RADIAL STRAIGHT SECTIONS IN SPIRAL SECTOR FFAG ACCELERATORS

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(presented by F. T. Cole)

I. INTRODUCTION

The geometry of RF accelerating cavities in spiral sector accelerators has raised serious problems. If the cavities are introduced along spirals, electric fields perpendicular to the equilibrium orbits are engendered and the resonance excitation of betatron oscillations by the accelerating voltage is greatly enhanced. On the other hand, radial straight sections for RF cavities destroy the scaling properties of the magnetic guide and focusing field. The straight sections appear at different phases relative to the field spirals at different energies and the numbers of betatron oscillations per revolution, ν_x and ν_y for radial and vertical motion, respectively, will vary periodically with particle energy so that resonances may be crossed.

The problem of calculating the changes of ν_x and ν_y with energy has been treated for special cases by several authors. Elfe and Kerst [1] found, by matrix methods that the changes in ν_x and ν_y decreased markedly when the number of radial straight sections per spiral sector was increased from 2 to 3. Ohkawa [2] confirmed this result with a smooth approximation estimate. The Harwell group found [3], with digital computation and approximate analytic treatments, disastrous changes of ν_x and ν_y with one radial straight section per spiral sector, but were able to reduce these changes greatly by putting five straight sections in four sectors. Of course, in this case the number of periods of the field per revolution is reduced by a factor 4, so that new stopbands are introduced, which must be avoided to preserve stable motion.

These results stimulated us to explore the numerology with as careful an analytic treatment as possible. We envisage a structure with N spiral sectors and P radial straight sections per revolution. G, the greatest common divisor of N and P, is the number of periods of the combined magnetic field per revolution. $Q = P/G$ is the number of radial straight sections and $R = N/G$ is the number of spirals per period of the magnetic field.

Our main result is the following: if we assume that the field with straight sections is generated from the spiral field without straight sections by multiplying it by a function of period $2\pi/P$ in the azimuthal angle θ representing the straight sections and if we neglect all harmonics n of the original spiral field such that $n \geq \frac{1}{2}Q$, then the linear betatron oscillation " frequencies " ν_x and ν_y are independent of energy. This result is independent of the form of the straight sections.

We believe that our assumption that the field with straight sections is generated by multiplication of two periodic functions is accurate and physically reasonable. Such multiplication produces a periodic field whose harmonics have the form $mN + nP$, with m and n integers or zero, which is in accord with intuition. Further, this multiplication gives a field which " bulges " out into the straight sections farther at a maximum of the spiral field than at a minimum, which is again in accord with intuition. What is neglected is the effect of the finite size of the forward and backward current windings around

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each spiral, which must join in the radial straight sections. We know from experience with the MURA electron accelerators that these effects of winding size can be very much reduced by carrying the windings in the straight sections rapidly away from the median plane.

In Section II, below, we outline an analytic treatment of the equilibrium orbit motion and linear betatron oscillations for general fields; in Section III we develop the form of the field with straight sections; in Section IV we prove the result stated above and in Section V we give digital computer evidence relating to our result.

II. ANALYTIC ORBIT THEORY (*)

We expand the median plane field, which has only a vertical component, in powers of the relative deviation ξ from a reference circle of radius r_0 (ξ is defined by $r = r_0(1+\xi)$) and in Fourier series in θ. That is,

$$B_z = B_o \sum_{m=0}^{\infty} \sum_{n} Z_{m,n} \xi^m e^{in\theta}. \tag{2.1}$$

All sums whose limits are not given are to be taken to extend from $-\infty$ to ∞, as, for example, the sum over n in Eq. (2.1).

The field of a scaling FFAG accelerator

$$\begin{cases} B_z = B_o(1+\xi)^k \sum_{n=0}^{\infty} (g_n \cos n\Psi + f_n \sin n\Psi) \\ \Psi = K \ln(1+\xi) - N\theta \end{cases} \tag{2.2}$$

can be written in the notation of Eq. (2.1) by taking

$$Z_{m,nN} = \beta_n \prod_{r=0}^{m-1} (k_n - r), \tag{2.3}$$

where

$$\begin{cases} \beta_n = \tfrac{1}{2}(g_n + if_n), & n > 0, \\ \quad = g_0, & n = 0, \\ \quad = \tfrac{1}{2}(g_n - if_n), & n < 0, \\ k_n = k - inK. \end{cases} \tag{2.4}$$

The linearized equations of motion about the equilibrium orbit, which we wish to solve for ν_x and ν_y, are well known [4]. They are

$$\begin{cases} \dfrac{d^2x}{ds^2} + \dfrac{1-n}{\rho^2} x = 0 \\ \dfrac{d^2z}{ds^2} + \dfrac{n}{\rho^2} z = 0, \end{cases} \tag{2.5}$$

where x is the normal deviation from the equilibrium orbit in the median plane, z is the deviation from the equilibrium orbit normal to the median plane, s is the arc-length along the equilibrium orbit,

$$\rho = -\frac{cp}{eB_z}$$

is the radius of curvature of the equilibrium orbit of a particle of kinetic momentum p, c is the velocity of light, e is the charge of a proton, and

$$n = -\frac{\rho}{B_z} \frac{\partial B_z}{\partial x}.$$

Both ρ and n are to be evaluated on the equilibrium orbit. We rewrite Eq. (2.5) in a dimensionless form by measuring all lengths in units of R_0, the length of the equilibrium orbit divided by 2π. We define ζ, η and ϕ by

$$\begin{cases} x = R_0\zeta \\ z = R_0\eta \\ s = R_0\phi, \end{cases} \tag{2.6}$$

and Eqs. (2.5) become

$$\begin{cases} \dfrac{d^2\zeta}{d\phi^2} + \lambda[\eta_1 + \lambda\eta_o^2]\zeta = 0 \\ \dfrac{d^2\eta}{d\phi^2} - \lambda\eta_1\eta = 0, \end{cases} \tag{2.7}$$

where

$$\begin{cases} \eta_o = B_z/B_0 \\ \eta_1 = \dfrac{\partial B_z}{\partial \zeta}/B_0 \\ \lambda = eR_0B_0/cp. \end{cases} \tag{2.8}$$

(*) The treatment we outline here is not original; it is perhaps most accurately described as an extension of the work of G. Parzen. Significant contributions to the methods outlined here have also been made by D. L. Judd, L. J. Laslett and T. Ohkawa. Their work is recorded in various (unpublished) MURA reports.

λ is a dimensionless constant which for a given field and momentum is a measure of R_0, and η_0 and η_1 are to be evaluated on the equilibrium orbit. Therefore, we must find the equilibrium orbit in order to know η_0 and η_1 in terms of the given field coefficients $Z_{m,n}$.

The equilibrium orbit is the solution of the median plane equation of motion about the reference circle,

$$(\xi' Z)' = (1+\xi)Z - \alpha(1+\xi)B_z/B_0 , \qquad (2.9)$$

which has the period of B_z. In Eq. (2.9), primes denote derivatives with respect to the azimuthal angle θ and

$$\begin{cases} Z = [(1+\xi)^2 + \xi'^2]^{-\frac{1}{2}} \\ \alpha = -er_0 B_0/cp . \end{cases} \qquad (2.10)$$

α is a dimensionless constant which for a given field and momentum is a measure of r_0. We expand Eq. (2.9) in powers of ξ by expanding the Lagrangian from which it is derivable (in order to preserve the Hamiltonian character of the motion). Correct through second order in ξ and ξ', Eq. (2.10) is

$$\xi'' = 1 + \xi\xi'' + \tfrac{1}{2}\xi'^2 - \alpha \sum_n e^{in\theta}\{Z_{0,n} + (Z_{1,n}+Z_{0,n})\xi + (Z_{2,n}+Z_{1,n})\xi^2\} . \qquad (2.11)$$

The solution we seek has the form

$$\xi = \sum_n \xi_n e^{in\theta} . \qquad (2.12)$$

We substitute Eq. (2.12) in Eq. (2.11) and equate terms of the same frequency, obtaining an infinite set of algebraic equations :

$$\begin{aligned} -n^2\xi_n = \delta_{n0} - \alpha Z_{0,n} &- \alpha \sum_m (Z_{1,m}+Z_{0,m})\xi_{n-m} \\ &- \alpha \sum_{m,p} (Z_{2,m}+Z_{1,m})\xi_p \xi_{n-m-p} \\ &- \tfrac{1}{2}\sum_m m(n+m)\xi_m \xi_{n-m} , \\ &n = 0, \pm 1, \pm 2, \dots . \end{aligned} \qquad (2.13)$$

Eqs. (2.13) can be solved for the ξ_n by an approximation method, which assumes that the terms depending on the ξ_n on the right hand side are small compared to the terms independent of the ξ_n. This is equivalent to assuming that the change of field across the equilibrium orbit is small compared to the peak field on the equilibrium orbit. The p th approximation, $\xi_n^{(p)}$, is calculated by substituting $\xi_n^{(p-1)}$ on the right hand side of Eq. (2.13). There is a difficulty with ξ_0, whose size depends on the reference radius chosen. We circumvent this difficulty by choosing α such that r_0 is the average radius of the equilibrium orbit. Then $\xi_0 = 0$ and the $n = 0$ equation of (2.13) gives a value for α.

Our assumption is then

$$\xi_n^{(0)} = 0$$

and by substituting this on the right hand side of Eq. (2.13)

$$\xi_n^{(1)} = \frac{\alpha Z_{0,n}}{n^2}$$

$$\begin{aligned} \xi_n^{(2)} = \frac{\alpha}{n^2}\Big\{ Z_{0,n} &+ \alpha \sum_{m \neq n} \frac{(Z_{1,m}+Z_{0,m})Z_{0,n-m}}{(n-m)^2} + \\ &+ \alpha^2 \sum_{\substack{m,p \neq 0 \\ m+p \neq n}} \frac{(Z_{2,m}+Z_{1,m})Z_{0,p}Z_{0,n-m-p}}{p^2(n-m-p)^2} + \\ &+ \tfrac{1}{2}\alpha \sum_{m \neq 0,n} \frac{m+n}{m(m-n)} Z_{0,m}Z_{0,n-m} \Big\} . \end{aligned} \qquad (2.14)$$

α satisfies Eq. (2.13) with $n = 0$ and ξ_n substituted from Eq. (2.14). Correct through terms quadratic in α, we then have

$$1 - Z_{0,0}\,\alpha - \left[\sum_{m \neq 0} \frac{(Z_{1,m} + {}^3/_2 Z_{0,m})Z_{0,-m}}{m^2} \right] \alpha^2 = 0 . \qquad (2.15)$$

In practice $\xi_n^{(2)}$ agrees with computer experiments to within a few per cent, while $\xi_n^{(1)}$ differs from $\xi_n^{(2)}$ by 10 - 20%. The method of solution seems *a posteriori* to be justified.

Parenthetically, we may remark that the term of Eq. (2.15) linear in α is due to the bending of the equilibrium orbit by the average field, while the term quadratic in α describes the additional bending due to the fact that the oscillations of the equilibrium orbit carry a particle into regions of different field.

η_0 and η_1 can now be expressed in terms of the $Z_{m,n}$. In calculating η_1, we must take note of the fact that the ξ (radial) and ζ (normal to the equilibrium orbit) directions are not parallel. A little partial differentiation and geometrical exercise give

$$\eta_1 = \frac{R_0 Z}{r_0 B_0}\left\{ (1+\xi)\frac{\partial B_z}{\partial \xi} - \frac{\xi'}{1+\xi}\frac{\partial B_z}{\partial \theta} \right\} , \qquad (2.16)$$

with all quantities to be evaluated on the equilibrium orbit.

We must take note also of the fact that η_0 and η_1 are given as functions of θ. They can be converted to functions of ϕ simultaneously with their Fourier analysis. Thus

$$\eta_i = \sum_m \eta_{i,m} e^{im\phi} \tag{2.17}$$

and

$$\eta_{i,m} = \frac{1}{2\pi}\int_0^{2\pi}\left[\eta_i(\theta)e^{-im\phi}\frac{d\phi}{d\theta}\right]d\theta\,.$$

From the definition of ϕ,

$$\frac{d\phi}{d\theta} = \frac{r_0}{R_0}Z^{-1}\,,$$

which we can expand in powers of ξ and ξ' and integrate to give

$$\phi = \frac{r_0}{R_0}\left\{\theta\left[1+\tfrac{1}{2}\sum_{m\neq 0} m^2\xi_m\xi_{-m}+\dots\right]+\right. \tag{2.18}$$

$$\left. +\sum_{n\neq 0}\frac{e^{in\theta}-1}{in}\left[\xi_n-\tfrac{1}{2}\sum_{m\neq 0,n} m(n-m)\xi_m\xi_{n-m}+\dots\right]\right\}.$$

In a conventional spiral sector accelerator, the periodic terms are of order N^{-3} compared to unity and the coefficient of the term linear in θ differs from unity by terms of order N^{-2}, both of which are negligible in cases of interest. Eq. (2.18) also gives R_0 in terms of r_0, since ϕ and θ have the common values 0 and 2π. Thus

$$R_0 = r_0\left[1+\tfrac{1}{2}\sum_{m\neq 0} m^2\xi_m\xi_{-m}+\dots\right]. \tag{2.19}$$

Eqs. (2.18) and (2.19) can be expressed in terms of the $Z_{m,n}$ by substituting from Eq. (2.14). Approximate expressions for the $\eta_{i,n}$ are found to be

$$\eta_{0,n} = Z_{0,n}+\alpha\sum_{m\neq n}\frac{Z_{1,m}Z_{0,n-m}}{(n-m)^2}$$

$$\eta_{1,n} = Z_{1,n}+2\alpha\sum_{m\neq n}\left[\frac{(Z_{2,m}+\tfrac{1}{2}Z_{1,m})Z_{0,n-m}}{(n-m)^2}+\right.$$

$$\left. +\frac{mZ_{0,m}Z_{0,n-m}}{2(n-m)}\right] \tag{2.20}$$

These expressions can be recognized as being essentially expansions in powers of $\alpha\, kFN^{-2}$ or αKFN^{-2}, where $F = \left[\sum_{n\neq 0}\beta_n\beta_{-n}\right]^{\frac{1}{2}}$ is the flutter, quantities of order 0.2 in either radial or spiral sector accelerators, so that the neglected terms are only a few per cent of the leading terms.

More generally, we have given the first terms of an expansion of $\eta_{i,n}$ in terms of the $Z_{m,p}$. This expansion is a sum of products of the $Z_{m,p}$. In each product, the sum of the second (θ) subscripts of the $Z_{m,p}$ must be n (The θ-indices in α always sum to zero).

To calculate the betatron oscillation frequencies, we shall use a method developed by Walkinshaw [5]. For a Hill equation

$$\begin{cases} \dfrac{d^2u}{d\phi^2}+[\omega^2+n(\phi)]u = 0 \\ n(\phi) = \sum_{n\neq 0} a_n e^{in\phi}\,, \end{cases} \tag{2.21}$$

the solution can be given as a series

$$\begin{cases} u(\phi) = \sum_{m=0}^{\infty} P_m(\phi) \\ P_{m+1}(\phi) = \dfrac{1}{\omega}\displaystyle\int_0^{\phi} n(\alpha)\sin\omega(\alpha-\phi)P_m(\alpha)d\alpha \\ P_0(\phi) = A\cos\omega\phi + B\sin\omega\phi\,. \end{cases} \tag{2.22}$$

The convergence of this series has been proved by Vogt-Nilsen [6] under the assumption that $|n(\phi)|$ is bounded. We can expect it to give accurate results for stability zones higher than the first. Through the second order ($m = 2$), this method gives for the phase change per revolution $\Sigma = 2\pi\nu$ for the case $\omega \neq 0$ and 2ω different from any integer n for which $a_n a_{-n} \neq 0$, that is, when Σ is not at the edge of a stopband :

$$\cos\Sigma = \cos 2\pi\omega - \frac{\pi\sin 2\pi\omega}{\omega}\sum_{n\neq 0}\frac{a_n a_{-n}}{n^2-(2\omega)^2}\,. \tag{2.23}$$

III. FORM OF THE FIELD

As the energy of a particle changes, the phase of the straight sections relative to the spirals changes. Rather than examining the dependence of ν_x and ν_y on energy directly, we shall examine their dependence on this relative phase τ of the straight sections and the spirals. We take a straight section function of the form

$$S(\theta) = \sum_m \lambda_m e^{imP(\theta-\tau)}, \qquad (3.1)$$

with P periods per revolution. The field without straight sections, a conventional scaling field, has the form, from Eqs. (2.3) and (2.4),

$$B_z^{(0)} = B_0 \sum_n \beta_n (1+\xi)^{k_n} e^{inN\theta}, \qquad (3.2)$$

with N periods per revolution. We generate the field with straight sections by multiplying Eqs. (3.1) and (3.2). Then

$$B_z = B_0 \sum_{m,n} \lambda_m \beta_n (1+\xi)^{k_n} e^{i(mP+nN)\theta} e^{-imP\tau},$$

which we can write as

$$B_z = B_0 \sum_n e^{in\theta} \left\{ \sum_m \lambda_m \beta_{(n-mP)/N} (1+\xi)^{k_{(n-mP)/N}} e^{-imP\tau} \right\},$$

so that $Z_{m,n}$ has the form (3.3)

$$Z_{m,n} = \frac{1}{m!} \sum_r \lambda_r \beta_{(n-rP)/N} \left[\frac{\partial^m}{\partial \xi^m} (1+\xi)^{k_{(n-rP)/N}} \right]_{\xi=0} e^{-irP\tau}$$
$$= \sum_r e^{-irP\tau} f(m,n,r). \qquad (3.4)$$

IV. PROOF OF THE THEOREM

We assert that if $\beta_n = 0$ for $|n| \geq \frac{1}{2} Q$, then ν_x and ν_y are independent of τ and thus of energy. To prove this, we show first that if $\beta_n = 0$ for $|n| \geq \frac{1}{2} Q$, and $f(m, n, r) \neq 0$, then $f(m, n, r') \neq 0$ only if $r' = r$.

If $f(m, n, r) \neq 0$, then $\beta_{(n-rP)/N} \neq 0$. Then, since $\beta_m \neq 0$ only for integral $m, n-rP$ must be an integral multiple of N, say $n-rP = sN$, with s a positive or negative integer or zero. Similarly, if $f(m, n, r') \neq 0$, then $n-r'P = s'N$. Then

$$(r'-r)P = (s-s')N,$$

or

$$(r'-r)\frac{P}{G} = (s-s')\frac{N}{G},$$

or

$$(r'-r)Q = (s-s')R. \qquad (4.1)$$

But G is by definition the greatest common divisor of P and N, so that Q and R are relatively prime numbers. Since they are, the diophantine equation (4.1) has a solution only if $s-s'$ is an integral multiple of Q, say $(s-s') = tQ$, with t a positive or negative integer or zero. But $|s| < \frac{1}{2} Q$ and $|s'| < \frac{1}{2} Q$, since $\beta_n = 0$ for $|n| \geq \frac{1}{2} Q$. Therefore $|s-s'| < Q$ and t must be zero. Then $s = s'$ and $r = r'$.

Thus only one term of the sum (3.4) is different from zero. $Z_{m,n}$ is different from zero only for $n = tN+rP$, with t and r positive or negative integers or zero. Then if $\beta_n = 0$ for $|n| \geq \frac{1}{2} Q$,

$$Z_{m,\,tN+rP} = e^{-irP\tau} \lambda_r \beta_t \frac{1}{m!} \left[\frac{\partial^m}{\partial \xi^m} (1+\xi)^{k_t} \right]_{\xi=0}. \qquad (4.2)$$

$\cos \Sigma_x$ and $\cos \Sigma_y$ are given by Eq. (2.23) as sums of products over the $\eta_{i,m}$. In each product the sum of the ϕ-indices of the $\eta_{i,m}$ must be zero. But $\eta_{i,m}$ is a sum of products of the $Z_{p,n}$, with the θ-indices of each product summing to m. Then any product $\eta_{i,m}\, \eta_{j,-m}$ must be a sum of products of the $Z_{p,n}$ with the θ-indices of each product summing to zero. The most general sum which can appear is

$$\sum_{\substack{m_1, m_2, \ldots, m_T \\ n_1, n_2, \ldots, n_{(T-1)}}} Z_{m_1, n_1} Z_{m_2, n_2} \cdots Z_{m_{T-1}, n_{T-1}} Z_{m_T, -n_1-n_2-\ldots-n_{(T-1)}}. \qquad (4.3)$$

Since the $Z_{m,n}$ are different from zero only for $n = tN + rP$, this sum can be written

$$\sum_{\substack{m_1, m_2, \ldots, m_T \\ r_1, r_2, \ldots, r_{(T-1)} \\ t_1, t_2, \ldots, t_{(T-1)}}} Z_{m_1, r_1 P + t_1 N} Z_{m_2, r_2 P + t_2 N} \cdots \cdots Z_{m_{(T-1)}, r_{(T-1)}P + t_{(T-1)}N} Z_{m_T, -M}, \qquad (4.4)$$

where

$$M = (r_1 + r_2 + \ldots + r_{T-1}\ P) + (t_1 + t_2 + \ldots + t_{T-1})\ N.$$

If we now substitute the form (4.2), valid when $\beta_n = 0$ for $|n| \geq \frac{1}{2} Q$, it is clear that the form (4.4) is independent of τ and thus of energy. Since all terms of $\cos \Sigma_x$ and $\cos \Sigma_y$ have the form (4.4), Σ_x and Σ_y and therefore ν_x and ν_y are independent of τ and thus of energy.

We can use this result to interpret more clearly the earlier work [1-3]. In each case, the changes of ν_x and ν_y were reduced greatly when Q was increased, from 2 to 3 in the work of Elfe and Kerst and from 1 to 5 in the work of the Harwell group. Since the field harmonics β_n decrease with n at least as rapidly as n^{-1} in most accelerators, the first few harmonics are responsible for the major part of the change of ν_x and ν_y with energy.

In closing this section, we remark that in the fields we have discussed here, the phases of the spirals continue uninterrupted across the straight sections. We have proved the same theorem in the case where the spiral phases do not change at all across the straight sections, with the approximation that the straight section function (Eq. (3.1)) is a rectangular wave [7].

V. DIGITAL COMPUTER EVIDENCE

Orbits were integrated numerically on the IBM-704 through enough periods to find the equilibrium orbit and measure ν_x and ν_y, using the "Spirit" program developed for the purpose. This program multiplies the scaling field (Eq. (2.2)) by the straight section function (Eq. (3.1)) to find the field and integrates the exact equations of two-dimensional motion by the Runge - Kutta method.

The straight section function was chosen from approximate magnetostatic calculations. It is not claimed that our choice of $S(\theta)$ is necessarily realistic in all cases; it suffices for our purposes that it can give rise to changes of ν_x and ν_y with energy. The field drops to about 80% of its full value in straight sections and the total length of straight sections is about 10% of the length of the spiral sector, thus giving a total straight section length per spiral sector of about 1 m in a 10 GeV accelerator. Table I gives the Fourier coefficients of the straight section function.

TABLE I

Fourier coefficients of the straight section function

η	λ_η
0	1
± 1	−0.03105
± 2	−0.02274
± 3	−0.01499
± 4	−0.00872
± 5	−0.00477
± 6	−0.00242
± 7	−0.00117
± 8	−0.00053

The first scaling field we investigated had the parameters $N = 30$, $k = 53$, $K = 280$, $g_0 = 1$, $g_1 = 1$, and, when they were inserted, $g_2 = g_3 = 0.2$. In Table II, Q is the number of radial straight sections per period of the structure, R is the number of spirals per period, G is the number of periods of the structure per revolution, $n_{\max}$ is the maximum harmonic number of the scaling field, $\langle\nu_x\rangle$ and $\langle\nu_y\rangle$ are the mean values of ν_x and ν_y (averaged over τ or energy) and $\frac{\Delta\nu_x}{\langle\nu_x\rangle}$ and $\frac{\Delta\nu_y}{\langle\nu_y\rangle}$ are the maximum relative deviations of ν_x and ν_y in per cent. Each datum point $\left(\frac{\Delta\nu_x}{\langle\nu_x\rangle} \text{ and } \frac{\Delta\nu_y}{\langle\nu_y\rangle}\right)$ requires about 2 hours of running time on the computer. Points marked U were found to have unstable radial motion, due in all cases to the stopband near $\Sigma_x = \pi$ introduced by lowering the periodicity from 30 to 15. When the radial motion is unstable, it is quite difficult (and not very interesting) to investigate vertical motion.

From our experience we would judge that digital computation gives values of ν_x and ν_y with errors of about 1%. Values of $\frac{\Delta\nu}{\langle\nu\rangle}$ less than a few per cent may be regarded as negligible.

Because of the stopbands, which occurred at some points of interest, we have also investigated the same effects in an accelerator with $k = 30$, $K = 210$ and all other parameters unchanged from above. Table III gives the same quantities as Table II for these points.

TABLE II

Digital computation results on effects of radial straight sections

Q	R	G	$n_{\max}$	$\langle\nu_x\rangle$	$\langle\nu_y\rangle$	$\frac{\Delta\nu_x}{\langle\nu_x\rangle}(\%)$	$\frac{\Delta\nu_y}{\langle\nu_y\rangle}(\%)$
0	1	30	1	8.266	5.818	0	0
			2	8.261	6.379	0	0
			3	8.226	6.144	0	0
1	1	30	1	8.060	5.788	24.50	38.02
			2	8.060	6.283	29.89	33.20
2	1	30	1	8.204	5.832	12.24	4.04
			2	8.231	6.399	15.69	13.14
3	1	30	1	8.220	5.821	3.19	0.83
			2	8.256	6.383	7.90	1.64
			3	8.192	6.144	9.87	13.38
3	2	15	1	U			
			2	U			
			3	U			
4	1	30	1	8.223	5.820	0.58	0.13
			2	8.258	6.380	2.66	0.33
5	2	15	1	8.044	5.830	1.86	0.26
			2	U			
			3	U			
5	3	10	1	8.069	5.617	0.43	4.45
			2	8.027	6.294	0.94	4.45

TABLE III

Digital computation results on effects of radial straight sections

Q	R	G	n_{max}	$\langle \nu_x \rangle$	$\langle \nu_y \rangle$	$\frac{\Delta \nu_x}{\langle \nu_x \rangle}(\%)$	$\frac{\Delta \nu_y}{\langle \nu_y \rangle}(\%)$
0	1	30	1	5.891	4.391	0	0
1	1	30	1	5.703	4.338	27.02	42.46
2	1	30	1	5.886	4.399	10.40	2.86
3	2	15	1	5.916	4.397	0.92	1.06

The digital computer results show phenomena which bear out the theorem proved in this paper. In some cases it appears that the largest decrease in $\Delta \nu_y$ occurs when Q becomes greater than n_{max}, rather than when Q becomes greater than $2n_{max}$, as the theorem would predict. This effect is presumably due to some detailed cancellation of terms of cos Σ_y, but we have not yet gained an understanding of it.

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(*) see note on reports, p. 696.