \gtrsim 1$. Hence we regard Eq. (33) as the slow-roll condition.

We derive the consistent set of the slow-roll conditions. We again consider the ratio,

$$\beta = \frac{\ddot{\phi}}{3H\dot{\phi}}. \tag{35}$$

For slow-roll (thawing) models, we first assume that $\beta$ is an $\mathcal{O}(1)$ approximately constant quantity not equal to $-1$ in the sense $|\dot{\beta}| \ll H|\beta|$, and the consistency of the assumption will be checked later. In terms of $\beta$, using Eq. (31), $\dot{\phi}$ is rewritten as

$$\dot{\phi} = -\frac{V'_{\rm eff}}{3(1+\beta)H}, \tag{36}$$

and the condition Eq. (33) gives the first one of the slow-roll conditions

$$\epsilon := \frac{V'^2_{\rm eff}}{6H^2U} \ll 1, \tag{37}$$

where we have omitted $1 + \beta$ since it is an $\mathcal{O}(1)$ quantity and introduced the factor of 1/6 in $\epsilon$ so that $\epsilon$ coincides with the inflationary slow-roll parameter, $\epsilon = \frac{1}{2}\left(\frac{V'}{\kappa V}\right)^2$, if the scalar field dominates the expansion: $H^2 \simeq \kappa^2 V/3$ and $U \simeq V_{\rm eff} \simeq V$.

Similar to the case of inflation, the consistency of Eq. (35) and Eq. (31) should give the second slow-roll condition. In fact, from the time derivative of Eq. (36)

$$\ddot{\phi} = -\frac{\dot{H}}{H}\dot{\phi} - \frac{V''}{3(1+\beta)H}\dot{\phi} - \frac{F''H(1-3w_B)}{1+\beta}\dot{\phi} + \frac{3F'H^2(1-3w_B)}{1+\beta} - \frac{\dot{\beta}}{1+\beta}\dot{\phi}, \tag{38}$$

where we have used $(H^2(1-3w_B))^{\cdot} \simeq -3H^3(1-3w_B)$. On the other hand, from Eq. (35) and Eq. (36), $\dddot{\phi} = 3\beta H\ddot{\phi} = -\beta V'_{\rm eff}/(1+\beta)$, and so we obtain

$$\begin{aligned}\beta = \frac{\dddot{\phi}}{3H\ddot{\phi}} &\simeq -\frac{\dot{H}}{3H^2} - \frac{V''}{9(1+\beta)H^2} - \frac{F''(1-3w_B)}{3(1+\beta)} - \frac{V'_{\rm eff}-V'}{V'_{\rm eff}}, \\ &= \frac{w_B-1}{2} - \frac{V''}{9(1+\beta)H^2} - \frac{F''(1-3w_B)}{3(1+\beta)} + \frac{V'}{V'_{\rm eff}},\end{aligned} \tag{39}$$

where we have used $3F'H^2(1-3w_B) = V'_{\rm eff} - V'$ and $|\dot{\beta}| \ll H|\beta|$. While the left-hand-side of Eq. (39) is assumed to be an almost time-independent quantity, the terms other than the first in the right-hand-side are time-dependent quantities in general. Therefore the assumption is consistent if they are negligible:[1]

$$\eta := \frac{V''}{3H^2}; \qquad |\eta| \ll 1 \quad \text{and} \quad |F''(1-3w_B)| \ll 1 \quad \text{and} \quad \left|\frac{V'}{V'_{\rm eff}}\right| \ll 1, \tag{40}$$

so that $\beta$ becomes

$$\beta = \frac{w_B-1}{2}. \tag{41}$$

$\beta$ given by Eq. (41) is consistently an $\mathcal{O}(1)$ constant not equal to $-1$. Here the factor $1/3$ is introduced in $\eta$ so that $\eta$ coincides with the inflationary slow-roll parameter, $\eta = V''/\kappa^2 V$, if $H^2 \simeq \kappa^2 V/3$. The conditions in Eq. (40) are quintessence counterparts of the inflationary slow-roll condition $|V''|/\kappa^2 V \ll 1$.

Eq. (37) and Eq. (40) constitute the slow-roll conditions for extended quintessence during matter/radiation epoch. $\beta$ (Eq. (41)) is negative and is quite different from that for a minimally coupled scalar field (Eq. (11)) which is positive. Therefore, this can be a discriminating probe of the non-minimal coupling of the scalar field. Although it may be difficult to determine the thawing dynamics from distance measurements, the ratio $\beta$ may be determined by measuring the time variation of the fine structure constant $\alpha$ if $\phi$ induces such a variation and $\alpha$ depends linearly on $\phi$.

In Fig. 3, the evolution of $\beta$ is shown for a massive scalar field ($V = \frac{1}{2}m^2\phi^2$) with a non-minimal coupling $F = \frac{1}{2}\xi\phi^2$ with $\xi = 10^{-2}$. The evolution of $\beta$ agrees nicely with Eq. (41).

## 3.2 The Equation of State

Next we derive analytically the equation of state [9]. We consider the case that the non-minimal coupling is given by $F(\phi) = \frac{1}{2}\xi\phi^2$. In terms of $u$ defined in Eq. (14), the equation of motion Eq. (27) becomes

$$\ddot{u} + \left[-\frac{3}{2}\left(\dot{H} + \frac{3}{2}H^2\right) + 6\xi\left(\dot{H} + 2H^2\right)\right]u + \left[V' + 6\xi\phi_i\left(\dot{H} + 2H^2\right)\right]a^{\frac{3}{2}} = 0. \tag{42}$$

Since we are interested in the slow-roll motion of the quintessence field $\phi$, we may expand the potential $V(\phi)$ around the initial value $\phi_i$ up to the quadratic order as in Eq. (17). Moreover we assume that the scale factor $a(t)$ is well approximated by that in the $\Lambda$CDM model Eq. (19). Then the approximate solution, which is valid as far as $t \gg t_\Lambda$ and $t \gg t_i$, is given by [9]

$$\phi(t) = \phi_i + \frac{V'(\phi_i) + 4\xi\kappa^2\phi_i V(\phi_i)}{V''(\phi_i)}\left[\frac{\sinh(kt)\cosh\left(\frac{t_i}{t_\Lambda}\right)}{kt_\Lambda\sinh\left(\frac{t}{t_\Lambda}\right)} - 1\right], \tag{43}$$

with $k \equiv \sqrt{\left(\frac{3}{4} - 4\xi\right)\kappa^2 V(\phi_i) - V''(\phi_i)}$. The equation of state $w$ is then given by [9]

$$1 + w = (1+w_0)a^{3(K-1)}\left[\frac{(K - F(a))(F(a)+1)^K + (K + F(a))(F(a)-1)^K}{(K - \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2}+1)^K + (K + \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2}-1)^K}\right]^2, \tag{44}$$

[1] The exception is the case of $F'' = $ const. In this case $F''$ needs not to be small. For example, if $F = \frac{1}{2}\xi\phi^2$, then $\beta$ satisfies $\beta = -\frac{1}{3}$ during the radiation era and $\beta = -\frac{1}{2} - \frac{\xi}{3(1+\beta)}$ so that $\beta = \frac{-9+\sqrt{9-48\xi}}{12}$ during the matter era.

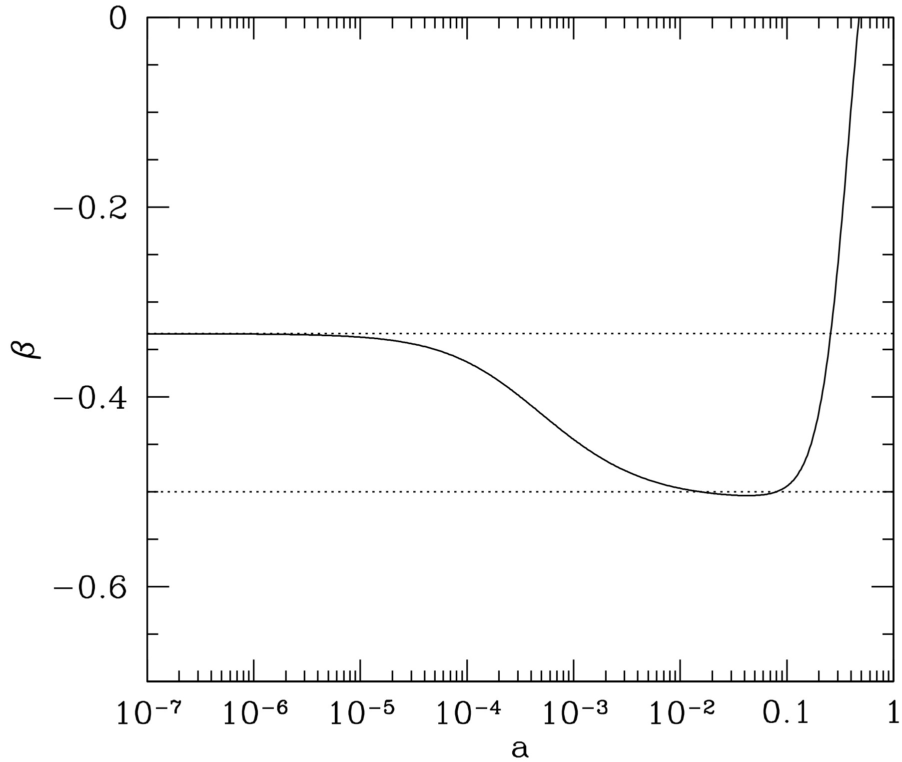


Figure 3: $\beta$ as a function of $a$ for a massive scalar field model with $F = \frac{1}{2}\xi\phi^2$ with $\xi = 10^{-2}$. The dotted lines are $\beta = -\frac{1}{3}, -\frac{1}{2}$, respectively.

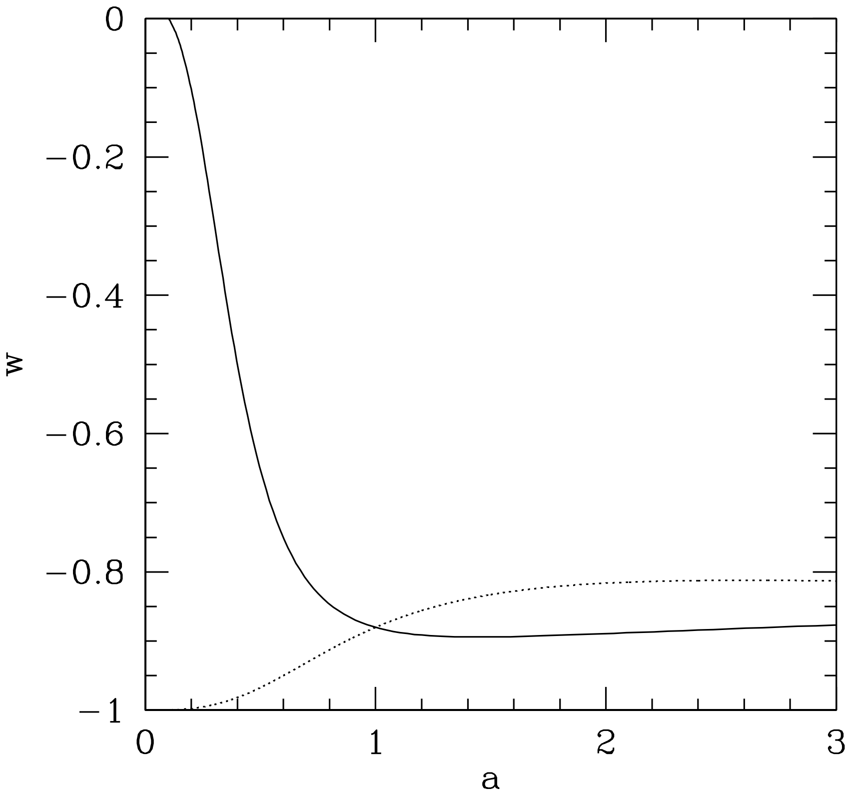


Figure 4: $w_\phi$ as a function of $a$. The solid line is the numerical solution, while the dotted line is the asymptotic solution Eq. (44).

where $K = kt_\Lambda = \sqrt{1 - \frac{16\xi}{3} - \frac{4}{3}\frac{V''(\phi_i)}{\kappa^2 V(\phi_i)}}$. We have normalized the expression to $w_{\phi 0}$ in Eq. (44). This expression completely coincides with that in the minimal coupling Eq. (21). It is noted, however, that this expression applies only for $t \gtrsim t_\Lambda$ and that the definition of $K$ is different but, when $\xi$ approaches 0, reduces to that of the minimally coupled scalar field.

In Fig. 4, $w_\phi$ is shown as a function of $a$. We find that apart from the slight offset $w_\phi$ approaches the asymptotic solution given by Eq. (44). This, together with [6], makes the functional form of $w_\phi(a)$, Eq. (21), even more useful.

## 4 Summary: What is the Kepler's law of Dark Energy?

We have derived slow-roll conditions for thawing quintessence models, Eq. (7) and Eq. (10). We have also solved the equation of motion of the slow-roll thawing quintessence and obtained the equation of state as a function of the scale factor $w(a)$, Eq. (21), which involves only two parameters. We have found that this $w(a)$ is in general not fit by a linear evolution in $a$ which is frequently used in the literature. We have also found that this $w(a)$ applies to quintessence models and to k-essence models with $w \approx -1$ and also to extended quintessence models. Applying this parametrization to SNIa data and BAO, we find that the present-day value of $w$ is constrained to lie near $-1$, while the curvature parameter $K$ is poorly constrained by the observations. Further, we see that the cosmological constant limit of these models is consistent with the current data. As an extension, we have also derived the slow-roll conditions for non-minimally coupled scalar field during the radiation/matter dominated epoch. We have also derived the slow-roll equation of motion of the scalar field and found that the ratio $\ddot{\phi}/3H\dot{\phi}$ becomes constant but negative, in sharp contrast to the result for the minimally coupled scalar field. This ratio can be a discriminating probe of the non-minimal coupling of the scalar field.

Finally, I would like to ask a provocative question. When I teach classical mechanics to freshman, I am always impressed by the role which the Kepler's laws of the planetary motion played in formulating the Newtonian mechanics. As we know, every law is essential in establishing the universal attractive force of gravity: the second law shows that the force depends only on the distance between the planet and the Sun (central force); from the first law, the force is found to be proportional to inverse square of the distance; the third law establishes the force depends only on the mass of each body with the universal constant: the Newton's constant. Being impressed by the success of Kepler's law as a phenomenological law, I cannot help asking "What is the Kepler's law of dark energy (or the Universe)?".

Truly finally, I would like to say "Happy birth day to Prof. Maeda and Prof. Nakamura!"

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