

Numerical simulation of quantum gravity

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Abstract

We performed a nonperturbative sum over geometries in (2+1)-dimensional causal dynamical triangulation through the help of Monte Carlo simulations, where the spacial topologies are assumed to be S^2 for simplicity. We also studied a case where the spacial topologies are T^2 . The numerical results imply these emerging quantum universes can be described by solutions of the Einstein equation at macroscopic scales. We also found that spectral dimensions of these spacetime are roughly 3.

1 Introduction

Causal dynamical triangulation (CDT) is a discrete regularization method for gravitational path integral [1]. In CDT, Lorentzian spacetime geometry is approximated by a simplicial manifold. The simplicial manifold is constructed by gluing “simplex” in such a way so as to have global proper-time foliation (cf. fig. 1). The simplex is the Lorentzian tetrahedron (in 2+1-dim) in which some edges are fixed to be space-like and the others are time-like (cf. fig. 2). The squared length of space-like links is $l_s^2 = a^2$ and the squared length of time-like links is $l_t^2 = -\alpha a^2$ ($\alpha > 0$), where α is a relative scaling of space-like and time-like links. Because the spacial topology change induces causality violation, we forbid the spacial topology to change in time. Then, in (2+1)-dimension the spacetime topology may be written as a product $M = I \times \Sigma^{(2)}$, where I denotes an interval and $\Sigma^{(2)}$ denotes two dimensional surfaces. For convenience, we chose $\Sigma^{(2)} = S^2$ or T^2 , where S^2 is a two-sphere and T^2 is a torus. Moreover, we set periodic boundary conditions in the time direction. Therefore, we assume spacetime topology to be $M = S^1 \times S^2$ or $S^1 \times T^2$.

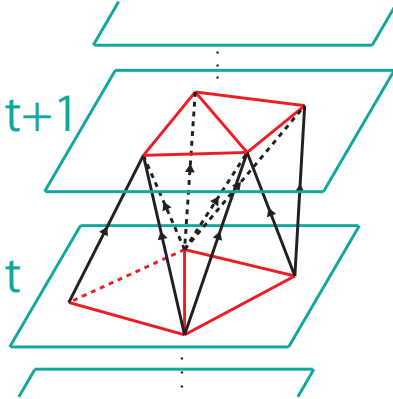


Figure 1: Global proper-time foliation in 3-dimension.

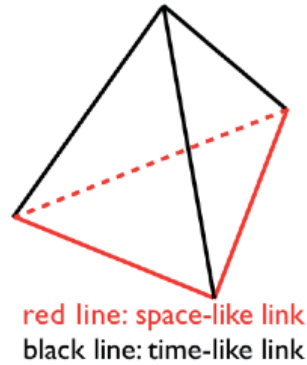


Figure 2: A 3-dimensional simplex with space-like links and time-like links.

In CDT the continuum path integral for gravity is represented by a sum over all discrete spacetime geometries T .

$$Z_L = \int \mathcal{D}g e^{iS_L[g]} \rightarrow Z_L = \sum_T e^{iS_L^{\text{Regge}}[T]}, \quad (1.1)$$

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where S_L denotes the Einstein Hilbert action and S_L^{Regge} denotes that of discretized version. In an analogy with quantum field theory, one might hope to define the gravitational path integral by rotating to the Euclidean signature [2]

$$\alpha \rightarrow -\alpha, \quad S_E^{\text{Regge}} := -iS_L^{\text{Regge}}, \tag{1.2}$$

$$Z_E = \sum_T e^{-S_E^{\text{Regge}}[T]}. \tag{1.3}$$

For a technical reason, we have performed the simulations with (approximately) constant three-volume. We choose $\alpha = 1$, and the Euclidean Regge action can be written in the following form [3]

$$S_E^{\text{Regge}} = -\kappa_0 N_0 + \kappa_3 N_3, \tag{1.4}$$

$$\kappa_0 = \frac{a}{4G}, \quad \kappa_3 = \frac{a}{4G} \left(\frac{3}{\pi} \arccos \frac{1}{3} - 1 \right) + \frac{a^3 \Lambda}{48\sqrt{2}\pi G}, \tag{1.5}$$

where N_0 is the number of vertices, N_3 is the number of simplices, G and Λ are respectively Newton's gravitational constant and the cosmological constant. Just like statistical mechanics, one can compute various geometric quantities from this partition function Z_E in principle. We computed the partition function Z_E by performing the Monte Carlo simulation.

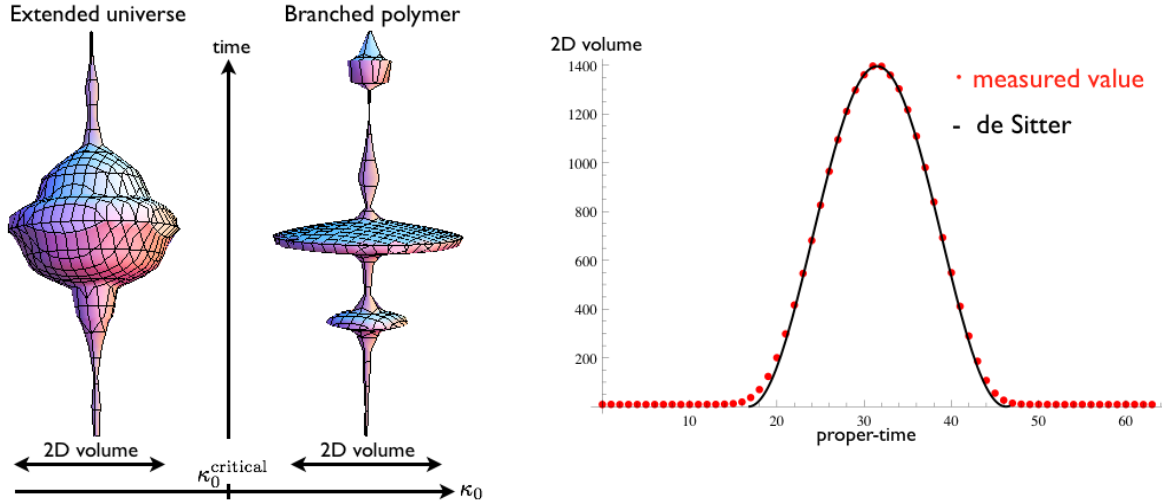


Figure 3: Monte Carlo snapshot of a typical sphere universe

Figure 4: Averaged spacial volume in the extended universe phase, measured for $N_3 = 64000$ and $\kappa_0 = 0.4$.

2 Emergence of de Sitter spacetime

We assume spacial topologies S^2 . Therefore the relevant spacetime topology is $S^1 \times S^2$. In 3D CDT, there appears two phases (cf. fig. 3) [4]. Which one emerges depends on the value of κ_0 (inverse of gravitational coupling constant). For sufficiently small κ_0 , the the averaged 2D spacial volume $\langle V_2(t) \rangle$ at proper time t can be described by that of the de Sitter instanton (cf. fig. 4)

$$\langle V_2(t) \rangle \propto \cos^2\left(\frac{t}{B}\right), \tag{2.1}$$

where B is a constant. We provide further evidence that the emergent universe in the extended universe phase is de Sitter spacetime at macroscopic scales. By studying a diffusion process on the discrete spacetime, we obtain an effective dimension, that is, the spectral dimension D_S . We will check whether the large-scale effective dimension is consistent with 3 in the extended universe phase (left part of fig. 3). The measuring method is described in detail in [5], [6]. We consider a randomly walking test particle in a given discrete spacetime configuration. The particle starts from a given tetrahedron and moves to nearest neighbor tetrahedra at a step. We measure the return probability $P(\sigma)$, which is the probability that the particle returns to the initial tetrahedron at σ steps.

We average over the return probabilities with randomly choosing tetrahedra and on different discrete spacetime configurations. When the diffusion step σ is not so large, the return probability for diffusion on fractal geometry is well-studied and is given by the formula as follows

$$P(\sigma) \propto \sigma^{-D_S/2}. \tag{2.2}$$

We can estimate the spectral dimension D_S by taking the logarithmic derivative

$$D_S = -2 \frac{d \log P(\sigma)}{d \log \sigma} + \text{finite-size correction}, \tag{2.3}$$

as long as the diffusion step σ is not so large. The ansatz, $D_S = a - be^{-c\sigma}$, gives the best fit with the measured spectral dimension $D_S(\sigma)$, where a, b, c are the fitting parameters. For the simulation with $N_3 = 64000$ and $\kappa_0 = 0.4$, the best fit become

$$D_S = 2.808 - 0.710e^{-0.026\sigma}. \tag{2.4}$$

Then, we obtained the extrapolating values

$$D_S(\sigma = \infty) = 2.81, \quad D_S(\sigma = 0) = 2.10. \tag{2.5}$$

The spectral dimension in the extended universe phase is approximately 3 at macroscopic scales.

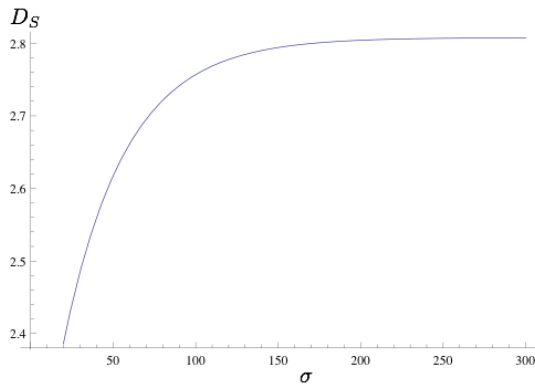


Figure 5: Spectral dimension in the extended universe phase, measured for $N_3 = 64000$ and $\kappa_0 = 0.4$.

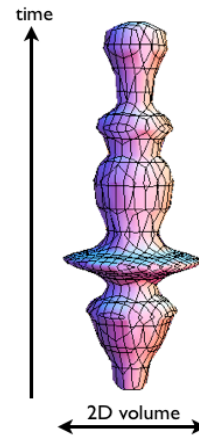


Figure 6: Monte Carlo snapshot of a typical torus universe, measured for $N_3 = 64000$ and $\kappa_0 = 0.4$.

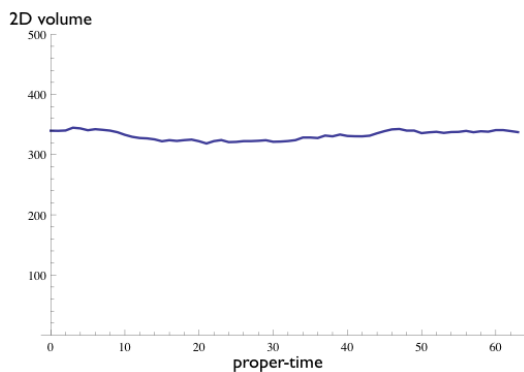


Figure 7: Averaged spatial volume of the torus universe, measured for $N_3 = 64000$ and $\kappa_0 = 0.4$.

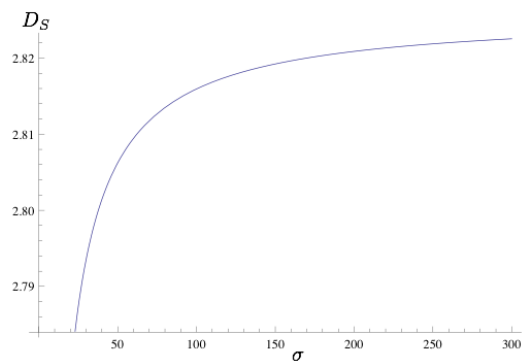


Figure 8: Spectral dimension of the torus universe, measured for $N_3 = 64000$ and $\kappa_0 = 0.4$.

3 The torus universe

Now, we consider a universe with spacial topology T^2 . Monte Carlo snapshot of a typical torus universe and the averaged 2D spacial volume $\langle V_2(t) \rangle$ at proper time t are shown in Figure 6, 7. This spacetime corresponds to the following metric at macroscopic scales

$$ds^2 = d\tau^2 + e^{2\{\alpha(\tau)+\beta(\tau)\}} dx^2 + e^{2\{\alpha(\tau)-\beta(\tau)\}} dy^2. \quad (3.1)$$

The corresponding Euclidean Einstein equations are as follows

$$\dot{\alpha}^2 - \dot{\beta}^2 = -\frac{\Lambda}{2}, \quad \ddot{\alpha} + 2\dot{\alpha}^2 = -\frac{\Lambda}{2}, \quad \ddot{\beta} + 2\dot{\alpha}\dot{\beta} = 0, \quad (3.2)$$

where $\dot{}$ denotes a proper-time derivative. Our simulation result seems to correspond to $\alpha = \text{const}$, $\beta = \text{const}$. So the effective cosmological constant $\Lambda_{\text{eff}} = 0$ is expected.

In this case, the following fit agrees with the data (cf. fig. 8)

$$D_S = a - \frac{b}{c + \sigma} = 2.826 - \frac{1.001}{1.001 + \sigma}, \quad (3.3)$$

where a, b and c are the fitting parameters. We obtained the asymptotic value for $N_3 = 64000$ and $\kappa_0 = 0.4$

$$D_S(\sigma = \infty) = 2.83, \quad D_S(\sigma = 0) = 1.83. \quad (3.4)$$

The spectral dimension of the torus universe is approximately 3 at macroscopic scales.

4 Summary

We found that the quantum universe emerging from a nonperturbative sum over geometries can be described the solutions of Euclidean Einstein equation on large scales in both cases of spacetime topologies are $S^1 \times S^2$ and $S^1 \times T^2$. Moreover, we obtained the effective dimension $D_S \approx 3$ by studying the diffusion process on the discrete spacetime ensemble.

References

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