

A New Perspective on Relativistic Quantum Mechanics

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Abstract. Based on a linear realization formulation of a quantum relativity, — proposed relativity for 'quantum space-time', we introduce the new Poincaré-Snyder relativity and Snyder relativity as relativities in between the latter and the well known Galilean and Einstein cases. While there is supposed to be not separate notion of classical and quantum mechanics at the level of the very unconventional quantum relativity, the Poincaré-Snyder relativity is more like a mathematically extended form of Einstein relativity on which we can write down a formal canonical classical and quantum mechanics. We discuss how the Poincaré-Snyder relativity may provide a stronger framework for the description of the usual (Einstein) relativistic quantum mechanics and present a first look of the interesting picture from the new perspective.

1. Introduction

Our title may give the impression that our topic is just about (Einstein) relativistic quantum mechanics. We are, however, presenting the new perspective from the consideration of relativities beyond the usual (Einstein) relativity. To put it in another way, Einstein relativity symmetry maybe considered a part of a bigger symmetry, the full consideration of which gives a boarder perspective on the usual relativistic quantum mechanics. The full symmetry is a symmetry for what we called the Poincaré-Snyder relativity [1]. The the new Poincaré-Snyder relativity, and Snyder relativity, are introduced as relativities in between the well known Galilean and Einstein cases and a proposed Quantum Relativity [2]. The latter is supposed to be the relativity for the ‘quantum space-time’.

Our Quantum Relativity idea is based on the following aspects of our new approach to think about a plausible formulation of the the physics of ‘quantum space-time’ [2, 3] and eventually ‘quantum gravity’. We consider there being a true microscopic/quantum structure of space-time itself and seek a direct description of that. Such a perspective of quantum gravity being beyond a quantized version of a classical theory of gravity has been advocated by many theorists, including particularly speakers at this Workshop [4]. It may then be just natural consider the ‘quantum space-time’ has its own relativity symmetry — the Quantum Relativity. The latter idea maybe considered as first introduced by Snyder in 1947, with the perspective that the Einstein Relativity has to be modified or deformed to accommodate a reference frame independent quantum scale [5]. Such an issue of a deformed special relativity theory rekindled quite some attention at the beginning of the century [6]. We added two important idea to the basic approach towards the theory of Quantum Relativity. Firstly, the ‘quantum space-time’ maybe described as something beyond ‘space-time’ [3]. We introduced the u -coordinates to supplement the s (pace) and t (ime) coordinates for the classical geometric arena for formulating the Quantum Relativity [3, 7]. As space is the arena for Galilean Relativity and space-time the one for Einstein Relativity, ‘space-time- u ..’ will be the one for the Quantum Relativity. In ref.[2], we introduced further the idea that a single ‘quantum scale’ is not enough to get to the Quantum Relativity. We have the well known quantum scale as one scale through assuming \hbar . To build also \hbar itself into the framework, we need both a quantum momentum and a quantum length parameters. We are only talking about the special theory of Quantum Relativity though, and we will need the general theory for the ‘quantum gravity’.

The picture we sketch is more than a simple speculation. It is given by the mathematics of Lie algebra deformation/stabilization [8] and the simple linear realization [3]. From a pure mathematical point of view, stabilization gives a new symmetry that has the old one as a contraction limit. The Lorentz symmetry is exactly a stabilization/deformation of the Galilean counterpart, with the deformation parameter given by $1/c^2$. Philosophically, one can argue that only stable symmetries can be scientifically verified to be correct. The contracted unstable limit is a singular

point on the ‘parameter space’, requiring infinite experimental precision to be confirmed against the stable symmetry which admits other parameter values. And typically, the symmetry is independent of the exactly value of the parameter, so long as its nonzero. Following the line of thinking, we implemented in Ref.[3] a linear realization perspective. The linear realization scheme, applied to the Einstein Relativity as the deformation of Galilean Relativity on 3D space, is nothing other than the 4D Minkowski space-time picture. Such a mathematically conservative approach, however, leads to a very radical physics perspective, that space-time is to be described as part of something bigger [3], what we called the quantum world in Ref.[2] with the ‘space-time- $u..$ ’ coordinates. In the latter paper we identify the Quantum Relativity symmetry as $SO(2, 4)$, with a linear realization on a 6D classical geometry beyond space-time providing a description of a 4D noncommutative (quantum) space-time. The quantum world is really the coset space $SO(2, 4)/SO(2, 3)$, *i.e.* the hypersurface $\eta_{MN}X^MX^N = 1$ [$\eta_{MN} = (-1, 1, 1, 1, 1, -1)$]. For the lack of a better terminology, we denote it by ‘AdS₅’. ‡ With the $SO(2, 4)$ relativity, fundamental constants \hbar and G are essentially putting on exactly the same footing as c . §

2. Getting to the Quantum Relativity Symmetry $SO(2, 4)$

In summary, we have the Lie algebra sequence of stabilizations and extensions by translations :

$$\begin{aligned} ISO(3) &\rightarrow SO(1, 3) \hookrightarrow ISO(1, 3) \\ &\rightarrow SO(1, 4) \hookrightarrow ISO(1, 4) \rightarrow SO(2, 4) \end{aligned}$$

The $ISO(3)$ algebra corresponds to the symmetry for 3D space. It is unstable, with $SO(1, 3)$ a possible stabilization result. The only other mathematical option is $SO(4)$. It is the physics picture of the deformation parameter being identified as invariant speed that fix the right choice. A linear realization of the $SO(1, 3)$ group adopts the 4D Minkowski space-time M^4 over that of 3D space \mathbb{R}^3 as the arena for the relativity theory; $SO(1, 3)$ is the isometry of M^4 . With the linear realization comes the natural extension of the Lorentz symmetry to Poincaré symmetry $ISO(1, 3)$, by adding the coordinate translations. The latter algebra is again an unstable symmetry, to be stabilized to $SO(1, 4)$, with the Planck momentum κc as an energy-momentum invariant. The physics behind the stabilization step as well as the last one is summarized in Table 1.

‡ Note that we used a different metric sign convention in our earlier papers, refs.[2, 3], and partly during this talk presentation. We apologize for the confusion.

§ We do believe the generic noncommutative geometry, as the natural mathematical generalization of the classical/commutative geometry [9], should be the right mathematical setting for quantum gravity. In more exact terms, we believe, or postulate that, *Non-Commutative Geometry is to Quantum Gravity as Non-Euclidean Geometry is to Classical (Einstein) Gravity*. We see our ‘AdS₅’ quantum world as providing an alternative description of the asymptotic flat limit of the 4D noncommutative geometry, somewhat similar to the embedding of a non-Euclidean geometry into a higher dimensional Euclidean geometry.

Table 1. Physics of the Relativity Deformations Summarized:- The first column shows the familiar Galilean to Einstein case. Having an invariant speed c change the velocity from a unconstrained three-vector v^i to a constrained four-vector u^μ , matched to a 4D arena with an extra coordinate ct . Mathematically, the zero commutator between Galilean boosts is deformed to a rotation. The latter two columns similarly summarized the physics of the two further steps towards the quantum relativity of Ref.[2]. For example (2nd column), having an invariant magnitude κc for $p^\mu = \frac{dx^\mu}{d\sigma}$ change the unconstrained four-vector to a constrained five-vector π^A , to be matched to a 5D arena with an extra coordinate $\kappa c\sigma$. The originally zero commutator between two momentum boosts, to be described before imposing the constraint as $\Delta x^\mu = p^\mu \sigma$ on 4D space-time, is deformed to a Lorentz transformation. The introduction of the σ parameter/coordinate and the momentum boosts before the deformation are the key features behind the Poincaré-Snyder relativity introduced in this letter.

$\Delta x^i(t) = v^i \cdot t$	$\Delta x^\mu(\sigma) = p^\mu \cdot \sigma$	$\Delta x^A(\rho) = z^A \cdot \rho$
$ v^i \leq c$	$\sqrt{-\eta_{\mu\nu} p^\mu p^\nu} \leq \kappa c$	$ z^A \leq \ell$
$\eta_{ij} v^i v^j = c^2 \left(1 - \frac{1}{\gamma^2}\right)$	$\eta_{\mu\nu} p^\mu p^\nu = -\kappa^2 c^2 \left(1 - \frac{1}{\Gamma^2}\right)$	$\eta_{AB} z^A z^B = \ell^2 \left(1 + \frac{1}{G^2}\right)$
$M_{0i} \equiv N_i \sim P_i$ $[N_i, N_j] \longrightarrow M_{ij}$	$J_{\mu 4} \equiv O_\mu \sim P_\mu$ $[O_\mu, O_\nu] \longrightarrow M_{\mu\nu}$	$J_{A5} \equiv O'_A \sim P_A$ $[O'_A, O'_B] \longrightarrow J_{AB}$
$\vec{u}^4 = \frac{\gamma}{c}(c, v^i)$ $-\eta_{\mu\nu} u^\mu u^\nu = 1$	$\vec{\pi}^5 = \frac{\Gamma}{\kappa c}(p^\mu, \kappa c)$ $\eta_{AB} \pi^A \pi^B = 1$	$\vec{X}^6 = \frac{G}{\ell}(z^A, \ell)$ $\eta_{MN} X^M X^N = 1$
$\mathbb{R}^3 \rightarrow SO(1, 3)/SO(3)$	$M^4 \rightarrow SO(1, 4)/SO(1, 3)$	$M^5 \rightarrow SO(2, 4)/SO(2, 3)$

As in the case of ct in going from \mathbb{R}^3 to M^4 , a linear realization asks to have a fifth coordinate $\kappa c\sigma$ be incorporated to form the arena for the $SO(1, 4)$ relativity. The σ -coordinate is quite peculiar. It has a space-like geometric signature and a physics dimension of $\left(\frac{\text{time}}{\text{mass}}\right)$. It is neither space or time. After all, whether the extra coordinate possibly has the physics meaning of a space-time one is a question of how one formulate the physics theory. For example, taking the metric of the 5D geometry as a gravitational field will be giving σ a space-time (indeed space rather than time) interpretation. The latter kind of ready-to-be-formulated theories will be in conflict with the role of σ as seen from the relativity symmetry stabilization physics, or equivalently for having the right Einstein limit. The linear realization does suggest a radical departure of our physics thinking. It provokes even the question of if we could still formulate dynamics in the way we used to. Anyway, we have again a natural extension of the symmetry algebra to $\mathbf{ISO}(1, 4)$, and again a stabilization awaits. It may look like such a sequence of stabilization followed by extension will go on forever. However, as illustrated in Table 1, we take the last stabilization as corresponding a ‘length’ invariant, as terminate the sequence. The idea is that the quantum scale as usually described equally effectively by Planck mass κ and Planck length ℓ assumes \hbar . To retrieve \hbar from the symmetry stabilization picture similar to c , one should rather use both Planck momentum and Planck length for the two deformations following the familiar one with c , getting \hbar from the identity $\hbar = \kappa c\ell$. The deformation gives an $\mathbf{SO}(2, 4)$ algebra. The c constraint

enforces the velocity space to be the coset $SO(1,3)/SO(3)$. Similarly, the second deformation curves the momentum space and the last curves the ‘quantum space-time’ arena itself. Hence, though the Quantum Relativity symmetry $SO(2,4)$ is to be linearly realized as an isometry of a 6D (beyond space-time) geometry, the relevant quantum world is only the hypersurface given by $\eta_{\mathcal{M}\mathcal{N}}X^{\mathcal{M}}X^{\mathcal{N}} = 1$. Coordinate translations are simply not admissible symmetries. ||

We have the Quantum Relativity symmetry obtained through the the Lie algebra stabilization scheme with quite limited physics inputs. To really construct a theory to be tested experimentally, we need to take it beyond kinematics. The radical beyond space-time picture posts a daunting challenge, as we have to figure out the role of the new σ and ρ coordinates (the u -coordinates) in any picture of dynamics. At the most conventional level of classical physics, dynamics is a study of motion and motion is characterized by change of spatial position with respect to time. The σ and ρ coordinates would have apparently nothing to do with motion then. Obviously, understanding the physics role of σ and ρ will be a key to confront the theoretical challenge ahead.

3. The Poincaré-Snyder Relativity

In order to develop thinking about a plausible formulation of the challenging dynamics of the Quantum Relativity, we first take a step background. We want to work on intermediate limiting theories that are more similar to the familiar one, yet allow up to probe the the new conceptual structures.

In ref.[1], we introduced explicitly the intermediate relativity theories, which are summarized in Table 2. In particular, we focus on the Poincaré-Snyder Relativity the classical and quantum mechanics of which are studied in refs.[1, 10]. The symmetry algebra $\mathbf{G}(1,3)$ for the relativity is the Inönü-Wigner contraction limit from $\mathbf{ISO}(1,4)$ and $\mathbf{SO}(2,4)$. It has, besides the $\mathbf{ISO}(1,3)$ generators, generators for the σ -dependent momentum boosts in the commuting limit. It is a Galilean-like symmetry on the 4D Minkowski space-time with an external absolute σ -parameter taking the role of Galilean time. As $dt = \frac{dx}{v}$, we can start thinking about σ through $p^\mu = \frac{dx^\mu}{d\sigma}$. While this change the conventional definition of momentum, it actually allow a standard canonical formulation

|| For the readers who are not familiar with the idea of symmetry stabilization, we present here a simply argument for an easy appreciation of the $\mathbf{ISO}(3)$ to $\mathbf{SO}(1,3)$ story. Firstly, note that the only difference in the two algebras is the commutator of two velocity boosts. Commutators of Lorentz boost generators are rotation generators, as given by $[N_i, N_j] = -i \frac{1}{c^2} M_{ij}$, where $\frac{1}{c^2}$ has been explicitly shown. The commuting algebra of Galilean boosts is retrieved at the $\frac{1}{c^2}$ goes to zero limit. The latter is unstable, as taking any small change in value of the zero structural constant changes the algebra. The Lorentz algebra is stable; different values of c give isomorphic algebra connected by a simple scaling. A more direct way is see it is to realize that the mathematics of symmetry algebra sees no units in physics. The value of c depends, though, on our choice of units, we can make it 3×10^{-7} (km ps⁻¹) or 10^{28} (A yr⁻¹). The symmetry of space-time is of course independent of what units physicists cooked up to measure things inside. The value of c tells only when we would be in a regime where the Galilean symmetry, as an approximation of the Lorentz symmetry, is good enough to describe physics.

Table 2. The various relativities – matching the generators : The table matches out the generators for the various relativity symmetries from a pure mathematical point of view. Note as algebras, the mathematical structures of translations (denoted by P) or the boosts (denoted by K and K' – the so-called Lorentz boosts not included as they are really space-time rotations) in relation to rotations J , are the same. Algebraically, translation and boost generators are distinguished only by the commutation with the Hamiltonian (H and H'). Successive contractions retrieve $G(1, 3)$ and $ISO(1, 4)$ from $SO(2, 4)$, similar to the more familiar $G(3)$ and $ISO(1, 3)$ from $SO(1, 4)$. In the physics picture under discussion, however, $SO(1, 4)$ part of our so-called Snyder relativity $ISO(1, 4)$ is *different* from the usual de-Sitter $SO(1, 4)$ contracting to $ISO(1, 3)$. We consider simply keeping only the P_μ and $J_{\mu\nu}$ generators to reduce from our Poincaré-Snyder $G(1, 3)$ to the Einstein $ISO(1, 3)$.

Relativity Symmetry	Quantum $SO(2, 4)$	Snyder $ISO(1, 4)$	Poincaré-Snyder $G(1, 3)$	Einstein $ISO(1, 3)$	Galilean $G(3)$
Arena	'AdS ₅ '	M^5	M^4 (with σ)	M^4	\mathbb{R}^3 (with t)
SO(1,4) part	J_{ij}	J_{ij}	J_{ij}	J_{ij}	J_{ij}
	J_{i0}	J_{i0}	J_{i0}	J_{i0}	K_i
	J_{40}	J_{40}	K'_0	P_0	H
	J_{4i}	J_{4i}	K'_i	P_i	P_i
	J_{50}	P_0	P_0		
	J_{5i}	P_i	P_i		
	J_{54}	P_4	H'		

of Hamiltonian dynamics as σ -evolution. The single particle (σ -)Hamiltonian has the form

$$H' = \frac{1}{2} p_a^\mu p_{a\mu} + V(x^\mu). \quad (1)$$

The new definition for momentum is admissible so long as the physics theory guarantees $p^\mu = m c u^\mu$, the Einstein energy-momentum at the proper limit. From the latter requirement, one retrieves for such cases $\sigma = \frac{\tau}{m}$, the proper time over the rest mass of an Einstein particle. That is very interesting for a coordinate with a space-like geometric signature indeed. As a solid complement, the straightforward canonical formulation of the classical σ -Hamiltonian dynamics for the free particle case can be reduced to the dynamics of an Einstein particle with $p^\mu = m c u^\mu$. The classical Poincaré-Snyder dynamics looks hence like a generalization of Einstein particle dynamics which in fact gets around the undesirable situation with the no-interaction theorem [15].

It is interesting to note that parameter having a similar role to that of σ has been used quite a lot in various approaches to (Einstein) relativistic quantum mechanics in somewhat ambiguous ways [11]. A major difference in our new perspective as given from the Poincaré-Snyder Relativity is the idea that σ should not be taken as a parameter providing some measure of time at all. If we want a time-evolution picture, we have to go to a relational framework. That is to say, the σ -dynamics gives definite relation

between time, as a σ -dynamics variable, and the other space variables. some examples have been given in ref.[10].

4. Quantum Mechanics as σ -Hamiltonian Dynamics

Aldaya and de Azcárraga [12] presented a particularly nice approach to geometric quantization in which the quantum dynamical description of the system can be obtained with the symmetry group as the basic starting point. In fact, while the approach gives an elegant presentation for the quantization of the Galilean system, its application to the case of Einstein Relativity is less than equally appealing. For the former case, the group to be considered is a $U(1)$ central extension of the symmetry for the corresponding classical system — $G(3)$. The essentially unique nontrivial central extension is depicted by the modified commutator $[K_i, P_j] = m\delta_{ij}\Xi$, where Ξ is a central charge (m the particle mass). The $ISO(1, 3)$ symmetry, however, admits no such nontrivial central extension. However, representation for a $U(1)$ central extension is simply equivalent to projective representation as required in quantum mechanics. So, the above looks like a short-coming for Einstein Relativity itself. It is easy to see from Table 2 that the $G(1, 3)$ symmetry for the Poincaré-Snyder case has a mathematical structure mostly parallel to that of the Galilean $G(3)$. Hence, we should have the same kind of nontrivial $U(1)$ central extension available for the implementation of such a quantization scheme.

The standard action of $G(1, 3)$ on the Minkowski space-time (x^μ) with the extra, external, parameter σ is given by

$$\begin{aligned}x'^\mu &= \Lambda^\mu{}_\nu x^\nu + p^\mu \sigma + A^\mu, \\ \sigma' &= \sigma + b.\end{aligned}\tag{2}$$

An element of our extended $G(1, 3)$ group may be written as $g = (b, A^\mu, p^\mu, \Lambda^\mu{}_\nu, e^{i\theta})$, with group product rule given by

$$\begin{aligned}A''^\mu &= \Lambda'^\mu{}_\nu A^\nu + p'^\mu b + A'^\mu, & b'' &= b' + b, \\ p''^\mu &= \Lambda'^\mu{}_\nu p^\nu + p'^\mu, & \Lambda''^\mu{}_\nu &= \Lambda'^\mu{}_\rho \Lambda'^\rho{}_\nu,\end{aligned}\tag{3}$$

and the nontrivial $U(1)$ extension of given by

$$\theta'' = \theta' + \theta + z \left[A'_\mu \Lambda'^\mu{}_\nu p^\nu + b(p'^\mu \Lambda'^\mu{}_\nu p_\nu + \frac{1}{2} p'^\mu p'_\mu) \right].\tag{4}$$

The last term is the cocycle the exact choice of which is arbitrary up to a coboundary [14]; z corresponds to a value of the central charge is taken as an arbitrary constant at this point.

The right-invariant vector fields are given by

$$\begin{aligned}\mathbf{X}_b^R &= \frac{\partial}{\partial b}, & \mathbf{X}_{A^\mu}^R &= \frac{\partial}{\partial A^\mu} + zp_\mu \frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta}, \\ \mathbf{X}_{p^\mu}^R &= b \frac{\partial}{\partial A^\mu} + \frac{\partial}{\partial p^\mu} + zb p_\mu \frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta}, \\ \mathbf{X}_{\omega^{\mu\nu}}^R &= \widetilde{\mathbf{X}}_{\omega^{\mu\nu}}^R + A_\nu \frac{\partial}{\partial A^\mu} - A_\mu \frac{\partial}{\partial A^\nu} + p_\nu \frac{\partial}{\partial p^\mu} - p_\mu \frac{\partial}{\partial p^\nu},\end{aligned}$$

$$\mathbf{X}_\zeta^R = \frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta}, \quad (5)$$

where we skip the details of $\widetilde{\mathbf{X}}_{\omega^{\mu\nu}}^R$, the invariant vector field for the $SO(1,3)$ subgroup [with $\Lambda(\omega) = e^{\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}}$] [1]. Note that $\zeta = \exp(\frac{i}{\hbar}\theta)$ with $\frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \theta}$ locally. The left-invariant vector fields are given by

$$\begin{aligned} \mathbf{X}_b^L &= \frac{\partial}{\partial b} + p^\mu \frac{\partial}{\partial A^\mu} + \frac{z}{2} p^\mu p_\mu \frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta}, \\ \mathbf{X}_{A^\mu}^L &= \frac{\partial}{\partial A^\nu} \Lambda^\nu{}_\mu, & \mathbf{X}_{p^\mu}^L &= \frac{\partial}{\partial p^\nu} \Lambda^\nu{}_\mu + z A_\nu \Lambda^\nu{}_\mu \frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta}, \\ \mathbf{X}_{\omega^{\mu\nu}}^L &= \widetilde{\mathbf{X}}_{\omega^{\mu\nu}}^L, & \mathbf{X}_\zeta^L &= \frac{i\zeta}{\hbar} \frac{\partial}{\partial \zeta}; \end{aligned} \quad (6)$$

again we skip explicit expression for the $SO(1,3)$ vector field $\widetilde{\mathbf{X}}_{\omega^{\mu\nu}}^L$. We have the quantization form given by the left-invariant 1-forms conjugate to \mathbf{X}_ζ^L , the vertical 1-form; explicitly

$$\Theta = -z A^\nu \Lambda_\nu{}^\mu dp_\mu - \frac{z}{2} p^\mu p_\mu db + \frac{\hbar d\zeta}{i\zeta}. \quad (7)$$

The characteristic module is defined through the conditions $i_{\mathbf{X}}\Theta = 0$ and $i_{\mathbf{X}}d\Theta = 0$, characterizing the differential system given by the vector field \mathbf{X}_b^L . We have the equations of motion given by

$$\begin{aligned} \frac{db}{d\sigma} &= 1, & \frac{dA^\mu}{d\sigma} &= p^\mu, \\ \frac{dp^\mu}{d\sigma} &= 0, & \frac{d\Lambda^\mu{}_\nu}{d\sigma} &= 0 \quad \left(\frac{d\omega^{\mu\nu}}{d\sigma} = 0 \right), \\ \frac{d\zeta}{d\sigma} &= \frac{z}{2} p^\mu p_\mu \frac{i\zeta}{\hbar}. \end{aligned} \quad (8)$$

Identifying the integration parameter as σ gives

$$\begin{aligned} b &= \sigma, & A^\mu &= p^\mu \sigma + K^{\nu\mu} \Lambda_\nu{}^\mu, & p^\mu &= P^\mu, \\ \zeta &= \zeta_o \exp\left(\frac{iz}{2\hbar} p^\mu p_\mu \sigma\right). \end{aligned} \quad (9)$$

Naturally, A^μ is to be identified as x^μ giving p^μ as $\frac{dx^\mu}{d\sigma}$, showing consistence with our original introduction of the momentum boosts (see table 1) as the extra symmetry transformations to supplement $SO(1,3)$ and hence getting to $G(1,3)$. We have constants of motion, $K^{\nu\mu} \Lambda_\nu{}^\mu$, P^μ , and ζ_o which parameterize the quantum manifold. Passing to the latter, Θ takes the form

$$\Theta_P = -z K^\mu dP_\mu + \frac{\hbar d\zeta_o}{i\zeta_o}. \quad (10)$$

The symplectic 2-form is given by $\omega = d\Theta$. Taking $z = 1$, $H' = \frac{1}{2} p^\mu p_\mu$, and $K^{\nu\mu} = A^\nu \Lambda_\nu{}^\mu$ we have

$$\omega = d\Theta_P = -dK^{\nu\mu} \wedge dp_\mu, \quad (11)$$

where we have taken $z = 1$ corresponding to $H' = \frac{1}{2} p^\mu p_\mu$, which gives the right form for the classical σ -Hamiltonian [10]. The expression suggested the identification of

(K'^μ, b) as particle ‘position’ variables (x^μ, σ) and H' as the σ -Hamiltonian generating ‘evolution’ in the absolute parameter σ . Note that we also need $z = 1$ for the extraction of the Heisenberg commutation relation (see also ref.[2, 3]). The prequantum operator associated with a real function f on the classical phase space acting on wavefunction ψ is given by

$$\hat{f}\psi \equiv -i\hbar\tilde{X}f \cdot \psi = -i\hbar X_f \cdot \psi + [f - \Theta(X_f)]\psi, \quad (12)$$

where $i_{X_f}\omega = -df$. In particular,

$$\begin{aligned} \hat{K}'^\mu &= i\hbar \frac{\partial}{\partial P_\mu}, & \hat{P}_\mu &= -i\hbar \frac{\partial}{\partial K'^\mu} + P_\mu, \\ \hat{\sigma} &= i\hbar \frac{\partial}{\partial H'}, & \hat{H}' &= i\hbar \frac{\partial}{\partial \sigma}, \end{aligned} \quad (13)$$

where the an extra negative sign is adopted in \hat{H}' . The operators K'^μ and P_μ can also be obtained from $\mathbf{X}_{A^\mu}^R$ and $\mathbf{X}_{p^\mu}^R$. The full polarization subalgebra can be taken as spanned by $\{\mathbf{X}_b^L, \mathbf{X}_{A^\mu}^L, \mathbf{X}_{\omega^{\mu\nu}}^L\}$, giving the momentum space wavefunction $\phi(P_\mu)$, *i.e.* the wavefunction as dependent only on P_μ but not K'^μ . We have then simply $\hat{P}_\mu\phi(P_\mu) = P_\mu\phi(P_\mu)$. \mathbf{X}_b^L generates the Euler-Lagrange equation

$$i\hbar\partial_\sigma\psi + \frac{\hbar^2}{2}\partial_\mu\partial^\mu\psi = 0 \quad (14)$$

for the Fourier transform ψ of ‘momentum’ space wavefunction ϕ . Note that \hat{H}' and \hat{P}_μ constitute a complete set of commuting observables for the configuration space wavefunction. The equation expresses an operator form of $H' = \frac{1}{2}p^\mu p_\mu$ with \hat{P}_μ reduced to just $-i\hbar\frac{\partial}{\partial K'^\mu}$, *i.e.* $-i\hbar\frac{\partial}{\partial x'^\mu}$. Eq.(14) is of the same form as the so-called (Lorentz) covariant Schrödinger equation studied in the literature[11], except with the parameter σ replacing the proper time τ (or rather τ/m). The equation, with again essentially the proper time as evolution parameter, is also what is obtained in Ref.[13]. One can see that the rest mass, or $m^2/2$ to be exact, of an Einstein particle is just the \hat{H}' eigenvalue. Without considering the spin degree of freedom, the usual (Einstein) relativistic quantum mechanics corresponds to the σ independent eigenvalue equation for \hat{H}' , obtainable from Eq.(14) separation of the σ variable from the x^μ variables. The eigenvalue equation is the Klein-Gordon equation. Recall that for an Einstein particle, *i.e.* taking the Poincaré-Snyder free particle to the Einstein relativistic limit, we have $\sigma \rightarrow \tau/m$.

5. Final remarks

We have outlined a scheme to think about the physics of the Quantum Relativity. We consider the subject a sensible and very interesting theoretical endeavor, though conceptually quite radical in comparison to other approaches in the literature. It demands creative but careful thinking about physics beyond the usual framework. The subject is in a very primitive stage, and the challenge ahead is formidable. We introduced the Poincaré-Snyder Relativity as a limit of the Quantum Relativity at the level of the

familiar Einstein Relativity. It gives the perspective that the latter maybe just a part of a bigger symmetry, the full usage of which allows a formulation of a richer machines naturally bypassing the no-interaction theorem. We focus here more of the quantum mechanics. The latter gives a new perspective onto relativistic quantum mechanics in a Lorentz covariant form, with a formal evolution parameter beyond space-time. Further theoretical and maybe experimental investigations of the mechanics will help to shed light on our physics understanding of the evolution parameter σ , which will be very useful in our final task for the dynamics of the Quantum Relativity.

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