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Titre :

Géométrie des simplexes et modèles de mousses de spin.

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RESUME en français :

Dans cette thèse nous présenterons une construction pour l'amplitude quantique associée à un 4-simplex Lorentzian, en modifiant une construction antérieure par Barrett et Crane. Nous utiliserons cette amplitude ensuite pour construire une intégrale de chemin représentant une somme sur des géométries simpliciales pour une triangulation fixe de l'espace-temps. Comme résultat, nous obtenons une description de l'espace quantique au bord de la triangulation donnée par des réseaux de spin, en établissant ainsi une connexion entre l'approche des mousses de spin et la Gravité Quantique à Boucles. Finalement, nous analyserons la limite semiclassique de l'amplitude pour un 4-simplex et obtenons comme résultat que la contribution dominante est donnée par l'exponentielle de l'action de Regge pour des données au bord décrivant bien une géométrie Lorentzienne.

TITRE en anglais : Spinfoams from simplicial geometry.

RESUME en anglais :

In this thesis we present a construction of the quantum amplitude associated to a Lorentzian 4-simplex, modifying a previous construction by Barrett and Crane. This 4-simplex amplitude is further used to construct a path integral defining a sum over simplicial geometries for a fixed triangulation of space-time. As a result we obtain a boundary state space given by spin-networks, establishing a connection between spin foams and Loop Quantum Gravity. Finally, we perform the semiclassical analysis for a single 4-simplex amplitude and find that for a set of Lorentzian boundary data, the leading order is given by the exponential of the Regge action.

DISCIPLINE : Physique théorique.

MOTS-CLES : Mousses de spin ; gravité quantique à boucles ; géométrie simpliciale.

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“Espero que este livro seja detestado. Isso não prova que ele seja bom, mas me liberta. O maior castigo do artista é ser gostado.

...

É verdade que muito eu já tenho recomeçado ... Só que nunca me veio uma sensação tão livre de recomeço.”

Mário de Andrade

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This thesis has traveled quite a lot, and my thanks go to the many friends that received me in these last months. Fabiana, Francesca, Kristen, Simone, Daniele. Each chapter is attached to a different place, to different memories. Ideas, places and memories are linked by the words written here.

Many thanks to the group in Marseille, for the amazing scientific atmosphere. I have profited immensely from the discussions with the people there. To Jon Engle, for a collaboration that led to many of the results presented in this manuscript. To Daniele Oriti for a careful reading of the manuscript. To Michael Reisenberger for very helpful remarks and suggestions.

To Carlo Rovelli, for his guiding, patience and physical insight.

Preface

Le travail présenté ici a eu comme principale motivation la compréhension de l'espace de bord associé à des modèles de mousses de spin en gravité quantique. Une telle compréhension est de grande importance pour le calcul des observables, notamment le propagateur du graviton (Rovelli 2006). Le modèle couramment utilisé pour ces calculs a été le modèle de Barrett et Crane (BC) paru dans (Barrett et Crane 1998 et 2000). Des inconsistances dans le calcul du propagateur en utilisant le modèle BC ont été reportés dans (Alesci et Rovelli 2007). En suivant ce travail, nous nous sommes mis à réviser la construction du modèle BC et cela a donné lieu à la construction d'une nouvelle classe de modèles. La définition et étude de ces modèles a été présentée dans une série d'articles :

[1] Engle J., Pereira R., and Rovelli C. (2007) The Loop-quantum-gravity vertex-amplitude *Phys. Rev. Lett.* **99** 161301.

[2] Engle J., Pereira R., and Rovelli C. (2008) Flipped spinfoam vertex and loop gravity *Nucl. Phys.* B**798** 251-290.

[3] Pereira R. (2008) Lorentzian LQG vertex amplitude *Class. Quant. Grav.* **25** 085013.

[4] Engle J., Livine E., Pereira R. and Rovelli C. (2008) LQG vertex with finite Immirzi parameter *Nucl. Phys.* B**799** 136-149.

[5] Engle J. and Pereira R. (2008) Coherent states, constraint classes, and area operators in the new spin-foam models *Class. Quant. Grav.* **25** 105010.

[6] Engle J. and Pereira R. (2009) Regularization and finiteness of the Lorentzian LQG vertices *Phys. Rev.* D **79** 084034.

[7] Barrett J.W., Dowdall R.J., Fairbairn W.J., Hellmann F. and Pereira R. (2009) Lorentzian spin foam amplitudes : Graphical calculus and asymptotics. `arXiv:0907.2440 [gr-qc]`.

Les modèles ont été définis d'abord pour le cas de signature Euclidienne, dans [1] et [2], et généralisés ensuite pour le cas de signature Lorentzienne dans [3]. Les premiers modèles définis ont été nommés *Flipped* et comme on verra plus tard correspondent au cas où le paramètre de Immirzi est fixé à zéro. Les modèles pour un paramètre de Immirzi arbitraire ont été donnés dans [4]. Dans ce manuscrit nous nous concentrerons dans le cas de signature Lorentzienne.

L'organisation de ce manuscrit est la suivante. Dans le premier chapitre nous donnerons une introduction au domaine des mousses de spin. L'idée est de donner quelques éléments essentiels pour la compréhension de la suite du texte et au même temps introduire le travail présenté ici dans le contexte plus large du domaine. Dans le deuxième chapitre nous présenterons la construction de l'amplitude pour un 4-simplexe, en décrivant l'espace de phase classique qui lui est associé. On finira ce chapitre avec une preuve que cette amplitude est finie, résultat qui est paru dans [6]. Les différences entre notre construction et le modèle BC seront discutées au fur et à mesure. Dans le chapitre 3 nous construirons l'amplitude pour une triangulation arbitraire. En utilisant la notion d'état cohérent nous présenterons cette amplitude sous la forme d'une somme sur des histoires classiques. Des modèles de mousses de spin construits de cette façon ont été considérés en premier par (Livine et Speziale 2007) dans le contexte de la théorie BF SU(2) et ensuite par (Freidel et Krasnov 2008) pour des modèles de la gravité en 4 dimensions en signature Euclidienne. Voir aussi (Conrady et Freidel 2008a) pour la construction de l'intégrale de chemin. Ce chapitre consiste donc d'une adaptation au modèle Lorentzian présenté ici de leur construction. Dans le chapitre 4 l'analyse semi-classique de l'amplitude d'un 4-simplexe est présentée. Cela nous permettra de relier cette amplitude à une géométrie de Regge et aussi à vérifier quelques hypothèses dans la procédure de quantification faites au chapitre 2. Ces résultats sont parus dans [7]. Le dernier chapitre est consacré à une conclusion où nous discuterons quelques problèmes laissés en ouvert.

Chapitre 1

Introduction

Dans ce chapitre nous réviserons la littérature importante pour la compréhension du travail présenté dans cette thèse avec le but de motiver les résultats présentés dans les chapitres suivants. La présentation suivra un point de vue historique en essayant de mettre en contexte le travail présenté ici. Nous essayerons d'emphatiser la pluridisciplinarité caractéristique au domaine et comment le concept de géométrie quantique joue le rôle de point de rencontre de différentes approches. Quelques sous sections seront plus techniques car elles contiennent des résultats qui seront importants pour la suite.

Préliminaires

Cette thèse appartient à un domaine caractérisé par la pluralité des influences et l'échange d'idées parmi différents domaines de la physique et des mathématiques. L'approche dite des mousses de spin pour la gravité quantique peut être vue comme venant des Théories des Champs Topologiques, où la gravité est formulée comme une théorie BF soumise à des contraintes. Les mousses de spin peuvent être considérées aussi comme une version sur réseau de la Relativité Générale, dans l'esprit de la Gravité Quantique à Boucles, et représente dans ce contexte un essai de construction de l'opérateur Hamiltonien pour cette théorie. Une troisième possibilité c'est de considérer les mousses de spin comme une réécriture du calcul de Regge, avec des variables différentes.

En suivant ce dernier point de vue, l'introduction historique que l'on prétend donner dans ce chapitre demeure avec le papier de Regge (Regge 1961), qui a proposé une nouvelle voie pour traiter la RG classique. L'idée essentielle était d'enlever le rôle prédominant que les transformations par difféomorphismes avait dans les théories des champs, dont le titre du papier "General Relativity without coordinates". Il est raisonnable de penser que ce saut conceptuel puisse avoir des conséquences fondamentales pour la quantification de la théorie. Le fait que les difféomorphismes doivent être traités différemment des autres symétries de jauge présentes dans la nature est probablement l'idée la plus remarquable de ce papier.

Le calcul de Regge a une histoire pleine de détours, en ayant expérimenté à la fois des moments de grand enthousiasme et à la fois des moments de complet abandon. Son importance a été remarquée par Wheeler, non seulement conceptuellement mais aussi comme un outil pour la relativité numérique, dans son cours à l'école aux Houches (Wheeler 1964). Dans ce même cours il propose l'idée de la mousse de l'espace-temps, selon laquelle l'espace-temps paraît continu à des grandes échelles mais doit avoir des courbures importantes avec éventuellement des différentes topologies à des échelles petites. Cela arriverait car à ces très petites échelles on doit attendre des déviations importantes de la platitude et donc l'occurrence de collapse gravitationnel. Ce concept de la mousse d'espace-temps a influencé quasiment toutes les approches pour la gravité quantique jusqu'au moment.

Ces deux idées, celle du calcul de Regge et celle de la mousse de l'espace-temps seront reconsidérées plus tard par Hawking (Hawking 1978) comme une façon d'implémenter l'intégrale de chemin (Euclidienne) pour la gravité quantique. L'avantage d'utiliser le calcul de Regge est qu'il est naturellement adapté à des extensions à des différentes topologies.

Minisuperespace simplicial

La construction de l'intégrale de chemin pour le calcul de Regge a été décrite par Hartle (Hartle 1985). Nous la révisons ici et cela nous permettra d'établir le programme pour la suite quand on traitera les mousses de spin. Hartle s'est concentré sur la gravité Euclidienne et nous adapterons ici la construction pour la signature Lorentzienne. Des questions importantes dans le contexte Euclidien, comme par exemple le fait que l'action gravitationnelle n'a pas de minimum dû à des transformations conformes, ne seront pas touchées ici.

On s'intéresse principalement à l'intégrale fonctionnelle :

$$\Psi(\text{3-geometries } h) = \sum_{\text{4-geometries } g} e^{i\frac{S(g)}{\hbar}}. \tag{1.1}$$

Les géométries 4-dimensionnelles g sur lesquelles on somme doivent être restreintes aux géométries 3-dimensionnelles h au bord de la région d'espace-temps considérée. Dans le cas Euclidien, la fonctionnelle $\Psi(h)$ peut être interprétée comme la fonction d'onde de l'univers (Hartle and Hawking 1983) [1].

1. Une intégrale de chemin pour la gravité a été proposée en premier dans (Misner 1957) d'après une suggestion par Wheeler. La définition de l'expression au dessus est en fait un problème dans les cours de Wheeler aux Houches.

Tout le problème réside dans une définition précise de cette somme. La stratégie suivie ici est de restreindre la somme à des géométries simpliciales, ce que l'on appelle une approximation de minisuperespace simplicial pour l'intégrale fonctionnelle. Pour bien spécifier une configuration géométrique, considérons un réseau des simplexes Σ, c-à-d, considère un ensemble de vertex du réseau et les combinaisons permettant de former les autres simplexes de la triangulation : segments, triangles, tétraèdres et 4-simplexes. On les dénotera e, t, τ et σ, respectivement, en suivant la notation anglaise. On pourra aussi utiliser la triangulation duale Σ^*, où les vertexes v sont duaux aux 4-simplexes, les segments (duaux) e^* sont duaux aux tétraèdres et les faces f sont duales aux triangles.

Ensuite, on associe des longueurs carrées l_e^2 à chaque segment du réseau. Selon le signe de l_e^2 le segment peut être du type espace, temps ou nul. Ces longueurs ne sont pas toutes indépendantes et satisfont à un certain nombre de contraintes qui garantissent que des 4-simplexes géométriques peuvent bien être reconstruits [2]. L'intégrale fonctionnelle s'écrit :

$$\Psi(l_e^2, e \subset \partial\Sigma_1) = \int_{\mathcal{C}} d\mu(l_e^2, e \subset int\Sigma_1)\, e^{i\frac{S_R(l_e^2)}{\hbar}}. \tag{1.2}$$

$\partial\Sigma$ et $int\Sigma$ dénotent respectivement le bord et l'intérieur du réseau simplicial Σ. $d\mu(l_e)$ est une certaine mesure d'intégration sur les longueurs des segments à l'intérieur de la triangulation. L'intégrale au dessus n'est pas en général bien définie et un certain choix de contour d'intégration $\mathcal{C}$ doit être fait. L'action S_R est donnée par :

$$S_R = \sum_{t\subset\Sigma_2} A(t)\varepsilon(t). \tag{1.3}$$

$A(t)$ est l'aire du triangle t. $\varepsilon(t)$ est l'angle de déficit autour ce triangle et est défini par :

$$\varepsilon(t) := \sum_v \theta(t, v)\,, \tag{1.4}$$

où la somme est sur les 4-simplexes v partageant le triangle t et $\theta(v, t)$ est l'angle diédral entre les deux tétraèdres dans v partageant t (voir la figure 1 pour un exemple en deux dimensions). L'angle dihédral sera défini au chapitre 2.

Remarque que, supposant la mesure et le contour d'intégration bien définis, l'expression au dessus nous fournit directement une évaluation numérique de l'intégrale fonctionnelle. Aussi, l'approximation simpliciale intègre de façon très naturelle des topologies différentes dans la somme, ce qui permet une discussion du rôle que la topologie peut

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2. Une condition suffisante est que la métrique $g_{\mu\nu}(v)$ associée au 4-simplexe v, et reconstruite avec les 10 longueurs associées aux segments formant ce simplexe, ait une signature $(-, +, +, +)$ (voir Sorkin 1974)

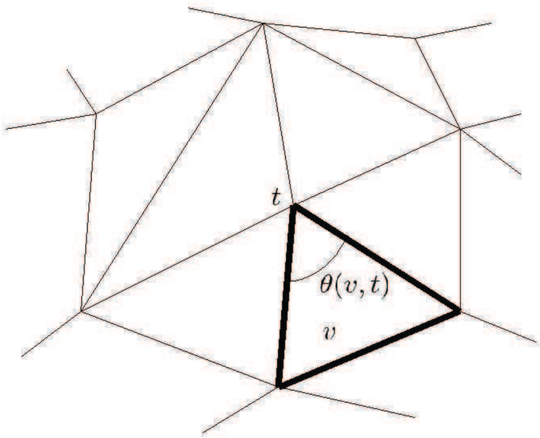

FIGURE 1.1 – Exemple d'une triangulation simpliciale en 2d.

jouer en gravité quantique. L'espoir est qu'une telle stratégie puisse donner des résultats physiquement raisonnables, même avec une approximation grossière. L'avantage de cette approche est que, en principe, on pourrait améliorer l'approximation autant que l'on veut en prenant des triangulations de plus en plus fines.

Les questions manquantes sont la mesure et le contour d'intégration. Une condition pour le choix de la mesure est que la loi de composition pour les amplitudes quantiques soit satisfaite. Le contour d'intégration doit être choisi de façon que l'intégrale soit convergente et t.q. elle représente correctement une somme sur des géométries compactes.

Un choix naturel pour la mesure pourrait être :

$$d\mu(l_e^2) = \mu(l_e^2) \prod_{e \subset int\Sigma} dl_e^2, \tag{1.5}$$

où dl_e^2 est la mesure de Lebesgue et

$$\mu(l_e^2) = \begin{cases} 1 & \text{si inégalités simpliciales satisfaites} \\ 0 & \text{sinon} \end{cases} \tag{1.6}$$

Un bon choix de la mesure est cependant un problème en ouvert dans le domaine. On verra plus tard que l'approche des mousses de spin suggère naturellement une classe de mesures. Le choix de contour est également problématique et on y reviendra plus tard quand on essayera de définir l'intégrale de chemin avec des mousses de spin.

De l'autre coté, on pourrait envisager qu'un choix de contour ne soit pas absolument nécessaire. En effet, éventuellement nous sommes intéressés au calcul des valeurs moyennes d'observables au bord de la triangulation Σ. On s'intéresse à des observables au bord car, à la fin, ce n'est que des mesures faites au bord d'une région de l'espace-temps qui seront accessibles à des expériences. La valeur moyenne d'un observable est donnée par :

$$\langle\mathcal{O}\rangle := \frac{\int_{\mathcal{C}} d\mu(l_e^2, e \subset \partial\Sigma_1)\Psi(l_e^2)\mathcal{O}(l_e^2)\Psi_0(l_e^2)}{\int_{\mathcal{C}} d\mu(l_e^2, e \subset \partial\Sigma_1)\Psi(l_e^2)\Psi_0(l_e^2)}. \tag{1.7}$$

$\Psi_0(l_e^2)$ représente un certain état du vide - ou du bord - et son choix dépend de l'observation que l'on veut faire.

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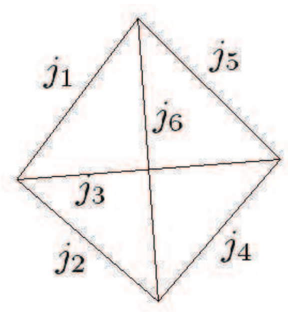
Un modèle pour la gravité, en utilisant le calcul de Regge - et dans le contexte de la gravité à 3 dimensions - a été proposé par Regge lui même en collaboration avec Ponzano (Ponzano et Regge 1968), en suivant un chemin tout à fait inattendu. Ce modèle représente le premier exemple d'une mousse de spin pour la gravité.

Modèle de Ponzano-Regge

Le point de départ pour Ponzano et Regge a été de considérer le comportement asymptotique du $6j$ symbole de Wigner :

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

A fin d'énoncer leur résultat, on représente un $6j$ symbole comme un tétraèdre, chaque spin j_i étant associé à un segment de ce tétraèdre :



Les longueurs géométriques sont données par $j_i + \frac{1}{2}$. Le comportement asymptotique du $6j$ pour des spins larges peut être séparé en deux cas différents, selon le signe du volume carré V^2 du tétraèdre construit avec les 6 spins. Si $V^2 \geq 0$ alors un vrai tétraèdre géométrique peut être construit. Le cas $V^2 < 0$ peut être vu comme un extension à la signature Lorentzienne (voir Barrett et Foxon 1994). Pour $V^2 > 0$, Ponzano et Regge obtienne la formule suivante :

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_e \left(j_e + \frac{1}{2}\right)\theta_e + \frac{\pi}{4}\right) \tag{1.8}$$

Pour $V^2 < 0$, la formule asymptotique est donnée par :

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{2\sqrt{12\pi|V|}} \cos\Phi \exp\left(-|\sum_e \left(j_e + \frac{1}{2}\right) Im\theta_e|\right) \tag{1.9}$$

où $\Phi := \sum_e j_e Re\theta_e$. L'exponentielle représente un effet tunnel vers la région $V^2 < 0$. Le cas d'intérêt pour nous est celui où un vrai tétraèdre peut être construit et donc $V^2 > 0$. Le comportement semiclassique du $6j$ dans ce cas est décrit par l'action de Regge pour le tétraèdre :

$$S_{Regge} = \sum_e \left(j_e + \frac{1}{2}\right)\theta_e. \tag{1.10}$$

Ce résultat surprenant a conduit les auteurs à proposer le $6j$ comme le point de départ pour la construction d'un modèle pour la gravité quantique en trois dimensions. Le modèle est spécifié par une amplitude associée à chaque réseau simplicial :

$$Z_{PR} = \sum_{j_e} (-1)^\chi \prod_e (2j_e + 1) \prod_\tau \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_\tau \tag{1.11}$$

où e et τ sont des segments et des tétraèdres de la triangulation. Il est important de remarquer que l'amplitude ainsi définie est en général infinie et doit être régularisée. Ponzano et Regge proposent une procédure de régularisation et d'autres procédures sont aussi possibles.

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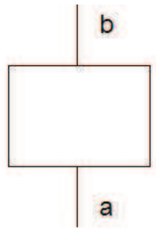
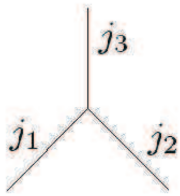
Le fait que la théorie des représentations du groupe SU(2) puisse être utile pour la gravité quantique a été remarqué par Penrose (Penrose 1971), en suivant une approche complètement indépendante. Son intuition était que l'espace-temps doit être construit à partir de structures complètement combinatoires, données par des réseaux de spin.

Réseaux de spin

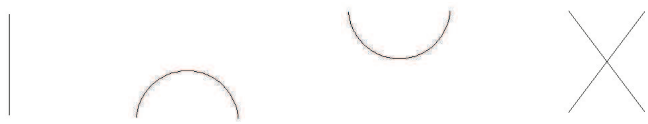
Un réseau de spin est donné par un graphe dans le plan, chaque ligne du graphe portant une représentation irréductible du groupe SU(2). Chaque réseau de spin encode un certain calcul avec des tenseurs invariants sous l'action de SU(2). On suivra ici (Barrett et Naish-Guzman 2009). Considère deux représentations de SU(2) a et b, pas nécessairement irréductibles. Un opérateur d'entrelacement entre a et b est représenté dans un diagramme par une boite connectant deux lignes externes portant les représentations a et b :



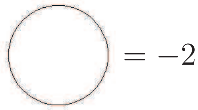
Un exemple est donné par l'opérateur d'entrelacement trois-valent :



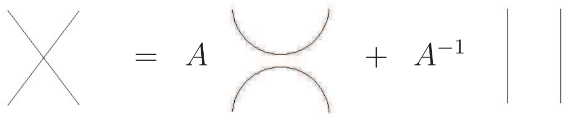
où $a = j_1 \otimes j_2$ et b=j_3.
Des opérateurs d'entrelacement peuvent être composés, soit horizontalement, et dans ce cas on représentera les deux diagrammes dans la même ligne, soit verticalement, où les lignes doivent se rencontrer et former des nouvelles lignes continues. Tout diagramme peut être construit à partir de la représentation fondamentale 1/2 et une ligne sans label portera toujours la représentation fondamentale. Tout diagramme peut être construit à partir des diagrammes de base suivants :



soit, l'identité, le max, le min et le croisement, resp. Les lois pour le calcul diagrammatique ont été introduites par Penrose, sous la forme de son calcul binoriel, et généralisées plus tard par Kauffman (Kauffman 1994) pour le cas q-déformé. Kauffman introduit un paramètre de déformation A, $A^2 = q$. Le calcul binoriel correspond à $A = -1$. Les lois de base sont les suivantes :



et



Ces lois peuvent être comprises comme des opérations avec des tenseurs transformant sous certaines représentations de SU(2). Le diagramme min est représenté par le tenseur antisymétrique avec des indices en bas ϵ_{mn} et le max avec le même tenseur avec des indices en haut ϵ^{mn}. La première identité vient donc du calcul suivant : $\epsilon_{mn}\epsilon^{mn} = -\delta^m_m = -2$. En évaluant la partie à droite de la deuxième identité, pour $A^2 = 1$, on obtient : $A(\epsilon_{mn}\epsilon^{kl} + \delta^k_m\delta^l_n) = A\delta^k_n\delta^l_m$. Ce qui nous apprend comment défaire un croisement dans un diagramme. Cela définit une matrice $\mathcal{R} : a \otimes b \to b \otimes a$ donnée par l'application qui interchange les deux arguments fois A.

Cela définit un calcul pour des diagrammes dessinés dans le plan. On pourrait aussi considérer des diagrammes dessinés librement en trois dimensions en gardant pour cela l'ordre des arguments en chaque vertex.

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L'utilisation des réseaux de spin a été proprement justifiée par l'analyse canonique de la Gravité Quantique à Boucles (Rovelli et Smolin 1988), où les réseaux de spins apparaissent comme une base pour l'espace d'Hilbert cinématique invariant par l'action des difféomorphismes. La stratégie en GQB a été d'implémenter sur un réseau les nouvelles variables pour la RG découvertes par Ashtekar (Ashtekar 1986), qui on rendu la gravité beaucoup plus proche des théories de jauge. Il est important de remarquer que les similarités avec les théories de jauge regardent seulement le group de Lorentz interne et pas les transformations par difféomorphisme. Toute la magique de la GQB est précisement de traiter ces deux groupes de façons complètement différentes. Le groupe de Lorentz est traité plus ou moins comme un groupe de jauge interne en Théorie de Jauge sur Réseau. Les difféos sont traités directement par la discrétisation, ce qui nous fait revenir à l'intuition de Regge.

La GQB nous fournit une procédure de quantification précise et générale, en établissant une nouvelle façon de penser la quantification canonique en améliorant la procédure standard de Dirac et autres (Dirac 1964).

Le problème majeur de quantifier la gravité n'a cependant pas encore été résolu. La difficulté est liée à la construction de l'opérateur Hamiltonien, plus précisement aux plusieurs ambiguités avec lesquelles nous sommes confrontées en essayant de le définir. Le point est que peut être la voie canonique pour quantifier la gravité n'est pas le choix le plus naturel, la gravité étant en essence covariante.

Une approche plus covariante serait donc plus appropriée pour le problème en question. Une telle approche est donnée par l'intégrale de chemin de Feynman. Comme discuté avant, cette approche a été suivie par Hawking et collaborateurs et là le calcul de Regge a été utilisé pour décrire le concept de la mousse d'espace-temps. Les mousses de spin nous fournissent un nouveau regard à l'intégrale de chemin pour la gravité. C'est un nouveau regard parce que des variables différentes sont utilisées pour décrire des géométries simpliciales, telles que la gravité est reformulée comme une théorie de jauge sur réseau. Cela suggère naturelement une connexion avec la GQB. En effet les modèles de mousses de spin en 4d ont été introduits en premier par Reisenberger (Reisenberger 1994, voir aussi Iwasaki 1994) justement avec la motivation d'introduire un opérateur Hamiltonien pour la GQB. Ceux ci utilisent des techniques venant des modèles topologiques sur réseau introduits plut tôt par Ooguri (Ooguri 1992, voir aussi Boulatov 1992) dans le contexte des théories des champs topologiques. Le point de départ pour Reisenberger a été la théorie de Plebanski (Plebanski 1997), où la gravité est formulée comme une théorie BF (Horowitz 1989) contrainte. Car les mousses de spin sont dérivées à partir des théories topologiques, une mesure naturelle est suggerée.

La procédure proposée par Reisenberger a été d'abord de quantifier la partie topologique, à la Ooguri, et après d'imposer les contraintes, dans le même esprit de la recette de Dirac pour l'imposition de contraintes pour la quantification des systèmes canoniques.

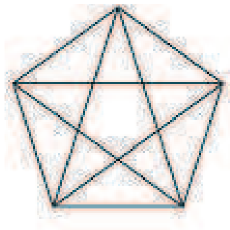
Un modèle topologique sur réseau est donné par une somme sur des représentations irréductibles de certaines amplitudes associées à des simplexes formant le réseau en question. Cela définit une fonction de partition associée à chaque complexe Σ :

$$Z_\Sigma = \sum_{\{j\}} \prod_t A_t \prod_\tau A_\tau \prod_v A_v \tag{1.12}$$

Les contraintes qui réduisent la théorie BF à la gravité vont donc restreindre les repésentations sur lesquelles on somme, ce qui brise l'invariance topologique du modèle et introduit des dégrés de libertés locaux. Cette procédure générale, il faut bien le remarquer, est encore tentative. La question qui se pose est si les dégrés de liberté physiques de la théorie continue sont bien récupérés après l'imposition de contraintes et qu'une certaine limite continue soit définie.

La difficulté principale avec l'approche de Reisenberger a été l'imposition des contraintes. Cela est dû au fait que les contraintes de la théorie SU(2) qu'il était en train d'utiliser était trop compliquées pour être résolues exactement. Reisenberger a ainsi proposé une imposition plus faible des contraintes, mais un modèle - au sens décrit ci dessus - n'a pas pu être écrit (Reisenbereger 1997, voir aussi Reisenberger et Rovelli 1997).

Un chemin différent a été suivi par Barrett et Crane (Barrett et Crane 1998). L'idée était de commencer avec le tétrahèdre quantique de Barbieri (Barbieri 1998, voir aussi Baez et Barrett 1999) en 3 dimensions et cette idée a été utilisée pour la quantification du 4-simplexe en quatre dimensions. Les auteurs ont été capables de définir un modèle de mousse de spin, d'abord pour signature Euclidienne et ensuite généralisé pour la signature Lorentzienne (Barrett et Crane 2000). En focalisant sur le cas Lorentzien, leur modèle est basé sur une amplitude de vertex donnée par un symbole $10j$, labelé par dix nombres réels p_{ab}, associés à des segments duaux d'un graph avec 5 vertex labelés par $a, b = 1...5$:



Il est donné explicitement par :

$$10j(p_{ab}) := \int_{Q_1^{\times 4}} \prod_{a=1}^{4} \mathrm{d}x_a \prod_{ab} K_{p_{ab}}(x_a, x_b). \tag{1.13}$$

Q_1 est l'hyperboloïde dans l'espace de Minkowski et le propagateur $K_{p_{ab}}(x_a, x_b)$ est défini par

$$K_p(x, y) := \frac{\sin p\, r(x, y)}{p \sinh r(x, y)}, \tag{1.14}$$

où $r(x, y)$ est la distance hyperbolique entre x et y.

Nous suivrons ici un chemin similaire. Dans le prochain chapitre nous commencerons par réviser la construction de Barbieri pour le tétraèdre quantique en trois dimensions et cela motivera la discussion sur le 4-simplex. Nous identifierons l'espace de phase associé à ce système classique et nous lui proposérons une certaine quantification. Le résultat final sera une amplitude pour le 4-simplex jouant le rôle du symbole $10j$ dans le modèle BC. Cette amplitude de vertex sera utilisée comme base pour la construction du modèle de mousse de spin dans le chapitre suivant. Toujours quand approprié nous remarquerons les différences entre notre construction et celle du modèle BC.

Contents

Preface

The main motivation of the work presented here was to understand the boundary space related to spin foam models for gravity, this understanding being of great importance for the computation of observables, such as the graviton propagator (Rovelli 2006). The model used in these calculations was the Barrett-Crane (BC) model (Barrett and Crane 1998 and 2000) and some inconsistencies in computing components of the graviton propagator using the BC model were reported in (Alesci and Rovelli 2007). This has led to a review of the BC construction and to the proposition of a new class of models. The definition and study of these models have been given in a series of articles listed below:

[1] Engle J., Pereira R., and Rovelli C. (2007) The Loop-quantum-gravity vertex-amplitude *Phys. Rev. Lett.* **99** 161301.

[2] Engle J., Pereira R., and Rovelli C. (2008) Flipped spinfoam vertex and loop gravity *Nucl. Phys.* B**798** 251-290.

[3] Pereira R. (2008) Lorentzian LQG vertex amplitude *Class. Quant. Grav.* **25** 085013.

[4] Engle J., Livine E., Pereira R. and Rovelli C. (2008) LQG vertex with finite Immirzi parameter *Nucl. Phys.* B**799** 136-149.

[5] Engle J. and Pereira R. (2008) Coherent states, constraint classes, and area operators in the new spin-foam models *Class. Quant. Grav.* **25** 105010.

[6] Engle J. and Pereira R. (2009) Regularization and finiteness of the Lorentzian LQG vertices *Phys. Rev.* D **79** 084034.

[7] Barrett J.W., Dowdall R.J., Fairbairn W.J., Hellmann F. and Pereira R. (2009) Lorentzian spin foam amplitudes: Graphical calculus and asymptotics. `arXiv:0907.2440 [gr-qc]`.

The models were first defined for the Euclidean signature, in [1] and [2], and later generalized to the Lorentzian case in [3]. The first models defined for both signatures were called Flipped, and were later generalized with the introduction of the Immirzi parameter γ in [4]. We will see that the Flipped models correspond to the limit $\gamma \to 0$. We will focus here on the case of Lorentzian signature.

The outline of the manuscript is the following. In chapter 1 we give a historical introduction to the spin foam field, giving the basic background material for the understanding of the rest of the manuscript. The aim is to put the work presented here in the larger context of the field. In chapter 2 we give our construction for the 4-simplex amplitude. We present the classical phase space associated to a 4-simplex and propose a quantization procedure. We conclude the chapter with a proof of fineteness for the amplitude, given in [6]. The differences to the BC construction are pointed out when appropriate. In chapter 3 we construct an amplitude for a general triangulation by gluing simplices together. With the use of coherent sates we are able to write this amplitude as a sum over classical histories. Spin foam models using coherent states were first considered in (Livine and Speziale 2007) in the case of SU(2) BF theory and in (Freidel and Krasnov 2008) in the context of Euclidean gravity. See (Conrady and Freidel 2008a) for the construction of the path integral. We adapt their construction to the Lorentzian model presented here. In chapter 4 we perform the semiclassical analysis of the 4-simplex amplitude presented in chapter 2, and relate it to a Regge 4-simplex geometry, testing some of the assumptions made in the construction of this amplitude in chapter 2. The results of this chapter appeared in [7]. Finally we conclude and point out some open problems.

Chapter 1

Introduction and overview

In this chapter, we review the literature relevant for the work of this thesis and try to motivate the results presented in the following chapters. The presentation will be somehow historical, in order to put in context the work presented here. We try to emphasize the multi-disciplinary history of the field and how the concept of quantum geometry is the meeting point of very different approaches. Some subsections will be given a detailed technical description, as they represent important background material for the rest of the manuscript.

Preliminaries

The work of this thesis belongs to a field known for the plurality of influences and interchange between different domains of physics and mathematics. The spin foam approach to gravity can be seen as coming from Topological Field Theories, gravity being constructed as a constrained BF theory. It can be seen as well as a lattice version of General Relativity, in the spirit of Loop Quantum Gravity, representing in this context a tentative construction for a Hamiltonian evolution. A third possibility is to consider spin foams as a different look at Regge calculus, written in different variables.

From that perspective, the historical overview that we pretend to give in this chapter starts with Regge's paper (Regge 1961), where a conceptually new way of treating (classical) GR was proposed. It is conceptually new because it takes away the importance of continuous diffeomorphisms (diffeos) from the construction of the theory, hence the title "General Relativity without coordinates". It is reasonable to believe that this conceptual shift might have strong consequences for the quantization of the theory. The fact that diffeos should be treated differently than other gauge symmetries present in nature is probably the most remarkable idea of Regge's paper and, we believe, should be kept in mind.

Regge calculus has a long and twisted history, having followed times of great enthusiasm and times of abandon during the years. Its importance was first advertised by Wheeler, conceptually but also as a tool for numerical relativity, in his lecture notes at Les Houches (Wheeler 1964). He further introduces in the same course the idea of a space-time foam, according to which space-time appears smooth on large scales but is highly curved with possibly different topologies on very short scales. This is because, on these very short scales, one should expect a high deviation from flatness and therefore the appearance of wormholes and other forms of gravitational collapse. The idea of a space-time foam is a very influential concept and nearly every approach to quantum gravity aims at consistently reconstruct this scenario.

Both ideas, Regge calculus and the concept of the space-time foam, were reconsidered later by Hawking (Hawking 1978) as a way to implement a path integral for (Euclidean) quantum gravity. The advantage of Regge calculus is that it allows very naturally an extension to different topologies and hence fits well with the foam concept.

Simplicial minisuperspace

The construction of a path integral using Regge calculus was nicely described by Hartle (Hartle 1985). We review it here as it sets up the program we intend to follow later with spin foams. His construction was for Euclidean quantum gravity, and we will make small adjustments to deal with the Lorentzian signature. Important questions in the Euclidean context, such as the unboundedness of the gravitational action due to conformal transformations (Gibbons, Hawking and Perry 1978) will be sidelined.

The main object of interest is the functional integral:

$$\Psi(\text{3-geometries } h) = \sum_{\text{4-geometries } g} e^{i\frac{S(g)}{\hbar}}. \tag{1.1}$$

The 4-geometries g over which one sums over should agree with the 3-geometries h on the boundary of the region of space time considered. In the Euclidean case, the functional $\Psi(h)$ can be interpreted as the wave function of the universe (Hartle and Hawking 1983) [1].

Properly defining the sum above is the main goal of any path integral approach for gravity. The strategy here is to restrict the sum to simplicial geometries only, defining hence what Hartle calls a simplicial minisuperspace approximation for the functional integral. To specify a geometrical configuration, first fix a simplicial net Σ, that is, specify the vertices of the net and the combinations that make up the higher simplices

[1] The first appearance of a path integral for gravity was in (Misner 1957) after a suggestion by Wheeler. The definition of the expression above is actually a problem in Wheeler's lecture notes!

of the triangulation: edges, triangles, tetrahedra and 4-simplices. We will note them e, t, τ and σ respectively. We may also work with the dual graph Σ^*, where vertices v are dual to 4-simplices, (dual) edges $e*$ are dual to tetrahedra and faces f are dual to triangles.

Next, assign (squared) lengths l_e^2 to the edges of the net. According to the sign of l_e^2 the edge can be spacelike, timelike or null. The edge lengths are not all independent and satisfy a number of constraints guaranteeing that geometrical four simplices can be reconstructed out of them[2]. The functional integral takes the form:

$$\Psi(l_e^2, e \subset \partial\Sigma_1) = \int_{\mathcal{C}} d\mu(l_e^2, e \subset int\Sigma_1)\, e^{i\frac{S_R(l_e^2)}{\hbar}}. \tag{1.2}$$

$\partial\Sigma$ and $int\Sigma$ denote respectively the boundary and interior of the simplicial net Σ. The subscript denotes the subset of 1-simplices of Σ. In general Σ_n denotes the subset of n-simplices of Σ. We may drop the subscript when the context is clear. $d\mu(l_e)$ is a certain measure on the interior edge lengths. The integral above is generally not well defined and a certain contour of integration $\mathcal{C}$ has to be given. The action S_R is given by:

$$S_R = \sum_{t\subset\Sigma_2} A(t)\varepsilon(t). \tag{1.3}$$

$A(t)$ is the area of the triangle t. $\varepsilon(t)$ is the deficit angle around this triangle and is defined by:

$$\varepsilon(t) := \sum_v \theta(t, v)\ , \tag{1.4}$$

where the sum is over 4-simplices sharing the triangle t and $\theta(v, t)$ is the dihedral angle between the two tetrahedra in v sharing t (see figure 1 for a two dimensional example). The dihedral angles will be defined in chapter 2.

Some remarks are in order. With the measure and the contour of integration properly defined, the expression above leads directly to a numerical evaluation of the functional integral. Also, the simplicial approximation integrates easily different topologies in the sum, allowing for a discussion of the role of topology in quantum gravity. The hope is that it may give physically correct results even with very crude approximations. The great advantage of the simplicial minisuperspace approximation is that, in principle, one could refine the approximation as much as needed by taking finer and finer triangulations.

The missing points are the measure and the contour of integration. A condition for the choice of measure is that the composition law for quantum amplitudes should

[2]A sufficient condition is that the metric $g_{\mu\nu}(v)$ associated to a 4-simplex v and constructed out of the 10 edge lengths forming v should have signature $(-,+,+,+)$ (see Sorkin 1974).

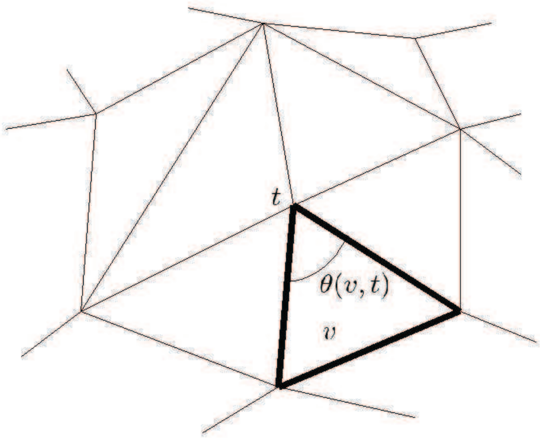


Figure 1.1: Example of a simplicial triangulation in 2d.

be satisfied. The contour of integration should be chosen such that the integral is convergent and s.t. it correctly represents a sum over compact geometries.

A natural choice for the measure could be:

$$d\mu(l_e^2) = \mu(l_e^2) \prod_{e \subset int\Sigma} dl_e^2, \tag{1.5}$$

where dl_e^2 is the Lebesgue measure and

$$\mu(l_e^2) = \begin{cases} 1 & \text{if simplicial inequalities satisfied} \\ 0 & \text{if not} \end{cases} \tag{1.6}$$

The correct choice of measure is still an open problem though. We will see that the spin foam approach to quantum gravity suggests a natural measure, or at least reduces considerably the ambiguities. The choice of contour is likewise problematic and we shall not discuss this further. We will discuss a possible choice of contour when dealing with the spin foam path integral.

The interesting thing to notice is that one might not need to define the contour completely. In fact, eventually one is interested in computing expectation values of observables. We are interested in observables defined on the boundary of Σ. This is because at the end only observations taken on the boundary of a region of space time are accessible for experiences. The expectation value of an observable is then given by:

$$\langle \mathcal{O} \rangle := \frac{\int_{\mathcal{C}} d\mu(l_e^2, e \subset \partial\Sigma_1) \Psi(l_e^2) \mathcal{O}(l_e^2) \Psi_0(l_e^2)}{\int_{\mathcal{C}} d\mu(l_e^2, e \subset \partial\Sigma_1) \Psi(l_e^2) \Psi_0(l_e^2)}. \tag{1.7}$$

$\Psi_0(l_e^2)$ represents a certain vacuum - or boundary - state that should be chosen according to the problem in hand. Hartle interprets the functional integral $\Psi(l_e^2)$ itself as the vacuum state and then, in his definition of the expectation value of an observable Ψ_0 and Ψ are identified. This more general definition we use here is given by the general boundary formulation of (Oeckl 2003).

The idea of computing directly expectation values is that possible infinities that would come out independently from the numerator and denominator of the above expression might cancel out, given a clever choice of observable. We will come back to this after we define the spin foam path integral.

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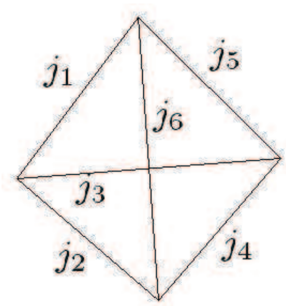
A quantum model for gravity, using Regge calculus and in the context of 3d gravity, was actually first proposed by Regge himself, together with Ponzano (Ponzano and Regge 1968), following a quite unexpected route. This is the first example of a spin foam model for gravity.

Ponzano-Regge model

The starting point for Ponzano and Regge was to consider the asymptotic behavior of Wigner's $6j$ symbol for large spins:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

To state their result, represent the $6j$ symbol as a tetrahedron, each spin j_i being associated to an edge in this tetrahedron:



The geometric edge lengths are given by $j_i + \frac{1}{2}$. The asymptotic behavior of the $6j$ for large spins can be separated in two different cases, depending on the sign of the volume squared V^2 of the tetrahedron constructed out of the six spins. If $V^2 \geq 0$ a true geometric tetrahedron can be constructed. The $V^2 < 0$ case can be seen as an extension to Lorentzian signature (see Barrett and Foxon 1994). For $V^2 > 0$, Ponzano and Regge obtain the following formula:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_e (j_e + \frac{1}{2})\theta_e + \frac{\pi}{4} \right) \tag{1.8}$$

where the sum is over the edges of the tetrahedron and θ_e is the dihedral angle on the edge e. For $V^2 < 0$, the asymptotic formula is given by:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \sim \frac{1}{2\sqrt{12\pi|V|}} \cos\Phi \exp\left(-|\sum_e (j_e + \frac{1}{2}) Im\theta_e| \right) \tag{1.9}$$

where $\Phi := \sum_e j_e Re\theta_e$. The exponential decrease represents a tunnel effect into the region $V^2 < 0$.

The interest for us is that in the geometrical case $V^2 > 0$, the asymptotic behavior of the $6j$ symbol is described by the 3d Regge action for a tetrahedron:

$$S_{Regge} = \sum_e (j_e + \frac{1}{2})\theta_e. \tag{1.10}$$

This surprising result led Ponzano and Regge to propose the use of $6j$ symbols as the building block of a quantum model for 3d gravity. The model is specified by an amplitude associated to each simplicial geometry:

$$Z_{PR} = \sum_{j_e} (-1)^\chi \prod_e (2j_e + 1) \prod_\tau \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}_\tau \tag{1.11}$$

where e and τ label edges and tetrahedra of the triangulation, and χ is a certain linear function on the spins (for a precise definition of the model, including signs, see Barrett and Naish-Guzman 2009). It should also be noted that the amplitude as defined above is generally infinite and should be regularized in some way. Ponzano and Regge give a prescription for regularizing it, and other prescriptions are also possible, but we will leave that for latter. Understanding properly the spin foam quantization of gravity in 3d is a necessary step for understanding the physically relevant 4d case, as most of the steps one has to go through in 3d are repeated in 4d. One crucial difference remains. In 3d the model is independent of the triangulation while in 4d it is not. This is a simplicial counter part of the fact that continuum 3d gravity can be described by a topological field theory (Witten 1988) while 4d gravity can be described by a constrained topological field theory. The constraints break the topological invariance of the theory.

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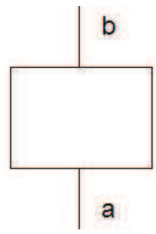
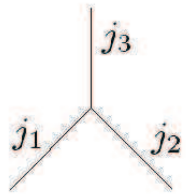
That gravity could be approached from such an unexpected road should come as a surprise, or maybe not. The fact that gravity may be related to the representation theory of SU(2) was also noticed by Penrose (Penrose 1971) from a completely different perspective. His intuition was that space-time should be constructed from purely combinatorial structures. The inspiration was his twistor program for quantum gravity. According to Penrose, space-time should be constructed out of spin-networks.

Spin networks

A spin network is given by a planar graph, each line of the graph carrying an irreducible representation of SU(2). Each spin network encodes a certain calculation with invariant tensors under the action of SU(2). We will follow closely (Barrett and Naish-Guzman 2009). Consider two, not necessarily irreducible, representations of SU(2) a and b. An intertwining operator from a to b is represented diagrammatically by a box connecting two external lines carrying the representations a and b:



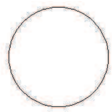
A basic example is given by the three-valent intertwiner:



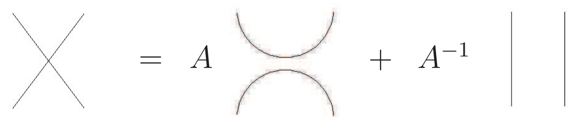
where $a = j_1 \otimes j_2$ and b=j_3. A triple of spins (j_1, j_2, j_3) is called admissible if j_3 is in the Clebsch-Gordan decomposition of $j_1 \otimes j_2$. To each triple (j_1, j_2, j_3) there is a canonical choice of intertwining operator (for instance, s.t. the Clebsch-Gordan coefficients are real).

Intertwiners can be composed together, either horizontally, where one just represent the two diagrams on the same line, or vertically, where lines have to meet to form continuous lines. Any diagram can be built out of the fundamental representation 1/2 and a line without a label will always carry the spin 1/2 representation. The basic building blocks in the fundamental are the following diagrams:

denoting the identity, max, min and crossing diagrams respectively. The basic rules for the diagrammatic calculus were first introduced by Penrose, in his binor calculus, and generalized later by Kauffman (Kauffman 1994) to the q-deformed case. Kauffman introduces a deformation parameter A, $A^2 = q$. The basic rules are:

 $= -2$

and



The important case of interest for us here is $A^2 = 1$, to which we restrict from now on. Note that Penrose's binor calculus corresponds to $A = -1$.These rules can be understood in terms of operations with tensors transforming in certain representations of SU(2). The min diagram is represented by the antisymmetric tensor with indices down ϵ_{mn} and the max by the antisymmetric tensor with indices up ϵ^{mn}. The loop identity then comes directly from the computation: $\epsilon_{mn}\epsilon^{mn} = -\delta^m_m = -2$. By evaluating the r.h.s. of the crossing identity: $A(\epsilon_{mn}\epsilon^{kl} + \delta^k_m\delta^l_n) = A\delta^k_n\delta^l_m$, one learns how to undo a crossing in a diagram. This defines an $\mathcal{R}$ matrix : $a \otimes b \to b \otimes a$ given by the flip map times A.

This defines the diagrammatic calculus represented as a diagram in the plane, keeping track of up and down indices. One could alternatively represent it with a graph freely drawn in three dimensions plus the ordering of legs on each vertex. The amplitude is not a property of the graph alone.

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Spin-networks have been given a proper justification via the canonical approach of Loop Quantum Gravity (Rovelli and Smolin 1988), where spin-nets appear as a basis for the kinematical diff-invariant Hilbert space of theory. The strategy in LQG is to implement in a lattice way the new variables for GR discovered by Ashtekar (Ashtekar 1986), which made gravity much closer to Gauge Theories. It should be noted that the similarities with Gauge Theories regard the internal Lorentz group and not the diffeomorphism group. The magic of LQG is precisely to treat the two groups in completely different ways. The Lorentz group is dealt with pretty much the same way as one deals with the internal gauge groups of standard matter in Lattice Gauge Theory, while the diffeos are taken care of through the discretization, bringing us back again to Regge's idea.

LQG provides us with a precise and general quantization procedure. It establishes a new way of thinking about quantization, at least in the canonical perspective, refining the standard quantization procedure of Dirac and others (Dirac 1964).

The main problem of quantizing gravity has not yet been solved though. The main difficulty is related to the construction of the Hamiltonian constraint, more precisely in the many ambiguities one is faced with when trying to define it. The point is that perhaps to quantize gravity from a canonical perspective might not be the most natural way, gravity being in essence covariant. It does not mean it cannot be done, just that the canonical splitting introduces far too many ambiguities, from which a physicist cannot really choose.

A more covariant, relativistic approach should be more suitable for the task. This is given by Feynman's path integral approach to Quantum Mechanics. As discussed above, this was followed by Hawking and collaborators and Regge calculus was used to describe the idea of a space time foam. Spin foams provide a new look at the path integral approach for gravity. It is a new look because simplicial geometry is written with different variables, such that gravity is reformulated as a (lattice) gauge theory[3]. This suggests naturally a connection with LQG. In fact, spin foam models for 4d gravity were first introduced by Reisenberger (Reisenberger 1994, see also Iwasaki 1994) - at the name of a lattice worldsheet formulation of gravity - with the motivation to construct an evolution operator between initial and final spin-network states. And this was to be used as an implementation of the Hamiltonian operator in LQG. It uses heavily the machinery of lattice topological models introduced earlier by Ooguri (Ooguri 1992; see also Boulatov 1992) in the context of topological field theories. The starting point for Reisenberger was Plebanski theory (Plebanski 1977), where gravity is formulated as a constrained BF theory (Horowitz 1989). Because spin foams are derived as constrained lattice topological models, a natural measure for the path integral is suggested, solving a severe problem of standard quantum Regge Calculus.

The procedure proposed by Reisenberger was to first quantize the topological piece, à la Ooguri, and then impose the constraints, in the same spirit as Dirac's recipe for imposing constraints in the canonical picture.

A topological lattice model is given by a sum over (irreducible) representations of certain amplitudes associated to simplices of which the lattice is formed. It defines a partition function associated to a given simplicial complex Σ:

[3]Hence the name "spin" replacing spacetime, spins being duals to group elements.

$$Z_\Sigma = \sum_{\{j\}} \prod_t A_t \prod_\tau A_\tau \prod_v A_v \tag{1.12}$$

The constraints that reduce BF theory to gravity should then restrict the representations over which one sums over, breaking the topological invariance of the model and introducing local degrees of freedom. This general procedure is, one should note, still tentative. Whether or not the local degrees of freedom on the continuum theory are recovered is still an open question, and a difficult one.

Because it is an important tool and also because it helps setting up the program for spin foams, let us take the time to describe Ooguri's model and to relate it to continuum BF theories.

Ooguri's model

Ooguri defines a lattice statistical model on a triangulated manifold in four dimensions associated to a group G. In the case of $G = \mathrm{SU}(2)$ the building block is the $15j$ symbol and the model can be seen as a four dimensional version of the Ponzano-Regge model. The model is shown to be a topological invariant and, for an orientable manifold, can be seen as a lattice version of BF theory. Ooguri also considers the extension to the q-deformed case, extending the Turaev-Viro (Turaev and Viro 1992) model to four dimensions.

Let us start by defining the model. We will later prove triangulation independence and relate it to continuous BF theory. Because of topological invariance, the continuum limit is trivial and the lattice version is actually exact. The model can be defined for any Lie group G as long as it is compact. For non-compact groups with a Plancherel decomposition (for instance semi-simple Lie groups, in particular $\mathrm{SL}(2,\mathbb{C})$) the model can be formally defined and needs to be regularized afterwards. In order not to overload the equations, we will restrict here to the case of $G = \mathrm{SU}(2)$ as the representation theory is quite simple, but we will try to keep the discussion as general as possible, such that the adaptation of the formulas presented to other groups should be an easy exercise. We will be mainly interested to the $G = \mathrm{SL}(2,\mathbb{C})$ case in the manuscript.

Start with a real-valued[4] function of four variables in G, $\phi(g_1, g_2, g_3, g_4) = \phi(g_i)$, $g_i \in G$, $i = 1, .., 4$. The Plancherel decomposition of this function gives:

$$\Phi(g_i) = \sum_{j_i, m_i, n_i} \Phi^{\{j_i\}}_{\{m_i n_i\}} \prod_i D^{j_i}_{m_i n_i}(g_i) \tag{1.13}$$

We require invariance under the right action of G:

[4] This restriction is not really necessary and we could consider simply complex functions.

$$\Phi(g_1U, g_2U, g_3U, g_4U) = \Phi(g_1, g_2, g_3, g_4), \tag{1.14}$$

with $U \in G$. This is equivalent to

$$\Phi(g_i) = \int_G dU\, \Phi(g_iU). \tag{1.15}$$

dU is the normalized Haar measure on G. This expression does not really make sense for a non compact group and should be regarded as formal in that case, as long as an invariant measure can be defined, as is the case of semi-simple Lie groups.

Let us now define the action:

$$\begin{aligned} S &= \frac{1}{2}\int \prod_i^4 dg_i\, \Phi^2(g_1, g_2, g_3, g_4) + \frac{\lambda}{5!}\int \prod_i^{10} dg_i\, \Phi(g_1, g_2, g_3, g_4) \\ &\quad \Phi(g_4, g_5, g_6, g_7)\Phi(g_7, g_3, g_8, g_9)\Phi(g_9, g_6, g_2, g_{10})\Phi(g_{10}, g_8, g_5, g_1). \end{aligned} \tag{1.16}$$

The motivation for this action is as follows. Each Φ is associated to a tetrahedron in a given triangulated manifold, with each group element associated to a triangle in this tetrahedron. The kinetic term $\sim \Phi^2$ in the action dictates the gluing of tetrahedra while the interaction term $\sim \Phi^5$ dictates the gluing of faces of tetrahedra to form a 4-simplex.

Some formulas for the integration of representation matrices will be useful in what follows:

$$\begin{aligned} \int_{\mathrm{SU(2)}} \bar{D}^{j_1}_{m_1n_1}(U) D^{j_2}_{m_2n_2}(U) &= \frac{\delta_{j_1j_2}}{d_{j_1}}\, \delta_{m_1m_2}\delta_{n_1n_2} \\ \int_{\mathrm{SU(2)}} \prod_{i=1}^{3} D^{j_i}_{m_in_i}(U) &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \\ \int_{\mathrm{SU(2)}} \prod_{i=1}^{4} D^{j_i}_{m_in_i}(U) &= \sum_{\iota} \bar{\iota}^{m_1m_2m_3m_4}\iota^{n_1n_2n_3n_4} \end{aligned} \tag{1.17}$$

$d_j := 2j + 1$ is the dimension of the representation j. $\left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ is Wigner's $3j$ symbol defined in terms of the Clebsch-Gordan coefficients by:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{d_{j_3}}} \langle j_1, j_2; m_1, m_2 | j_3, m_3\rangle. \tag{1.18}$$

We have also defined intertwining operator between the irreps $\{j_i\}$:

$$\iota^{\{m_i\}} := \sum_{mm'} \begin{pmatrix} j_1 & j_2 & \iota \\ m_1 & m_2 & m \end{pmatrix} g_{\iota}^{mm'} \begin{pmatrix} \iota & j_3 & j_4 \\ m' & m_3 & m_4 \end{pmatrix} \tag{1.19}$$

$g_j^{mm'}$ is generically the Killing-Cartan metric for the group G, but in the case of SU(2) is just the delta on the j representation $\delta^{(j)}_{mm'}$.
Using the formulas above and invariance under the right action of the group, the decomposition of $\Phi(g_i)$ can be written as:

$$\begin{aligned}
\Phi(g_i) &= \sum_{j_i m_i n_i} \Phi^{j_i}_{m_i n_i} \int dU \prod_i^4 D^{j_i}_{m_i n_i}(g_i U) = \\
&= \sum_{j_i m_i n_i} \Phi^{j_i}_{m_i n_i} \prod_i \int dU \sum_{p_i p_i'} D^{j_i}_{m_i p_i}(g_i) g^{p_i p_i'}_{j_i} D^{j_i}_{p_i' n_i}(U) = \\
&= \sum_{j_i m_i n_i} \Phi^{j_i}_{m_i n_i} \prod_i \sum_{p_i p_i'} D^{j_i}_{m_i p_i}(g_i) g^{p_i p_i'}_{j_i} \sum_{\iota} \overline{\iota}^{\{p_i'\}} \iota^{\{n_i\}} = \\
&= \sum_{\iota j_i m_i} \left(\sum_{n_i} \Phi^{j_i}_{m_i n_i} \iota^{\{n_i\}} \right) \sum_{p_i p_i'} \prod_i D^{j_i}_{m_i p_i}(g_i) g^{p_i p_i'}_{j_i} \overline{\iota}^{\{p_i'\}} = \\
&=: \sum_{\iota j_i m_i} \left(M^{j_i \iota}_{m_i} \right) \sum_{p_i p_i'} \prod_i D^{j_i}_{m_i p_i}(g_i) g^{p_i p_i'}_{j_i} \overline{\iota}^{\{p_i'\}}.
\end{aligned} \tag{1.20}$$

Reality of $\Phi(g_i)$ imposes the following condition on $M^{j_i \iota}_{m_i}$:

$$\bar{M}^{j_i \iota}_{m_i} = (-1)^{\sum_i (j_i - m_i)} M^{j_i \iota}_{-m_i} \tag{1.21}$$

We further require $\Phi(g_i)$ to be invariant under cyclic permutations of any three of its arguments, implying the following recoupling condition on $M^{j_i \iota}_{m_i}$:

$$M^{j_3 j_1; \iota; j_2 j_4}_{m_3 m_1; m_2 m_4} = \sum_{\iota'} (-1)^{j_1+j_2+j_3+j_4} \sqrt{d_\iota d_{\iota'}} \left\{ \begin{array}{ccc} j_1 & j_2 & \iota' \\ j_3 & j_4 & \iota \end{array} \right\} M^{j_1 j_2; \iota'; j_3 j_4}_{m_1 m_2; m_3 m_4} \tag{1.22}$$

After performing the group integrations the action can be written in terms of dual variables $M^{j_i \iota}_{m_i}$. The partition function is defined as:

$$Z = \int \prod_{\iota j_i m_i} dM^{\iota j_i}_{m_i} \, e^{-S(M)} \tag{1.23}$$

This can be seen as a generalization of matrix models to higher rank tensors and receives the name of Group Field Theory (GFT) in the literature (see Oriti 2001 for a review). The partition function can be further expressed as a perturbative series on the parameter λ:

$$Z = \sum_C \frac{1}{N_{sym}} \lambda^{N_4(C)} Z_C, \tag{1.24}$$

where the sum is over four dimensional simplicial complexes, N_{sym} is the rank of symmetries of C and $N_4(C)$ is the number of 4-simplices in C. Z_C is given by:

$$Z_C = \sum_{\{j\}} (-1)^{\chi} \prod_t d_{j_t} \prod_\tau \{6j\}_\tau \prod_v \{15j\}_v, \tag{1.25}$$

where χ is again a certain linear function on the spins. We see that the full partition function is obtained through a sum over triangulations, each triangulation being associated with a certain amplitude. Knowing the behavior of Z_C under a change of triangulations constitute a necessary step to controlling the sum in the definition of Z. A special class of lattice models is for which Z_C depends only on the combinatorial class of C. We will now show that this is the case for Ooguri's model, when suitably regularized. We will use a powerful mathematical result by Pachner (Pachner 1991) stating that any two simplicial complexes are combinatorially equivalent if and only if they are connected by a sequence of certain basic moves. In four dimensions there are five basic moves. It will then be enough to study the behavior of Z_C under these five moves. The moves are the following:

$1 \to 5$: Consider a 4-simplex and add a point at its center. Then draw five edges connecting this points to the original vertices of the 4-simplex. This decomposes the original 4-simplex into five 4-simplices.

$2 \to 4$: Consider two 4-simplices sharing a tetrahedron. There are two vertices not belonging to the common tetrahedron. Then draw an edge between these two vertices. This decomposes the original two 4-simplices into four 4-simplices.

$3 \to 3$: Consider three 4-simplices sharing one triangle, and three tetrahedra, each shared by two 4-simplices. There are three edges belonging to only one of the 4-simplices. Now, consider the triangle formed by these three edges. Then reorganize the edges of the original triangulation such that this triangle is shared by three new 4-simplices. This move recombines the original three 4-simplices into three different 4-simplices.

$5 \to 1$ and $4 \to 2$: These are obtained by reversing the moves $1 \to 5$ and $2 \to 4$.

To study the behavior of Z_C under these moves, first let us rewrite it as:

$$Z_C = \sum_{\{j,m,n\}} \prod_t d_{j_t} \prod_\tau \int dU_\tau \, D^{j_{1,\tau}}(U_\tau) \dots D^{j_{4,\tau}}(U_\tau) \tag{1.26}$$

Using the resolution of identity over G:

$$\delta(U) = \sum_j d_j \, Tr[U], \tag{1.27}$$

we see that Z_C can be written for a complex C without boundary as:

$$Z_C = \int \prod_\tau dU_\tau \prod_t \delta(U_{\tau_1(t)} \dots U_{\tau_n(t)}). \tag{1.28}$$

When C has a boundary there will be extra factors corresponding to triangles on the boundary of C. Let us now see how Z_C transforms on each move:

$1 \to 5$: This creates ten new triangles and ten new tetrahedra, introducing ten new spins $j_1 ... j_{10}$ and ten new group variables $U_1 ... U_{10}$. Summing over these new spin variables generates ten delta functions, enforcing the group elements U_i to be equal to the identity. The set of conditions imposed by the delta functions are not all independent, and four of them are redundant, giving an overall factor $\delta(0)^4$. Thus $Z_C \to \delta(0)^4 Z_C$ under this move. Notice that the number of redundant delta functions is equal to the number of new edges $N_1(C)$ minus the number of added vertices $N_0(C)$.

$2 \to 4$: Under this move, four triangles are added, generating four delta functions, three of which are independent. We then have $Z_C \to \delta(0) Z_C$. Notice that again the number of redundant delta functions is equal to $N_1(C) - N_0(C)$.

The move $3 \to 3$ does not introduce any redundant delta functions and Z_C is invariant under it. The moves $4 \to 2$ and $5 \to 1$ are just the inverse of the first two moves above and the partition function goes as $Z_C \to \delta(0)^{-4} Z_C$ and $Z_C \to \delta(0)^{-1} Z_C$ respectively. Combining these results, we have that:

$$Z_C = \delta(0)^{N_1(C) - N_0(C)} \chi_G(C), \tag{1.29}$$

where $N_1(C)$ and $N_0(C)$ are the number os edges, resp. vertices, in C and $\chi_G(C)$ depends only on the combinatorial class of C.

We would like now to come back to expression (1.28) and relate it to continuum BF theories (Horowitz 1989). A BF theory in four dimensions for a group G is defined by the action:

$$S_{BF} = \int_{\mathcal{M}} \mathrm{Tr}[B \wedge F(A)], \tag{1.30}$$

where $\mathcal{M}$ is a given four dimensional manifold, $B \in \Omega^2(\mathcal{M}) \otimes \mathfrak{g}$ is a Lie algebra valued two form, $A \in \Omega^1(\mathcal{M}) \otimes \mathfrak{g}$ is a connection for the group G and $F(A)$ is the associated curvature. The action is invariant under the following transformations:

$$\begin{aligned} \delta A &= D\lambda \\ \delta B &= [B, \lambda] + D\omega, \end{aligned} \tag{1.31}$$

for $\lambda \in \Omega^0(\mathcal{M}) \otimes \mathfrak{g}$ and $\omega \in \Omega^1(\mathcal{M}) \otimes \mathfrak{g}$. D denotes the covariant curvature for the connection A. The partition function for this action can be formally defined as:

$$Z_{BF} = \int DBDA\, e^{iS_{BF}}. \tag{1.32}$$

Formally integrating over B, one gets:

$$Z_{BF} \sim \int DA\, \delta(F(A)) \tag{1.33}$$

To relate the expression above with the expression (1.28) for Z_C, we need to discretize the action S_{BF}. The connection A is discretized by path ordered holonomies supported on dual edges of the combinatorial complex C and the two form B is integrated over triangles of C. The curvature $F(A)$ is discretized by closed holonomies around dual faces of C, giving the identification between the Z_C and Z_{BF}. Because Z_C (or better, a regularized version of it) is invariant under a refinement of the triangulation, the continuum limit is trivial and the discrete version (1.28) is actually exact, once enough care is taken with the regularization procedure.

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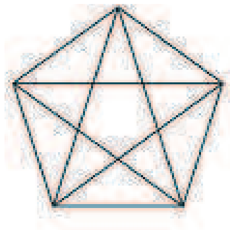
In his model Reisenberger ultimately does not impose the constraints that correspond to Plebanski's constraints strictly, but rather weights the histories so that the ones violating these constraints are supressed in the partition function. The resulting model turns out to be somewhat cumbersome to work with (Reisenberger 1997, see also Reisenberger and Rovelli 1997).

A different route was followed by Barrett and Crane (Barrett and Crane 1998). The idea was to start with the quantum tetrahedron of Barbieri (Barbieri 1998, see also Baez and Barrett 1999) in three dimensions and then use it to quantize directly a 4-simplex in 4d. They were able to define a state sum model. They did it first for the Euclidean signature case and then generalized the construction for the Lorentzian case (Barrett and Crane 2000). Restricting to the Lorentzian case, the model is based on a vertex amplitude given by a $10j$ symbol, labeled by a ten real numbers p_{ab}, one per edge of a complete graph with 5 vertices, $a, b = 1...5$ label the vertices of this graph:



It is given explicitly by:

$$10j(p_{ab}) := \int_{Q_1^{\times 4}} \prod_{a=1}^{4} \mathrm{d}x_a \prod_{ab} K_{p_{ab}}(x_a, x_b). \tag{1.34}$$

Q_1 is the unit upper hyperboloid in Minkowski space and the propagator $K_{p_{ab}}(x_a, x_b)$ is defined by

$$K_p(x, y) := \frac{\sin p\, r(x, y)}{p \sinh r(x, y)}, \tag{1.35}$$

where $r(x, y)$ is the hyperbolic distance between x and y.

In this thesis we will follow a similar route. In the next chapter we start by reviewing Barbieri's construction for the quantum tetrahedron, and this will motivate the discussion on the 4-simplex. We will identify the phase space associated to this classical system and propose a certain quantization of it. The final result will be a quantum amplitude for the 4-simplex, playing the role of the $10j$ symbol in the BC model. This 4-simplex amplitude will then be used as a building block for the construction of the spin foam amplitude in the following chapter. Whenever appropriate we will point out the differences between our construction and the BC model.

Chapter 2

Quantum geometry

"La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caràtteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caràtteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto."

Galileo Galilei

2.1 The quantum tetrahedron

A classical tetrahedron in 3 dimensions is described by a set of four normals n^i_f, $f = 1...4$ and $i = 1...3$. Each normal describes the embedding of a plane geometry in 3d. The norm of each vector gives the area of the associated triangle. The condition for these normals to describe a true tetrahedron in 3d is that they should close:

$$\sum_{f=1...4} n^i_f = 0 \ \ \forall\, i. \tag{2.1}$$

A classical tetrahedron geometry can be alternatively described by its 6 edge lengths, or 4 areas and 2 dihedral angles, tottalling 6 (classical) degrees of freedom. When described by normals, the same degrees of freedom are recovered after the constraint above and the rotation SO(3) gauge freedom are considered. In fact, we have $4 \times 3 - 3 - 3 = 6$ degrees of freedom, as it should be. The interesting remark by Barbieri is that, after quantization, only five of these can be determined simultaneously, reflecting the Heisenberg uncertainty relations for this system.

To quantize the system we need to introduce a symplectic structure. Barbieri's prescription is to associate to each normal n^i_f a generator of the algebra of SO(3), s.t.:

$$\{n^i_f, n^j_f\} = \epsilon^{ij}{}_k\, n^k_f$$
$$\{n^i_f, n^j_{f'}\} = 0 \ \ \forall i, j \ ; \ f \neq f'. \tag{2.2}$$

For the moment, we have four independent classical angular momentum systems. The constraint (2.1) will introduce the interactions and will be imposed at the quantum level. A nice framework for quantization is given by geometric quantization (see for eg. online review Blau 1992). Each angular momentum describes a two sphere $\mathcal{S}^2_f$, of radius given by the norm of the vector n^i_f, i.e. the area of the face f. The prequantization condition is that the area should be quantized, that is $A_f = j_f \in \mathbb{N}$[1]. For a fixed area j_f, the Hilbert space associated to the face f, $\mathcal{H}_{j_f}$, is spanned by vectors of the form $|j_f, m_f\rangle$, $|m_f| \leq j_f$, as is usual from the theory of angular momenta. The full kinematical Hilbert space is given by:

$$\mathcal{K}_3 = \bigoplus_{\{j_f\}} \left(\otimes_f \mathcal{H}_{j_f}\right) \tag{2.3}$$

The last step is to impose the closure constraint, the result of which is to reduce each tensor product $\otimes_f \mathcal{H}_{j_f}$ to the invariant subspace under the action of the group. Each invariant subspace $\mathrm{Inv}\big(\otimes_f \mathcal{H}_{j_f}\big)$ is in turn spanned by a base of intertwiners between the four representations j_f, each element of the base being labelled by a spin i. We then have that a quantum state is completely specified by the four areas j_f and the intertwiner i, leaving us with 5 quantum numbers, instead of the 6 original degrees of freedom of the classical system. A simple way to state this result is the following. Construct the operators $\hat{n}^2_f$ and $\hat{n}_{ff'} := \hat{n}_f \cdot \hat{n}_{f'}$, $f \neq f'$. It is now easy to see that while the $\hat{n}^2_f$ commute between themselves, the operators $\hat{n}_{ff'}$ and $\hat{n}_{ff''}$, for $f' \neq f''$, do not. More geometrically, it means that the two dihedral angles necessary to describe the classical geometry cannot be determined simultaneously in the quantum theory.

2.2 The quantum 4-simplex

We now move one dimension up. The starting point to construct the tetrahedron was to embed a plane geometry in 3 dimensions. We will proceed analogously in 4d. We want to construct a 4-simplex out of tetrahedra, the same way as a tetrahedron is constructed

[1] We are considering here representations of SO(3). Because the algebras are the same, we could as well consider representations of its universal cover SU(2) in which case half integer values for the areas would be allowed.

out of triangles. To do that consider a 4-normal N^I_τ to the tetrahedron τ, transforming under the four dimensional isometry group. If we want to consider Euclidean space, the group is SO(4), while, for Minkowiski space the group is SO(3, 1)[2]. We may also use the notation N^I_a, with $a = 1...5$ labeling one of the five tetrahedra forming the 4-simplex and $I = 0...3$.

The five normals N^I_a satisfy as well a closure condition:

$$\sum_{a=1...5} N^I_a = 0 \ \ \forall I \tag{2.4}$$

Note that with this closure condition we have the correct classical degrees of freedom for a 4-simplex: $5 \times 4 - 4 - 6 = 10$. In fact a 4-simplex has 10 edges and knowing these ten edge lengths is enough to specify the geometry of the simplex. We could organize these degrees of freedom as the five volumes of each tetrahedron and 5 dihedral angles, for example. The five normals N^I_a subject to the closure condition above specifies a 4-simplex geometry completely up to isometries.

At this point we would like to follow the strategy employed in 3d and quantize this classical system. To each normal is associated a phase space isomorphic to a 3 dimensional hypersurface of Minkowiski space, Q_ε, with $\varepsilon = -1, 0, 1$. These correspond resp. to the unit positive timelike hyperboloid, the positive light cone, and the single-sheeted unit spacelike hyperboloid, depending whether N^I_a is a timelike, spacelike or null vector. In this thesis we will restrict ourselves to the case where the normals are timelike, and consequently all tetrahedra will be spacelike. The other two cases are of course interesting and should be given a proper study, but we leave this to further investigation. In addition, each N^I_τ can be either future pointing and thus identified with a point in Q_1, or past pointing, and thus identified with a point in the negative timelike hyperboloid.

The quantization of this classical system was described by Mukunda (Mukunda 1993). Each point in Q_1 is parametrized by the variables q^I, with $q^I q_I = 1$ and $q^0 \geq 0$. We will consider the cotangent bundle T^*Q_1. This is freely parameterized by the spatial components q_i and conjugated momenta p_j, $i, j = 1, 2, 3$, subject to canonical Poisson brackets $\{q_i, p_j\} = \delta_{ij}$. The time component q_0 is determined by $q_0 = \sqrt{1 + q^2} \geq 1$. We want to give a covariant description of this phase space. Define then:

$$S_{jk} = q_j p_k - q_k p_j \tag{2.5}$$

$$S_{0j} = q_0 p_j \tag{2.6}$$

[2]Note that we could have made the same distinction in 3d and consider normals transforming for eg. under SO(2, 1). Under the embedding in 4 dimensions, the SO(3) tetrahedra constructed above correspond to spacelike tetrahedra.

We can now take S^{IJ} and q^I as basic variables. The Poisson brackets in terms of these variables are the following:

$$\begin{aligned}
&\{q^I, q^J\} = 0\\
&\{S^{IJ}, q^K\} = \eta^{K[J} q^{I]}\\
&\{S^{IJ}, S^{KL}\} = \eta^{[J[K} S^{I]L]},
\end{aligned} \tag{2.7}$$

where $\eta = (-,+,+,+)$ is the Minkowiski metric on $\mathbb{R}^{3,1}$, with which indexes are lowered and raised. Thus the S^{IJ} act on q^I as generators of SO$(3,1)$.

The variables (S, q) are not all independent and satisfy the following constraints:

$$\begin{aligned}
&q^I q_I - 1 = 0\\
&(\star S)_{IJ} q^J = 0\\
&(\star S)_{IJ} S^{IJ} = 0
\end{aligned} \tag{2.8}$$

where $(\star S)_{IJ} := \frac{1}{2}\epsilon^{IJ}{}_{KL}S^{KL}$, $\epsilon^{0123} = 1$. One can check that this set of constraints is first class, that is, the Poisson bracket of any two of them vanishes on the constraint surface.

The associated Hilbert space $\mathcal{H}_1$ is identified with the space of square integrable functions over Q_1, with norm given by:

$$||\psi||_1^2 = \int_{\mathbb{R}^3} \frac{d^3q}{\sqrt{1+q^2}}\, |\psi(q)|^2 \tag{2.9}$$

A Lorentz transformation Λ is represented by a unitary operator $\mathcal{U}_1(\Lambda)$ on this Hilbert space:

$$\mathcal{U}_1(\Lambda)\psi = \psi' \;\;,\;\; \psi'(q) = \psi(\Lambda^{-1}q) \tag{2.10}$$

This representation of the Lorentz group can be decomposed in irreducibles, and one finds for Q_1, because of the constraints (2.8) above:

$$L^2(Q_1) = \bigoplus_p d\mu_p\,(0,p), \tag{2.11}$$

where $(0,p)$, $p \in \mathbb{R}$, labels a certain irreducible unitary representation of the Lorentz group (see appendix A). We see then that each tetrahedron is labeled by a real number. To finalize the construction one needs to impose the closure condition on the normals N^I_τ. Because the normals cannot be identified directly with generators of Lorentz transformations, as was the case in three dimensions, the implementation of the closure condition seems more involved in four dimensions. The strategy will be to enlarge the classical phase space, in order to induce a more natural quantization. We will do that in the next section.

2.3 Bivector geometry

We have seen that a 4-simplex geometry is completely determined by five normals N_a^I satisfying a closure condition. The idea - and this is the fundamental idea in the Barrett-Crane construction - is to consider bivectors B_{ab}^{IJ} describing the geometry of the triangle $t_{(ab)}$ (or just (ab) for short) shared by the tetrahedra a and b. We may shift when needed to the more general notation B_t or $B_t(\tau)$ when the orientation is important. Thus $B_{ab} = B_{t_{ab}}(\tau_a)$ and $B_{ba} = B_{t_{ab}}(\tau_b)$.

Let $\Lambda^2(\mathbb{R}^{3,1})$ be the space of Lorentzian bivectors. A pair of vectors $N, M \in \mathbb{R}^{3,1}$ determines a simple bivector $N \wedge M$ which can be considered as the antisymmetric tensor

$$N \wedge M = N \otimes M - M \otimes N. \tag{2.12}$$

The above equation fixes our conventions for the wedge product of two vectors. The norm $|B|$ of a bivector B in $\Lambda^2(\mathbb{R}^{3,1})$ is defined by

$$|B|^2 = \frac{1}{2} B^{IJ} B_{IJ}, \tag{2.13}$$

where again $I, J, K = 0, ..., 3$ label the components of the antisymmetric tensor, and indices are raised and lowered with the standard Minkowski metric $\eta = (-, +, +, +)$ on $\mathbb{R}^{3,1}$. A bivector is said to be space-like (resp. time-like) if $|B|^2 > 0$ (resp. $|B|^2 < 0$). We will use the fact that the space $\Lambda^2(\mathbb{R}^{3,1})$ can be identified as a vector space with the Lie algebra $\mathfrak{so}(3, 1)$ of the Lorentz group using the isomorphism $\varsigma : \Lambda^2(\mathbb{R}^{3,1}) \to \mathfrak{so}(3, 1)$, $B \mapsto Id \otimes \eta\,(B)$, with the metric regarded as a map $\eta : \mathbb{R}^{3,1} \to (\mathbb{R}^{3,1})^*$. Hence, if B is viewed as an anti-symmetric four-by-four matrix, the identification with a Lorentz algebra element yields

$$\varsigma : \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & r_1 & r_2 \\ -b_2 & -r_1 & 0 & r_3 \\ -b_3 & -r_2 & -r_3 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & r_1 & r_2 \\ b_2 & -r_1 & 0 & r_3 \\ b_3 & -r_2 & -r_3 & 0 \end{bmatrix}. \tag{2.14}$$

Starting from the normals to tetrahedra N_a the bivector B_{ab} describing the geometry of the triangle t_{ab} is given by the following expression:

$$B_{ab} = A_{ab} \frac{\star N_a \wedge N_b}{|\star N_a \wedge N_b|}. \tag{2.15}$$

A_{ab} is the area of the triangle (ab) and $\star$ is the Hodge operator acting on internal indices: $(\star B)^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL} B^{KL}$, $\star^2 = -1$.

Barrett and Crane noticed that the four simplex geometry is in fact completely determined - up to isometries - by the bivectors, subject to a number of constraints:

- Orientation:
$$B_{ab} = -B_{ba} \tag{2.16}$$

- Closure:
$$\sum_{b \neq a} B_{ab}^{IJ} = 0 \ , \ \forall a \tag{2.17}$$

- Diagonal simplicity:
$$(\star B_{ab})_{IJ} B_{ab}^{IJ} = 0 \ , \ \forall (ab) \tag{2.18}$$

- Cross simplicity:
$$(\star B_{ab})_{IJ} B_{ac}^{IJ} = 0 \ , \ \forall a, b \neq c \tag{2.19}$$

- 3d non-degeneracy: Each tetrahedron geometry is non degenerate.

- 4d non-degeneracy: The 4-simplex is non degenerate, that is, for six triangles sharing a common vertex, the six corresponding bivectors are linearly independent.

The fact that this set of constraints determine a 4-simplex geometry appeared first in (Barrett and Crane 1998). The theorem was originally designed for Euclidean geometry, but as it makes no use of the metric, it can also be applied to the Lorentzian case.

Theorem 1. *(Barrett and Crane) Each geometric 4-simplex determines a set of bivectors satisfying the constraints above, and each set of bivectors satisfying these constraints determines a geometric 4-simplex unique up to parallel translation and inversion through the origin.*

Proof. The diagonal simplicity constraint implies that each bivector is simple, that is, of the form $B = u \wedge v$. Each simple bivector determines a plane through the origin in $\mathbb{R}^{3,1}$. Cross simplicity implies that any two of these planes on a given tetrahedron belong to the same three dimensional hyperplane.

Given a geometric 4-simplex, the bivectors constructed out of its triangles satisfy the closure condition by Stoke's theorem and the non-degeneracy conditions by assumption, which proves the first part of the theorem.

Now consider a set of bivectors B_t satisfying the constraints above. The simplicity constraints and 4d non-degeneracy imply that the four planes of a tetrahedron lie in a common hyperplane or share a common direction. The 3d non-degeneracy condition rules out the case where they share a common direction.

To construct the geometric 4-simplex, shift one of the five hyperplanes away from the origin by parallel translation. The hyperplanes now bound a geometric 4-simplex, with bivectors $B'_t = \lambda_t B_t$ proportional to the ones we started with. The closure and orientation conditions imply that the λ_t are all equal. Moving the hyperplane scales the

4-simplex and therefore one can fix all the λ_t to 1 or -1. Hence $B'_t = \mu B_t$, for $\mu = \pm 1$. Because μ is only determined up to a sign, the geometric 4-simplex is then only determined up to an inversion through the origin. ■

A major role is played by both diagonal and cross simplicity constraints. In the Barrett-Crane construction, the bivectors are associated to Lie algebra elements and these constraints, being quadratic in the bivectors, translate into constraints on representations associated to triangles and tetrahedra. As we saw in the proof above, they imply that the planes associated to faces of a given tetrahedron all lie in the same three dimensional hyperplane. The same condition can be restated using directly the normals N_a^I[3]:

- Simplicity:

$$\forall a \ , \ \exists \ N_a \text{ s.t. } N_{a\,I} B_{ab}^{IJ} = 0 \ \forall b \neq a. \tag{2.20}$$

The formula (2.15) for the bivector in terms of the normals is then trivially obtained. In fact, because of the orientation condition the bivector B_{ab} is orthogonal to both normals N_a and N_b, thus proportional to $\star N_a \wedge N_b$.

Let us further explore the 4-simplex geometry. First define the dihedral angle between two tetrahedra (see Barrett and Foxon 1994). For a 4-simplex in Minkowski space with all tetrahedra space-like, the dihedral angles are all boost parameters. The a-th tetrahedron has a outward-pointing timelike normal vector $\hat{N}_a$, and the dihedral angle at the intersection of two tetrahedra is determined up to sign by

$$\cosh \Theta_{ab} = |\hat{N}_a \cdot \hat{N}_b|, \tag{2.21}$$

and can be viewed as a distance on the unit hyperboloid.

The sign of the dihedral angle is more delicate. One could define them all to be positive, but this would lead to additional signs in the formula for the Regge action. It is much better to take account of the nature of the triangle where the two tetrahedra meet. The tetrahedra come in two types: the outward normals are either future-pointing or past-pointing. The triangles are then classified into two types: *thin wedge*, where one of the incident tetrahedra is future and the other one past, and *thick wedge*, where both are either future or past (see figure 2.1). The dihedral angle is defined to be positive for a thin wedge and negative for a thick wedge.

An important object in what follows is the dihedral boost. Define F_a the future pointing normal associated to the tetrahedron a, that is, $F_a = \varepsilon_a \hat{N}_a$, where $\varepsilon_a = +1$ if $\hat{N}_a$ is future pointing and $\varepsilon_a = -1$ if $\hat{N}_a$ is past pointing. Then the dihedral boost D_{ab}

[3]Note that from this point and on we start to diverge from the work of Barrett and Crane.

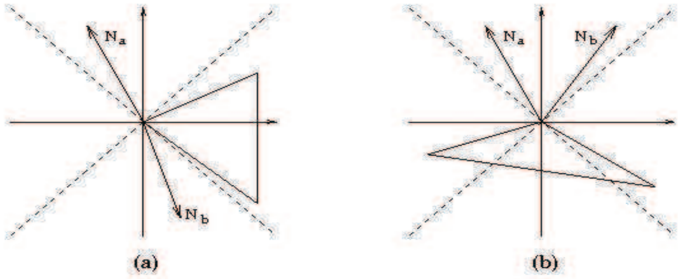


Figure 2.1: Example of a thin wedge (a), and of a thick wedge (b).

from the tetrahedron a to the tetrahedron b is defined as the Lorentz transformation mapping F_a to F_b and preserving the bivector B_{ab}:

$$D_{ab}\,F_a = F_b \ \text{ and } \ D_{ab} \otimes D_{ab}\,B_{ab} = B_{ab}. \tag{2.22}$$

It is given explicitly by the following formula:

$$D_{ab} = \exp\left(\Theta_{ab}\varsigma(\star\tilde{B}_{ab})\right), \tag{2.23}$$

where $\varsigma : \Lambda^2(\mathbb{R}^{3,1}) \to \mathfrak{so}(3,1)$ is the map from bivectors into Lie algebra elements defined before. $\tilde{B}_{ab}$ is the normalized bivector : $\tilde{B}_{ab} := B_{ab}/|B_{ab}|$, and Θ_{ab} is the dihedral angle between tetrahedra a and b defined above.

This formula is proved as follows. Since B_{ab} is a simple spacelike bivector, then D_{ab} is a Lorentz transformation which stabilises a space-like plane. The bivector in the exponent is

$$\Theta_{ab} * \tilde{B}_{ab} = -\Theta_{ab}\frac{N_a \wedge N_b}{|*N_a \wedge N_b|} = |\Theta_{ab}|\frac{F_a \wedge F_b}{|*F_a \wedge F_b|},$$

using the sign convention in the definition of a dihedral angle. This bivector acts in the plane spanned by F_a and F_b and the boost parameter has the right magnitude. It just remains to check that it maps F_a to F_b, and not vice-versa. To first order in small Θ, one has

$$\exp\left(|\Theta_{ab}|\frac{\varsigma(F_a \wedge F_b)}{|*F_a \wedge F_b|}\right) F_a \simeq F_a + \frac{|\Theta_{ab}|}{|\sinh\Theta_{ab}|}\left((F_b \cdot F_a)F_a - (F_a \cdot F_a)F_b\right) \simeq F_b.$$

This calculation uses the convention replacing wedge products with bivectors, and the fact that $F^2 = -1$.

——— * ——— * ———

With an eye on quantization, we would like to introduce a different parametrization of the classical system determining the geometry of the 4-simplex. Associate then to each tetrahedron a Lorentz transformation $X_a \in \mathrm{SO}^{\uparrow}(3,1)$ such that $X_a \mathcal{T} = F_a$, where $\mathcal{T}$ is the unit timelike vector $(1,0,0,0)$. This condition specifies X_a up to a rotation, that we keep arbitrary for the moment. Then, parallel transport the bivectors to the origin of the hyperboloid:

$$b_{ab} := X_a^{-1} \otimes X_a^{-1} \triangleright B_{ab}. \tag{2.24}$$

We might refer to the bivectors b_{ab} as being in the reference frame of the tetrahedron t_a and the bivectors B_{ab} as being in the reference frame of the 4-simplex. The orientation condition for the original bivectors B_{ab} translate into the following constraint for the transported bivectors b_{ab}:

$$b_{ab} = -X_{ab} \otimes X_{ab} \triangleright b_{ba}, \tag{2.25}$$

where we have defined $X_{ab} := X_a^{-1} X_b$. Using the expression (2.15) for the bivector B_{ab} we have that

$$b_{ab} = A_{ab} \star (\mathcal{T} \wedge \mathcal{N}_{ab}), \tag{2.26}$$

where $\mathcal{N}_{ab} =: (0, \hat{n}_{ab})$ and $\hat{n}^i_{ab}$ is given by normalizing the vector $\varepsilon_a \varepsilon_b (X_{ab})^i{}_0$. Furthermore we have

$$b^{ij}_{ab} = -A_{ab} \epsilon^{ij}{}_k \hat{n}^k_{ab} \Leftrightarrow \hat{n}^k_{ab} = -\frac{1}{2A_{ab}} \epsilon^k{}_{ij} b^{ij}_{ab}. \tag{2.27}$$

The simplicity constraint for the bivector b_{ab} translates as:

$$b^{0i}_{ab} = 0 \ , \ \forall i, \tag{2.28}$$

which is equivalent to stating that there exists a vector $\hat{n}_{ab}$ such that the bivector b_{ab} is given by the formula (2.26). The closure condition translates into a closure condition for the normals $\hat{n}_{ab}$:

$$\sum_{a \neq b} A_{ab} \, \hat{n}_{ab} = 0. \tag{2.29}$$

The normals $\hat{n}_{ab}$ and areas A_{ab} determine completely the geometry of a tetrahedron and the 3d non-degeneracy condition is just a condition on this tetrahedron being non degenerate. The 4d non-degeneracy condition is more complicated and involves the group elements X_a as well.

Conversely, starting with the set of areas A_{ab} and normals $\hat{n}_{ab}$ determining five non-degenerate tetrahedron geometries, construct bivectors b_{ab} through equation (2.26). Then if there exist group elements $X_a \in \mathrm{SO}^{\uparrow}(3,1)$ such that the bivectors satisfy the constraints (2.25), reconstruct the original bivectors $B_{ab} = X_a \otimes X_a \triangleright b_{ab}$. These bivectors satisfy the Barrett-Crane constraints and then determine a 4-simplex geometry provided it is non-degenerate, according to Theorem 1.

Boundary data

At this point we see that the geometric data has been separated in two sets of variables $\{A_{ab}, \hat{n}_{ab}\}$ and $\{X_a\}$. The first set, given by areas and normals describes the intrinsic 3d geometry of tetrahedra on the boundary of the simplex, while the second set describes the embedding of these 3d geometries in 4d, thus describing the extrinsic geometry of the 4-simplex. We will refer to the set $\{A_{ab}, \hat{n}_{ab}\}$ as the boundary data of the 4-simplex. Note that this definition can be generalized to the boundary of any triangulation. We will come back to this in the next chapter.

Now we may as well reverse the problem and ask ourselves under what conditions a given boundary data determines the geometry of the 4-simplex. In different words, which conditions normals and areas need to satisfy in order to be an admissible boundary data for a geometric 4-simplex. To start understanding this question, define $g_{ab} := X_b^{-1} D_{ab} X_a$. Because of the definition of the dihedral boost (2.22) we have that g_{ab} preserves the vector $\mathcal{T}$ and then belongs to the SO(3) subgroup. Now, from the parallel transport condition on the bivectors b_{ab} and the fact that the dihedral boost D_{ab} preserves the bivector B_{ab} we have that:

$$\begin{aligned} &D_{ab} \otimes D_{ab} \triangleright B_{ab} = -B_{ba} \Leftrightarrow D_{ab}X_a \otimes D_{ab}X_a \triangleright b_{ab} = -X_b \otimes X_b \triangleright b_{ba} \\ &\Leftrightarrow g_{ab} \otimes g_{ab} \triangleright b_{ab} = -b_{ba} \Leftrightarrow g_{ab} \cdot \hat{n}_{ab} = -\hat{n}_{ba}. \end{aligned} \tag{2.30}$$

Furthermore, from the formula of the dihedral boost (2.23) and the definition of the transported bivectors, one has:

$$D_{ab} = e^{\Theta_{ab}\varsigma(\star\tilde{B}_{ab})} = X_a e^{\Theta_{ab}\varsigma(\star\tilde{b}_{ab})} X_a^{-1} = X_a e^{\Theta_{ab}\pi(\mathbf{K}\cdot\hat{n}_{ab})} X_a^{-1}, \tag{2.31}$$

where $\pi : \mathfrak{so}(3,1) \to \mathrm{End}\mathbb{R}^{3,1}$ is the vector representation of the Lorentz algebra defined in appendix A and $\mathbf{K}$ are the boost generators. Finally from the definition of g_{ab}, one has that

$$X_{ba} = g_{ab} e^{-\Theta_{ab}\pi(\mathbf{K}\cdot\hat{n}_{ab})}, \tag{2.32}$$

which states that g_{ab} is the rotation part of the Lorentz transformation X_{ba}.

Conversely, the boundary data determines the geometry of each tetrahedron independently, following the discussion on the quantum tetrahedron. The question now is if these tetrahedra can be glued appropriately to reconstruct a 4-simplex geometry. Consider a triangle t_{ab}. The two normals $\hat{n}_{ab}$, $\hat{n}_{ba}$ associated to this triangle define a rotation

$$g_{ab} \cdot \hat{n}_{ab} = -\hat{n}_{ba}, \tag{2.33}$$

up to a SO(2) rotation preserving the vector $\hat{n}_{ab}$. This SO(2) rotation is essential and guarantees that the triangle geometries Δ_{ab} and Δ_{ba} reconstructed independently on each tetrahedron are mapped correctly one into the other:

$$g_{ab} \cdot \Delta_{ab} = \Delta_{ba} \tag{2.34}$$

A consistent choice of SO(2) rotations for all the triangles exists if and only if the boundary data is the boundary data of a geometrical 4-simplex. This is equivalent to stating that there exist group elements X_a associated to tetrahedra such that the rotations g_{ab} are given by $g_{ab} = X_b^{-1} D_{ab} X_a$. If this is the case, we call the boundary data Regge-like. This notion will be important when discussing the asymptotics of the 4-simplex amplitude in chapter 4.

2.4 Quantization

Note. From now on we replace the part of the Lorentz group connected to the identity $\mathrm{SO}^{\uparrow}(3,1)$ by its double cover $\mathrm{SL}(2,\mathbb{C})$, to which we refer loosely as the Lorentz group. We review the relation between these two groups in appendix A.

Our starting point for quantization will be the classical system described by the set of variables (b_{ab}, X_{ab}) and subject to the constraints described above, that we recollect here:

- Parallel transport:

$$b_{ab} = -X_{ab} \otimes X_{ab} \triangleright b_{ba} = -X_{ab} b_{ba} X_{ba} \tag{2.35}$$

- Closure:

$$\sum_{b \neq a} b_{ab}^{IJ} = 0 \ , \ \forall a \tag{2.36}$$

- Simplicity:

$$b_{ab}^{0i} = 0 \ , \ \forall i \tag{2.37}$$

- Non-degeneracy: The 3d and 4d geometries are non-degenerate;
- Flatness:

$$X_{ab} X_{bc} X_{ca} = 1 \ , \forall (abc) \Leftrightarrow \exists X_a \in \mathrm{SL}(2,\mathbb{C}) \, \text{s.t.} \, X_{ab} = X_a^{-1} X_b \tag{2.38}$$

The flatness condition implies that the X_{ab} are of the form $X_{ab} = X_a^{-1} X_b$. The X_a are defined up to a global Lorentz transformation. One can then use the group elements X_a to reconstruct the bivectors $B_{ab} = X_a \otimes X_a \triangleright b_{ab}$. That these conditions are sufficient to reconstruct a geometric 4-simplex then follows from the theorem of Barrett and Crane above.

The advantage of introducing group variables is that one can use standard techniques for the quantization of group manifolds (Isham at Les Houches 1983). At each triangle these variables (b_t, X_t) parameterize the cotangent space over the Lorentz group

$T^*\mathrm{SL}(2,\mathbb{C})$. Ignoring the constraints for the moment, one needs to introduce a symplectic structure for this space. A natural choice would be given by the following Poisson brackets:

$$\begin{aligned}\{X_t, X_t\} &= 0,\\ \{b_t^{IJ}, X_t\} &= X_t L^{IJ},\\ \{b_t^{IJ}, b_t^{KL}\} &= \eta^{[J[K} b_t^{I]L]}.\end{aligned} \tag{2.39}$$

L^{IJ} are the generators of $\mathrm{SL}(2,\mathbb{C})$. We see that the bivectors b_t^{IJ} are represented by left invariant vector fields J_t^{IJ} on the copy of the group associated to the triangle t. For each triangle, the kinematical Hilbert space is given by the space of square integrable functions over $\mathrm{SL}(2,\mathbb{C})$, $\mathcal{H}_t := L^2\left(\mathrm{SL}(2,\mathbb{C})\right)$. The unconstrained Hilbert space for the 4-simplex is given by the tensoring together ten copies of it:

$$\mathcal{K}_4 := \bigotimes_t \mathcal{H}_t \tag{2.40}$$

The choice of a symplectic structure for a given classical phase space is in general not unique, even though it is usually suggested by the canonical analysis of a given classical action. We would like to consider a modification of the symplectic structure given above. Consider the following identification between b_t and J_t:

$$J_t \rightsquigarrow \star b_t - \frac{1}{\gamma}\, b_t, \tag{2.41}$$

which, after inverting, implies:

$$b_t \rightsquigarrow -\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{\gamma} J_t + \star J_t\right). \tag{2.42}$$

The parameter γ mimics the Immirzi parameter (Immirzi 1997) used in LQG, and the symplectic structure defined by the identification above can be given a motivation from a a certain action principle, see appendix B. The kinematical Hilbert space associated to each triangle is still $L^2\left(\mathrm{SL}(2,\mathbb{C})\right)$, but each b_t is represented according to the identification with left invariant vector fields given in the last equation.

The space $L^2\left(\mathrm{SL}(2,\mathbb{C})\right)$ can be decomposed in irreducible representations of the Lorentz group, using the Plancherel decomposition for this group (see appendix A). Irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ are labeled by a pair of numbers $(k,p) \in \mathbb{Z} \times \mathbb{R}$ and the Plancherel decomposition is given by:

$$L^2\left(\mathrm{SL}(2,\mathbb{C})\right) = \bigoplus_{(k,p)} \bar{\mathcal{H}}_{(k,p)} \otimes \mathcal{H}_{(k,p)} \tag{2.43}$$

We would like now to impose the constraints on this kinematical Hilbert space. Let us take a look first at the simplicity constraints. Using the representation of bivectors in terms of left invariant vector fields given by equation (2.42), the simplicity constraint on the triangle t reads:

$$\frac{1}{\gamma}J_t^{0i} + \frac{1}{2}\epsilon^{0i}{}_{jk}J_t^{jk} =: \frac{1}{\gamma}K_t^i + H_t^i = 0 \Rightarrow K_t^i = -\gamma H_t^i. \tag{2.44}$$

We have introduced the boost/rotation decomposition of the Lorentz algebra in the definitions: $K^i := J^{0i}$ and $H^i := \frac{1}{2}\epsilon^{0i}{}_{jk}J^{jk} = \frac{1}{2}\epsilon^i{}_{jk}J^{jk}$.

A proper implementation of the constraints taking the full algebra into account from the start is still lacking and would constitute a valuable step in understanding better the construction of the spin foam models we present in this manuscript. The essential problem is that we are dealing with a second class constraint system (Henneaux and Teitelboim 1994 and Dirac 1964) and these are usually very difficult to quantize. The standard procedure is to define Dirac brackets and then construct the quantum space as a representation of those. In practical, finding a representation of the Dirac brackets is a very difficult task, as there are few systems one actually knows how to quantize exactly. Usually the best solution for such a system is simply to find a different classical parametrization of the phase space, such that if there are still constraints to be solved, they are first class.

We will follow here a more general and less standard procedure based on the concept of a Master constraint, introduced by Klauder (Klauder 1997) and developed further by Thiemann (Thiemann 2006, see also Dittrich and Thiemann 2006). We will do so only for the simplicity constraints. The closure, flatness and parallel transport conditions will be treated separately.

Later in the text, we will test these assumptions by looking at the semiclassical approximation of the quantum amplitude associated to a 4-simplex. We will see that indeed the leading order for this amplitude is consistent with a Regge geometry for this simplex.

The idea is to consider, instead of the three constraints $C_t^i := \frac{1}{\gamma}K_t^i + H_t^i$, a single constraint given by the sum of the squares:

$$M_t := \sum_{ij} \delta_{ij} C_t^i C_t^j = \sum_i \left(\frac{1}{\gamma}K_t^i + H_t^i\right)^2. \tag{2.45}$$

This constraint is quadratic in the generators and can therefore be written in terms of the Casimir operators for the Lorentz group and the rotation subgroup (see appendix A):

$$M_t = \frac{1}{\gamma^2}\left(K_t^2 - H_t^2\right) + \left(1 + \frac{1}{\gamma^2}\right)H_t^2 + \frac{2}{\gamma}K_t \cdot H_t, \tag{2.46}$$

which imply the following equation for the representation labels associated to this triangle:

$$\left(\frac{p}{\gamma} - k\right)^2 + \left(1 + \frac{1}{\gamma^2}\right)(j^2 - k^2) = 0. \tag{2.47}$$

In the last equation we have used a classical ordering for the Casimir operator $H^2 = j^2$, for j half integer, and we have suppressed the constant term in the first Casimir for the Lorentz group. This choice of ordering is such that it allows for a large class of solutions to this equation. These are given by:

$$p = \gamma k \text{ and } |k| = j \tag{2.48}$$

To understand the second condition, consider the decomposition of $\mathcal{H}_{(k,p)}$ in irreducibles of the rotation subgroup:

$$\mathcal{H}_{(k,p)} = \bigoplus_{j \geq |k|} \mathcal{H}_j. \tag{2.49}$$

We see that the simplicity constraints restrict this decomposition to the lowest SU(2) irreducible in the tower.

The implementation of these conditions is as follows. As we saw before, the unconstrained Hilbert space is given by the tensor product of ten copies of $L^2(\mathrm{SL}(2,\mathbb{C}))$. A state in this space is a function of ten group elements, each one associated to a given triangle in the 4-simplex. Denote it $\psi(X_{ab})$. The simplicity constraints imply that only representations satisfying the equations (2.48) will appear in the Plancherel decomposition of ψ, provided that each $\mathcal{H}_{(k,p)}$ in (2.43) is further decomposed according to (2.49). A basis for this space is given by the tensor product of representation matrices, one for each triangle in the simplex:

$$\psi^{\{j_{ab}\}}_{\{m_{ab}\}}(X_{ab}) := \bigotimes_{(ab)} D^{(j_{ab},\gamma j_{ab})}_{j_{ab}m_{ab},j_{ab}m_{ba}}(X_{ab}). \tag{2.50}$$

Note that we have chosen k to be positive, and we will do so from now on. This is a choice and one could in principle keep the two sectors of solutions to the simplicity constraints. We have used the canonical basis (cf. appendix A) to write down the representation matrices as it makes the imposition of the second condition in (2.48) evident.

We would like next to consider the flatness condition. This will be taken care of by projecting the amplitude $\psi(X_{ab})$ on the solutions of the constraint. Let us do so for an element of the basis (2.50). This defines an amplitude in terms of the spins j_{ab} and the magnetic numbers m_{ab}. Denote it $A(j_{ab}, m_{ab})$:

$$A(j_{ab}, m_{ab}) = \int \prod_a dX_a \prod_{(ab)} dX_{ab} \prod_{(ab)} \delta(X_a X_{ab} X_b^{-1}) \, \psi^{\{j_{ab}\}}_{\{m_{ab}\}}(X_{ab}) \tag{2.51}$$

Note that we have chosen an orientation for each triangle (ab) in the 4-simplex, for eg. $a < b$. Choosing a different orientation would give a slightly different definition of the amplitude.

We will not deal with the parallel transport and closure conditions at this point. They will reappear later as critical point equations for the semiclassical analysis of the 4-simplex amplitude. This means that the quantum amplitude is picked on the classical solutions allowing at the same time for some fluctuations around them. We could in principle try to impose these conditions strongly in the definition of the amplitude, and this is suggested by some recent papers (Bonzom 2009, Oriti 2009). This would complicate considerably the state sum model and we thus prefer not to impose these constraints.

A note on the closure constraint is in order. By the identification of the bivectors with generators, imposing the closure is equivalent to imposing invariance of the amplitude $A(j_{ab}, m_{ab})$ under the diagonal action of the Lorentz group in each tetrahedron. We see though that after the implementation of the simplicity constraints the amplitude is invariant only under the action of the rotation subgroup. In fact the tensor $A(j_{ab}, m_{ab})$ with the group acting on the magnetic numbers is an invariant tensor. This comes from the choice of a canonical reference system for the boundary data. Indeed the tetrahedra are normal to the canonical vector $\mathcal{T}$, which reduces the action of the Lorentz group to the action of the rotation subgroup leaving this vector invariant.

Because $A(j_{ab}, m_{ab})$ is an invariant tensor, it can be expressed in terms of intertwiner states by a change of basis. Denoting $|\{j_{ab}\}, i_a\rangle = \sum_{\{m_{ab}\}} C^{i_a}_{\{m_{ab}\}} \int_{\mathrm{SU}(2)} dg\, g \cdot (\otimes_{b\neq a} |j_{ab}, m_{ab}\rangle)$ this new basis and $\tilde{A}(j_{ab}, i_a)$ the transformed amplitude, we see that we are left with 15 degrees of freedom specifying the geometry of the 4-simplex.

It is probably a good point to take some time and compare the quantization of the 4-simplex with the quantum tetrahedron described earlier. The situation is that, while for the tetrahedron we were able to completely quantize the classical system and find the true quantum degrees of freedom, in the 4-simplex case we are not there yet. We have been able to impose exactly some of the constraints defining the geometric 4-simplex, and this allowed us to get down to 15 quantum degrees of freedom. As we have not imposed exactly the parallel transport and the closure constraints, we are not able to tell exactly how many independent quantum numbers are necessary to specify the state of a quantum 4-simplex. All we can do is associate a quantum amplitude to a given state.

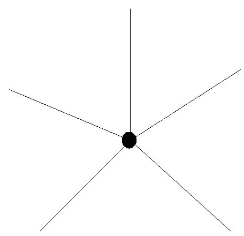
2.5 Regularization

The amplitude (2.51) is not well defined as it is divergent, related to the fact that the Lorentz group is non-compact. This divergence can be regularized in a straight forward way. To see where it comes from let us rewrite the expression (2.51) in a more explicit form:

$$A(j_{ab}, m_{ab}) = \int \prod_a dX_a \prod_{(ab)} D^{(j_{ab}, \gamma j_{ab})}_{j_{ab} m_{ab}, j_{ab} m_{ba}} (X_a^{-1} X_b), \tag{2.52}$$

where we have reintroduced the D matrices in the definition of $\psi^{\{j_{ab}\}}_{\{m_{ab}\}}$ and solved the delta functions. Now, because the integrand depends on the integration variables only through the combinations $X_a^{-1} X_b$, it is invariant under the the change of variables $X_a \to \tilde{X}_a := G\, X_a$, where $G \in \mathrm{SL}(2, \mathbb{C})$ is an arbitrary element of the group. Because the measure, being the Haar measure, is also invariant under this transformation, one sees that the full amplitude is invariant. Choosing G equal to any of the X_a, say $G = X_5$, fixes $\tilde{X}_5$ to the identity and $A(j_{ab}, m_{ab})$ is then proportional to the infinite factor $\int_{\mathrm{SL}(2,\mathbb{C})} dX_5$. The regularization procedure will be to simply drop one of the integrations, exactly as proposed for the Barrett-Crane model (Barrett and Crane 1998). We would like to show here that the amplitude regularized like that is in fact finite.

A nice way to see this regularization procedure is to view it as a gauge fixing of a Lorentz symmetry acting on the center of the 4-simplex, as one would do in Lattice Gauge Theory (Creutz 1985). In fact consider the dual graph of a 4-simplex:



The gauge symmetry acts on the central vertex and the gauge fixing procedure consists in fixing to the identity the group elements of a maximal tree. The maximal tree for the graph above is just one edge, and gauge fixing is thus equivalent to dropping one of the integrations in the amplitude above. We will later come back to this when constructing the amplitude for general triangulations.

Define then the regularized amplitude:

$$A^{reg}(j_{ab}, m_{ab}) = \int \prod_a dX_a\, \delta(X_5) \prod_{(ab)} D^{(j_{ab},\gamma j_{ab})}_{j_{ab}m_{ab},j_{ab}m_{ba}}(X_a^{-1}X_b). \tag{2.53}$$

The first thing to do is to decompose the group variables X_a in boosts and rotations:

$$X_a = B(x_a)R_a, \tag{2.54}$$

where x_a denotes a point in the hyperboloid Q_1. Now, we have the following result:

$$\begin{aligned} D^{(j,\gamma j)}_{jm,jm'}(X_a^{-1}X_b) &= D^{(j,\gamma j)}_{jm,jm'}(R_a^{-1}B(x_a)^{-1}B(x_b)R_b) = \\ &= \sum_{kn,k'n'} D^{(j,\gamma j)}_{jm,kn}(R_a^{-1})D^{(j,\gamma j)}_{kn,k'n'}(B(x_a)^{-1}B(x_b))D^{(j,\gamma j)}_{k'n',jm'}(R_b) = \\ &= \sum_{n,n'} D^j_{mn}(R_a^{-1})D^{(j,\gamma j)}_{jn,jn'}(B(x_a)^{-1}B(x_b))D^j_{n'm'}(R_b) \end{aligned} \tag{2.55}$$

In the calculation above we have used the property of D matrices when restricted to an element of the rotation subgroup:

$$D^{(k,p)}_{jm,j'm'}(R) = \delta_{jj'}D^j_{mm'}(R), \tag{2.56}$$

where $D^j_{mm'}(R)$ is the representation matrix for SO(3) on the representation j. Because this is an unitary representation and because the group SO(3) is compact, the matrices $D^j_{mm'}(R)$ are bounded by 1 and the sum on (n, n') above is finite. We can then forget the rotations R_a, and finiteness of $A^{reg}(j_{ab}, m_{ab})$ is thus equivalent to finiteness of the corresponding expression by replacing the X_a with the boosts $B(x_a)$. Because the price to pay is minimum, we will look at a more general expression, for which the simplicity constraints have not been imposed yet. To state our results we will use the notion introduced in (Baez and Barrett 2001) of the integrability of a labeled graph. Given a graph Γ, assign an irreducible representation in the principal series (k_{ij}, p_{ji}) to each link (ij), where i, j denote nodes of the graph and thus a pair denote the link between them. Also to each end of the link assign a couple (l_{ij}, m_{ij}) and (l_{ji}, m_{ji}) of half integers and magnetic numbers, specifying thus an angular momentum state at that end of the link. This labeled graph is what is called a projected spin network (Livine 2002), and we denote it (Γ, χ), χ denoting collectively the set of labels. Given such a labelled graph choose an arbitrary node t_* in Γ, and number the nodes starting with t_*, $1, \ldots, N$ for convenience. Then (Γ, χ) is said to be an integrable graph if the following quantity is finite:

$$F^\Gamma(\chi) := \left[\prod_{i=1}^N \int_{x_i\in Q_1} \mathrm{d}x_i\right] \prod_{\substack{i,j\in\{1,\ldots,N\}\\ i<j}} K^{k_{(ij)},p_{(ij)}}_{l_{ij}m_{ij},l_{ji}m_{ji}}(x_i, x_j), \tag{2.57}$$

where we have defined:

$$K^{k_{ij},p_{ij}}_{l_{ij}m_{ij},l_{ji}m_{ji}}(x_i, x_j) := D^{k_{ij},p_{ij}}_{l_{ij}m_{ij},l_{ji}m_{ji}}(B(x_i)^{-1}B(x_j)) \tag{2.58}$$

We see that the 4-simplex amplitude defined in (2.53) is a special case of the expression (2.57) above when the labels are restricted by the simplicity constraints and the graph Γ is the (dual of) the boundary of a 4-simplex. We want to prove that the expression (2.57) is finite. To do that we will follow closely (Baez and Barrett 2001). We start by adapting a lemma in (BB 2001) to our construction, Lemma 2 below. The importance of Lemma 2 is two fold. In the first place, it is important in the proof of Theorem 2 below, which states that the tetrahedron graph is integrable. Secondly, and more importantly, it guarantees that, given an integrable graph, every other graph constructed from it by adding a node with at least three legs will also be integrable. This is the first part of Theorem 3, which we borrow with no changes from (BB 2001). These two conclusions then imply that the 4-simplex graph is integrable, which we state as a corollary.

Before starting, let us take a closer look at the propagator $K^{k_{ij},p_{ij}}_{l_{ij}m_{ij},l_{ji}m_{ji}}(x_i,x_j)$. First rewrite the composition of boosts $B(x_i)^{-1}B(x_j)$ as:

$$B(x_i)^{-1}B(x_j) = R(x_i,x_j)B_z(r(x_i,x_j))R'(x_i,x_j), \tag{2.59}$$

for some two rotations $R(x_i,x_j)$ and $R'(x_i,x_j)$ and a boost $B_z(r(x_i,x_j))$ in the z direction. $r(x_i,x_j)$ denotes the rapidity of the boost. We can always choose this decomposition such that $r(x_1,x_2)$ is positive, and we do so. $r(x_i,x_j)$ is in fact the hyperbolic distance between x_i and x_j. To see this, we recall that the hyperbolic distance, or hyperbolic angle, between two points $x_i,x_j \in Q_1$ is defined by

$$d(x_i,x_j) := \cosh^{-1}(x_i,x_j) \tag{2.60}$$

where $(\cdot,\cdot)$ denotes the Minkowski metric. We thus have

$$\begin{aligned}\cosh d(x_i,x_j) &= (x_i,x_j) = (B(x_i)e,B(x_j)e) = (e,B(x_i)^{-1}B(x_j)e)\\ &= (e,R(x_i,x_j)B_z(r(x_i,x_j))R'(x_i,x_j)e)\\ &= (R(x_i,x_j)^{-1}e,B_z(r(x_i,x_j))R'(x_i,x_j)e)\\ &= (e,B_z(r(x_i,x_j))e) = \cosh r(x_i,x_j)\end{aligned} \tag{2.61}$$

so that $r(x_i,x_j) = d(x_i,x_j)$, proving $r(x_i,x_j)$ is the hyperbolic distance, as claimed.

Now, let us consider the matrix elements of $B_z(r)$ in a given representation (k,p) in the principal series. Because the generator $K_z := J^{03}$ of z-boosts commutes with H_z, we have

$$D^{k,p}_{lm,l'm'}(B_z(r)) = \delta_{mm'}d^{k,p}_{ll'm}(r) \tag{2.62}$$

for some function $d^{k,p}_{ll'm}(r)$. As shown in appendix A, the behavior of $d^{k,p}_{ll'm}(r)$ in the $r \to \infty$ limit is of the form

$$d^{k,p}_{ll'm}(r) \propto e^{-\lambda_{k,m}r} \tag{2.63}$$

where

$$\lambda_{k,m} = 1 + |m+k| \geq 1. \tag{2.64}$$

This in particular implies that for any $\epsilon > 0$,

$$\lim_{r\to\infty} e^{(1-\epsilon)r} d^{k,p}_{ll'm}(r) = 0. \tag{2.65}$$

Because $e^{(1-\epsilon)r} d^{k,p}_{ll'm}(r)$ is furthermore continuous, we know $e^{(1-\epsilon)r} d^{k,p}_{ll'm}(r)$ is bounded on $r \in [0,\infty)$, so that there exists $C^{k,p,\epsilon}_{ll'm} \in \mathbb{R}^+$ such that

$$\begin{aligned} e^{(1-\epsilon)r} d^{k,p}_{ll'm}(r) &< C^{k,p,\epsilon}_{ll'm} \\ \Rightarrow \qquad d^{k,p}_{ll'm}(r) &< C^{k,p,\epsilon}_{ll'm} e^{-(1-\epsilon)r} \end{aligned} \tag{2.66}$$

for all $r \in [0,\infty)$.

We then have the following bound:

$$\begin{aligned} \left| D^{k,p}_{lm,l'm'} \left(B(x_i)^{-1} B(x_j) \right) \right| &\leq \sum_{m''} \left| D^l_{mm''}(R) d^{k,p}_{ll'm''}(r(x_i,x_j)) D^{l'}_{m''m'}(R') \right| \\ &= \sum_{m''} \left| D^l_{mm''}(R) \right| \left| d^{k,p}_{ll'm''}(r(x_i,x_j)) \right| \left| D^{l'}_{m''m'}(R') \right| \\ &\leq \sum_{m''} \left| d^{k,p}_{ll'm''}(r(x_i,x_j)) \right| \\ &< \left(\sum_{m''} C^{k,p,\epsilon}_{ll'm''} \right) e^{-(1-\epsilon)r}, \end{aligned} \tag{2.67}$$

where we have used again that the rotation representation matrices are bounded by 1. Defining $C^{k,p,\epsilon}_{ll'} := \sum_m C^{k,p,\epsilon}_{ll'm}$, which is finite because the sum is finite, we thus have

$$\left| K^{k,p}_{ll',mm'} \right| = \left| D^{k,p}_{lm,l'm'} \left(B(x_i)^{-1} B(x_j) \right) \right| < C^{k,p,\epsilon}_{ll'} e^{-(1-\epsilon)r} \tag{2.68}$$

for all $r \in [0,\infty)$. To prepare Lemma 2 below, we need the following result on hyperbolic geometry, that we also borrow from (BB 2001):

Lemma 1. *Suppose $x_1, x_2, ..., x_n \in Q_1$. Then there exists a point q s.t. for any point $x \in Q_1$ we have:*

$$d(x,q) \leq \frac{1}{n} \left(d(x_1,x) + \cdots + d(x_n,x) \right) \tag{2.69}$$

Lemma 2. *If $n \geq 3$, the integral*

$$J := \int_{Q_1} \mathrm{d}x \; \left| K^{k_1,p_1}_{l_1 l_1', m_1 m_1'}(x,x_1) \right| ... \left| K^{k_n,p_n}_{l_n l_n', m_n m_n'}(x,x_n) \right|$$

converges and for any $0 < \epsilon < 1/3$ there exists $C^\epsilon(\{k_i,p_i,l_i,l_i'\})$, $i = 1...n$, function of the representation labels, such that for any $(x_1,...,x_n)$,

$$J \leq C^\epsilon(\{k_i,p_i,l_i,l_i'\}) \exp\left(-\frac{n-2-n\epsilon}{n(n-1)} \sum_{i<j} r_{ij} \right),$$

where $r_{ij} := d(x_i,x_j)$.

Proof. First, using (2.68) one has:

$$\left|K^{k_1,p_1}_{l_1 l'_1,m_1 m'_1}(x,x_1)\right| ... \left|K^{k_n,p_n}_{l_n l'_n,m_n m'_n}(x,x_n)\right| \leq \left(\prod_{i=1}^{n} C^{k_i,p_i,\epsilon}_{l_i l'_i}\right) e^{-(1-\epsilon)\sum r_i}, \tag{2.70}$$

where $r_i := d(x,x_i)$. Define $\tilde{C} := \prod_{i=1}^n C^{k_i,p_i,\epsilon}_{l_i l'_i}$, then one has

$$J \leq 4\pi\tilde{C}\int_0^\infty \sinh^2 r \mathrm{d}r e^{-(1-\epsilon)\sum r_i}, \tag{2.71}$$

where r is defined as the distance of x from the barycentre of the points $(x_1, ...x_n)$. The fact that it exists is object of Lemma 1 above. From the same lemma, one has

$$\sum r_i \geq nr. \tag{2.72}$$

In addition, defining

$$M := \frac{1}{n} min_x \sum_i r_i(x), \tag{2.73}$$

one has

$$\sum r_i \geq nM. \tag{2.74}$$

Both inequalities can be used to prove the following bound for J:

$$J \leq 4\pi\tilde{C}C' \; e^{-(n-2-n\epsilon)M}, \tag{2.75}$$

for some positive constant C' depending only on ϵ and n. From the triangle inequality, one has

$$\sum r_i \geq \frac{1}{n-1}\sum_{i<j} r_{ij}, \tag{2.76}$$

and

$$M \geq \frac{1}{n(n-1)}\sum_{i<j} r_{ij}, \tag{2.77}$$

which then implies the lemma with $C = 4\pi\tilde{C}C'$. ■

Theorem 2. *The tetrahedron graph, with any labelling, is integrable.*

Proof. We will show that the following quantity (for any fixed $x_1 \in Q_1$ and independent of it) is finite:

$$\begin{aligned} I \;\; := \;\; & \int_{Q_1^3} \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \; |K^{\chi_{12}}(x_1,x_2)K^{\chi_{13}}(x_1,x_3)K^{\chi_{14}}(x_1,x_4) \\ & K^{\chi_{23}}(x_2,x_3)K^{\chi_{24}}(x_2,x_4)K^{\chi_{34}}(x_3,x_4)| \, , \end{aligned} \tag{2.78}$$

where χ_{ij} denotes, for short, the set of labels $(k_{(ij)}, p_{(ij)}, l_{ij}, m_{ij}, l_{ji}, m_{ji})$. Start by integrating over x_4 using Lemma 2,

$$I \leq C^{\epsilon}(\{\chi_{ij}\}) \int_{Q_1^2} \mathrm{d}x_2 \mathrm{d}x_3 \; e^{-\frac{1}{6}(1-3\epsilon)(r_{12}+r_{13}+r_{23})} \left| K^{\chi_{12}}(x_1,x_2) K^{\chi_{13}}(x_1,x_3) K^{\chi_{23}}(x_2,x_3) \right|, \tag{2.79}$$

where $r_{ij} = d(x_i, x_j)$. Next, we integrate over x_3. Consider the quantity

$$L := \int_{Q_1} \mathrm{d}x_3 \; e^{-\frac{1}{6}(1-3\epsilon)(r_{13}+r_{23})} \left| K^{\chi_{13}}(x_1,x_3) K^{\chi_{23}}(x_2,x_3) \right|. \tag{2.80}$$

By (2.68), one has

$$L \leq \tilde{C} \int_{Q_1} \mathrm{d}x_3 \; e^{-(r_{13}+r_{23})(\frac{7}{6}-\frac{3\epsilon}{2})} \tag{2.81}$$

Now, introduce the new coordinate system (k, l, ϕ), where:

$$k = \frac{1}{2}(r_{13} + r_{23}) \;\; ; \;\; l = \frac{1}{2}(r_{13} - r_{23}), \tag{2.82}$$

and ϕ is the angle between the plane containing x_1, x_2, x_3 and a given plane containing x_1 and x_2. Their ranges are: $k \in [\frac{r_{12}}{2}, \infty)$, $l \in [-\frac{r_{12}}{2}, \frac{r_{12}}{2}]$ and $\phi \in [0, 2\pi)$. The measure $\mathrm{d}x_3$ on Q_1 in this coordinate system reads (see appendix of BB 2001):

$$\mathrm{d}x_3 = 2 \, \frac{\sinh r_{13} \sinh r_{23}}{\sinh r_{12}} \, \mathrm{d}k \; \mathrm{d}l \; \mathrm{d}\phi. \tag{2.83}$$

In terms of these new coordinates, we have:

$$\begin{aligned} L &\leq \frac{\tilde{C}}{\sinh r_{12}} \int_0^{2\pi} \mathrm{d}\phi \int_{-\frac{r_{12}}{2}}^{\frac{r_{12}}{2}} \mathrm{d}l \int_{\frac{r_{12}}{2}}^{\infty} \mathrm{d}k \; e^{-k(\frac{1}{3}-3\epsilon)} \\ &\leq \frac{2\pi r_{12} \tilde{C}}{\sinh r_{12}} \; e^{-r_{12}(\frac{1}{6}-\frac{3\epsilon}{2})} \end{aligned} \tag{2.84}$$

for $\epsilon < 1/9 < 1/3$. Plugging this in the evaluation of I, we get:

$$\begin{aligned} I &\leq C' \int \mathrm{d}r \; r \sinh r \; e^{-r(\frac{4}{3}-3\epsilon)} \\ &\leq C' \int \mathrm{d}r \; r \; e^{-r(\frac{1}{3}-3\epsilon)}, \end{aligned} \tag{2.85}$$

which is finite for $0 < \epsilon < 1/9$ and some constant C' depending on the representation labels $\{\chi_{ij}\}$. ■

Theorem 3. *(Baez and Barrett) A graph obtained from an integrable graph by connecting an extra vertex to the existing labeled graph by at least three edges, with arbitrary labeling, is integrable. A graph obtained from an integrable graph by adding extra edges, with arbitrary labeling, is integrable. A graph constructed by joining two disjoint integrable graphs at a vertex is integrable.*

Corollary 1. *The 4-simplex graph, with any labeling, is integrable.*

The BC and Flipped models

Before concluding this chapter we would like to take a look at two special values for the Immirzi parameter: $\gamma = 0$ and $\gamma = \infty$. The case for $\gamma = 0$ has originated the Flipped model and was defined prior to the model presented here, which applies for arbitrary values of γ. In this case the simplicity constraints (2.48) restrict the representation labels to be of the form $(k, 0)$.

More interesting is the case where γ is set to infinity. In this case, representations are restricted to be of the form $(0, p)$ and $j = 0$. The amplitude is a function of the representation labels p_{ab} which are now free parameters. It is given by the formula:

$$A(p_{ab}) = \int_{\mathrm{SL}(2,\mathbb{C})^{\times 5}} \prod_a \mathrm{d}X_a \, \delta(X_5) \prod_{ab} D^{(0,p_{ab})}_{0\,0,0\,0}(X_a^{-1} X_b). \tag{2.86}$$

Because of the restriction to the invariant SU(2) irreducible, $j_{ab} = 0$, the D matrix in the formula above does not really depend on the rotation parts of the group elements X_a. We then have that

$$D^{(0,p_{ab})}_{0\,0,0\,0}(X_a^{-1} X_b) = D^{(0,p_{ab})}_{0\,0,0\,0}(B(x_a)^{-1} B(x_b)) = d^{(0,p_{ab})}_{000}(r(x_a, x_b)) = \frac{\sin p_{ab}\, r(x_a, x_b)}{p_{ab} \sinh r(x_a, x_b)}.$$

We recognize the propagator $K_{p_{ab}}(x_a, x_b)$ defined in (1.35), and the 4-simplex amplitude in this limit is just the 10j symbol defined before:

$$A(p_{ab}) = 10j(p_{ab}). \tag{2.87}$$

This limit for γ is especially singular. In fact, we see that the boundary data has a different structure, as the amplitude does not depend anymore on magnetics numbers. This will have an important consequence in trying to construct a path integral by gluing simplices together. In particular it makes it difficult to extract semiclassical information out of the boundary data. This statement will become clearer in the next chapter.

Chapter 3

Spin foam

In the last chapter we have considered the quantization of a geometric 4-simplex, and as a result we have associated to it a quantum amplitude depending on a set of boundary data labeled by spins and magnetic numbers. In this chapter we would like to consider more general triangulations and construct a quantum amplitude for these. By gluing simplices properly we will be able to write this amplitude as a sum over classical histories, each history being determined by a certain simplicial configuration. In the process we will consider a different set of boundary variables, where magnetic numbers are replaced by normals to triangles.

The procedure is to start with a canonical description of the quantum system at hand and then reconstruct the sum over histories out of it, in the same spirit as Feynman's original derivation of the path integral for a non relativistic system. We will revise his construction with a somewhat alternative point of view, following Klauder (Klauder 1997), where coherent states play a key role.

3.1 Feynman's path integral

The idea one should have in mind is that of a Feynman path integral. There the path integral gives a transition amplitude between initial and final states living in the Hilbert space $\mathcal{H}_t$ associated to a given time slice Σ_t. The full kinematical Hilbert space for a system defined between the times t_i and t_f is then given by $\mathcal{K}_{t_i,t_f} = \bar{\mathcal{H}}_{t_i} \otimes \mathcal{H}_{t_f}$, to be compared with our kinematical space defined before (2.40). We will suppose later a certain set of constraints $\hat{\Phi}_i$ to be imposed on this kinematical Hilbert space.

For the moment suppose a free system described by a Hamiltonian H and a certain set of variables parameterizing the phase space (p_j, q_j). We want to define the probability amplitude that a initial state $\psi_i \in \mathcal{H}_{t_i}$ evolves into a final state $\psi_f \in \mathcal{H}_{t_f}$, given by the matrix element of the evolution operator $\hat{U}(T) = e^{-i\hat{H}T}$:

$$K(\psi_f, \psi_i) := \langle \psi_f | e^{-i\hat{H}(t_f - t_i)} | \psi_i \rangle. \tag{3.1}$$

To construct the path integral, first choose a certain foliation of the space-time region between Σ_{t_i} and Σ_{t_f}, such that the total time $T = t_f - t_i$ is subdivided in small intervals of duration $\epsilon = T/(N+1)$, for large N. On each time slice, insert a resolution of the identity of $\mathcal{H}_t$. The choice of variables for writing this resolution of the identity is in principle arbitrary, and different choices will lead to different path integrals. We will follow Klauder (Klauder 1997) and use a coherent state basis. Coherent states are designed to extract the classical information out of quantum systems, and this is exactly what we are trying to do in passing from the canonical expression (3.1) to a path integral in terms of classical trajectories.

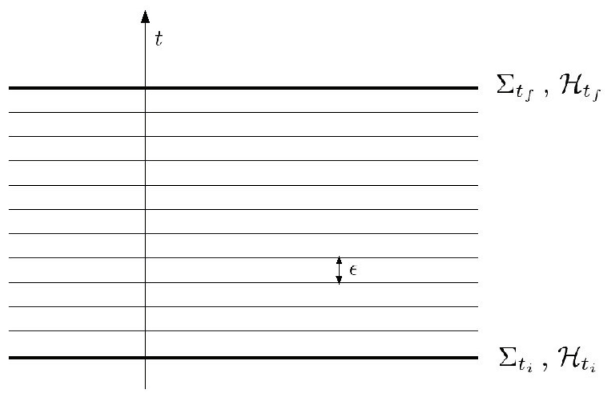


Figure 3.1: Foliation of space time in slices.

If $\hat{P}_j$ and $\hat{Q}_j$ are canonical operators obtained from quantizing the classical variables (p_j, q_j) describing our phase space, then a coherent state $|p, q\rangle$ is defined by:

$$|p, q\rangle := e^{i\varphi(p,q)} e^{-iq^j \hat{P}_j} e^{ip^j \hat{Q}_j} |\eta\rangle. \tag{3.2}$$

In the definition above, the pair (p, q) labels collectively the set of variables (p_j, q_j). $\varphi(p, q)$ is a phase that we may choose conveniently. The operators $\hat{P}_j$, $\hat{Q}_i$ satisfy the canonical commutation relations and $|\eta\rangle$ is a certain fiducial vector. It can be given by the ground state of the system, for example.

The definition is such that the expectation values of $\hat{P}_j$ and $\hat{Q}_j$ are given by the corresponding coherent state labels p_j and q_j. The definition given above is restricted to the case where a canonical pair of variables is available, but can be generalized to other cases, in special when the variables describing the phase space are quantized as generators of a certain symmetry group, as will be the case of interest for us later. For now, let us continue exploring this simpler system.

The crucial property of coherent states for us is that they provide a resolution of the identity:

$$\mathbb{1} = \int \mathrm{d}\mu(p,q)|p,q\rangle\langle p,q| \tag{3.3}$$

By inserting this resolution of identity at each time slice, the amplitude for initial and final states given in the coherent state basis is given by:

$$\begin{aligned}
K(p'',q'';p',q') &= \langle p'',q''|e^{-i\hat{H}T}|p',q'\rangle = \langle p'',q''|e^{-i\hat{H}\epsilon}\cdots e^{-i\hat{H}\epsilon}\cdots|p',q'\rangle = \\
&= \int \prod_{l=1}^{N} \mathrm{d}\mu(p_l,q_l) \prod_{l=0}^{N} \langle p_{l+1},q_{l+1}|e^{-i\hat{H}\epsilon}|p_l,q_l\rangle \\
&= \lim_{\epsilon\to 0} \int \prod_{l=1}^{N} \mathrm{d}\mu(p_l,q_l) \prod_{l=0}^{N} \Big[\langle p_{l+1},q_{l+1}|p_l,q_l\rangle - i\epsilon\langle p_{l+1},q_{l+1}|\hat{H}|p_l,q_l\rangle\Big] \qquad (3.4)
\end{aligned}$$

A formal limiting procedure allows then to write the standard path integral

$$\mathcal{N}\int D\mu(p,q)e^{i\int\left[p_j\dot{q}^j - H(p,q)\right]dt}, \tag{3.5}$$

where $\mathcal{N}$ is a certain normalization factor coming from the limiting procedure. It is important to emphasize that the expression for the path integral as a sum over classical histories is only formal, and one should be careful when using it for practical calculations. The proper definition is given by the expression (3.4), and uses directly the structure of the quantum space. The passage to a classical description is taken care of by the coherent states.

Let us now consider the case of a system subject to a set of constraints Φ_i. From a classical point of view we would insert Lagrange multipliers in the action such that the path integral becomes

$$\int D\mu(p,q,\lambda)\, e^{i\int\left[p_j\dot{q}^j - H(p,q) - \lambda^i\Phi_i(p,q)\right]}. \tag{3.6}$$

We would like to recover the above expression for the path integral from canonical methods, in parallel to what we did before in the unconstrained case. To that aim, define projectors from the kinematical Hilbert space $\mathcal{H}_t$ to a subspace satisfying the constraints. A projection operator $\mathbb{E}$ is such that $\mathbb{E}^2 = \mathbb{E}$ and $\mathbb{E}^\dagger = \mathbb{E}$. As an example, consider the case where the constraints satisfy a certain Lie algebra and in addition form a first class system with the Hamiltonian:

$$\begin{aligned}
[\,\hat{\Phi}_i\,,\,\hat{\Phi}_j\,] &= i\,c_{ij}{}^{k}\,\hat{\Phi}_k. \\
[\,\hat{\Phi}_i\,,\,\hat{H}\,] &= i\,h_i^j\,\hat{\Phi}_j. \qquad (3.7)
\end{aligned}$$

Then the projector $\mathbb{E}$ can be defined by:

$$\mathbb{E} := \int e^{-i\xi^i \hat{\Phi}_i} \mathrm{d}\mu(\xi), \tag{3.8}$$

where $\mathrm{d}\mu(\xi)$ is the Haar invariant measure on the group, that we suppose compact for simplicity [1]. Using the invariance properties of the Haar measure, one can check that indeed $\mathbb{E}$ defined above squares to itself and is hermitian. One can also check that

$$e^{-i\tau^i \hat{\Phi}_i} \mathbb{E} = \mathbb{E}, \tag{3.11}$$

which is equivalent to stating that $\mathbb{E}$ projects into the subspace where $\hat{\Phi}_i = 0$. Also, because it is a first class system,

$$e^{-i\hat{H}T} \mathbb{E} = \mathbb{E} e^{-i\hat{H}T} = \mathbb{E} e^{-i\hat{H}T} \mathbb{E}. \tag{3.12}$$

Now, let us construct the constrained path integral. Because the constraints form a first class system with the Hamiltonian, the constraints commute with the evolution and it is sufficient to impose them only on the initial time slice. On the coherent state basis, we define:

$$K(p'', q'', p', q') = \langle p'', q'' | e^{-iT\hat{H}} \mathbb{E} | p', q' \rangle. \tag{3.13}$$

As before, introduce a foliation of the region of space time between the initial and final time slices. We then have:

$$K(p'', q'', p', q') = \langle p'', q'' | \mathbb{E} e^{-i\epsilon \hat{H}} \mathbb{E} e^{-i\epsilon \hat{H}} \mathbb{E} \cdots \mathbb{E} e^{-i\epsilon \hat{H}} \mathbb{E} | p', q' \rangle, \tag{3.14}$$

where we have used the property (3.12) of the projector operator. Now, introduce resolutions of the identity on each time slice:

$$\langle p'', q'' | e^{-iT\hat{H}} \mathbb{E} | p', q' \rangle = \lim_{\epsilon \to 0} \int \langle p_{N+1}, q_{N+1} | e^{-i\epsilon \lambda_N^i \hat{\Phi}_i} | p_N, q_N \rangle \times$$
$$\times \prod_{l=0}^{N-1} \langle p_{l+1}, q_{l+1} | e^{-i\epsilon \hat{H}} e^{-i\epsilon \lambda_l^i \hat{\Phi}_i} | p_l, q_l \rangle \prod_{l=1}^{N} \left(\mathrm{d}\mu(p_l, q_l) \mathrm{d}\epsilon \lambda_l \right) \tag{3.15}$$

---

[1] In the case where the group is non compact one could introduce a smearing function in the definition of the projector (see Klauder 1997):

$$\mathbb{E} := \int e^{-i\xi^i \hat{\Phi}_i} f(\xi) \mathrm{d}\mu(\xi). \tag{3.9}$$

The smearing function has to satisfy the following properties:

$$\int \mathrm{d}\mu(\xi')\, f(\xi') f(\xi'^{-1} \cdot \xi) = f(\xi) \;\text{ and }\; \bar{f}(\xi^{-1}) \mathrm{d}\mu(\xi^{-1}) = f(\xi) \mathrm{d}\mu(\xi), \tag{3.10}$$

as can be readily verified.

A formal limiting procedure can then be performed, leading to the path integral

$$\int D\mu(p,q) DE(\lambda)\, e^{i\int \left(p_j \dot{q}^j - H(p,q) - \lambda^i \Phi_i\right)dt}, \tag{3.16}$$

where $DE(\lambda)$ is a certain measure on the Lagrange multipliers induced by the limiting procedure.

Up to now, we have been considering first class constraints, but the construction can be readily generalized to second class systems. The essential difference in this case is that the projectors do not commute any more with the evolution operator, and it is not sufficient to impose the constraints only on the initial time slice. All one needs to do is to start directly with the expression (3.14), taken now as the definition of the amplitude. Classically, the main difference between the two cases is that for first class systems the Lagrange multipliers $\lambda^i(t)$ are arbitrary functions, while for second class systems, they are at least partially determined by the equations of motion. The role of Lagrange multipliers is to ensure that points remain on the constraint surface, as the Hamiltonian evolution tends to force them out of it in the case of second class constraints. In the definition of the path integral (3.14) that is exactly what we are doing, the projector operators on each time slice enforcing the histories to remain on the constraint surface. As a consequence of that we have that for first class systems the path integral (3.16) does not depend on the measure $DE(\lambda)$. Any measure would do the job, provided it is correctly normalized. This is not the case for second class systems, where the measure is determined by the choice of projectors and the limiting procedure.

Now, back to simplicial geometry. We want to follow an analogous procedure to the one described above. The foliation of space time will be replaced by a simplicial decomposition of space time, each 4-simplex playing the role of a time step ϵ [2]. We will have a free evolution given by the unconstrained 4-simplex amplitude and the simplicity constraints will be imposed at common boundaries of simplices. We will do so everywhere, in parallel to the case of second class systems discussed before, thus ensuring that the constraints are imposed in every point of space time. 4-simplices meet on tetrahedra, and thus the equivalent of the Hilbert space $\mathcal{H}_t$ associated to a given time slice will be the Hilbert space $\mathcal{K}_3$ associated to tetrahedra. Following Klauder's procedure, we need to introduce coherent states describing this kinematical Hilbert space.

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[2] Because we are dealing with a relativistic system the Hamiltonian evolution is itself a constraining operation. In our case, the unconstrained 4-simplex amplitude is just a projection on solutions of the the flatness constraint $X_{ab} = X_a^{-1} X_b$.

3.2 Coherent tetrahedra

The aim of this section is to give a semiclassical description to the quantum tetrahedron described earlier. The ideas presented here were first introduced in (Livine and Speziale 2007), where the authors used the notion of coherent states for the SU(2) group defined earlier by Perelomov (Perelomov 1986).

We collect here the essential facts concerning Perelomov's construction. A coherent state for SU(2) is labeled by a unit vector $\hat{n}$ in $\mathcal{S}^2 \sim \mathrm{SU}(2)/\mathrm{U}(1)$ and a spin j. To the normal $\hat{n}$ associate a group element $g(\hat{n}) \in \mathrm{SU}(2)/\mathrm{U}(1)$ that maps the unit vector in the z direction e_z into the normal $\hat{n}$. This rotation is defined up to a phase labeling a rotation around the z direction. A phase $\varphi \in (0, 2\pi)$ will label the representative of $g(\hat{n})$ in SU(2), that we denote $g(\hat{n}, \varphi) = g(\hat{n}) e^{i\varphi\sigma_z}$, where σ_z is the Pauli matrix on the z direction. We define $g(\hat{n})$ such that the diagonal components are real. Parameterizing each normal by the two spherical angles (θ, ϕ), we have:

$$g(\hat{n}) = g(\theta, \phi) = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 e^{-i\phi} \\ -\sin\theta/2 e^{i\phi} & \cos\theta/2 \end{pmatrix} \tag{3.17}$$

The coherent state is defined by the action of $g(\hat{n}, \varphi)$ on the maximum weight state $|j, +j\rangle$[3]:

$$|j, \hat{n}, \varphi\rangle := g(\hat{n}, \varphi) \triangleright |j, +j\rangle = e^{i\varphi j} g(\hat{n}) \triangleright |j, +j\rangle. \tag{3.18}$$

The phase φ will play an important role in the construction of boundary states. The decomposition of this state in the usual basis $|j, m\rangle$ is given by:

$$|j, \hat{n}, \varphi\rangle = \sum_m D^j_{m,+j}(g(\hat{n}, \varphi)) \, |j, m\rangle. \tag{3.19}$$

These are coherent states in the sense that they minimize the uncertainty ΔH^2. The expectation value of the generators is given by:

$$\langle j, \hat{n}, \varphi| \vec{H} \, |j, \hat{n}, \varphi\rangle = j\hat{n} \tag{3.20}$$

A different parametrization of coherent states is given by the identification between unit spinors and unit vectors in $\mathbb{R}^3$:

$$\xi \otimes \xi^\dagger = \frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \hat{n}_\xi). \tag{3.21}$$

[3]Alternatively one could define a coherent state by acting on the lowest weight state $|j, -j\rangle$. The difference is immaterial and we will always use the definition above.

$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, the dot $\cdot$ is the scalar product in $\mathbb{R}^3$ and $\dagger$ stands for Hermitian conjugation. Also, if the vector $\hat{n}_\xi$ is associated to the spinor ξ then the opposite vector $-\hat{n}_\xi$ is associated to the spinor $J\xi$, where J is the anti-linear map:

$$J \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} -\bar{\xi}_1 \\ \bar{\xi}_0 \end{pmatrix} \tag{3.22}$$

Note that for a given vector $\hat{n}$ the associated spinor is only defined up to a phase, and the phase φ can be absorbed into the definition of ξ. Explicitly, for $\hat{n} = (\theta, \phi)$:

$$\xi(\theta, \phi, \varphi) = e^{i\varphi} \begin{pmatrix} \cos\theta/2 \\ -\sin\theta/2 e^{i\phi} \end{pmatrix} \tag{3.23}$$

Define the map $g : \mathbb{C}^2 \to \mathrm{GL}(2, \mathbb{C})$

$$g(\xi) = \begin{pmatrix} \xi_0 & -\bar{\xi}_1 \\ \xi_1 & \bar{\xi}_0 \end{pmatrix}, \tag{3.24}$$

such that the definition of $\xi(\theta, \phi, \varphi)$ is consistent with the definition of $g(\hat{n}, \varphi)$ before. The coherent state $|j, \xi\rangle$ is defined analogously:

$$|j, \xi\rangle := g(\xi) \triangleright |j, +j\rangle \tag{3.25}$$

The important property of coherent states is that they provide a (over)complete basis for the carrying space $\mathcal{H}_j$:

$$\mathbb{1}_j = d_j \int_{S^2} d^2\hat{n}\, |j, \hat{n}, \varphi\rangle\langle j, \hat{n}, \varphi| = \sum_m |j, m\rangle\langle j, m|. \tag{3.26}$$

Note that the phases coming from the bra and ket states cancel out in the expression above and there is no point in integrating over them. The semiclassical geometry of a tetrahedron τ is described by the state (Livine and Speziale 2008)

$$|\tau\rangle_{j_t, \hat{n}_t, \varphi_t} = \int_{\mathrm{SU}(2)} dg\, g \cdot \left(\otimes_{t \subset \tau} |j_t, \hat{n}_t, \varphi_t\rangle\right). \tag{3.27}$$

This state belongs to the invariant subspace under the diagonal action of SU(2), $\mathrm{Inv}\big(\otimes_{j_f} \mathcal{H}_{j_f}\big)$, as can be easily verified. It can in particular be decomposed in the usual intertwiner basis for the group. The states defined above provide an over complete basis to this invariant space. An interesting property is that the subset satisfying a closure condition

$$\sum_{t \subset \tau} j_t \hat{n}_t = 0 \tag{3.28}$$

is still a basis of this space. This property is a result of a theorem in (Guillemin and Sternberg 1982) and is largely discussed in (Conrady and Freidel 2009).

Boosted tetrahedra

We would like now to understand how to embed tetrahedra in 4d such that it describes the boundary of a four dimensional triangulation. In other words, we want to understand how SL$(2,\mathbb{C})$ group elements act on the coherent states describing a boundary tetrahedron geometry. This is given by identifying states for the SU(2) group $|j,m\rangle$ with the canonical basis for the Lorentz group $|(k,p);j,m\rangle$ discussed before when defining the vertex amplitude. A coherent state is then given by the action of a rotation $g(\hat{n},\varphi)$ on the maximum weight state:

$$|(k,p);j,\hat{n},\varphi\rangle := g(\hat{n},\varphi) \triangleright |(k,p);j,+j\rangle. \tag{3.29}$$

We will later give an explicit expression for this state. For the moment, let us see how this state describes the embedded geometry of a triangle. We are interested in the expectation value of the quantized bivector $\hat{b}_t$ describing the triangle t. Using the identification with generators (2.42), we have that

$$\langle j,\hat{n},\varphi;(k,p)|\,\hat{b}_t\,|(k,p);j,\hat{n},\varphi\rangle = -\frac{\gamma^2}{1+\gamma^2}\langle\frac{1}{\gamma}J_t + \star J_t\rangle. \tag{3.30}$$

Separating in boost and rotation components, we have:

$$\langle\hat{b}^{0i}\rangle = -\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{\gamma}\langle K^i\rangle + \langle H^i\rangle\right) \tag{3.31}$$

and

$$\langle\hat{b}^{ij}\rangle = -\frac{\gamma^2}{1+\gamma^2}\epsilon^{ij}{}_k\left(\frac{1}{\gamma}\langle H^k\rangle - \langle K^k\rangle\right). \tag{3.32}$$

The expectation values of the boost and rotation generators on the coherent state $|(k,p);j,\hat{n},\varphi\rangle$ are given by (see appendix A):

$$\langle\vec{H}\rangle = j\hat{n} \;\text{ and }\; \langle\vec{K}\rangle = -j\beta_{(j)}\hat{n}, \tag{3.33}$$

where $\beta_{(j)}$ is given by

$$\beta_{(j)} := \frac{kp}{j(j+1)}. \tag{3.34}$$

When the simplicity constraints (2.48) are satisfied and for a classical ordering of the SU(2) Casimir L^2, we have that $\beta_{(j)}$ is constant and equal to the Immirzi parameter γ. This implies that $\langle\vec{K}\rangle = -\gamma\langle\vec{H}\rangle$ on this state, which in turn implies that $\langle\hat{b}^{0i}\rangle = 0$, as it should, and $\langle\hat{b}^{ij}\rangle = -\gamma\epsilon^{ij}{}_k\langle H^k\rangle = -\gamma j\epsilon^{ij}{}_k\hat{n}^k$. Now, remembering the formula

(2.27) relating classical bivectors and normals, we see that the boosted coherent states describe correctly the semiclassical geometry of a triangle, with area given by:

$$A_t = \gamma j_t. \tag{3.35}$$

Let us take a look at the boundary data for the BC model. As described at the end of the last chapter, this can be obtained setting γ to ∞. The representation labels k and j are set to zero and p is left free. Because j is set to zero the boundary states are all set to the singlet and the boundary data is specified uniquely by the representation labels $\{p_t\}$. The area A_t of a triangle is identified to p_t. To see that, remark that A_t is just the norm of the bivector b_t, which after quantization is identified in this limit for γ with $(- \star J_t)$ by eq. (2.42). This norm is given by the Casimir C_1 (cf. appendix A). Dropping the constant term in this Casimir we have:

$$A_t = p_t. \tag{3.36}$$

The Flipped model, obtained in the limit where γ is set to zero has a curious geometric content. Even though it preserves spins and normals as boundary data, the physical areas are all zero, according to equation (3.35). This supports the claim that this model corresponds to the degenerate sector of Plebanski theory made in (Freidel and Krasnov 2008). A precise analysis of this relationship between continuum and discrete theories would be very interesting but we leave this to further investigation.

Boundary state

The coherent tetrahedra defined here will have a double role in the construction of the spin foam amplitude. In one hand they give us a suitable resolution of the identity on intermediate Hilbert spaces associated to tetrahedra, allowing to rewrite the amplitude as a sum over classical histories, as discussed in the last section. On the other hand, for triangulations with boundary, they can be used to construct semiclassical states describing the boundary geometry.

A boundary state $\psi(j, \hat{n}, \varphi)$ is given by the tensor product of states of the form (3.27), one for each tetrahedron on the boundary triangulation $\partial\Sigma$:

$$\psi(j, \hat{n}, \varphi) := \otimes_{\tau \subset \partial\Sigma} \, |\tau\rangle_{j_t, \hat{n}_t, \varphi_t}. \tag{3.37}$$

We will be interested in the case where this boundary data describes a Regge geometry for the boundary triangulation. If the boundary data $(\gamma j, \hat{n})$ is Regge like then, there exist generically unique rotations $\hat{g}_{ab} \in \mathrm{SO}(3)$ gluing the geometry of the triangle t_{ab} seen by tetrahedra τ_a and τ_b:

$$\hat{g}_{ab} \Delta_{ab} = \Delta_{ba} \tag{3.38}$$

and

$$\hat{g}_{ab}\hat{n}_{ab} = -\hat{n}_{ba} \tag{3.39}$$

Then choose a spin lift g_{ab} for each rotation $\hat{g}_{ab}$. For Regge boundary data, there is a canonical choice of phases given such that the following gluing conditions are satisfied:

$$g_{ab}J\xi_{ab} = \xi_{ba}. \tag{3.40}$$

Remember that the choice of phases is hidden in the definition of the spinors ξ_{ab}. We call a state defined with Regge like boundary data $(\gamma j, \hat{n})$ and with this canonical choice of phases a Regge state.

3.3 Gluing 4-simplices

The coherent tetrahedra of the last section provide us with a nice semiclassical description for the kinematical space associated to a single tetrahedron. The key property is given by the resolution of the identity (3.26) on the carrying space $\mathcal{H}_{j_t}$ associated to a triangle and for a fixed spin j_t. We would like to see this resolution of the identity as a projector imposing the simplicity constraints on that triangle. To do so we would need a resolution of the identity involving all quantum numbers associated to that triangle, (k_t, p_t) and j_t, and a prescription to restrict this sum to the solutions of the constraints. The precise construction of this projector is not clear to us and could in principle fix the measure of the sum over the spins. We will consider here an ansatz for the measure consisting of a factor $d_{j_t}^{\alpha}$ for each triangle, where α is an arbitrary integer. The motivation for this choice is that this is the measure that comes out for BF theory.

The same way as general Regge triangulations are constructed by gluing together different 4-simplices, we will construct an amplitude for an arbitrary triangulation by gluing together amplitudes for single 4-simplices. In doing this, the gluing conditions are essential. In classical Regge calculus, the gluing conditions are that the lengths of the edges shared by two simplices should agree.

We see that the gluing of neighbouring simplices is taken care of by the insertion of resolutions of the identity in terms of coherent states describing the semiclassical geometry of triangles and tetrahedra. When gluing two simplices through the tetrahedron τ, we demand that areas and normals should be identified. As we saw in the discussion on the quantum tetrahedron, the set of areas and normals specifies completely the tetrahedron geometry, thus guaranteeing the correct gluing between simplices.

In the BC model this gluing procedure would have the following classical interpretation. According to the discussion in the last section, the boundary data there is restricted to the areas of the triangles. When restricted to a single 4-simplex this is not really a problem. In fact, in this case the geometry is specified by 10 edge lenghts. As there are also 10 triangles in a 4-simplex, the edge lenghts can be expressed in terms of areas, up to discrete ambiguities (see Barrett, Rocek and Williams 1999). When gluing two simplices, the problem appears. The gluing conditions are that areas should be identified, but because areas do not specify the tetrahedron geometry completely - one would need two extra angles to do it - the two simplices are not glued correctly and remain uncorrelated.

As a result of introducing coherent states we now have the 4-simplex amplitude as a function of the boundary normals instead of the magnetic numbers. Start then with the amplitude $A(j_{vt}, m_{v\tau t})$ associated to a given simplex v. Let us write it in terms of the coherent state basis defined above. The change of basis is given by the formula (3.19). This defines an amplitude $A(j_{vt}, \hat{n}_{v\tau t})$ that now depends on normals and spins as boundary variables for the 4-simplex. More concretely, remember that we have chosen an orientation for each triangle in the 4-simplex. Then, to each triangle associate normals $-\hat{n}_{v\tau t}$ and $\hat{n}_{v\tau' t}$ if τ is at the source of the triangle t and resp. τ' at the end of the triangle, according to the orientation chosen before. The idea behind this definition is that the normals should represent always outward normals to tetrahedra. We have then:

$$A(j_{vt}, \hat{n}_{v\tau t}) = \int \prod_{\tau \subset v} \mathrm{d}X_\tau \prod_{t \subset v} \langle j_{vt}, -\hat{n}_{v\tau t}; (j_{vt}, \gamma j_{vt}) | X_{v\tau}^{-1} X_{v\tau'} | (j_{vt}, \gamma j_{vt}); j_{vt}, \hat{n}_{v\tau' t} \rangle \tag{3.41}$$

The gluing is given by identifying the normals in two neighbouring 4-simplices v and v', $\hat{n}_{v\tau t} = -\hat{n}_{v'\tau t} = \hat{n}_{\tau t}$, and the spins, $j_{vt} = j_{v't} = j_t$. In doing that we are forced to choose compatible orientations for the two neighbouring 4-simplices, meaning that the same triangle seen by two different simplices have opposite orientations, thus the minus sign in the gluing condition for the normals.

As we want to sum over all possible histories, we further integrate over normals and spins associated to interior triangles. Note that because of the resolution of the identity (3.26), we could have kept the original definition of the amplitude in terms of magnetic numbers $m_{v\tau t}$, and identified the two $m_{v\tau t}$, $m_{v'\tau t}$ coming from different simplices, and this would have been equivalent to the prescription given in terms of coherent states. Coherent states are somehow more natural as the geometric picture is clearer. The true advantage of using coherent states is that they allow to reconstruct a simplicial classical action, and thus construct a path integral as a sum over classical trajectories.

We are now able to write an amplitude for a certain general triangulation Σ:

$$Z_\Sigma = \sum_{\{j_t\}} d_{j_t}^\alpha \int \prod_{vt} dX_{v\tau} \prod_{\tau t} d^2\hat{n}_{\tau t} \prod_t \prod_{v \supset t} \langle j_t, \hat{n}_{\tau t}; (j_t, \gamma j_t) | X_{v\tau}^{-1} X_{v\tau'} | (j_t, \gamma j_t); j_t \hat{n}_{\tau' t} \rangle. \tag{3.42}$$

τ and τ' in the last expression denote the two tetrahedra sharing the triangle t, and the product $v \supset t$ is over the 4-simplices in the link of the triangle t. $\alpha \in \mathbb{Z}$ labels the choice for the measure in the sum over spins, as discussed at the beginning of this section. In the expression above only the normals associated to interior triangles are integrated over. For faces in the interior of Σ the product over $(v \supset t)$ goes around t and comes back to the same normal, while for faces on the boundary of Σ, it starts on a given tetrahedron on the boundary of the triangulation and ends on the other tetrahedron sharing the same triangle.

The boundary data is specified by the set of spins and normals living on the boundary tetrahedra, as discussed in the last section. We further suppose that it is Regge like and that a canonical choice of phases has been made.

The state sum model is obtained from the following considerations. First introduce a redundant integration over SU(2) per pair of dual vertices and tetrahedra $(v\tau)$. Noting this extra variable $g_{v\tau}$, make the following change of variables: $X_{v\tau} \to X_{v\tau} g_{v\tau}$. It is clear that the integration over $g_{v\tau}$ can be reabsorbed by using invariance under right multiplication of the measure over $X_{v\tau}$. Now $g_{v\tau}$ appears four times, one per face of the tetrahedron τ. Performing this integration we get up to a sign[4]

$$\begin{aligned}
&\int_{\mathrm{SU}(2)} \mathrm{d}g_{v\tau} \, \otimes_{t\subset\tau} \left(g_{v\tau} \triangleright |(j_t, \gamma j_t); j_t, m_{\tau t}\rangle\right) = \\
&= \sum_{m'_{\tau t}} \int_{\mathrm{SU}(2)} \mathrm{d}g_{v\tau} \, \otimes_{t\subset\tau} \left(D^{j_t}_{m'_{\tau t} m_{\tau t}}(g_{v\tau}) |(j_t, \gamma j_t); j_t, m'_{\tau t}\rangle \right) = \\
&= \sum_{m'_{\tau t}} \sum_{i_{v\tau}} \bar{i}^{\{m'_{\tau t}\}}_{v\tau} \, i^{\{m_{\tau t}\}}_{v\tau} \otimes_{t\subset\tau} |(j_t, \gamma j_t); j_t, m'_{\tau t}\rangle. \qquad (3.43)
\end{aligned}$$

Inserting this result in the amplitude (3.42), rewriting the resolutions of the identity (3.26) in terms of magnetic numbers and remembering the definition of the vertex amplitude in the intertwiner basis $\tilde{A}(j_{vt}, i_{v\tau})$ we can rewrite Z_Σ as:

$$\begin{aligned}
Z_\Sigma &= \sum_{\{j_t, i_{v\tau}\}} \prod_t d^\alpha_{j_t} \prod_\tau i_{v\tau} \cdot \bar{i}_{v'\tau} \prod_v \tilde{A}(j_{vt}, i_{v\tau}) \\
&= \sum_{\{j_t, i_\tau\}} \prod_t d^\alpha_{j_t} \prod_v \tilde{A}(j_{vt}, i_\tau), \qquad (3.44)
\end{aligned}$$

where in passing to the second line we have used the orthonormality of intertwiners: $i_{v\tau} \cdot \bar{i}_{v'\tau} = \delta_{i_{v\tau}, i_{v'\tau}}$ for the two simplices v and v' sharing τ.

Now, the definition of the partition function (3.42) above is only formal, being divergent as it is stated. There are two sources of divergences. The first comes from the integration over the Lorentz transformations, which can be dealt with in exact the same way as we saw in the last section for one single simplex. This is because the Lorentz

[4]The sign depends on the choice of orientations for the faces in each 4-simplex. The following formula is for the case where all four triangles of the tetrahedron τ are outward or inward.

group elements X_{vt} integrated independently for each four simplex. We thus choose one tetrahedron per simplex and drop the integration of the corresponding group element. From a Lattice Gauge Theory point of view, this can be seen as a strange sort of gauge fixing, where the gauge has been reduced in each edge to SU(2) after the simplicity constraints have been imposed, and the original Lorentz invariance is conserved at the dual vertices. Because of this breaking of symmetry, one cannot really fix completely the elements on a maximal tree as in LGT[5] see figure 3.3.

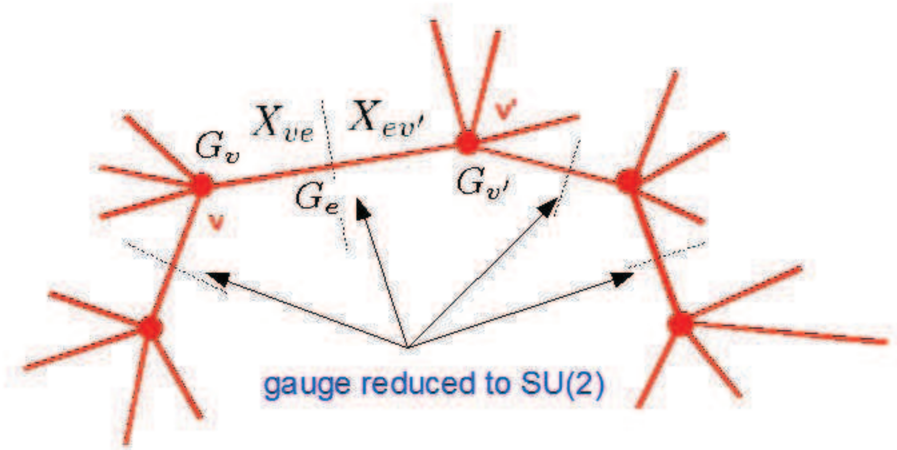

Figure 3.2: Gauge fixing in the dual of a triangulation.

The second source of divergence comes from the sum over spins, and this is more delicate. In order to understand this point, we need to take a closer look at the amplitude, with the aim of writing it as a sum over classical histories.

Rewriting the amplitude

Before constructing the path integral, we would like to write the amplitude (3.41) in a more concrete form. This will allow in particular to rewrite it in an exponentiated form

[5] What one can do is to choose a maximal tree and gauge fix the rotation part of the group elements living on that tree. This does not help us with the divergence problem, and we thus prefer to keep the full group elements.

and to reconstruct a simplicial action. In the next chapter we will relate this action to the Regge action, in the case of a single simplex.

The main object we want to look at here is the propagator:

$$P_{vt} := \langle j_t, J\xi_{t\tau}; (k_t, p_t)| X_{v\tau}^{-1} X_{v\tau'} |(k_t, p_t); j_t, \xi_{t\tau'}\rangle, \tag{3.45}$$

where $k_t = j_t$ and $p_t = \gamma j_t$. This is such that the 4-simplex amplitude is given by:

$$A(j_t, \xi_{t\tau}) = \int \prod_{v\tau} \mathrm{d}X_{v\tau}\, \delta(X_{v\tau*}) \prod_{vt} P_{vt}. \tag{3.46}$$

We have chosen an arbitrary tetrahedron $\tau*$ in v and gauge fixed the corresponding group element $X_{v\tau*}$.

To study the expression (3.45), we need first to go through a bit of representation theory for the Lorentz group. We review the representation theory in an appendix, but give here the basic facts necessary to understand the following construction. A representation on the principal series (k, p) is defined on the space of homogeneous functions of two complex variables $z = (z_0, z_1)$:

$$f(\lambda z) = \lambda^{-1+ip+k} \bar{\lambda}^{-1+ip-k} f(z), \tag{3.47}$$

for λ any complex number.

The inner product on this space is given by

$$(f, g) := \int_{\mathbb{CP}^1} \bar{f} g\, \Omega, \tag{3.48}$$

where we have used the standard 2-form on $\mathbb{C}^2 - \{0\}$:

$$\Omega = \frac{i}{2}(z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0) \wedge (\bar{z}_0 \mathrm{d}\bar{z}_1 - \bar{z}_1 \mathrm{d}\bar{z}_0). \tag{3.49}$$

To perform the integration above, one chooses a section. The standard choice is $z = (\zeta, 1)$, for which the measure reduces to the standard measure on the complex plane, $\Omega = \frac{i}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} = \mathrm{d}x \wedge \mathrm{d}y$, $\zeta = x + iy$.

An element $X \in \mathrm{SL}(2, \mathbb{C})$ acts on this space by:

$$(Xf)(z) = f(X^T z) \tag{3.50}$$

In order to understand the expression for the propagator (3.45), we need to write the state $|(k, p); j, \hat{n}\rangle$ as a homogeneous function. We use the standard notation (Naimark 1962) $f^j_m(z)^{(k,p)}$ for an element of the canonical basis represented in the space of homogeneous functions. They can be given and explicit expression using hypergeometrical functions. It will be more useful however to use homogeneity and scale the argument such that it is normalized, $\langle z, z\rangle = 1$. Given a normalized spinor ξ, construct the SU(2) matrix using the map (3.51):

$$g(\xi) = \begin{pmatrix} \xi_0 & -\bar{\xi}_1 \\ \xi_1 & \bar{\xi}_0 \end{pmatrix}. \tag{3.51}$$

The canonical basis when restricted to normalized spinors is identified with SU(2) representation matrices:

$$f^j_m(\xi)^{(k,p)} = \sqrt{\frac{d_j}{\pi}}\, D^j_{m\,k}(g(\xi)). \tag{3.52}$$

The following formula for a representation matrix of $GL(2,\mathbb{C})$, restricted to matrices of the form (3.51), will be useful in what follows:

$$\begin{aligned} D^j_{m\,k}(g(\xi)) \;&=\; \left[\frac{(j+m)!(j-m)!}{(j+k)!(j-k)!}\right]^{\frac{1}{2}} \sum_n \begin{pmatrix} j+k \\ n \end{pmatrix} \begin{pmatrix} j-k \\ j+m-n \end{pmatrix} \times \\ &\times\; (\xi_0)^n \left(-\bar{\xi}_1\right)^{j+m-n} (\xi_1)^{j+k-n} \left(\bar{\xi}_0\right)^{n-m-k}. \end{aligned} \tag{3.53}$$

The sum over is over values of n such that the binomial coefficients do not vanish. Evaluating $f^j_m(z)^{(k,p)}$ on non-normalized spinors and using homogeneity, we get the following formula:

$$f^j_m(z)^{(k,p)} = \sqrt{\frac{d_j}{\pi}}\, \langle z, z\rangle^{ip-1-j}\, D^j_{m\,k}(g(z)), \tag{3.54}$$

One can check that the function defined above has the correct homogeneity.

A coherent state $|(k,p); j, \xi\rangle$ will then be represented by the homogeneous function $f^j_\xi(z)^{(k,p)}$, defined by:

$$f^j_\xi(z)^{(k,p)} := g(\xi) \triangleright f^j_{+j}(z)^{(k,p)} = f^j_{+j}(g(\xi)^T z)^{(k,p)} = f^j_{+j}\left(\langle \bar{z}, \xi\rangle, \left[\bar{\xi}, z\right]\right)^{(k,p)}, \tag{3.55}$$

where the notation $f^j_{+j}(z) = f^j_{+j}(z_1, z_2)$ has been used in the last step. We now restrict to the case $j = k$ with k positive. We have that

$$f^k_\xi(z)^{(k,p)} = \sqrt{\frac{d_k}{\pi}}\, \langle z, z\rangle^{ip-1-k}\, \langle \bar{z}, \xi\rangle^{2k} \tag{3.56}$$

With the machinery introduced above, the propagator (3.45) can be written as:

$$
\begin{aligned}
P_{vt} &= \Big(X_{v\tau} f^{k_t}_{J\xi_{t\tau}}(z)^{(k_t,p_t)}, X_{v\tau'} f^{k_t}_{\xi_{t\tau'}}(z)^{(k_t,p_t)}\Big) \\
&= \int_{\mathbb{CP}^1} \Omega_{z_{vt}} \, \overline{f^{k_t}_{J\xi_{t\tau}}(X^T_{v\tau} z_{vt})} \, f^{k_t}_{\xi_{t\tau'}}(X^T_{v\tau'} z_{vt}) \\
&= \frac{d_{k_t}}{\pi} \int_{\mathbb{CP}^1} \Omega_{z_{vt}} \, \langle X^\dagger_{v\tau}\bar{z}_{vt}, X^\dagger_{v\tau}\bar{z}_{vt}\rangle^{-ip_t-1-k_t} \langle J\xi_{t\tau}, X^\dagger_{v\tau}\bar{z}_{vt}\rangle^{2k_t} \times \\
&\quad \times \langle X^\dagger_{v\tau'}\bar{z}_{vt}, X^\dagger_{v\tau'}\bar{z}_{vt}\rangle^{ip_t-1-k_t} \langle X^\dagger_{v\tau'}\bar{z}_{vt}, \xi_{t\tau'}\rangle^{2k_t} \\
&= \frac{d_{k_t}}{\pi} \int_{\mathbb{CP}^1} \Omega_{z_{vt}} \, \langle X^\dagger_{v\tau} z_{vt}, X^\dagger_{v\tau} z_{vt}\rangle^{-ip_t-1-k_t} \langle J\xi_{t\tau}, X^\dagger_{\tau} z_{vt}\rangle^{2k_t} \times \\
&\quad \times \langle X^\dagger_{v\tau'} z_{vt}, X^\dagger_{v\tau'} z_{vt}\rangle^{ip_t-1-k_t} \langle X^\dagger_{v\tau'} z_{vt}, \xi_{t\tau'}\rangle^{2k_t}
\end{aligned}
\tag{3.57}
$$

Now, define

$$
Z_{vt\tau} = X^\dagger_{v\tau} z_{vt}, \tag{3.58}
$$

such that, the propagator can be written as

$$
P_{vt} = \frac{d_{k_t}}{\pi} \int_{\mathbb{CP}^1} \Omega_{vt} \left(\frac{\langle Z_{vt\tau'}, Z_{vt\tau'}\rangle}{\langle Z_{vt\tau}, Z_{vt\tau}\rangle} \right)^{ip_t} \left(\frac{\langle J\xi_{t\tau}, Z_{vt\tau}\rangle \langle Z_{vt\tau'}, \xi_{t\tau'}\rangle}{\langle Z_{vt\tau}, Z_{vt\tau}\rangle^{1/2} \langle Z_{vt\tau'}, Z_{vt\tau'}\rangle^{1/2}} \right)^{2k_t},
$$

and

$$
\Omega_{vt} := \frac{\Omega_{z_{vt}}}{\langle Z_{vt\tau}, Z_{vt\tau}\rangle \langle Z_{vt\tau'}, Z_{vt\tau'}\rangle} \tag{3.59}
$$

is a measure on $\mathbb{CP}^1$.

We see that the propagator can then be written as an integral weighted by the exponential of a simplicial action:

$$
P_{vt} = \int_{\mathbb{CP}^1} \frac{d_{k_t}}{\pi} \Omega_{vt} \, e^{S_{vt}}, \tag{3.60}
$$

with the action given by:

$$
S_{vt}[j, X, \xi, z] = k_t \log \frac{\langle J\xi_{t\tau}, Z_{vt\tau}\rangle^2 \langle Z_{vt\tau'}, \xi_{t\tau'}\rangle^2}{\langle Z_{vt\tau}, Z_{vt\tau}\rangle \langle Z_{vt\tau'}, Z_{vt\tau'}\rangle} + ip_t \log \frac{\langle Z_{vt\tau'}, Z_{vt\tau'}\rangle}{\langle Z_{vt\tau}, Z_{vt\tau}\rangle}. \tag{3.61}
$$

The first term is complex, defined mod $2\pi i$, and the second term purely imaginary.

3.4 The path integral

The full partition function can then be rewritten as:

$$Z_\Sigma = \sum_{\{j_t\}} d_{j_t}^\alpha \int \left(\prod_{vt} dX_{v\tau} \prod_v \delta(X_{v\tau*}) \prod_{\tau t} \mathrm{d}\xi_{\tau t} \prod_{vt} \Omega_{vt} \right) e^{\sum_{vt} S_{vt}[j_t, X_{v\tau}, \xi_{t\tau}, z_{vt}]}. \quad (3.62)$$

In the expression above, for each 4-simplex, we have chosen an arbitrary tetrahedron $\tau*$ and gauge fixed the corresponding group element $X_{v\tau*}$ to the identity.

We see that the partition function is expressed as a path integral with classical action given by $S = \sum_{vt} S_{vt}$. The important property of this action is that it is homogeneous in the representation labels j_t, $S_{vt} =: j_t s_{vt}$. We can now come back to the sum over spins and the possible divergences coming out of it. By formally inverting the order of integrations and sums, one has for each t a (almost) geometric sum:

$$P_t := \sum_{j_t} d_{j_t}^\alpha \, a_t^{j_t}, \quad (3.63)$$

where we have defined $a_t := e^{\sum_{v \supset t} s_{vt}}$. Because the divergent part of this sum is controlled by the large spin sector, we consider the approximation $d_j \sim 2j$. Performing the sum, we get:

$$P_t \sim \frac{Q(a_t)}{(1 - a_t)^{\alpha+1}}. \quad (3.64)$$

$Q(a_t)$ is a polynomial of order α in a_t such that $Q(1) \neq 0$.

We would like now to make an analogy with the path integral for a scalar relativistic particle in flat space time (see for instance Weinberg 1995). There, for a given Feynman diagram we associate for each edge e in this diagram the propagator $D_e^F = i(p_e^2 + m_e^2 + i\varepsilon)^{-1}$. The Feynman propagator has naive poles for $p_e^2 = -m_e^2$, corresponding to a classical relativistic particle propagating freely. The prescription for adding $i\varepsilon$ defines how one should deform the integration contour to correctly take these poles into account. Putting propagators together and the delta functions to ensure the preservation of momenta then allows to identify the real propagating modes at a given order of perturbation theory.

In the simplicial path integral (3.62) defined above, we associate to each triangle a propagator P_t defined in (3.64). The naive poles for this propagator are given by configurations satisfying the equation $a_t = 1$. After taking into account the measure of integration, we would hope that the true poles be related to classical Regge configurations, satisfying the Regge equations of motion.

The analogy becomes clearer when expressing the Feynman propagator in the Schwinger proper-time representation:

$$D_e^F = \int_0^\infty \mathrm{d}s \; e^{is(p_e^2 + m^2 + i\varepsilon)}. \quad (3.65)$$

This expression is to be compared with eq. (3.63), the sum over j_t playing the role of the integral over the proper time s. We see that the prescription for adding the $i\varepsilon$ term has the effect of regulating the sum over s. We might as well do the same for our simplicial path integral and regulate the sum over spins:

$$P_t^\varepsilon \sim \sum_{j_t} j_t^\alpha \, e^{-\varepsilon j_t} \, a_t^{j_t}. \tag{3.66}$$

This can in turn be inserted back into (3.62), providing us with a regulated expression for the partition function Z_Δ^ε:

$$\begin{aligned} Z_\Sigma^\varepsilon &= \int \prod_{vt} dX_{v\tau} \prod_v \delta(X_{v\tau *}) \prod_{\tau t} \mathrm{d}\xi_{\tau t} \prod_{vt} \Omega_{vt} \prod_t P_t^\varepsilon[\xi, X, z] \\ &= \int \prod_{vt} dX_{v\tau} \prod_v \delta(X_{v\tau *}) \prod_t P_t^\varepsilon[X], \end{aligned} \tag{3.67}$$

where we have defined $P_t^\varepsilon[X] := \prod_{\tau t} \mathrm{d}\xi_{\tau t} \prod_{vt} \Omega_{vt} \, P_t^\varepsilon[\xi, X, z]$. The propagator $P_t[X]$ is of course a very complicated object but understanding its structure, in particular understanding its pole structure, is of key importance to extracting physical information out of the amplitude defined here. We see that the prescription (3.66) corresponds to a certain choice of contour of integration regulating the a priori ill defined expression (3.62).

———— * ———— * ————

It is probably a good point to take a look back at Hartle's simplicial minisuperspace program, and check at which step of it we are standing. We have been able to define the analog of the wave function $\Psi(l_e^2)$ depending on boundary edge lengths. The difference is on the boundary data: for us it is specified by a certain set of spins, representing the areas of triangles and unit normals to these triangles.

Remember that two questions were left open in Hartle's program. The first one was the choice of the measure. We have seen that in the spin foam context, while the arbitrariness remains, a certain class of measures is suggested by the BF path integral.

A second question was related to a proper choice of contour of integration. We have proposed such a choice, in analogy to the theory of propagators in flat space. Of course this is only an analogy and one needs to check if this choice is physically reasonable: one needs to check if the poles dominating the amplitude correspond to classical simplicial geometries allowed by the boundary data. We leave this analysis to further investigation.

Let us explore the definition of observables for the spin foam model given here. We want to define the equivalent of equation (1.7) in the context of the simplicial minisuperspace.

To do that we need two inputs: a vacuum, or boundary, state Ψ_0 and an observable $\mathcal{O}$ acting on the boundary space. The boundary data can be specified by spins and intertwiners, equivalently by spins and normals, or finally by spins and spinors, as explained before. Any of this set of variables provides a basis for the boundary state space. In turn, the action of an observable on the vacuum state can be decomposed in this basis. To compute observables it is thus sufficient to compute the path integral for a boundary state given by an element of this basis. Any basis would do the job, and the best choice depends on the observable chosen. Note this element Ψ_∂ and the path integral with this boundary state $Z^\varepsilon_\Sigma(\psi_\partial)$. The basic observable is then given by

$$S^\varepsilon_\Sigma := \frac{Z^\varepsilon_\Sigma(\Psi_\partial)}{Z^\varepsilon_\Sigma(\Psi_0)} \tag{3.68}$$

and plays the role of the S matrix in field theories. Borrowing some intuition from LQG, a natural choice of vacuum state would be $\Psi_0 = \prod_t \delta_{j_t,0}$ [6], which corresponds simply to setting all the boundary areas to zero.

A discussed in chapter 1, one hopes that the expression above is better behaved than the path integral alone, as divergences might cancel between the numerator and the denominator. The question now is to define the limit $\varepsilon \to 0$. We leave that and the many other questions discussed earlier for further investigation. We will recollect all these issues in the concluding chapter.

[6] In passing to dual variables this corresponds to the constant functional $\Psi(g_t) = 1$, see for instance (Ashtekar and Lewandowski 2004).

Chapter 4

Asymptotics

In the last chapter we have defined an amplitude for certain boundary data defined on a given simplicial decomposition of space time. This is in principle a very difficult expression to compute with and one needs to find ways to extract physical information out of it. In this chapter we would like to explore the properties of the amplitude defined in the last chapter using for that purpose semiclassical methods. We will use a stationary phase approximation to the path integral, which is equivalent to the standard WKB approximation for quantum mechanics.

Other than extracting observable information, we are also interested in testing some of the assumptions that led us to construct the path integral. Remember that, in discussing the implementation of the classical constraints we made a choice to leave some of them free. The correct semiclassical behavior for the amplitude will, if not validate, give us confidence in the choices made when quantizing the classical system.

We will restrict ourselves to the case of a single 4-simplex. This is, one might argue, a too simple system, as a single simplex is always flat, with curvature appearing only after gluing simplices around a given face. Understanding a triangulation with internal faces - and controlling the potential divergences appearing there - is of course a crucial question that we leave open.

As we saw before, the classical action is homogeneous in the spin variables. The semiclassical regime will then be probed by a large spin limit. More precisely, scale uniformly all the spins $j_t \to \lambda j_t$. The semiclassical approximation is given for λ large.

The discussion of this chapter follows closely (Barrett, Dowdall, Fairbairn, Gomes and Hellmann 2009) and (Barrett, Fairbairn and Hellmann 2009) hereby denoted (BDFGH 2009) and (BFH 2009) respectively.

4.1 Posing the problem

In this section we would like to state the method in its full generality (see Hormander 1983). Let then D be a closed manifold of dimension n, and S and a be smooth complex valued functions on D such that Re$S \leq 0$. We are interested in evaluating the following expression:

$$f(\lambda) = \int_D \mathrm{d}x\, a(x)\, e^{\lambda S(x)}. \tag{4.1}$$

We call a critical point a stationary point of the action $S(x)$ for which Re$S(x) = 0$. We assume the stationary points to be isolated and non degenerate, i.e., the Hessian H of S has non zero determinant on these points. If S has no critical points then for large λ the function f decreases faster than any power of λ^{-1}, that is,

$$f(\lambda) = o(\lambda^{-N})\ ,\ \forall N \geq 1. \tag{4.2}$$

In the presence of critical points, then f is given for large λ by a sum over contributions of the form

$$f(\lambda) = \sum_{x_c} a(x_c) \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{1}{\sqrt{\det H}} e^{\lambda S(x_c)} \left[1 + O(1/\lambda)\right]. \tag{4.3}$$

One can give an explicit expression for the sub leading terms hidden in $O(1/\lambda)$, and these will be important in computing observables (see Bianchi, Magliaro and Perini 2009). We restrict ourselves here to the leading order terms.

To pose the problem, let us recall the classical action associated to a single 4-simplex:

$$S[j, X, \xi, z] = \sum_{a<b} k_{ab} \log \frac{\langle J\xi_{ab}, Z_{ab}\rangle^2 \langle Z_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle} + ip_{ab} \log \frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}. \tag{4.4}$$

We have shifted back to the previous notations, $a, b = 1...5$ labeling the tetrahedra of the 4-simplex and a pair (ab) labeling the triangle between the two tetrahedra. Analogously, $Z_{ab} := X_a^\dagger z_{ab}$ and $Z_{ba} := X_b^\dagger z_{ab}$. We are interested in applying the stationary phase method described in the last section to the following function:

$$f_4(\lambda) := A(\lambda j_{ab}, \xi_{ab}) = \int_{\mathrm{SL}(2,\mathbb{C})^{\times 5}} \prod_a \mathrm{d}X_a\, \delta(X_5) \int_{(\mathbb{CP}^1)^{\times 10}} \prod_{ab} \frac{d_{k_{ab}}}{\pi} \Omega_{ab}\, e^{S[\lambda j, X, \xi, z]} \tag{4.5}$$

All we have to do then is to find the stationary points of the action (4.4) with vanishing real part. We will do that in the next section. In the following of this chapter we will then classify the solutions to the critical point equations and relate them to geometric configurations for the 4-simplex.

The boundary data is given by the set (j_{ab}, ξ_{ab}) and we suppose it is Regge like, following the discussion on the boundary state in the last chapter. In particular the tetrahedra constructed out of this boundary data are all non-degenerate. Also, because it does not cost us any extra work, we will not impose the simplicity constraint fixing p to be proportional to k. Surprisingly, we will see that the critical point equations will admit solutions only for p proportional to k.

4.2 Critical points

In this section we look for critical points of the action (4.4), first looking at the equations enforcing the real part to vanish and then at the stationarity of the action.

Condition on the real part of the action

The real part of the action

$$\mathrm{Re}\, S = \sum_{a<b} k_{ab} \log \frac{|\langle Z_{ab}, J\xi_{ab}\rangle|^2 |\langle \xi_{ba}, Z_{ba}\rangle|^2}{\langle Z_{ab}, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle}, \tag{4.6}$$

satisfies $\mathrm{Re}\, S \leq 0$ and is hence a maximum where it vanishes. It vanishes if and only if, on each triangle ab, $a < b$, the following condition holds

$$\frac{\langle Z_{ab}, J\xi_{ab}\rangle \langle J\xi_{ab}, Z_{ab}\rangle \langle \xi_{ba}, Z_{ba}\rangle \langle Z_{ba}, \xi_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle} = 1. \tag{4.7}$$

This equation admits solutions if the coherent states $J\xi_{ab}$ and ξ_{ba} are proportional to Z_{ab} and Z_{ba} respectively. Considering the fact that the coherent states are normalized, the most general solution to the above equation can be written

$$J\xi_{ab} = \frac{e^{i\phi_{ab}}}{\parallel Z_{ab} \parallel} X_a^\dagger\, z, \quad \text{and} \quad \xi_{ba} = \frac{e^{i\phi_{ba}}}{\parallel Z_{ba} \parallel} X_b^\dagger\, z, \tag{4.8}$$

where $\parallel Z_{ab} \parallel$ is the norm of Z_{ab} induced by the Hermitian inner product, and ϕ_{ab} and ϕ_{ba} are phases. Eliminating z, and introducing the notation $\theta_{ab} = \phi_{ab} - \phi_{ba}$, we obtain the first of the equations for a critical point

$$(X_a^\dagger)^{-1}\, J\xi_{ab} = \frac{\parallel Z_{ba} \parallel}{\parallel Z_{ab} \parallel} e^{i\theta_{ab}} (X_b^\dagger)^{-1}\, \xi_{ba}, \tag{4.9}$$

for each $a < b$. We now turn to the variational problem for the action.

Stationary and critical points

We now compute the critical points of the action by evaluating the first derivative of the action of the configurations satisfying the condition (4.7). The action (4.4) is a function of the $\mathrm{SL}(2,\mathbb{C})$ group variables X and of the spinors z. We start by considering stationarity with respect to the spinor variables.

There is a spinor z_{ab} for each triangle ab, $a<b$, and the variation of the action with respect to these complex variables gives two spinor equations for each triangle. For the triangle ab, the variation with respect to the corresponding z variable leads to the following (co-)spinor equation

$$\begin{aligned}\delta_z S &= ip_{ab}\left(\frac{1}{\langle Z_{ba},Z_{ba}\rangle}(X_bZ_{ba})^\dagger-\frac{1}{\langle Z_{ab},Z_{ab}\rangle}(X_aZ_{ab})^\dagger\right)\\ &\quad+k_{ab}\left(\frac{2}{\langle J\xi_{ab},Z_{ab}\rangle}(X_aJ\xi_{ab})^\dagger-\frac{1}{\langle Z_{ab},Z_{ab}\rangle}(X_aZ_{ab})^\dagger-\frac{1}{\langle Z_{ba},Z_{ba}\rangle}(X_bZ_{ba})^\dagger\right),\end{aligned}$$

while the variation with respect to $\bar z$ yields the spinor equation displayed below

$$\begin{aligned}\delta_{\bar z} S &= ip_{ab}\left(\frac{1}{\langle Z_{ba},Z_{ba}\rangle}X_bZ_{ba}-\frac{1}{\langle Z_{ab},Z_{ab}\rangle}X_aZ_{ab}\right)\\ &\quad+k_{ab}\left(\frac{2}{\langle Z_{ba},\xi_{ba}\rangle}X_b\xi_{ba}-\frac{1}{\langle Z_{ab},Z_{ab}\rangle}X_aZ_{ab}-\frac{1}{\langle Z_{ba},Z_{ba}\rangle}X_bZ_{ba}\right).\end{aligned}$$

Evaluating the above variations on the motion (4.8) and equating them to zero leads to the following two equations

$$(X_a\,J\xi_{ab})^\dagger=\frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{-i\theta_{ab}}(X_b\,\xi_{ba})^\dagger,\quad \text{and}\quad X_a\,J\xi_{ab}=\frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{i\theta_{ab}}X_b\,\xi_{ba},\tag{4.10}$$

using the assumption that $(k_{ab},p_{ab})\neq(0,0)$. The two equations above are related by Hermitian conjugation and there is therefore only one relevant equation extracted from the stationarity of the spinor variables. Thus, our second critical equation is the following

$$X_a\,J\xi_{ab}=\frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{i\theta_{ab}}X_b\,\xi_{ba}.\tag{4.11}$$

Finally, we consider stationarity with respect to the group variables. The right variation of an arbitrary $\mathrm{SL}(2,\mathbb{C})$ element X and its Hermitian conjugate are given by

$$\delta X=X\circ L,\quad \text{and}\quad \delta X^\dagger=L^\dagger\circ X^\dagger,\tag{4.12}$$

where L is an arbitrary element of the real Lie algebra $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$.

Because $\mathfrak{sl}(2,\mathbb{C})=\mathfrak{su}(2)^{\mathbb{C}}$, there exists a vector space isomorphism $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\cong\mathfrak{su}(2)\oplus i\mathfrak{su}(2)$. Using this isomorphism, we can decompose L into a rotational and boost part

$L = \alpha^i H_i + \beta^i K_i$, with α^i, β^i in $\mathbb{R}$ for all $i = 1, 2, 3$. Conventions for the spinor representation of the rotation and boost generators respectively are such that $\mathbf{H} = \frac{i}{2}\boldsymbol{\sigma}$ and $\mathbf{K} = \frac{1}{2}\boldsymbol{\sigma}$ (see appendix A).

The variation of the action with respect to the group variable[1] X_a, $a = 1, ..., 4$, yields

$$\begin{aligned} \delta_{X_a} S \;=\; & -\sum_{b:b\neq a} \left[i p_{ab} \left(\frac{\langle Z_{ab}, L\, Z_{ab}\rangle}{\langle Z_{ab}, Z_{ab}\rangle} + \frac{\langle Z_{ab}, L^\dagger\, Z_{ab}\rangle}{\langle Z_{ab}, Z_{ab}\rangle} \right) \right. \\ & \left. + k_{ab} \left(\frac{\langle Z_{ab}, L\, Z_{ab}\rangle}{\langle Z_{ab}, Z_{ab}\rangle} + \frac{\langle Z_{ab}, L^\dagger\, Z_{ab}\rangle}{\langle Z_{ab}, Z_{ab}\rangle} - 2 \frac{\langle J\xi_{ab}, L^\dagger\, Z_{ab}\rangle}{\langle J\xi_{ab}, Z_{ab}\rangle} \right) \right]. \end{aligned} \tag{4.13}$$

Now, evaluating this first derivative on the points satisfying the condition on the real part of the action (4.8) and equating the result to zero leads to

$$\sum_{b:b\neq a} i p_{ab} \left(\langle J\xi_{ab}, L\, J\xi_{ab}\rangle + \langle J\xi_{ab}, L^\dagger\, J\xi_{ab}\rangle \right) + k_{ab} \left(\langle J\xi_{ab}, L\, J\xi_{ab}\rangle - \langle J\xi_{ab}, L^\dagger\, J\xi_{ab}\rangle \right) = 0.$$

Finally, we use the fact that the spinors ξ_{ab} determine SU(2) coherent states, that is,

$$\langle J\xi_{ab}, \mathbf{H}\, J\xi_{ab}\rangle = -\frac{i}{2}\, \mathbf{n}_{ab}, \quad \text{and} \quad \langle J\xi_{ab}, \mathbf{K}\, J\xi_{ab}\rangle = -\frac{1}{2}\, \mathbf{n}_{ab}, \tag{4.14}$$

where $\mathbf{n} := \mathbf{n}_\xi \in \mathbb{R}^3$ is the unit vector corresponding to the coherent state ξ. This leads immediately to the following two equations

$$\sum_{b:b\neq a} p_{ab} \mathbf{n}_{ab} = 0, \quad \text{and} \quad \sum_{b:b\neq a} k_{ab} \mathbf{n}_{ab} = 0, \tag{4.15}$$

because α^i and β^i are arbitrary. These two equations can only be satisfied if there is a restriction on the representation labels. We have that $p_{ab} = \gamma_a k_{ab}$ for some arbitrary constant γ_a at the a-th tetrahedron. However, since the equations hold for each tetrahedron, $\gamma_a = \gamma_b = \tilde{\gamma}$ and the representations are related by a global parameter $p_{ab} = \tilde{\gamma} k_{ab}$. By identifying this global parameter with the Immirzi parameter used in the construction of the 4-simplex amplitude, we recover the full simplicity constraints (2.48).

With this condition, the two equations collapse to a single stationary point equation

$$\sum_{b:b\neq a} k_{ab} \mathbf{n}_{ab} = 0. \tag{4.16}$$

---

[1]The variation with respect to the variable X_a performed here corresponds to a vertex which is the source of all its edges. The variation for a vertex which is the target of some, or all, of its edges will require varying with respect to X_b. This proceeds similarly to the above and leads to the same stationary point equations.

To summarise, we have obtained three critical point equations given by expressions (4.9), (4.11) and (4.16). Solutions to these equations dominate the asymptotic formula for the Lorentzian 4-simplex amplitude.

4.3 Geometrical interpretation

In this section, we show how geometrical structures emerge from the critical point equations.

The geometry of the critical points is based on the identification between spinors and null vectors (see Penrose and Rindler 1986). Let $\Gamma : \mathbb{R}^{3,1} \to \mathbb{H}$; $x \mapsto \Gamma(x) = x^0 \mathbb{1} + x^i \sigma_i$ be the vector space isomorphism between Minkowski space $\mathbb{R}^{3,1}$ and the space of two-by-two hermitian matrices $\mathbb{H}$. Through this isomorphism, the action of a Lorentz group element $\hat{X}$ on $\mathbb{R}^{3,1}$ lifts to the action of an $\mathrm{SL}(2,\mathbb{C})$ element X on $\mathbb{H}$ as follows $\Gamma(\hat{X}x) = X\Gamma(x)X^{\dagger}$. Using this isomorphism, we can map spinors to null vectors through the following procedure. Let

$$\zeta : \mathbb{C}^2 \to \mathbb{H}_0^+, \quad z \mapsto \zeta(z) = z \otimes z^{\dagger}, \tag{4.17}$$

be the standard map between spin space and the space of degenerate two-by-two Hermitian matrices with positive trace

$$\mathbb{H}_0^+ = \{h \in \mathbb{H} \mid \det h = 0, \text{ and } \operatorname{Tr} h > 0\}. \tag{4.18}$$

Note that this non-linear map is obviously not injective and satisfies $\zeta(re^{i\theta}z) = r^2\zeta(z)$. Finally, the space $\mathbb{H}_0^+$ can be identified via Γ to the future pointing null cone C^+ in Minkowski space and the map $\iota = \Gamma^{-1} \circ \zeta : \mathbb{C}^2 \to \mathbb{R}^{3,1}$ maps spinors to null vectors.

Following the above construction, we can therefore associate the null vector

$$\iota(\xi) = \frac{1}{2}(1, \mathbf{n}), \tag{4.19}$$

to the coherent state ξ by using equation (3.21). In fact, we can associate a second null vector to the coherent state ξ by using the antilinear structure J:

$$\iota(J\xi) = \frac{1}{2}(1, -\mathbf{n}). \tag{4.20}$$

The two spinors ξ and $J\xi$ form a spin frame because $[\xi, J\xi] = \langle J\xi, J\xi \rangle = 1$.

From this spin frame, we can construct bivectors in the vector representation as follows. Define the following time-like and space-like vectors

$$\mathcal{T} = (\iota(\xi) + \iota(J\xi)) = (1, \mathbf{0}), \quad \text{and} \quad \mathcal{N} = (\iota(\xi) - \iota(J\xi)) = (0, \mathbf{n}). \tag{4.21}$$

From these two vectors construct the space-like bivector

$$\tilde{b} = \star\mathcal{T} \wedge \mathcal{N} = -2 \star \iota(\xi) \wedge \iota(J\xi), \tag{4.22}$$

where the star $\star$ is the Hodge operator acting on the space $\Lambda^2(\mathbb{R}^{3,1})$. In the four-by-four matrix representation, b is given explicitly by

$$\tilde{b} = \star \begin{bmatrix} 0 & n^1 & n^2 & n^3 \\ -n^1 & 0 & 0 & 0 \\ -n^2 & 0 & 0 & 0 \\ -n^3 & 0 & 0 & 0 \end{bmatrix}. \tag{4.23}$$

This bivector is space-like because $\mathcal{T}\wedge\mathcal{N}$ is time-like and the Hodge operator is an anti-involution. It is also simple by construction and satisfies a cross-simplicity condition

$$\mathcal{T}^I \tilde{b}^{JK} \eta_{IJ} = 0. \tag{4.24}$$

Following this construction for every coherent state ξ_{ab} consequently leads to a collection of bivectors $\tilde{b}_{ab}$ living in the hyperplane $\mathcal{T}^\perp$, with $\mathcal{T}$ the reference point of the future hyperboloid Q_1.

We now show that our critical point equations allow to reconstruct a 4-simplex geometry out of these bivectors. We will check each of the conditions (2.4) leading to the reconstruction Theorem 1. First define $b_{ab} = \gamma k_{ab} \tilde{b}_{ab}$. According to the discussion on the boundary state in the last chapter, γk_{ab} has the interpretation of the area A_{ab} of the triangle t_{ab}. The bivectors satisfy simplicity by construction. The critical equation (4.16) implies closure. 3d non-degeneracy comes from the assumption on the boundary data. It remains to check the parallel transport condition and deal with the 4d non-degeneracy condition.

We will see that the critical equations (4.9) and (4.11) are just the expression of the parallel transport condition on the fundamental representation. To see that, we need to write the spinorial equations (4.9) and (4.11) in the vector representation. We use the map ι defined above. Note that

$$\iota(Xz) = \Gamma^{-1}\left(Xz \otimes (Xz)^\dagger\right) = \Gamma^{-1}\left(Xz \otimes z^\dagger X^\dagger\right) = \hat{X}\,\iota(z). \tag{4.25}$$

Equation (4.11) reads:

$$\hat{X}_a\,\iota(J\xi_{ab}) = \frac{\parallel Z_{ab} \parallel^2}{\parallel Z_{ba} \parallel^2}\,\hat{X}_b\,\iota(\xi_{ba}) \tag{4.26}$$

and equation (4.9):

$$\hat{X}_a\,\iota(\xi_{ab}) = \frac{\parallel Z_{ba} \parallel^2}{\parallel Z_{ab} \parallel^2}\,\hat{X}_b\,\iota(J\xi_{ba}). \tag{4.27}$$

To get the last equation we have used the action of the J structure on $\mathrm{SL}(2,\mathbb{C})$: writing an arbitrary $\mathrm{SL}(2,\mathbb{C})$ element X as $X = \exp \alpha^i H_i + \beta^i K_i$, with α^i, β^i real, it is immediate to see that

$$JXJ^{-1} = (X^\dagger)^{-1}. \tag{4.28}$$

The restriction of the above equation to the unitary subgroup states that J commutes with SU(2) as expected.

Wedging these two vector equations leads to the bivector equation

$$\hat{X}_a \otimes \hat{X}_a \,\iota(\xi_{ab}) \wedge \iota(J\xi_{ab}) = -\hat{X}_b \otimes \hat{X}_b \,\iota(\xi_{ba}) \wedge \iota(J\xi_{ba}), \tag{4.29}$$

which implies the parallel transport condition for the bivectors b_{ab} and b_{ba}.

The last condition we need to deal with is the 4d non-degeneracy. We use Lemma 3 from (BDFGH 2009). The lemma states that a bivector geometry is either fully non-degenerate or fully contained in a 3-dimensional hyperplane. The lemma was stated in the context of Euclidean geometry but did not use the metric at all and thus applies unaltered to our case.

In the non-degenerate case, we can use Theorem 1 to reconstruct a 4-simplex geometry. Solutions to the critical point equations are therefore related to geometric 4-simplices up to inversion. In particular, for every solution that is non-degenerate, there exists a parameter μ which takes the value either 1 or -1, and an inversion-related pair of Lorentzian 4-simplexes σ. These are such that the bivectors (of either of the two simplexes) $B_{ab}(\sigma)$ satisfy

$$B_{ab}(\sigma) = \mu B_{ab} = \mu \,\hat{X}_a \otimes \hat{X}_a \,b_{ab}. \tag{4.30}$$

A key subtlety in the geometric interpretation of our equations arises due to the fact that $\mathrm{SL}(2,\mathbb{C})$ maps only to the connected component of $\mathrm{SO}(3,1)$ and takes future pointing vectors into future pointing vectors, meaning that the inversion map is not in $\mathrm{SL}(2,\mathbb{C})$.

If the solutions fall into the fully degenerate case this implies all F_a are pointing in the same direction. As we have fixed $X_5 = 1$ this means we have $F_a = X_a \mathcal{T} = \mathcal{T}$, for all a. That is the X_a are in the SU(2) subgroup that stabilizes $\mathcal{T}$. As such the two distinct critical and stationary point equations (4.9) and (4.11) reduce to the single equation:

$$X_a \,J\xi_{ab} = e^{i\theta_{ab}} X_b \,\xi_{ba} \tag{4.31}$$

The solutions to these equations have been studied in (BDFGH 2009 and BFH 2009). A solution determines a geometrical structure called a vector geometry. This is a set of vectors $\mathbf{v}_{ab} \in \mathbb{R}^3$ satisfying closure, $\sum_a k_{ab}\mathbf{v}_{ab} = 0$, and orientation, $\mathbf{v}_{ab} = -\mathbf{v}_{ab}$. In this case, $\mathbf{v}_{ab} = X_a \mathbf{n}_{ab}$.

In particular it was shown that for Regge like boundary data equation (4.31) admits solutions exactly if the boundary is that of a Euclidean 3- or 4-dimensional 4-simplex.

In the 3-dimensional case $X_a \mathbf{n}_{ab}$ are the normals to the faces of the 4-simplex, in the 4-dimensional case there are two solutions corresponding to the self-dual and anti self-dual parts of the Euclidean bivector geometry.

4.4 Symmetries and classification of the solutions

An important input for the asymptotic formula is the classification of the solutions to the critical points. To start with, we need to consider their symmetries.

Symmetries induced by the symmetries of the action

It is straightforward to see that the amplitude (4.5), and also the action (4.4) (modulo $2\pi i$), admit three types of symmetry.

- *Lorentz.* A global transformation $X_a \to Y X_a$, $z_{ab} \to (Y^\dagger)^{-1} z_{ab}$, for Y in $\mathrm{SL}(2,\mathbb{C})$, acting on all the variables simultaneously.
- *Spin lift.* At each vertex a, the transformation $X_a \to -X_a$.
- *Rescaling.* At each triangle $a < b$, the transformation $z_{ab} \to \kappa z_{ab}$ for $0 \neq \kappa \in \mathbb{C}$

Note however that the Lorentz symmetry does not affect the asymptotic problem because the amplitude is gauge-fixed such that $X_5 = 1$. The spin lift symmetry then only acts at the vertices $a = 1, 2, 3, 4$.
These symmetries of the action map critical points to critical points, except that some of the symmetries are broken by the gauge-fixing that is used to define the non-compact integrals.

Parity

An additional symmetry of the critical points that is not determined by a symmetry of the action is the parity operation, given by the inversion of the spatial coordinates of Minkowski space. Using the SU(2) antilinear map J, one can construct a map acting on Minkowski vectors through their identification with two-by-two Hermitian matrices. Most importantly, because the J map commutes with SU(2), it necessarily anticommutes with Hermitian matrices which implies that

$$J\Gamma(x)J^{-1} = x^0 \mathbb{1} - x^i \sigma_i = \Gamma(Px), \tag{4.32}$$

where P is the mapping $(x^0, \mathbf{x}) \mapsto (x^0, -\mathbf{x})$ on $\mathbb{R}^{3,1}$. Now, we have seen that the J map has a well defined action (4.28) on $\mathrm{SL}(2,\mathbb{C})$. The corresponding parity transformation

P is of key importance since it shows that given a solution to the critical point equations (4.9) and (4.11), the transformation

$$\begin{aligned} X_a &\to P(X_a) = J X_a J^{-1} && \forall a, \\ z_{ab} &\to X_a X_a^\dagger z_{ab} && \forall a < b \end{aligned} \tag{4.33}$$

leaves the critical point equations unchanged, because $\| Z_{ab} \| / \| Z_{ba} \|$ is mapped to $(\| Z_{ab} \| / \| Z_{ba} \|)^{-1}$, and is an involution. Therefore, P, together with the above transformation on z is a symmetry of the critical points. It is not a symmetry of the action, but a prescription to construct a solution out of a solution.

An important feature of the action of P is that it flips the orientation parameter μ to $-\mu$. The remainder of this section shows that this is the case.

Firstly, from the definition (2.15) of the bivectors of a 4-simplex, it follows that

$$P B_{ab}(\sigma) = -B_{ab}(P\sigma). \tag{4.34}$$

This is because the the normal vectors N_a transform as vectors under P, but $\star P = -P\star$. Another way of saying this is that the N_a are determined by the metric only but $\star$ requires an orientation.

Secondly, the bivector b_{ab} has only space-space components, so is unaffected by the parity operation. This means that in the equation (4.30) defining μ,

$$P B_{ab}(\sigma) = \mu \, P(\hat{X}_a) \otimes P(\hat{X}_a) \, b_{ab} \tag{4.35}$$

and hence

$$\mu_\sigma = -\mu_{P\sigma}. \tag{4.36}$$

Classification

Using the above results we can now classify the solutions to the critical point equations for different types of boundary data. In this classification, solutions which are related by the symmetries of the action are regarded as the same solution.

Regge-like boundary data. Given a Regge-like boundary geometry, a flat metric geometry for the 4-simplex is specified completely up to rigid motion. In particular the metric on the interior is uniquely fixed by knowing all the edge lengths, and this is fixed by the boundary data. One way to see this is to note that there is a linear isomorphism between the set of square edge lengths and set of interior metrics. Note that $l_{ab}^2 = v_{ab}^\mu v_{ab}^\nu g_{\mu\nu}$ for a 4-simplex with edge vectors v and lengths l is indeed linear and that choosing the edge vectors as basis vectors one can calculate their inner products using only the other edge lengths. But knowing the inner products and the lengths of the basis vectors is equivalent to knowing the metric. Now as the tetrahedra are

all Euclidean and non-degenerate, the metric of the four-simplex must be of signature $(-,+,+,+)$ or $(+,+,+,+)$.
The solutions can now be classified according to the boundary data.

- *Lorentzian 4-simplex:* If the boundary data is that of a non-degenerate Lorentzian 4-simplex, then two distinct critical points exist, related by the parity transformation P in section 4.4. Since the boundary data determines the metric of the 4-simplex σ, there are only four possibilities which are unrelated by the action of $\mathrm{SL}(2,\mathbb{C})$, corresponding to the four connected components of the group $\mathrm{O}(3,1)$. These are σ, its inversion partner $-\sigma$ and the parity-related $P\sigma$ and $-P\sigma$. The solutions correspond to inversion-related pairs, thus it is clear that the two solutions given by $(\sigma,-\sigma)$ and $(P\sigma,-P\sigma)$ exhaust all the possibilities.

- *Euclidean 4-simplex:* If the boundary data describes a Euclidean 4-simplex, then there will be exactly two critical points, $\{X_a^+\}$ and $\{X_a^-\}$, with all matrices in SU(2). These can be used to reconstruct a Euclidean 4-simplex σ_E, as in (BDFGH 2009). These critical points can also be used to construct the parity related 4-simplex $P\sigma_E$.

- *4-simplex in $\mathbb{R}^3$:* If the boundary data corresponds to a degenerate 4-simplex in $\mathbb{R}^3$ then there will be a single SU(2) critical point. This determines a vector geometry. A second critical point cannot exist or we would be able to construct a non-degenerate Euclidean 4-simplex, which is not possible with this boundary data.

Furthermore it was established in (BFH 2009) that no other vector geometries exist for Regge like boundary data.

Non Regge-like boundary data. If the boundary data is not Regge-like then the remaining possibilities are to obtain exactly one critical point in SU(2) which determines a vector geometry, or no critical points at all.

Regge action

In this section, the action (4.4) at a critical point is expressed in terms of the underlying geometry. On all critical points, the real part vanishes and we are left with the imaginary part

$$S = i\sum_{a<b} p_{ab}\,\log\frac{\parallel Z_{ba}\parallel^2}{\parallel Z_{ab}\parallel^2} - 2k_{ab}\,\theta_{ab}. \qquad (4.37)$$

In the case where the critical points determine a non-degenerate 4-simplex, the following discussion shows that the argument of the logarithm is related to the dihedral angle Θ_{ab} at the ab triangle.

If F_a and F_b are the future pointing normals determining the two hyperplanes intersecting along the triangle ab, the corresponding dihedral angle Θ_{ab} obeys

$$\cosh \Theta_{ab} = -F_a \cdot F_b = \frac{1}{2}\mathrm{Tr}\ (\Gamma(F_a)^{-1}\Gamma(F_b)). \tag{4.38}$$

To see how this relates to the critical points, couple equations (4.9) and (4.11) and eliminate $J\xi_{ab}$. This leads to the following eigenvalue equation

$$X_a^{-1}X_bX_b^\dagger(X_a^\dagger)^{-1}\ J\xi_{ab} = \frac{\parallel Z_{ba}\parallel^2}{\parallel Z_{ab}\parallel^2}\ J\xi_{ab} \tag{4.39}$$

and the corresponding J transformed equation:

$$X_a^{-1}X_bX_b^\dagger(X_a^\dagger)^{-1}\ \xi_{ab} = \frac{\parallel Z_{ab}\parallel^2}{\parallel Z_{ba}\parallel^2}\ \xi_{ab}. \tag{4.40}$$

The matrix $X_a^{-1}X_bX_b^\dagger(X_a^\dagger)^{-1}$ in this equation is Hermitian, so it follows that it has eigenvalues

$$e^{r_{ab}} = \frac{\parallel Z_{ab}\parallel^2}{\parallel Z_{ba}\parallel^2}. \tag{4.41}$$

and the inverse of this, $e^{-r_{ab}}$. Moreover, the eigenvectors are orthogonal. This trace is the same as the trace in (4.38), and so

$$|\Theta_{ab}| = |r_{ab}|. \tag{4.42}$$

Therefore, the parameter r_{ab} is the dihedral angle up to a sign.

To solve this sign ambiguity, it will prove useful to obtain an exponentiated form of this matrix. This is achieved by noting that since the spinors ξ_{ab} are SU(2) coherent states, they satisfy

$$(\mathbf{H}.\mathbf{n})\,\xi = \frac{i}{2}\,\xi, \quad \text{and} \quad (\mathbf{K}.\mathbf{n})\,\xi = \frac{1}{2}\,\xi. \tag{4.43}$$

Hence, the Hermitian matrix $X_a^{-1}X_bX_b^\dagger(X_a^\dagger)^{-1}$ can be written as a pure boost

$$X_a^{-1}X_bX_b^\dagger(X_a^\dagger)^{-1} = e^{2r_{ab}\,\mathbf{K}.\mathbf{n}_{ab}} = g(\mathbf{n}_{ab})\begin{pmatrix} e^{r_{ab}} & 0 \\ 0 & e^{-r_{ab}} \end{pmatrix} g(\mathbf{n}_{ab})^{-1}, \tag{4.44}$$

where $g(\mathbf{n}_{ab})$ is a unitary matrix.

Using the above expression, we can now overcome the sign ambiguity of (4.42), by using the definition of the dihedral angle of a Lorentzian 4-simplex σ in terms of the parameter of the dihedral boost:

$$\hat{D}_{ab} = \exp\left(\Theta_{ab}\,\varsigma(\star\tilde{B}_{ab}(\sigma))\right), \tag{4.45}$$

Now, the expression (4.30) of the geometric bivectors in terms of the the boundary data leads to the following equality

$$\varsigma(*\tilde{B}_{ab}(\sigma)) = -\mu\,\hat{X}_a \begin{bmatrix} 0 & n^1_{ab} & n^2_{ab} & n^3_{ab} \\ n^1_{ab} & 0 & 0 & 0 \\ n^2_{ab} & 0 & 0 & 0 \\ n^3_{ab} & 0 & 0 & 0 \end{bmatrix} \hat{X}_a^{-1} = \mu\,\hat{X}_a\,\pi(\mathbf{n}_{ab}\cdot\mathbf{K})\,\hat{X}_a^{-1}, \qquad (4.46)$$

where $\pi : \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \to \mathrm{End}(\mathbb{R}^{3,1})$ is the vector representation of the Lorentz algebra (see appendix A).

This implies that a lift of the dihedral boost to $\mathrm{SL}(2,\mathbb{C})$ is explicitly given by

$$D_{ab} = X_a\,e^{\mu\Theta_{ab}\mathbf{K}.\mathbf{n}_{ab}}\,X_a^{-1}, \qquad (4.47)$$

K being here a 2×2 matrix. The sign choice of the lift plays no role in what follows.

Next, we use a property of the dihedral boost just established,

$$\Gamma(\hat{D}_{ab}\,F_a) = \Gamma(F_b). \qquad (4.48)$$

This can be written

$$D_{ab}X_aX_a^{\dagger}D_{ab}^{\dagger} = X_bX_b^{\dagger}, \qquad (4.49)$$

which implies

$$X_a\,e^{2\mu\Theta_{ab}\mathbf{K}.\mathbf{n}_{ab}}\,X_a^{\dagger} = X_b\,X_b^{\dagger}. \qquad (4.50)$$

Comparing the last equality and (4.44) finally gives that

$$e^{2\mu\Theta_{ab}\mathbf{K}.\mathbf{n}_{ab}} = e^{2r_{ab}\mathbf{K}.\mathbf{n}_{ab}}, \qquad (4.51)$$

which implies that $\mu\Theta_{ab} = r_{ab}$. This solves the sign ambiguity.

In this last part, we show how the action at critical point simplifies for the case of a Regge state. The computation uses the critical point equations combined with the definition of the boundary state (3.40), as follows

$$\begin{aligned} X_a\,J\xi_{ab} &= \frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{i\theta_{ab}}X_b\,\xi_{ba}, \\ &= \frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{i\theta_{ab}}X_b\,g_{ab}\,J\xi_{ab}, \\ &= \frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{i\theta_{ab}}X_b\,g_{ab}\,X_a^{-1}\,X_a\,J\xi_{ab}, \\ &= \frac{\parallel Z_{ab}\parallel}{\parallel Z_{ba}\parallel}e^{i\theta_{ab}}D_{ab}\,X_a\,J\xi_{ab}, \end{aligned} \qquad (4.52)$$

As D_{ab} is a pure boost, that is, has eigenvalues that are strictly positive, it follows that $\theta_{ab} = 0$.

Therefore the action on the critical points corresponding to a non-degenerate 4-simplex yields the Regge action for the Lorentzian 4-simplex determined by the boundary data

$$S = -i\mu \sum_{a<b} \gamma k_{ab} \Theta_{ab}. \tag{4.53}$$

4.5 Asymptotic formula

We can now state the asymptotic formula for the 4-simplex amplitude. A formula is called asymptotic if the error term is bounded by a constant times one more power of λ^{-1} than that stated in the asymptotic formula.

Given a set of boundary data, then in the limit $\lambda \to \infty$ and for $p_{ab} = \gamma k_{ab}$

1. If the boundary state is the Regge state of the boundary geometry of a non-degenerate Lorentzian 4-simplex we obtain:

$$f_4 \sim \left(\frac{1}{\lambda}\right)^{12} \left[N_+ \exp\left(i\lambda\gamma \sum_{a<b} k_{ab}\Theta_{ab} \right) + N_- \exp\left(-i\lambda\gamma \sum_{a<b} k_{ab}\Theta_{ab} \right) \right] \tag{4.54}$$

   $N_\pm$ are independent of λ and are given below.

2. If the boundary state is the Regge state of the boundary geometry of a non-degenerate Euclidean 4-simplex we obtain:

$$f_4 \sim \left(\frac{1}{\lambda}\right)^{12} \left[N_+ \exp\left(i\lambda \sum_{a<b} k_{ab}\Theta^E_{ab} \right) + N_- \exp\left(-i\lambda \sum_{a<b} k_{ab}\Theta^E_{ab} \right) \right] \tag{4.55}$$

   Θ^E_{ab} is the dihedral angle of the Euclidean 4-simplex.

3. If the boundary state is not that of the boundary of a non-degenerate 4-simplex but allows a single vector geometry as solution, then for an appropriate phase choice the asymptotic formula is:

$$f_4 \sim \left(\frac{2\pi}{\lambda}\right)^{12} N \tag{4.56}$$

   The number N is independent of λ.

4. For a set of boundary data that is neither a non-degenerate 4-geometry nor a vector geometry, the amplitude is suppressed for large λ.

$$f_4 = o(\lambda^{-M}) \quad \forall M \tag{4.57}$$

In the Lorentzian case the two contributions to the asymptotics correspond to the parity related reconstructions of this 4 simplex geometry. Calling $N_\pm$ the constants for the sector with $\mu = \mp 1$ respectively they are given by

$$\begin{aligned} N_\pm &= (2\pi)^{22} \frac{2^4}{\sqrt{\det H_\pm}} \prod_{a<b} \frac{2k_{ab}}{\pi} \Omega_{ab}|_{\text{crit}} \\ &= 2^{36}\pi^{12} \frac{1}{\sqrt{\det H_\pm}} \prod_{a<b} k_{ab}\Omega_{ab}|_{\text{crit}} \end{aligned} \tag{4.58}$$

The factor $(2\pi)^{22}$ comes from the stationary phase formula as the integral has 6×4 dimensions coming from the $\mathrm{SL}(2,\mathbb{C})$ integrations and 20 dimensions from the z variables. Since the formula is asymptotic, we have used $d_{\lambda k} \sim 2\lambda k$ and cancelled the scaling from the coefficients. The additional factor 2^4 comes from the fact that both spin lifts at the critical points give the same contribution to the action. $H_\pm$ is the Hessian matrix of the action (4.4) evaluated at the critical points (+) and the parity related critical points (-), this is evaluated in appendix C. $\Omega_{ab}|_{\text{crit}}$ is the measure term evaluated at the critical points. A choice of coordinate must be made to evaluate this, however the ratio Ω_{ab} with $\sqrt{\det H_\pm}$ is invariant of this choice of coordinates.

For the Euclidean case we get contributions from the self dual and anti self dual part of the bivector geometry which combine to give the full 4-dimensional Euclidean bivector geometry. The phase part of the action for Euclidean boundary data is evaluated in (BFH to appear). $N_\pm$ are the same as above but evaluated at the appropriate critical points. The dihedral angle Θ^E_{ab} of a Euclidean 4-simplex arises in the following way. For the case of Euclidean boundary data, there are two $\mathrm{SU}(2)$ solutions, say X^+, X^-, to the critical point equations (4.31). For these solutions the boost parameter $r_{ab} = 0$ but the phase term $\theta_\pm$ remains. The interpretation of the critical points is a non-degenerate Euclidean 4-simplex as described in (BDFGH 2009) and the Regge phase choice implies that $\theta_+ + \theta_- = 0$ and $\theta_+ = \frac{1}{2}\Theta^E_{ab}$. Combining this gives case 2.

The case of a single vector geometry proceeds analogously. An appropriate phase choice and the geometry of these solutions is described in (BFH 2009). In particular it was shown that no such vector geometries exist for Lorentzian boundary data there.

For the final case, no critical points exist and the stationary phase theorem tells us that the amplitude is suppressed.

By the classification of critical points this concludes our asymptotic analysis of the amplitude.

Chapter 5

Conclusions

In this concluding chapter we will take a look back and review what we have been able to understand while at the same time pointing out the many problems left open.

Our starting point was the classical phase space associated to a 4-simplex. We have seen that a straightforward quantization following the strategy used in three dimensions to construct the quantum tetrahedron looks complicated, the complication coming apparently from an accident of dimensions: in three dimensions the normals to triangles can be quantized as generators of the symmetry group, while in four dimensions this is no longer the case. We have then introduced bivectors satisfying a certain number of constraints insuring that they describe correctly the geometry of a 4-simplex. This classical setting was the starting point of the Barrett-Crane construction.

We have further introduced different reference frames in the 4-simplex and parallel transported bivectors to frames associated to tetrahedra. The idea behind this additional extension of the phase space was to separate the classical data into two different sets: one corresponding to the intrinsic geometry of the boundary tetrahedra and the other corresponding to the embedding of these tetrahedra in four dimensions to form together a 4-simplex. The advantage of this reformulation is that it puts the classical system into a more canonical form, intrinsic geometrical data being conjugated to extrinsic data, as is well known from canonical general relativity.

The intrinsic geometric data of triangles was described by bivectors $b_t(\tau)$ on the reference frame of the tetrahedra, or equivalently by areas A_t and normals $n_{t\tau}$. The extrinsic data was described by $\mathrm{SL}(2,\mathbb{C})$ group elements $X_{\tau\tau'} = X_{\tau v}X_{v\tau'}$. With this parametrization, the classical phase space is identified with ten copies of the cotangent space to the group manifold, and could be quantized by identifying bivectors with generators of the group, or more precisely to a general linear combination of generators and their duals. This linear combination is controlled by the Immirzi parameter γ, borrowed from the continuous theory, as explained in appendix B. The introduction of this parameter turns out to be essential to understand the boundary state structure: for finite γ we were able to identify the boundary states of a 4-simplex with spin-networks for the dual graph - describing the full quantum geometry of the boundary tetrahedra - while for the BC model the boundary data is specified only by the areas of triangles. The situation is somehow similar to what happens in LQG, where one is able to formulate a theory of connections only after the Immirzi parameter is introduced.

This understanding of the boundary states, other than drawing a closer connection with the canonical theory, is crucial in defining the gluing between simplices in the construction of the path integral. Because boundary states encode the full geometric data of tetrahedra, these can be used to identify the geometries of two neighbouring 4-simplices. The gluing is such that the intrinsic geometry is constrained to agree as seen from different simplices, while the extrinsic geometry is left free.

In constructing the 4-simplex amplitude a number of simplifications and choices have been made. First, we have concentrated our efforts to the case where all tetrahedra are space-like. The extension to time-like tetrahedra should be possible and we leave this to further investigation. In dealing with the constraints we have not been able to impose all of them at the quantum level. We have proposed a procedure to deal with the simplicity constraints - based on the idea of a Master constraint - and the flatness condition, which is taken care of by inserting delta functions in the amplitude. We have left both the closure and the parallel transport conditions free at that point. Both conditions are recovered later as critical point equations in the semiclassical analysis of the 4-simplex amplitude. A proper constraint analysis of the classical phase space associated to a 4-simplex is still lacking. We were able nevertheless to construct a consistent picture, such that in the semiclassical regime all the constraints are recovered and thus correctly describe the classical geometry we started with.

We have also attempted at a construction of the path integral for a fixed triangulation. As discussed in the introduction this corresponds to the spin foam version of Hartle's simplicial minisuperspace program for Regge calculus. A number of issues regarding this expression remain unsolved. We still do not have a full control over the measure of integration. We have left a freedom in the measure for the sum over spins, allowing for an arbitrary power of the dimension d_j associated to each face in the triangulation, but more complicated measures even for the other variables in the sum should be allowed in principle. The reason we restrict to this class of measures is that we are able to write this path integral as a state sum model, gravity being identified as a constrained BF theory.

More work is needed to properly understand and even define the expression for the path integral. The semiclassical analysis of a single 4-simplex amplitude is the starting point for understanding larger triangulations. One needs to understand which configurations dominate the sum. A proper choice of contour will be essential in relating these configurations to Regge configurations, satisfying the equations of motion in the discrete setting. When discussing the 4-simplex asymptotics, some assumptions on the boundary data were made. In particular we have assumed 3d non-degeneracy. Also, we have seen that we had at first order contributions from both Lorentzian and Euclidean boundary data. When considering triangulations with interior faces, one would expect to sum over all possible internal configurations, including degenerate and Euclidean data. We leave these and the many other questions listed above to future research.

Appendix A

Lorentz transformations

In this appendix we give an overview on the Lorentz group and its representations. We start by defining the group and its algebra, describe the unitary representations used in the main text to construct the spin foam amplitude and conclude the appendix with some properties of d matrices used in the proof of finiteness of the vertex amplitude.

The group

The Lorentz group is defined as the group of linear transformations on Minkowski space $\mathbb{R}^{3,1}$ preserving the scalar product

$$(x, y) := \vec{x} \cdot \vec{y} - x^0\, y^0 = \eta_{IJ} x^I y^J \tag{A.1}$$

for $I, J = 0, ..3$ and η is the diagonal metric $\eta = diag(-1, 1, 1, 1)$. Let Λ be a Lorentz transformation, then by definition one has that:

$$\Lambda^t \eta \Lambda = \eta. \tag{A.2}$$

If Λ and M satisfy (A.2) then clearly also ΛM and Λ^{-1} do, which implies that Lorentz transformations form a group, that we note $\mathrm{O}(3,1)$. It has four connected components determined by the signs of $\det \Lambda$ and $\operatorname{sgn} \Lambda^0_0$. Clearly one has $\det \Lambda = \pm 1$ from eq. (A.2). Also it is a continuous function on the components of Λ and cannot map one connected component into another. For the sign of Λ^0_0 by taking the 00 component of equation (A.2), one has

$$(\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2 = 1, \tag{A.3}$$

which implies that $|\Lambda^0_0| \geq 1$ and then the two components for the two different signs of Λ^0_0 cannot be mapped continuously one into another. The four components are usually noted in the literature as $\mathrm{O}^{\uparrow,\downarrow}_{\pm}(3,1)$ according to the combinations of signs. In this conventions $\uparrow$ and $\downarrow$ correspond to $\operatorname{sgn} \Lambda = +1$ and $\operatorname{sgn} \Lambda = -1$ resp. and the $\pm$

subscript to the sign of $\det\Lambda$. We may also use the notation $\mathrm{SO}^{\uparrow,\downarrow}(3,1)$ for the components with $\det\Lambda = +1$. Each component contains a special transformation: $\mathrm{O}^{\uparrow}_{+}(3,1)$ contains the identity transformation $\mathbb{1}$; $\mathrm{O}^{\uparrow}_{-}(3,1)$ contains the parity transformation P acting on vectors as $P(x^0,\vec{x}) = (x^0,-\vec{x})$; $\mathrm{O}^{\downarrow}_{+}$ contains the the time inversion transformation $T(x^0,\vec{x}) = (-x^0,\vec{x})$; and $\mathrm{O}^{\downarrow}_{-}$ contains the the inversion transformation given by $-\mathbb{1} = PT$. Using these spacial transformations one can map any component into the other. It is thus sufficient to study the component connected to the identity and the special transformations P, T and PT.

In order to study the component $\mathrm{SO}^{\uparrow}(3,1)$ it is useful to go to its double cover $\mathrm{SL}(2,\mathbb{C})$. First define the map $\Gamma : \mathbb{R}^{3,1} \to \mathbb{H}$ from vectors in the Minkowski space to the space of Hermitian matrices, given by $\Gamma(x) = x^0\mathbb{1} + x^i\sigma_i$. Define $\sigma^I = (\mathbb{1},\sigma^i)$. Then the inverse map is given by $(\Gamma^{-1}(X))^I = \frac{1}{2}\mathrm{Tr}(X\sigma^I)$ for $X \in \mathbb{H}$.

The action of an element $\hat{X} \in \mathrm{SO}^{\uparrow}(3,1)$ on $\mathbb{R}^{3,1}$ lifts to the action of $X \in \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}$ following

$$\Gamma(\hat{X}x) = X\Gamma(x)X^\dagger \tag{A.4}$$

for x a vector in Minkowski space. The map $X \mapsto \hat{X}$ is two to one, as $+X$ and $-X$ map to the same Lorentz transformation $\hat{X}$. The fact that $\mathrm{SL}(2,\mathbb{C})$ maps only to the connected component to the identity comes from a continuity argument. In fact in the action defined above (A.4) one can map continuously X to the identity and the corresponding $\hat{X}$ varies continuously to the identity as well.

The $\mathrm{SL}(2,\mathbb{C})$ group has various useful decompositions (see Ruhl). One of special interest for us is given by

$$X = R\, d(r)\, R'\ , \tag{A.5}$$

where $R, R' \in SU(2)$ and

$$d(r) = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \tag{A.6}$$

for $r > 0$. The Haar measure in this decomposition reads

$$\mathrm{d}X = \frac{1}{4\pi}\,\mathrm{d}R\mathrm{d}R' \sinh^2 r\mathrm{d}r. \tag{A.7}$$

The algebra

In this section, we summarise the conventions used throughout this paper regarding the Lorentz algebra. The Lie algebra $\mathfrak{so}(3,1)$ of the Lorentz group is a real, semi-simple Lie algebra of dimension six. A basis of $\mathfrak{so}(3,1)$ is provided by the generators $(L_{\alpha\beta})_{\alpha<\beta=0,\ldots,3}$. The Lie algebra structure is coded in the brackets

$$[L_{\alpha\beta}, L_{\gamma\delta}] = -\eta_{\alpha\gamma}\, L_{\beta\delta} + \eta_{\alpha\delta}\, L_{\beta\gamma} + \eta_{\beta\gamma}\, L_{\alpha\delta} - \eta_{\beta\delta}\, L_{\alpha\gamma}, \tag{A.8}$$

where η is the standard Minkowski metric with signature $-+++$.

It is convenient to decompose any (infinitesimal) Lorentz transformation into a purely spatial rotation and a hyperbolic rotation, or boost. This is achieved by introducing the rotation and boost generators respectively given by

$$H_i = \frac{1}{2}\,\epsilon_i{}^{jk} L_{jk}, \quad \text{and} \quad K_i = L_{i0}, \qquad i,j,k = 1,2,3, \tag{A.9}$$

where ϵ_{ijk} is the three-dimensional Levi-Cevita tensor.

Using the Lie algebra structure of $\mathfrak{so}(3,1)$ displayed above, it is immediate to check the following commutation relations between the rotation and boost generators

$$[H_i, H_j] = -\epsilon_{ij}{}^{k} H_k, \qquad [H_i, K_j] = -\epsilon_{ij}{}^{k} K_k, \qquad [K_i, K_j] = \epsilon_{ij}{}^{k} H_k. \tag{A.10}$$

The finite dimensional representations of the Lorentz algebra used in this paper are the spinor and vector representations. In the spinor representation $\rho : \mathfrak{so}(3,1) \to \operatorname{End}\mathbb{C}^2$, the rotation and boost generators are given explicitly in terms of the Hermitian Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

by the following expressions

$$\rho(H_i) = \frac{i}{2}\sigma_i, \qquad \text{and} \qquad \rho(K_i) = \frac{1}{2}\sigma_i. \tag{A.11}$$

This is immediate to check by using the property $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^{k}\sigma_k$ of the Pauli matrices. Throughout the text, the map ρ is kept implicit when there is no possible confusion.

In fact, the above presentation gives the explicit isomorphism between the Lorentz algebra $\mathfrak{so}(3,1)$ and the realification $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ of the Lie algebra of the two-dimensional complex unimodular group $\mathrm{SL}(2,\mathbb{C})$ because

$$\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} = (\mathfrak{su}(2)^{\mathbb{C}})_{\mathbb{R}} \cong \mathfrak{su}(2) \oplus i\mathfrak{su}(2), \tag{A.12}$$

where the direct sum is at the level of vector spaces.

Finally, we also used explicitly the vector representation $\pi : \mathfrak{so}(3,1) \to \operatorname{End}\mathbb{R}^{3,1}$ of the Lorentz algebra in which the matrix elements of the $L_{\alpha\beta}$ generators are given by

$$\pi(L_{\alpha\beta})^{I}{}_{J} = \delta^{I}_{\alpha}\eta_{\beta J} - \eta_{\alpha J}\delta^{I}_{\beta}, \qquad I, J = 0, ..., 3. \tag{A.13}$$

From the above expression, it is immediate to compute the matrix elements of the image of the rotation and boost generators in the vector representation:

$$\pi(H_i)^{I}{}_{J} = \epsilon_i{}^{I}{}_{J}, \qquad \text{and} \qquad \pi(K_i)^{I}{}_{J} = \delta^{I}_{i}\eta_{0J} - \eta_{iJ}\delta^{I}_{0}. \tag{A.14}$$

Therefore, an arbitrary element L in the Lorentz algebra expressed in the rotation/boost basis as follows $L = \mathbf{r}\cdot\mathbf{H} + \mathbf{b}\cdot\mathbf{K}$ is given by the four-by-four matrix

$$\pi(L) = \begin{bmatrix} 0 & -b^1 & -b^2 & -b^3 \\ -b^1 & 0 & r^3 & -r^2 \\ -b^2 & -r^3 & 0 & r^1 \\ -b^3 & r^2 & -r^1 & 0 \end{bmatrix}, \qquad \text{(A.15)}$$

in the vector representation.

Unitary representations

The principal series of irreducible unitary representations of the Lorentz group $\mathrm{SL}(2,\mathbb{C})$ are labelled by two parameters (k,p), with k a half-integer and p a real number (Gelfand et al. 1966, Naimark 1962 and Ruhl 1970). The representations are in a Hilbert space $\mathcal{H}_{(k,p)}$, the labels (k,p) indicating the action of the group in this space. The hermitian inner product is denoted (ψ,χ) for vectors $\psi,\chi \in \mathcal{H}_{(k,p)}$.
Some standard facts about these representations are as follows:

- The (k,p) representation splits into the irreducible representations V_j of the $\mathrm{SU}(2)$ subgroup as

$$\mathcal{H}_{(k,p)} = \bigoplus_{j=|k|}^{\infty} V_j \qquad \text{(A.16)}$$

  with j increasing in steps of 1.

- There is a unitary intertwiner of representations

$$\mathcal{A}\colon \mathcal{H}_{(k,p)} \to \mathcal{H}_{(-k,-p)} \qquad \text{(A.17)}$$

- There is an anti-linear map

$$\mathcal{J} : \mathcal{H}_{(k,p)} \to \mathcal{H}_{(k,p)} \qquad \text{(A.18)}$$

  which commutes with the group action, and is unitary in the sense that $(\mathcal{J}\psi, \mathcal{J}\chi) = (\chi,\psi)$. It satisfies $\mathcal{J}^2 = (-1)^{2k}$.

- There is an invariant bilinear form β on $\mathcal{H}_{(k,p)}$ defined by

$$\beta(\psi,\chi) = (\mathcal{J}\psi,\chi). \qquad \text{(A.19)}$$

  The properties of $\mathcal{J}$ show immediately that $\beta(\psi,\chi) = (-1)^{2k}\beta(\chi,\psi)$, so that β is symmetric or antisymmetric as $2k$ is even or odd.

Both $\mathcal{A}$ and $\mathcal{J}$ respect the decomposition into SU(2) subspaces; clearly $\mathcal{A}$ restricts to a multiple of the identity operator on the j-th subspace, whilst $\mathcal{J}$ restricts to a phase times the standard antilinear map J on an SU(2) representation. Explicit formulae fixing these constants are given below.

The standard representation is to take $\mathcal{H}_{(k,p)}$ to be the space of functions of two complex variables $z = (z_0, z_1)$ that are homogeneous

$$f(\lambda z) = \lambda^{-1+ip+k}\bar{\lambda}^{-1+ip-k} f(z). \tag{A.20}$$

The inner product on this vector space is defined using the standard invariant 2-form on $\mathbb{C}^2 - \{0\}$ given by

$$\Omega = \frac{i}{2}(z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0) \wedge (\bar{z}_0 \mathrm{d}\bar{z}_1 - \bar{z}_1 \mathrm{d}\bar{z}_0). \tag{A.21}$$

For $f, g \in \mathcal{H}_{(k,p)}$, the form $\bar{f}g\Omega$ has the right homogeneity to project down to $\mathbb{CP}^1$. The inner product is given by integrating this 2-form.

$$(f, g) = \int_{\mathbb{CP}^1} \bar{f}\, g\, \Omega. \tag{A.22}$$

Alternatively, one can do the integration on a section of the bundle $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$. The standard choice is given by $z = (\zeta, 1)$, for which the integration measure reduces to the standard measure on the plane, $\Omega = \frac{i}{2}\mathrm{d}\zeta \wedge \bar{\mathrm{d}}\zeta = \mathrm{d}x \wedge \mathrm{d}y$, with $\zeta = x + iy$.

The element $X \in \mathrm{SL}(2, \mathbb{C})$ acts on a homogeneous function by

$$(Xf)(z) = f(X^T z), \tag{A.23}$$

which uses the transpose matrix X^T. This gives the unitary representation in the principal series.

Our standard notation for structures on $\mathbb{C}^2$ is as follows. The Hermitian inner product is

$$\langle z, w \rangle = \bar{z}_0 w_0 + \bar{z}_1 w_1, \tag{A.24}$$

the SU(2) structure map

$$J \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} -\bar{z}_1 \\ \bar{z}_0 \end{pmatrix}, \tag{A.25}$$

and the antisymmetric bilinear form is

$$[z, w] = z_0 w_1 - z_1 w_0 = \langle Jz, w \rangle. \tag{A.26}$$

The unitary isomorphism $\mathcal{A}$ for $(k, p) \neq (0, 0)$ is defined up to a scalar multiple by the map:

$$\mathcal{A}f(w) = c' \int_{\mathbb{CP}^1} [w, z]^{-1-k-ip}\overline{[w, z]}^{-1+k-ip} f(z)\, \Omega. \tag{A.27}$$

We fix the constant $c' = \sqrt{k^2+p^2}/\pi$ for convenience.

There is also an anti-unitary isomorphism $\mathcal{H}_{(k,p)} \to \mathcal{H}_{(-k,-p)}$ given by complex conjugation of functions. Combining these two gives the antilinear stucture map

$$\mathcal{J} f = \overline{\mathcal{A} f}. \tag{A.28}$$

We have that:

$$\mathcal{A}\mathcal{A} f = (-1)^{2k} f, \tag{A.29}$$

mapping from $\mathcal{H}_{(k,p)}$ to $\mathcal{H}_{(-k,-p)}$ and back. A short calculation shows that this implies also $\mathcal{J}^2 = (-1)^{2k}$, this time both mappings being on the same space. These relations will be verified again explicitly below, using coherent states.

Using these definitions, the bilinear form is

$$\beta(\psi,\chi) = c' \int_{\mathbb{CP}^1\times\mathbb{CP}^1} [w,z]^{-1-k-ip}\overline{[w,z]}^{-1+k-ip} \psi(z)\chi(w)\,\Omega_z\,\Omega_w. \tag{A.30}$$

These integrals are a little tricky since each power of $[z,w]$ does not separately exist as a function on the whole plane; one has to combine them. Writing $[w,z] = re^{i\theta}$, the integrand contains

$$r^{-2-2ip} e^{-2ik\theta}, \tag{A.31}$$

which is well-defined. It is also clear from this formula that $\beta(\psi,\chi) = (-1)^{2k}\beta(\chi,\psi)$.

The canonical basis

Given a carrying space $\mathcal{H}_{(k,p)}$ the canonical basis is given by the basis diagonalizing simultaneously the the Casimir operators $J\cdot J$, $\star J\cdot J$, L^2 and L_z (see below for these operators). We note an element of this basis $|(k,p);j,m\rangle$. We may also use the standard notation (Naimark 1962) $f^j_m(z)^{(k,p)}$ for an element of the canonical basis represented in the space of homogeneous functions. They can be given an explicit expression using hypergeometrical functions. It will be more useful however to use homogeneity and scale the argument such that it is normalized, $\langle z,z\rangle = 1$. Given a normalized spinor ξ, construct the SU(2) matrix using the map (3.51):

$$g(\xi) = \begin{pmatrix} \xi_0 & -\bar{\xi}_1 \\ \xi_1 & \bar{\xi}_0 \end{pmatrix}. \tag{A.32}$$

The canonical basis when restricted to normalized spinors is identified with SU(2) representation matrices[1]:

$$f^j_m(\xi)^{(k,p)} = \sqrt{\frac{d_j}{\pi}}\, D^j_{m\,k}(g(\xi)). \tag{A.33}$$

[1]The definition used here is slightly different from the one used in the literature (Ruhl 1970). The difference amounts to a change of variables $z \mapsto Jz$.

Evaluating $f^j_m(z)^{(k,p)}$ on non-normalized spinors and using homogeneity, we get the following formula:

$$f^j_m(z)^{(k,p)} = \sqrt{\frac{d_j}{\pi}}\,\langle z, z\rangle^{ip-1-j}\; D^j_{m\,k}(g(z)), \tag{A.34}$$

One can check that the function defined above has the correct homogeneity.

Matrix elements of generators in the canonical basis and Casimir operators

In this subsection we give the action of the rotation and boost generators of the Lorentz algebra on the canonical basis. The state $|(k,p); j, m\rangle$ is noted simply $|j, m\rangle$. The definitions are that $H_\pm = H_x \pm iH_y$ and $K_\pm = K_x \pm iK_y$.

$$\begin{aligned}
H_z|j,m\rangle &= m|j,m\rangle \\
H_+|j,m\rangle &= \sqrt{(j+m+1)(j-m)}|j,m+1\rangle \\
H_-|j,m\rangle &= \sqrt{(j-m+1)(j+m)}|j,m-1\rangle \\
K_z|j,m\rangle &= -\gamma_{(j)}\sqrt{(j^2-m^2)}|j-1,m\rangle - \beta_{(j)}m|j,m\rangle + \gamma_{(j+1)}\sqrt{(j+1)^2-m^2}|j+1,m\rangle \\
K_+|j,m\rangle &= -\gamma_{(j)}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle - \beta_{(j)}\sqrt{(j-m)(j+m+1)}|j,m+1\rangle - \\
&- \gamma_{(j+1)}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle \\
K_-|j,m\rangle &= \gamma_{(j)}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle - \beta_{(j)}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle + \\
&+ \gamma_{(j+1)}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle
\end{aligned} \tag{A.35}$$

where

$$\beta_{(j)} = \frac{kp}{j(j+1)} \quad\text{and}\quad \gamma_{(j)} = \frac{i}{j}\sqrt{\frac{(j^2-k^2)(j^2+p^2)}{4j^2-1}}. \tag{A.36}$$

The Casimir operators and their action on elements of the basis are given by:

$$\begin{aligned}
C_1 &= J\cdot J = 2(H^2-K^2) = 2\left(k^2-p^2-1\right) \\
C_2 &= \star J\cdot J = -4H\cdot K = 4kp \\
L^2 &= j(j+1).
\end{aligned} \tag{A.37}$$

Formulas for d matrices

We complete this appendix with some explicit formulas for the matrices $d^{k,p}_{ll'm}(r)$, referred to in the main text. In particular, we show that the asymptotic behavior (2.63) holds. We follow closely section (4-5) of (Ruhl 1970). We start with the following useful expression:

$$d^{k,p}_{ll'm}(r) = \{\cdots\}^{\frac{1}{2}}\;(2\sinh r)^{-l-l'}\sum_{\nu,\mu} c_{\nu\mu}\, e^{\mu r}\,\frac{\sinh\left(r(ip+\nu)\right)}{(ip+\nu)\sinh r}\;, \tag{A.38}$$

where $\nu + l'$ and $\mu + l'$ are integers, and

$$\{\cdots\} = \left\{(2l+1)(2l'+1) \times \frac{(l+k)!\,(l-k)!\,(l'+k)!\,(l'-k)!}{(l+m)!(l-m)!(l'+m)!(l'-m)!}\right\}. \tag{A.39}$$

To define the coefficients $c_{\nu\mu}$, it is useful to redefine the summation labels $(\nu, \mu) \to (a, b)$, while introducing a new sum over integers (n_1, n_2):

$$\begin{aligned} 2b &= \mu + \nu + l' - l + k + 2n_1 - m & \text{(A.40)} \\ 2a &= \nu - \mu + l' + l + m - k - 2n_1. & \text{(A.41)} \end{aligned}$$

The sum over (ν, μ) can then be traded by a sum over (n_1, n_2, a, b):

$$\sum_{\nu\mu} c_{\nu\mu}\,(\cdots) = \sum_{n_1,n_2} \binom{l+m}{n_1}\binom{l-m}{n_1-m+k}\binom{l'+m}{n_2}\binom{l'-m}{n_2-m+k}$$
$$\times \sum_{a,b} (-1)^{a+b+m-k} \binom{l+l'-n_1-n_2+m-k}{a}\binom{n_1+n_2-m+k}{b}\,(\cdots)$$

where all summations extend over the domain where the binomial coefficients do not vanish. From eq. (A.38), one sees that the asymptotic behavior for $r \to \infty$ is of the form:

$$d^{k,p}_{ll'm}(r) \sim e^{-r(l+l'+1-(\mu+\nu)_{max})}, \tag{A.42}$$

for $(\mu + \nu)$ taking its maximal value. One can check that this maximal value is given by:

$$(\mu + \nu)_{max} = l + l' - |m + k| \tag{A.43}$$

which then gives the asymptotic behavior

$$d^{k,p}_{ll'm}(r) \sim e^{-r(1+|m+k|)} \tag{A.44}$$

as advertised in the main text.

Appendix B

Plebanski theory

In this appendix we will show that the bivector geometry described in chapter two can be seen as coming from discretizing continuous gravity, written in the Plebanski formulation (Plebanski 1977). Plebanski's paper deals with the self dual theory. Here we will consider a generalization of the construction for the full Lorentz group (see Reisenberger 1998, De Pietri and Freidel 1999). This will also motivate the introduction of the Immirzi parameter γ in the choice of symplectic structure.

We start with Einstein-Cartan theory with an extra term referred to as the Holst term in the literature (Holst 1996). The action is given by:

$$S_{EC\gamma} = \int {}^{\star}(e \wedge e)_{IJ} \wedge R(\omega)^{IJ} - \frac{1}{\gamma} \int (e \wedge e)_{IJ} \wedge R(\omega)^{IJ}. \tag{B.1}$$

The second term in the action is the Holst term and is identically zero on solutions to the equations of motion. ω^{IJ} is the spin connection one-form and $R(\omega)^I J$ the curvature two-form. e^I is the (co)tetrad one form. The canonical analysis of this action leads to the canonically conjugated pair of variables (A^i_a, E^b_j), where E^b_j is the densitized triad and and the Barbero-Immirzi connection $A^i_a := \frac{1}{2}\epsilon_{ijk}\omega^{kl}_a + \gamma\omega^{0i}_a$, where ω^{kl}_a and ω^{0i}_a are components of the pull back of the spin connection to the three dimensional slice. The introduction of the Immirzi parameter is essential so that the canonical variable is indeed a connection. If it is set to $\gamma = i$ we have the original self dual theory (Ashtekar 1986).

The idea of Plebanski is to define the bivector two-forms $B = e \wedge e$ and consider them as basic variables for the theory, supplementing the action with the appropriate constraints such that the original relation with co-tetrads can be recovered. These so called simplicity constraints are given by

$$B^{IJ} \wedge B^{KL} = \frac{1}{4!}\epsilon^{IJKL}(B \wedge \star B). \tag{B.2}$$

We have defined $(B \wedge \star B) = \epsilon_{IJKL} B^{IJ} \wedge B^{KL}$. The above constraints can be derived from an action principle by adding a term $\phi_{IJKL} B^{IJ} \wedge B^{KL}$. We demand the Lagrange multiplier ϕ_{IJKL} to be trace free: $\epsilon_{IJKL}\phi^{IJKL} = 0$. The action is now given as a constrained BF theory:

$$S_P = \int (\star B - \frac{1}{\gamma}B)_{IJ} \wedge R(\omega)^{IJ} + \phi_{IJKL}B^{IJ} \wedge B^{KL}. \tag{B.3}$$

As a side note, this motivates the quantization procedure used in chapter 2 (see eq. 2.41). After discretization, the unconstrained action will become $\sum_t (\star B_t - \frac{1}{\gamma}B_t)U_t$, where U_t is the holonomy around the face dual to t, cf. discussion on Ooguri's model. We see that the variable conjugated to U_t is $(\star B_t - \frac{1}{\gamma}B_t)$, and these ae quantized as invariant vector fields acting on this copy of the group, as is usual from spin foam quantization.

There are two families of solutions to the simplicity constraints (Reisenberger 1998, De Pietri and Freidel 1999):

$$B^{IJ} = \pm e^I \wedge e^J \;\; \text{and} \;\; B^{IJ} = \pm \star (e \wedge e)^{IJ}. \tag{B.4}$$

Now, given non-degenerate configurations, s.t. $V := \frac{1}{4!}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta}B^{IJ}_{\mu\nu}B^{KL}_{\alpha\beta} \neq 0$, one can see that the constraints (B.2) are equivalent to the following set of constraints:

$$\epsilon_{IJKL}B^{IJ}_{\mu\nu}(x)B^{KL}_{\alpha\beta}(x) = \epsilon_{\mu\nu\alpha\beta}V(x). \tag{B.5}$$

The usefulness of this reformulation is that now the constraints are ready to be discretized. For that, choose a simplicial decomposition of space-times and integrate on each triangle of this simplicial complex the two-form bivector B. This defines to each triangle the bivector $B^{IJ}_t = \int_t B^{IJ}$. The constraints (B.5) are discretized analogously. There are three different cases according to the two pairs of indices $(\mu\nu)$ and $(\alpha\beta)$ are integrated on the same triangle, on different triangles but on the same tetrahedron, and finally on different triangles and different tetrahedra. Because the constraint (B.5) is localized on a point x of space time, we demand that the triangles belong to the same 4-simplex. The three cases correspond to the following constraints:

- Diagonal simplicity:
$$(\star B_t)_{IJ}B^{IJ}_t = 0 \tag{B.6}$$

- Cross simplicity: (for t and t' sharing an edge)
$$(\star B_t)_{IJ}B^{IJ}_{t'} = 0 \tag{B.7}$$

- Volume simplicity: (for t and t' on the same 4-simplex but on different tetrahedra)
$$(\star B_t)_{IJ}B^{IJ}_{t'} = \tilde{V} \tag{B.8}$$

In the volume simplicity condition $\tilde{V}$ is proportional to the 4-volume of the 4-simplex. This condition imposes that the volume of the 4-simplex computed from different pairs of triangles is the same. In addition, from Stokes theorem one has the closure:

- Closure:

$$\sum_{t\subset\tau} B_t^{IJ} = 0. \tag{B.9}$$

Also, in defining B_t one has a choice of orientation for the triangle. Different orientations are related by the condition:

- Orientation:

$$B_t(\tau) = -B_t(\tau'). \tag{B.10}$$

Adding to these conditions non-degeneracy conditions, we see that we have recovered the Barrett-Crane conditions defining a 4-simplex geometry. By using repeatedly the orientation and the closure conditions, the volume simplicity condition is automatically satisfied. To see that, let us come back to the notation B_{ab}, $a = 1...5$ and consider the pair of faces (12) and (34). Then compute the volume from this pair of faces $\epsilon_{IJKL}B_{12}^{IJ}B_{34}^{KL}$. By using closure, orientation and the first two simplicity conditions, one has:

$$\epsilon_{IJKL}B_{12}^{IJ}B_{34}^{KL} = -\epsilon_{IJKL}B_{12}^{IJ}B_{54}^{KL} = -\epsilon_{IJKL}B_{14}^{IJ}B_{32}^{KL} = \cdots \tag{B.11}$$

which is what we wanted to show.

We can now use the Theorem 1, which implies the existence of tetrads N_τ such that $B_{\tau\tau'} \propto \star(N_\tau \wedge N_{\tau'})$. To appreciate better the connection of this result with the continuum reconstruction result (B.4), it will be useful to define discrete co-tetrads (see Conrady and Freidel 2008b).

Discrete co-tetrads are obtained by integrating the continuous co-tetrad e_μ^I along an edge l of the triangulation: $E_l^I = \int_l e^I$. To state the relation between tetrads and co-tetrads, let us introduce some notation. First note that an edge l in a given 4-simplex is identified by the two tetrahedra that do not contain this edge in the 4-simplex. One may note the discrete co-tetrad by this pair of tetrahedra $E_{\tau\tau'}^I$. The tetrads N_τ are associated to tetrahedra as they represent four-normals to them. The relation between discrete tetrads and discrete co-tetrads can then be stated as follows. Given a set of co-tetrads $E_{\tau_a\tau_b}^I$ on a given 4-simplex, associate the tetrads as the unique set of vectors N_τ such that:

$$U_{\tau\, I}E_{\tau''\tau'}^I = \delta_{\tau''\tau} - \delta_{\tau'\tau} \tag{B.12}$$

In this formula, the U_τ is proportional to N_τ such that its norm is given by

$$|U_\tau| = \frac{V_3(\tau)}{|V_4|}. \tag{B.13}$$

$V_3(\tau)/3!$ is the 3d volume of the tetrahedron τ and $V_4/4!$ is the oriented volume of the 4-simplex. It is given by:

$$V_4 := \det(E_{\tau_2\tau_1}, ..., E_{\tau_5\tau_1}), \tag{B.14}$$

where we have chosen an orientation for the tetrahedra in the 4-simplex $(\tau_1 ... \tau_5)$. The relation between $U's$ and $E's$ is given by the following formula:

$$U_{\tau_2} = \frac{1}{V_4} \star (E_{\tau_3\tau_1} \wedge E_{\tau_4\tau_1} \wedge E_{\tau_5\tau_1}), \tag{B.15}$$

and cyclically. In the last expression the dualization is defined by $\star(E_1 \wedge ... \wedge E_n)_{I_1...I_{4-n}} := \epsilon_{I_1...I_4} E_1^{I_{5-n}} ... E_n^{I_4}$. Conversely:

$$E_{\tau_2\tau_1} = V_4 \star (U_{\tau_3} \wedge U_{\tau_4} \wedge U_{\tau_5}). \tag{B.16}$$

From these formulas we have the following identification for bivectors:

$$\star(E_{\tau_1\tau_2} \wedge E_{\tau_2\tau_3}) = V_4(U_{\tau_4} \wedge U_{\tau_5}). \tag{B.17}$$

The last formula implies that the bivectors B_{ab} can be expressed in terms of the cotetrads such that, for example, $B_{45} \propto E_{12} \wedge E_{23}$, and cyclically. All the constraints are invariant under the transformation $B \to \star B$, and thus all the statements are true for bivectors replaced by their duals (the correct geometric sector in Theorem 1 is obtained by the non-degeneracy conditions). We see that we have recovered the analogue of the continuous result (B.4) in the discrete setting.

Appendix C

The Hessian

Here we calculate the Hessian matrix required in the stationary phase formula.

The Hessian is defined as the matrix of second derivatives of the action where the variable X_5 has been gauge fixed to the identity. We split the Hessian matrix into derivatives with respect to the X_a variables and derivatives with respect to the z_{ab}. The Hessian will then be a 44×44 matrix of the form

$$H = \begin{pmatrix} H^{XX} & H^{Xz} \\ H^{zX} & H^{zz} \end{pmatrix} \tag{C.1}$$

We will now describe each block of this matrix. H^{XX} is a 24×24 matrix containing only derivatives with respect to the X_a. Note that due to the form of the action, derivatives with respect to two different variables will be zero and it will be block diagonal

$$H^{XX} = \begin{pmatrix} H^{X_1X_1} & 0 & 0 & 0 \\ 0 & H^{X_2X_2} & 0 & 0 \\ 0 & 0 & H^{X_3X_3} & 0 \\ 0 & 0 & 0 & H^{X_4X_4} \end{pmatrix} \tag{C.2}$$

Each $H^{X_1X_1}$ is a 6×6 matrix. The variation has been performed by splitting the $\mathrm{SL}(2,\mathbb{C})$ element into a boost and a rotation generator. This gives

$$H^{X_aX_a} = \begin{pmatrix} H^{X_a^RX_a^R} & H^{X_a^BX_a^R} \\ H^{X_a^RX_a^B} & H^{X_a^BX_a^B} \end{pmatrix} \tag{C.3}$$

H^{zz} is a 20×20 matrix. The derivatives are with respect to the spinor variables z_{ab} on each of the ten edges ab of the amplitude. To perform these derivatives we must choose a section for z_{ab}.

Next the mixed spinor and $\mathrm{SL}(2,\mathbb{C})$ derivatives. We have arranged the derivatives in the order of the orientation $a < b$, ie $z_{12}, z_{13}, z_{14}, z_{15}, z_{23}, z_{24}, z_{25}, z_{34}, z_{35}, z_{45}$. The matrix $H^{X\bar{z}}$ is a 24×20 matrix with the following non-zero entries

$$H^{X\bar{z}} =$$
$$\begin{pmatrix} H^{X_1\bar{z}_{12}} & H^{X_1\bar{z}_{13}} & H^{X_1\bar{z}_{14}} & H^{X_1\bar{z}_{15}} & 0 & 0 & 0 & 0 & 0 & 0 \\ H^{X_2\bar{z}_{12}} & 0 & 0 & 0 & H^{X_2\bar{z}_{23}} & H^{X_2\bar{z}_{24}} & H^{X_2\bar{z}_{25}} & 0 & 0 & 0 \\ 0 & H^{X_3\bar{z}_{13}} & 0 & 0 & H^{X_2\bar{z}_{23}} & 0 & 0 & H^{X_3\bar{z}_{34}} & H^{X_3\bar{z}_{35}} & 0 \\ 0 & 0 & H^{X_4\bar{z}_{14}} & 0 & 0 & H^{X_4\bar{z}_{24}} & 0 & H^{X_4\bar{z}_{34}} & 0 & H^{X_4\bar{z}_{45}} \end{pmatrix}$$

These derivatives also require a choice of section for z_{ab}.

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"De tudo, ficaram três coisas: a certeza de que ele estava sempre começando, a certeza de que era preciso continuar e a certeza de que seria interrompido antes de terminar."

Fernando Sabino