Group theory and de Sitter QFT

The Concept of Mass

Master's Thesis

December 2013

Author:
Marco Boers

Supervisor:
Dr. Diederik Roest

Abstract

The concept of 'mass' has a quite obscure status in de Sitter space. For example, in Minkowski spacetime the absence of a mass term, conformal invariance and light cone propagation are all synonymous, while in de Sitter space this is not the case. In this thesis we investigate whether a group theoretical approach might bring resolvement to this issue. It will be similar to what is done in Minkowski spacetime; the unitary irreducible representation (UIR) spaces of the isometry group are associated with complete sets of one-particle states (elementary systems), and the Casimir operators of the group are associated with the invariants of the quantum-mechanical system (rest mass and spin in Minkowski spacetime). Since the notion 'mass' is not well defined in de Sitter space, we will consider it in reference to Minkowski spacetime. This reference is inferred by using group contractions. We review a particular (Garidi) mass definition, given in terms of the parameters labeling the UIRs of the de Sitter isometry group $SO(4,1)$, and for the scalar field we compare this mass with an alternative definition.

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University of Groningen
Faculty of Mathematics and Natural Sciences
Theoretical High-Energy Physics

rijksuniversiteit
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Introduction

Soon after Einstein famously added the cosmological constant term to his theory of general relativity, de Sitter described the vacuum solution with constant positive curvature [1], and the geometric space it gave rise to was readily named after him. The amount of symmetry of this space, the same as Minkowski spacetime, didn't go unnoticed, and in the years that followed the discovery of relativistic quantum mechanics, Dirac made the first effort to formulate a quantum mechanical theory in such a space [2]. However, issues associated with e.g. the absence of a lower bound on the energy operator drove most researchers in the field towards studying the just as symmetric constant negatively curved anti-de Sitter space, or to turn away from the subject completely.

Things changed in the 1980's, when Guth proposed his theory of inflation in an attempt to resolve the cosmological horizon and flatness problems [3]. During the inflationary epoch the universe would exponentially expand and thus be described by the de Sitter geometry. This led to a revival of interest in the subject of quantum field theory in such a space. The quite recent discovery that the universe is not only expanding, but actually undergoing an accelerated expansion [4, 5] again raised interest in the formulation of a consistent quantum field theory in de Sitter space. Another reason researchers got interested in studing field theory in this space is the fact that the curvature parameter might serve as a natural cutoff for infrared and other divergences in the process of regularization in flat spacetime.

A large amount of work has been carried out in trying to formulate a consistent field theory in de Sitter space, and many advances have been made. However, the non-trivial problems already arising in the quantization of fields satisfying the simple Klein-Gordon type equation, for example the non-uniqueness of the vacuum state [6], can be seen as a portent for the technical and interpretive difficulties that arise when working in this constantly curved space. One such interpretation issue concerns the notion of 'mass'.

At the present, it seems that the free fields have been succesfully quantized, for example by translating the Wightman approach to quantum field theory in Minkowski spacetime [7] (also known as *constructive* or *axiomatic* QFT) to de Sitter space (see [8] and references therein). Research is now being done on including interactions and finding out what their implications are, which turn out to be quite peculiar. For example, it is found that 'massive' particles are inherently unstable in first order perturbation theory [9, 10], which should arouse one's suspicion with regard to a Minkowskian interpretation of the concept of mass. Another interesting recent discovery with regard to this notion is the fact that de Sitter space allows the existence of local, covariant tachyonic fields that admit a de Sitter invariant physical space, in contrast to quantum field theory in Minkowski space [11, 12].

In e.g. [13] it is argued that de Sitter space itself is in fact unstable when interactions are included. The central point of the argument is that an inertially moving charged particle accelerates with respect to another inertial observer in that space, and thus emits radiation. This radiation happens at the cost of the decrease of the curvature, and one asymptotically finds him- or herself in a universe with non-accelerating expansion (see also [14, 15, 16]). Such a theory with a dynamical cosmological 'constant' might shed some light on the flatness problem. However, we will not discuss these theories in this work, but restrict ourselves to free fields in a constant positively curved background.

The (still open) question of the interpretation of the notion of mass in de Sitter space will be the main focus of this thesis. We will investigate how group theory might add to this discussion; in any curved spacetime it is always possible to consider the mass of a particle as its rest mass as it should locally hold in the tangent Minkowski spacetime, but in the de Sitter case we are dealing with a maximally symmetric space, which allows for a different approach. This 'different approach' will be a generalization of the method introduced by Wigner in [17], which focusses on the unitary irreducible representations (UIRs) of the isometry group of the spacetime in which the field theory is formulated. In Minkowski spacetime this is the Poincaré group; the group of translations and Lorentz transformations. The representation spaces of these UIRs form so called *elementary systems*, which are identified as the Hilbert spaces of the quantum mechanical one-particle states. Invariants of the Poincaré group can then be linked to invariants of the quantum mechanical systems. These group theoretical invariants are the Casimir operators, and they are linked to the physical notions 'rest mass' and 'spin'. Our main objective is to investigate if we can extrapolate this method to de Sitter space in order to get a better understanding of the notion of mass in this space, which brings us to our research question:

In how far does the group theoretical approach to quantum field theory, in terms of associating UIRs of the spacetime isometry group to quantum mechanical elementary systems, lead to a better understanding of the concept of 'mass' in de Sitter spacetime?

In order to answer this question we will first have to review the basics of groups and representation theory, after which we will look at the state of affairs in flat spacetime in detail. The general idea is that one should start with the classification of the unitary irreducible representations, and then link these representations to the field equations. As we will see, this linking-process is not quite as straightforward in de Sitter space compared to Minkowski spacetime.

We will follow the argumentation of Garidi [18] (see also [19]) leading to a particular mass definition. The main axiomatic point is that in de Sitter space, 'mass' has meaning *only* in reference to Minkowski space. This reference is established by considering how the different representations behave under group contraction as the curvature of the space tends to zero; de Sitter representations that have as limit the Poincaré representations associated to massive particles will be called massive.

The concept of 'masslessness' is somewhat more involved. In Minkowski space, $m = 0$, conformal invariance, light cone propagation, gauge invariance and the presence of two helicity states (for $s \neq 0$) are basically all synonymous. In curved space this is not the case. For example, as was shown in [20], fields associated to the wave equation with no mass term do in fact propagate inside the light cone. A particular choice for calling de Sitter fields 'massless' is considered, namely when their unitary irreducible representations can be extended to the conformal group, since those contract to the Poincaré representations associated to massless particles. However, for scalar fields this leads to some peculiar results, and we propose a formula different from Garidi's for the mass of the scalar field, which seems to be more in line with the mass parameter used in certain inflationary theories (see e.g. [21]).

The organization of this thesis is as follows: in chapter 1 we review the basic properties of groups and algebras that will be important throughout the rest of this work. Chapter 2 is devoted to the basics of representation theory. Next, chapter 3 deals with the representations of specific groups, where we conclude with the classification of the unitary irreducible representations of the Poincaré and de Sitter isometry group. Chapter 4 is devoted to group contractions, in order to establish how the de Sitter and Poincaré unitary irreducible representations are related. The final chapter's purpose is to link the group theoretical content described in the previous chapters to quantum field theory. We first cover the Minkowski case and then make an effort to generalize to de Sitter space. The link between the unitary irreducible representations and the field equations will be established using the ambient space formalism, after which we will review the arguments leading to Garidi's mass definition. We conclude with proposing a different mass definition for the scalar field, and a review of the application of Garidi's formula to higher-spin fields.

Chapter 1

Basics of Group Theory

We start by giving a brief overview of the basic notions from group theory that will be important throughout this thesis. The sections are based on the information given in [22] and [23], but any other standard work on basic (Lie) group theory could in principle have been used.

1.1 What is a group?

Since the aim of this thesis is to investigate the connection between quantum theory and group theory, it is important to establish precisely what we mean when we talk about these mathematical entities called groups. The definition is as follows:

Definition 1.1.1. *A set G with an operation $*$ forms a group if it satisfies the following axioms:*

1. *closure:* $\forall g_1, g_2 \in G, \quad g_1 * g_2 \in G,$
2. *associativity:* $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3),$
3. *identity:* $\exists e \in G$, *such that* $g * e = e * g = g, \quad \forall g \in G,$
4. *inverse:* $\forall g \in G, \exists \hat{g} \in G$ *such that* $g * \hat{g} = \hat{g} * g = e.$

The group operation $*$ is often called multiplication. From hereon we will omit the symbol $*$ for brevity, so $g_1 * g_2 \equiv g_1 g_2$. G is said to be *Abelian* if $\forall g_1, g_2 \in G$ we have $g_1 g_2 = g_2 g_1$, i.e. if the multiplication law is commutative. Most groups we will encounter do not have this property, for example, the quite basic group of rotations in three dimensions is clearly non-Abelian (see section 3.1).

Let us consider mappings between groups. A homomorphism is a mapping f of a group G_1 into G_2, such that $\forall g, h \in G_1$ we have $f(gh) = f(g)f(h)$. When the homomorphism is one-to-one it is called an isomorphism. If there exists an isomorphism between two groups, then they are called isomorphic, which basically means that they have the same properties and no group theoretical distinction need be made. The notation for indicating that two groups G_1 and G_2 are isomorphic is $G_1 \sim G_2$.

Groups often possess sets of elements which again form groups. These sets are known as *subgroups*. To be more mathematically precise: a subset H of the group G is said to be a subgroup of G if it satisfies the group axioms under the multiplication law $*$ of G. A subgroup is called *invariant* if for $\forall g \in G$ and $\forall h \in H$ we have $gh\hat{g} \in H$. Every group G has at least two invariant subgroups, namely the trivial one: $\{e\}$ and G itself. When a group does not contain any non-trivial invariant subgroup it is called *simple*. The term *semi-simple* is used for groups that do not contain any Abelian invariant subgroup.

Let us now introduce the *direct product* of two groups. This is most easily done if we consider two subgroups H_1 and H_2 of a group G, satisfying the following:

1. $h_1 h_2 = h_2 h_1 \quad \forall h_1 \in H_1, h_2 \in H_2$
2. every element of G can be written uniquely as $g = h_1 h_2$ where $h_1 \in H_1$ and $h_2 \in H_2$

Then G is said to be a *direct product group* and can be written as $G = H_1 \otimes H_2$. It is clear that H_1 and H_2 are in fact invariant subgroups of G.

Related to direct product grous are *factor groups*, but in order to define them, we must first define *cosets*. Let $H = \{h_1, h_2, ...\}$ be a subgroup of G and let p be an element of G which is not in H. The set of elements $pH = \{ph_1, ph_2, ...\}$ is called a left coset of H. $Hp = \{h_1 p, h_2 p, ...\}$ is called a right coset. Now, if H is an invariant subgroup of G, the factor (or quotient) group is defined by the set of cosets endowed with the law of multiplication $pH \cdot qH = (pq)H$ where $p, q \in G$ and $p, q \notin H$. It is denoted by G/H.

1.2 Lie groups

So far we have kept the concept of a group very general, however it is not hard to see that it is possible to identify different classes of groups, for example, one can discriminate between discrete and continuous groups. Consider three dimensional Euclidean space. The set of reflections in the three orthogonal planes is obviously a discrete group, while the set of rotations around the axes is a continuous group. Our focus will be on groups of the latter kind, or to be more specific, on Lie groups. We will not give the precice definition, since it involves elements of topology and differential geometry.

Loosly speaking, a Lie group is an infinite group for which the operations of multiplication and inversion are smooth. The elements can be labeled by a set of continuous parameters, and the number of linearly independent parameters used to label them is the dimension of the Lie group. These groups are used (among other things) for describing continuous symmetries of mathematical objects and structures, and that is what we will be doing in this thesis. More specific, we will use Lie groups for describing the continuous symmetries of the particular spacetime in which we are formulating our quantum field theory (e.g in section 3.5 we consider the symmetry group for Minkowski spacetime; the Poincaré group).

Any n-dimensional Lie group G can be parametrized in such a way that it can be described in terms of n subgroups, each of which is labeled by one parameter. Let us write this explicitly. Stating that G is n-dimensional is stating that the elements $g \in G$ can be labeled by n parameters:

$$g = g(\alpha_1, ..., \alpha_n) \tag{1.2.1}$$

We can choose the parametrization such that the sets of elements of the form

$$g_k(t) = g(0, ..., 0, t, 0, ..., 0) \tag{1.2.2}$$

are one-parameter subgroups of G, where t is in the k^{th} position and $1 \leq k \leq n$. The condition that $\{g_k(t);\ t \in \mathbb{R}\}$ forms a subgroup of G can be stated as follows:

$$g_k(t) g_k(s) = g_k(t + s) \tag{1.2.3}$$

We will now briefly discuss some topological properties of Lie groups, namely *connectedness* and *compactness*. Since Lie groups carry the structure of real- or complex-analytic manifolds, we can talk about such topological notions.

A Lie group is said to be connected if it is not the union of two disjoint nonempty open sets of elements. If a line connecting any two elements can be continuously transformed into every other possible line between those two elements (while staying within the group), then the group is said to be *simply connected*.

Roughly speaking, a Lie group is said to be (non-)compact if the set of parameters used to label the group elements is (non-)compact. A more general method of establishing whether a group is (non-)compact is discussed in section 1.3.4. Compactness of a group becomes very important when we will consider representations. For example, it can be shown that for non-compact groups, all unitary irreducible representations are infinite dimensional. The fact that the symmetry groups of the spacetimes we are going to study are indeed non-compact should stress this importance (see e.g. section 3.6).

1.2.1 Linear Lie groups

Making the distinction between discrete and continuous is certainly not the only discrimination possible for groups, and to state that we are focussing on Lie groups in this thesis is not quite specific enough. We will restrict ourselves to *linear* Lie groups, i.e. specific subgroups of the real (or complex) General Linear group $GL(n, \mathbb{R})$, the group of all non-singular linear transformations of some n-dimensional real (or complex) space, whose elements are non-singular matrices.

There are a number of subgroups of $GL(n, \mathbb{R})$ which have extremely important applications in modern physics. We will give a few examples which we will encounter several times throughout this work.

$SL(n, \mathbb{R})$

The set of matrices $g \in GL(n, \mathbb{R})$ with $\det g = 1$ forms an invariant subgroup which is denoted by $SL(n, \mathbb{R})$. A physically important example is the group $SL(2, \mathbb{C})$: it is associated with a particular manifestation of the group of Lorentz transformations and is used to construct the Dirac spinors.

$O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$

The group of all real orthogonal matrices acting on $\mathbb{R}^n$ is denoted by $O(n, \mathbb{R})$ (the entry $\mathbb{R}$ is usually dropped for brevity). We can easily see that $O(n)$ consists of two components. Consider $a \in O(n)$, meaning $aa^T = I_n$, which implies $(\det a)^2 = 1$, so there is one component with $\det a = 1$ and one with $\det a = -1$. The former coincides with the group $SO(n)$. The group $SO(3)$ has an important role in almost all of physics, since it is the group of rotations in three spatial dimensions. Another example of its application can be found in atomic physics; it is closely connected to the spherical harmonics used to describe the orbitals of electrons.

$U(n)$ and $SU(n)$

The group of all unitary matrices is denoted by $U(n)$. The subgroup $\{u \in U(n) | \det u = 1\} = U(n) \cap SL(n, \mathbb{C})$ is denoted by $SU(n)$. The (special) unitary groups are of huge importance in modern physics. For example, the gauge group of the standard model is given by $SU(3) \times SU(2) \times U(1)$.

Pseudo- groups

Consider in $\mathbb{R}^n$ the following form:

$$[\mathbf{x}, \mathbf{y}]_{p,q} = x_1 y_1 + ... + x_p y_p - x_{p+1} y_{p+1} - ... - x_{p+q} y_{p+q} \tag{1.2.4}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $p + q = n$. The group of linear transformations leaving this form invariant is denoted by $O(p, q)$. $SO(p, q)$ is defined by $O(p, q) \cap SL(p + q, \mathbb{R})$. We call these groups pseudo-orthogonal. The main part of this thesis is concerned with pseudo-orthogonal groups, for example $SO(3, 1)$: the group of Lorentz transformations, and $SO(4, 1)$: the group associated with de Sitter space.

Pseudo-unitary groups can be defined in the same way as above, but now considering in $\mathbb{C}^n$ the form

$$[\mathbf{z}, \mathbf{w}]_{p,q} = z_1 \overline{w_1} + ... + z_p \overline{w_p} - z_{p+1} \overline{w_{p+1}} - ... - z_{p+q} \overline{w_{p+q}} \tag{1.2.5}$$

where the bar implies Hermitian conjugation.

This concludes our introduction to the beautiful subject of Lie groups. However, for the most part we will not work with the groups, but rather with the so-called *Lie algebras*. The remainder of this chapter will be devoted to these structures and their relation to the groups.

1.3 Lie algebras

Lie algebras are probably the favorite group theoretical tool for physicists, since they are much easier to work with than the groups themselves. When we come to the classification of the unitary irreducible representations of specific groups in chapter 3, we will mainly be working with the corresponding Lie algebras. Section 1.3.1 is based on [24] and section 1.3.2 on [25], but just as for the first part of this chapter, we could have used any other standard work.

1.3.1 Definition of Lie algebras

Before we come to discussing the connection between the groups and algebras, it is convenient to give the precise mathematical definition of the Lie algebra structure.

Definition 1.3.1. *Let V be some finite-dimensional vector space over $\mathbb{C}$ (or $\mathbb{R}$). Let $X, Y \in V$. V is said to be a Lie algebra over $\mathbb{C}$ (or $\mathbb{R}$) if there is a composition rule $[X, Y]$ in V, satisfying $\forall X, Y, Z \in V$:*

1. *linearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ for $\alpha, \beta \in \mathbb{C}$ (or $\mathbb{R}$),*
2. *antisymmetry: $[X, Y] = -[Y, X]$,*
3. *Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.*

The operation $[\quad , \quad]$ is called Lie multiplication.

We say that a Lie algebra is Abelian or commutative if $\forall X, Y \in V$ we have $[X, Y] = 0$. Upon inspecting section 1.3.2 it will become apparent that this notion of 'being Abelian' is equivalent to the one for groups introduced earlier in this chapter.

Some more definitions are needed here. A subspace N of V is called a subalgebra if[1] $[N, N] \subset N$. N is an ideal if $[V, N] \subset N$. The ideal N for which we have $[V, N] = 0$ is called the center of V.

[1] A note on notation: let A and B be algebras, then by $[A, B]$ we mean all possible combinations $[a, b]$ where $a \in A$ and $b \in B$.

If an operator C commutes with all elements of V, i.e. $[C, V] = 0$, then it is called a *Casimir operator*. These special operators turn out to be of great importance for the physical applications of the algebras: they will be associated with invariants of the theory. For example, rest mass is invariant under spacetime transformations in Minkowski space, i.e. under transformations of the Poincaré group, and we shall see in section 3.5 how it is related (or equated) to the Casimir operator of this group.

When reading this last sentence, one might have noted that the Lie algebraic 'Casmir operator'-notion is used in reference to the group, not to the algebra. This might seem wrong, but it is extremely common. It turns out that in most cases there is no reason to make a fuss about this, since from a physicist's point of view there is no important difference between groups and algebras (in relation to this, see section 2.7). But let us first review how particular groups are linked to particular algebras.

1.3.2 Connection between Lie groups and algebras

In order to make clear the connection between Lie groups and algebras we will use the concept of matrix representations, which we will properly define in section 2.1. Let us consider an n-dimensional Lie group G whose elements g are labeled by a set of parameters $\alpha = \{\alpha_1, ..., \alpha_n\}$, i.e. $g = g(\alpha)$, such that $g(0) = e$. We assume that the group actions can be represented by $d \times d$ matrices denoted by D (this is always possible for linear groups), such that $D(g(\alpha)) = D(\alpha)$ and $D(0) = \mathbb{1}$. We can expand D around $\mathbb{1}$:

$$D(\delta\alpha) = \mathbb{1} + i\delta\alpha_a X^a + ... \tag{1.3.1}$$

where $\delta\alpha$ is an infinitesimal α and a runs over the number of parameters. X^a are called the *generators* of the group[2] and are defined as:

$$X^a \equiv -i\frac{\partial}{\partial\alpha_a}D(\alpha)\bigg|_{\alpha=0} \tag{1.3.2}$$

so the generators are in fact tangent vectors at the identity element of the group.
For a compact group one can write all group actions in terms of the generators and group parameters, even far away from the identity. In order to show this we write $\delta\alpha_a$ as α_a/k and we raise $D(\delta\alpha)$ to some large power k:

$$D(\alpha) = \lim_{k\to\infty}\left(\mathbb{1} + i\frac{\alpha_a X^a}{k}\right)^k = \exp(i\alpha_a X^a). \tag{1.3.3}$$

When we assume that there are no superfluous parameters α then the X^a are linearly independent and span a vector space. This vector space, together with a Lie multiplication defined above, is called the Lie algebra $\mathfrak{g}$ of G. The dimension n of $\mathfrak{g}$ is equal to the number of generators, i.e. the number of parameters used to label the elements of G. Since $\{X^1, ..., X^n\}$ is a basis for $\mathfrak{g}$ we can write every element of $\mathfrak{g}$ as a linear combination of the X^a. In particular, we can write every Lie product of two generators as a linear combination of generators:

$$[X^i, X^j] = c_k^{ij} X^k \tag{1.3.4}$$

[2]Some authors prefer to use the terms 'infinitesimal generators' or 'infinitesimal operators'. The mathematical definitions also differ throughout the literature; some choose to exclude the factor i, some choose to exclude the minus sign, others choose to exclude both. It should be clear that this does not change the theory, but one must take notice! For example, Hermiticity in one convention is translated into anti-Hermiticity in the other.

where $i, j, k = 1, .., n$. The c_k^{ij}'s are called *structure constants* and they carry all information about the algebra-, and thus group structure.

As mentioned before, we will be interested in groups and algebras associated with spacetime symmetries. There is a very elegant link between the generators of these groups and the structure of the particular spacetime, which is made explicit with the Killing vector formalism.

1.3.3 Isometries and Killing vectors

Up until now we have not given a definition of the term 'spacetime symmetry', and thus we intend to give one now. Consider $\mathbb{R}^n$ endowed with a particular metric, specifying the spacetime, e.g. $\text{diag}(1, -1, -1, -1)$ for 4-dimensional Minkowski spacetime. Now consider a distance-preserving bijective map from this metric space to itself; it is called an *isometry*[3] (in terms of 4-dimensional Minkowski spacetime, this is a map which preserves the length of any 4-vector). Now, the group of spacetime symmetries is defined as the group of all isometries of the spacetime in question.

To find an explicit form for the generators of these isometry groups we can use the Killing vector formalism, where 'Killing vector' is basically synonymous to 'generator of the isometry group'. The following rigorous definition is given in [26]:

Definition 1.3.2. *Let ϕ_t be a C^∞ map, forming a one-parameter group of diffeomorphisms from $\mathbb{R} \times M \to M$ such that for some fixed $t \in \mathbb{R}$ we have $\phi_t : M \to M$, and for all $t, s \in \mathbb{R}$ we have $\phi_t \circ \phi_s = \phi_{t+s}$. Note that this last statement implies that $\phi_{t=0}$ is the identity map. We now associate a vector field v to ϕ_t in the following way: for any fixed $x \in M$, $\phi_t(x) : \mathbb{R} \to M$ is a curve passing through x at $t = 0$. This curve is called an* orbit *of ϕ_t. We define $v|_x$ as the tangent to this curve at $t = 0$. So we see that associated to every one-parameter (sub)group of isometry transformations of M is a vector field v, and the so-called* Killing vector *$v|_x$ can be viewed as the infinitesimal generator of these transformations.*

One can find the explicit coordinate form of the Killing vectors, and thus the generators of the isometry group, by using the following formula:

$$K_{\alpha\beta} = i \left(x_\alpha \frac{\partial}{\partial x^\beta} - x_\beta \frac{\partial}{\partial x^\alpha} \right), \tag{1.3.5}$$

which can be quite helpful in explicit calculations. Let us now turn our attention to another helpful tool, which again carries the name of Wilhelm Killing. This tool can be used to establish the (non-)compactness of Lie groups.

1.3.4 Killing form

Let us define, on the Lie algebra $\mathfrak{g}$, the linear map $\text{ad}X : \mathfrak{g} \to \mathfrak{g}$ by:

$$\text{ad}X(Y) \equiv [X, Y], \tag{1.3.6}$$

and next define a bilinear form on $\mathfrak{g}$ by:

$$B(X, Y) \equiv \text{Tr}(\text{ad}X \text{ad}Y) \tag{1.3.7}$$

which has the following properties:

[3] Actually, it is a *global* isometry, where the prefix points to the bijective character of the mapping.

1. symmetry: $B(X, Y) = B(Y, X)$,
2. bilinearity: $B(\alpha X + \beta Y, Z) = \alpha B(X, Z) + \beta B(Y, Z)$ for all $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$,
3. $B(\mathrm{ad}X(Y), Z) + B(Y, \mathrm{ad}X(Z)) = 0$.

The bilinear form defined in (1.3.7) is called the *Killing form*. It is closely related to the structure constants of (1.3.4) (for details, see [23]):

$$B(X_i, X_j) = \mathrm{Tr}(\mathrm{ad}X_i \mathrm{ad}X_j) = c_{im}^k c_{jk}^m \equiv b_{ij}. \tag{1.3.8}$$

The Killing form is a great tool for establishing if a group is (non-)compact. A Lie algebra (and its corresponding group) is said to be *semisimple* if $\forall Y \in \mathfrak{g}$, $B(X, Y) = 0$ implies $X = 0$. If a group is semisimple we can use the form to find out if the group in question is compact. We state, without proof, *Cartan's compactness criterion*: if the Killing form of a semisimple Lie algebra $\mathfrak{g}$ is strictly negative and G is the associated connected Lie group with finite center, then G is compact.

Again, we stress the importance of compactness of groups. As one will read in the following chapter, if a group is compact, then one automatically knows that all irreducible representations are finite dimensional and unitary, while non-compact groups have no finite dimensional unitary irreducible representations at all!

With this we conclude this introductory chapter containing the required basics of group theory. Again, it is based on other introductions to this subject found in [22, 23, 24, 25]. The next chapter will cover the fundamentals of representation theory, which is the basis for the physical applications of the mathematical structures described above.

Chapter 2

Basics of Representation Theory

One can argue that representations of groups (and algebras) are *the* mathematical objects that make group theory relevant for, and applicable to physics. They are concrete manifestations of the more abstract notion of 'a group'. In this chapter we review the basic notions of representation theory that will be used throughout the rest of this thesis.

2.1 Definition of a representation

A *representation* T of a group G in a linear space $\mathfrak{L}$ over a field $\kappa = \mathbb{R}, \mathbb{C}, ...$ (the space of the representation) is the homomorphism $T : G \to GL(\mathfrak{L}, \kappa)$. $GL(\mathfrak{L}, \kappa)$ is the group of non-singular linear transformations of $\mathfrak{L}$. It satisfies

1. $T(g_1 g_2) = T(g_1)T(g_2)$,
2. $T(e) = E$,

where E is the identity operator in $\mathfrak{L}$. The dimension of a representation is equal to the dimension of $\mathfrak{L}$. We shall mainly deal with matrix representations, in that case $\mathfrak{L} = \mathbb{R}^n$ and the dimension of T is n. We say that a representation is *faithful* if the homomorphism is also an isomorphism.
Every group has a trivial representations, namely

$$T(g) = E, \quad \forall g \in G. \tag{2.1.1}$$

Next we take a closer look at matrix representations. Consider a n-dimensional vector space with a basis $\{\hat{\mathbf{e}}_i, i = 1, ..., n\}$. The operators $T(g)$ will be $n \times n$ matrices working on the basis vectors as follows:

$$T(g) |e_i\rangle = |e_j\rangle D(g)^j{}_i. \tag{2.1.2}$$

We can show that the matrices $D(g)$ obey the same rules of multiplication as the operators $T(g)$:

$$\begin{aligned} T(g_1)T(g_2) |e_i\rangle &= T(g_1) |e_j\rangle D(g_2)^j{}_i = |e_k\rangle D(g_1)^k{}_j D(g_2)^j{}_i \\ &= T(g_1 g_2) |e_i\rangle = |e_k\rangle D(g_1 g_2)^k{}_i \end{aligned} \tag{2.1.3}$$

so we see that

$$D(g_1)D(g_2) = D(g_1 g_2) \tag{2.1.4}$$

where we implicitly use matrix multiplication.

Let $T(G)$ be a representation of a group G on a vector space $\mathfrak{L}$. Let S be some non-singular operator on $\mathfrak{L}$, then

$$T'(G) = ST(g)S^{-1} \tag{2.1.5}$$

also is a representation of G on $\mathfrak{L}$. $T(G)$ and $T'(G)$ are said to be connected by a *similarity transformation* S. Representations that can be connected by a similarity transformation are called *equivalent*.

2.2 Irreducible representations

We will now introduce the concept of *(ir)reducibility*. In order to do so, we first define an *invariant subspace*. Let $T(G)$ be a representation of G on $\mathfrak{L}$, and let $\mathfrak{L}_1 \subset \mathfrak{L}$ such that $T(g)\,|x\rangle \in \mathfrak{L}_1$ for all $\mathbf{x} \in \mathfrak{L}_1$ and $g \in G$. Then $\mathfrak{L}_1$ is called an invariant subspace of $\mathfrak{L}$ with respect to $T(G)$. It is called *trivial* when it consists of the whole space or if it only contains the null vector and it is said to be *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $T(G)$.
When the space $\mathfrak{L}$ has no invariant subspaces with respect to the representation $T(G)$, the latter is said to be irreducible. If there is at least one invariant subspace $\mathfrak{L}_1$, the representation is said to be reducible. When the orthogonal complement[1] of $\mathfrak{L}_1$, say $\mathfrak{L}_2$, is also invariant with respect to $T(G)$, then $T(G)$ is called *decomposable* or *fully reducible*.

We will see that irreducible representations (irreps from now on) are physically of most interest. Another (physically important) property that a representation can have is *unitarity*. We say that a representation is unitary if the representation space $\mathfrak{L}$ is a *Hilbert space* (or *inner product space*), and the operators $T(g)$ are unitary[2] for all $g \in G$.

2.3 Schur's lemma

For later use we will state without proof Schur's lemma.

Let $T(G)$ be an irrep of the group G on the space $\mathfrak{L}$ and A some operator on $\mathfrak{L}$. If $AT(g) = T(g)A$ for all $g \in G$, then $A = \lambda E$ where $\lambda \in \mathbb{R}$ and E the identity operator on $\mathfrak{L}$.

Another later important statement is the following: for compact groups, all irreps are finite dimensional and unitary.

2.4 Direct product representations

Next we will focus on *direct product representations*. Analogous to defining irreps, we first define the space on which the representation will work: the *direct product space*. Let $\mathfrak{U}$ and $\mathfrak{V}$ be Hilbert spaces with orthonormal bases $\{\hat{\mathbf{u}}_i; i = 1, ..., n_u\}$ and $\{\hat{\mathbf{v}}_j; j = 1, ..., n_v\}$ respectively. Then the direct product space $\mathfrak{W} = \mathfrak{U} \times \mathfrak{V}$ is made out of all linear combinations of the orthonormal basis vectors $\{\hat{\mathbf{w}}_k; k = (i, j); i = 1, ..., n_u; j = 1, ..., n_v\}$ where $\hat{\mathbf{w}}_k$ can be regarded

[1]The orthogonal complement of a space $\mathfrak{L}_1 \subset \mathfrak{L}$ is defined as $\mathfrak{L}_2 = \{\mathbf{x} \mid \langle \mathbf{x}|\mathbf{y}\rangle = 0\ \forall \mathbf{y} \in \mathfrak{L}_1\}$. If $\mathfrak{L}$ is finite-dimensional, $\mathfrak{L}_2$ is a subspace of $\mathfrak{L}$, and we write $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$.

[2]The operators $T(g)$ are said to be unitary if $T(g)T(g)^\dagger = T(g)^\dagger T(g) = E$. The most important property of unitary operators is that they leave the inner product invariant, and as a consequence they leave lengths of vectors and angles between vectors invariant.

as the formal product $\hat{\mathbf{w}}_k = \hat{\mathbf{u}}_i \cdot \hat{\mathbf{v}}_j$. The dimension of $\mathfrak{W}$ is obviously $n_u \times n_v$. By definition we have

1. $\langle w^{k'} | w_k \rangle = \delta^{k'}_{\ k} = \delta^{i'}_{\ i} \delta^{j'}_{\ j}$,
2. $\mathfrak{W} = \{\mathbf{x}; |x\rangle = |w_k\rangle\, x^k\}$ with $x^k \in \mathbb{C}$ the components of $\mathbf{x}$,
3. $\langle x|y \rangle \equiv x^{\dagger}_k y^k$.

Let us now consider the operators A on $\mathfrak{U}$ and B on $\mathfrak{V}$, whose product $D = A \times B$ is defined on $\mathfrak{W} = \mathfrak{U} \times \mathfrak{V}$ by the action on the basis vectors $\{\hat{\mathbf{w}}_k\}$:

$$D\,|w_k\rangle = |w_{k'}\rangle\, D^{k'}_{\ k}, \qquad D^{k'}_{\ k} \equiv A^{i'}_{\ i} B^{j'}_{\ j} \tag{2.4.1}$$

where $A^{i'}_{\ i}$ is the matrix element of A on the subspace $\mathfrak{U}$ with respect to the basis $\{\hat{\mathbf{u}}_i\}$ and $B^{j'}_{\ j}$ is the matrix element of B on the subspace $\mathfrak{V}$ with respect to the basis $\{\hat{\mathbf{v}}_j\}$, and $k = (i, j)$, $k' = (i', j')$.
We now have the tools to define direct product representations. Let $D^{\mu}(G)$ be a representation[3] of G on $\mathfrak{U}$ and $D^{\nu}(G)$ a representation of G on $\mathfrak{V}$. Then the operators $D^{\mu\times\nu}(g) = D^{\mu}(g) \otimes D^{\nu}(g)$ form a representation of G on $\mathfrak{W}$. $D^{\mu\times\nu}(G)$ is called a direct product representation.

2.5 Decomposition of direct product representations

$D^{\mu\times\nu}(G)$ is in general reducible, and it can be decomposed as a direct sum of irreps on $\mathfrak{W}$:

$$D^{\mu} \otimes D^{\nu} = S\left(\bigoplus_{\lambda} a^{\mu\nu}_{\lambda} D^{\lambda}\right) S^{-1} \tag{2.5.1}$$

where D^{μ}, D^{ν} and D^{λ} are irreps of G on respectively $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W} = \mathfrak{U} \times \mathfrak{V}$. S is a non-singular operator providing the similarity transformation.
This means that the space $\mathfrak{W}$ consists of invariant subspaces $\mathfrak{W}^{\lambda}_{\alpha}$, where λ is the label of the irrep and $\alpha = 1, ..., a^{\mu\nu}_{\lambda}$ labels the different spaces correpsonding to the same λ, i.e.

$$\mathfrak{W} = \bigoplus_{\lambda,\alpha} \mathfrak{W}^{\lambda}_{\alpha}. \tag{2.5.2}$$

We can choose a new basis $\{\hat{\mathbf{w}}^{\lambda}_{\alpha l}; l = 1, ..., n_{\lambda}\}$ for $\mathfrak{W}$ such that the first n_1 basis vectors span $\mathfrak{W}^1_1$, the next n_1 basis vectors span $\mathfrak{W}^1_2$, and so on, until we have a complete orthonormal basis. This basis is linked to the old basis $\{\hat{\mathbf{w}}_{(i,j)}\}$ by a unitary transformation. We write

$$|w^{\lambda}_{\alpha l}\rangle = \sum_{i,j} |w_{(i,j)}\rangle \,\langle i, j(\mu,\nu)\alpha, \lambda, l\rangle \tag{2.5.3}$$

where $\langle i, j(\mu,\nu)\alpha, \lambda, l\rangle$ are the matrix elements of the transformation matrix, with (i, j) labeling the rows and (α, λ, l) the columns. They are called *Clebsch-Gordan coefficients*.
The inverse relation is given by

$$|w_{(i,j)}\rangle = \sum_{\alpha,\lambda,l} |w^{\lambda}_{\alpha l}\rangle \,\langle \alpha, \lambda, l(\mu,\nu) i, j\rangle\,. \tag{2.5.4}$$

[3]Note that these greek indices label the irreps; they do not represent matrix- or vector entries.

We now apply an operator from the representation to the two different bases, according to (2.4.1) we have:

$$D^{\mu\times\nu}(g)\,|w_{(i,j)}\rangle = |w_{(i',j')}\rangle\, D^{\mu}(g)^{i'}{}_{i} D^{\nu}(g)^{j'}{}_{j} \tag{2.5.5}$$

and

$$D^{\mu\times\nu}(g)\,|w^{\lambda}_{\alpha l}\rangle = |w^{\lambda}_{\alpha l'}\rangle\, D^{\lambda}(g)^{l'}{}_{l} \tag{2.5.6}$$

where summation over repeated indices is understood.
For brevity, in the following we will denote the different basis vectors as $|i,j\rangle$ and $|\alpha,\lambda,l\rangle$, and we will replace the (μ,ν) label in the Clebsch-Gordan coefficients by $|$. We now combine (2.5.4), (2.5.5) and (2.5.6) to write

$$\begin{aligned} D^{\mu\times\nu}(g)\,|i,j\rangle &= D^{\mu\times\nu}(g)\,|\alpha,\lambda,l\rangle\,\langle\alpha,\lambda,l|i,j\rangle \\ &= |\alpha,\lambda,l'\rangle\, D^{\lambda}(g)^{l'}{}_{l}\,\langle\alpha,\lambda,l|i,j\rangle \\ &= |i',j'\rangle\,\langle i',j'|\alpha,\lambda,l'\rangle\, D^{\lambda}(g)^{l'}{}_{l}\,\langle\alpha,\lambda,l|i,j\rangle\,. \end{aligned} \tag{2.5.7}$$

We now have all the tools to explicitly write the similarity transformation of (2.5.1) composed of Clebsch-Gordan coefficients. The following matrix relations hold:

$$D^{\mu}(g)^{i'}_{1} D^{\nu}(g)^{j'}{}_{j} = \langle i',j'|\alpha,\lambda,l'\rangle\, D^{\lambda}(g)^{l'}{}_{l}\,\langle\alpha,\lambda,l|i,j\rangle \tag{2.5.8a}$$

$$\delta^{\alpha'}_{\alpha}\delta^{\lambda}_{\lambda'} D^{\lambda}(g)^{l'}{}_{l} = \langle\alpha',\lambda',l'|i',j'\rangle\, D^{\mu}(g)^{i'}{}_{i} D^{\nu}(g)^{j'}{}_{j}\,\langle i,j|\alpha,\lambda,l\rangle \tag{2.5.8b}$$

The first of the above equations is basically (2.5.1). The second equation is the reciprocal of the first, and it makes explicit the block-diagonal form of the direct product representation in the new basis.

2.6 Irreducible operators and the Wigner-Eckart theorem

We begin by defining *irreducible vectors* on a vector space $\mathfrak{L}$. Let $\mathfrak{L}_\mu$ be an invariant subspace of $\mathfrak{L}$ with respect to some representation $T(G)$ of the group G. Any set of vectors $\{\hat{\mathbf{e}}^{\mu}_{i}; i = 1, ..., n_\mu\}$ transforming under $T(G)$ as

$$T(g)\,|e^{\mu}_{i}\rangle = |e^{\mu}_{j}\rangle\, D^{\mu}(g)^{j}{}_{i} \tag{2.6.1}$$

where $D^{\mu}(G)$ is an irreducible matrix representation of G, is said to be an irreducible set transforming under the μ-representation.
Next we will define *irreducible operators*. They, together with the theorem following the definition, will be of great importance later.
Consider a set of operators $\{O^{\mu}_{i}; i = 1, ..., n_\mu\}$ on a vector space $\mathfrak{L}$. They are said to be irreducible operators if they transform under actions of the group G as follows:

$$T(g)O^{\mu}_{i}T(g)^{-1} = O^{\mu}_{j} D^{\mu}(g)^{j}{}_{i} \tag{2.6.2}$$

where $T(G)$ is some unitary representation of G on $\mathfrak{L}$ and $D^{\mu}(G)$ an irreducible matrix representation.
Now we are interested in how the combination of irreducible operators and vectors will transform under the action of G. It is easy to show, using (2.6.1) and (2.6.2), that we have:

$$\begin{aligned} T(g)O^{\mu}_{i}\,|e^{\nu}_{j}\rangle &= T(g)O^{\mu}_{i}T(g)^{-1}T(g)\,|e^{\nu}_{j}\rangle \\ &= O^{\mu}_{k}\,|e^{\nu}_{l}\rangle\, D^{\mu}(g)^{k}{}_{i} D^{\nu}(g)^{l}{}_{j} \end{aligned} \tag{2.6.3}$$

so we see that the combination transforms according to the direct product representation $D^{\mu\times\nu}$. This means that, by using (2.5.4), we can express it as:

$$O_i^\mu \left|e_j^\nu\right\rangle = \sum_{\alpha,\lambda,l} \left|w_{\alpha l}^\lambda\right\rangle \left\langle \alpha, \lambda, l(\mu,\nu) i, j\right\rangle . \tag{2.6.4}$$

We can now compute the matrix element $\langle e_\lambda^l | O_i^\mu | e_j^\nu \rangle$ and state the *Wigner-Eckhart theorem*: let $\{O_i^\mu\}$ be a set of irreducible operators as defined in (2.6.2). We then have:

$$\langle e_\lambda^l | O_i^\mu | e_j^\nu \rangle = \sum_\alpha \left\langle \alpha, \lambda, l(\mu,\nu) i, j\right\rangle \left\langle \lambda | O^\mu | \nu \right\rangle_\alpha \tag{2.6.5}$$

where

$$\left\langle \lambda | O^\mu | \nu \right\rangle_\alpha \equiv \frac{1}{n_\lambda} \sum_k \langle e_\lambda^k | w_{\alpha k}^\lambda \rangle \tag{2.6.6}$$

is called the *reduced matrix element*. All the i, j and l dependence is now in the Clebsch-Gordan coefficients. The result of (2.6.5) will be of great importance later, in section 3.4.

2.7 Representations of Lie algebras

So far we only discussed representations of groups. We will now focus on representations of Lie algebras. They tend to be much easier to classify.
Let T_G be a representation[4] of the n-dimensional group G. The corresponding representation of the algebra $\mathfrak{g}$ can be obtained by taking the differential of T_G (along the same line as (1.3.2)). Let $X \in \mathfrak{g}$ correspond to some one-parameter subgroup $g(t)$, then the representation $T_\mathfrak{g}$ of $\mathfrak{g}$ is given by:

$$T_\mathfrak{g}(X) = \left.\frac{dT_G(g(t))}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{T_G(g(t)) - E}{t}. \tag{2.7.1}$$

The $T_\mathfrak{g}$ operators are called *infinitesimal operators*. In total we can define n of such operators, one for every one-parameter subgroup, who together define a representation for $\mathfrak{g}$. For all $X^i \in \mathfrak{g}$, where $i = 1, ..., n$ we have:

$$T_G\left(\exp \alpha_i X^i\right) = \exp\left(\alpha_i T_\mathfrak{g}(X^i)\right). \tag{2.7.2}$$

There are two more important statements that we shall not prove here. Firstly, if two representations of G are equivalent, then the corresponding representations of $\mathfrak{g}$ are equivalent as well. Secondly, if $T_\mathfrak{g}$ is irreducible, then so is T_G. The converse of both statements holds only when G is (simply) connected.

[4]In the following we implicitly assume that T_G is finite dimensional. It becomes more complex if we consider infinite dimensional representations, because then the limit in (2.7.1) can not exist on all vectors of the representation space on which $T_\mathfrak{g}$ acts. However, it can be done, and the results will be the same.

Chapter 3

Representations of Specific Groups

In this chapter we consider representations (and their representation spaces) of specific groups, namely: $SO(3)$, $SU(2)$, the Lorentz group $SO(3,1)$, the Poincaré group, and finally the group of isometries of de Sitter space $SO(4,1)$. For the last three of these groups we give the classification of the unitary irreducible representations (UIRs). The classification for the Poincaré group is of great interest for physicists, since the representation spaces of the UIRs can be linked to the Hilbert spaces of the quantum mechanical particle states. We explicitly carry out the classification of the (physically less interesting) UIRs of the Lorentz group, since it is analogous to the $SO(4,1)$ case, but less tedious. All sections are based on [22], except for section 3.6.

3.1 $SO(3)$

We will firstly take a look at the group of rotations in 3-dimensions. It is the group of linear transformations in 3-dimensional Euclidean space which leave the length of any vector invariant. Let $\mathbf{x}$ be such a vector. With use of an orthonormal basis $\{\hat{\mathbf{e}}_i; i = 1, 2, 3\}$ we can write the vector as $\mathbf{x} = \hat{\mathbf{e}}_i x^i$. A rotation will be represented by the matrix R. When applied to $\mathbf{x}$, the components change as

$$x'^i = R^i{}_j x^j \tag{3.1.1}$$

We require that the length of the vector is invariant; $x_i x^i = x'_i x'^i$. This leads to the constraint on R:

$$RR^T = R^T R = \mathbb{1} \tag{3.1.2}$$

so we can conclude that $R \in O(3)$; the 3-dimensional version of the orthogonal group first introduced in section 1.2.1. We already mentioned that $O(N)$ consists of two components, distinguished by the sign of the determinant of its elements. Since all physical rotations can be reached continuously from the 'identity rotation' ($R = \mathbb{1}$, which has $\det R = \det \mathbb{1} = 1$) we conclude that the group of rotations in 3-dimensions is in fact $SO(3)$.

The continuous parameters which label the group elements R of $SO(3)$ can be chosen in an infinite number of different ways. We will use the angles of rotation around the three orthogonal axes. We can readily distinguish the three one-parameter subgroups $R_n(\alpha)$ where $n = 1, 2, 3$. They are rotations in the plane orthogonal to the axis of rotation, i.e. $SO(2)$ subgroups. A matrix representation is given by

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \tag{3.1.3a}$$

$$R_2(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \tag{3.1.3b}$$

$$R_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1.3c}$$

With every one-parameter subgroup we can associate a generator of the corresponding algebra. We write

$$R_n(\alpha) = \exp(-i\alpha J_n) \tag{3.1.4}$$

where J_n are the three generators. A representation for the generators can be found by using (1.3.2) or (2.7.1). We find:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1.5}$$

One can prove that the generators satisfy the following Lie algebra:

$$[J_k, J_l] = i\epsilon_{klm} J^m \tag{3.1.6}$$

where ϵ_{klm} is the *Levi-Civita symbol*; it is $+1$ if (k, l, m) is an even permutation of $(1, 2, 3)$, -1 when the permutation is odd, and it is 0 if indices are repeated.

Next we will build irreps for the algebra $\mathfrak{so}(3)$ defined by (3.1.6). In section 2.2 we stated that the representation space $\mathfrak{L}$ of an irrep T is a proper invariant space under the group action. The strategy we will use is to construct this proper invariant space by starting out with a convenient vector and generate the rest of the vectors in an irreducible basis by repeatedly applying certain selected operators.

The basis vectors of $\mathfrak{L}$ will be chosen such that they are eigenvectors of a commuting set of operators, since only commuting operators can have complete sets of eigenvectors. We already encountered operators that commute with all elements of a Lie algebra; Casimir operators. It is straightforward to check that the operator $J^2 = (J_1)^2 + (J_2)^2 + (J_3)^2$ commutes with all J_k's. In short:

$$[J_k, J^2] = 0, \quad k = 1, 2, 3 \tag{3.1.7}$$

Due to Schur's lemma (see section 2.3), when the J_k's form an irrep, J^2 will be a multiple of the unit matrix. This means that all vectors of the irrep space are eigenvectors of J^2, all with the same eigenvalue.

Conventionally, the basis vectors of $\mathfrak{L}$ are chosen to be simultaneous eigenvectors of J^2 and J_3. We will use the so called *raising* and *lowering operators* to construct the basis vectors from the starting vector. They are given by

$$J_+ = J_1 + iJ_2 \tag{3.1.8a}$$

$$J_- = J_1 - iJ_2 \tag{3.1.8b}$$

which can be shown to have the following properties:

$$[J_3, J_\pm] = \pm J_\pm \tag{3.1.9a}$$

$$[J_+, J_-] = 2J_3 \tag{3.1.9b}$$

$$J^2 = {J_3}^2 - J_3 + J_+J_- = {J_3}^2 + J_3 + J_-J_+ \tag{3.1.9c}$$

$${J_\pm}^\dagger = J_\mp \tag{3.1.9d}$$

We will denote our starting vector for the representation space $\mathfrak{L}$ by $|m\rangle$. It is an eigenvector of J_3 (and J^2) with eigenvalue m:

$$J_3 \, |m\rangle = |m\rangle \, m \tag{3.1.10}$$

By using (3.1.9a) we can show that:

$$J_3 J_+ \, |m\rangle = [J_3, J_+] \, |m\rangle + J_+ J_3 \, |m\rangle = J_+ \, |m\rangle \, (m+1) \tag{3.1.11}$$

This means that $J_+ \, |m\rangle$ is again an eigenvector of J_3, now with eigenvalue $(m + 1)$. We will denote $J_+ \, |m\rangle$ by $|m + 1\rangle$ (after normalizing it to unity). By repeatedly applying J_+ to the vectors we can generate new eigenvectors of J_3. Due to the fact that the representation space is finite dimensional (the group is compact) the process must terminate at some vector. Let's call this vector $|j\rangle$:

$$J_3 \, |j\rangle = j \tag{3.1.12a}$$

$$J_+ \, |j\rangle = 0 \tag{3.1.12b}$$

so, by using (3.1.9c) we know that

$$J^2 \, |j\rangle = |j\rangle \, j(j+1) \tag{3.1.12c}$$

Next we reverse the process: we start out with $|j\rangle$ and apply J_- to it, again using (3.1.9a):

$$J_3 J_- \, |j\rangle = J_- \, |j\rangle \, (j-1) \tag{3.1.13}$$

Again, after normalizing, the vector $J_- \, |j\rangle$ is written as $|j - 1\rangle$. All vectors produced in this manner will have the same eigenvalue for J^2, namely $j(j + 1)$. Again, because the irrep space will be finite dimensional, the process must terminate at some vector $|l\rangle$:

$$J_- \, |l\rangle = 0 \tag{3.1.14}$$

Let us write:

$$\begin{aligned} 0 &= \langle l|{J_-}^\dagger J_-|l\rangle = \langle l|J_+J_-|l\rangle = \langle l|J^2 - {J_3}^2 + J_3|l\rangle \\ &= j(j+1) - l(l-1) \end{aligned} \tag{3.1.15}$$

which means that we must have $l = -j$. We applied J_- an integer number of times to get from $|j\rangle$ to $|-j\rangle$, so $2j$ must be an integer. This leads to the conclusion that j can take on the following values:

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ... \tag{3.1.16}$$

By construction, the dimension of $\mathfrak{L}$ is $2j + 1$.

Let us summarize the results derived above. The orthonormal basis vectors of the irreducible representation space $\mathfrak{L}$ for the irreps J of the Lie algebra $\mathfrak{so}(3)$ are specified by the following equations:

$$J^2 \, |jm\rangle = |jm\rangle \, j(j+1) \tag{3.1.17a}$$

$$J_3 \, |jm\rangle = |jm\rangle \, m \tag{3.1.17b}$$

$$J_\pm \, |jm\rangle = |jm \pm 1\rangle \, [j(j+1) - m(m\pm 1)]^{1/2} \tag{3.1.17c}$$

where the normalization factor in the last equation can be derived along the same line as (3.1.15). This basis is often refered to as the *canonical basis*.

Next we extend the result from the algebra $\mathfrak{so}(3)$ to the group $SO(3)$. We first note that, with use of the one-parameter subgroups, every group element can be written as

$$\begin{aligned} R(\alpha,\beta,\gamma) &= R_3(\alpha)R_2(\beta)R_3(\gamma) \\ &= e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3} \end{aligned} \tag{3.1.18}$$

Now consider an operator T representing the rotation $R(\alpha,\beta,\gamma)$, acting on $\mathfrak{L}$:

$$T(\alpha,\beta,\gamma) \, |jm\rangle = |jm'\rangle \, D^j(\alpha,\beta,\gamma)^{m'}_{\ \ m} \tag{3.1.19}$$

where

$$D^j(\alpha,\beta,\gamma)^{m'}_{\ \ m} = e^{-i\alpha m'} d^j(\beta)^{m'}_{\ \ m} e^{-i\gamma m} \tag{3.1.20}$$

and

$$d^j(\beta)^{m'}_{\ \ m} = \langle jm'| e^{-i\beta J_2} |jm\rangle \tag{3.1.21}$$

Note that J_2 will change the value of m; it can be written in terms of J_+ and J_-. It can be shown that for half integer values of j, rotations of an odd number of complete revolutions are not mapped to E, but to $-E$ (for example a rotation of 2π around a certain axis). When the number of complete revolutions is even, they are indeed mapped to E. For integer values of j this behavior is absent; all number of complete revolutions are mapped to E. We can make this explicit by plugging in a 2π rotation around the 3-axis into (3.1.20):

$$D^j[R_3(2\pi)]^{m'}_{\ \ m} = D^j[e^{-i2\pi J_3}]^{m'}_{\ \ m} = \delta_{m'm} e^{-i2m\pi} = \delta_{m'm} e^{-i(2j\pi)} = (-1)^{2j} \delta_{m'm} \tag{3.1.22}$$

where in the second to last equality we used the fact that $(j-m)$ is an integer. This result can be generalized to 2π rotations in any direction. We say that irreps with half integer j are *double-valued*.

3.2 Direct product representations of $SO(3)$

In this section we will discuss the direct product representations of $SO(3)$ and their decomposition into irreps. This will be of importance when we come to the Lorentz group.

We start with two irreps D^j and $D^{j'}$ of $SO(3)$. They act on the spaces $\mathfrak{L}$ and $\mathfrak{L}'$ respectively. The direct product representation $D^{j\times j'}$ acts on the direct product space $\mathfrak{L} \times \mathfrak{L}'$ which has dimension $(2j+1)(2j'+1)$. As a basis we take:

$$|m, m'\rangle = |jm\rangle \times |j'm'\rangle \tag{3.2.1}$$

on which the group action is by definition

$$T(R) \, |m, m'\rangle = |n, n'\rangle \, D^j(R)^n_{\ \ m} D^{j'}(R)^{n'}_{\ \ m'} \tag{3.2.2}$$

This representation T is in general reducible. We now turn our attention to the generators; we want to know how the representations of the generators on $\mathfrak{L} \times \mathfrak{L}'$ are linked to the ones on $\mathfrak{L}$ and $\mathfrak{L}'$. We differentiate, just as in (1.3.1), both the left- and right-hand side of

$$D^j[R_n(d\alpha)] D^{j'}[R_n(d\alpha)] = D^{j\times j'}[R_n(d\alpha)] \tag{3.2.3}$$

where R_n is defined by (3.1.4). We find, up to first order in $d\alpha$:

$$E^j \times E^{j'} - id\alpha[J_n{}^j \times E^{j'} + E^j \times J_n{}^{j'}] = E^{j\times j'} - id\alpha J_n{}^{j\times j'} \tag{3.2.4}$$

leading to

$$J_n{}^{j\times j'} = J_n{}^j \times E^{j'} + E^j \times J_n{}^{j'} \tag{3.2.5}$$

(or in short $J_n{}^j + J_n{}^{j'}$) and we conclude that the generators of the direct product representation are given by the sum of the corresponding generators of the two irreps we started out with. In the following we will omit the superscript representation labels, since we will only be concerned with the $J_n{}^{j\times j'}$ generators.

It is obvious that $|m, m'\rangle$ is an eigenvector of J_3:

$$J_3 \, |m, m'\rangle = |m, m'\rangle \, (m + m') \tag{3.2.6}$$

We know that the maximum values for m and m' are j and j' respectively, so the highest eigenvalue of J_3 is $(j + j')$, corresponding to the eigenvector $|j, j'\rangle$. Note that there are two eigenvectors corresponding to $(j+j'-1)$, namely $|j-1, j'\rangle$ and $|j, j'-1\rangle$. In the same way there are three vectors corresponding to $(j+j'-2)$, four for $(j+j'-3)$, and so on. When continuing this process, we will at some point encounter the maximal number of eigenvectors with the same eigenvalue. This will be the case for all eigenvalues between $(j - j')$ and $(-j + j')$. Still continuing, we find that the opposite of the first scenario happens; the number of eigenvectors goes down as the eigenvalues go down, until we reach the lowest value: $(-j - j')$, which again corresponds to only one vector.

Next we will (implicitly) go through the same procedure as the one leading to (3.1.17), and by doing so we construct an invariant subspace of $\mathfrak{L} \times \mathfrak{L}'$ spanned by simultaneous eigenvectors of J^2 and J_3, with eigenvalues $J(J+1)$ and M respectively. We already noted that there is only one vector with eigenvalue $M = j + j'$, and that it is the highest member of the irreducible basis, labeled by $J = j + j'$. We write:

$$|J = j + j', M = j + j'\rangle_{\text{new}} = |j, j'\rangle_{\text{old}} \tag{3.2.7}$$

where the subscript indicates if the vector belongs to the 'new' $|J, M\rangle$ or 'old' $|m, m'\rangle$ basis. Next we construct the rest of the new basis vectors by repeated application of J_-. Applying it once gives:

$$\begin{aligned} J_- \, |J = j + j', M = j + j'\rangle_{\text{new}} &= |J = j + j', M = j + j' - 1\rangle_{\text{new}} \\ &= J_- \, |j, j'\rangle_{\text{old}} \\ &= |j-1, j'\rangle_{\text{old}} + |j, j'-1\rangle_{\text{old}} \end{aligned} \tag{3.2.8}$$

where we have left out all normalization factors for brevity. The process will terminate when $M = -j - j'$. At that point we have constructed $2J + 1$ basis vectors which span an invariant subspace of $\mathfrak{L} \times \mathfrak{L}'$ corresponding to $J = j + j'$.

As noted before, there are two linearly independent eigenvectors of J_3 corresponding to $M = j + j' - 1$. One of those two is in the invariant subspace we just constructed, see (3.2.8). The other one will now be the new starting point for the same procedure; we construct another invariant subspace, now associated with $J = j + j' - 1$.

This process is repeated, until we end up with an invariant subspace corresponding to $J = |j - j'|$. Then, the whole space $\mathfrak{L} \times \mathfrak{L}'$ is spanned by the vectors $|J, M\rangle$ where $|j - j'| \leq J \leq j + j'$ and $-J \leq M \leq J$.

We can write the new basis vectors in terms of the old ones, and vice versa, using the methods described in section 2.5:

$$|J, M\rangle = |m, m'\rangle \langle mm'(jj')JM\rangle \tag{3.2.9}$$

$$|m, m'\rangle = |J, M\rangle \langle JM(jj')mm'\rangle \tag{3.2.10}$$

where summation over repeated indices is implied. The values of the Clebsch-Gordan coefficients are known and can be easily looked up. We state here without proof two useful general properties of these coefficients. Firstly

$$\langle mm'(jj')JM\rangle = 0 \tag{3.2.11}$$

except for the case when $m + m' = M$ and $|j - j'| \leq J \leq j + j'$. And secondly

$$\langle JM(jj')mm'\rangle \langle mm'(jj')J'M'\rangle = \delta^J_{J'}\delta^M_{M'} \tag{3.2.12}$$

Finally, for completeness, we write the equation that links the irreps D^j and $D^{j'}$ of $SO(3)$ on $\mathfrak{L}$ and $\mathfrak{L}'$ respectively, to the irreps D^J on invariant subspaces of $\mathfrak{L} \times \mathfrak{L}'$. It is a specific case of the general result (2.5.8a).

$$D^j(R)^m_{\ n} D^{j'}(R)^{m'}_{\ n'} = \sum_{J,M,N} \langle mm'(jj')JM\rangle\, D^J(R)^M_{\ N} \langle JN(jj')nn'\rangle \tag{3.2.13}$$

3.3 $SU(2)$

The next group we will consider is $SU(2)$. It is locally isomorphic to $SO(3)$, so they share the same Lie algebra. $SU(2)$ is compact and simply connected, and all irreps of the algebra are single-valued irreps of the group (in contrast to $SO(3)$). $SU(2)$ is in fact the universal covering group of $SO(3)$. We will make clear the connection between $SU(2)$ and rotations in 3 dimensions by using the Hermitian traceless *Pauli matrices* σ_i:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3.1}$$

Every element of the vector $\mathbf{x} = (x^1, x^2, x^3)$ is combined with a Pauli matrix, and we define the new coordinates

$$X = \sigma_i x^i \tag{3.3.2}$$

for which we have

$$\det X = -|\mathbf{x}|^2. \tag{3.3.3}$$

Let $U \in SU(2)$. It induces a linear transformation on X:

$$X \to X' = UXU^{-1} \tag{3.3.4}$$

X' again is traceless and Hermitian, thus it can be associated with a coordinate vector x' as in (3.3.2). Note that $\det X' = \det X$, from which it follows that $|\mathbf{x}| = |\mathbf{x}'|$. So we see that the $SU(2)$ transformation in (3.3.4) induces an $SO(3)$ transformation in 3 dimensional Euclidian space.

Note that the correspondence between an element U of $SU(2)$ and an element R of $SO(3)$ is two-to-one; $-U$ obviously corresponds to the same rotation as U. This is very closely related to the double-valuedness of the j-half-integer irreps of $SO(3)$.

3.4 The Lorentz Group $SO(3,1)$

In this section we will classify the irreducible representations of the Lorentz group[1] $SO(3,1)$; the group of 4×4 matrices, orthogonal with respect to the metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and with $\det = 1$. It is the group of continuous linear transformations Λ that leave the length of 4-vectors invariant (we presume that the reader is familiar with Lorentz transformations). It is the first group we consider that is non-compact; the parameters labeling the group elements corresponding to Lorentz boosts are not bounded. The most important consequence of the non-compactness is that all unitary representations will be infinite dimensional. We begin by taking a look at the Lie algebra of the Lorentz group.

The generators for a transformation $\Lambda \in SO(3,1)$ parametrized by ω can be defined along the same line as (1.3.1). We write:

$$\Lambda(\delta\omega) = \mathbb{1} - \frac{i}{2}\delta\omega^{\mu\nu} J_{\mu\nu} \tag{3.4.1}$$

where $\delta\omega^{\mu\nu} = -\delta\omega_{\mu\nu}$ are anti-symmetrical infinitesimal parameters, and $J_{\mu\nu}$ are the covariant generators of the group. The contravariant generators are found by raising the indices with the metric tensor:

$$J^{\mu\nu} = \eta^{\mu\lambda} J_{\lambda\sigma} \eta^{\sigma\nu} \tag{3.4.2}$$

so we see that $J^{mn} = J_{mn}$ and $J^{0m} = -J_{0m} = J_{m0}$ for $m = 1,2,3$. The generators J^{mn} generate rotations in the (m,n) plane. Because there are only 3 spatial dimensions, this can also be interpreted as a rotation around the k-axis, where (k,m,n) is some permutation of $(1,2,3)$. We can write:

$$J^k = \frac{1}{2}\epsilon^{kmn} J_{mn} \tag{3.4.3a}$$

and inversely

$$J^{mn} = \epsilon^{mnk} J_k \tag{3.4.3b}$$

Lorentz boosts are generated by

$$K_m \equiv J_{m0} \tag{3.4.4}$$

Now, we can summarize the Lie algebra of the Lorentz group in one equation:

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i(\eta_{\mu\sigma} J_{\lambda\nu} - \eta_{\mu\lambda} J_{\sigma\nu} + \eta_{\nu\sigma} J_{\mu\lambda} - \eta_{\nu\lambda} J_{\mu\sigma}) \tag{3.4.5}$$

or, in terms of the rotation J_m and boost K_m operators:

$$[J_m, J_n] = i\epsilon^{mnl} J_l \tag{3.4.6a}$$

$$[K_m, J_n] = i\epsilon^{mnl} K_l \tag{3.4.6b}$$

$$[K_m, K_n] = -i\epsilon^{mnl} J_l \tag{3.4.6c}$$

Consider the following basis transformation ($m = 1,2,3$):

$$M_m = (J_m + iK_m)/2 \tag{3.4.7a}$$

$$N_m = (J_m - iK_m)/2 \tag{3.4.7b}$$

[1] Actually, we are only interested in Lorentz transformations which preserve the direction of time. This subgroup is usually denoted by $SO_0(3,1)$. But since this abuse of notation is quite common, we will omit the subscript for brevity.

This transformation reduces the algebra of the Lorentz group to a product of two subalgebras. In fact, these subalgebras are $\mathfrak{su}(2)$ $(= \mathfrak{so}(3))$ algebras:

$$[M_m, M_n] = i\epsilon^{mnk} M_k \tag{3.4.8a}$$

$$[N_m, N_n] = i\epsilon^{mnk} N_k \tag{3.4.8b}$$

$$[M_m, N_n] = 0 \tag{3.4.8c}$$

and we say that the Lorentz algebra is identical to the $\mathfrak{su}(2)_M \times \mathfrak{su}(2)_N$ algebra. Finite dimensional irreps of this algebra are obtained from those of $\mathfrak{su}(2) = \mathfrak{so}(3)$ by using the methods described in sections 2.5 and 3.2. Note that the $SU(2)_M \times SU(2)_N$ group is compact, while the $SO(3,1)$ group is not, in spite of the fact that they share the same Lie algebra (if complexified). The first group corresponds to the exponentiation (see (1.3.3)) of $\{iM_m, iN_m\}$, and the second one to the exponentiation of $\{iJ_m, iK_m\}$. We note that in order for the representations to be unitary, we need the generators to be Hermitian:

$$\begin{aligned} T(g)T(g)^\dagger &= e^{-i\omega^{\mu\nu} J_{\mu\nu}} (e^{-i\omega^{\mu\nu} J_{\mu\nu}})^\dagger \\ &= e^{-i\omega^{\mu\nu} J_{\mu\nu}} e^{i\omega^{\mu\nu} J_{\mu\nu}^\dagger} \\ &= e^{-i\omega^{\mu\nu} (J_{\mu\nu} - J_{\mu\nu}^\dagger)} \\ &= \mathbb{1}, \qquad \text{if } J_{\mu\nu} = J_{\mu\nu}^\dagger. \end{aligned} \tag{3.4.9}$$

Due to the factors i in (3.4.7), the two different sets of generators can not simultaneously be Hermitian, and the finite dimensional representations of the Lorentz group will not be unitary.

Next we will construct the finite dimensional irreps of the Lorentz group and see that they are non-unitary. We know from the earlier sections that the representations of the direct product algebra are labeled by the two numbers $(u, v;\ 2u, 2v = 0, 1, 2, ...)$ such that $u(u+1)$ and $v(v+1)$ are eigenvalues of the Casimir operators M^2 and N^2 respectively. The natural choice of basis for the representation space is $\{|kl\rangle\,; -v \leq k \leq v; -u \leq l \leq u\}$ (corresponding to (3.2.1) in section 3.2), on which the generators act as follows:

$$J_3 |kl\rangle = (M_3 + N_3) |kl\rangle = |kl\rangle\, (k + l) \tag{3.4.10a}$$

$$\begin{aligned} J_\pm |kl\rangle = (M_\pm + N_\pm) |kl\rangle = &|k \pm 1 l\rangle \,[u(u+1) - k(k \pm 1)]^{1/2} \\ &+ |kl \pm 1\rangle\, [v(v+1) - l(l \pm 1)]^{1/2} \end{aligned} \tag{3.4.10b}$$

$$K_3 |kl\rangle = i(N_3 - M_3) |kl\rangle = |kl\rangle\, i(l - k) \tag{3.4.10c}$$

$$\begin{aligned} K_\pm |kl\rangle = i(N_\pm - M_\pm) |kl\rangle = &|kl \pm 1\rangle\, i[v(v+1) - l(l \pm 1)]^{1/2} \\ &- |k \pm 1 l\rangle\, i[u(u+1) - k(k \pm 1)]^{1/2} \end{aligned} \tag{3.4.10d}$$

where we have kept the normalization factors explicitly in the formulas. We can now diagonalize with respect to J^2 and J_3, and the representation space will become a direct sum of invariant subspaces labeled by $j = |u - v|, |u - v + 1|, ..., u + v$ with bases $\{|j, m\rangle\,;\ -j \leq m \leq j\}$ (corresponding to (3.2.9) from section 3.2).

It is easy to see that the boost generators are non-Hermitian in the $|kl\rangle$ basis; (3.4.10c) and (3.4.10d) show that they have imaginary eigenvalues. K_3 is in fact anti-Hermitian. We conclude that the finite dimensional irreps of the Lorentz group are non-unitary, as expected.

We now turn our attention to the unitary irreps of the Lorentz group. Since the group is non-compact they will be infinite dimensional. The irreps defined above were labeled by (u, v); we will now label them by (j_0, j_1), where $j_0 = |u - v|$ and $j_1 = u + v$. We shall find out what the constraint of unitarity implies by using the canonical basis $\{|j_0, j_1; j, m\rangle\}$.

The action of the rotational generators $\{J_m\}$ on the basis vectors $|j, m\rangle$ is the same as described in section 3.1 and they do not change the value of j. To find the action of the boost generators $\{K_m\}$ we will use the Wigner-Eckart theorem (see section 2.6). When we express the Wigner-Eckart theorem (2.6.5) in terms of symbols used in this section, it reads:

$$\langle j'm'|O^s_\lambda|jm\rangle = \langle j'm'(s,j)\lambda m\rangle \, \langle j'||O^s||j\rangle \tag{3.4.11}$$

where s is the 'angular momentum' of the operators O^s_λ, determining under which irrep they transform (corresponding to μ in section 2.6) and $\lambda = -s, ..., s$. The matrix elements vanish unless:

1. $|j - s| \leq j' \leq j + s$,
2. $m' = \lambda + m$.

We will be interested in the reduced matrix elements $\langle j'||O^s||j\rangle$; the Clebsch-Gordan coefficients can be looked up in the literature.

$\{K_m\}$ is a set of irreducible operators transforming as a 3-vector under rotations of $SO(3)$, see (3.4.6b). From (3.4.11) it follows that

$$\langle j'm'|K_3|jm\rangle = A^{j'}{}_j \, \langle j'm'(1,j)0m\rangle \tag{3.4.12a}$$

$$\langle j'm'|K_\pm|jm\rangle = \mp\sqrt{2}A^{j'}{}_j \, \langle j'm'(1,j)\pm 1m\rangle \tag{3.4.12b}$$

where the factor $\mp\sqrt{2}$ is included, because $\{K_3, -2^{-1/2}K_+, 2^{-1/2}K_-\}$ forms a normalized set of irreducible operators. We can find the constraints on $A^{j'}{}_j$ by requiring that the commutation relations for the K's are satisfied, in particular:

$$[K_\pm, K_3] = \pm J_\pm \tag{3.4.13a}$$

and

$$[K_+, K_-] = -2J_3 \tag{3.4.13b}$$

We will explicitly show how this goes, since for (for example) $SO(4,1)$ the procedure is completely analogous, but far more tedious. We start with a redefinition of the $A^{j'}{}_j$'s:

$$A^+_j = A^{j+1}{}_j/[(j+1)(2j+1)]^{1/2} \tag{3.4.14a}$$

$$A_j = A^j{}_j/[j(j+1)]^{1/2} \tag{3.4.14b}$$

$$A^-_j = A^{j-1}{}_j/[j(2j+1)]^{1/2} \tag{3.4.14c}$$

Next, we rewrite (3.4.12) by using the above redefinitions and the Clebsch-Gordan coefficients from the literature:

$$\begin{aligned} K_3\,|jm\rangle = &|j-1m\rangle\,[(j+m)(j-m)]^{1/2}A^-_j \\ &+ |jm\rangle\, mA_j \\ &+ |j+1m\rangle\,[(j+m+1)(j-m+1)]^{1/2}A^+_j \end{aligned} \tag{3.4.15a}$$

$$\begin{aligned} K_\pm \left|jm\right\rangle = &\mp \left|j-1 m\pm 1\right\rangle \left[(j\mp m)(j\mp m+1)\right]^{1/2} A_j^- \\ &+ \left|jm\pm 1\right\rangle \left[(j\mp m)(j\pm m+1)\right]^{1/2} A_j \\ &\pm \left|j+1 m\pm 1\right\rangle \left[(j\pm m+1)(j\pm m+2)\right]^{1/2} A_j^+ \end{aligned} \tag{3.4.15b}$$

If we plug the above results into (3.4.13), we will find the following conditions:

$$\begin{aligned} \left[(j-1)A_{j-1} - (j+1)A_j\right] A_j^- &= 0 \\ \left[(j+2)A_{J+1} - jA_j\right] A_j^+ &= 0 \\ (2j-1)A_j^- A_{j-1}^+ - {A_j}^2 - (2j+3)A_{j+1}^- A_j^+ &= 1 \end{aligned} \tag{3.4.16}$$

Remember we denoted the lowest value of j by j_0. We deduce from (3.4.15) that $A_{j_0}^- = 0$. In general $A_j^\pm \neq 0$, and we can deduce the following recursion formula from (3.4.16):

$$A_{j+1} = \frac{jA_j}{j+2} \tag{3.4.17}$$

with solution

$$A_j = i\frac{\nu j_0}{j(j+1)} \tag{3.4.18}$$

where $\nu \in \mathbb{C}$ is an arbitrary constant. We substitute (3.4.18) into the last equation of (3.4.16) to find for ${B_j}^2 \equiv -A_j^- = -A_{j-1}^+$:

$$(2j+3)B_{j+1}^2 = (2j-1){B_j}^2 + 1 - \left[\frac{\nu j_0}{j(j+1)}\right]^2 \tag{3.4.19}$$

We know that $B_{j_0} = 0$, and we can solve the above equation and find:

$${B_j}^2 = \frac{(j^2 - {j_0}^2)(j^2 - \nu^2)}{j^2(4j^2-1)} \tag{3.4.20}$$

and

$$A_j^- = B_j \xi_j \qquad A_{j-1}^+ = -B_j \xi_j^{-1} \tag{3.4.21}$$

with ξ_j arbitrary for the moment. Next we must see what the allowed values are for ν and ξ_j. They are restricted by demanding unitarity, from which follows that K_3 must be Hermitian and that $K_+ = K_-^\dagger$. Using (3.4.15) we find that these conditions translate into:

$$A_j = {A_j}^* \tag{3.4.22a}$$

$$A_j^- = -{A_{j-1}^+}^* \tag{3.4.22b}$$

We now plug in (3.4.18) and (3.4.21) into the above, and find:

$$j_0(\nu + \nu^*) = 0 \tag{3.4.23a}$$

$$|B_j|(|\xi_j|^2 - e^{-2i\beta_j}) = 0 \tag{3.4.23b}$$

where we used the fact that $B_j = |B_j|e^{i\beta_j}$. Assuming that $|B_j| \neq 0$, (3.4.23b) implies that $|\xi_j|^2 = 1$ and $\beta_j = 0$ (so that ${B_j}^2 > 0$). We choose, by convention, $\xi_j = 1$ for all j. The condition in (3.4.23a) implies that there are two distinct classes of irreps:

1. *Principal series*: $\nu = -\nu^*$. In this case we shall write $\nu = -iw$, where $w \in \mathbb{R}$. We have:

$$A_j = \frac{wj_0}{j(j+1)} \tag{3.4.24a}$$

$$B_j{}^2 = \frac{(j^2 - j_0^2)(j^2 + w^2)}{j^2(4j^2 - 1)} \tag{3.4.24b}$$

2. *Complementary series*: $j_0 = 0$. We then have:

$$A_j = 0 \tag{3.4.25a}$$

$$B_j{}^2 = \frac{j^2 - \nu}{4j^2 - 1} \tag{3.4.25b}$$

   where $B_0 = B_{j_0} = 0$. The condition that $B_j{}^2$ is real and positive leads to the following condition: $-1 \leq \nu \leq 1$.

To summerize the above, we state the following: the irreps in the principal series are labeled by $\nu = -iw$ where $w \in \mathbb{R}$ and $j_0 = 0, 1/2, 1, ...$; the irreps in the complementary series are labeled by $j_0 = 0$ and $-1 \leq \nu \leq 1$. These two series of irreps classify all unitary irreducible representations of $SO(3,1)$. The matrix elements of the rotational generators $\{J_m\}$ are given by the canonical form of section 3.1, and those of the boost generators $\{K_m\}$ are given by (3.4.12), where the $A^k{}_j$'s (with $k = j, j \pm 1$) are given by:

$$\begin{aligned} A^j{}_j &= i\frac{\nu j_0}{[j(j+1)]^{1/2}} \\ A^j{}_{j-1} &= -[j(2j-1)]^{1/2} B_j \\ A^{j-1}_j &= [j(2j+1)]^{1/2} B_j \end{aligned} \tag{3.4.26}$$

with

$$B_j{}^2 = \frac{(j^2 - j_0{}^2)(j^2 - \nu^2)}{j^2(4j^2 - 1)} \tag{3.4.27}$$

This concludes our classification of the UIRs of the Lorentz group. The results are conveniently summarized in the following diagram:

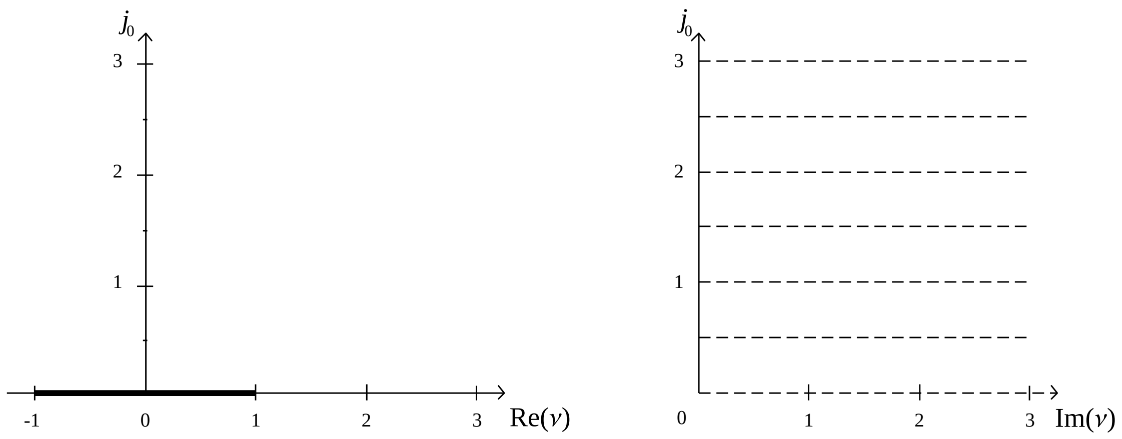

Figure 3.1: This diagram summarizes the classification of the UIRs of the Lorentz group. The dashed lines represent the principal series UIRs, and the fat line represents the complementary series UIRs.

3.5 The Poincaré Group

We now come to the Poincaré group; it is the group of all translations and Lorentz transformations in Minkowski spacetime. It is sometimes referred to as the *inhomogeneous Lorentz group*. We will again be interested in the irreps of this group, which we will construct by using the method of *induced representations*. We start by examining the Lie algebra of the group.

3.5.1 Lie algebra

We already know the algebra of the Lorentz group (which is obviously a subgroup of the Poincaré group), so we only have to find the generators for the translations, and see how they commute with the boosts and rotations. This is again done by taking an infinitesimal translation:

$$T(\delta b) = E + i\delta b^{\mu} P_{\mu} \tag{3.5.1}$$

It is obvious that translations commute; the space is flat, so we have $[P_{\mu}, P_{\nu}] = 0$. One can show that the generators P_{μ} transform as unit 4-vectors under the action of the Lorentz group, and we have:

$$[P_{\mu}, J_{\lambda\sigma}] = i(P_{\lambda}\eta_{\mu\sigma} - P_{\sigma}\eta_{\mu\lambda}) \tag{3.5.2}$$

Writing this in terms of the J_m's and K_m's introduced in section 3.4 we find:

$$[P^0, J_n] = 0 \tag{3.5.3a}$$

$$[P_m, J_n] = i\epsilon^{mnl} P_l \tag{3.5.3b}$$

$$[P_m, K_n] = i\delta_{mn} P^0 \tag{3.5.3c}$$

$$[P^0, K_n] = iP_n \tag{3.5.3d}$$

where $m, n, l = 1, 2, 3$. Together with (3.4.6) these equations form the algebra of the Poincaré group.

3.5.2 Induced representation method

The induced representation method that we will be using for constructing the irreps is only applicible when the group G has an Abelian invariant subgroup A. In the case of the Poincaré group this is the group of translations T_4. The method relies on constructing a basis for the representation space consisting of simultaneous eigenvectors of the generators of the Abelian invariant subgroup and of some other specific operators.

Let us define a vector space $\mathfrak{A}$ consisting of eigenvectors of the generators of A. Let us take some 'standard vector' $\bar{\mathbf{p}}$, which consists of the eigenvalues of the different generators. The basis vectors of $\mathfrak{A}$ which have as eigenvalue this standard vector form a subspace $\mathfrak{A}' \subset \mathfrak{A}$. We define the so called *little group* by all group elements in the factor group G/A which leave the subspace $\mathfrak{A}'$ corresponding to the standard vector $\bar{\mathbf{p}}$ invariant. The irreps of the little group will induce irreps of G. We will not prove this result, but we will see that it works in the case of the Poincaré group.

We can readily identify the factor group of Poincaré/T_4; it is the Lorentz group. So all little groups we will encounter in the following will be subgroups of $SO(3,1)$. The basis vectors of the representation space will be eigenvectors of P^{μ} and of commuting operators from the little group.

3.5.3 Casimir operators

One can show that the quadratic combination of the translation generators commutes with all generators; it is a Casimir operator:

$$C_1 \equiv P_\mu P^\mu = {P_0}^2 - \mathbf{P}^2 \tag{3.5.4}$$

The eigenvalues $c_1 = p_\mu p^\mu$ of C_1 are not positive definite, so we will consider the three cases (the reason for these names will become clear later):

1. Time-like p: $c_1 > 0$
2. Light-like p: $c_1 = 0$ and $\mathbf{p} \neq 0$
3. Space-like p: $c_1 < 0$

We will construct unitary irreducible representations for all three cases, but before we go into that, we will define a second Casimir operator. Let

$$W^\lambda \equiv \epsilon^{\lambda\mu\nu\sigma} \frac{J_{\mu\nu} P_\sigma}{2} \tag{3.5.5}$$

be the so called *Pauli-Lubanski vector*[2]. The second Casimir operator is defined as:

$$C_2 = W_\lambda W^\lambda \tag{3.5.6}$$

One can show that C_2 commutes with all generators.

3.5.4 Time-like case

We start with the standard vector $p_t^\mu = (p_0, \mathbf{p}) = (m, \mathbf{0})$, so that $c_1 = m^2$. The maximal subgroup of $SO(3,1)$ which leaves p_t^μ invariant is obviously the group acting only on the 3-vector part of p_t^μ; the group of rotations in three spacial dimensions $SO(3)$.

Let $\{|\mathbf{0}\lambda\rangle\}$ be the basis vectors corresponding to the eigenvalues p_t^μ of P^μ, such that:

$$P^\mu |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle\, p_t^\mu \tag{3.5.7a}$$

$$J^2 |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle\, s(s+1) \tag{3.5.7b}$$

$$J_3 |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle\, \lambda \tag{3.5.7c}$$

The $SO(3)$ invariant space spanned by this (canonical) basis is a subspace[3] of the complete representation space that we are constructing. In order to build the remaining basis vectors of the complete space, we have to operate on $|\mathbf{0}\lambda\rangle$ with the remaining transformations of the factor group. In this case, those are the Lorentz boosts. Consider such a boost in the 3-direction, parametrized by ξ. We define the action by:

$$|p\hat{\mathbf{e}}_3\ \lambda\rangle \equiv L_3(\xi)\, |\mathbf{0}\lambda\rangle \tag{3.5.8}$$

with $p = m \sinh \xi$. In order to get a general result we follow this boost by a rotation:

$$|\mathbf{p}\lambda\rangle \equiv R(\alpha, \beta, 0)\, |p\hat{\mathbf{e}}_3\ \lambda\rangle \equiv H(p)\, |\mathbf{0}\lambda\rangle \tag{3.5.9}$$

[2] This vector is quite special; it can be shown that the independent components of $\{W^\lambda\}$ are in fact the generators of the little group corresponding to the standard vector $\bar{\mathbf{p}}$.

[3] It is the space $\mathfrak{A}'$ defined in section 3.5.2.

The Lorentz transformation $H(p)$ is the general transformation that takes the vector p_t^μ to any vector p^μ. The space spanned by $\{|\mathbf{p}\lambda\rangle\}$ is invariant under the action of the whole Poincaré group, that is to say:

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{ib^\mu p_\mu} \tag{3.5.10a}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\mathbf{p}'\lambda'\rangle\, D^s[R(\Lambda, p)]^{\lambda'}{}_{\lambda} \tag{3.5.10b}$$

where $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ and $D^s[R]$ is the representation matrix of $SO(3)$. We will now explicitly show that we can represent every Lorentz transformation on $|\mathbf{p}\lambda\rangle$ by a rotation. We have:

$$\Lambda\,|\mathbf{p}\lambda\rangle = \Lambda H(p)\,|\mathbf{0}\lambda\rangle = H(p')[H^{-1}(p')\Lambda H(p)]\,|\mathbf{0}\lambda\rangle \tag{3.5.11}$$

Now consider the action of the term in brackets on the standard vector:

$$\begin{aligned} H^{-1}(p')\Lambda H(p)p_t^\mu &= H^{-1}(p')\Lambda p^\mu \\ &= H^{-1}(p')p'^\mu \\ &= p_t^\mu \end{aligned} \tag{3.5.12}$$

so we can conclude that $[H^{-1}(p')\Lambda H(p)p_t^\mu] \equiv R(\Lambda, p) \in SO(3)$ and we see that (3.5.11) indeed leads to (3.5.10b).

We know that the space spanned by $\{|\mathbf{p}\lambda\rangle\}$ is irreducible, because all vectors are generated from one starting vector $|\mathbf{0}\lambda = s\rangle$ by repeatedly applying $J_\pm$ and $H(p)$; by construction there are no nontrivial invariant subspaces.

Since all generators are Hermitian on this basis, and the representation matrices on the right-hand side of (3.5.10) are unitary, we know that the representations constructed in this way are not only irreducible, but also unitary.

Note that we can do exactly the same for $p_t^\mu = (p_0, \mathbf{p}) = (-m, \mathbf{0})$. The unitary irreps constructed in that way are called 'negative energy' irreps, and they have the same defining properties as the 'positive energy' irreps constructed above. We shall collectively denote the positive energy unitary irreps by $\mathcal{P}^+(m, s)$ and the negative energy ones by $\mathcal{P}^-(m, s)$.

Before we move on to the next case, we want to draw the attention to the second Casimir operator. The action of the Pauli-Lubanski vector on the subspace corresponding to the standard vector p_t^μ is given by:

$$W^\lambda = \epsilon^{\lambda\mu\nu\sigma}\frac{J_{\mu\nu}p_{t\sigma}}{2} \tag{3.5.13a}$$

so that

$$W^0 = 0 \tag{3.5.13b}$$

and

$$W^i = \frac{m}{2}\epsilon^{ijk}J_{jk} = mJ^i \tag{3.5.13c}$$

so we see in fact that the Casimir operator C_2/m^2 has eigenvalues $s(s+1)$ and we can use it, together with C_1, to label the irreps: (m, s) labels the irrep, while $\mathbf{p}$ and λ label the basis vectors of the representation space.

3.5.5 Light-like case

Next we turn our attention to the so called light-like case ($c_1 = 0$ while $\mathbf{p} \neq 0$). For the standard vector we will use

$$p_l^\mu = (\omega_0, 0, 0, \omega_0) \tag{3.5.14}$$

We transform this vector into any vector $p^\mu = (\omega, \mathbf{p})$, where $\mathbf{p} = \omega\hat{\mathbf{p}}$, by applying the group action $H(p) = R(\alpha, \beta, 0)L_3(\xi)$. One can show that the little group is the group of translations and rotations in two dimensions, i.e. the *Euclidean group* E_2. One way of showing this is by using the theorem that the components of the Pauli-Lubanski vector are the generators of the little group. Let us check:

$$W^\lambda = \epsilon^{\lambda\mu\nu\sigma}\frac{J_{\mu\nu}p_{l\sigma}}{2} \tag{3.5.15}$$

so that

$$\begin{aligned} W^0 &= W^3 = \omega_0 J_{12} = \omega_0 J_3 \\ W^1 &= \omega_0(J_{23} + J_{20}) = \omega_0(J_1 + K_2) \\ W^2 &= \omega_0(J_{31} - J_{10}) = \omega_0(J - 2 - K_1) \end{aligned} \tag{3.5.16}$$

and the Lie algebra is given by:

$$\begin{aligned} \left[W^1, W^2\right] &= 0 \\ [W^2, J_3] &= iW^1 \\ [W^1, J_3] &= -iW^2 \end{aligned} \tag{3.5.17}$$

which is indeed the algebra of the Euclidean group in two dimensions, where W^1 and W^2 generate the translations, while J_3 generates the rotation. The Casimir operator C_2 is given by $W_\mu W^\mu = (W^1)^2 + (W^2)^2$, and it has the eigenvalue $w^2 \geq 0$. We will give a short description of the unitary irreps of E_2. Consider the canonical basis $\{|w\lambda\rangle\}$ which consist of simultaneous eigenvectors of C_2 and J_3, such that:

$$C_2 |w\lambda\rangle = |w\lambda\rangle\, w^2 \tag{3.5.18}$$

$$J_3 |w\lambda\rangle = |w\lambda\rangle\, \lambda \tag{3.5.19}$$

where $\lambda = 0, \pm1, \pm2, ...$, and we have:

$$\begin{aligned} \langle w\lambda|W_\pm^\dagger W_\pm|w\lambda\rangle &= \langle w\lambda|W_\mp W_\pm|w\lambda\rangle \\ &= \langle w\lambda|W_\mu W^\mu|w\lambda\rangle \\ &= w^2\,\langle w\lambda|w\lambda\rangle \\ &= w^2 \end{aligned} \tag{3.5.20}$$

Now we see that the representation space for $w = 0$ is one-dimensional since $W_\pm|0\lambda\rangle = 0$, and we have (denoting $|0\lambda\rangle$ by $|\lambda\rangle$):

$$\begin{aligned} J_3 |\lambda\rangle &= |\lambda\rangle\,\lambda \\ R(\theta)|\lambda\rangle &= |\lambda\rangle\, e^{-i\lambda\theta} \\ T(\mathbf{b})|\lambda\rangle &= |\lambda\rangle \end{aligned} \tag{3.5.21}$$

where $R(\theta)$ represents a rotation over an angle θ, and $T(\mathbf{b})$ a translation by an amount $\mathbf{b}$. This one dimensional irrep is *degenerate*, as opposed to *faithful*; it maps E_2 to the subgroup of rotations $SO(2)$.

Now consider the case where $w^2 > 0$. The representation space is now infinite dimensional and has the basis $\{|w\lambda\rangle\,;\ \lambda = 0, \pm1, \pm2, ...\}$, which can be constructed by applying $W_\pm$

to some starting vector. These representations labeled by w are faithful, unitary and irreducible.

We now return to the light-like irreps of the Poincaré group. Starting out from any of the irreps of E_2 we can generate a basis that forms an invariant space under the full Poincaré group by applying the Lorentz transformations $H(p)$ to the basis vectors of the E_2-irrep space. These Poincaré irreps will be labeled by $m(=0)$ and w.

Let us focus on the degenerate E_2 irrep. The subspace of eigenvectors with as eigenvalue the standard vector p_l^μ is one dimensional, and for the one basis vector $|\mathbf{p}_l\lambda\rangle$ we have:

$$\begin{aligned} P^\mu\,|\mathbf{p}_l\lambda\rangle &= |\mathbf{p}_l\lambda\rangle\,p_l^\mu \\ J_3\,|\mathbf{p}_l\lambda\rangle &= |\mathbf{p}_l\lambda\rangle\,\lambda \\ W^i\,|\mathbf{p}_l\lambda\rangle &= 0 \qquad\qquad \text{where } i=1,2 \end{aligned} \tag{3.5.22}$$

It turns out that all integer-λ irreps are single-valued, while the half-integer-λ irreps are double-valued. The basis vectors of the full representation space are defined by:

$$|\mathbf{p}\lambda\rangle = H(p)\,|\mathbf{p}_l\lambda\rangle \tag{3.5.23}$$

and the action of the Poincaré group on them is given by:

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\,e^{ib^\mu p_\mu} \tag{3.5.24a}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\Lambda\mathbf{p}\lambda\rangle\,e^{-i\lambda\theta(\Lambda,p)} \tag{3.5.24b}$$

where $\theta(\Lambda,p)$ can be obtained in a similar way as was used for $R(\Lambda,p)$ in the time-like case. The irreps are labeled by $(m=0,\lambda)$

Since the irreps of the Poincaré group induced by the $w>0$ irreps of E_2 have less physical significance for us we will omit them here.

Similar to the time-like case, we can construct negative energy light-like irreps by starting with the standard vector $p_l^\mu = (-\omega_0, 0, 0, -\omega_0)$.

3.5.6 Space-like case

We will complete the discussion of the Poincaré irreps by building the space-like irreps ($c_1 < 0$). We start again by choosing a standard vector: $p_s^\mu = (0,0,0,Q)$. We find the generators of the little group associated with this vector by using the Pauli-Lubanski vector:

$$\begin{aligned} W_0 &= QJ_3 \\ W_1 &= QJ_{20} = QK_2 \\ W_2 &= QJ_{01} = -QK_1 \end{aligned} \tag{3.5.25}$$

who form the following Lie algebra:

$$\begin{aligned} [K_2, J_3] &= iK_1 \\ [J_3, K_1] &= iK_2 \\ [K_1, K_2] &= -iJ_3 \end{aligned} \tag{3.5.26}$$

and the second Casimir operator is given by

$$C_2 = Q^2({K_1}^2 + {K_2}^2 - {J_3}^2) \tag{3.5.27}$$

Note that the Lie algebra is very similar to the one associated with $SO(3)$ if we make the substitution $K_1 \to J_1$ and $K_2 \to J_2$. The only difference is in the minus sign in the last equation. The corresponding group in this case is $SO(2,1)$ and is non-compact.

We state here the unitary irreducible representations of $SO(2,1)$. They consist of two classes:

1. $c_2 > 0$; in this case c_2 can take on continuous values, and the infinite basis vectors of the representation space are labeled by $|p_s\lambda\rangle_{c_2}$, where $\lambda = 0, \pm 1, ...$

2. $c_2 \leq 0$; in this case c_2 can only take on the discrete values $-j(j+1)$, where $j = 0, 1/2, 1, ...$ (the half integer j irreps are double-valued). The basis vectors of this representation space are again labeled by $|p_s\lambda\rangle_{c_2}$, but now $\lambda = j+1, j+2, ...$ or $\lambda = -j-1, -j-2, ...$

Given one of these irreps of $SO(2,1)$ one can generate the irreducible representation space of the full Poincaré group by applying Lorentz transformations (that are not from the little group) to the basis vectors of the irrep of $SO(2,1)$:

$$|\mathbf{p}\lambda\rangle = H(p)\,|p_s\lambda\rangle \tag{3.5.28}$$

where in the space-like case we can take $H(p) = R_3(\alpha)L_1(\zeta)L_3(\xi)$. The group acts on these vectors in the now familiar way:

$$\begin{aligned} T(b)\,|\mathbf{p}\lambda\rangle &= |\mathbf{p}\lambda\rangle\, e^{ib^\mu p_\mu} \\ \Lambda\,|\mathbf{p}\lambda\rangle &= |\Lambda\mathbf{p}\lambda'\rangle\, D^{c_2}[H^{-1}(\Lambda p)\Lambda H(p)]^{\lambda'}_{\lambda} \end{aligned} \tag{3.5.29}$$

where D^{c_2} is the representation matrix for the $SO(2,1)$.

It is a common misunderstanding that the Poincaré group has no space-like (i.e. imaginary mass) unitary irreducible representations. The above construction shows that they in fact do exist.

3.6 $SO(4,1)$

We now turn our attention to the unitary irreducible representations of the group of isometries of *de Sitter space*; $SO(4,1)$. It is a non-compact semisimple group, so all its unitary irreps will be infinite dimensional. A first attempt to classify these irreps dates back to the 1940's [27, 28], and it was completed in a rigorous way in the 60's [29, 30]. Some later papers use other manifestations of the group [31] and one can argue about the accessiblity of the different methods used. A general method for finding all unitary irreps for $SO(N,1)$ can be found in [32].

Let us first of all give a definition of the de Sitter space. It can be viewed as the surface of a hyberboloid embedded in 5-dimensional Minkowski space:

$$\mathfrak{M}_{dS} = \{x \in \mathbb{R}^5 |\ x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}\} \tag{3.6.1}$$

where $\alpha, \beta = 0, 1, 2, 3, 4$ and $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. This Minkowskian metric induces a metric $g^{dS}_{\mu\mu}$ on the intrinsic 4-dimensional coordinates X^μ:

$$ds = \eta_{\alpha\beta}dx^\alpha dx^\beta = g^{dS}_{\mu\nu}dX^\mu dX^\nu \tag{3.6.2}$$

where $\mu, \nu = 0, 1, 2, 3$.

The generators of the Lie algebra of $SO(4,1)$ obey the general commutation rules for (special) orthogonal groups (compare with (3.4.5)):

$$[K_{\alpha\beta}, K_{\gamma\delta}] = i\left(\eta_{\alpha\delta}K_{\beta\gamma} - \eta_{\alpha\gamma}K_{\beta\delta} + \eta_{\beta\gamma}K_{\alpha\delta} - \eta_{\beta\delta}K_{\alpha\gamma}\right) \tag{3.6.3}$$

We can define two Casimir operators in the following way (see e.g. [27]):

$$Q^{(1)} = -\frac{1}{2}K_{\alpha\beta}K^{\alpha\beta} \tag{3.6.4}$$

$$Q^{(2)} = -W_\alpha W^\alpha \tag{3.6.5}$$

where

$$W^\alpha = \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta\zeta}K_{\beta\gamma}K_{\delta\zeta} \tag{3.6.6}$$

and $\epsilon^{\alpha\beta\gamma\delta\zeta}$ is the Levi-Civita symbol in five dimensions: it equals $+1/-1$ if $(\alpha,\beta,\gamma,\delta,\zeta)$ is an even/odd permutation of $(0,1,2,3,4)$, and is zero when indices are repeated. Because these Casimirs (by definition) commute with all generators, they are constant on each unitary irrep, and they will be used to label these irreps. This is analogous to the way we labeled the irreps of the Poincaré group in section 3.5.

For the Lorentz group we found that there are two series of unitary irreps, and we expect something similar for $SO(4,1)$. It was shown by Dixmier in 1961 [29] that there are in fact three series for this group. We will not repeat the calculation here, since it is analogous to the $SO(3,1)$ case of section 3.4, but more tedious. Every unitary irrep is labeled by a pair of parameters (p,q) with $2p \in \mathbb{N}$ and $q \in \mathbb{C}$. The eigenvalues of the Casimirs are linked to these parameters in the following way:

$$\begin{aligned} Q^{(1)} &= [-p(p+1) - (q+1)(q-2)]\mathbb{1} \\ Q^{(2)} &= [-p(p+1)q(q-1)]\mathbb{1} \end{aligned} \tag{3.6.7}$$

The three series that can be distinghuised are the following:

1. *Principle series.* Also known as 'massive' representations, for reasons that will become clear later. They are labeled by $(p,q) = (p, \frac{1}{2} + i\nu)$ with $\nu \in \mathbb{R}$, and we will denote the unitary irreps from this series by $U_{p,\nu}$. We have

$$\begin{aligned} &p = 0,1,2,... \quad \text{and} \quad \nu \geq 0 \quad \text{or,} \\ &p = \frac{1}{2}, \frac{3}{2}, ... \quad \text{and} \quad \nu > 0 \end{aligned} \tag{3.6.8}$$

   The eigenvalues of the Casimir operators take on the form:

$$\begin{aligned} Q^{(1)} &= \left[\left(\frac{9}{4} + \nu^2\right) - p(p+1)\right]\mathbb{1} \\ Q^{(2)} &= \left[\left(\frac{1}{4} + \nu^2\right)p(p+1)\right]\mathbb{1} \end{aligned} \tag{3.6.9}$$

2. *Complementary series.* The representations of this series are labeled by $(p,q) = (p, \frac{1}{2} + \nu)$ with $\nu \in \mathbb{R}$ and are denoted by $V_{p,\nu}$. We have

$$\begin{aligned} &p = 0 \quad \text{and} \quad 0 < |\nu| < \frac{3}{2} \quad \text{or,} \\ &p = 1,2,3,... \quad \text{and} \quad 0 < |\nu| < \frac{1}{2} \end{aligned} \tag{3.6.10}$$

The eigenvalues of the Casimir operators take on the form:

$$\begin{aligned} Q^{(1)} &= \left[\left(\frac{9}{4} - \nu^2\right) - p(p+1)\right] \mathbb{1} \\ Q^{(2)} &= \left[\left(\frac{1}{4} - \nu^2\right) p(p+1)\right] \mathbb{1} \end{aligned} \tag{3.6.11}$$

3. *Discrete series.* The representations of this series are labeled by (p, q) and are denoted by $\Pi_{p,0}$ and $\Pi^{(\pm)}_{p,q}$ (the $\pm$ will be discussed later; it is linked to helicity). We have

$$\begin{aligned} &p = 1, 2, 3, ... \quad \text{and} \quad q = p, p-1, ..., 0 \quad \text{or,} \\ &p = \frac{1}{2}, \frac{3}{2}, ... \quad \text{and} \quad q = p, p-1, ..., \frac{1}{2} \end{aligned} \tag{3.6.12}$$

and the Casimir eigenvalues remain of the form (3.6.7).

This completes the general classification of the unitary irreps of $SO(4, 1)$. The results are conveniently summarized in the following diagram:

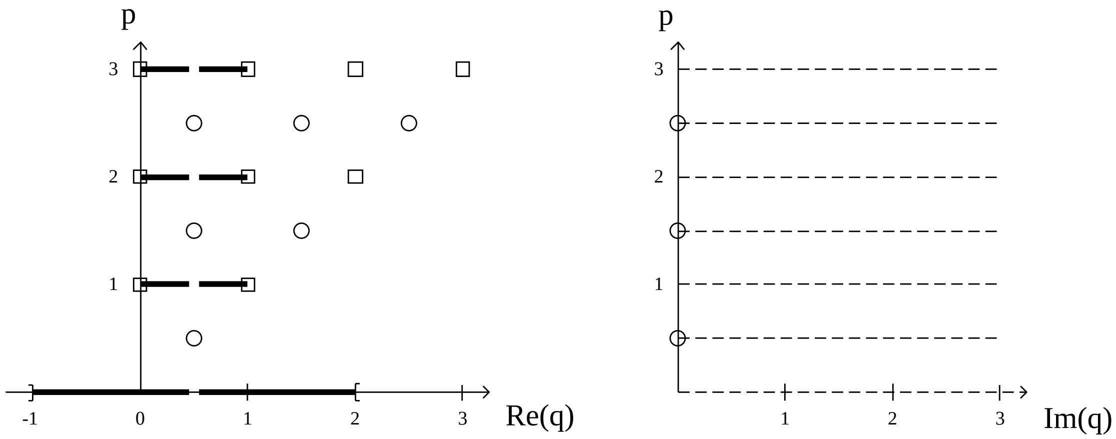

Figure 3.2: This diagram summarizes the classification of the UIRs of $SO(4, 1)$ (based on [18]). The dashed lines represent the principal series UIRs, and the fat lines represents the complementary series UIRs. The discrete series UIRs with p integer and half-integer are represented by the squares and circles respectively. Note that the p-axis of the right diagram coincides with the $\text{Re}(q) = 1/2$ vertical of the left diagram.

For later purposes we give an explicit form for the generators, using the Killing vector formalism (see section 1.3.3). Since the group $SO(4, 1)$ is 10-dimensional there will be ten independent Killing vectors, and they are given by:

$$K_{\alpha\beta} = i \left(x_\alpha \frac{\partial}{\partial x^\beta} - x_\beta \frac{\partial}{\partial x^\alpha} \right) \tag{3.6.13}$$

Note that there are indeed ten vectors: $K_{\alpha\beta}$ is anti-symmetric, thus it contains $n(n-1)/2$ independent elements.

Chapter 4

Group Contractions

In this chapter we will describe the process of *group contraction*; starting out with some group, one obtains a new group by 'contracting' the former to the latter. It was first described by Inönü and Wigner in 1953 [33, 34], and it will be of great importance for the rest of this thesis. We will review two distinct ways of defining a contraction; one that refers to the infinitesimal operators, and one that refers to the elements of the group. Thereafter we will briefly discuss the contraction of representations. The final section of this chapter is devoted to two examples of group contractions: $SO(3) \to E_2$ for getting a feeling for the procedure, and Poincaré $\to SO(4,1)$ which is relevant for the rest of this thesis.

4.1 General procedure

Let us begin with the mathematical description of the first method. It states that a contraction is the operation of obtaining a new group by a singular transformation of the infinitesimal elements of the old group, and the former is said to be a contraction of the latter.

Consider a Lie group G with n parameters α^i and infinitesimal operators X_i. The Lie algebra is given by these operators and the structure constants, linked by the following relation[1]:

$$[X_i, X_j] = c_{ij}{}^k X_k \tag{4.1.1}$$

We can apply a linear non-singular transformation U to the infinitesimal operators. We write:

$$Y_i = X_j U^j{}_i \tag{4.1.2}$$

which corresponds to the transformation of the group parameters:

$$\alpha^i = U^i{}_j \beta^j \tag{4.1.3}$$

The structure of the group will remain the same, but the structure constants do change. The new structure constants $C_{ij}{}^k$ are defined by:

$$[Y_i, Y_j] = C_{ij}{}^k Y_k \tag{4.1.4}$$

and can be expressed in terms of the old ones:

$$C_{ij}{}^k = U_i{}^l U_j{}^m c_{lm}{}^n (U^{-1})_n{}^k \tag{4.1.5}$$

[1]Summation over repeated indices, when not explicit, is implied everywhere in this section.

In order to make this transformation a contraction, we have to make U singular. This is done by a limiting process. We write U as a sum of an ϵ-independent singular and an ϵ-dependent (non-)singular part:

$$U^i{}_j = u^i{}_j + \epsilon w^i{}_j \tag{4.1.6}$$

By definition:

$$\det U \begin{cases} \neq 0 & \text{for} \quad 0 < \epsilon < \epsilon_0 \\ = 0 & \text{for} \quad \epsilon = 0 \end{cases}$$

It is possible[2] to get the $n \times n$ matrices u and w of the following form:

$$u = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} v & 0 \\ 0 & \mathbb{1}_{n-r} \end{pmatrix} \tag{4.1.7}$$

We can split the infinitesimal operators up into two parts:

$$\begin{aligned} Y_{1\mu} &= X_{1\mu} + \epsilon v^\nu{}_\mu X_{1\nu} \\ Y_{2\lambda} &= \epsilon X_{2\lambda} \end{aligned} \tag{4.1.8a}$$

where $\mu = 1, 2, ..., r$ and $\lambda = 1, 2, ..., n-r$. The corresponding group parameter transformations are:

$$\begin{aligned} \alpha_{1\mu} &= \beta_{1\mu} + \epsilon v^\nu{}_\mu \beta_{1\nu} \\ \alpha_{2\lambda} &= \epsilon \beta_{2\lambda} \end{aligned} \tag{4.1.8b}$$

By bringing u and w to this particular diagonal form, we have transformed the infinitesimal elements, and therefore also the structure constants. With explicit summation, we have:

$$[X_{a\mu}, X_{b\nu}] = \sum_{\sigma=1}^{r} c_{a\mu,b\nu}{}^{1\sigma} X_{1\sigma} + \sum_{\sigma=1}^{n-r} c_{a\mu,b\nu}{}^{2\sigma} X_{2\sigma} \tag{4.1.9}$$

with $a, b = 1, 2$. For the operators $Y_{1\mu}$ we find:

$$\begin{aligned} [Y_{1\mu}, Y_{1\nu}] &= [X_{1\mu}, X_{1\nu}] + \epsilon(v^\rho_\mu \delta^\sigma_\nu + \delta^\rho_\mu v^\sigma_\nu + \epsilon v^\rho_\mu v^\sigma_\nu)[X_{1\rho}, X_{1\sigma}] \\ &= c_{1\mu,1\nu}{}^{1\kappa} Y_{1\kappa} + \frac{1}{\epsilon} c_{1\mu,1\nu}{}^{2\kappa} Y_{2\kappa} + \mathcal{O}(\epsilon) \end{aligned} \tag{4.1.10}$$

so we see that the commutator will only converge in the $\epsilon \to 0$ limit if we have:

$$c_{1\mu,1\nu}{}^{2\kappa} = 0 \tag{4.1.11}$$

which means that the $X_{1\mu}$ form a subalgebra. If we assume that (4.1.11) is satisfied, then we can immediately derive that we must have:

$$\begin{aligned} C_{1\mu,1\nu}{}^{1\kappa} &= c_{1\mu,1\nu}{}^{1\kappa} \\ C_{1\mu,2\lambda}{}^{2\rho} &= c_{1\mu,2\lambda}{}^{2\rho} \\ C_{1\mu,1\nu}{}^{2\lambda} &= c_{1\mu,1\nu}{}^{2\lambda} = 0 \\ C_{1\mu,2\lambda}{}^{1\kappa} &= 0 \\ C_{2\lambda,2\rho}{}^{1\kappa} &= C_{2\lambda,2\rho}{}^{2\sigma} = 0 \end{aligned} \tag{4.1.12}$$

[2]In some special cases, which we will not encounter, it is not possible. The condition to ensure that it's possible can be found in [35].

where $\mu, \nu, \kappa = 1, ..., r$ and $\lambda, \rho, \sigma = 1, ..., n-r$. One can explicitly check that the new structure constants satisfy

$$C_{ij}{}^{k} C_{kl}{}^{m} + C_{jk}{}^{l} C_{il}{}^{m} + C_{ki}{}^{l} C_{jl}{}^{m} = 0 \tag{4.1.13}$$

for all values of $i, j, k, m = 1, ..., n$, so that we have a new n-dimensional group G' with structure constants $C_{ij}{}^{k}$ which is not isomorphic to G.

In [33], Inönü and Wigner summerized this by stating the following theorem:

> *Every Lie group can be contracted with respect to any of its continuous subgroups and only with respect to these. The subgroup with respect to which the contraction is undertaken will be called S. The contracted infinitesimal elements form an Abelian invariant subgroup A of the contracted group G'. The subgroup S with respect to which the contraction was undertaken is isomorphic with the factor group of this invariant subgroup in G', i.e. $S \sim G'/A$. Conversely, the existence of an Abelian invariant subgroup in G' and the possibility to choose from each of its cosets an element so that these form a subgroup S, is a necessary condition for the possibility to obtain the group G' from another group G by contraction.*

Let us now review the second method of defining a contraction. It was first stated by Mickelsson and Niederle in 1972 [36], and it makes no use of the infinitesimal operators. We will see that this definition is mathematically more precise, but some will find it less insightful. All the groups involved are Lie groups.

Let V be a neighbourhood of the identity e of G, such that $V^2 = \{g_1 g_2 | g_1, g_2 \in V\}$ is defined. Let there be a family of mappings $f_\epsilon : V^2 \to G^c$ where $\epsilon \in (0, 1]$. We say that the group G^c is a contraction of G, if the following conditions are satisfied:

1. f_ϵ is a homeomorphism from V^2 onto $f_\epsilon(V^2)$ for $\forall \epsilon \in (0, 1]$,
2. if $g^c \in G^c$, then $\exists \epsilon_0 \in (0, 1]$ such that $g^c \in f_\epsilon(V)$ if $\epsilon < \epsilon_0$. In other words, $f_\epsilon^{-1}(g^c)$ is defined and belongs to V when $\epsilon < \epsilon_0$,
3. $f_\epsilon(e)$ is the identity in G^c for $\forall \epsilon \in (0, 1]$,
4. if $g_1^c, g_2^c \in G^c$, then $g_1^c g_2^c = \lim_{\epsilon \to 0} f_\epsilon \left(f_\epsilon^{-1}(g_1^c) \cdot f_\epsilon^{-1}(g_2^c) \right)$.

This defines the contraction G^c of the group G.

In the next section we will look at how representations behave under group contraction.

4.2 Representations under contraction

From the results of the previous section we can immediately deduce that there will be some problems with the representations of the group that is contracted by using the first method. Look at equation (4.1.8a), and consider the infinitesimal operators X to be the generators for some representation of G. When $\epsilon \to 0$, we have that $Y_{2\mu} \to 0$, so the resulting representation of G' cannot be faithful. In fact, the resulting space spanned by the Y's will be a representation for the factor group G'/A. There are (at least) two methods for obtaining faithful representations. The first one is to apply an ϵ-dependent similarity transformation before contracting. The second method is to consider a sequence of representations which converges to a faithful representation, and for which the infinitesimal elements converge to finite values. In the next section we will encounter an example of the second method.

Mickelsson and Niederle do not encounter this problem, due to the fact that they do not work with infinitesimal operators. Let us give their definition of a contraction of a representation.

> *Consider a one-parameter family of continuous unitary representations $D^\epsilon(g)$ of G in representation spaces $\mathfrak{H}^\epsilon$, with $\epsilon \in (0,1]$. Let A_ϵ be a continuous linear mapping from $\mathfrak{H}^\epsilon$ to $\mathfrak{H}$, such that $A_\epsilon : \mathfrak{H}^\epsilon \to A_\epsilon(\mathfrak{H}^\epsilon)$ is unitary, and for $\forall\psi \in \mathfrak{H}$, $\exists\epsilon_\psi$ such that when $\epsilon < \epsilon_\psi$ we have $\psi \in A_\epsilon(\mathfrak{H}^\epsilon)$. If the limit of $A_\epsilon D^\epsilon(f_\epsilon^{-1}(g^c))A_\epsilon^{-1}\psi$ exists for $\forall\psi \in \mathfrak{H}, \forall g^c \in G^c$ as $\epsilon \to 0$ and is continuous in g^c, and the homomorphism $g^c \to D(g^c)$ defined by*
>
> $$D(g^c)\psi = \lim_{\epsilon\to 0} A_\epsilon D^\epsilon(f_\epsilon^{-1}(g^c))A_\epsilon^{-1}\psi \tag{4.2.1}$$
>
> *is a unitary representation of G^c, then the representation $g^c \to D(g^c)$ of G^c is said to be a contraction of the representation $g \to D^1(g)$ of G.*

Next we will look at some examples of contraction of groups and representations.

4.3 Examples of contractions

We will consider two examples: the first one is to get a feeling for the contraction procedure, and the second one we will use for the field theory in de Sitter space. We start with the contraction that we are able to visualize.

4.3.1 $SO(3) \to E_2$

By now we have seen quite a bit of $SO(3)$; the group of rotations in three spacial dimensions. Before we start calculating, it is easy to see that this group will contract to the symmetry group of the plane. Just imagine a sphere whose radius $R \to \infty$. In this limit the surface of the sphere becomes a flat surface. Symmetry transformations of the sphere then become translations and rotations in the plane.
Let us now do the contraction explicitly. $SO(3)$ has three group generators which obey the following commutation relations:

$$\begin{aligned}[J_1, J_2] &= iJ_3\\ [J_2, J_3] &= iJ_1\\ [J_3, J_1] &= iJ_2\end{aligned} \tag{4.3.1}$$

We will contract with respect to the one-parameter subgroup spanned by J_3. We make the following substitution [22]:

$$\begin{aligned}Y_1 &= \epsilon J_2\\ Y_2 &= -\epsilon J_1\\ Y_3 &= J_3\end{aligned} \tag{4.3.2}$$

As we take the limit $\epsilon \to 0$, we obtain:

$$\begin{aligned}[Y_1, Y_2] &= -\epsilon^2[J_2, J_1] = i\epsilon^2 J_3 = i\epsilon^2 Y_3 \to 0\\ [Y_2, Y_3] &= -\epsilon[J_1, J_3] = i\epsilon J_2 = iY_1\\ [Y_3, Y_1] &= \epsilon[J_3, J_2] = -i\epsilon J_1 = iY_2\end{aligned} \tag{4.3.3}$$

We can now easily identify these commutation relations with those of E_2, where $Y_{1,2}$ are translations and Y_3 is the rotation. Instead of $\epsilon \to 0$ we could have taken the radius of the sphere

$R = 1/\epsilon \to \infty$. We see that the mathematical description yields the intuitive result.

Let us now consider the contraction of the unitary representations of $SO(3)$. We will use the converging sequence method. Recall the D^j representations from section 3.1. The generators ($-i \times$ infinitesimal operators) will have eigenvalues (omitting the j label):

$$\begin{aligned}
\langle m'|J_3|m\rangle &= m\delta_{mm'} \\
\langle m'|J_2|m\rangle &= -\frac{i}{2}[j(j+1)-m(m+1)]^{1/2}\delta_{m'm+1} + \frac{i}{2}[j(j+1)-m(m-1)]^{1/2}\delta_{m'm-1} \\
\langle m'|J_1|m\rangle &= \frac{1}{2}[j(j+1)-m(m+1)]^{1/2}\delta_{m'm+1} + \frac{1}{2}[j(j+1)-m(m-1)]^{1/2}\delta_{m'm-1}
\end{aligned} \tag{4.3.4}$$

where m, m' take on integer values between $-j$ and j. Now we take the same contraction as above, but at the same time we let $j \to \infty$, such that

$$\lim_{\substack{\epsilon\to 0\\ j\to\infty}} \epsilon j = K \tag{4.3.5}$$

where K is a finite constant. In this limit we get for the eigenvalues of the new generators:

$$\begin{aligned}
\langle m'|Y_3|m\rangle &= m\delta_{mm'} \\
\langle m'|Y_2|m\rangle &= -\frac{K}{2}(\delta_{m'm+1} + \delta_{m'm-1}) \\
\langle m'|Y_1|m\rangle &= -\frac{iK}{2}(\delta_{m'm+1} - \delta_{m'm-1})
\end{aligned} \tag{4.3.6}$$

where m, m' now take on integer values between $-\infty$ and $+\infty$. By checking the commutation relations one can show that these elements form a faithful unitary representation of E_2. In the limit, the squared operator J^2 eigenvalue equation

$$\langle m'|J^2|m\rangle = j(j+1) \tag{4.3.7}$$

takes on the following form:

$$\langle m'|({Y_1}^2 + {Y_2}^2)|m\rangle = K^2 \tag{4.3.8}$$

This completes the analysis of the contraction of the unitary representations of $SO(3)$ to the faithful unitary representations of E_2. Let us take a look at another example.

4.3.2 $SO(4,1) \to$ Poincaré

We shall now look at the contraction of the group of isometries of de Sitter space. As we take the limit $H \to 0$ in the definition (3.6.1) of this space, we end up with flat 3+1 spacetime. Therefore we expect this group to contract to the group of isometries of flat spacetime; the Poincaré group. First we will use the Wigner-Inönü method to make it plausible, then we will proof it by using the second method.

The generators of the $SO(4,1)$ group are given by (3.6.3). Consider the generators of the one-parameter subgroup of hyperbolic rotations in the $(i,4)$ plane: K_{i4}, where $i = 0,1,2,3$. The remaining six generators span the Lorentz group $SO(3,1)$ which corresponds to the subgroup S in the theorem of Wigner and Inönü. We can contract the four K_{i4} generators and they will form an Abelian invariant subgroup A of the contracted group G'. We can identify A with the subgroup of translations due to its Abelian nature. Now, since we know that $S \sim G'/A$, it is clear that G' is the group of translations and Lorentz transformations, i.e. the Poincaré group.

Next we follow the proof given in [36] that the Poincaré group is indeed a contraction of $SO(4,1)$. The structure is as follows: firstly we construct a neighbourhood V of the identity e, then we define a family of mappings f_ϵ, and finally we show that these mappings satisfy the four conditions given in the definition of a contraction. Let

$$A' = \left\{ p | p \in G,\ p = e^{i\alpha^j K_{j4}} \right\} \tag{4.3.9}$$

where $j = 0, 1, 2, 3$. K_{j4} again are the non-compact generators of the hyperbolic rotations in the $(j, 4)$ plane. Let

$$V' = \{ g | g \in G,\ g = k \cdot p,\ k \in SO_0(3,1),\ p \in A' \} \tag{4.3.10}$$

then there exists $\Delta > 0$ such that if

$$V = \{ g | g \in V',\ g = k \cdot p(\alpha),\ -\Delta < \alpha_j < \Delta \} \subset V' \tag{4.3.11}$$

then

$$V^2 \subset V' \tag{4.3.12}$$

As a family of functions we take

$$f_\epsilon(k \cdot p(\alpha)) = k \cdot r\left(\frac{1}{\epsilon}\alpha\right) \tag{4.3.13}$$

where $r(\alpha)$ is a translation. Condition 1), 2) and 3) are clearly satisfied. Checking condition 4) requires some more work:

$$\begin{aligned}
&\lim_{\epsilon \to 0} f_\epsilon(f_\epsilon^{-1}(k^{(1)} \cdot r(\alpha^{(1)})) \cdot f_\epsilon^{-1}(k^{(2)} \cdot r(\alpha^{(2)}))) \\
&= \lim_{\epsilon \to 0} f_\epsilon(k^{(1)} \cdot p(\epsilon\alpha^{(1)}) \cdot k^{(2)} \cdot p(\epsilon\alpha^{(2)})) \\
&= \lim_{\epsilon \to 0} f_\epsilon(k^{(1)} \cdot k^{(2)} \cdot (k^{(2)})^{-1} \cdot p(\epsilon\alpha^{(1)}) \cdot k^{(2)} \cdot p(\epsilon\alpha^{(2)}))
\end{aligned} \tag{4.3.14}$$

Let us denote the algebra of $SO(3,1)$ by K and $\{M_{j4}, j = 0, 1, 2, 3\}$ by P. We know that $[P, K] \subset P$ and $[P, P] \subset K$. This means that we can make the approximation, up to first order in ϵ:

$$\begin{aligned}
(k^{(2)})^{-1} \cdot p(\epsilon\alpha^{(1)}) \cdot k^{(2)} \cdot p(\epsilon\alpha^{(2)}) &\approx p(\epsilon k_{ji}^{(2)} \alpha_j^{(1)}) \cdot p(\epsilon\alpha^{(2)}) \\
&\approx p(\epsilon k_{ji}^{(2)} \alpha_j^{(1)} + \epsilon\alpha_i^{(2)})
\end{aligned} \tag{4.3.15}$$

where

$$\epsilon k_{ji}^{(2)} \alpha_j^{(1)} \equiv (\epsilon k_{j0}^{(2)} \alpha_j^{(1)}, \epsilon k_{j1}^{(2)} \alpha_j^{(1)}, ..., \epsilon k_{j3}^{(2)} \alpha_j^{(1)}) \tag{4.3.16}$$

We can now combine (4.3.14) and (4.3.15) to arrive at

$$\begin{aligned}
\lim_{\epsilon \to 0} f_\epsilon(f_\epsilon^{-1}(k^{(1)} \cdot r(\alpha^{(1)})) \cdot f_\epsilon^{-1}(k^{(2)} \cdot r(\alpha^{(2)}))) &= k^{(1)} \cdot k^{(2)} \cdot r(k_{ji}^{(2)} \alpha_j^{(1)} + \alpha_i^{(2)}) \\
&= k^{(1)} \cdot r(\alpha^{(1)}) \cdot k^{(2)} \cdot r(\alpha^{(2)})
\end{aligned} \tag{4.3.17}$$

which completes the proof that the Poincaré group is a contraction of the de Sitter group $SO(4,1)$.

The procedure of contracting the representations of $SO(4,1)$ is more involved. We will not give the calculations here; we will just state the results for the different series, and where the derivations can be found.

Massive limits

In general, the unitary irreps $U_{p,\nu}$ from the principal series contract to the direct sum of positive and negative energy time-like unitary irreps of the Poincaré group [36]:

$$U_{p,\nu} \to c_+\mathcal{P}^+(m,s) \oplus c_-\mathcal{P}^-(m,s) \tag{4.3.18}$$

as $H \to 0$ and $\nu \to \infty$, while $\nu H = m$, the Poincaré mass, is kept constant. It can be shown that for certain manifestations of these UIRs one of the constants c_+ and c_- can be set to unity, while the other one vanishes (see e.g. [37]).

Massless limits

Next we consider the irreps which contract to light-like, i.e. massless, Poincaré UIRs. These light-like UIRs have the property that they can be *uniquely extended* (we will explain this point) to UIRs of the so called *conformal group*: $SO(4,2)$ [38], whose double-covering group can be shown to be $SU(2,2)$. Let us first review some properties of these groups.

The generators of $SO(4,2)$ are given by:

$$K_{AB}, \quad A, B = 0, 1, 2, 3, 4, 5. \tag{4.3.19}$$

They obey the by now familiar commutation relations:

$$[K_{AB}, K_{CD}] = i\,(\eta_{AD}K_{BC} - \eta_{AC}K_{BD} + \eta_{BC}K_{AD} - \eta_{BD}K_{AC}) \tag{4.3.20}$$

where $\eta_{AB} = \mathrm{diag}(1, -1, -1, -1, -1, 1)$. The group has two $SO(4,1)$ subgroups; one with generators

$$K_{ab}, \quad a, b = 1, 2, 3, 4, 5, \tag{4.3.21}$$

and one with generators

$$K_{\alpha\beta}, \quad \alpha, \beta = 0, 1, 2, 3, 4. \tag{4.3.22}$$

The Poincaré group too is a subgroup of $SO(4,2)$. Its algebra is spanned by the generators

$$K_{\mu\nu}, \quad P_\mu \equiv K_{\mu 4} + K_{\mu 5}, \tag{4.3.23}$$

where $\mu, \nu = 0, 1, 2, 3$. The operator K_{45} is called the dilation operator.

We will not give the complete classification of the UIRs of the conformal group, since it is more involved than for $SO(4,1)$, and we will not need it. For our purposes it is sufficient to know that the UIRs are constructed in a similar way as we did for the Poincaré group; by using the method of induced representations. For the conformal group we can use the maximal compact subgroup $SU(2) \times SU(2) \times U(1)$ for inducing the representations. This means that the UIRs can be labeled by (E_0, j_1, j_2), where $j_1, j_2 \in \mathbb{N}/2$ label the $SU(2)$ parts, and $E_0 \in \mathbb{R}$ (the conformal energy) labels the $U(1)$ part. The UIRs will be denoted by $\mathcal{C}(E_0, j_1, j_2)$. For a complete classification see [39, 40, 41].

Now, let D be some representation of $SU(2,2)$, such that the restriction $D|\mathcal{P}$ of D to the Poincaré group is unitary and irreducible. It can be shown [42, 43] that in that case D is in fact unitary and irreducible, and its form is uniquely determined by $D|\mathcal{P}$. This is what is meant by saying that a UIR $D|\mathcal{P}$ of the Poincaré group can be uniquely extended to a UIR D of the conformal group. It has been shown that only the light-like UIRs of the Poincaré group can be uniquely extended to UIRs of the conformal group (see [42] and references therein).

It turns out that the extension of the light-like Poincaré UIRs can be restricted to the de Sitter group $SO(4,1)$, and that the obtained UIRs of $SO(4,1)$ can then be contracted to the Poincaré group, yielding the same UIRs that we started out with [38]. Not all de Sitter UIRs are taking part in this process; only one particular UIR of the complementary series with $(p,q) = (0,1)$, and a certain class of UIRs from the discrete series; namely the ones for which $p = q$. We will call these the massless de Sitter UIRs, and we summerize the results in the following scheme, where '$\hookrightarrow$' indicates unique extendability:

$$V_{0,1/2} \quad \hookrightarrow \quad \begin{array}{c} \mathcal{C}(1,0,0) \\ \oplus \\ \mathcal{C}(-1,0,0) \end{array} \quad \xrightarrow{H\to 0} \quad \begin{array}{ccc} \mathcal{C}(1,0,0) & \hookleftarrow & \mathcal{P}^{+}(0,0) \\ \oplus & & \oplus \\ \mathcal{C}(-1,0,0) & \hookleftarrow & \mathcal{P}^{-}(0,0) \end{array}$$

$$\Pi^{(+)}_{s,s} \quad \hookrightarrow \quad \begin{array}{c} \mathcal{C}(s+1,s,0) \\ \oplus \\ \mathcal{C}(-s-1,s,0) \end{array} \quad \xrightarrow{H\to 0} \quad \begin{array}{ccc} \mathcal{C}(s+1,s,0) & \hookleftarrow & \mathcal{P}^{+}(0,s) \\ \oplus & & \oplus \\ \mathcal{C}(-s-1,s,0) & \hookleftarrow & \mathcal{P}^{-}(0,s) \end{array}$$

$$\Pi^{(-)}_{s,s} \quad \hookrightarrow \quad \begin{array}{c} \mathcal{C}(s+1,0,s) \\ \oplus \\ \mathcal{C}(-s-1,0,s) \end{array} \quad \xrightarrow{H\to 0} \quad \begin{array}{ccc} \mathcal{C}(s+1,0,s) & \hookleftarrow & \mathcal{P}^{+}(0,-s) \\ \oplus & & \oplus \\ \mathcal{C}(-s-1,0,s) & \hookleftarrow & \mathcal{P}^{-}(0,-s) \end{array}$$

The explanation for the fact that the de Sitter UIRs extend to two conformal group UIRs, while the Poincaré UIRs extend to only one can be found in [44, 43]. Note that the $(\pm)$ label on the discrete series of $SO(4,1)$ indicates the sign of the 'helicity', while the $\pm$ label on the Poincaré UIRs indicates the sign of the energy.
All other $SO(4,1)$ UIRs either contract to the space-like Poincaré UIRs, or have no contraction limit at all. We will not discuss them here.

Chapter 5

Link between Quantum Field Theory and Group Theory

In this chapter we will investigate the connection between group theory and quantum field theory. We are assuming that the reader is familiar with (relativistic) quantum mechanics and the basics of quantum field theory. Nevertheless, in the first section we will recapitulate the basics of the quantization of the scalar field in flat 3+1 Minkowski spacetime in order to make the generalization to (constantly) curved spacetime more insightful. Next we will link the different notions of quantum theory (such as *wave functions*, *field operators*, *particle states*, *wave equations*, etc.) to group theoretical concepts (such as invariance, UIRs, etc.). In this way we hope to make clear how very important physical notions such as *mass* and *spin*, which often seem to be 'axiomatic' starting points in field theory, are actually very closely linked (emergent almost) to properties of the isometry group of the spacetime the field is described in. However, this link between properties of the isometry group and physical notions gets obscured when one is trying go make the generalization to field theory in curved spacetime. One problem is that due to the absence of a global time-like Killing vector it is not possible to define a state which is interpreted as 'the vacuum' for all inertial observers, which might undermine the concept of a particle. Another problem is that there is an ambiguity in introducing the mass in the field theory: the field can have an arbitrary coupling to the gravitational field (i.e. curvature) and this coupling term has the same dimensions as the mass term, which makes them hard (if not impossible) to distinguish. We will look at some possible resolutions, or circumventions to these problems, in which group theory will play an important role.

5.1 Flat spacetime

5.1.1 Quantization of the scalar field

We begin with a short recapitulation of the basics of the (second/canonical) quantization of the scalar field. We follow the book by Birrell and Davies [45] which gives a nice summary. More details can be found in [46] or any other standard text on quantum field theory.
We start with the Lagrangian density for a free scalar field $\phi(x)$:

$$\mathcal{L}(x) = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right) \tag{5.1.1}$$

We then construct the action

$$S = \int d^4x\mathcal{L}(x) \tag{5.1.2}$$

and demand that the variation in the action with respect to small variations in ϕ vanishes:

$$\delta S = 0. \tag{5.1.3}$$

This leads to the Klein-Gordon equation:

$$(\Box + m^2)\phi = 0 \tag{5.1.4}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and m will be the mass of the field quanta when the theory is quantized. One set of solutions to this equation is given by the *plane wave solutions*, or *modes* (we left out the normalization factor; it is a matter of convention):

$$u_{\mathbf{p}}(x) \propto e^{-ipx} \tag{5.1.5}$$

where

$$p^0 = \sqrt{\mathbf{p}^2 + m^2} \tag{5.1.6}$$

We call these modes *positive frequency* or *positive energy* with respect to time. They are eigenfunctions of the operator $i\partial_0$:

$$i\partial_0 u_{\mathbf{p}}(x) = p^0 u_{\mathbf{p}}(x) \tag{5.1.7}$$

where $p^0 > 0$. Note that the operator $i\partial_0$ is in fact the time-like Killing vector, associated with infinitesimal translations in the time direction. Negative energy modes are also solutions to (5.1.4). They are the complex conjugates of the positive energy modes and can be distinguished from them by inspecting the eigenvalue of the time-like Killing vector.

Let us define a scalar product:

$$\begin{aligned}(\phi_1, \phi_2) &= -i\int d^3x\{\phi_1(x)\partial_0\phi_2^*(x) - [\partial_0\phi_1(x)]\phi_2^*(x)\} \\ &= -i\int d^3x \phi_1(x)\overleftrightarrow{\partial_0}\phi_2^*(x)\end{aligned} \tag{5.1.8}$$

where the integration is performed over a spacelike hyperplane of constant time t. With the proper normalization the modes (5.1.5) satisfy:

$$(u_{\mathbf{p}}, u_{\mathbf{p}'}) = \delta^3(\mathbf{p} - \mathbf{p}') \tag{5.1.9}$$

We can now start with the actual quantization. We interpret the field $\phi(x)$ and its canonical conjugate $\pi = \partial_0\phi$ as operators and impose the equal time commutation relations:

$$\begin{aligned}&[\phi(x), \phi(y)] = 0 \\ &[\pi(x), \pi(y)] = 0 \\ &[\phi(x), \pi(y)] = i\delta^3(\mathbf{x} - \mathbf{y})\end{aligned} \tag{5.1.10}$$

Since the positive and negative energy modes together form a complete orthonormal basis with respect to the scalar product (5.1.8), we can expand the field operator as a linear combination of the modes:

$$\phi(x) = \sum_{\mathbf{p}}[a_{\mathbf{p}}u_{\mathbf{p}}(x) + a_{\mathbf{p}}^\dagger u_{\mathbf{p}}^*(x)] \tag{5.1.11}$$

Then the equal time commutation relations for ϕ and π reduce to:

$$\begin{aligned}&[a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0 \\ &[a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0 \\ &[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = \delta_{\mathbf{p}\mathbf{p}'}\end{aligned} \tag{5.1.12}$$

We can construct the Hilbert space of quantum states by starting with the vacuum state, which has the defining property:

$$a_{\mathbf{p}}\,|0\rangle = 0, \quad \forall \mathbf{p} \tag{5.1.13}$$

(where $a_{\mathbf{p}}$ is called the *annihilation operator*) and repeatedly acting on it with the so called *creation operator*:

$$|\mathbf{p}\rangle = a^{\dagger}_{\mathbf{p}}\,|0\rangle \tag{5.1.14}$$

5.1.2 Interpretation of the particle states

Next, let us investigate how these particle states $|\mathbf{p}\rangle$ are linked to group theory. The notation of section 3.5.4 already implied that there might be a link between these states and UIRs of the Poincaré group, and it turns out that they are exactly the basis vectors of the representation space of the time-like Poincaré UIRs with spin 0. The following paragraphs are meant to make this plausible, by showing what happens when we apply a translation $T(b)$ (that is: $x^\mu \to x^\mu + b^\mu$) to a wave function $\Phi(x)$ of a particle state $|\Phi\rangle$. The wave function can be expressed as a matrix element of the field operator:

$$\Phi(x) = \langle 0|\phi(x)|\Phi\rangle \tag{5.1.15}$$

We know that for some definite momentum $\mathbf{p}$ the wave function will be of the form of $u_{\mathbf{p}}(x)$, as we can derive using the formula's given above[1]:

$$\begin{aligned}
\langle 0|\phi(x)|\mathbf{p}\rangle &= \sum_{\mathbf{p}'} \langle 0|[a_{\mathbf{p}'}u_{\mathbf{p}'}(x) + a^{\dagger}_{\mathbf{p}'}u^{*}_{\mathbf{p}'}(x)]a^{\dagger}_{\mathbf{p}}|0\rangle \\
&= \sum_{\mathbf{p}'} \Big(\langle 0|a_{\mathbf{p}'}u_{\mathbf{p}'}a^{\dagger}_{\mathbf{p}}|0\rangle + \langle a^{\dagger}_{\mathbf{p}'}u^{*}_{\mathbf{p}'}a^{\dagger}_{\mathbf{p}}|0\rangle\Big) \\
&= \sum_{\mathbf{p}'} \Big(\langle 0|u_{\mathbf{p}'}\delta_{\mathbf{p}\mathbf{p}'}|0\rangle - \langle 0|u_{\mathbf{p}'}a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}'}|0\rangle + 0\Big) \\
&= u_{\mathbf{p}}(x)
\end{aligned} \tag{5.1.16}$$

Now let's apply the translation. Assuming that $|\mathbf{p}\rangle$ is indeed a basis vector for the representation space of section 3.5.4, we know how it will transfrom, namely as:

$$T(b)\,|\mathbf{p}\rangle = e^{ibp}\,|\mathbf{p}\rangle \tag{5.1.17}$$

The field operator will simply have a shift in the coordinate. Let's see what happens:

$$\begin{aligned}
\Phi'(x+b) &= \langle 0|\phi(x+b)|\Phi'\rangle \\
&= \langle 0|\phi(x+b)e^{ibp}|\mathbf{p}\rangle \\
&= \sum_{\mathbf{p}'} \langle 0|u_{\mathbf{p}'}(x+b)\delta_{\mathbf{p}\mathbf{p}'}e^{ibp}|0\rangle \\
&= u_{\mathbf{p}}(x+b)e^{ibp}
\end{aligned} \tag{5.1.18}$$

where we have left out some steps we did explicitly in (5.1.16). Recall that

$$u_{\mathbf{p}}(x+b) \propto e^{-ip(x+b)} \tag{5.1.19}$$

so we can conclude:

$$\Phi'(x+b) = u_{\mathbf{p}}(x) = \Phi(x). \tag{5.1.20}$$

i.e. that the wave function is invariant under translations, as we expected. Note that this is due to the fact that the particle state $|\mathbf{p}\rangle$ is a basis vector of the representation space of the time-like UIR of the Poincaré group.

[1] We implicity assume that the vacuum state is normalized, i.e. $\langle 0|0\rangle = 1$.

5.1.3 Relativistic wave functions, field operators and wave equations

In this section, following [22], we will give a more general description of the group theoretical aspects of the different mathematical objects we encountered when we were dealing with the free scalar field. The wave function we encountered there is the simplest example of this class of mathematical objects. Let us give a general, group theoretical definition here, followed by some examples. A wave function $\{\Psi^\alpha(x)\}$ is a set of n complex-valued functions of spacetime which transform under a Lorentz transformation Λ as follows:

$$\Psi'^\alpha(x') = D[\Lambda]^\alpha{}_\beta \Psi^\beta(x) \tag{5.1.21}$$

where $D[\Lambda]$ is an n-dimensional matrix representation of $SO(3,1)$ and $x' = \Lambda x$. Let us review some examples of the relativistic wave function:

- *The Klein-Gordon wave function.* This is the one from the previous section. Again, it's the simplest example; it has one component, and it transforms as the $(u,v) = (0,0)$ irreducible representation of $SO(3,1)$.
- *The electromagnetic field tensor $F^{\mu\nu}(x)$.* It transforms as the $(1,0) \oplus (0,1)$ representation of $SO(3,1)$.
- *The vector potential $A^\mu(x)$.* It describes a spin-1 particle; the vector boson. This function transforms as the $(1/2, 1/2)$ representation.
- *The Dirac wave function.* It has four components and is a solution to the Dirac equation. It transforms as the $(0,1/2) \oplus (1/2,0)$ representation.

Next, consider the relativistic field operator $\{\psi^\alpha(x)\}$. It is a set of n operator valued functions of spacetime, transforming under a Lorentz transformation Λ as follows:

$$T[\Lambda]\psi^\alpha(x)T[\Lambda^{-1}] = D[\Lambda^{-1}]^\alpha{}_\beta \psi^\beta(\Lambda x) \tag{5.1.22}$$

where $D[\Lambda]$ again is an n-dimensional matrix representation of $SO(3,1)$, and $T[\Lambda]$ is an operator representing the transformation Λ, and it is acting on the same Hilbert space as ψ. The second-quantized versions of the wave functions given above are examples of relativistic field operators.

In order for the field operator ψ to represent a particle with mass m and spin s, it must satisfy a wave equation. We already encountered one example: the Klein-Gordon equation for spin-0 particles:

$$(\Box + m^2)\phi(x) = 0, \tag{5.1.23}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, and we assume that the reader is familiar with the famous equation for spin-1/2 particles, the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0. \tag{5.1.24}$$

Note that we have omitted the Lorentz indices of the γ^μ's and ψ. More general, we can write the differential equation as

$$\Pi(m,\partial)^\alpha{}_\beta \psi^\beta(x) = 0 \tag{5.1.25}$$

where Π is a linear differential operator, and a matrix with respect to the Lorentz indices. We can take the Fourier tranform of $\psi(x)$: $\widetilde{\psi}(p)$, and convert (5.1.25) into an algebraic equation:

$$\Pi(m,p)\widetilde{\psi}(p) = 0 \tag{5.1.26}$$

where we have ommited the Lorentz indices. One can check that $\widetilde{\psi}(p)$ has the same transformation properties as $\psi(x)$. We have to put some restrictions on $\Pi(m,p)$ in order for it to represent a suitable relativistic wave equation for particles of mass m and spin s. First of all, it must be relativistically covariant, i.e. we want

$$\Pi(m,\Lambda p)\widetilde{\psi}(\Lambda p)=0 \tag{5.1.27}$$

to hold. This implies that $\Pi(m,p)$ must transform under a Lorentz transformation as

$$T[\Lambda]\Pi(m,p)T[\Lambda^{-1}]=\Pi[m,\Lambda p] \tag{5.1.28}$$

Secondly, we want the mass-shell condition $p^2=m^2$ to hold. Thirdly and finally, if $\psi(p)$ transforms as the (u,v) representation of $SO(3,1)$, then its spin content is $|u-v|\leq j\leq u+v$.

We will now consider the plane wave expansion of the field operator ψ. For brevity we will only focus on the positive energy part. Consider the rest frame of a particle of mass m. In that case $p=p_t=(m,\mathbf{0})$. For a spin-s particle, we want the positive energy part of equation (5.1.26) to have $2(s+1)$ independent solutions, corresponding to the $\lambda=-s,-s+1,...,s$ distinct spin states of the particle in rest. Let us denote these solutions[2] by $u(\mathbf{0}\lambda)$:

$$\Pi(m,p_t)^{\alpha}{}_{\beta}u^{\beta}(\mathbf{0}\lambda)=0 \tag{5.1.29}$$

for $\lambda=-s,...,s$. The elementary solutions can now be given by boosting the frame, just as we did for p_t in section 3.5.4:

$$u^{\alpha}(\mathbf{p}\lambda)=D[H(p)]^{\alpha}{}_{\beta}u^{\beta}(\mathbf{0}\lambda) \tag{5.1.30}$$

and by using (5.1.28) we can easily check that these are indeed solutions. We call them *plane wave solutions*, and they are concrete realizations of the basis states $\{|\mathbf{p}\lambda\rangle\}$ of the time-like Poincaré UIRs from section 3.5.4. Note that these plane wave solutions are labeled by a Lorentz index α (whose range is determined by the Lorentz representation label (u,v)), and the Poincaré label $(\mathbf{p},\lambda)$ (where m from the Poincaré UIR label (m,s) is used to define the mass-shell condition for p_t, and s determines the range of λ). The plane wave expansion of the field operator is given by:

$$\psi^{\alpha}(x)=\sum_{\lambda}\int\tilde{dp}[a(\mathbf{p}\lambda)u^{\alpha}(\mathbf{p}\lambda)e^{-ipx}+\text{negative energy term}] \tag{5.1.31}$$

where $\tilde{dp}$ is an invariant measure on the mass-shell, the e^{-ipx} factor is present due to the Fourier transform and $a(\mathbf{p}\lambda)$ is an operator-valued expansion coefficient, readily identified as the annihilation operator.

The plane wave expansion provides a link between representations of the Lorentz and the Poincaré group. The field operator $\psi^{\alpha}(x)$ transforms as a certain finite dimensional non-unitary representation of the Lorentz group. The annihilation operators $a(\mathbf{p}\lambda)$ transform as a UIR of the Poincaré group, labeled by (m,s). The connection between these two is provided by the plane wave solutions (modes) $u^{\alpha}(\mathbf{p}\lambda)e^{-ipx}$, which carry both Lorentz and Poincaré labels.
The fact that the field operators are in a non-unitary representation is not a problem; generaly they do not correspond to physical observables. The particle states however do correspond to physical states, and we see that they are indeed in unitary representations.

We will now try to generalize these results to a field theory in curved spacetime.

[2] Note that here we are working in momentum space, in contrast to the modes in section 5.1.1.

5.2 Curved spacetime

Just as in the flat case, let us first look at the free scalar field quantization, and see where the differences are.

5.2.1 Quantization of the scalar field

Again, we follow [45]. For now, we do not specify which curved spacetime we are dealing with, since there are certain issues that are inherent to field theories in all curved spacetimes.

Let us start with the Lagrangian density:

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g(x)}\{g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - [m^2 + \xi R(x)]\phi^2(x)\} \tag{5.2.1}$$

where $g \equiv |\det g_{\mu\nu}|$, ϕ is the scalar field, m is the mass of the field quanta, R is the Ricci scalar and ξ is a numerical factor. Note that the main difference with (5.1.1) is the coupling to the gravitational field; the term $\xi R\phi^2$. It is the only possible term of this kind that has the correct dimensions. We define an action, apply the variational principle, and arrive at the field equation:

$$(\Box_x + m^2 + \xi R)\phi = 0 \tag{5.2.2}$$

where

$$\Box_x\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = (-g)^{-\frac{1}{2}}\partial_\mu\left[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\phi\right] \tag{5.2.3}$$

We want to draw some attention to the coupling constant ξ. Usually, two distinct values are said to be of particular interest. The first one is where $\xi = 0$; it is called the *minimally coupled case*. The other one is where:

$$\xi = \frac{1}{4}\left(\frac{n-2}{n-1}\right) \tag{5.2.4}$$

where n is the dimension of the spacetime; it is called the *conformally coupled case*. The name stems from the fact that if ξ has this value and $m = 0$ in (5.2.2), then the wave equation is invariant under conformal transformations[3]. Note though that there is an ambiguity here; it is not clear how to distinguish between $m^2\phi$ and $\xi R\phi$. For that reason some authors choose to work with an 'effective' mass: $\widetilde{m}^2 \equiv m^2 + \xi R$, but then the notions of 'minimally coupled' and 'conformally coupled' become somewhat obscured. We will come back to this important point in section 5.4.2.

It is possible to define a generalization of the scalar product (5.1.8). It uses the notion of a *Cauchy surface*. Intuitively, it is a hyperplane in spacetime which can be viewed as an instant of time. The product is defined as:

$$(\phi_1, \phi_2) = -i\int \phi_1(x)\overleftrightarrow{\partial_\mu}\phi_2^*(x)[-g_\Sigma(x)]^{\frac{1}{2}}d\Sigma^\mu \tag{5.2.5}$$

where $d\Sigma^\mu = n^\mu d\Sigma$ with n^μ being the future-directed unit vector orthogonal to the spacelike (Cauchy) hypersurface Σ and $d\Sigma$ is the volume element in Σ. It can be shown that the value of the scalar product is independent of Σ [45].

Consider a complete set of elementary solutions to (5.2.2) that is orthonormal with respect to (5.2.5):

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0 \tag{5.2.6}$$

[3] A conformal transformation is a rescaling of the metric by some continuous, non-vanishing, finite, real function $\Omega(x)$: $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

where the indices schematically represent the labels that must be used to characterize these modes. Since the set is complete, we can use it to make an expansion of the field operator $\phi(x)$, just as in (5.1.11):

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*] \tag{5.2.7}$$

We quantize the theory by implementing the familiar commutation relations:

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij} \tag{5.2.8}$$

Then the vacuum and particle states are constructed in the same way as before: $a_i \left|0\right\rangle = 0$ and $\left|1_i\right\rangle = a_i^\dagger \left|0\right\rangle$, where $\left|1_i\right\rangle$ represents a one-particle state characterized by whatever the label i stands for. This completes the quantization of the field. However, there is a inherent ambiguity in choosing the modes in (5.2.6).

5.2.2 Bogoliubov transformations and different vacua

Consider another complete set of orthonormal solutions to (5.2.2): $\bar{u}_j(x)$. Again, we may expand the field operator:

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)] \tag{5.2.9}$$

In this way we can define a new vacuum state:

$$\bar{a}_j \left|\bar{0}\right\rangle, \quad \forall j \tag{5.2.10}$$

Since both sets are complete orthonormal sets, we can expand the new modes in terms of the old ones, and the other way around:

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*) \tag{5.2.11a}$$

$$u_i = \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*) \tag{5.2.11b}$$

These relations are called *Bogoliubov transformations*, and α_{ij}, β_{ij} are the Bogoliubov coefficients. From the transformation relations we can deduce that:

$$\alpha_{ij} = (\bar{u}_i, u_j), \quad \beta_{ij} = -(\bar{u}_i, u_j^*) \tag{5.2.12}$$

We will state some more relations connected to the Bogoliubov transformations, which can be derived by equating the different expansions of the field operators and using the properties of the different modes given above. For the annihilation operators one finds:

$$a_i = \sum_j (\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger) \tag{5.2.13a}$$

$$\bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger) \tag{5.2.13b}$$

and for the coefficients:

$$\sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij} \tag{5.2.14a}$$

$$\sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) = 0 \tag{5.2.14b}$$

Now, note that the two different Hilbert spaces based on the two diffent choices of modes are distinct whenever $\beta_{ji} \neq 0$; for example the annihilation operator of (5.2.13a) does not annihilate the 'barred' vacuum:

$$a_i |\bar{0}\rangle = \sum_j \beta_{ji}^* |\bar{1}_j\rangle \neq 0 \tag{5.2.15}$$

or, to take this one step further, consider the expectation value of the operator $N_i = a_i^\dagger a_i$ indicating the number of u_i modes in the state $|\bar{0}\rangle$:

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \tag{5.2.16}$$

which means that the vacuum of the $\bar{u}_j$ modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode.

The absence of a unique vacuum is not only the case in curved spacetime: the same story is applicable to the Minkowski case. However, there we have a natural set of modes which are closely related to the natural rectangular coordinate system (t, x, y, z), which in turn is associated with the Poincaré group. Minkowski vacua that are defined by $\bar{u}_j$ modes which mix positive en negative energy natural u_i modes are not Poincaré invariant. Bogolubov transformations that do not mix positive and negative energy modes (i.e. $\beta_{ji} = 0$) are called *trivial* or *non-mixing*.

The natural modes can be identified because of the presence of a time-like Killing vector; it is orthogonal to the spacelike hypersurfaces used to define the scalar product, which in turn is used to establish the orthonormality of the modes. These modes are also eigenfunctions of this Killing vector, which is closely related to the positivity of the energy of the modes.

On the contrary, in curved spacetime one usually does not have a time-like Killing vector, leading to a more pressing ambiguity in choosing the modes for defining the vacuum.

5.2.3 De Sitter space and α-vacua

Let us now see how this works in de Sitter space. It is one of the maximally symmetric curved spacetimes; there are 10 Killing vectors. The only other spacetimes which have this level of symmetry are the anti-de Sitter space (AdS) and our familiar Minkowski spacetime (note that the Poincaré group has 10 generators). However, among the Killing vectors of de Sitter space there is no global time-like Killing vector, so we have to deal with some of the complications discussed in the previous section. Fortunately, due to the large amount of symmetry, the choice of modes (and thus vacuum) will not be as ambiguous as in the general curved case.

In the following, we will not give complete mathematical descriptions of all the steps that are being taken, since they tend to be quite tedious. As concise as we can, we will try to give the references to where the calculations can be found in detail. Another note: we will only be concerned with de Sitter invariant vacua, i.e. vacua that are invariant under the transformations of $SO(4,1)$.

For convenience, let us again state the definition of de Sitter space as an embedded hyperboloid:

$$\mathfrak{M}_{dS} = \{x \in \mathbb{R}^5 |\ x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\} \tag{5.2.17}$$

where $\alpha, \beta = 0, 1, 2, 3, 4$ and $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, -1)$. The conventional choice for the natural modes is the one defining the *Euclidean vacuum* (sometimes referred to as the *Bunch-*

Davies vacuum). The modes are found by first making the analytical continuation[4] from de Sitter space dS_4 to the 4-sphere S^4, then the two-point function[5] is found on the sphere and the final step is to analytically continue this function back to de Sitter space (see e.g. [47]).

Starting from the modes, one can define a set of de Sitter invariant vacua with the use of Bogolubov transformations. This was carried out in [48], and later described in e.g. [6]. Let u_i be the modes corresponding to the Euclidean vacuum. The new modes are defined by a specific Bogolubov transformation:

$$\bar{u}_i = Au_i + Bu_i^* \tag{5.2.18}$$

The first property of this transformation is that the complex Bogolubov coefficients A and B are independent of the specific modes in a set (compare with (5.2.11a), but then $j = i \ \forall j$ and the sum disappears). It assures that the new vacuum will be de Sitter invariant [48]. The second property is that it preserves orthonormality of the modes, so that it will define a new vacuum. We have

$$\begin{aligned} (\bar{u}_i, \bar{u}_j) &= (|A|^2 - |B|^2)(u_i, u_j) \\ &= (|A|^2 - |B|^2)\delta_{ij} \end{aligned} \tag{5.2.19}$$

so that we find the condition for A and B:

$$|A|^2 - |B|^2 = 1 \tag{5.2.20}$$

The most general solution for complex A and B gives for the new modes (an overall phase factor is discarded, since it is of no physical interest):

$$\bar{u}_i = (\cosh\alpha)u_i + (e^{i\beta}\sinh\alpha)u_i^* \tag{5.2.21}$$

where the range of the real parameters is given by $\alpha \in [0, \infty)$, $\beta \in (-\pi, \pi)$. The Euclidean vacuum corresponds to $\alpha = 0$. It can be shown [6] that all vacuum states with $\beta \neq 0$ are not time-reversal invariant. For that reason the de Sitter vacuum states are usually said to be a one-real-parameter set of vacua, known as *α-vacua*.

Let us now focus our attention on some more group theoretical issues.

5.2.4 The role of group theory

Recall that in the flat spacetime case the particle states are labeled by the Poincaré label (p, λ), whose values are closely related to the UIR label (m, s); m is used to define the mass-shell condition and s determines the range of λ. We identified m with the mass of the field quanta, and s with the spin. In a group theoretical sense, we associate these parameters with the eigenvalues of the Casimir operators (see section 3.5). Since the notion of mass is not as clear in de Sitter space as in Minkowski space in a field theoretical sense, we may ask ourselves if the Casimir operators of $SO(4, 1)$ might bring resolvement to the issue. There is a very straightforward link between the Casimir operators and the wave equation, which we shall investigate by using the *ambient space formalism* in the next section. That is also where we will consider fields with spin content in de Sitter space; something that we did not bother ourselves with yet.

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[4] This is done by making the coordinate x^0 purely imaginary; it is clear that (5.2.17) then defines a sphere with radius R.

[5] The two-point correlation function, i.e. Green's function, contains all information regarding the free field and its modes.

5.3 Ambient space formalism

The ambient space formalism is a very useful tool for making the link between quantum field theory in (anti-)de Sitter space and the group theoretical approach. The general idea is that we can reduce the wave equation to a Casimir eigenvalue equation. This connection was established for field theory in flat space in e.g. [49, 50], and it lead to the so called canonical form of the covariant particle equations. In this way, the connection between the Wigner UIRs of the Poincaré group and the solutions to the wave equation was made very explicit. It seems that the ambient space formalism approach has its roots in these papers and the references therein.

One of the differences between Minkowski and de Sitter space is that the isometry group of the former, Poincaré, has a clear description in 4 dimensions, while the isometry group of the latter, $SO(4,1)$, is described in 5 dimensions. For this reason we must have a mathematical framework allowing us to go from the Casimir eigenvalue equation in 5-dimensional ambient space coordinates x^α to the wave equation in 4-dimensional intrinsic de Sitter coordinates X^μ, and vice versa.

Consider a tensor field[6] $\mathcal{K}_{\eta_1 \ldots \eta_r}(x)$ on $\mathbb{R}^5$ endowed with the metric $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1,-1)$. It can be viewed as a homogeneous[7] function of arbitrary degree n. In order to make sure that $\mathcal{K}(x)$ is on $\mathfrak{M}_{dS}$ we require transversality:

$$x \cdot \mathcal{K} = 0, \quad x \in \mathfrak{M}_{dS} \tag{5.3.1}$$

Our goal is to link $\mathcal{K}(x)$ to the fields defined in de Sitter space, which shall be indicated by $h_{\mu_1 \ldots \mu_r}(X)$ and we want these fields h to carry a specific $SO(4,1)$ UIR. Consequently, we want the fields $\mathcal{K}$ to carry this specific UIR. We will see how we can make sure that $\mathcal{K}$ is in a particular UIR of choice. Let us start by looking into a particular description of the generators of $SO(4,1)$, and their action of the field $\mathcal{K}(x)$.

5.3.1 Action of the generators

The generators of $SO(4,1)$ where already described in section 3.6. Here we will modify them slightly by adding an operator to each generator. This operator is dealing with the spin[8] aspect of the field $\mathcal{K}$ [51, 52]. This is closely related to its rank, which one could compare to the Minkowski case, where objects with integer spin s are described by tensors of rank $r = s$, take for example the quanta of the gravitational field $G_{\mu\nu}$: gravitons have spin 2.

Half-integer spin s objects are described by spinor-tensors of tensor-rank $r = s - \frac{1}{2}$ and with an additional spinorial index i. In these fermionic cases one usually deals with gamma matrices (more general; the Clifford algebra), and we will shortly see where they enter the story.

Let us state the form of the slightly modified generators:

$$L_{\alpha\beta}^{(s)} = M_{\alpha\beta} + S_{\alpha\beta}^{(s)} \tag{5.3.2}$$

[6]We will often omit the tensorial indices, for brevity.

[7]A homogeneous function f of degree n has the following property: $f(ax) = a^n f(x)$. It is assumed that $\mathcal{K}(x)$ is homogeneous in order to extend the domain of definition of the physical field, so that we can give meaning to the operation ∂_α in the ambient space [2].

[8]The usage of the word 'spin' is not really justifiable, since it is not always clear how to extrapolate the Minkowskian notion of spin to curved spacetime. However, we will use the word from time to time in relation to de Sitter space, since it is conventional.

where $M_{\alpha\beta}$ stands for the 'orbital' part[9] and $S^{(s)}_{\alpha\beta}$ for the 'spinorial' part. The orbital part is of the usual form:

$$M_{\alpha\beta} = i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) \tag{5.3.3}$$

The second order Casimir operator is of the usual form:

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} \tag{5.3.4}$$

where we can isolate the purely scalar part:

$$Q^{(1)}_0 = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} \tag{5.3.5}$$

In order to specify the spinoral part we must distinguish between integer and half-integer spin fields. Integer spin-r fields will be represented by tensors of rank r, and the action of $S_{\alpha\beta}$ is (see e.g. [53]):

$$S_{\alpha\beta}\mathcal{K}_{\eta_1\ldots\eta_r} = -i\sum_i \left(\eta_{\alpha\eta_i}\mathcal{K}_{\eta_1\ldots(\eta_i\leftrightarrow\beta)\ldots\eta_r} - \eta_{\beta\eta_i}\mathcal{K}_{\eta_1\ldots(\eta_i\leftrightarrow\alpha)\ldots\eta_r}\right) \tag{5.3.6}$$

where the notation $(\eta_i \leftrightarrow \beta)$ indicates that the i^{th} index must be replaced by β. Now consider half-integer spin fields with spin $s = r + \frac{1}{2}$. The fields are represented by a four component spinor-tensor $\mathcal{K}^i_{\eta_1\ldots\eta_r}$ where $i = 1, 2, 3, 4$. The spinorial action splits up into two parts;

$$S^{(s)}_{\alpha\beta} = S_{\alpha\beta} + S^{(\frac{1}{2})}_{\alpha\beta} \tag{5.3.7}$$

where the first part is just (5.3.6) which acts on the tensorial indices. The second part is acting on the spinor index of the field, and it is defined as (see e.g. [54, 55]):

$$S^{(\frac{1}{2})}_{\alpha\beta} = -\frac{i}{4}[\gamma_\alpha, \gamma_\beta] \tag{5.3.8}$$

where the five gamma matrices are determined by the standard relations:

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha\beta}, \quad \gamma^{\alpha\dagger} = \gamma^0\gamma^\alpha\gamma^0 \tag{5.3.9}$$

5.3.2 Representation of $\mathcal{K}(x)$

We will now explain how we can make sure that $\mathcal{K}(x)$ is in some particular UIR. In order to make this more insightful we briefly switch to anti-de Sitter space, but the conclusion will be equally valid for de Sitter space. The reason for this switch is that most detailed literature on this subject deals with AdS (see for example [51, 56, 57, 58, 54, 53, 59]). Note that the only difference between AdS and dS in the description above is the metric of the ambient space[10], and a + sign in front of 'inverse radius' H of the space. An additional important difference is in the classification of the UIRs. In the AdS case the isometry group is $SO(3, 2)$ and a UIR is denoted by $D(E_0, s)$, where s is interpreted as the spin.

Now, we want to describe a field that carries the particular UIR $D(E_0, s)$. In the ambient space formalism we do this by starting with the tensor field $\mathcal{K}_{\mu_1\ldots\mu_s}(x)$. But we note something very important: the generators belong to a non-compact group and the representation space is finite dimensional (note that every tensorial index represents a 5-dimensional space). Hence, the

[9]'Orbital' as in the orbits used to define the Killing vectors, see section 1.3.3.

[10]For AdS one uses $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, 1)$.

representation of the space that the spinorial operators (5.3.6) are acting on cannot be unitary. This means that $\mathcal{K}_{\mu_1 \ldots \mu_s}(x)$ is not in a UIR. In fact, it is in the direct product representation:

$$D(E_0, 0) \otimes D(s) \tag{5.3.10}$$

where the orbital part $M_{\alpha\beta}$ of the generators is associated with $D(E_0, 0)$ and the spinoral part $S^{(s)}_{\alpha\beta}$ is associated with $D(s)$. Here $D(s)$ is a suitably chosen finite-dimensional irreducible component with highest weight $(0, s)$ contained in $\otimes^s D(1)$, where $D(1)$ is the 5-dimensional irrep of $SO(3, 2)$ [58]. Take for example a spin-1 field $\mathcal{K}_\alpha(x)$. The operator $S^{(1)}_{\alpha\beta}$ clearly acts on a 5-dimensional space spanned by $\{\mathcal{K}_\alpha(x)|\alpha = 0, 1, 2, 3, 4\}$, i.e. $\mathcal{K}_\alpha(x)$ is in the representation $D(1)$.

The point is that we can reduce the direct product (5.3.10) to a sum of irreps (see section 2.5) in the following recursive way [53]:

$$\begin{aligned} D(E_0, s-1) \otimes D(1) =& D(E_0, s) \oplus D(E_0, s-1) \oplus D(E_0, s-2) \\ & \oplus D(E_0 - 1, s-1) \oplus D(E_0 + 1, s-1) \end{aligned} \tag{5.3.11}$$

We can now write the Casimir eigenvalue equation in ambient space that will be linked to the field equation in intrinsic coordinates. What it essentially does is select the UIR $D(E_0, s)$ appearing in the decomposition of the direct product. It is given by:

$$\left(Q^{(1)} - \langle Q^{(1)} \rangle\right) \mathcal{K}(x) = 0 \tag{5.3.12}$$

where $\langle Q^{(1)} \rangle$ is the eigenvalue of the Casimir operator $Q^{(1)}$ for the specific UIR $D(E_0, s)$.

It may be clear that this will work in exactly the same way in de Sitter space. The eigenvalues of $Q^{(1)}$ that we will be using are given in section 3.6. The next section will be devoted to finding a concrete form for the operator $Q^{(1)}$ acting on $\mathcal{K}$ in the direct product representation.

5.3.3 Action of the Casimir operator

Before we can state the concrete form of $Q^{(1)}$, we must give a couple of definitions. The trace $\mathcal{K}'$ of the tensor $\mathcal{K}$ of rank r is defined as:

$$\mathcal{K}' \equiv \mathcal{K}_{\mu_1 \ldots \mu_{r-2}} = \eta^{\mu_{r-1}\mu_r} \mathcal{K}_{\mu_1 \ldots \mu_r} \tag{5.3.13}$$

and we note that $\mathcal{K}'$ is of rank $r-2$. Next we define the non-normalized symmetrization operator $\mathcal{S}_p$. It acts on a tensor product of two symmetric tensors ζ and ω (of rank p and $s-p$ resp.) in the following way:

$$(\mathcal{S}_p \zeta \omega)_{\alpha_1 \ldots \alpha_s} = \sum_{i_1 < i_2 < \ldots < i_p} \zeta_{\alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_p}} \omega_{\alpha_1 \ldots \not\alpha_{i_1} \ldots \not\alpha_{i_2} \ldots \not\alpha_{i_p} \ldots \alpha_s} \tag{5.3.14}$$

where the slash through an index means that it must be removed.

Now, writing out the formula (5.3.4) for $Q^{(1)}$, acting on a rank-r tensor field $\mathcal{K}_{\mu_1 \ldots \mu_r}(x)$:

$$Q^{(1)} \mathcal{K}(x) = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} \mathcal{K}(x) - \frac{1}{2} S^{(r)}_{\alpha\beta} S^{\alpha\beta(r)} \mathcal{K}(x) - M_{\alpha\beta} S^{\alpha\beta(r)} \mathcal{K}(x) \tag{5.3.15}$$

For integer spin one can show [53, 18]:

$$\frac{1}{2} S^{(r)}_{\alpha\beta} S^{\alpha\beta(r)} \mathcal{K}(x) = r(r+3)\mathcal{K}(x) - 2\mathcal{S}_2 \eta \mathcal{K}'(x) \tag{5.3.16a}$$

$$M_{\alpha\beta}S^{\alpha\beta(r)}\mathcal{K}(x) = 2\mathcal{S}_1\partial x\cdot\mathcal{K}(x) - 2\mathcal{S}_1 x\partial\cdot\mathcal{K}(x) - 2r\mathcal{K}(x) \tag{5.3.16b}$$

so we arrive at the form for the second order Casimir for integer spin:

$$Q^{(1)}\mathcal{K}(x) = \left(Q_0^{(1)} - r(r+1)\right)\mathcal{K}(x) + 2\mathcal{S}_2\eta\mathcal{K}'(x) + 2\mathcal{S}_1 x\partial\cdot\mathcal{K}(x) - 2\mathcal{S}_1\partial x\cdot\mathcal{K}(x) \tag{5.3.17}$$

and for half-integer spin we must add the following term to this expression [55, 18]:

$$\left(\frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} - \frac{5}{2}\right)\mathcal{K}(x) + \mathcal{S}_1\gamma(\gamma\cdot\mathcal{K}(x)) \tag{5.3.18}$$

We now know the action $Q^{(1)}$ on the (spinor-)tensor field $\mathcal{K}(x)$. The next step is to determine how $\mathcal{K}(x)$ is linked to the field $h(X)$ in intrinsic coordinates.

5.3.4 Link between $\mathcal{K}(x)$ and $h(X)$

To be able to make this link, we must first give some more definitions. Let us start by defining the *transverse projection operator* $\theta_{\alpha\beta}$:

$$\theta_{\alpha\beta} \equiv \eta_{\alpha\beta} + H^2 x_\alpha x_\beta \tag{5.3.19}$$

satisfying

$$\theta_{\alpha\beta}x^\alpha = \theta_{\alpha\beta}x^\beta = 0 \tag{5.3.20}$$

With the use of the transverse projection operator we define the tangential/transverse derivative $\bar{\partial}$ on the de Sitter space:

$$\bar{\partial}_\alpha \equiv \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + H^2 x_\alpha x\cdot\partial \tag{5.3.21}$$

and we can easily see that indeed $x\cdot\bar{\partial} = 0$.

Next we will provide the framework for establishing the link between tensor fields in ambient space notation and intrinsic notation. The transformation law is the usual tensorial one:

$$h_{\mu_1\ldots\mu_r}(X) = \frac{\partial x^{\alpha_1}}{\partial X^{\mu_1}}\ldots\frac{\partial x^{\alpha_r}}{\partial X^{\mu_r}}\mathcal{K}_{\alpha_1\ldots\alpha_r}(x) \tag{5.3.22}$$

The covariant derivatives acting on the intrinsic tensor field are transformed in the following way:

$$\nabla_\mu\ldots\nabla_\nu h_{\lambda_1\ldots\lambda_r}(X) = \frac{\partial x^\alpha}{\partial X^\mu}\ldots\frac{\partial x^\beta}{\partial X^\nu}\frac{\partial x^{\eta_1}}{\partial X^{\lambda_1}}\ldots\frac{\partial x^{\eta_r}}{\partial X^{\lambda_r}}\mathrm{Trpr}\bar{\partial}_\alpha\ldots\mathrm{Trpr}\bar{\partial}_\beta\mathcal{K}_{\eta_1\ldots\eta_r}(x) \tag{5.3.23}$$

where the Trpr is the generalization of the transverse projection operator:

$$(\mathrm{Trpr}\mathcal{K})_{\lambda_1\ldots\lambda_r} = \theta^{\eta_1}_{\lambda_1}\ldots\theta^{\eta_r}_{\lambda_r}\mathcal{K}_{\eta_1\ldots\eta_r} \tag{5.3.24}$$

Let us first of all note that for a scalar field $\mathcal{K}(x) = \phi(x)$ we can express the Casimir operator in terms of the tangential derivative:

$$Q_0^{(1)}\phi(x) = -H^{-2}\bar{\partial}^2\phi(x) \tag{5.3.25}$$

and we can easily see that we can express the intrinsic d'Alembertian operator $\Box_H = \nabla_\mu\nabla^\mu$ in terms of $Q_0^{(1)}$:

$$\Box_H\phi(X) = \bar{\partial}^2\phi(x) = -H^2 Q_0^{(1)}\phi(x) \tag{5.3.26}$$

where the subscript H on $\Box_H$ is there to remind us that the d'Alembertian is curvature dependent.

Now, it is proven in [18] that the general expression for the d'Alembertian for rank-r tensor fields is given by:

$$\begin{aligned}\Box_H h_{\mu_1\ldots\mu_r}(X) =& \frac{\partial x^{\beta_1}}{\partial X^{\mu_1}}\cdots\frac{\partial x^{\beta_r}}{\partial X^{\mu_r}}\Big[- H^2\left(Q_0^{(1)}+r\right)\mathcal{K}_{\beta_1\ldots\beta_r} \\ &+ 2H^4\sum_{j=1}^{r} x_{\beta_j}\sum_{i<j} x_{\beta_i}\mathcal{K}'_{\beta_1\ldots\not\beta_i\not\beta_j\ldots\beta_r} \\ &- 2H^2\sum_{i=1}^{r} x_{\beta_i}\left(\bar\partial\cdot\mathcal{K}_{\beta_1\ldots\not\beta_i\ldots\beta_r} - H^2 x\cdot\mathcal{K}_{\beta_1\ldots\not\beta_i\ldots\beta_r}\right)\Big]\end{aligned} \tag{5.3.27}$$

In the next section we will see how we can explicitly make the link between the wave equation in intrinsic coordinates and the Casimir eigenvalue equation in ambient space coordinates using the expressions we have given in the sections above.

5.3.5 Explicit link between intrinsic wave equation and ambient space Casimir equation

Let us start with the most simple case: the scalar field. Then, (5.3.12) reduces to:

$$\left(Q_0^{(1)} - \langle Q_0^{(1)}\rangle\right)\mathcal{K}(x) = 0 \tag{5.3.28}$$

and the expression for the d'Alembertian in terms of $Q_0^{(1)}$ was already given by (5.3.25). Thus we can conclude:

$$\left(\Box_H + H^2\langle Q_0^{(1)}\rangle\right)h(X) = 0 \tag{5.3.29}$$

This equation gives the explicit link between the field theory wave equation and the UIR the field carries.

It should be obvious from the general expressions for $Q^{(1)}$ and $\Box_H$ that this link is not quite as trivial in the spinorial case as it is for scalar fields. However, there are cases where we can put certain conditions on the fields such that the expressions simplify greatly.

The case where this is most explicit is when we are dealing with massive fields with integer spin. By massive we mean fields that carry a de Sitter UIR that contracts to a massive Poincaré UIR; fields whose UIR belongs to the principal series (see sections 3.6 and 4.3.2 for details). Let us consider such a massive rank-r tensor field $\mathcal{K}_{\mu_1\ldots\mu_r}(x)$. Physical arguments lead to the condition that $\mathcal{K}$ must be divergenceless, i.e. $\partial\cdot\mathcal{K} = 0$ (see e.g. [60, 61]). Together with the transverse condition $x\cdot\mathcal{K} = 0$ this implies that $\mathcal{K}$ is traceless: $\mathcal{K}' = 0$. These conditions allow one to constrain the number of propagating degrees of freedom to $2r+1$.
The Casimir eigenvalue equation then reduces to:

$$\left(Q_0^{(1)} - r(r+1) - \langle Q^{(1)}\rangle\right)\mathcal{K}_{\mu_1\ldots\mu_r}(x) = 0 \tag{5.3.30}$$

The expression (5.3.27) for the d'Alembertian takes on the simple form:

$$\Box_H h_{\mu_1\ldots\mu_r}(X) = -\frac{\partial x^{\alpha_1}}{\partial X^{\mu_1}}\cdots\frac{\partial x^{\alpha_r}}{\partial X^{\mu_r}}\left(H^2 Q_0^{(1)} + H^2 r\right)\mathcal{K}_{\alpha_1\ldots\alpha_r}(x) \tag{5.3.31}$$

Now we can give the expression that links the eigenvalue of the second order Casimir operator directly to the field equation in intrinsic coordinates:

$$\left(\Box_H + H^2 r(r+2) + H^2 \langle Q^{(1)} \rangle\right) h_{\mu_1 \ldots \mu_r}(X) = 0 \tag{5.3.32}$$

where $\langle Q^{(1)} \rangle$ is the eigenvalue belonging to a principal series UIR.

For fields that carry UIRs different from the principal series, it is in general not possible to impose the traceless and divergenceless conditions. In those cases the strategy is again to write the field equation in terms of $Q^{(1)}$, and then search for the relevant physical subspace corresponding to a massless UIR of $SO(4,1)$. From section 4.3.2 we know that the UIRs that have a massless limit are given by the discrete series UIRs with $p = q = s$. Their Casimir eigenvalue is $\langle Q^{(1)}_{p=q} \rangle = -2(s^2 - 1)$, so the field equation takes on the form:

$$\left(\Box_H + H^2 r(r+2) - 2H^2(s^2 - 1)\right) h_{\mu_1 \ldots \mu_r}(X) + G(x) = 0 \tag{5.3.33}$$

where $G(x)$ depends on traces and divergences of h. In [61] it was found that for massless spin-1 fields this physical subspace is given by:

$$\left(\Box_H + 3H^2\right) h_{\mu_1}(X) = 0 \tag{5.3.34}$$

and in [62] it was shown that for massless spin-2 fields it is determined by:

$$\left(\Box_H + 2H^2\right) h_{\mu_1 \mu_2}(X) = 0 \tag{5.3.35}$$

In the next section we will be focussing on the interpretation of the mass parameter in de Sitter space, with the use of the ambient space formalism.

5.4 Mass in de Sitter space

As we mentioned a few times before: the concept of mass, and consequently masslessness, is quite obscure in field theory in de Sitter space. One way this problem reveals itself is when we consider massless fields. In flat spacetime, the properties that characterize massless $m = 0$ fields are gauge invariance, light cone propagation, conformal invariance and the presence of two helicity states (for $s > 0$); these things are essentially synonymous. This is not the case in curved spacetime. For example, consider the wave equation for a scalar field with no mass term:

$$\Box_H \phi(X) = 0 \tag{5.4.1}$$

It was shown in [20] that in a gravitational background such a field does not only propagate on, but also inside the light cone; a property usually associated with massive fields.

Another way in which the obscurity of the mass concept in (A)dS establishes itself is in the *partially massless fields*; for these fields a certain gauge invariance allows one to reduce the number of propagating degrees of freedom (pdof), but not all the way to two. They were first discovered by Deser and Nepomechie in 1984 [63], and they occur for spin $s \geq 3/2$.

Closely related to the partially massless fields are the *forbidden mass ranges*. The masses in this range lead to negative norm states, i.e. non-unitarity. For example, it is shown in [64] that for spin-2 fields the forbidden mass range is $0 < m^2 < 2H^2$.

Our goal in the next section is to review the approach of Garidi [18] to this state of affairs. In an attempt to reconcile group and field theoretical notions, he proposes a mass definition in terms of the parameters labeling the $SO(4,1)$ UIRs.

5.4.1 Garidi's mass definition

Garidi's most important axiom is that the notion of mass in de Sitter space is only defined in reference to mass in Minkowski space. This reference is inferred by following the dS UIRs under group contraction, and deciding whether a dS UIR is massive or massless by examining the Poincaré UIR it contracts to. This implies that the only dS UIRs to which the notion of mass applies are the ones belonging to the principal series, and the ones extendable to UIRs of the conformal group (see section 4.3.2). Another axiom is that the mass parameter must be real-valued for the fields carrying these UIRs. The mass definition he provides can also be used for fields carrying a dS UIR with no contraction limit, but he insists that in that case the proposed parameter cannot be interpreted as a mass. We will repeat the arguments given in his article [18], and define the mass parameter.

We first consider fields with spin $s \neq 0$. We would like to associate the mass to the Casimir eigenvalue $\langle Q^{(1)} \rangle$. The lowest value for the mass ($m = 0$) will be taken in reference to the dS UIR that contracts to the massless Poincaré UIR for a given value of s. The UIRs that contract to the massless Poincaré UIRs are those belonging to the discrete series with $p = q = s$, and we can check that the eigenvalues $\langle Q^{(1)}_{p=q} \rangle$ are indeed the lowest possible ones[11]. The masses for the fields carrying other UIRs are then constructed in reference to this lowest value. The definition of the mass for a given value of $s = p$ is then given by:

$$m_H^2 \equiv H^2 \left(\langle Q^{(1)} \rangle - \langle Q^{(1)}_{p=q} \rangle \right) = \langle Q^{(1)} \rangle H^2 + 2(p^2 - 1)H^2 = [(p-q)(p+q-1)]H^2 \qquad (5.4.2)$$

Note that $m_H^2 \geq 0$, by construction. In the contraction limit, we let $H \to 0$ and $\nu \to \infty$ while $\nu H = m$, where m is the Poincaré mass. This means that in the limit, q scales as $q \sim \frac{im}{H}$, and we see that $m_H^2 \to m^2$ as $H \to 0$.

With the use of this mass parameter we can write the wave equation for massive integer spin fields (5.3.32) in the following way:

$$\left(\Box_H + [2 - s(s-2)]H^2 + m_H^2 \right) h_{\mu_1 \ldots \mu_s}(X) = 0 \qquad (5.4.3)$$

One can easily check that for $s = 1, 2$ and $m_H^2 = 0$, this equation indeed leads to the 'field-theoretical correct' massless equations given in (5.3.34) and (5.3.35).

Next we consider the scalar fields. In section 4.3.2 we stated that the de Sitter UIR contracting to the massless scalar Poincaré UIR belongs to the complementary series, and is labeled by $(p, q) = (0, 1)$. But now we must draw the attention to the concept of *Weyl equivalence*. Two UIRs are said to be Weyl equivalent if the two different labels lead to the same eigenvalues of the Casimir operators. Note that this is the case for $(p, 1-q)$. So we see that the complementary UIR labeled by $(0, 1)$ is Weyl equivalent to $(0, 0)$. Therefore, Garidi argues, again we can use $\langle Q^{(1)}_{p=q} \rangle$ as the lowest value for the mass, so that the definition (5.4.2) also holds for the scalar fields. There are however a few points that need some more attention.

[11] In fact, this is *the* condition we need for the mass to be real-valued, and unfortunately we will see that this is not the case for scalar fields, which complicates matters.

Consider the massless scalar UIR with $(p, q) = (0, 1)$, which is extendable to the conformal group. The Casimir eigenvalue equation reduces to:

$$\left(Q_0^{(1)} - 2\right) \phi(x) = 0 \tag{5.4.4}$$

which, by using (5.3.26), gives rise to the wave equation in intrinsic coordinates:

$$\left(\Box_H + 2H^2\right) \phi(X) = 0 \tag{5.4.5}$$

At this point we have not made any field theoretical assumptions: we started with the Casimir eigenvalue equation, then used the ambient space formalism to change to intrinsic coordinates, and ended up with this equation. Now let us compare it to the field theoretical equation for the scalar field:

$$(\Box_H + m_H^2 + \xi R)\phi(X) = 0 \tag{5.4.6}$$

Note that we must implicitly set $\xi = \frac{1}{6}$ to recover (5.4.5) (the Ricci scalar has the value $R = 12H^2$ in de Sitter space [45]), which is known as the conformally coupled case, and indeed, the equation is invariant under conformal transformations (see section 5.2.1 and [45]).

Let us now consider the following scalar field equation:

$$\Box_H \phi(X) = 0 \tag{5.4.7}$$

This field has been shown to carry the discrete series UIR with label $(1, 0)$ [65]. Note that for these values of p and q we again find $m_H^2 = 0$. Therefore we must set $\xi = 0$, which is known as the minimally coupled case. However, also note that this UIR does not have a contraction limit, and therefore we cannot interpret m_H as a mass and $p = 1$ (!) as spin.

We already briefly mentioned that the reference Casimir value $\langle Q^{(1)}_{p=q=0}\rangle = 2$ is not the lowest possible eigenvalue for $p = 0$; the eigenvalues for $p = 0$ and $-1 < q < 0$ (and their Weyl equivalent $1 < q < 2$ UIRs) are smaller. This implies imaginary values for m_H. Garidi argues that this is not a problem, since the UIRs with these values do not have a contraction limit.

These last paragraphs might be somewhat confusing to the reader, and arouse questions like: 'Are we allowed to change the value of ξ and still work with the same mass formula?' or 'Isn't it a problem that there still are allowed imaginary values for m_H?'. For this reason we will devote the following section solely to scalar fields and elaborate on these points.

5.4.2 Scalar fields

A large amount of literature on dS fields is restricted to scalars. For example, in most inflationary theories, the fields involved (e.g. the inflaton and isocurvaton) do not have spin. For a given theory, the masses of these fields have a large influence on the results it produces. In slow-roll models the mass parameter of the inflaton must be small ($m \ll H$), just as in quasi-single field models where there is an aditional field, the isocurvaton, with a mass parameter $m \sim H$ (see e.g. [21]). The fact that the theories depend heavily on this parameter, and that the concept of mass for scalars in dS is still somewhat obscure[12] motivates us to write this following section. We will begin with critically recapitulating Garidi's views on the scalar field, and after that we propose a different mass formula for the scalars.

[12]Especially for the fields with no Minkowskian limit.

We first note that Garidi explicitly only wants to use the term 'mass' in reference to Minkowski space, where the notion is properly defined. However, the fields with a small mass compared to H used in inflationary theories are in the complementary series, which does not have a limit in Minkowski space. Nevertheless, Garidi feels that his mass parameter can be used, even when it can not be interpreted as a 'mass'.

Secondly, recall that there is an ambiguity in the definition of the mass parameter in the Klein-Gordon equation, due to the coupling of the field to the background curvature:

$$(\Box_H + m_H^2 + \xi R)\phi = 0 \tag{5.4.8}$$

Garidi implicitly puts $\xi = 1/6$, so that $m_H^2 = 0$ leads to conformal invariance. He points out that his mass formula also yields $m_H^2 = 0$ for the UIR corresponding to the massless minimally coupled field, where $\xi = 0$, which has no Minkowski limit. It seems somewhat arbitrary to use different values of ξ in the same mass definition, since in principle we cannot differentiate between m_H^2 and an effective mass $m_H^2 + \xi R$.

Thirdly, we note that Garidi chooses to define the mass in reference to the complementary series UIR with $(p, q) = (0, 1) = (0, 0)$, corresponding to the massless conformally coupled field (recall that UIRs with (p, q) are Weyl equivalent to $(p, 1 - q)$), which does have a Minkowski limit. This allows for imaginary values for the mass, which is conveniently displayed in the following figure:

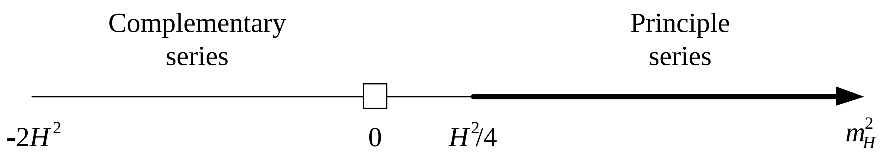

Figure 5.1: Scalar field mass relation for Garidi's definition. The square indicates the massless minimally coupled discrete series UIR. Located at the same spot is the massless conformally coupled complementary series UIR. One might note that in [18] the part of the complementary series left of zero is not displayed, but we feel that it should be included.

We argue that in the case of the scalar field there is a different possibility for a mass formula, at least equally convenient. It does not define the mass parameter in reference to the conformally coupled case, but to the minimally coupled case. In a way, this means that we are working with the effective mass, and it might be a more natural choice if one does not put the emphasis on the (non)existence of a Minkowski limit. The field equation for the minimally coupled case is

$$\Box_H \phi = 0 \tag{5.4.9}$$

while the field equation for the conformally coupled case is

$$(\Box_H + 2H^2)\phi = 0 \tag{5.4.10}$$

from which is seems more natural to take the minimally coupled case as the reference point for a mass parameter definition. On the contrary, the massless minimally coupled field does propagate inside the lightcone [20] and is not conformally invariant, in contrast to the massless conformally coupled field.

Let us give the definition of the new mass parameter:

$$\widetilde{m}_H^2 \equiv H^2 \left(\langle Q^{(1)} \rangle - \langle Q^{(1)}_{p=1,q=0} \rangle \right) = H^2 \langle Q^{(1)} \rangle = \left[-p(p+1) - (q+1)(q-2) \right] H^2 \qquad (5.4.11)$$

and repeat Garidi's mass formula:

$$m_H^2 \equiv H^2 \left(\langle Q^{(1)} \rangle - \langle Q^{(1)}_{p=q} \rangle \right) = \langle Q^{(1)} \rangle H^2 + 2(p^2-1)H^2 = [(p-q)(p+q-1)]H^2 \qquad (5.4.12)$$

We will now review the different series and particular UIRs, and compare the values for the different mass definitions.

- *Massless minimally coupled case*
  This UIR belongs to the discrete series with[13] $(p,q) = (1,0)$ and is denoted by $\Pi_{1,0}$. It has no Minkowski limit.

  $m_H^2 = \widetilde{m}_H^2 = 0.$

- *Massless conformally coupled case*
  This UIR belongs to the complementary series with $(p,q) = (0,1)$ and is denoted by $V_{0,1}$. It contracts to the massless Poincaré UIR as $H \to 0$.

  $m_H^2 = 0$, while $\widetilde{m}_H^2 = 2H^2$.

- *Principle series*
  The UIRs from the principal series contract to the massive Poincaré UIRs and they are denoted by $U_{p,q}$, where $p = 0$ and $q = \frac{1}{2} + i\nu$, with $\nu \geq 0$.

  $m_H^2 = (\frac{1}{4} + \nu^2)H^2$, while $\widetilde{m}_H^2 = (\frac{9}{4} + \nu^2)H^2$.

- *Complementary series*
  The UIRs from the complementary series do not have a Minkowski limit, except for $(p,q) = (0,1)$. They are denoted by $V_{p,q}$, where $p = 0$ and $q = \frac{1}{2} + \nu$, with $0 < |\nu| < \frac{3}{2}$.

  $m_H^2 = (\frac{1}{4} - \nu^2)H^2$, while $\widetilde{m}_H^2 = (\frac{9}{4} - \nu^2)H^2$.

  Note that $-2H^2 < m_H^2 < \frac{1}{4}H^2$, while $0 < \widetilde{m}_H^2 < \frac{9}{4}H^2$.

- *Discrete series*
  The UIRs from the discrete series do not have a Minkowski limit. They are denoted by $\Pi_{p,q}$, where $p = 1, 2, ...$ and $q = 0$.

  $m_H^2 = p(p-1)H^2$, while $\widetilde{m}_H^2 = (-p(p+1)+2)H^2$.

The difference between the two formulas is most striking if we compare figure 5.1 and 5.2. We see that the principal and complementary series are simply shifted by $2H^2$, while the relative position of the massless minimally coupled field has changed. This is also easily seen in formulas (5.4.11) and (5.4.12): for $p = 0$ there is a difference of $2H^2$, while for $p = 1$ they coincide. One might argue that $\widetilde{m}_H$ is favorable over m_H, since the latter has the same mass value for two distinct[14] UIRs. On the contrary, $\widetilde{m}_H \neq 0$ for the UIR that can be extended to the conformal group.

[13] We will not point out the Weyl equivalent UIRs for brevity.

[14] Distinct in the sense that they have different values for (p,q), and are not Weyl equivalent.

Figure 5.2: Scalar field mass relation for the alternative definition.

We conclude this section with the observation that, since the interpretation of the concept of mass in de Sitter spacetime is still very much an open question, the mass parameter is merely a tool for keeping track of the group theoretical content of our field theoretical equations. The vagueness of the interpretation of the concept reveals itself upon going through the recent literature. For example, it is shown in [9, 10] that when interactions are included, de Sitter symmetry does not prevent massive scalars from the principal series from decaying into pairs of heavier particles. Another example is the discovery of tachyonic fields which are local and allow for a de Sitter invariant physical space [11]. They carry UIRs from the discrete series, and are tachyonic in the sense that they have a negative squared mass.

The obscurity of the mass concept is not restricted to scalar fields; next we will review an example for non-zero spin fields. In a series of papers on gauge invariance and partially massless higher spin fields Deser and Waldron concluded that for half-integer spin ($s \geq 3/2$), the strictly massless fields actually are anti-de Sitter instead of de Sitter fields [66, 67, 68, 69]. The following section is devoted to this claim, and how the application of Garidi's mass formula leads to a different result. Our discussion will be largely based on [18]; there Garidi explains the inconsistency between his mass definition and the one used by Deser and Waldron.

5.4.3 Gauge invariant fields

In Minkowski field theory, massive fields with spin s have $2s+1$ propagating degrees of freedom, while massless fields (with $s > 0$) have two: the helicity states $\pm s$. The reduction of propagating degrees of freedom is established by introducing gauge invariance. As was mentioned at the beginning of section 5.4, it turns out that for massive fields in de Sitter space with $s > 1$ a certain gauge invariance allows us to reduce the number of pdof to intermediate values, giving rise to the so called partially massless fields. Closely related to these fields are the forbidden mass ranges; these masses lead to negative norm states, i.e. non-unitarity. Forbidden mass ranges and (partially) massless fields are usually depicted in the plane spanned by m^2 and Λ, resulting in a phase diagram (here $\Lambda = 3H^2$ is the cosmological constant). For example, we already stated that for spin-2 fields the forbidden mass range is $0 < m^2 < 2H^2$, and they become partially massless at $m^2 = 2H^2$. The corresponding phase diagram is given in figure 5.3.

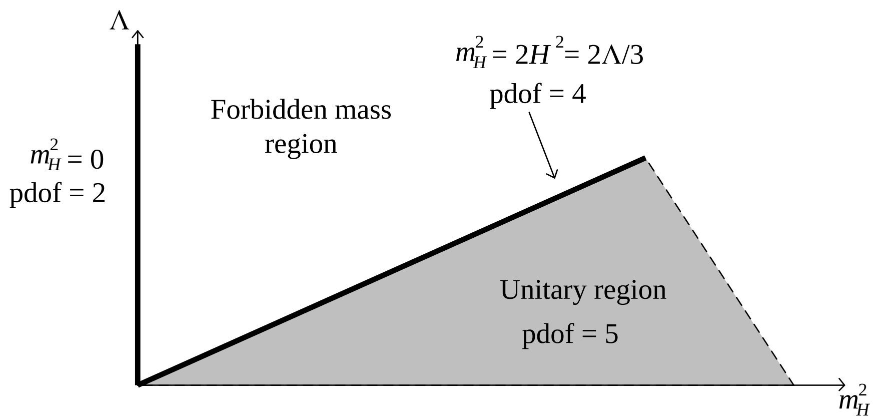

Figure 5.3: Phase diagram for spin-2 fields, based on [18].

A different kind of diagram is given in figure 5.4; by using Garidi's mass definition we can deduce which UIR series the different masses belong to, and in that way it complements figure 5.3. Similar figures can be made for $s = 3$ fields, only there we have an additional partially massless field (see [18] and [66]).

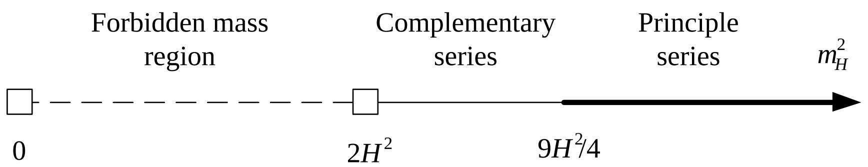

Figure 5.4: Mass relation for spin-2 fields, based on [18].

For these integer spin values Garidi agrees with Deser and Waldron. This changes when we consider half-integer spins $s \geq 3/2$. In [66] it was found that for $s = 3/2$ fields, gauge invariance allows one to reduce the number of pdof to 2 at the mass value $m^2 = -\Lambda/3$, which corresponds to the discrete series UIR with $p = q = 3/2$ uniquely extendable to UIRs of the conformal group. The minus sign lead the authors to the conclusion that these fields are actually anti-de Sitter fields (where $\Lambda < 0$, so that $m^2 > 0$), i.e. that there are no strictly massless fields in de Sitter with $s = 3/2$. However, in [18] it was noted that, for a given spin, this mass parameter is defined relatively to the first terms of the discrete series with $q = 1/2$, i.e.

$$m_{DW}^2 = H^2 \left(\langle Q^{(1)} \rangle - \langle Q^{(1)}_{p,q=1/2} \rangle \right) = \frac{\Lambda}{3} \left[-q(q-1) - \frac{1}{4} \right] \tag{5.4.13}$$

rather than with respect to the $p = q$ UIRs contracting to the Minkowski massless fields. This has as a consequence that, even though the strictly massless fields (those with only two helicity states) are still located at $s = p = q$, they do not have $m_{DW}^2 = 0$, but instead $m_{DW}^2 = -\Lambda/3$ for $s = 3/2$.

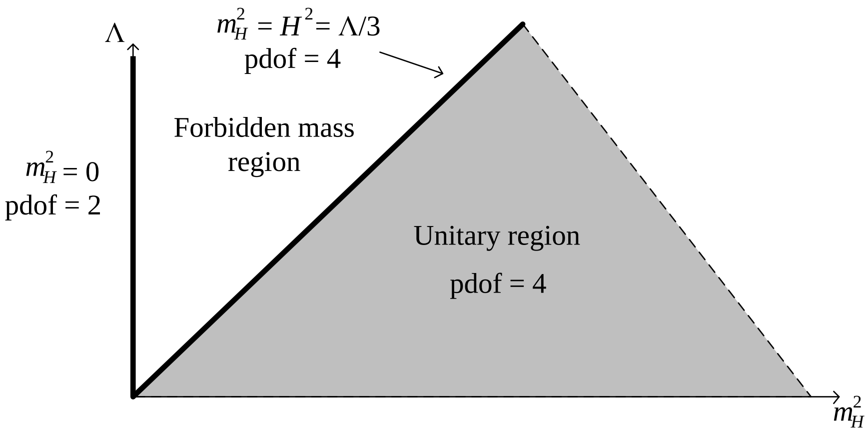

Figure 5.5: Phase diagram for spin-3/2 fields, based on [18].

Garidi claims that it is not right to say that the strictly massless field is an anti-de Sitter field, but rather that the mass is not correctly defined. If we use his mass definition (5.4.2), we see that the strictly massless case $p = q$ does yield $m_H^2 = 0$, leading to the phase diagram and mass relation shown in figure 5.5 and 5.6. We note that Garidi's mass formula gives $m_H^2 = \Lambda/3$ for $(p, q) = (3/2, 1/2)$, which are the values used by Deser and Waldron to define their masses relative to. In a similar way, it is easy to check that for $s = 5/2$, the strictly massless field with $p = q = 5/2$ has $m_{DW}^2 = -4\Lambda/3$ and $m_H^2 = 0$.

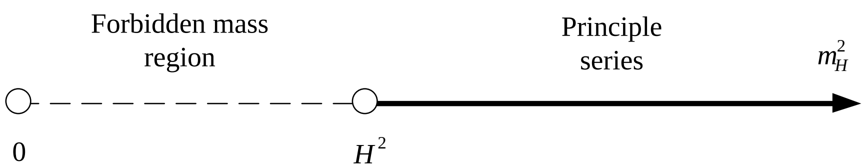

Figure 5.6: Mass diagram for spin-3/2 fields. A circle represents a particular discrete series UIR, based on [18].

We conclude with the following remarks. It seems more consistent to use the same mass formula for integer and half-integer spin fields. Moreover, the fact that Garidi's mass formula yields $m_H^2 = 0$ for the strictly massless fields, whose UIR contract to the massless Minkowski UIR in the $H \to 0$ limit, pleads in favour of using this formula.

Mass relation diagrams like 5.4 and 5.6 can be easily deduced from the $SO(4, 1)$ UIR classification diagram given at the end of section 3.6. Let us present the diagram in a slightly different form:

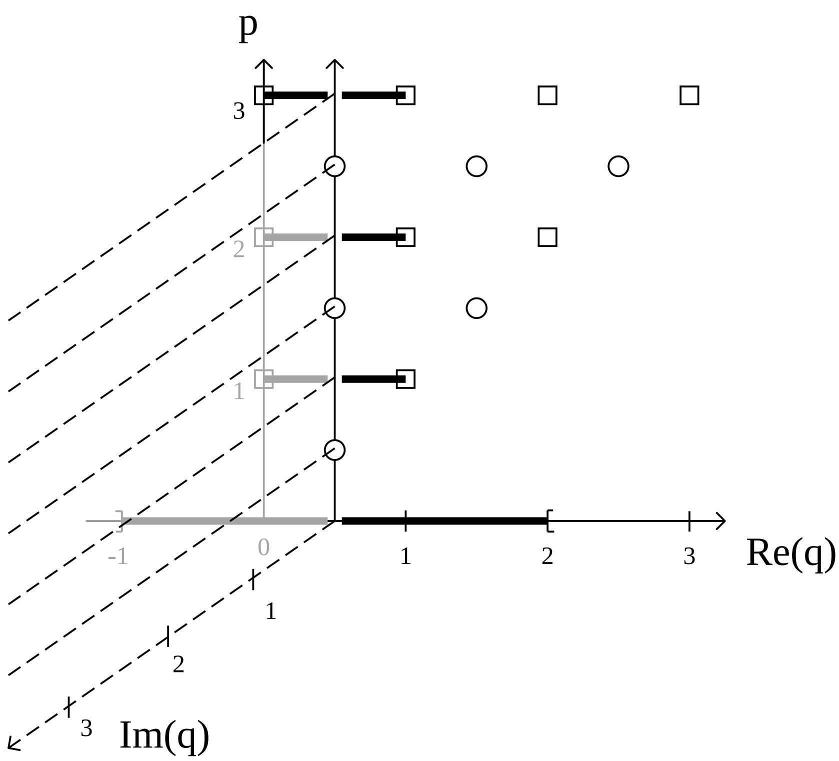

Figure 5.7: This diagram summarizes the classification of the UIRs of $SO(4,1)$. The dashed lines represent the principal series UIRs, and the fat lines represents the complementary series UIRs. The discrete series UIRs with p integer and half-integer are represented by the squares and circles respectively.

Now consider the spin-2 case; we have $p = 2$. The massless UIR is located on the diagonal where $p = q$, since that is the reference value for the Casimir operator in the mass definition. If we start in the point $(2,2)$ and move to the left in the diagram, we note that there are no UIRs there; this is the forbidden mass region (compare with figure 5.4). Moving further left, we find another discrete series UIR, and after that we come upon the complementary series. Continuing going left, we arrive at the principal series, and as the mass increases we move along the dashed line. In this way we can easily deduce the mass relation diagram for any non-zero spin field.

Conclusion

In this thesis we examined the question: "*In how far does the group theoretical approach to quantum field theory, in terms of associating UIRs of the spacetime isometry group to quantum mechanical elementary systems, lead to a better understanding of the concept of 'mass' in de Sitter spacetime?*" In order to answer this question we started by looking at the state of affairs in flat 3+1 Minkowski spacetime, whose isometry group is the Poincaré group, consisting of translations and Lorentz transformations. Due to the presence of the Abelian invariant subgroup of translations, we are able to use the induced representation method for constructing the UIRs of the group. There turn out to be three distinct classes of UIRs, known as the time-like (or massive), light-like (or massless) and space-like cases, which can be distinguished by looking at the eigenvalue of the first Casimir operator (positive, zero, and negative resp.), which is associated with the rest mass of the field carrying such a particular UIR.

Since the concept of mass in de Sitter space is quite obscure from the field theoretical perspective, we have looked at the possibility of resolving this issue by considering the Casimir operator of its isometry group, $SO(4,1)$, in conformity to the link between Minkowskian rest mass and the Poincaré Casimir. However, it is not possible to make the direct conceptual analogy between the $SO(4,1)$ and Poincaré Casimir; the latter is the square of the translation generators (i.e. momentum operators) and is used to define the mass-shell condition, while there are no analogous symmetries to translations in de Sitter space, since $SO(4,1)$ does not possess such an Abelian invariant subgroup.

In order to be able to make a connection between the Casimir operator of $SO(4,1)$ and the mass parameter in the field theoretical equations we followed the argumentation presented by Garidi in [18]. The basis of the idea is that the concept of mass is only applicable to de Sitter field theory in reference to Minkowski spacetime. This reference is inferred by using group contractions; we find out which dS UIRs contract to which Poincaré UIRs as the curvature tends to zero. Together with this, we use the ambient space formalism to convey the group theoretical information to the field theoretical equations in de Sitter spacetime. This is necessary, since the symmetries are most conveniently described in the 5-dimensional ambient space, while the intrinsic 4-dimensional coordinates are more suited for the field theory, in contrast to the Poincaré group, which has a clear 4-dimensional description. With the use of this formalism we have obtained wave equations in 4-dimensions containing parameters used for labeling the UIRs, by starting from the simple Casimir eigenvalue equation in 5-dimensions (see section 5.3).

The classification of $SO(4,1)$ UIRs reveals that there are three types, belonging either to the so-called principal, complementary or discrete series. In the zero curvature limit the principal series contracts to the massive Poincaré UIRs, one particular complementary series UIR contracts to the massless scalar Poincaré UIR, and a certain class of the discrete series contracts to the massless nonzero-spin Poincaré UIRs (see section 4.3.2 for details). These UIRs contracting to the massless Poincaré UIRs are uniquely extendable to UIRs of the conformal group $SO(4,2)$. All other $SO(4,1)$ UIRs have no (or no physically interesting) link to the Poincaré group in the zero curvature limit. This last observation excludes a range of de Sitter fields from having a Minkowskian interpretation of their mass parameter in this sense.

An interesting explicit formula for the mass parameter was proposed in [18]; for a given spin, we define the mass as the eigenvalue of the Casimir operator, in reference to the eigenvalue of the UIR contracting to the massless Poincaré UIR for that given spin. That is to say: $m_H^2 \sim (\langle Q \rangle - \langle Q_{\text{massless}} \rangle)$. This parameter has the property that for all nonzero spin fields it is real and vanishes only for the fields carrying the UIR contracting to the massless Poincaré UIR. However, it turned out that for part of the complementary series of the scalar field, this mass formula can take on imaginary values. It was also noted that the mass formula not only gives zero for the UIR contracting to the massless Poincaré UIR (the massless conformally coupled case), but again gives zero for the massless minimally coupled case (see section 5.4.2). These points might be considered as posing a threat to attractiveness of this definition, though Garidi argues that it must only be interpreted as a mass when applied to fields carrying UIRs that have a Minkowskian limit.

If we abandon the view that we need to have a Minkowskian interpretation of the de Sitter mass, and embrace the view that the mass parameter can (only) be used as a tool for keeping track of the group theoretical content of the field theoretical wave equations, then there are more possibilities for defining the mass. We considered a different formula for the scalar fields. Instead of using the Casimir eigenvalue of the massless conformally coupled case, we used the eigenvalue of the massless minimally coupled case as the reference value. This leads to a mass parameter which is real for all scalar UIRs, and zero only for the minimally coupled case (see section 5.4.2). This particular definition seems to be more in accordance with the mass parameter used in literature on inflationary models, which depend heavily on this parameter.

We conclude by noting that the status of the notion of mass in de Sitter spacetime is still a point of debate. For example, it has recently been shown that when interactions are included, $SO(4,1)$ symmetry does not prevent particles carrying a principal series UIR from decaying into pairs of heavier particles. With regard to the question posed at the beginning of this section we can conclude that, even though the isometry group does not provide us with a comparably clear notion of mass as it does in Minkowski spacetime, the ambient space formalism, together with group contractions, allows us to keep track of the group theoretical content of the field theoretical equations, and in particular allows us to link the mass parameter to the parameters labeling the de Sitter UIRs.

Acknowledgement

The author would like to thank dr. Diederik Roest for the helpful discussions, and the enthousiastic and motivating way in which he talks about physics in general, which has been a large source of inspiration.

Bibliography

[1] W. de Sitter, "Einstein's Theory of Gravitation and its Astronomical Consequences. Third Paper," *Monthly Notices of the Royal Astronomical Society*, vol. 78, pp. 3–28, 1917.

[2] P. A. M. Dirac, "The electron wave equation in de-Sitter space," *Annals of Mathematics*, vol. 36, no. 3, pp. 657–669, 1935.

[3] A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," *Physical Review D*, vol. 23, pp. 347–356, 1981.

[4] A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astronomical Journal*, vol. 116, pp. 1009–1038, 1998. arXiv:astro-ph/9805201.

[5] S. Perlmutter *et al.*, "Measurements of Ω and Λ from 42 high redshift supernovae," *Astrophysical Journal*, vol. 517, pp. 565–586, 1999. arXiv:astro-ph/9812133.

[6] B. Allen, "Vacuum states in de Sitter space," *Physical Review D*, vol. 32, no. 12, pp. 3136–3149, 1985.

[7] R. F. Streater and W. A. S., *PCT, spin and statistics, and all that.* The Benjamin/Cummings Publishing Company, Inc., 1964.

[8] J. P. Gazeau and M. Lachièze Rey, "Quantum field theory in de Sitter space: A Survey of recent approaches," *PoS*, vol. IC2006, p. 007, 2006. arXiv:hep-th/0610296.

[9] J. Bros, H. Epstein, and U. Moschella, "Lifetime of a massive particle in a de Sitter universe," *Journal of Cosmology and Astroparticle Physics*, vol. 0802, p. 003, 2008. arXiv:hep-th/0612184.

[10] J. Bros, H. Epstein, and U. Moschella, "Particle decays and stability on the de Sitter universe," *Annales Henri Poincaré*, vol. 11, pp. 611–658, 2010. arXiv:0812.3513.

[11] J. Bros, H. Epstein, and U. Moschella, "Scalar tachyons in the de Sitter universe," *Letters in Mathematical Physics*, vol. 93, pp. 203–211, 2010. arXiv:1003.1396.

[12] U. Moschella, "Infrared surprises in the de Sitter universe," 2012. arXiv:1210.4815.

[13] E. T. Akhmedov and P. V. Buividovich, "Interacting Field Theories in de Sitter Space are Non-Unitary," *Physical Review D*, vol. 78, p. 104005, 2008. arXiv:0808.4106.

[14] I. Antoniadis, P. O. Mazur, and E. Mottola, "Cosmological dark energy: Prospects for a dynamical theory," *New Journal of Physics*, vol. 9, p. 11, 2007. arXiv:gr-qc/0612068.

[15] A. M. Polyakov, "Decay of Vacuum Energy," *Nuclear Physics B*, vol. 834, pp. 316–329, 2010. arXiv:0912.5503.

[16] A. M. Polyakov, "De Sitter space and eternity," *Nuclear Physics B*, vol. 797, pp. 199–217, 2008. arXiv:0709.2899.

[17] E. P. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group," *Annals of Mathematics*, vol. 40, no. 1, pp. 149–204, 1939.

[18] T. Garidi, "What is mass in de Sitterian physics?," 2003. arXiv:hep-th/0309104.

[19] J. P. Gazeau and M. Novello, "The question of mass in (anti-) de Sitter spacetimes," *Journal of Physics A*, vol. 41, p. 304008, 2008.

[20] B. S. DeWitt and R. W. Brehme, "Radiation Damping in a Gravitational Field," *Annals of Physics*, vol. 9, pp. 220–259, 1960.

[21] X. Chen and Y. Wang, "Quasi-Single Field Inflation and Non-Gaussianities," *Journal of Cosmology and Astroparticle Physics*, vol. 1004, p. 027, 2010. arXiv:0911.3380.

[22] W. Tung, *Group theory in physics.* World Scientific, 1985.

[23] N. J. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions, Volume 1.* Kluwer Academic Publishers, 1991.

[24] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications.* PWN - Polish Scientific Publishers - Warszawa, 1977.

[25] D. Boer, "Symmetry in Physics." lecture notes from the course Symmetry in Physics, University of Groningen, 2011.

[26] R. M. Wald, *General Relativity.* The University of Chicago Press, 1984.

[27] L. H. Thomas, "On unitary representations of the group of de Sitter space," *Annals of Mathematics*, vol. 42, no. 1, pp. 113–126, 1941.

[28] T. D. Newton, "A note on the representations of the de Sitter group," *Annals of Mathematics*, vol. 51, no. 3, pp. 730–733, 1950.

[29] J. Dixmier, "Représentations intégrables du groupe de De Sitter," *Bulletin de la S. M. F.*, vol. 89, pp. 9–41, 1961.

[30] R. Takahashi, "Sur les représentations unitaires des groupes de Lorentz généralisés," *Bull. Soc. math. France*, vol. 91, pp. 289–433, 1963.

[31] S. Ström, "Induced representations of the (1+4) de Sitter group in an angular momentum basis and the decomposition of these representations with respect to representations of the Lorentz group," *Annales de l'I. H. P., section A*, vol. 13, pp. 77–98, 1970.

[32] F. Schwarz, "Unitary Irreducible Representations of the Groups SO(n,1)," *Journal of Mathematical Physics*, vol. 12, pp. 131–139, 1971.

[33] E. Inönü and E. P. Wigner, "On the contraction of groups and their representations," *Proceedings of the National Academy of Sciences*, vol. 39, pp. 510–524, 1953.

[34] E. Inönü, "Contraction of Lie Groups and their Representations," in *Group Theoretical Concepts and Methods in Elementary Particle Physics* (F. Gürsey, ed.), Gordon and Breach, 1962.

[35] E. Inönü and E. P. Wigner, "On a particular type of convergence to a singular matrix," *Proceedings of the National Academy of Sciences*, vol. 40, pp. 119–121, 1954.

[36] J. Mickelsson and J. Niederle, "Contractions of representations of de Sitter groups," *Communications in Mathematical Physics*, vol. 27, pp. 167–180, 1972.

[37] T. Garidi, E. Huguet, and J. Renaud, "DeSitter waves and the zero curvature limit," *Physical Review D*, vol. 67, p. 124028, 2003. arXiv:gr-qc/0304031.

[38] E. Angelopoulos and M. Laoues, "Masslessness in n-dimensions," *Reviews in Mathematical Physics*, vol. 10, pp. 271–300, 1998. arXiv:hep-th/9806100.

[39] T. Yao, "Unitary Irreducible Representations of SU(2,2). I," *Journal of Mathematical Physics*, vol. 8, no. 10, pp. 1931–1954, 1967.

[40] T. Yao, "Unitary Irreducible Representations of SU(2,2). II," *Journal of Mathematical Physics*, vol. 9, no. 10, pp. 1615–1626, 1968.

[41] T. Yao, "Unitary Irreducible Representations of SU(2,2). III. Reduction with Respect to an Iso-Poincaré Subgroup," *Journal of Mathematical Physics*, vol. 12, no. 3, pp. 315–342, 1971.

[42] E. Angelopoulos and M. Flato, "On unitary implementability of conformal transformations," *Letters in Mathematical Physics*, vol. 2, no. 5, pp. 405–412, 1978.

[43] E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, "Massless particles, conformal group, and de Sitter universe," *Physical Review D*, vol. 23, no. 6, pp. 1278–1289, 1981.

[44] A. O. Barut and A. Böhm, "Reduction of a Class of O(4,2) Representations with Respect to SO(4,1) and SO(3,2)," *Journal of Mathematical Physics*, vol. 11, pp. 2938–2945, 1970.

[45] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge University Press, 1982.

[46] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory*. Westview Press, 1995.

[47] J. S. Dowker and R. Critchley, "Scalar effective Lagrangian in de Sitter space," *Physical Review D*, vol. 13, no. 2, pp. 224–234, 1976.

[48] N. A. Chernikov and E. A. Tagirov, "Quantum theory of scalar field in de Sitter space-time," *Annales de l'I. H. P., section A*, vol. 9, no. 2, pp. 109–141, 1968.

[49] L. L. Foldy, "Synthesis of Covariant Particle Equations," *Physical Review*, vol. 102, no. 2, pp. 568–581, 1956.

[50] A. Chakrabarti, "Canonical Form of the Covariant Free-Particle Equations," *Journal of Mathematical Physics*, vol. 4, no. 10, pp. 1215–1222, 1963.

[51] C. Fronsdal, "Elementary Particles in a Curved Space," *Review of Modern Physics*, vol. 37, no. 1, pp. 221–224, 1965.

[52] P. Roman and J. J. Aghassi, "Classical Field Theory and Gravitation in a de Sitter World," *Journal of Mathematical Physics*, vol. 7, no. 7, pp. 1273–1283, 1966.

[53] J. P. Gazeau and M. Hans, "Integral-spin fields on (3+2)-de Sitter space," *Journal of Mathematical Physics*, vol. 29, no. 12, pp. 2533–2552, 1988.

[54] J. Fang and C. Fronsdal, "Massless, half-integer-spin fields in de Sitter space," *Physical Review D*, vol. 22, no. 6, pp. 1361–1367, 1980.

[55] M. Lesimple, "Construction of homogeneous propagators for massive half-integer spin fields in 3+2 De Sitter space, with implications for the massless limit," *Letters in Mathematical Physics*, vol. 15, no. 2, pp. 143–150, 1988.

[56] C. Fronsdal, "Elementary particles in a curved space. II," *Physical Review D*, vol. 10, no. 2, pp. 589–598, 1974.

[57] C. Fronsdal and R. B. Haugen, "Elementary particles in a curved space. III," *Physical Review D*, vol. 12, no. 12, pp. 3810–3818, 1975.

[58] C. Fronsdal, "Singletons and massless, integral-spin fields on de Sitter space," *Physical Review D*, vol. 20, no. 4, pp. 848–856, 1979.

[59] J. P. Gazeau, M. Hans, and R. Murenzi, "Invariant bilinear forms on 3+2 de Sitter space," *Classical and Quantum Gravity*, vol. 6, pp. 329–348, 1989.

[60] A. Hindawi, B. A. Ovrut, and D. Waldram, "Consistent spin-two coupling and quadratic gravitation," *Physical Review D*, vol. 53, no. 10, pp. 5583–5596, 1996. arXiv:hep-th/9509142.

[61] B. Allen and T. Jacobson, "Vector Two-Point Functions in Maximally Symmetric Spaces," *Communications in Mathematical Physics*, vol. 103, pp. 669–692, 1986.

[62] I. Antoniadis, J. Iliopoulos, and T. N. Tomaras, "One loop effective action around de Sitter space," *Nuclear Physics B*, vol. 462, pp. 437–452, 1996. arXiv:hep-th/9510112.

[63] S. Deser and I. Nepomechie, "Gauge Invariance versus Masslessness in de Sitter Spaces," *Annals of Physics*, vol. 154, pp. 396–420, 1984.

[64] A. Higuchi, "Forbidden mass range for spin-2 field theory in de Sitter spacetime," *Nuclear Physics B*, vol. 282, pp. 397–436, 1987.

[65] J. P. Gazeau, J. Renaud, and M. V. Takook, "Gupta-Bleuler quantization for minimally coupled scalar fields in de Sitter space," *Classical and Quantum Gravity*, vol. 17, pp. 1415–1434, 2000. arXiv:gr-qc/9904023.

[66] S. Deser and A. Waldron, "Gauge Invariances and Phases of Massive Higher Spins in (Anti-) de Sitter Space," *Physical Review Letters*, vol. 87, no. 3, p. 031601, 2001. arXiv:hep-th/0102166.

[67] S. Deser and A. Waldron, "Partial masslessness of higher spins in (A)dS," *Nuclear Physics B*, vol. 607, pp. 577–604, 2001. arXiv:hep-th/0103198.

[68] S. Deser and A. Waldron, "Stability of massive cosmological gravitons," *Physics Letters B*, vol. 508, pp. 347–353, 2001.

[69] S. Deser and A. Waldron, "Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations," *Physics Letters B*, vol. 513, pp. 137–141, 2001. arXiv:hep-th/0105181.