Some properties of energies in nonsynchronous reference frames in cosmology

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Abstract. It is shown that use of non synchronous reference frame in cosmology can lead to unusual behavior of the energy of particles: it can have not only positive but also negative and zero energies. A special example of the model with the scale factor $a(t) \sim t$ is considered.

1. Introduction

Physics of rotating black holes leads to use of the Boyer-Lindquist [1] coordinates in which the metrical tensor has nondiagonal terms. The consequence of this form of the tensor is existence of negative and zero energies of particles in the region close to the horizon of the black hole called ergosphere [2]. This leads to a well known Penrose effect [3, 4] in decays and collisions of particles. But nondiagonal term in metrical tensor occurs also if one considers the expanding Universe in nonsynchronous reference frames. Differently from the rotating Kerr's black holes this nondiagonal term corresponds to a product not of time and the angle, but of time and the radial coordinate. In our papers [5, 6] it was shown that similarly to the black holes one can have particles with negative and zero energies in this case also. In this paper we continue to study this property of the energy of particles. After studying the general case of of the nonstationary metric of the cosmological model we specially investigate the case with the scale factor $a(t) \sim t$. One of the well known examples of such metric is the Milne's metric [7], i.e. Minkowski space-time in Milne's coordinates.

2. Negative and zero energies in nonsynchronous reference frame in cosmology

The metric of homogeneous isotropic cosmological model can be written as

$$ds^2 = c^2 dt^2 - a^2(t) \left(d\chi^2 + f^2(\chi) d\Omega^2 \right), \tag{1}$$

where c is velocity of light, $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$, the coordinate χ is changing from 0 to π in the closed model with $f(\chi) = \sin\chi$ and from 0 to $+\infty$ for quasi-Euclidean model with $f(\chi) = \chi$, and $f(\chi) = \sinh\chi$ for the open model. The radial distance between points $\chi = 0$ and χ in metric (1) is $D = a(t)\chi$. If t is fixed then in closed model the maximal value of D is $D_{\max} = \pi a(t)$. In open and flat models D is non limited.

Take the new coordinates t, D, θ, φ (see also [8, 9]). Then

$$dD = HD\,dt + a\,d\chi, \qquad d\chi = \frac{1}{a}\,(dD - H\,Ddt) \tag{2}$$

where $H = \dot{a}(t)/a(t)$ is a Hubble parameter, dot above the symbol means the derivative with respect to time t. In new coordinates the interval (1) becomes

$$ds^2 = \left(1 - \left(\frac{HD}{c}\right)^2\right) c^2 dt^2 + 2HD\,dtdD - dD^2 - a^2 f^2\left(\frac{D}{a}\right)\,d\Omega^2. \tag{3}$$

Let $D_s = c/|H(t)|$. The value of D_s defines the boundary of the region out of which the physical bodies cannot be at rest similar to the value of the static limit in the case of rotating black hole. Really, from the condition $ds^2 \geq 0$ one comes to conclusion that no physical object can be at rest in this coordinate system for $D > D_s$. If the universe is expanding then $\dot{a} > 0$ and for $D > D_s$ for the observer using coordinates t, D, θ, φ all bodies move in the direction of growing D (for them $dD > 0$), in case of contraction of the universe one has $dD < 0$.

Note that the value of the static limit D_s in general case is changing in time. The exclusion is the case of constant Hubble parameter H, i.e. scale factor $a \sim e^{Ht}$. In this case the static limit coincides with the event horizon defined by the formula

$$D_H = a(t)c\int_t^\infty \frac{dt}{a(t)}. \tag{4}$$

For $f(\chi) = \chi$ the metric (1) with such scale factor corresponds to de Sitter space-time.

Similar to the ergosphere of rotating black hole in the region out the static limit one can have particles with non-positive energies. Let us show this.

As it was shown in our paper [6] the energy of the particle with mass m for the observer in the reference frame with metric g_{ik} is

$$E = mc^2 g_{0k}\frac{dx^k}{ds}. \tag{5}$$

Note that the energy of the particle in metrics (1), (3) is not conserved because metrics depends on time.

In case with the interval (3) the energy is

$$E = E'\left(1 - \frac{D^2}{D_s^2} + \frac{HD}{c^2}\frac{dD}{dt}\right), \tag{6}$$

where $E' = mc^2 dt/d\tau$ is the energy in comoving coordinate system, τ is the proper time of moving particle. For the analysis of possible values of particle energies one can express its value through the radial velocity in comoving coordinates

$$v_\chi = a\frac{d\chi}{dt}. \tag{7}$$

Condition $ds^2 \geq 0$ leads to

$$\left|a\frac{d\chi}{dt}\right| \leq c, \tag{8}$$

i.e. $-c \leq v_\chi \leq c$. Using notification

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

write (6) in the form

$$E = E' \left(1 + \frac{HD}{c^2} \frac{a\, d\chi}{dt} \right) = E' \left(1 + \operatorname{sgn}(\dot a) \frac{D}{D_s} \frac{v_\chi}{c} \right). \tag{9}$$

So negative energies for particles in the reference frame t, D, θ, φ occur if

$$E < 0 \quad \Rightarrow \quad D > D_s, \quad c \geq |v_\chi| > c\frac{D_s}{D}, \quad \operatorname{sgn}(v_\chi) = -\operatorname{sgn}(\dot a), \tag{10}$$

and one has movement in direction of decreasing of the coordinate χ in case of expanding Universe and in direction of increasing radial coordinate χ in case of contracting cosmological model. To have the negative energy close to the static limit the particle must be close to the the velocity of light in comoving coordinates. Far from the static limit $D \gg D_s$ one can have negative energies for small velocities.

From (9) one can see that zero energies are possible for $D \geq D_s$:

$$E = 0 \quad \Rightarrow \quad D \geq D_s, \ \ \frac{d\chi}{dt} = -\frac{c^2}{D\dot a}. \tag{11}$$

3. Negative and zero energies in nonsynchronous reference frame in the model with $a(t) = \sigma ct$

Consider the case of expanding cosmological model with a scale factor $a(t) = \sigma ct$. This model is interesting because if $f(\chi) = \sinh \chi$ and $\sigma = 1$ it describes flat space-time in curvilinear coordinates (Milne Universe).

Using conformal time η, so that $cdt = ad\eta$ one obtains instead of metric (1) the form

$$ds^2 = a^2(\eta) \left(d\eta^2 - d\chi^2 - f^2(\chi) d\Omega^2 \right). \tag{12}$$

In this model

$$a = \sigma ct = \sigma ct_0 e^{\sigma(\eta - \eta_0)}, \qquad \eta = \eta_0 + \frac{1}{\sigma} \log \frac{t}{t_0}, \qquad t = t_0 e^{\sigma(\eta-\eta_0)}, \tag{13}$$

where η_0 is arbitrary constant fixing point of the origin of conformal time $t(\eta_0) = t_0$.

Einstein's equations

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda g_{ik} = -8\pi \frac{G}{c^4} T_{ik}, \tag{14}$$

where Λ is cosmological constant, G is the gravitational constant, T_{ik} is energy-momentum tensor of background matter, in metric (12) take form

$$T_0^0 = \frac{3c^2}{8\pi G} \left(\frac{\dot a^2 + Kc^2}{a^2} - \frac{c^2}{3} \Lambda \right), \tag{15}$$

$$T_\alpha^\beta = \delta_\alpha^\beta \frac{c^2}{4\pi G} \left(\frac{\ddot a}{a} + \frac{\dot a^2 + Kc^2}{2a^2} - \frac{c^2}{2} \Lambda \right), \tag{16}$$

where $K = 1, 0, -1$ for closed, quasi-Euclidean and open model respectively. So for the scale factor $a(t) = \sigma ct$ the components of the energy-momentum tensor of background matter are

$$T_i^k = \mathrm{diag}\,(\varepsilon, -p, -p, -p), \tag{17}$$

and energy ε and pressure p satisfy the relations

$$p - \frac{c^4}{8\pi G}\Lambda = -\frac{1}{3}\left(\varepsilon + \frac{c^4}{8\pi G}\Lambda\right) = \frac{-c^2}{8\pi G}\frac{(\sigma^2 + K)}{\sigma^2 t^2}. \tag{18}$$

Note that ε and p are the energy density and pressure of background matter in coordinate frame t, D (see [10]). The scalar curvature for a scale factor $a(t) = \sigma ct$ is

$$R = \frac{6}{a^2}\left[\frac{a\ddot{a}}{c^2} + \left(\frac{\dot{a}^2}{c^2} + K\right)\right] = \frac{6}{c^2 t^2}\left(1 + \frac{K}{\sigma^2}\right), \tag{19}$$

and it is either zero for $K = -1$, $\sigma = 1$ or decreases in time. Existence of cosmological constant in the model does not lead to accelerated expansion ($\ddot{a} = 0$) and is compensated by a background matter with equation of state (18).

The static limit in coordinates t, D in this model is equal to $D_s = ct$. The region out of the static limit at the fixed moment t is defined by the inequality $\chi > 1/\sigma$. Particle energy in the reference frame t, D, θ, φ is

$$E = E'\left(1 + \sigma^2 t\chi\frac{d\chi}{dt}\right) = E'\left(1 + \sigma\chi\frac{d\chi}{d\eta}\right). \tag{20}$$

Particle with conserved zero energy during some time must move (noninertially and not on the geodesic line!) according to the law

$$E = 0 \quad \Rightarrow \quad \chi = \sqrt{\chi_0^2 - \frac{2}{\sigma^2}\log\frac{t}{t_0}} = \sqrt{\chi_0^2 - \frac{2}{\sigma}(\eta - \eta_0)}, \qquad \chi_0 \geq \frac{1}{\sigma}. \tag{21}$$

Possible interval of such movement is

$$t \in \left[t_0,\ t_0 \exp\frac{\sigma^2\chi_0^2 - 1}{2}\right], \qquad \eta \in \left[\eta_0,\ \eta_0 + \frac{\sigma}{2}\left(\chi_0^2 - \frac{1}{\sigma^2}\right)\right]. \tag{22}$$

If $\sigma = 1$ than in new coordinates

$$T = t\cosh\chi, \qquad r = ct\sinh\chi, \qquad cT > r > 0 \tag{23}$$

the interval (1) with our scale factor $a(t) = ct$ and $f(\chi) = \sinh\chi$ has the form of the interval in Minkowski space-time [11]

$$ds^2 = c^2 dT^2 - dr^2 - r^2\, d\Omega^2. \tag{24}$$

The arising space-time (Milne universe) with coordinates $t \geq 0$, $\chi \geq 0$ is projected on the future cone in coordinates cT, r.

Let us look what is the region in new coordinates which corresponds to the region out of the static limit in coordinates t, D and what are features of particle movement in flat coordinates if in t, D coordinates frame they have zero and negative energies. Note that in new coordinates $cT/r = \coth\chi$. So the region out of the static limit at the fixed moment t, i.e. the region $\chi \geq 1$ is limited in coordinates cT, r by rays $cT = \coth 1$ and $cT = 1$. This region is shown on Fig. 1

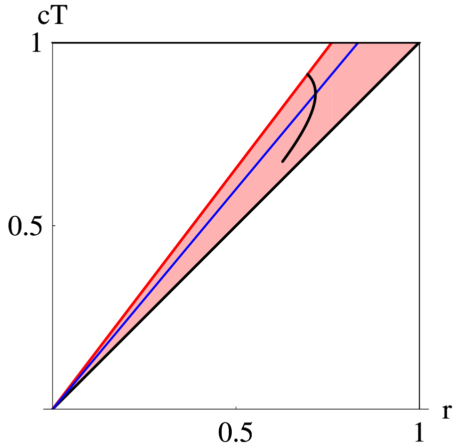

Figure 1. Region out of the static limit for $a = ct$ in coordinates T, r.

by pink color. The curve in this region shows the trajectory of the particle with zero energy (21) for $\chi_0 = 1.65$, $t_0 = 0.25$.

For velocities one obtains

$$\frac{dr}{dT} = \frac{\dfrac{\partial r}{\partial t} + \dfrac{\partial r}{\partial \chi}\dfrac{d\chi}{dt}}{\dfrac{\partial T}{\partial t} + \dfrac{\partial T}{\partial \chi}\dfrac{d\chi}{dt}} = c\frac{\tanh\chi + t\dfrac{d\chi}{dt}}{1 + t\tanh\chi\dfrac{d\chi}{dt}}, \qquad t\frac{d\chi}{dt} = \frac{\dfrac{dr}{dT} - c\tanh\chi}{c - \dfrac{dr}{dT}\tanh\chi}. \tag{25}$$

Particle energy will be negative in the frame t, D, θ, φ if $\chi > 1$ and $t\chi\frac{d\chi}{dt} < -1$ (see (20)). From (25) one can see that this is possible as for positive, as for negative and zero value of the velocity dr/dT in coordinates r, T of the Minkowski space-time. Especially if $r =$ const and $\chi \geq \chi_1$, where $\chi_1 \approx 1.1997$ is the positive root of equation

$$\chi \tanh\chi = 1,$$

then the energy (20) is equal to zero for a particle at rest in coordinates r, T. The region $\chi \geq \chi_1$ where particles at rest in coordinates r, T have negative energy is located on Fig. 1 below the blue line in a pink sector.

4. Conclusion

Our analysis shows that in nonsynchronous reference frame in cosmology due to existence of nondiagonal term in metrical tensor one can find region out of the static limit where particles with negative and zero energies exist like it occurs in the ergosphere of the rotating black hole. However differently from the case of Kerr metric here due to nonstationary form of the metric in cosmology particle energies are not conserved. In special case of the scale factor $a = \sigma ct$ we analysed the trajectories of particles which have for some interval negative and zero energies.

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