LIMITATION ON THE RATE OF DECREASE OF AMPLITUDES IN VARIOUS PROCESSES

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It has become apparent recently that, the asymptotic behaviour of the S-matrix element $A(s, t)$ for the transformation of two particles into two particles at high energies s and at a fixed momentum transfer t is determined by the singularities of the partial waves amplitudes $f_l(t)$ as functions of the angular momentum in the channel where t represents the energy [1-5]. If the singularity of $f_l(t)$ with the largest value of Re l is a Regge pole at $l = l(t)$ then the invariant amplitude behaves as $s^{l(t)}$. In the case of elastic processes for small values of t such a pole is the vacuum pole which for $t = 0$ has $l(0) = 1$. As one increases the momentum transfer $\sqrt{-t}$, $l(t)$ may become negative. This gives the impression that for a sufficiently large negative value of t the amplitude can decrease as s increases arbitrarily fast. We propose to show that in relativistic theory the partial wave amplitudes $f_l(t)$ have for any value of t singularities when Re $l \geqq -1$ and accordingly that the amplitude $A(s, t)$ cannot decrease faster than $1/s$ whatever the value of t. This conclusion is valid for the amplitude for any two particle process. The existence of such singularities is due to the existence in relativistic amplitudes of 3 Mandelstam spectral functions, which give rise to singularities in the neighbourhood of negative integral values of l. These singularities appear to be poles concentrated about these points, i.e. the points themselves are essential singularities. To prove this let us consider the expression for the partial wave amplitude:

$$f_l(t) = \frac{2}{\pi}\int_{z_0}^{\infty} Q_l(z) A_1(s, t)\mathrm{d}z \qquad (1)$$

where A_1 is the absorptive part of A,

$$z = 1+\frac{2s}{t-4\mu^2} \text{ and } z_0 = 1+\frac{8\mu^2}{t-4\mu^2}.$$

To simplify matters we consider the case of identical particles of mass μ. If Re $l > l_0$, where l_0 is determined by the maximum number of subtractions needed in dispersion formulas for $A(s, t)$, then as shown in [6]

$$\phi_l(t) = f_l(t)(t-4\mu^2)^{-l}$$

as a function of t satisfies a dispersion relation of the form

$$\phi_l(t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\mathrm{Im}\,\phi_l(t')}{t'-t}\mathrm{d}t' + \frac{1}{\pi}\int_{-\infty}^{0}\frac{\Delta\phi_l(t')}{t'-t}\mathrm{d}t' \qquad (2)$$

where

$$\Delta\phi_l(t) = \int_{4\mu^2}^{4\mu^2-t} P_l\left(\frac{2s}{4\mu^2-t}-1\right)A_1(s,\ t-i\varepsilon)\frac{4\mathrm{d}s}{(4\mu^2-t)^{l+1}}$$
$$+\frac{4}{\pi}\int_{s_1(t)}^{s_2(t)} Q_l\left(\frac{2s}{4\mu^2-t-i\varepsilon}-1\right)\rho(s,u,)\frac{\mathrm{d}s}{(4\mu^2-t)^{l+1}} \qquad (3)$$

The first integral in (3) is taken along a line such as AC or $A'C'$ (see Fig. 1); the second integral, which exists only in relativistic theory, is taken along a line such as $abcd$ or $a'd'$, in the region where the Mandelstam spectral function $\rho(s, u)$ is different from zero. In the dispersion relation (2) it is understood that the necessary number of subtractions have been made.

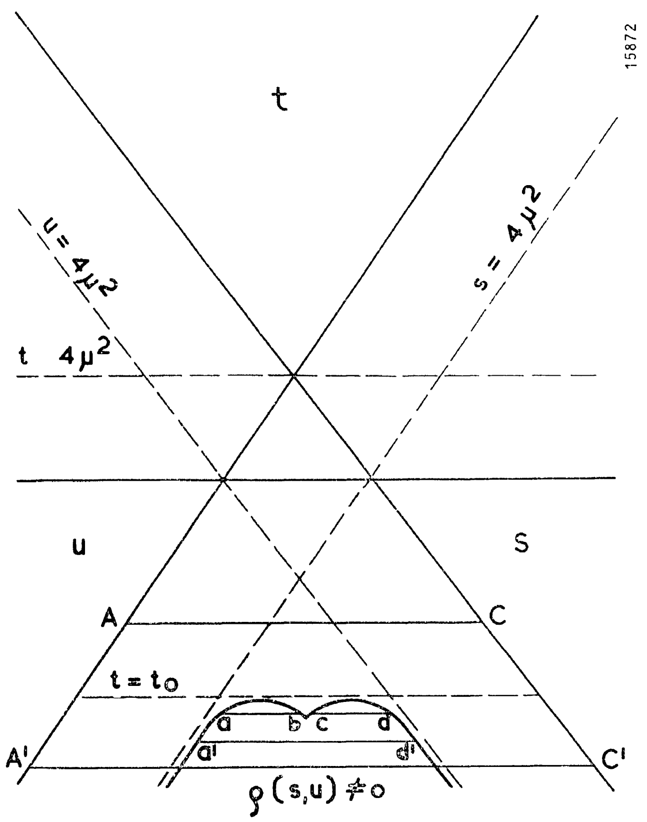

Fig. 1 The Mandelstam *s-t-u* plane.

All the above mentioned statements are based on the fact that the Legendre function $Q_l(z)$ has poles for all negative integral values of l. Although there are poles in $Q_l(z)$ it does not follow from formula (1) that the partial wave amplitude $f_l(t)$ has poles at these points. This is due to the fact that the representation of $f_l(t)$ in the form (1) is valid only when $\mathrm{Re}\, l > m$ where $A(s, t) \lesssim s^m$ for large s. If $m > -1$, the integral has no meaning for negative integral values of l and the question does not arise. If $m < -1$, the residue at the pole is zero, for instance for $l = -1$ it is equal to $4/\pi \int A_1(s, t)dz$ which must be equal to 0 in accordance with Cauchy's theorem for $A(s, t)$. This agrees with the fact that there are usually no singularities in non-relativistic theory for negative integral values of l.

In relativistic theory the situation is different since, according to (3) $Q_l(z)$ also comes into the expression for the discontinuity in $\phi_l(t)$ on the left hand cut. As already mentioned in [6] expression (3) for $\Delta\phi_l(t)$ has a meaning for any complex value of l, since it is determined by integrals over a finite region of analytic functions. This is why $\Delta\phi_l$ in a relativistic theory has poles for negative integral values of l.

Let us consider the question whether the residues at these poles cannot become 0. As the residue of $Q_l(z)$ at the pole at $l = -n-1$ is equal to $\pi P_n(z)$, the residue of $\Delta\phi_l(t)$ at this pole is $\int_{-z_0}^{+z_0} P_n(z)\rho(s, u)dz$, and $|z_o|$ is less than 1. Because of the completeness of the Legendre polynomials, these residues are all zero only if ρ is identically zero. Furthermore, we see that the residue of the pole at $l = -1$ cannot become zero at least for the range of t where the line *abcd* is in a region where the Mandelstam spectral function is positive (such a region always exists in the neighbourhood of the boundary of existence of ρ). It should be noted that in the case of the scattering of identical particles there is no singularity for even values of l since $\rho(s, u)$ is an even function of z. If one considers the dispersion relation (2) and the unitarity condition to be an equation determining $\phi_l(t)$, then the discontinuity on the left hand cut $\Delta\phi_l$ plays the role of the inhomogeneous term of the problem (i.e. it is equivalent to a potential). Then it follows from previous considerations that the amplitude $\phi_l(t)$ has singularities for integral negative values of l, at least for a range of values t. In order to know exactly what happens to $\phi_l(t)$ for these values of l let us refer to dispersion relation (2). Let us continue this equation into the region $l < l_0$ along the real axis. There may now be additional singularities of $\phi_l(t)$ beside the cuts in Eq. (2), and the equation must be revised to include these singularities. Let us suppose to start with that these singularities are moving poles, i.e. poles whose angular momentum l changes as t changes. For $l > l_0$ such poles if they exist are on unphysical sheets of the t-plane. As l decreases they may cross the right-hand branch cut and enter the physical sheet, for example by going through the branch point at $t = 4\mu^2$. If they cross this cut for $t > 4\mu^2$, the residue at the pole must go through zero as the pole crosses the cut, due to unitarity. These poles cannot come on to the physical sheet across the left-hand branch cut because $\Delta\phi_l(t)$ is analytic (see the discussion in [6]). The dispersion Eq. (2) now has the form:

$$\phi_l(t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\mathrm{Im}\, \phi_l(t')}{t'-t}dt' + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\Delta\phi_l(t')}{t'-t}dt' + \sum_{n=1}^{k} \frac{r_{n(l)}}{t_{n(l)}-t} \tag{4}$$

where k is the number of poles on the physical sheet $t_{n(l)}$ the location and $r_{n(l)}$ the residues of these poles.

Now let l approach -1. In this limit, $\Delta\phi_l(t)\to\infty$, and if the number k of pole terms in Eq. (4) remains bounded, $\phi_l(t)$ will also go to ∞ for any value of t. But for $t > 4\mu^2$, $\phi_l(t)$ is bounded, due to unitarity. This gives a contradiction; hence we must expect k to approach ∞ as l approaches -1. Furthermore the locations of the poles must become everywhere dense on the left hand cut $t < t_0$, (see Fig. 1) for $l\to-1$. If this is not the case the contributions of the pole terms and the left hand cut will have different analytic properties in t and will not compensate each other. Thus we expect that for a fixed $t < t_0$, there will be an infinite number of poles in any neighbourhood of $l = -1$, e.g. $\phi_l(t)$ has an essential singularity in l at $l = -1$. This essential singularity occurs for all values of t since it occurs for all values of t on the cut $t < t_0$.

Let us see whether the situation changes if there are singularities other than moving poles (Regge poles) for $-1 < l < l_0$. If such a singularity is formed by a branch point whose position does not depend on energy t then the limitation on the asymptotic behaviour of $A(s, t)$, will be even stronger. The analytic properties of $\phi_l(t)$ as a function of t do not change in this case. Only the unitarity condition changes when l is to the left of the branch point. The unitarity condition written in the form:

$$\frac{1}{2i}(\phi_l(t)-[\phi_{l*}(t)]^*) = (t-4\mu^2)^{l+\frac{1}{2}}t^{-\frac{1}{2}}\phi_l\phi_{l*}^* \qquad (5)$$

(if $4\mu^2 < t < 16\mu^2$) is valid for any value of l but left of the branch point does not mean that $|\phi_l(t)|$ is bounded because ϕ_{l*} is not equal to ϕ_l, due to the existence of a cut in the l plane. However it follows from the unitarity relation (5) that ϕ_l cannot be unbounded on both sides of the cut. If $\phi_l\to\infty$ on one side then on the other side it must be equal to $\pm\frac{1}{2i}(t-4\mu^2)^{-l-\frac{1}{2}}t^{\frac{1}{2}}$. If we consider l on the side of the cut where ϕ_l is finite then all the above considerations remain unchanged and so does the conclusion regarding the existence of an essential singularity at $l = -1$.

If in the interval mentioned we meet a branch point whose position depends on t, then in any case for $t < t_0$ the limitation on the asymptotic behaviour of A will be even more pronounced. We do not consider this question in detail since we do not understand how moving cuts can occur in the l plane.

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DISCUSSION

CHEW: I was not clear about the degree of certainty with which you established that the only l dependent singularities reached through multiparticle cuts will be poles.

GRIBOV: First the conclusion concerned only the three particle threshold. Secondly to prove this I suppose that the partial wave amplitude is an analytical function of l if the real part of l is sufficiently large. I suppose some kind of Mandelstam representation for the inelastic amplitude but of course with complex contour. What can be proved rigorously is that if I consider integral l and three particles inelastic unitarity, there are no singularities depending on l on the second sheet, except poles. That is if l is integral and the unitarity condition is simple. Our proof is based on the unitarity condition, but the unitarity condition cannot be proved for complex l nor for non integral l.