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A smooth transition towards a Tracy–Widom distribution for the largest eigenvalue of interacting *k*-body fermionic embedded Gaussian ensembles

Ernesto Carro[1,*], **Luis Benet**[2] and **Isaac Pérez Castillo**[3]

[1] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Av. Universidad s/n, Col. Chamilpa, Cuernavaca CP 62210, Morelos, Mexico
[2] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Av. Universidad s/n, Col. Chamilpa, Cuernavaca CP 62210, Morelos, Mexico
[3] Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, San Rafael Atlixco 186, Ciudad de México 09340, Mexico
E-mail: jecarro@icf.unam.mx

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Abstract. In spite of its simplicity, the central limit theorem captures one of the most outstanding phenomena in mathematical physics, that of universality. While this classical result is well understood, it is still not very clear what happens to this universal behavior when the random variables become correlated. A fruitful mathematically laboratory to investigate the rising of new universal properties is offered by the set of eigenvalues of random matrices. In this regard, a lot of work has been done using the standard random matrix ensembles and focusing on the distribution of extreme eigenvalues. In this case, the distribution of the largest or smallest eigenvalue departs from the Fisher–Tippett–Gnedenko

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[*] Author to whom any correspondence should be addressed.

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theorem yielding the celebrated Tracy–Widom distribution. One may wonder, yet again, how robust this new universal behavior captured by the Tracy–Widom distribution is when the correlation among eigenvalues changes. Few answers have been provided to this poignant question, and our intention in the present work is to contribute to this interesting unexplored territory. Thus, we study numerically the probability distribution for the normalized largest eigenvalue of the interacting k-body fermionic orthogonal and unitary embedded Gaussian ensembles in the diluted limit. We find a smooth transition from a slightly asymmetric Gaussian-like distribution, for small k/m, to the Tracy–Widom distribution as $k/m \to 1$, where k is the rank of the interaction and m is the number of fermions. Correlations at the edge of the spectrum are stronger for small values of k/m and are independent of the number of particles considered. Our results indicate that subtle correlations toward the edge of the spectrum distinguish the statistical properties of the spectrum of interacting many-body systems in the dilute limit, from those expected for the standard random matrix ensembles.

Keywords: extreme value statistics, large deviation theory, Tracy–Widom distribution, interacting many-body systems

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Contents

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1. Introduction

An important goal of basic sciences is to boldly search for universal behavior, a common denominator that underlies the description of phenomena based on simple ingredients, either in the physical or mathematical reality. When this occurs they tend to be a paradigm-shifting moment: Newton's, Maxwell's, Einstein's, and Boltzmann's seminal works in physics come to mind, as well as those of Descartes, Fermat, Klein, Hamilton, Riemann, Langlands, and many other seminal works in the realm of mathematics. Focusing on the physical reality, whatever that means, we all would agree that many, if not all, physical systems can be modeled as a set of interacting random variables. This is apparent when describing the macroscopic behavior of systems composed of a large number of constituents. Here, the main goal of statistical mechanics is, by relating the macrostate of a system with all the microstates compatible to it, to explain all of the phases of matter and the transitions between them. While descriptions of pure phases rely on the central limit theorem, close to a phase transition, when the microscopic constituents of the system become more and more correlated, the law of large numbers fails miserably, so other techniques, such as the renormalization group, must be used in its stead. Obviously, studying one physical system after another, identifying their basic constituents and their interactions, and seeking suitable mathematical and physical techniques to analyze them can be a tall order and, at times, quite frankly, tiring. In the last couple of decades, a more mundane and basic approach has been used: look for those mathematical models in which correlations of random variables can be easily modeled and manipulated, and study the emergence of new universal laws in these systems. Due to this, random matrix theory (RMT) has been at the forefront in the study of the emergent behavior of correlated random variables. Originally introduced to deal with the complexities of heavy nuclei Hamiltonian systems [1] —and historically also to deal with noisy linear systems of equations [2]— RMT has grown to be an extremely successful theory with a surprisingly wide range of applications [3–9]. Its three main symmetry classes, the so-called canonical ensembles, were originally unveiled in references [10, 11] and several others [12] —such as the Wishart [13], circular, or non-Hermitian ensembles—have become more relevant over the years.

It was Tracy and Widom [14–16] who first dared to look at how the probability distribution of the largest eigenvalue typically behaves and whether its distribution departed from the one described by the extreme value theorem of Fisher–Tippett–Gnedenko for independent and identically distributed random variables [17, 18]. It was noticed that a new emergent distribution appears, the now celebrated Tracy–Widom distribution. After this, a flurry of research followed suit that originated from the seminal work of Dean and Majumdar [19], focusing primarily on understanding the large deviation properties of extreme eigenvalues in standard ensembles of random matrices. By exploiting Dyson's log-gas analogy and shrewdly using saddle-point techniques, they managed to obtain the left and right rate functions of extreme eigenvalues. Importantly, they noticed that the deviations to the left of, say, the largest eigenvalue are markedly different to the ones on the right, scaling differently with the system size. More research was done along these lines for other standard random matrix ensembles [20–29], in diluted ensembles of random matrices [30–35], and for generalizations based on Dyson's log-gas analogy [36–38], among many others, to ascertain the robustness of this new emergent law on the statistics of extreme values; for a recent review, see [39].

While RMT has been a fruitful mathematical laboratory in this particular endeavor, it was also recognized that its ensembles are somewhat unrealistic in the sense that they assume interactions to involve all Hilbert-space states, whereas typical forces in nature involve two-, three-, or a few-body interactions. This criticism led to the introduction of the two-body ensembles [40–43], which eventually were generalized to the k-body embedded ensembles by French and Mon [44]. Applications of these ensembles include nuclear physics [3, 44, 45], atoms [46], disordered systems [47], and in general, finite interacting many-body systems [48, 49]. Some results are known analytically for the k-body embedded ensembles of Gaussian random matrices, which mostly focus on the properties of the mean-level density [48, 50–53]; for instance, it is known that there is a transition in the mean-level density from a Gaussian shape to a semi-circle when varying

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the rank of the interaction. Regarding the fluctuation properties of the spectrum, while it has not been proven that they are of the RMT type for the fermionic two-body embedded ensembles in the limit of large matrices, there is vast numerical evidence that points in that direction, at least for the central part of the spectrum; see, e.g. [42, 43, 45, 48, 54–56]. Not much is known on the behavior at the edge of the spectrum for these ensembles, aside from the early numerical investigations on the gap distribution using matrices of small dimensionality [3, 43]. The main goal of the present paper is to address the fluctuation properties of the largest eigenvalue of the fermionic k-body embedded ensembles of Gaussian random matrices in terms of its parameters. As, in contrast to the standard matrix ensembles, the number of mathematical techniques to tackle statistical problems in embedded ensembles is rather limited [44, 48, 53], our analysis is solely based on numerical methods. The results presented here, in spite of being purely numerical, are of interest since they contribute to understand the properties of this ensemble, and may motivate further investigations that could pave the way for some analytical work.

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2. Matrix model and definitions

Recall that the k-body fermionic embedded Gaussian ensemble (fEGE) is the family of matrix representations of quantum Hamiltonian systems consisting of m interacting spinless fermions, which can occupy any of ℓ possible degenerate single-particle states. The interaction among them is taken to be a k-body operator. More concretely, following [53], we introduce the operator $\Psi^\dagger_{k;\rho} \equiv \Psi^\dagger_{j_1\ldots j_k} = \prod_{s=1}^{k} a^\dagger_{j_s}$ that creates $k \leqslant m$ particles. Here, $a^\dagger_{j_s}$ is the fermionic creation operator of the single-particle state j_s, while the label ρ represents the set of indices $(j_1, j_2, \ldots, j_k)$, with the ordering $1 \leqslant j_1 < \ldots < j_k \leqslant \ell$. A similar definition, using the annihilation operators a_{j_s}, defines the k-body annihilation operator $\Psi_{k;\rho}$. Being fermionic operators, a_{j_r} and $a^\dagger_{j_s}$ satisfy the usual anticommutator relations, $\{a_{j_r}, a_{j_s}\} = \{a^\dagger_{j_r}, a^\dagger_{j_s}\} = 0$ and $\{a_{j_r}, a^\dagger_{j_s}\} = \delta_{j_r, j_s}$.

In the number operator representation, the k-body interaction is then given by:

$$V_k^{(\beta)} = \sum_{\rho,\sigma} v^{(\beta)}_{k;\rho,\sigma} \Psi^\dagger_{k;\rho} \Psi_{k;\sigma}, \tag{1}$$

where the coefficients $v^{(\beta)}_{k;\rho,\sigma}$, while obeying that $v^{(\beta)}_{k;\rho,\sigma} = [v^{(\beta)}_{k;\sigma,\rho}]^\star$, are independently distributed Gaussian random variables with zero mean and a constant variance, which henceforth is fixed to one. Finally, depending on Dyson's β parameter, the set of coefficients $v^{(\beta)}_{k;\rho,\sigma}$ is either real (for $\beta = 1$) or complex (for $\beta = 2$). We refer to k in equation (1) as the rank of the interaction. The m-particle Hilbert space is spanned by a basis created by filling up m states out of ℓ, that is, $|\mu\rangle = \Psi^\dagger_{m;\mu}|0\rangle$, which readily implies that this space has dimension $N = \binom{\ell}{m}$. Using this basis, the matrix representation of the k-body interaction has entries given by $\langle\mu|V_k^{(\beta)}|\nu\rangle = \langle 0|\Psi_{m;\mu} V_k^{(\beta)} \Psi^\dagger_{m;\nu}|0\rangle$. The particular case of $k = m$ coincides with the classical Gaussian ensembles of RMT, while for $m > k$ the matrix elements $\langle\mu|V_k^{(\beta)}|\nu\rangle$ display correlations or may even be identically zero [53]. To illustrate the appearance of correlations in the fEGEs, consider for simplicity the case $\ell = 6$, $m = 3$ and $k = 2$, and the many-body states $|\mu_1\rangle = \Psi^\dagger_{1,2,3}|0\rangle$, $|\mu_2\rangle = \Psi^\dagger_{3,4,5}|0\rangle$, $|\mu_3\rangle = \Psi^\dagger_{1,2,6}|0\rangle$, $|\mu_4\rangle = \Psi^\dagger_{4,5,6}|0\rangle$, $|\mu_5\rangle = \Psi^\dagger_{1,3,4}|0\rangle$, and $|\mu_6\rangle = \Psi^\dagger_{1,4,6}|0\rangle$. The two matrix elements $\langle\mu_1|V_{k=2}|\mu_2\rangle$ and $\langle\mu_3|V_{k=2}|\mu_4\rangle$ are identically $v_{(1,2);(4,5)}$, and thus are completely correlated. In turn, $\langle\mu_1|V_{k=2}|\mu_3\rangle = v_{(13);(16)} + v_{(23);(26)}$, and $\langle\mu_5|V_{k=2}|\mu_6\rangle = v_{(13);(16)} + v_{(34);(46)}$, showing a partial correlation. Finally, $\langle\mu_1|V_{k=2}|\mu_4\rangle = \langle\mu_2|V_{k=2}|\mu_3\rangle = 0$, showing the possibility of sparsity of the matrix $V_{k=2}$.

The limit $N\to\infty$ for the fEGEs can be attained either as $\ell\to\infty$ for fixed m or by fixing the filling factor m/ℓ and taking the limits $m,\ell\to\infty$. These limits have different properties in terms of the moments of the density of states [48, 50]. In the present study, we shall focus on the limit where m remains constant, which is more advantageous from a numerical point of view. This is the so-called dilute limit. In this limit, the results are expected also to hold for the bosonic ensemble [53].

Some results are known for this ensemble. By definition, as stated above, the case $k = m$ is identical to the classical Gaussian ensembles of RMT, and hence the average level density is a semicircle, all fluctuation properties follow the classical RMT results, and the distribution of the normalized largest eigenvalue is the Tracy-Widom distribution [14–16]. For $k \ll m$ in the dilute limit, the binary correlation method yields a Gaussian average level density [3, 44, 53]. From the kurtosis of the average level density, the transition from the semicircle to a Gaussian was identified at $k = m/2$ [50]; further insight into this transition was provided by studying higher even moments, uncovering interesting transition points at $m = rk$, for the $2\,r$ moment [57]. While vast amount of numerical work indicates for $k = 2$ that the spectral fluctuations are of RMT type, an analytical proof is still missing: mathematical convergence issues do not permit a straightforward application of the binary correlation method; as stated by Sredniki, the spectral fluctuations of the fEGE remains an unsolved problem [52]. Aside from the case $k = m$, not much is known about the properties at the edge of the spectrum; the gap (normalized spacing among the two largest eigenvalues) for the case $k = 2$ was studied for very small matrices [3, 43].

To study the probability distribution of the largest eigenvalue $\lambda_{\rm N}$ for the fEGEs, we first remove the N dependence on its mean and variance with respect to the ensemble average, which will allow for a proper characterization of the distribution; see e.g. [39]. We assume that its mean value can be written as $\langle\lambda_{\rm N}\rangle = \sqrt{2\beta}N^\alpha$, while its standard deviation behaves as $\sigma_{\lambda_{\rm N}} \sim N^\gamma$. Here, $\langle(\cdots)\rangle$ corresponds to the ensemble average in the embedded ensemble. The power-law assumption has been numerically asserted by direct inspection of the dependence of $\langle\lambda_{\rm N}\rangle$ and $\sigma_{\lambda_{\rm N}}$ on N, as illustrated in figure 1. The normalized largest eigenvalue, denoted as $\tilde\lambda_{\rm N}$, is then given by

$$\tilde\lambda_{\rm N} = \frac{\lambda_{\rm N} - \sqrt{2\beta}N^\alpha}{N^\gamma}. \tag{2}$$

As it is well known, for the standard Gaussian ensembles, which correspond in our case to $k = m$, the celebrated Tracy–Widom distribution corresponds to having scaling exponents $\alpha = 1/2$ and $\gamma = -1/6$ [14–16].

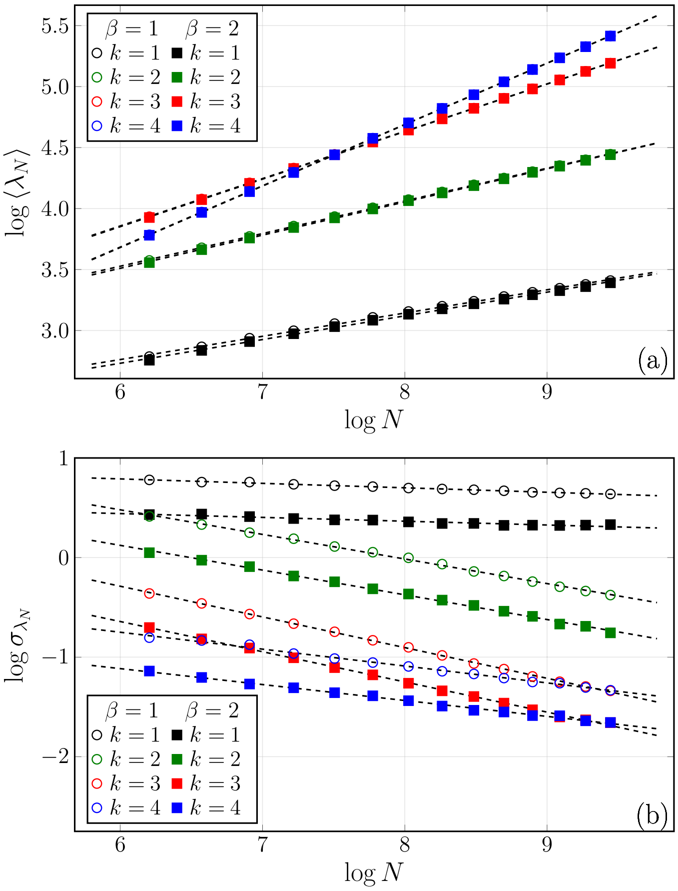

Figure 1. Scaling properties of (a) $\langle \lambda_{\mathrm{N}} \rangle$ and (b) $\sigma_{\lambda_{\mathrm{N}}}$ in terms of N, for $\beta = 1$ (empty circles) and $\beta = 2$ (filled squares), for $m = 4$, $k = 1, 2, 3, 4$, and various values of ℓ. The power-law scaling in terms of N is apparent. Notice that the slopes of the $\beta = 1$ and $\beta = 2$ cases are close to each other.

3. Results

For the fEGEs, the scaling exponents remain unknown. As an exact mathematical analysis is rather difficult, we henceforth rely on numerical methods in order to infer α and γ in terms of the parameters defining the EGEs. We use a simple random sampling method to generate 10^4 matrices belonging to the fEGEs for both $\beta = 1$ and $\beta = 2$ and for different number of fermions (parameter m) and single-particle states (controlled by the parameter ℓ). More concretely, we take $m = 4$ with values of $\ell \in \{8, 9, \ldots, 28\}$; $m = 5$ with $\ell \in \{10, 11, \ldots, 20\}$; $m = 6$ and $\ell \in \{12, 13, \ldots, 18\}$; $m = 7$ with $\ell \in \{14, 15, \ldots, 18\}$; and, finally, $m = 8$ with values $\ell \in \{16, \ldots, 18\}$. Moreover, in all cases the rank takes values $k = 1, \ldots, m$. Each choice of m and ℓ fixes the linear dimension N of the matrix, and by varying ℓ we can see how the scaling exponents behave with N. This is then repeated for different ranks k of the interaction.

Figure 2 summarizes the results for α (top figure) and γ (bottom figure) as a function of the ratio k/m for $\beta = 1$ (represented by circles) and $\beta = 2$ (represented by squares

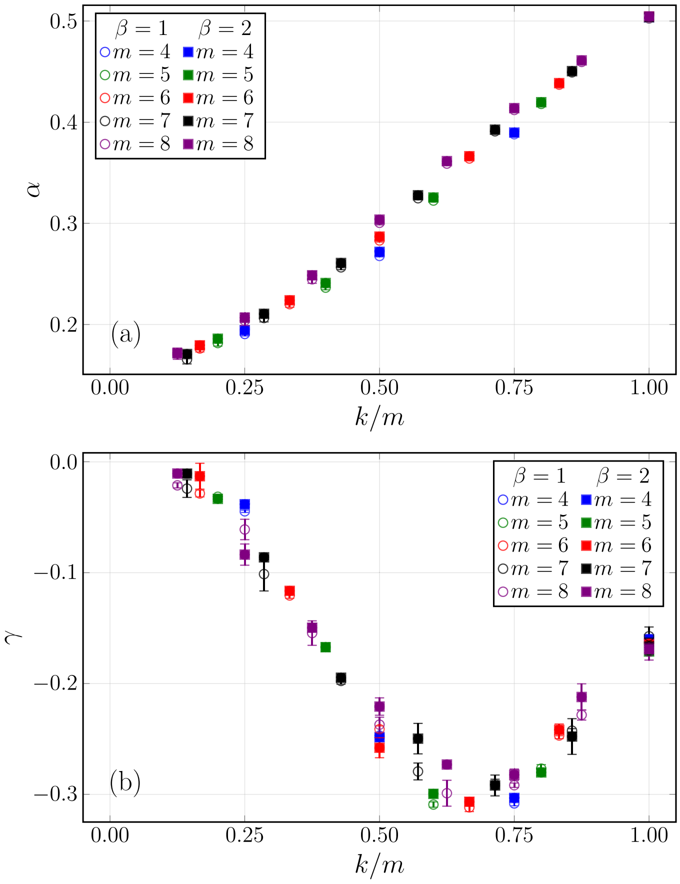

Figure 2. Scaling exponents (a) α and (b) γ for the mean and standard deviation of $\lambda_{\rm N}$ for the fEGEs, in terms of k/m. Data points represented by circles correspond to the orthogonal ensembles ($\beta = 1$), while data for the unitary case ($\beta = 2$) are represented by squares.

in the same figure). The error bars are the result of estimating the exponents by linear regression. As can be appreciated, both exponents, when plotted as a function of k/m, seem to coalesce into a curve, independent of the value of β. While the exponent α is monotonously increasing, the exponent γ that controls the amplitude of the fluctuations has, rather surprisingly, a minimum at around $k/m \sim 0.7$. Moreover, when $k/m = 1$ we recover the expected values of both exponents according to the Tracy–Widom distribution.

To dive deeper into the probability density function of the largest eigenvalue, denoted as $D(\tilde{\lambda}_{\rm N})$, let us take, for the sake of discussion, the values of $m = 7$ and $\ell = 14$ and investigate how the shape of the distribution changes as we vary the rank of the interaction. This is precisely shown in figure 3; the top figure corresponds to $\beta = 1$, while the bottom one is for the unitary case $\beta = 2$. To better quantify the profile of $D(\tilde{\lambda}_{\rm N})$, in figure 4 we also show the behavior of its variance, skewness, and excess kurtosis, normalized with respect to the ones of the Tracy–Widom, as a function of k/m. In the diluted limit we are working on (fixed m and large values of ℓ), we observe a smooth

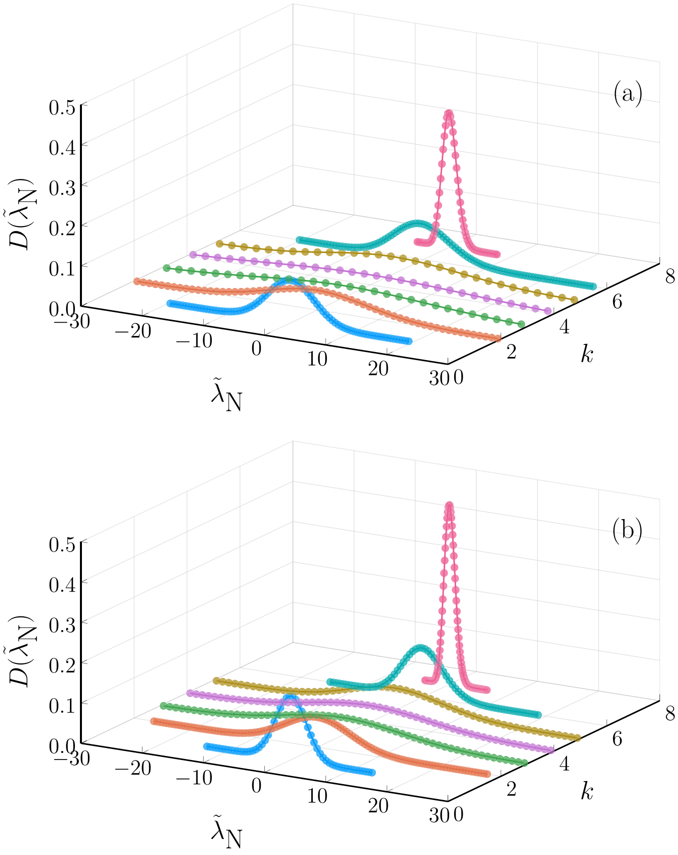

Figure 3. Probability distribution $D(\tilde{\lambda}_{\mathrm{N}})$ for the normalized largest eigenvalue in terms of k, for $m=7$ and $\ell=14$ $(f=1/2)$ for (a) $\beta=1$ and (b) $\beta=2$. The resulting distributions display a transition from an almost-symmetric Gaussian distribution for $k=1$, to the Tracy–Widom distribution for $k=m$. Different colors are used for different values of k.

transition from an almost symmetric Gaussian distribution for small values of k/m, to a more spread and asymmetric distribution as k/m is increased, to finally arriving at the Tracy–Widom distribution for $k/m=1$. Moreover, at around $k\simeq m/2$ the width of the distribution ceases to grow. These results indicate that the correlations at the edge of the spectrum are different in terms of k/m. While this statement is trivial for $k/m=1$, our findings for the one-body interaction $k=1$ are actually rather surprising. Here, according to [5] or [50], we would expect the spectral statistics in the bulk to correspond to an uncorrelated Poisson distribution, which should naïvely yield, in turn, to an extreme value distribution according to the Fisher–Tippett–Gnedenko theorem [58–60]. More precisely, since for $k=1$ the mean-level density is Gaussian [40, 48, 50, 53], an uncorrelated spectrum should yield a Gumbel distribution [58–61], but this is not the result obtained for $k=1$. Instead, the distribution is closer to a slightly asymmetric Gaussian distribution. Thus, there must remain important correlations occurring toward the edge of the spectrum in this case.

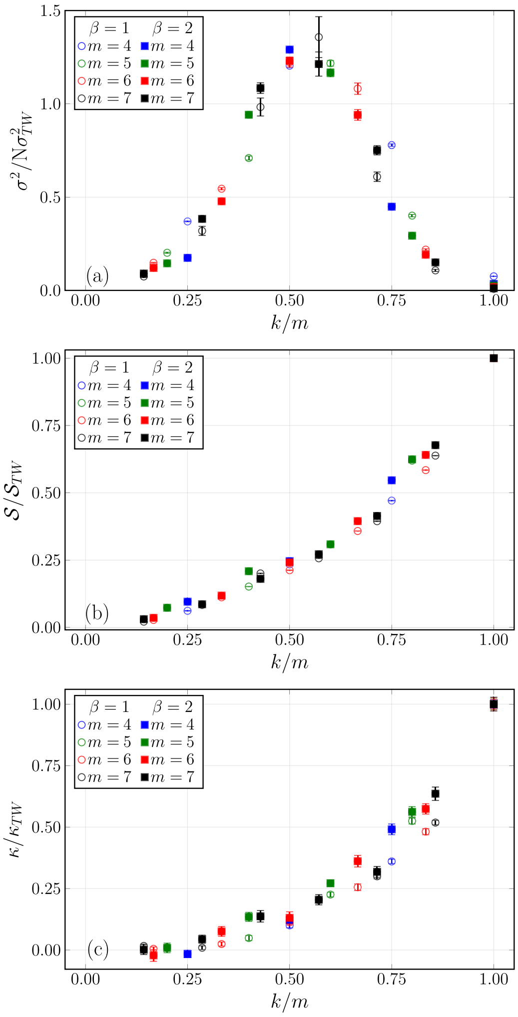

Figure 4. (a) Variance (divided by N), (b) skewness, and (c) excess kurtosis, normalized to the corresponding values of the Tracy–Widom distribution, for $D(\tilde{\lambda}_{\text{N}})$ in terms of k/m, for $\ell = 14$. Circles correspond to the results obtained for $\beta = 1$ and squares to the case $\beta = 2$.

To unveil this oddity, we proceed to estimate the correlation coefficient between the largest and second largest eigenvalues considering the formula:

$$c_{\text{N,N-1}} = \langle \tilde{\lambda}_{\text{N}} \tilde{\lambda}_{\text{N-1}} \rangle - \langle \tilde{\lambda}_{\text{N}} \rangle \langle \tilde{\lambda}_{\text{N-1}} \rangle \,. \tag{3}$$

In addition, we shall also estimate the distribution of the normalized distance $s = (\lambda_{\text{N}} - \lambda_{\text{N-1}})/\langle \lambda_{\text{N}} - \lambda_{\text{N-1}} \rangle$ between these two eigenvalues.

In figure 5(a), which shows the correlation coefficient $c_{\text{N,N-1}}$, we notice it is a positive and monotonically decreasing function of k/m. Moreover, its behavior seems to be independent of Dyson's index β. Fairly interestingly, for small values of k/m, and in particular for $k = 1$, the correlations are essentially twice stronger than the

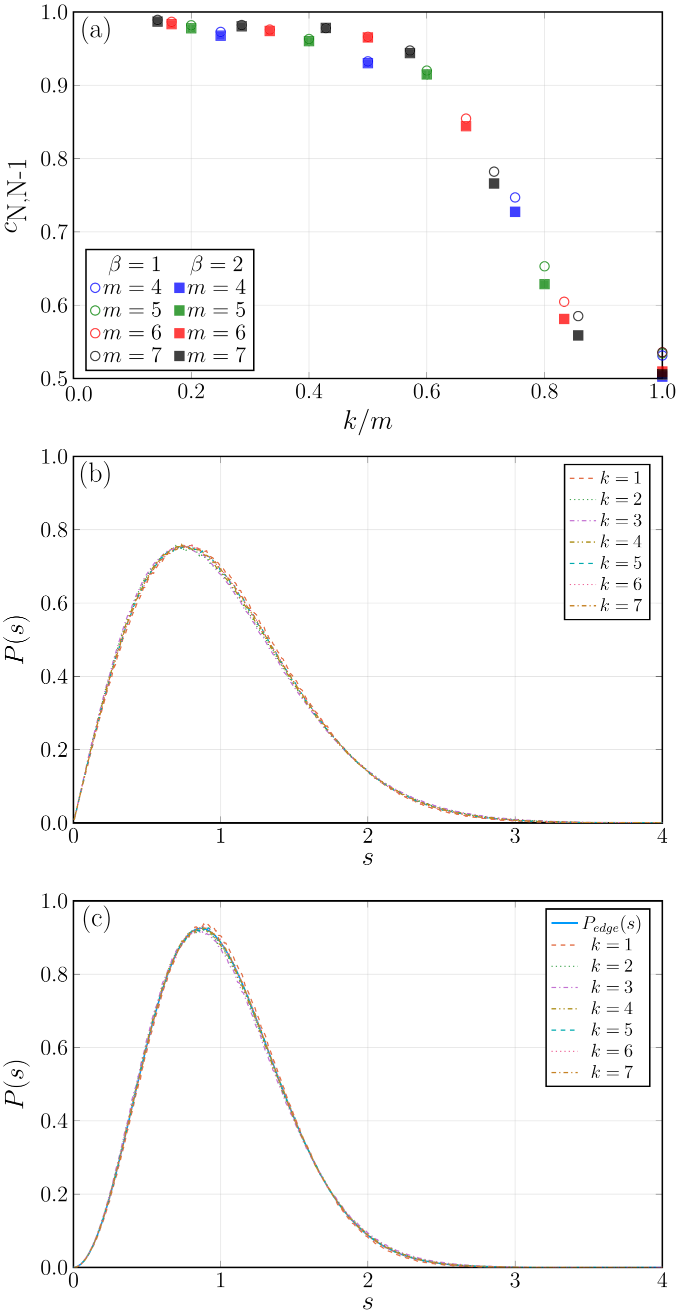

Figure 5. (a) Normalized correlation coefficient, equation (3), for the two largest eigenvalues of the fEGEs in terms of k/m with $\ell = 14$, for $\beta = 1$ (circles) and $\beta = 2$. (b) Distribution $P(s)$ of the (normalized) distance among the two largest eigenvalues for $m = 7$, $\ell = 14$, and $\beta = 1$, and (c) $\beta = 2$. The continuous curve (blue line) in (c) corresponds to the exact results for gap distribution at the edge of the GUE.

Tracy–Widom case of $k = m$. Moreover, the correlation coefficient seems to have a plateau—or to decay rather slowly—for values of $k/m \leqslant 1/2$, above which it starts to decay smoothly but rapidly toward the value predicted by the standard RMT ensembles.

In figures 5(b) and (c) we present the results for the distribution of the normalized distance s among the largest eigenvalues, for the orthogonal and unitary fEGEs, respectively, and various values of k, for $m=7$ and $\ell=14$. This quantity is analogous to the nearest-neighbor distribution used widely to characterize the spectral fluctuations at the bulk of the spectrum [5]. We observe that the distributions obtained are weakly dependent on the rank of the interaction k and depend on β exhibiting the usual level repulsion s^β. We compare the distributions for the case $\beta=2$ with the exact results obtained for the gap at the edge of the GUE [62, 63], which behaves as s^β for small s as expected, and as $\exp(-as^{3/2})$ for large values of s. The figure shows that the numerical results match the exact results for $k=m$, and due to the weak dependence on k, agree for all k values. Therefore, while the distributions of the largest eigenvalue depend on k, and the correlations $c_{\mathrm{N,N\text{-}1}}$ display certain k dependence, the normalized spacing among the two largest eigenvalues is weakly dependent on k, if it is dependent at all.

4. Summary and conclusions

In this paper, we have studied the statistical properties of the largest eigenvalue for the orthogonal and unitary k-body interacting fEGEs. We have presented numerical evidence showing that, in the dilute limit, where the results for fermions also apply to bosons [53], there is a smooth transition in the distribution of the largest eigenvalue from a slightly asymmetric Gaussian-like distribution for small values of k/m, to the Tracy–Widom distribution as k/m approaches one. This transition is such that both the normalized (with respect to the Tracy–Widom corresponding quantities) skewness and the excess kurtosis grow smoothly with respect to k/m, while the normalized width of the distribution displays a maximum and is somewhat asymmetric with respect to k/m. Based on these findings, it is clear that for $k \ll m$ the distribution of the largest normalized eigenvalue is different from the Tracy–Widom distribution [64]: while it is difficult to attain numerically the case $k/m \to 0$ in the dilute limit, our results indicate that the distribution of the largest normalized eigenvalue is close to a Gaussian, perhaps reaching the Gaussian limit in the large N limit. We also studied the correlation coefficient between the largest and second largest eigenvalues, finding it to be a strictly positive monotonically decreasing function of k/m, with different decay rates depending on the value k/m. The distribution of the distance among the two largest eigenvalues of the fEGEs is seemingly independent on k and thus agrees with the results for the classical ensembles of RMT. These results show that, while the statistics at the bulk of the spectrum for the fEGEs may coincide with those of the canonical ensembles, say for $k=2$, correlations that depend on k arise toward the edge of the spectrum.

The properties of the distributions of the largest eigenvalue for the fEGEs in the limit $N\to\infty$ with the constant filling factor remain an open problem as well as an analytical confirmation of our findings.

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