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Parallelizable manifold compactifications of $D = 11$ supergravity

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Abstract

In this thesis we present solutions of spontaneous compactifications of $D = 11$, $\mathcal{N} = 1$ supergravity on parallelizable manifolds S^1, S^3 and S^7. In Freund-Rubin compactifications one usually obtains AdS vacua in $4D$, these solutions usually sets the fermionic VEV's to zero. However giving them non zero VEV's allows us to define torsion given by the fermionic bilinears that essentially flattens the geometry giving us a vanishing cosmological constant on $\mathcal{M}_4$. We further give an analysis of the consistent truncation of the bosonic sector of $D = 11$ supergravity on a S^3 manifold and relate this to other known consistent truncation compactifications. We also consider the squashed S^7 where we check for surviving supersymmetries by analyzing the generalised holonomy, this compactification is of interest in phenomenology.

Populärvetenskaplig sammanfattning

Den bästa teorin vi har idag för att beskriva alla interaktioner mellan partiklar där gravitiation är så pass svag att den inte påverkar interaktionerna är Standard Modellen. Partikel acceleratorn på Cern har sedan början av 70 talet tagit fram experimentel data som stämmer överens med den teoretiskt beräknade data.

M teori är den ultimata teorin tror man idag, den beskriver alla fundamental krafters partiklar inkluderat graviton som är partikeln som medlar gravitationen. M teori är den teori som förenar alla 5 strängteorier: Typ IIA, IIB, $E_8 \times E_8$ heterotisk strängteori, $SO(32)$ heterotisk och Typ I, där supergravitation i $D = 11$ är den låg energi effektiva teorien på M-teori. De olika Strängteorier beskriver öppna eller slutna strängar. Alla dessa teorier baseras på supersymmetri, tyvärr har inte LHC hittat supersymmetriska partiklar men man tror idag att supersymmetrin i lägre dimensioner är en bruten symmetri. Till skillnad på den första strängteorin som bara var bosonisk där partiklarna lever på en 2 dimensionell världsyta och vibrerar i de 24 andra dimensioner. Om man introducerar supersymmetri så visar det sig att strängteori lever i 10 dimensioner. Men i verkliga livet ser vi inte dessa extra dimensioner för att de är så pass små, men de måste finnas där för att teorin ska beskriva konsistenta interaktioner mellan partiklar.

Att reducera ner dimensionerna är något man måste göra för att beskriva effektiva teorier i 4 dimensioner, det visar sig att siffran för alla lösningar på dimensionella reduktioner är ungefär 10^{500}. Detta är en väldigt hög siffra och man refererar till dessa lösningar som landskapet av strängteori. Genom observationer man gjort i astrofysik där man insett att universum accelererar, sätter vissa vilkor på vad för reduktion vi behöver göra om vi vill beskriva just vårat universum. Ett öppet universum inom allmän relativitets teori kallas för de Sitter och ett stängt universum tex en sfär beskrivs som Anti de Sitter. Den beräknade kosmologiska konstanten är väldigt liten 10^{-100}, man kan approximera att universum är i princip platt. Alla dessa lösningar som kommer från reduktioner beskriver då ett specifikt universum och ett specifikt vakuum associerat till detta universum.

Supergravitation visar sig vara den teori efter dimensionell reduktion som kan innehålla Standard modellen, detta är pga av att den är 11 dimensionell och därmed kan innehålla geometrier som kan beskriva symmetrin i Standard modellen.

I denna uppsats går vi igenom dessa dimensionella reduktioner från två olika perspektiv, den första är genom att reducera ner Lagrangianer och den andra metoden är genom att lösa lägre dimensionella rörelseekvationer. Vi visar också att en förmodan om olika typer av reduktioner visar sig vara konsekventa med de lägre dimensionella teorierna. Vi fortsätter sedan med att göra reduktioner av en speciell typ där vi antar att vi har fermionska vakuuum värden som resulterar i att vi får en reduktion som beskriver ett platt universum. Vi går också igenom en deformerad sfär reduktion på en sju-sfär som visar sig behålla en supersymmetri detta visar sig vara relevant inom phenomenologi som försöker beskriva en mer realistik reduktion från M-teori.

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Chapter 1

Introduction

Supergravity is a theory of general relativity and supersymmetry, it was first developed in 1976 by D.Z Freedman, Sergio Ferrara and Peter Van Nieuwenhuizen. They formulated a four dimensional supersymmetric Lagrangian containing gravity and a gravitino field, the gravitino field is called Rarita-Schwinger and describes spin 3/2 particles.

Two years after this theory was presented Cremmer-Julia-Scherk [1] found a eleven dimensional supergravity Lagrangian that was invariant under supersymmetry transformations. Eleven dimensions is very special since that is the highest dimension one can have if one wants a single graviton in the theory, this was first shown by Nahm [2], higher dimensions would require higher spin particles than spin two. The reason we do not consider higher spins than two is because we do not have a consistent theory of gravity coupled to massless particles in supergravity theories. Supergravity in $D = 11$ is the low energy effective description of M-theory. Compactifications of $D = 11$ supergravity could be a good candidate for describing the Standard model symmetries $SU(3) \otimes SU(2) \otimes U(1)$ this was first considered by Witten [3], where he proposed manifolds that have large enough symmetry to contain the Standard Model particles.

CJS also presented us with the particle content of the compactified theory on a seven torus T^7 which is a maximally supersymmetric theory. In this thesis the main focus on the compactifications of $D = 11$ supergravity theory on different manifolds which are parallelizable and admit a torsion term.

In the early 80s supergravity was a very active field, much of the work was done in compactifying the eleven dimensional supergravity [4–7]. The main feature of these papers is that the equations of motion admit a spontaneous compactification $i.e$ it should admit a solution of the metric describing the product $\mathcal{M}_4 \times \mathcal{B}^7$ where $\mathcal{B}^7$ is a compact manifold. These spontaneous compactifications assume that we have a vanishing field strength on the compact manifold, with the exception of Englert's solution which had a non-vanishing field strength for both manifolds.

The solutions using a parallelizable manifold that admits a torsion term which flattens the geometry yielding a Minkowsi vacuum in four dimensions $\mathcal{M}_4$. The seven sphere S^7 received the most attention due having a large enough symmetry group to contain e.g quarks and leptons. Compactifications of $D = 11$ ungauged/gauged supergravity admits an enhanced symmetry often referred to as hidden symmetry. These enhanced symmetries allows one to show consistent truncations of the compactifications.

The Freund-Rubin ansatz in their configuration only allows for maximally symmetric spacetime such that the spontaneous compactification is described by $AdS_4 \times \mathcal{B}^7$, these compactifications preserve all the supersymmetry in $D = 4$. Englert's solution breaks all eight supersymmetries even after the choice of orientation on the seven sphere, this is feature of choosing a non-vanishing field strength for $\mathcal{B}^7$. For instance squashed S^7 preserves different supersymmetries depending on the orientation this was first published in [8]. Their choice of orientation on the squashed S^7 either preserved $\mathcal{N} = 1$ super-

symmetry or $\mathcal{N} = 0$, both solutions are of interest to phenomenology. The counting of supersymmetries after compactification has been discussed [8,9] where they used holonomy to count, since the number of unbroken supersymmetries is equal to the number of the spinors left invariant by the holonomy group $\mathcal{H}$. Compactification of other kinds in particular $K3 \times T^3$ [9] have been shown to preserve $\mathcal{N} = 4$ supersymmetries however one interesting fact is that this compactification yields a larger particle content the details of the particle content. The reason for this is because the particle content is not determined by the geometry of $\mathcal{M}_7$ but rather the topology of the manifold. The topology of the compactified manifold is important since one can wrap p-branes around closed loops on the manifold which acts as excitations of particles in lower dimensions. The p-cycles can be described by a homology group where the dimensions of the homology group is the Betti numbers, which determines the particle content just as one can see explicitly in $K3 \times T^3$.

Using holonomy to find Killing spinors have been proven to be very useful in particular when it comes to finding membrane solutions of $D = 11$ supergravity, membrane solutions were first formulated in the early 90s and very important features of eleven dimensional supergravity.

Checking whether the compactifications are consistent, *i.e* are the lower dimensional equations of motions also are solutions of the higher dimensional equations of motion is something one has to do. The problem of consistency was first considered when one did compactifications on $D = 5$ Einstein gravity on a S^1 where after compactification one obtained Einstein-Maxwell theory including a dilaton. Setting the dilaton field to zero would yield a unification of electrodynamics and gravity coming from a pure gravity theory in $D = 5$, this however is not allowed since setting the dilaton field to zero would not satisfy the equation of motion. This is because we have a source term that is associated with the dilaton field describing the interactions between the various lower dimensional fields, hence setting the dilaton to zero would not yield a consistent truncation of the theory and also defeats the purpose of having higher dimensional theories. One would need to work out the Fourier expansions in order to say something about the consistency of the equations of motion. Fortunately there is a more practical way of doing this where one uses group theoretical arguments. A consistent truncation retains all the group singlets and throws out the non singlets, this is a consistent truncation since we can not create non-singlets from multiplying singlets.

The consistency of Kaluza-Klein theory was very active in the start of the millennium, where most of the work was done in the $D = 11$ supergravity theory [10–17] and for type IIB on S^5 [18] and recently a consistent truncation of type IIA supergravity on S^6 [19] was also shown.

Generally consistent truncations on sphere reductions should be such that the isometry group of the sphere should retain all the singlets *i.e* gauge fields should be retained under $SO(n+1)$. The general construction is to describe the manifold as a coset manifold containing a large enough symmetry group to retain all the gauge fields.

The outline of this thesis is as following, we start with supergravity in $D = 4$, $\mathcal{N} = 1$ and show that it is invariant under supersymmetry transformations. In chapter 3 we discuss compactifications in particular solutions of spontaneous compactification yielding a Minkowski background. In chapter 4 we discuss the details of consistent truncations of supergravity where we show an explicit in detail for S^3 sphere reduced on a D-dimensional theory. The last chapter we review the generalised holonomy of the $M5$-brane solution and the squashed S^7. We also show the relation between the integrability conditions and the surviving supersymmetries.

Chapter 2

Tools for supergravity

In this chapter we useful topics that one will stumble upon when studying supergravity. We start off with Poincaré group and extend it to a super group where the fermions are related to the bosons by a group operation. We then continue with the Cartan formalism where we briefly introduce frame fields and its applications. From the frame fields we discuss the Rarita-Schwinger field which is a spin-3/2 field describing the gravitino. This field is a key component in supergravity because it gives rise to the supersymmetric partner namely the graviton. We then move forward to first and second order formalism of general relativity. The second order formalism refers to second order derivatives of the metric or the frame field in the gravitational field equations. The frame field will be the dynamical variable describing gravity, the frame field let us go from curved spacetime to flat spacetime where we now can define our spinors. Therefore using the frame fields one naturally describes the spinors on the manifold.

The first order formalism also known as the Palatini formalism where we have that the frame field $e^a{}_\mu$ and the spin connection $\omega_{\mu ab}$ are independent variables and the equations of motion are of first order derivatives. By solving the spin connection, one finds a connection with torsion. These formulations will be used later to discuss the invariance of supersymmetry transformations in supergravity which is the last part of this chapter. The key references [20], [21] [1].

2.1 Supersymmetry algebra

The Poincaré group $ISO(1,3)$ contains four translations, three rotations and three boosts. The group corresponds to basic special relativity symmetries that act on the spacetime coordinate

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{2.1}$$

where Λ is the Lorentz transformations and the a^μ are the translations, where the Lorentz transformations leaves the metric invariant. The algebra for the generators of the Poincaré group are given as

$$[P^\mu, P^\nu] = 0, \quad [M_{\mu\nu}, M^{\rho\sigma}] = -2\delta^{[\rho}_{[\mu} M_{\nu]}{}^{\sigma]}, \quad [P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]} \ , \tag{2.2}$$

where P is the generator of translations and M is the generator of Lorentz transformations. We can raise and lower the indices by multiplying with a metric η. Before we go further on with spinors let us first discuss the Lorentz group $\mathfrak{so}(1,3)$ and its irreducible representations. The Lorentz group is homeomorphic to $SU(2) \oplus SU(2)$ where the generators of $SU(2)$ correspond to J_i of rotations and K_i of Lorentz boosts defined as

$$J_i = -\frac{1}{2}\epsilon_{ijk} M_{jk}, \quad K_i = M_{0i} \tag{2.3}$$

[1] Most of this chapter has been discussed in detail in the author's previous work [22]

which satisfies the algebra above. Using the algebra above one can construct linear combinations of J_i and K_i which are given as

$$I_k = \frac{1}{2}(J_k - iK_k), \quad I'_k = \frac{1}{2}(J_k + iK_k) \tag{2.4}$$

which satisfies the commutation relation of two independent copies of the Lie algebra $\mathfrak{su}(2)$

$$[I_i, I_j] = \epsilon_{ijk} I_k, \quad [I'_i, I'_j] = \epsilon_{ijk} I'_k, \quad [I_i, I'_j] = 0 \ . \tag{2.5}$$

We can thus interpret the product of these two linear combinations of the generators of the $SU(2)$ group as the physical spin where the the lie algebra $\mathfrak{so}(1,3)$ is related to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

There exists a homeomorphism $SO(1,3) \cong SL(2,\mathbb{C})$ which gives rise to spinor representations, namely the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation which is the central treatment for fermions in quantum field theory. We can show that there is a homeomorphism between these two Lie groups by considering 2×2 complex matrices, we know that a Lorentz transformation acts on the four vectors in the following way

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \ . \tag{2.6}$$

We introduce Pauli matrices given as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.7}$$

which can be written as $\vec{x} = x_\mu \sigma^\mu$ and the determinant of this matrix is given as $\vec{x}^2 = -x^\mu \eta_{\mu\nu} x^\nu$. There is a relation between the four dimensional Minkowski space and the 2×2 hermitian matrices, in fact there is a isomorphism between them. Using the relation that $\mathrm{Tr}(\sigma^\mu \bar{\sigma}_\nu) = 2\delta^\mu{}_\nu$ and the Pauli matrices one finds that the matrix $\vec{x}$ can be written as

$$\vec{x} = \bar{\sigma}_\mu x^\mu, \quad x^\mu = \frac{1}{2}\,\mathrm{Tr}(\sigma^\mu \vec{x}) \tag{2.8}$$

this gives the explicit form of the isomorphism. Consider $A \in SL(2,\mathbb{C})$ and consider the linear map

$$\vec{x} \to \vec{x}' = A\vec{x}A^\dagger \tag{2.9}$$

due to linearity and the hermitian matrix one can obtain an explicit form of the Lorentz matrix Λ this must indeed be a Lorentz transformation

$$\Lambda^\mu{}_\nu = \frac{1}{2}\,\mathrm{Tr}(\sigma^\mu A \bar{\sigma}_\nu A^\dagger) \ . \tag{2.10}$$

This transformation preserves the determinant and hence the Minkowski norm is invariant. Therefore every Lorentz transformation has the representation given in (1.9) which is a two-to-one map since if $A, B \in SL(2,\mathbb{C})$ then a transformation $\Lambda(AB) = \Lambda(A)\Lambda(B)$ is indeed valid if one uses (1.10) this shows the group homomorphism. We have now considered some properties of the Poincaré group and its relations to spinor representations, let us further extend the group by introducing a graded algebra.

The Super-Poincaré group is an extension of the group with one more type of generator namely Q^A_α where α is the spacetime spinor index and $A = 1, \ldots, \mathcal{N}$ labels the supercharges. The graded algebra is defined as

$$O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a = i C^e{}_{ab} O_e \tag{2.11}$$

where η_a takes the value

$$\eta_a = \begin{cases} 0, & O_a : \text{bosonic generator} \\ 1, & O_a : \text{fermionic generator} \end{cases} \ . \tag{2.12}$$

The commutation relations are given as $[B,B]=B$, $\{B,F\}=F$ and $\{F,F\}=B$, thus the extended commutation relations for the Super-Poincaré algebra is given as

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta, \quad [Q_\alpha, P^\mu]=0, \quad \{Q_\alpha, Q_\beta\}=0, \quad \{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \ . \tag{2.13}$$

We will now demonstrate an important property of the super-algebra using the last commutation rule in (2.13) together with introducing a fermionic operator $(-1)^F = (-)^F$ defined via

$$(-)^F|B\rangle = |B\rangle, \quad (-)^F|F\rangle = -|F\rangle \ . \tag{2.14}$$

The new fermionic operator anticommutes with Q_α *i.e*

$$(-)^F Q_\alpha|F\rangle = (-)^F|B\rangle = |B\rangle = Q_\alpha|F\rangle = -Q_\alpha(-)^F|F\rangle \rightarrow \{(-)^F, Q_\alpha\} = 0 \tag{2.15}$$

evaluation of the commutator above and taking the trace of it gives us

$$\mathrm{Tr}\left\{(-)^F\{Q_\alpha, \bar{Q}_\beta\}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} p_\mu \,\mathrm{Tr}\{(-)^F\} \ . \tag{2.16}$$

The conclusion is thus

$$0 = \mathrm{Tr}\{(-)^F\} = \sum_{bosons} \langle B|(-)^F|B\rangle + \sum_{fermions} \langle F|(-)^F|F\rangle = n_b - n_f \tag{2.17}$$

this is of course only valid if the Super-Poincaré algebra holds. If one does supersymmetry theories there is one rule that must always be satisfied and that is that the degree's of freedom for bosons are equal to the degree's of freedom of the fermions.

2.2 Cartan Formalism

In field theories only containing bosonic fields using a non-coordinate basis also known as vielbein is unnecessary since the fields are generally vectors or tensors. However once one introduces fermionic fields using a non-coordinate basis is a must since the spinors are defined by their special transformations properties under Lorentz transformations. The key references to this chapter will be [20] and [23].

In the coordinate basis, the tangent bundle $T_p\mathcal{M}$ is spanned by $e_\mu = \partial/\partial x^\mu$ and the cotangent bundle $T_p^*\mathcal{M}$ by dx^μ. Consider the linear combination

$$\hat{e_\alpha} = e_a{}^\mu \frac{\partial}{\partial x^\mu}, \quad e_\alpha{}^\mu \in GL(m,\mathbb{R}) \tag{2.18}$$

where $e_\alpha{}^\mu$ is non-degenerate, the linear combination means that the frame of basis which is obtained by a $GL(m,\mathbb{R})$-rotation of the vielbein basis $e_\alpha{}^\mu$ preserving the orientation. We require that the vielbein is orthonormal with respect to the metric $g_{\mu\nu}$ such that

$$g(\hat{e}_\alpha, \hat{e}_\beta) = e_\alpha{}^\mu e_\beta{}^\nu g_{\mu\nu} = \delta_{\alpha\beta} \ . \tag{2.19}$$

This relation forms an orthonormal set of $T_p\mathcal{M}$ where we have one basis for each tangent space. If the manifold is Lorentzian the right hand side of the equation above can be written as η_{ab}, therefore we can reverse the relation above. This allows us to use vielbeins to transform tensors and vectors back and forth between coordinate basis of the curved spacetime and the locally flat spacetime.

Since a vector V is independent of the chosen basis one can write the vector fields transformations as

$$V^\mu = V^\alpha e_\alpha{}^\mu, \quad V^\alpha = e^\mu{}_\alpha V^\mu \ . \tag{2.20}$$

The dual basis is defined by $\langle \hat{\theta}^\alpha, \hat{e}_\beta \rangle = \delta^\alpha{}_\beta$ where the basis is given as

$$\hat{\theta}^\alpha = e^\alpha{}_\mu dx^\mu \ , \tag{2.21}$$

these two coordinate basis we have defined are called the non-coordinate bases. The vielbein has a non-vanishing Lie bracket

$$[\hat{e}_\alpha, \hat{e}_\beta] = -\Omega_{\alpha\beta}{}^\gamma \hat{e}_\gamma \tag{2.22}$$

the $\Omega_{\alpha\beta}{}^\gamma$ are called anholonomy coefficients which are defined as

$$\Omega_{\alpha\beta}{}^\gamma = e^\gamma{}_\nu (e_\alpha{}^\mu \partial_\mu e_\beta{}^\nu - e_\beta{}^\mu \partial_\mu e_\alpha{}^\nu) \ . \tag{2.23}$$

The anholonomy coefficients come up when one introduces the spin connection, so let us indeed continue with connections on the manifold using the non-coordinate basis.

We start of with the Levi-Civita connection denoted by ∇ on the manifold $\mathcal{M}$ which is a metric compatible connection $\nabla_X g = 0$. The Levi-Civita connection is also torsion free which tells us that components of the connection also known as the Christoffel symbols $\Gamma^\lambda{}_{\mu\nu}$ are symmetric in the lower indices.

We can relate the Levi-Civita connection to the spin connection by the vielbein postulate

$$\partial_\mu \hat{e}^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\sigma_{\mu\nu} \hat{e}^a_\sigma = 0 \ , \tag{2.24}$$

if we were to write it in terms of the spin connection $\omega_\mu{}^a{}_b$ one can consider this the definition of the spin connection. This is not entirely true since the spin connection and the Christoffel symbols are not equivalent *i.e* the spin connection contains more information than the Christoffel symbols hence does not satisfy the metric compatibility. The affine connection consist of two parts, one that is symmmetric and one that is anti symmetric where the anti symmetric part gives rise to a torsion. If we consider the equation $\nabla_b e^a{}_\nu = 0$ such that

$$\partial_b e^a{}_\nu + \Gamma_b{}^a{}_c e^c_\nu + \omega_{b\nu\sigma} e^{a\sigma} = 0 \ , \tag{2.25}$$

multiplying with $e_a{}^\lambda$, gives us the correct spin connection definition.

Using the basis defined in (2.18) and taking the exterior derivative one obtains a two-form, defining the one-form for the spin connection given as $\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\gamma\beta} \hat{\theta}^\gamma$. From a formal definition of the torsion tensor given as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \ , \tag{2.26}$$

using the frame field basis and the dual basis one can write this in the following way

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta} - \Omega_{\beta\gamma}{}^\alpha \ . \tag{2.27}$$

As we mentioned before we now see how the anholonomy coefficients are connected to the spin connection. The one-form spin connection should satisfy the first Cartan structure equation

$$d\hat{\theta}^\alpha + \omega^\alpha{}_\beta \wedge \hat{\theta}^\beta = T^\alpha \tag{2.28}$$

one can see this if one lets the left hand side act with the basis vectors $\hat{e}_\gamma$ and $\hat{e}_\delta$. Note that if one sets the torsion to zero we once again obtain the Levi-Civita connection There is a similar structure equation for the curvature given as

$$d\omega^\alpha{}_\beta + \omega^\alpha{}_\beta \wedge \omega^\gamma{}_\beta = R^\alpha{}_\beta \ . \tag{2.29}$$

Since the spin connection describes spinors on your manifold, one can see how vital the role of using frame fields is.

If $\omega^{\alpha}{}_{\beta}$ is defined under a local transformation

$$\omega'^{\alpha}{}_{\beta} = \Lambda^{\alpha}{}_{\gamma}\omega^{\gamma}{}_{\delta}(\Lambda^{-1})^{\delta}{}_{\beta} + \Lambda^{\alpha}{}_{\gamma}(d\Lambda^{-1})^{\gamma}{}_{\beta} \tag{2.30}$$

which implies that the torsion transforms as a Lorentz vector and the components of the spin connection transforms as a covariant vector under coordinate transformations. The transformation above implies that

$$\omega_{\prime\kappa}{}^{\alpha}{}_{\beta} = \Lambda^{-1\alpha}{}_{\gamma}\partial_{\kappa}\Lambda^{\gamma}{}_{\beta} + \Lambda^{-1\alpha}{}_{\gamma}\omega_{\kappa}{}^{\gamma}{}_{\delta}\Lambda^{\delta}{}_{\beta} \tag{2.31}$$

where κ transforms under coordinate changes and the indices $\alpha, \beta, \gamma, \delta$ transforms under local orthogonal rotations, these are the gauge transformation properties of Yang-Mills potential. Since the metric is invariant under rotations $\Lambda^{\alpha}{}_{\beta}$ we have that $\Lambda^{\alpha}{}_{\beta}\delta_{\alpha\delta}\Lambda^{\delta}{}_{\gamma} = \eta_{\beta\gamma}$ with the corresponding gauge group of the Lorentzian manifold $SO(D-1,1)$.

One cannot do supergravity without fermions in the theory so it is natural to introduce a spinor field, as we will see in the next chapter when introducing the Rarita-Schwinger equation which is a spin-3/2 field that has $2^{[D/2]}$ components, before we go on let us just mention the covariant derivative. Since the spinors are described in their local frame components with the following transformation

$$\Psi' = \exp\left(-\frac{1}{4}\lambda^{ab}(x)\gamma_{ab}\right)\Psi(x) \tag{2.32}$$

thus giving us the covariant derivative defined as

$$D_{\mu}\Psi(x) = \left(\partial_{\mu} + \frac{1}{4}\omega_{\mu ab}(x)\gamma^{ab}\right)\Psi(x) \ . \tag{2.33}$$

Hence the covariant derivative contains the spin connection which we showed above transforms under Lorentz transformations

2.3 Rarita-Schwinger field

We consider the free spin-3/2 field $\Psi_{\mu}(x)$ with the following gauge transformation

$$\Psi_{\mu}(x) \to \Psi_{\mu}(x) + \partial_{\mu}\epsilon(x) \ , \tag{2.34}$$

where we assume that spinor field and gauge parameter ϵ are complex spinors with $(D-3)2^{[D/2]}$ spinor components for a given D dimensional spacetime.

We will now consider an action that is Lorentz invariant and invariant under the gauge transformation above. The action should also be first order in derivatives, the action can be written as

$$S = -\int d^{D}x\bar{\Psi}_{\mu}\gamma^{\mu\nu\rho}D_{\nu}\Psi_{\rho} \ . \tag{2.35}$$

where the $\gamma^{\mu\nu\rho}$ are gamma matrices satisfying the Clifford algebra

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}\mathbf{1} \ . \tag{2.36}$$

The gauge field Ψ_{μ} should have an anti-symmetric derivative of the gauge potential $D_{\mu}\Psi_{\nu} - D_{\nu}\Psi_{\mu}$. The reason we should have these anti-symmetric derivatives is because we need two connections one for the Lorentz connection for the flat indices and one connection for the curved indices. The role of these connections ensures that $D_{\mu}\Psi_{\nu}$ transforms as a tensor and the other connection should transform as the covariant vector like the spin connection.

The equations of motion for this action is given as

$$\gamma^{\mu\nu\rho}D_{\nu}\Psi_{\mu} = 0 \tag{2.37}$$

we thus have $D-1$ components in the D-dimensional Minkowski spacetime together with the spinor components $2^{[D/2]}$. This yields us $(D-1)2^{[D/2]}$ independent equations and this is indeed enough to determine the components of the spinor field Ψ_μ. The difference in the spinor field components compare to the gauge parameter components is called off-shell degree's of freedom.

We are interested in the on-shell degree's of freedom since we want the equations of motion to be satisfied and only contain real particles. The spin content of this field Ψ_i where Ψ_i transforms as spin $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ under $SU(2)$, however the spin-1/2 particles are killed by the constraints as we will show below this is desirble since the gravitino transforms as a vector-spinor. We are indeed left with $(D-3)2^{[D/2]}$ degree's of freedom where our field only contains one gravitino.

In order to evaluate the on-shell degree's of freedom one has to impose a gauge condition.

$$\gamma^i \Psi_i = 0 \ . \tag{2.38}$$

We will rewrite the components of the equations of motion as $\nu \to 0$ and $\nu \to i$, this gives us two equations of motion

$$\gamma^{ij} \partial_i \Psi_j = 0 \ , \tag{2.39}$$

$$-\gamma^0 \gamma^{ij} (\partial_0 \Psi_j - \partial_j \Psi_0) + \gamma^{ijk} \partial_j \Psi_k = 0 \ . \tag{2.40}$$

Using the Clifford algebra one can rewrite the first equation of motion gives us the following gauge condition

$$\partial^i \Psi_i = 0 \tag{2.41}$$

and for the second equation of motion one can multiply with γ_i to obtain a third gauge condition. The first two terms in the second equation of motion is zero due to Ψ_0 being a harmonic function satisfying the Laplace equation. Thus the third condition is given as

$$\gamma^\mu \partial_\mu \Psi_i = 0 \ . \tag{2.42}$$

The components of the spinor field Ψ_i satisfies the Dirac equation and the classical degree's of freedom are after imposing the gauge conditions is given by $(D-3)2^{[D/2]}$.

2.4 4D Supergravity in its first and second order formalism

Let us now start with formulating the first and second order formalism of supergravity. Supergravity in its simplest form should contain a kinetic term for the graviton and the Rarita-Schwinger field describing the gravitino. The action is thus given as

$$S_{SG} = \frac{1}{2\kappa^2} \int d^D x e \left(e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) \ . \tag{2.43}$$

The covariant derivative is the same as (2.33), and the supersymmetry transformations for the frame field and the gravitino are given as

$$\delta e^a{}_\mu = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu, \quad \delta \psi_\mu = D_\mu \epsilon \ . \tag{2.44}$$

We can check that these SUSY transformations are indeed the correct one by checking that the commutator of the supersymmetry algebra is satisfied. Two supersymmetry transformations should give rise to a suitable combination of local supersymmetry transformations that yields spacetime dependent vector field $\bar{\epsilon} \gamma^\mu \epsilon$.

We consider the commutator of two infinitesimal transformations of the vielbein which is given as

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e^a{}_\mu = D_\mu (\bar{\epsilon}_2 \gamma^a \epsilon_1) \tag{2.45}$$

where we have made use of the susy transformation for the gravitino field and the Majorana flip relation *i.e* $\bar{\lambda}\gamma_a\chi = -\bar{\chi}\gamma_a\lambda$. In order for us to understand what these transformations are we define the spacetime vector field $\xi^\mu = \bar{\epsilon}_2 e_a{}^\mu \gamma^a \epsilon_1$ together with a covariant derivative acting as a spacetime derivative on ξ^μ gives us the following commutation relation

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \xi^\nu D_\mu e_\nu{}^a + e_\nu{}^a \partial_\mu \xi^\nu \ . \tag{2.46}$$

We first consider the second order formalism of supergravity where the spin connection and the covariant derivative is torsion-free thus the vielbein postulate is given as

$$D_{[\mu} e_{\nu]}{}^a = T_{\mu\nu}{}^a \ . \tag{2.47}$$

Writing out the covariant derivative in the commutator and exchanging the lower indices in the first term gives us

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a = \xi^\nu \partial_\nu e_\mu{}^a + e_\nu{}^a \partial_\mu \xi^\nu + \xi^\nu \omega_\nu{}^a{}_b e_\mu{}^b \tag{2.48}$$

the first and the second terms are the infinitesimal action of a diffeomorphism on the frame field and the last term is the infinitesimal internal Lorentz transformation $\delta_\Lambda e_\mu{}^a$ where $\Lambda^a{}_b = \xi^\nu \omega_\nu{}^a{}_b$. Note that the first two terms are the Lie derivatives which transforms vectors and tensors correctly.

We can now show that the local supersymmetry transformations leaves the action invariant up to a linear order, notice that we did not check the commutator of two gravitino transformations as these will contain higher order terms. We can however neglect these terms since we consider infinitesimal variations of any equations of motion must give a linear combination of equations of motion. This means that the susy algebra closes on-shell.

In the Rarita-Schwinger action we have higher order fermionic terms these terms are generally quite hard to work with however one can neglect them if one uses a different formalism namely 1.5 formalism. We have first, third and fifth order fermionic terms, the higher order terms cancel mutually due to the variation of the Rarita-Schwinger field w.r.t the vielbein that comes from the contact term. We will now show the invariance of supergravity in detail.

The variation of the gravitational action is given as

$$\delta S_2 = \frac{1}{2\kappa^2} \int d^D x e \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) (-\bar{\epsilon}\gamma^\mu \psi^\nu) \ . \tag{2.49}$$

we obtain this by reducing the frame field susy transformation to

$$\delta e_a^\mu = -\frac{1}{2}\bar{\epsilon}\gamma^\mu \psi_a, \quad \delta e = \frac{1}{2}(\bar{\epsilon}\gamma^\rho \psi_\rho) \tag{2.50}$$

and ignore the total covariant derivative of $\delta R_{\mu\nu ab}$. We see that this is the Einstein field equation where the conserved symmetric fermion stress tensor $T_{\mu\nu}$ is set to zero since the fermionic equations of motion are satisfied.

We now continue with the variation of the gravitino field, since we are doing second order formalism now one can use partial integration since our connection is torsion-free. The variation for the gravitino field is given as

$$S_{3/2} = -\frac{1}{2\kappa^2} \int d^D x e \bar{\epsilon} \overleftarrow{D}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho = \frac{1}{4\kappa^2} \int d^D x e \bar{\epsilon} \gamma^{\mu\nu\rho} R_{\mu\nu ab} \gamma^{ab} \psi_\rho \tag{2.51}$$

where we partial integrated and used the Ricci identity $[D_\mu, D_\nu]\psi = \frac{1}{4} R_{\mu\nu ab} \gamma^{ab} \psi$. We have to use gamma-matrix manipulations in order to evaluate the product of the gamma

matrices. Useful tools for gamma-matrix algebra can be found in the appendix. The gamma matrix contraction with the Riemann tensor is given as

$$\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab} = \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 2R_{\mu\nu}{}^{\rho}{}_{b}\gamma^{\mu\nu\rho} + 4R_{\mu\nu}{}^{\mu}{}_{b}\gamma^{\nu\rho b} + 4\gamma^{\mu}R_{\mu\nu}{}^{\rho\nu} + 2\gamma^{\rho}R_{\mu\nu}{}^{\nu\mu} \quad (2.52)$$

the first and second term vanishes due to the Bianchi identity given as

$$R_{\mu\nu\rho}{}^{a} + R_{\nu\rho\mu}{}^{a} + R_{\rho\mu\nu}{}^{a} = 0 \ . \quad (2.53)$$

The third term vanishes due to the symmetry clash of the Ricci tensor and the gamma matrix $\gamma^{\mu\nu\rho}$, we end up with the following action

$$\delta S_{3/2} = \frac{1}{2\kappa^2}\int d^D x e(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(\bar{\epsilon}\gamma^{\mu}\psi^{\nu}) \quad (2.54)$$

and we see that both variations cancel each other and we have shown that the local supersymmetry holds for the linear terms in ψ_μ.

Let us now continue with the first order formalism of supergravity, where we now have bilinears in the spin connection which is now given as

$$\omega_{\mu ab} = \omega_{\mu ab}(e) - \frac{1}{4}(\bar{\psi}_{\mu}\gamma_{\rho}\psi_{\nu} - \bar{\psi}_{\nu}\gamma_{\mu}\psi_{\rho} + \bar{\psi}_{\rho}\gamma_{\nu}\psi_{\mu}) \quad (2.55)$$

the second part is also known as the contorsion tensor. Before we continue with the variation of the action using the first order formalism we need to mention a few things about infinitesimal transformations of the spin connection $\delta\omega_{\mu ab}$. Taking the exterior derivative of the first Cartan structure equation without torsion, one obtains the following relation

$$e^{a}_{\nu}e^{b}_{\rho}\delta\omega_{\mu ab} = (D_{[\mu}\delta e^{a}_{\nu]})e_{\rho a} - (D_{[\nu}\delta e^{a}_{\rho]})e_{\mu a} + (D_{[\rho}\delta e^{a}_{\mu]})e_{\nu a} \ . \quad (2.56)$$

Using this infinitesimal variation and plugging it into the Cartan structure equation for the curvature one obtains the following relation

$$\delta R_{\mu\nu ab} = D_{\mu}\delta\omega_{\nu ab} - D_{\nu}\delta\omega_{\mu ab} \quad (2.57)$$

the variation of the gravitational action using this relation is then given as

$$\delta S_2 = \frac{1}{2\kappa^2}\int d^D x \ e \ e^{\mu}_{a}e^{\nu}_{b}(D_{\mu}\delta\omega_{\nu}{}^{ab} - D_{\nu}\delta\omega_{\mu}{}^{ab}) \ . \quad (2.58)$$

This infinitesimal variation of the spin connection transforms as a tensor, hence we can change the Lorentz covariant derivative with a fully covariant derivatives if the torsion correction is present which it indeed is. The correction term will be related to the torsion by the connection from the field equations of $\omega_{\mu ab}$, thus giving us the following variation

$$\delta S_2 = \frac{1}{2\kappa^2}\int d^D x \ e \ e^{\mu}_{a}e^{\nu}_{b}\left(2\nabla_{\mu}\delta\omega_{\nu}{}^{ab} + T_{\mu\nu}{}^{\rho}\delta\omega_{\rho}{}^{ab}\right) \ . \quad (2.59)$$

Using the vielbein postulate where the covariant derivatives commute with the vielbeins and using the partial integration properties of the torsion-free connection one finds

$$\int d^D x\sqrt{-g}\nabla_{\mu}V^{\mu} = -\int d^D x\sqrt{-g}K_{\nu\mu}{}^{\nu}V^{\mu} \quad (2.60)$$

where we omitted the boundary term this term is proportional to $K_{\nu\mu}{}^{\nu} = -T_{\nu\mu}{}^{\nu}$ which exactly the correction term one needs. Hence our variation of the gravitational action becomes

$$\delta S_2 = \frac{1}{2\kappa^2}\int d^D x \ e \ \left(T_{\rho a}{}^{\rho}e^{\nu}_{b} - T_{\rho b}{}^{\rho}e^{\nu}_{a} + T_{ab}{}^{\nu}\right)\delta\omega_{\nu}{}^{ab} \ . \quad (2.61)$$

The variation of the of the gravitino field is given as

$$\delta S_{3/2} = \frac{1}{2\kappa^2}\int d^D x \left(\bar{\psi}\gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho\right)\delta\omega_\nu{}^{ab} \tag{2.62}$$

the gamma matrices are of rank 5,3,1 note however that the rank 3 terms vanish because of the antisymmetry in the gravitino indices μ, ρ. The gamma matrix manipulation is thus given as

$$\bar{\psi}_\mu\gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho = \bar{\psi}_\mu\left(\gamma^{\mu\nu\rho}{}_{ab} + 6\gamma^{[\mu}e^{\nu}{}_{[b}e^{\rho]}{}_{a}\right)\psi_\rho \ , \tag{2.63}$$

solving the equation $\delta S_2 + \delta S_{3/2} = 0$ one obtains that the solution for the torsion is given as

$$T_{ab}{}^{\nu} = \frac{1}{2}\bar{\psi}_a\gamma^\nu\psi_b + \frac{1}{4}\bar{\psi}_\mu\gamma^{\mu\nu\rho}{}_{ab}\psi_\rho \ . \tag{2.64}$$

We now see that indeed we have a connection with the torsion in the first formalism. We can rewrite the first order formalism into the second order formalism which yields us a theory that contains four-point gravitino scattering. These fermionic fields are necessary otherwise the theory would not be locally supersymmetric. The second order formalism using the torsion is given as

$$S = \frac{1}{2\kappa^2}\int d^4x \left(R(e) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + \mathcal{L}_{SG,torsion}\right) \tag{2.65}$$

where

$$\mathcal{L}_{SG,torsion} = -\frac{1}{16}\left((\bar{\psi}^\rho\gamma^\mu\psi^\nu)(\bar{\psi}_\rho\gamma_\mu\psi_\nu + 2\bar{\psi}_\rho\gamma_\nu\psi_\mu) - 4(\bar{\psi}_\mu\gamma\cdot\psi)(\bar{\psi}_\mu\gamma\cdot\psi)\right) \ , \tag{2.66}$$

this was first published in [24] where they showed that including the variation of fermionic terms of higher order preserves the local supersymmetry. However this can be seen to vanish if one considers the 1.5 formalism which is the formalism we will use from now on.

The 1.5 formalism neglects all the variations of the spin connection where one includes the contorsion tensor, the reason we neglect the variation of the spin connection is that the value of $\omega(e,\psi)$ is determined by its field equations. One solves the algebraic equation $\delta S/\delta\omega = 0$ and therefore only need to consider the following variation of the functional $S[e,\omega,\psi]$

$$\delta S = \int d^D x \left(\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta\psi}\delta\psi\right) \ . \tag{2.67}$$

Let us now show how $D = 4$ $\mathcal{N} = 1$ supergravity is locally supersymmetric using the 1.5 order formalism. We will use the same susy transformations as in the beginning of this section namely

$$\delta e^a{}_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \quad \delta\psi_\mu = D_\mu\epsilon \ . \tag{2.68}$$

We can simplify the gravitino action by introducing the highest rank Clifford element $\gamma_* = -i\gamma_0\gamma_1\gamma_2\gamma_3$, using this we can express the third rank Clifford matrices as

$$\gamma^{abc} = -i\epsilon^{abcd}\gamma_*\gamma_d, \quad \gamma^{\mu\nu\rho} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma \ . \tag{2.69}$$

The latin indices are for the local frames and the greek indices are for the coordinate basis, using these matrices our gravitino can be written as

$$S_{3/2} = \frac{i}{2\kappa^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_*\gamma_\sigma D_\nu\psi_\rho \ . \tag{2.70}$$

Since we are working in the 1.5 order formalism our variations will be

$$\delta S = \delta S_2 + \delta S_{3/2,e} + \delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}} \tag{2.71}$$

the variation of the gravitational action was obtained in (2.49), and the second term in the variation is obtained in the same way as we did in the second order formalism and is given as

$$\delta S_{3/2,e} = \frac{i}{4\kappa^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}\gamma^a\psi_\sigma)(\bar{\psi}_\mu\gamma_*\gamma_a D_\nu\psi_\rho)\ . \tag{2.72}$$

Now we consider the new variations w.r.t to the spinor fields, the variation is given using the susy transformation for the spinor field as

$$\delta S_{3/2,\psi} = \frac{i}{16\kappa^2}\int d^4x\bar{\psi}_\mu\epsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma\gamma^{ab}R_{\mu\nu ab}(\omega)\epsilon \tag{2.73}$$

where we shifted the covariant derivative after partial integration and made use of the Ricci identity. The last term is the variation of the Rarita-Schwinger equation w.r.t $\bar{\psi}$ this is given as

$$\delta S_{3/2,\bar{\psi}} = \frac{i}{2\kappa^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\overleftarrow{D}_\nu\gamma_*\gamma_\sigma D_\mu\epsilon\ , \tag{2.74}$$

using the covariant

$$\bar{\psi}_\rho\overleftarrow{D}_\nu = \partial_\nu\bar{\psi}_\rho - \frac{1}{4}\bar{\psi}_\rho\omega_{\nu ab}\gamma^{ab}\ , \tag{2.75}$$

once again we use partial integration and shift the covariant derivatives we also made use of the Majorana flip relation. The variation of the $\bar{\psi}$ can now be written

$$\delta S_{3/2,\bar{\psi}} = -\frac{i}{2\kappa^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_*\left((D_\nu\gamma_\sigma) - \frac{1}{8}\gamma_\sigma\gamma^{ab}R_{\mu\nu ab}(w)\epsilon\right)\ , \tag{2.76}$$

we see that both the variation on the spinor field contains curvature tensors therefore we can write them together as

$$\delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}} = -\frac{i}{2\kappa^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_*\left(\frac{1}{2}T_{\nu\sigma}{}^a\gamma_a D_\mu\epsilon - \frac{1}{4}\gamma_\sigma\gamma_{ab}R_{\mu\nu}{}^{ab}\epsilon\right)\ . \tag{2.77}$$

Few remarks on the torsion term in the variation, we have that the covariant derivative of γ_ν vanishes this can be seen using the vielbein postulate and this holds for an affine connection if one makes use of the total covariant derivative. However since we only make use of the Lorentz covariant derivative we have to add the Christoffel symbols and use the antisymmetry in the lower indices to obtain the torsion.

We have to evaluate the gamma matrix manipulations now in order to evaluate the variation fully, note that the gamma matrices in front of the curvature tensor can be written as

$$\gamma_\sigma\gamma_{ab} = ie^d_\sigma\epsilon_{abcd}\gamma_*\gamma^c + 2e_{\sigma[a}\gamma_{b]}\ . \tag{2.78}$$

Using this relation one can simplify the curvature tensor term as following

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e^d_\sigma R_{\nu\rho}{}^{ab} = 4e\left(R_c{}^\mu(\omega) - \frac{1}{2}e^\mu_c R(\omega)\right)\ . \tag{2.79}$$

Using this and the Majorana flip relation $\bar{\lambda}\gamma_a\chi = -\bar{\chi}\gamma_a\lambda$ and the rewritten curvature term we see that this term cancels with the gravitational term (2.49). We can now shown that supergravity in $D = 4$ is locally supersymmetric to the linear order.

We are left with the higher order terms that needs to cancel out, the second term in the gamma matrix manipulation above can be written as

$$\epsilon^{\mu\nu\rho\sigma}R_{\nu\rho\sigma b}(\omega) = -\epsilon^{\mu\nu\rho\sigma}D_\nu T_{\rho\sigma b} \tag{2.80}$$

where we have used the modified Bianchi identity $R_{(\mu\nu\rho)}{}^a = -D_{(\mu}T_{\nu\rho)}{}^a$, where the derivative is a Lorentz derivative containing the spin connection. The surviving terms of the variation w.r.t the spinor fields is given as

$$\delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}} = -\frac{i}{4\kappa^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_*\gamma_a(T_{\rho\sigma}{}^a D_\nu\epsilon + (D_\nu T_{\rho\sigma}{}^a)\epsilon)\ . \tag{2.81}$$

We have to now evaluate the $\delta S_{3/2,e}$ variation using Fierz rearrangement given as

$$(\gamma^{\mu})_{\alpha}{}^{\beta}(\gamma_{\mu})_{\gamma}{}^{\delta} = \frac{1}{2^m}\sum_{A} v_A(\Gamma_A)_{\alpha}{}^{\beta}(\Gamma^A)_{\gamma}{}^{\beta} \ , \tag{2.82}$$

where the expansion coefficients $v_A = (-)^{r_A}(D - 2r_A)$, r_A is the tensor rank of the Clifford basis element Γ_A. Applying this to the variation gives us

$$(\bar{\epsilon}\gamma^a\psi_{\sigma})(\bar{\psi}_{\mu}\gamma_*\gamma_a D_{\nu}\psi_{\rho}) = (\bar{\epsilon}\gamma_*\gamma_a D_{\nu}\psi_{\rho})T_{\mu\nu}{}^{a} \ , \tag{2.83}$$

where we have used the first order formalism torsion tensor we obtained before and used the antisymmetry properties of $\mu\sigma$ in the variation of the action. Changing the indices $\mu\rho$ we can rewrite the $\delta S_{3/2,e}$ term as

$$\delta S_{3/2,e} = -\frac{i}{4\kappa^2}\int d^4x \epsilon^{\mu\nu\rho\sigma} T_{\rho\sigma}{}^{a}\bar{\psi}_{\mu}\overleftarrow{D}_{\nu}\gamma_*\gamma_a\epsilon \ , \tag{2.84}$$

together with the surviving terms of the variation $\delta S_{3/2,\psi}, \delta S_{3/2,\bar{\psi}}$ we have

$$\delta S = -\frac{i}{4\kappa^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\left(T_{\rho\sigma}{}^{a}(\bar{\psi}_{\mu}\gamma_*\gamma_a D_{\nu}\epsilon + \bar{\psi}_{\mu}\overleftarrow{D}_{\nu}\gamma_*\gamma_b\epsilon) + D_{\nu}T_{\rho\sigma}{}^{b}\bar{\psi}_{\mu}\gamma_*\gamma_b\epsilon\right) \ . \tag{2.85}$$

Plugging in the definitions of the covariant derivative we see that the spin connection components of the covariant derivative cancels among them self and we are left with a total derivative, hence

$$\delta S = -\frac{i}{4\kappa^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\partial_{\nu}\left(T_{\rho\sigma}{}^{b}\bar{\psi}_{\mu}\gamma_*\gamma_b\epsilon\right) = 0 \ . \tag{2.86}$$

This shows that the $D = 4$, $\mathcal{N} = 1$ supergravity theory is locally supersymmetric, the same can be shown in all dimensions of supergravity with a few modifications. This sums up supergravity and its tools.

Chapter 3

Compactifications

We will consider $\mathcal{N}=1$ supergravity in 11 dimensions, where the field content consist of vierbein e_M^A a Majorana anti-commuting spin-3/2 field ψ_M and a antisymmetric 3-form gauge tensor A_{KLM}. From supersymmetry we know that we must have equal amounts of fermions as bosons so therefore our field content should obey this rule. We can calculate the field content by counting the state *i.e*

$$e_\mu^A = \frac{D(D-3)}{2}, \quad \psi_M = \frac{(D-3)2^{[D/2]}}{2}, \quad A_{KLM} = \binom{D-2}{p} \tag{3.1}$$

where p is the rank of the p-form in our case a 3-form, notice that the anti-commuting field only takes integer values. By just counting the states for the 11 dimensional supergravity we see that the we have 128 fermionic degrees of freedom and 128 bosonic degrees of freedom, where we have 44 from the vierbein and 84 from the 3-form gauge potential. The full Lagrangian including fermions and bosons for the supergravity [1] is given as

$$\begin{aligned}\mathcal{L} = \sqrt{-g}\Big(&\frac{1}{4}g^{TS}R_{Ts}(\hat{\omega}) - \frac{1}{2}\bar{\Psi}_M\Gamma^{MNP}D_N\left(\frac{\hat{\omega}+\omega}{2}\right)\psi_P - \frac{1}{24}F_{MNPQ}F^{MNPQ} \\ &+\frac{2^2}{12^4}\varepsilon^{MNPQVWXYUSR}F_{MNPQ}F^{VWXY}A_{USR} \\ &+\frac{3}{4\cdot 12^2}\left(\bar{\Psi}_M\Gamma^{MNPQVW}\Psi_N + 12\bar{\Psi}^W\Gamma^{XY}\Psi^Z\right)\left(F_{PQVW}+\hat{F}_{PQVW}\right)\Big)\,.\end{aligned} \tag{3.2}$$

We can obtain the equations of motion for this Lagrangian by considering variations w.r.t metric the 3-form gauge potential and the anti-commuting gravitino field, the equations of motion are given as

$$\Gamma^{RST}\hat{D}_S(\hat{\omega})\Psi_T = 0 \tag{3.3}$$

$$D_T(\hat{\omega})\hat{F}^{TURS} = (24)^{-2}\varepsilon^{MNPQVWXYUSR}\hat{F}_{MNPQ}\hat{F}_{VWXY} \tag{3.4}$$

$$R_{TS}(\hat{\omega}) - \frac{1}{2}g_{TS}R(\hat{\omega}) = \frac{1}{24}\left(g_{TS}\hat{F}_{MNPQ}\hat{F}^{MNPT} - 8\hat{F}_{MNPT}\hat{F}_S^{MNP}\right)\,. \tag{3.5}$$

The super-covariant derivative and the covariant derivative is given as

$$\hat{D}_S(\hat{\omega})\Psi_T = D_S(\hat{\omega})\Psi_T + T_S^{MNPQ}\hat{F}_{MNPQ}\Psi_T \tag{3.6}$$

$$D_S(\hat{\omega})\Psi_T = \partial_S\Psi_T + \frac{1}{4}\hat{\omega}_{SAB}\Gamma^{AB}\Psi_T\,. \tag{3.7}$$

The notation

$$T^{SMNPQ} = (12)^{-2}\left(\Gamma^{SMNPQ} - 8\Gamma^{[MNP}\eta^{Q]S}\right) \tag{3.8}$$

used in the supercovariant derivative is coming from the gamma matrix manipulations in the fermionic part of the Lagrangian as one can see this only appears in the gravitino EOM. The supercovariant spin connection and the supercovariant field strengths are given as

$$\hat{\omega}_{MRS} = \omega_{MRS}(e_M^A) + \frac{i}{2}\left(2\bar{\Psi}_M\Gamma_{[S}\Psi_{R]} + \bar{\Psi}_S\Gamma_M\Psi_R\right) \tag{3.9}$$

$$\hat{F}_{MNPQ} = F_{MNPQ} - 3\bar{\Psi}_{[M}\Gamma_{NP}\Psi_{Q]} \tag{3.10}$$

where the un-hatted field strength tensor is just the regular 4-form. Looking at the spin-connection term (1.9) the second term is the contorsion tensor, the torsion tensor is given as

$$T_{MNA} = \frac{i}{2}\bar{\Psi}_B\Gamma_{MNA}{}^{BD}\Psi_D - i\bar{\Psi}_M\Gamma_A\Psi_N \ . \tag{3.11}$$

The first term is totally antisymmetric while the second term is antisymmetric in only M and N. The capital letters $(A, B, \ldots = 0, \ldots, 10)$ refer to the tangent space indices and $(K, L, \ldots = 0, \ldots, 10)$ refer to the world indices. We are also using the following metric signature $(+, -, \ldots, -)$, our eleven dimensional Γ-matrices are in the Majorana representation and they form a imaginary representation of the Clifford algebra in 11 dimensions. The Γ-matrices satisfy the following condition

$$\{\Gamma_A, \Gamma_B\} = 2\eta^{AB}\mathbf{1}_{32} \ , \tag{3.12}$$

where η^{AB} is the metric. The spinors in our Lagrangian satisfy the Majorana condition

$$\Gamma_M^\dagger\Gamma^0 = \Gamma_M^T C \ , \tag{3.13}$$

where C is the charge conjugation matrix. Since 11 dimensional supergravity is a maximally supersymmetric theory the Majorana spinor has 32 real components hence the identity matrix in (1.12). When we consider dimensional reductions of the 11 dimensional supergravity theory we split the spinor components in their respective submanifolds, this will be clearer when we consider compactifications including fermionic parts of the Lagrangian. The supergravity equations of motion are invariant under the local supersymmetry transformations

$$\delta e_M^A = -i\bar{\varepsilon}\Gamma^A\Psi_M \tag{3.14}$$

$$\delta\Psi_M = D_M(\hat{\omega})\varepsilon - \frac{i}{144}\left(\Gamma^{NPQR}{}_M + 8\Gamma^{PQR}\delta_M^N\right)\hat{F}_{NPQR}\varepsilon \tag{3.15}$$

$$\delta A_{MNP} = \frac{3}{2}\bar{\varepsilon}\Gamma_{[MN}\Psi_{P]} \ , \tag{3.16}$$

where ε is the fermionic supersymmetry transformation parameter.

3.1 Toroidal compactification

Let us consider a simple dimensional reduction, where we want to compactify the extra dimensions on a 7-torus which is a product of seven S^1. We will only consider the bosonic part of supergravity in this chapter the Lagrangian is given as

$$\mathcal{L} = R * \mathbb{1} - \frac{1}{48} * \hat{F}_4 \wedge \hat{F}_4 + \frac{1}{6}\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3 \ . \tag{3.17}$$

Let us first mention a few things about doing dimensional reductions on the field strength tensors assuming we have $(D+1)$ dimensions and want to reduce it to D dimensions. The field strength tensor with n-index is defined as $\hat{F}_n = d\hat{A}_{n-1}$ so for each time we

reduce the dimension we obtain a $(n-1)$ gauge potential which lives in the D dimensions spacetime and the $(n-2)$ potential lives on S^1 direction. The gauge potential can be expressed as

$$\hat{A}_{n-1}(x,z) = A_{n-1}(x) + A_{n-2}(x) \wedge dz \tag{3.18}$$

and the field strength tensor is given as

$$F_n = dA_{n-1} - dA_{n-2} \wedge \mathcal{A}_1 \ . \tag{3.19}$$

As a side note before actually considering the toroidal compactification on 11 dimensional supergravity, let us consider the $(D+1)$ bosonic Lagrangian which takes the form

$$\mathcal{L} = \hat{e}\mathcal{R} - \frac{1}{2}\hat{e}(\partial\phi)^2 - \frac{1}{2n}\hat{e}e^{\hat{a}\cdot\phi}\hat{F}_n^2 \ , \tag{3.20}$$

using the equations above we can obtain the dimensionally reduced Lagrangian given as

$$\begin{aligned}\mathcal{L} =& e\mathcal{R} - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{2}(\partial\psi)^2 - \frac{1}{4}ee^{-2(D-1)a\phi}\mathcal{F}^2 \\ &- \frac{1}{2n}ee^{-2(n-1)a\psi+\hat{a}\phi}F_n^2 - \frac{1}{2(n-1)}ee^{2(D-n)a\psi+\hat{a}\phi}F_{n-1}^2 \ .\end{aligned} \tag{3.21}$$

We can use this reduction on our 11 dimensional supergravity theory with some modifications, notice that the dilaton vectors appear which live on the S^1 direction. We can now consider the bosonic part of our supergravity theory (1.17) doing the dimensional reduction of our kinetic term yields us

$$\frac{1}{24}*F_4\wedge F_4 = \frac{1}{24}e^{\vec{a}\cdot\vec{\phi}}*F_4\wedge F_4 + \frac{1}{6}\sum_i e^{\vec{a}_i\cdot\vec{\phi}}*F_3^i\wedge F_3^i + \frac{1}{2}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}*F_2^{ij}\wedge F_2^{ij} + \sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}*F_1^{ijk}\wedge F_1^{ijk} \tag{3.22}$$

where the $\vec{a}$ terms are the dilaton vectors coming from the dimensional reduction of the p-forms. Dilaton vectors only appear when one does compactifications on the kinetic terms this is why we do not see them in the bosonic action in 11 dimensions. For each time we reduce the field strength tensor we obtain a n-form field strength from the n-form in $(D+1)$ dimension and an $(n-1)$-form from the n-form field strength. We see that the lower the p-form is from the reduction the more dilaton vectors appear which one can see in (2.22). The index of the dilaton vectors labels the internal $(11-D)$ indices in D dimensions. We also obtain vielbein dilaton scalars which come from the dimensional reduction of the vielbein, these are given as

$$\frac{1}{24} * F_4 \wedge F_4 = \frac{1}{2}\sum_i e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}_2^i \wedge \mathcal{F}_2^i + \sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} * \mathcal{F}_1^{ij} \wedge \mathcal{F}_1^{ij} \ . \tag{3.23}$$

The last term is the Chern-Simons term which one deals with separately. One can write out the dimensional reductions for each field strength and for the gauge potential which are given as

$$\hat{F}_4 = \tilde{F}_4 + \tilde{F}_3^i \wedge dz^i - \frac{1}{2}\tilde{F}_2^{ij} \wedge dz^i \wedge dz^j - \frac{1}{6}\tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \tag{3.24}$$

$$\hat{F}_4 = \tilde{F}_4 + \tilde{F}_3^l \wedge dz^l - \frac{1}{2}\tilde{F}_2^{lm} \wedge dz^l \wedge dz^m - \frac{1}{6}\tilde{F}_1^{lmn} \wedge dz^l \wedge dz^m \wedge dz^n \tag{3.25}$$

$$\hat{A}_3 = A_3 + A_2^p \wedge dz^p - \frac{1}{2}A_1^{pq} \wedge dz^p \wedge dz^q - \frac{1}{6}A_0^{pqr} \wedge dz^p \wedge dz^q \wedge dz^r \ . \tag{3.26}$$

This is a very messy calculation since we are dealing with 64 terms in total. In order to obtain all the Chern-Simons terms for each dimension going from 11 dimensions to 4 dimensions one can see that the p-forms should have the same rank as the dimension

we are evaluating for. Therefore we choose the appropriate p-forms for each reduction and perform a partial integration over the volume where one also uses properties for differential forms namely

$$d(w^p \wedge w^q) = dw^p \wedge w^q + (-1)^p w^p \wedge dw^q \ . \tag{3.27}$$

The Chern-Simons terms for all the dimensions are given in [25]. Either way our dimensionally reduced Lagrangian for the bosonic supergravity is given by

$$\begin{aligned}\mathcal{L} = R * \mathbb{1} - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{48}ee^{\vec{a}\cdot\vec{\phi}} * F_4 \wedge F_4 - \frac{1}{12}\sum_i e^{\vec{a}_i\cdot\vec{\phi}} * F_3^i \wedge F_3^i - \frac{1}{4}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} * F_2^{ij} \wedge F_2^{ij} \\ - \frac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}} * F_1^{ijk} \wedge F_1^{ijk} - \frac{1}{4}\sum_i e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}_2^i \wedge \mathcal{F}_2^i - \frac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} \mathcal{F}_1^{ij} \wedge \mathcal{F}_1^{ij} + \mathcal{L}_{FFA} \ .\end{aligned} \tag{3.28}$$

If we go all the way down to 4 dimensions which is ideally what we want, we obtain a maximally supersymmetric supergravity with 32 supercharges *i.e* $\mathcal{N} = 8$. The field content should give us 128 bosons and 128 fermions. We immediately obtain the correct amount of fermions from counting the states the rest of the field content will come from our gauge potential $A_{(3)}$ and the metric. From the metric we obtain one graviton $g_{\mu\nu}$, seven vectors $g_{\mu i}$ and 28 scalars g_{ij}. The Majorana field is decomposed into 8 gravitinos and 56 half spin fields. Like we have stated previously we need equal amounts of bosonic and fermionic degrees of freedom and the rest of the bosons are coming from the 3-form gauge potential and its dual potentials. The gauge potential $A_{\mu ij}$ gives us 21 scalars and A_{ijk} yields us 35 scalars where as the dual potentials of these fields gives us the corresponding pseudoscalars. The last 7 scalars are coming from the $A_{\mu\nu i}$ potential and this is the entire field content of the toroidal compactification of bosonic supergravity.

We have seen how Kaluza-Klein compactifications are done, however there are different compactifications which will be of more use to us namely Freund-Rubin compactifications.

3.2 Freund-Rubin compactifications

We will now consider Freund-Rubin compactifications where instead of doing compactifications starting from the Lagrangian like we saw in the previous chapter, we instead do compactifications at the level of the equations of motion [4].

We begin with a simple example where we consider Einstein-Maxwell theory in D dimensions the equations of motion is given as

$$R^{MN} - \frac{1}{2}g^{MN}R = -8\pi G\Theta^{MN} \tag{3.29}$$

$$\frac{1}{\sqrt{|g|}}\partial_M\left(\sqrt{|g|}F^{MN}\right) = 0 \tag{3.30}$$

where F^{MN} is a totally antisymmetric tensor of rank $s-1$. The corresponding antisymmetric field strength tensor will be of rank s, the energy-momentum tensor is then given as

$$\Theta^{MN} = F_{L_1\ldots L_{s-1}}{}^M F^{L_1\ldots L_{s-1}N} - \frac{1}{2s!}F_{L_1\ldots L_s}F^{L_1\ldots L_s}g^{MN} \ . \tag{3.31}$$

We are interesting in finding solutions of the equations of motion such that (1.29) can be split into a s-dimensional manifold and of a $(d-s)$-dimensional Riemann manifold

i.e $\mathcal{M} = \mathcal{M}_s \times \mathcal{M}_{(d-s)}$. The flux field is proportional to the Levi-Civita tensor on the $\mathcal{M}_s$ manifold. The Maxwell equation admits the following solution

$$F^{M_1 \dots M_s} = \frac{1}{\sqrt{|g_s|}} \varepsilon^{M_1 \dots M_s} f \tag{3.32}$$

$$\varepsilon^{M_1 \dots M_s} = \begin{cases} \varepsilon^{\mu_1 \dots \mu_s}, & \text{for} \quad M_1 = \mu_1, \dots, M_s = \mu_s \\ 0, & \text{otherwise} \ . \end{cases} \tag{3.33}$$

where f is a constant with the dimension mass squared. Notice that the second term in θ^{MN} becomes a constant cosmological term meanwhile the first term vanishes except for when $M = \mu$ and $N = \nu$ and is proportional to $g^{\mu\nu}$. The scalar term is given as

$$F_{L_1 \dots L_s} F^{L_1 \dots L_s} = s! f^2 \operatorname{sgn}(g_s) \ . \tag{3.34}$$

The source term will have different cosmological terms with different cosmological constants on our sub manifolds. We can now find what the Einstein equations reduces too by defining R_s and R_{d-s}. We begin with contracting the Einstein field equations with the metric g_{MN} let us also drop the constant $8\pi G$ for now

$$g_{MN} R^{MN} - \frac{1}{2} g^{MN} g_{MN} R = g_{MN} F_{L_1 \dots L_{s-1}}{}^{M} F^{L_1 \dots L_{s-1} N} - g_{GM} \frac{1}{2s!} F_{L_1 \dots L_s} F^{L_1 \dots L_s} g^{MN} \tag{3.35}$$

using equation (1.35) and $g_{MN} g^{MN} = D$ we obtain

$$R(2 - D) = 2 f^2 \operatorname{sgn}(g_s) \varepsilon_{\ell_1 \dots \ell_{s-1} \nu} \varepsilon^{\ell_1 \dots \ell_{s-1} \nu} - \frac{D}{s!} f^2 \operatorname{sgn}(g_s) \varepsilon_{\ell_1 \dots \ell_s} \varepsilon^{\ell_1 \dots \ell_s} \ . \tag{3.36}$$

Rewriting this and contracting the levi-civita tensor gives us

$$R = -f^2 \operatorname{sgn}(g) \frac{D - 2s!}{D - 2} \tag{3.37}$$

this is the Ricci scalar. We can contract the Einstein field equations with the metric ansatz using the Ricci scalar let where we first consider the $\mathcal{M}_{d-s}$ manifold

$$g_{mn} R^{mn}_{D-s} - \frac{1}{2} g_{mn} g^{mn} R = g_{mn} F_{L_1 \dots L_{s-1}}{}^{m} F^{L_1 \dots L_{s-1} n} - g_{mn} \frac{1}{2s!} F_{L_1 \dots L_{s-1}} F^{L_1 \dots L_{s-1}} g^{mn} \tag{3.38}$$

notice that the first term vanishes now since $M \neq \mu$ and $N \neq \nu$. We eventually end up with

$$R_{D-s} = f^2 \operatorname{sgn}(g_s) \frac{(s! - 1)(D - s)}{D - 2} \ , \tag{3.39}$$

and the Ricci scalar for the manifold $\mathcal{M}_s$ when contracting with $g_{\mu\nu}$ gives us

$$R_s = -f^2 \operatorname{sgn}(g_s) \frac{s!(d - s! - 1)}{d - 2} \ . \tag{3.40}$$

Notice that R_{D-s} and R_s have different signs this tells us that the manifold $\mathcal{M}_s$ is a compact manifold with negative scalar curvature this is because the time-dimension lives in this manifold. The manifold $\mathcal{M}_{D-s}$ has a euclidean signature which satisfies the condition that a Riemann manifold is positive definite. For $d > s + 1$ it follows that $G f^2 > 0$ which tells us that the Ricci scalars have different signs and R_s has the same sign as g_s, hence when $\mathcal{M}_s$ has the time dimension then $\mathcal{M}_{D-s}$ is compact and vice versa. In 11 dimensions we will have compactification on either 7 or 4 space-like dimensions where the large space-time *i.e* has one time and $(D - s)$ space dimensions.

Our solution to Maxwell equations only exists if we can split our manifold to $\mathcal{M}_s \times \mathcal{M}_{D-s}$. Without supersymmetry one could choose a arbitrary s-form and we would not

be able to obtain information about the compactified dimensions. In 11 dimensional supergravity one has a 4-form field strength tensor which assures the solution of the Maxwell equation. We are only interested in 4-dimensional theories which admits maximal symmetry vacuum solutions. Since we are using mostly minus metric the vacuum should be invariant under $SO(4,1)$ which is de Sitter space, $SO(3,2)$ which is anti de Sitter space and the Poincaré group is for Minkowski space.

A requirement of maximal symmetry is that the vacuum expectation value of any fermion field should vanish and we set it to

$$\langle \Psi^A \rangle = 0 \ . \tag{3.41}$$

Spontaneous compactifications of 11 dimensional supergravity is natural framework of Freund-Rubin compactifications, the vacuum for these type of compactifications suffer from huge cosmological constants due to the reason of only considering the bosonic sectors of the Lagrangian.

We can however introduce non-vanishing fermionic fields which will lead to flat Minkowski vacuum due to the introduction of the torsion that flattens the geometry. Since our gravitino field is anti-commuting its classical limit vanishes we cannot have a classical gravitino field. We can however form bilinears and quadlinears which has a classical interpretation therefore there is a possibility that some bilinears will be non-zero [5, 6] *i.e*

$$\langle \Psi^A \Gamma^{B_1} \dots \Gamma^{B_n} \Psi^C \rangle \neq 0 \ . \tag{3.42}$$

We will consider the compactifications that admit a Minkowski vacuum, which were found in [5, 6, 26], where one considers a direct product of seven dimensional compact space and four dimensional Minkowski spacetime.

The metric ansatz for these type of compactifications is given as

$$\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}, \quad \langle g_{mn} \rangle = g_{mn}, \quad \langle g_{m\nu} \rangle = \langle g_{\mu n} \rangle = 0 \tag{3.43}$$

where our space-time coordinates are $M = (\mu, m)$ where the Greek indices are the four dimensional world indices and the Latin indices are our compact seven dimensional world indices. The supercovariant 4-form field strength vanishes

$$\langle \hat{F}_{KLMN} \rangle = 0 \ , \tag{3.44}$$

this is a result from the Bianchi identity and the fact that the supercovariant field strength tensor transforms covariantly under supersymmetry. Note that (3.10) does not imply the vanishing of the supercovariant field strength due to Lorentz invariance in four dimensions however we have non-vanishing bilinears given as

$$\begin{aligned} \langle \bar{\Psi}_{[\mu} \Gamma_{\nu\rho} \Psi_{\sigma]} \rangle &= \frac{1}{3} \sqrt{|g_4|} \varepsilon_{\mu\nu\rho\sigma} \\ \langle \bar{\Psi}_{[m} \Gamma_{np} \Psi_{q]} \rangle &= \frac{1}{3} f_{mnpq} \ . \end{aligned} \tag{3.45}$$

We will split the gravitino field as

$$\Psi^A = (\Psi^\alpha, \Psi^a) \tag{3.46}$$

here the Latin indices are the 1/2-spin fermions and the Greek indices are the 3/2-spin fermions. We have now included fermions into our compactification therefore we need to have a non-vanishing torsion which is fully anti-symmetric on the internal space. Our torsion defined in (3.11) is such that the first term is totally anti-symmetric while the second term is only antisymmetric in M and N.

Before we investigate further on the torsion term let us first define a parallelizable connection since we are interested in obtaining Minkowski vacuum we have to set our curvature tensor to zero *i.e*

$$R_{MN}(\hat{\omega}) = 0 \ . \tag{3.47}$$

We consider the vielbein formalism in order to find the solution, we denote $W_{SK}{}^{RT}$ the tangent space connection associated with spin connection $\hat{\omega}$ the vielbein postulate is given as

$$\nabla_K e^R{}_L = \partial_K e^R_L - W_{KL}{}^R + W_{SKL}{}^R = 0 \tag{3.48}$$

the curvature of W_S is zero due to

$$R_{KLM}{}^N(\Gamma) = E^T_M E^N_U R_{KLT}{}^U(\omega) \ . \tag{3.49}$$

Notice now that the spin-connection will not necessarily be zero however our covariant derivative will take the form

$$\Gamma^M{}_{KL} = E^K_A \partial_K E^A_L \ . \tag{3.50}$$

This is the Weitzenböck connection whose curvature vanishes identically [27]. By setting the spin connection to zero one obtains that the spin connection is given entirely in terms of the gravitino bilinears

$$\omega_{ABC}(e^A{}_M) = -\frac{i}{2}\left(2\bar{\Psi}_A \Gamma_{[C}\Psi_{B]} + \bar{\Psi}_C \Gamma_A \Psi_B\right) \tag{3.51}$$

and we can write the spin connection in terms of its torsion tensor [26]

$$\omega_{ABC} = -(T_{ABC} - T_{ACB} + T_{CAB}) \ . \tag{3.52}$$

The main property of the Weitzenböck connection is that parallel transport is path-independent and it is possible to define parallelism of vectors at different spacetime points. Notice now that the torsion we defined in (3.11) is slightly different from the one we just obtained, the first term in the torsion we defined vanishes since it gets canceled with other terms in the 11 dimensional supergravity Lagrangian. Also notice that it does not appear in the equations of motion, one last remark about the first term is that it is trivial for a spin-connection when considering a compactification with Minkowski vacuum

$$T^A{}_{MN} = -i\bar{\Psi}_M \Gamma^A \Psi_N \tag{3.53}$$

The vanishing of the supercovariant spin-connection simplifies the equation of motion for the gravitino field, which is now given as

$$\Gamma^{ABC}\partial_B \Psi_C = 0 \ . \tag{3.54}$$

The solution for this equations of motion are provided by a constant anti-commuting Majorana fermions Ψ_A, *i.e*

$$\Psi_A(x) = \Psi_A \tag{3.55}$$

the equations of motion (3.5) is also satisfied because of (3.44). The gravitational equation is split into

$$R_{\mu\nu}(\hat{\omega}) = 0, \quad R_{mn}(\hat{\omega}) = 0 \tag{3.56}$$

this is always satisfied since the spin connection is zero. We are interested in a flat Minkowski metric that has a vanishing connection and torsion. Therefore we need to solve the curvature equation $R_{mn}(\hat{\omega})$ which will tell us something about the seven dimensional geometries. We will consider both the fermionic and bosonic degree's of freedom in our compactification.

The supersymmetry transformations for this framework is given as

$$\delta e_M^A = -i\bar{\varepsilon}\Gamma^A\Psi_M \ , \tag{3.57}$$
$$\delta\Psi_A = \partial_A\varepsilon \ , \tag{3.58}$$
$$\delta A_{ABC} = \frac{3}{2}\bar{\varepsilon}\Gamma_{[AB}\Psi_{C]} \ . \tag{3.59}$$

Supersymmetry is preserved when all the fermionic degree's of freedom are annihilated by the vacuum. Let us now investigate some solutions coming from the compactification.

3.3 Spontaneous compactifications

We consider the Freund-Rubin ansatz for the 3-form field in the external space, whereas the 4-form field strength takes the following form

$$F_{\alpha\beta\gamma\delta} = 6m_0\varepsilon_{\alpha\beta\gamma\delta} \ , \quad \alpha,\beta\ldots = 0,1,2,3 \tag{3.60}$$
$$F_{KLMN} = 0, \qquad \text{otherwise} \tag{3.61}$$

where m_0 is a constant. Using this ansatz one can plug this into (3.5) and obtain that the gravitational equation is reduced to

$$R_{\mu\nu} = -12m_0^2 g_{\mu\nu} \ , \tag{3.62}$$
$$R_{mn} = 6m_0^2 g_{mn} \tag{3.63}$$

this is what we obtained when deriving the ansatz for the Freund-Rubin compactificaiton when plugging in the appropriate s-form and choosing $D = 11$. This type of solution is yet again AdS-spacetime because of the different signs and is indeed the maximal symmetric solution. Notice that the seven dimensional surface is S^7 with the tangent space group $SO(7)$ [28]. Introducing the gravitino field and looking for $\mathcal{M}^4 \times \mathcal{B}^7$ compactifications where $\mathcal{M}^4$ is the Minkowski spacetime and $\mathcal{B}^7$ is the compact manifold. Since we are working in the Weitzenböck connection we only consider the manifold $\mathcal{B}^7$ to be parallelizable where the curvature tensor vanishes everywhere [29] therefore the only parallelizable spheres are S^1, S^3 and S^7.

Few remarks on the ansatz, we need to see that it indeed satisfies $\Gamma_M\Psi^M = 0$. Let us assume that it does one can obtain the ansatz in the following way

$$\begin{aligned} 11 \otimes 32\big|_{\psi=0} &= ((4_v,1)\oplus(1,7))\otimes(4_s,8)\big|_{\psi=0} \\ &= (\underbrace{4_v\otimes 4_s}_{H^\alpha},\underbrace{8}_{\theta}) \oplus (\underbrace{4_s}_{J},\underbrace{7\otimes 8}_{\theta^a}) \ . \end{aligned} \tag{3.64}$$

Since there are 32 Majorana components for 11 dimensional supergravity one splits them up in this way to obtain the ansatz where we have 8 gravitino fields with the greek indices and 56 spin-1/2 fields with the latin indices. Checking the $\psi = 0$ components now using the ansatz above we obtain

$$\begin{aligned} \Gamma_M\Psi^M &= (\Gamma_\alpha\Psi^\alpha + \Gamma_a\Psi^a) = (\gamma_\alpha\otimes\mathbf{1}_8)(H^\alpha\otimes\theta) + (\gamma^5\otimes\tau_a)(J\otimes\theta^a) \\ &= \gamma_\alpha\gamma^5\gamma^\alpha J\otimes\theta + \gamma^5 J\otimes\tau_a\theta^a = -4\gamma^5 J\otimes\theta + 7\gamma^5 J\otimes\theta \ , \end{aligned}$$

we see that we need to fix constants for the gravitino fields and the spin-1/2 choosing $\Psi^\alpha = 7(H^\alpha\otimes\theta)$ and $\Psi^a = 4(J\otimes\theta^a)$ fixes this problem and the condition is satisfied.

$\mathcal{M}^4 \times S^7$ solution

We will consider the following form of the gravitino Ψ_M [26]

$$\Psi^\alpha = 7(H^\alpha \otimes \theta), \quad \Psi^a = 4(J \otimes \theta^a) \tag{3.65}$$

where

$$\begin{aligned} J &= (a, a, c, c)^T \ , \\ H^\alpha &= \gamma^5 \gamma^\alpha J \ , \\ \theta &= (1, 0, 0, 0, 0, 0, 0, 0)^T \ , \\ \theta^a &= \tau^a \theta \ . \end{aligned} \tag{3.66}$$

The parameters a, c are constant real Grassman variables which guarantees that the graivitino satisfies the Majorana condition (3.13) since we are in odd dimensions we would not be able to impose a Weyl condition [30]. The vacuum expectation value for the super-covariant 4-form field strength as we put it should be equal to zero therefore equation (2.10)

$$F_{MNPQ} = 3\bar{\Psi}_{[M}\Gamma_{NP}\Psi_{Q]} \ . \tag{3.67}$$

Evaluating the right hand side of the equation for the 4D Minkowski space gives us

$$\begin{aligned} \bar{\Psi}_{[\alpha}\Gamma_{\beta\sigma}\Psi_{\rho]} &= \bar{H}_{[\alpha} \otimes \bar{\theta}\Gamma_{\beta\sigma}H_{\rho]} \otimes \theta \\ &= -\bar{J}\bar{\gamma}_{[\alpha}\gamma_5\gamma_\beta\gamma_\sigma\gamma_5\gamma_{\rho]}J \\ &= -\bar{J}C\gamma_{[\alpha}C^{-1}C\gamma_5C^{-1}\gamma_\beta\gamma_\sigma\gamma_5\gamma_{\rho]}J \\ &= -\bar{J}\gamma_{[\alpha}\gamma_\beta\gamma_\sigma\gamma_{\rho]}J \\ &= i\kappa\varepsilon_{\alpha\beta\sigma\rho}\bar{J}\gamma_5 J \ , \end{aligned} \tag{3.68}$$

where κ is a constant coming from the ansatz and we have used fully anti-symmetric gamma matrices can be expressed as a Levi-Civita density and the C are the charge conjugation matrices coming from the hermitian gamma matrix components. We have also made use of the Γ-matrix splitting

$$\Gamma^\alpha = \gamma^\alpha \otimes \mathbf{1}_8, \quad \Gamma^a = \gamma^5 \otimes \tau^a \tag{3.69}$$

Doing the matrix multiplication in the last row gives us $\bar{J}\gamma_5 J = 4ac$, therefore our RHS becomes

$$\bar{\Psi}_{[\alpha}\Gamma_{\beta\sigma}\Psi_{\rho]} = \kappa 4iac\varepsilon_{\alpha\beta\sigma\rho} \ . \tag{3.70}$$

We can now compare this result to our ansatz (3.61) and see that the constant takes the value $m_0 = 6iac$. We can also evaluate the torsion term in these solution by looking at (3.53), the calculating is straightforward

$$\begin{aligned} T_{abc} &= \frac{i}{2}\bar{\Psi}_a\Gamma_b\Psi_c \\ &= \frac{i}{2}(\bar{J} \otimes \bar{\theta}_a)(\gamma_5 \otimes \tau_b)(J \otimes \theta_c) \\ &= \frac{i}{2}\bar{J}\gamma_5 J\bar{\theta}\tau_a\tau_b\tau_c\theta \\ &= -\frac{\kappa m_0}{3}a_{abc} \ . \end{aligned} \tag{3.71}$$

We have used the octonionic structure constant $-a_{abc} = \theta^T\tau_a\tau_b\tau_c\theta$ and again our gravitino ansatz. This is the torsion for the compact manifold S^7 and is fully anti-symmetric in its indices and vanishes anywhere else. The spin-connection (3.51) is then given as

$$\omega_{abc}(e^a_m) = -\frac{\kappa m_0}{3}a_{abc} \tag{3.72}$$

the spin-connection vanishes for the external space $i.e$

$$\omega_{\alpha\beta\gamma}(e^{\alpha}_{\mu}) = 0 \tag{3.73}$$

so we choose it bo be the Minkowski spacetime $\mathcal{M}^4$. Since the spin-connection vanishes for the space we can set the metric to be of Minkowski metric and the curvature tensor vanishes. We have thus solved the equation

$$R_{mn}(e^{a}_{m}) = 0 \tag{3.74}$$

since we found a torsion that is fully anti-symmetric in its indices and is parallelizable on the compact manifold, therefore we have obtained a four dimensional theory with zero cosmological constant [5, 7].

$\mathcal{M}^4 \times S^3 \times T^4$ solution

Our Freund-Rubin ansatz for the gravitino field will take the same form as the first case, as we mentioned S^3 is also parallelizable together with T^4. In this compactification we define θ^a as

$$\theta^a = (\tau^1\theta, \tau^2\theta, \tau^3\theta, 0, 0, 0, 0) \ . \tag{3.75}$$

The field-strength tensor for the Minkowski space is given as

$$F_{\alpha\beta\sigma\rho} = 12i\kappa ac\varepsilon_{\alpha\beta\sigma\rho} \tag{3.76}$$

as one would expect. The torsion for the parallelizable S^3 manifold is given as

$$\begin{aligned} T_{\bar{a}\bar{b}\bar{c}} &= \frac{i}{2}\bar{\Psi}_{\bar{a}}\Gamma_{\bar{b}}\Psi_{\bar{c}} \\ &= \frac{i}{2}\bar{J}\otimes\bar{\theta}^{\bar{a}}\gamma_b J\otimes\theta^{\bar{c}} \\ &= \frac{i}{2}(\bar{J}\otimes\bar{\theta}^{\bar{a}})(\gamma_5\otimes\tau_b)(J\otimes\theta^{\bar{c}}) \\ &= -2i\kappa ac\varepsilon_{\bar{a}\bar{b}\bar{c}} \end{aligned} \tag{3.77}$$

where the indices are $\bar{a}, \bar{b}, \bar{c} = 1, 2, 3$ and we also used the octononic structure constants however this time with (3.75) as the parametrization. The Einstein equations that we have to solve are

$$R_{\mu\nu} = 0, \quad R_{mn} = 0, \quad R_{\bar{m}\bar{n}} = 0 \ . \tag{3.78}$$

The Minkowski space is once again satisfied since the spin-connection vanishes, the torsion for S^3 is parallelizable which again is satisfied by $R_{mn}(\hat{\omega}) = 0$. The maximally symmetric solution to the 4-torus curvature tensor is also satisfied therefore we have obtained the $\mathcal{M}^4 \times S^3 \times T^4$ compactification. The 4-torus admits no torsion since it is a flat metric.

$\mathcal{M}^4 \times T^7$ solution

This solution is also a solution where the gravitino components are non-vanishing, our ansatz for this solution is given as

$$\Psi^{\alpha} = (H^{\alpha}\otimes\theta) \tag{3.79}$$

$$\Psi^{a} = 0 \ . \tag{3.80}$$

These solution has a vanishing spin-connection since both the manifolds are flat hence

$$\omega_{ABC}(e^{A}_{M}) = 0 \ . \tag{3.81}$$

The torsion is automatically zero since no parallelization exists, and we have obtained a maximally symmetric case namely $\mathcal{M}^4 \times T^7$.

We haven't mentioned the supersymmetry transformations for these solutions. Let us consider the first solution, by plugging the Freund-Rubin ansatz into the supersymmetry transformation for the vielbein we obtain

$$\begin{aligned}\delta e_M^A &= -i\bar{\varepsilon}\Gamma^\alpha\Psi_\alpha - i\bar{\varepsilon}\Gamma^a\Psi_a \\ &= -i\bar{\varepsilon}(\gamma^\alpha \otimes \mathbf{1}_8)(H_\alpha \otimes \theta) - i\bar{\varepsilon}(\gamma^5 \otimes \tau^a)(J \otimes \theta_a) \\ &= -i\bar{\varepsilon}\gamma^\alpha\gamma^5\gamma_\alpha J\theta - i\bar{\varepsilon}\gamma^5 J\tau^a\theta_a\end{aligned}$$

where we have made use of (3.66), we see that there is no way to shift the spinor parameter ε such that this vanishes. The second supersymmetry transformation is given as

$$\delta\Psi_A = \partial_A\varepsilon \tag{3.82}$$

this is zero as long we choose a spinor parameter that is constant. The third one is given as

$$\delta A_{MNP} = \frac{3}{2}\bar{\varepsilon}\Gamma_{[\alpha\beta}\Psi_{\sigma]} + \frac{3}{2}\bar{\varepsilon}\Gamma_{[ab}\Psi_{c]} \tag{3.83}$$

once again we cannot choose the spinor parameter such that shifting the fields make it vanish. This tells us that supersymmetry is completely broken in the solution for $\mathcal{M}^4 \times S^7$, the same applies for the solution of $\mathcal{M}^4 \times S^3 \times T^4$. In order for the supersymmetry to be preserved the supersymmetric fermionic charges must annihilate the vacuum, which they clearly don't. The supersymmetry would also be preserved if one considered a commuting spinor parameter in general our supersymmetry transformation will not vanish since we have nonzero bilinears.

The solution $\mathcal{M}^4 \times T^7$ however preserves supersymmetry since there exists such a parameter that shifts all the fields to make them vanish *i.e*

$$\varepsilon = T \otimes \theta \tag{3.84}$$

where $T = (-a, a, c, -c)^T$. Thus the transformation for the vielbein becomes

$$\delta e_M^A = -i\bar{\varepsilon}\Gamma^\alpha\Psi_\alpha = -i(\bar{T} \otimes \bar{\theta})(H_\alpha \otimes \theta) = -i\bar{T}\gamma_5\gamma_\alpha J = 0$$

this is indeed zero if one works out the matrix multiplication. The second supersymmetry transformation $\delta\Psi_M$ is also zero since our spinor parameter is constant. The third supersymmetry transformation is given as

$$\delta A_{MNP} = \frac{3}{2}\bar{\varepsilon}\Gamma_{[\alpha\beta}\Psi_{\sigma]} = \frac{3}{2}\bar{\varepsilon}\gamma_{[\alpha}\gamma_\beta H_{\sigma]} \otimes \theta = \frac{3i}{2}\bar{\varepsilon}\varepsilon_{\alpha\beta\gamma}J = 0\ . \tag{3.85}$$

We have thus seen that choosing this spinor parameter for the 7-torus solution preserves $\mathcal{N} = 2$ supersymmetries.

Chapter 4

Consistency of sphere reductions

In this chapter we will consider the bosonic sector of the eleven dimensional supergravity Lagrangian where we now introduce Yang-Mills gauge fields that are associated with the isometry group of the n-sphere. There is several known consistent truncations of maximally supersymmetric supergravity, *e.g* for eleven dimensional supergravity compactified on S^7 and S^4 [11, 13], S^5 for Type-IIB supergravity [18] and for Type-IIA supergravity the compactified sphere S^6 gives rise to consistent truncation [19].

We have mentioned the toroidal reduction the bosonic sector of eleven dimensional supergravity, where we have that the global symmetry of the torus reduction is $GL(n, \mathbb{R})$ which is enhanced when we reduce the dimensions. This is the result of scalars coming from the Kaluza-Klein reduction of the metric and the higher dimensional fields. In particular if one considers a Lagrangian containing gravity and a p-form the enhancement of the global symmetry can occur if (D, p) is equal to $(11, 4)$ and $(11, 7)$ for eleven dimensional supergravity and $(10, 5)$ and $(10, 4)$ for the Type-IIA supergravity. Since a 4-form is dual to a 7-form the corresponding enhanced symmetries are given as $SL(8, \mathbb{R})/SO(8)$ in $D = 4$ and $SL(5, \mathbb{R})/SO(5)$ in $D = 7$. In the $(10, 5)$ case we have that our p-form is self-dual and the corresponding enhanced symmetry is given as $SL(6, \mathbb{R})/SO(6)$. Further condition is needed in order to have the enhanced symmetry of $D = 4$ on T^7 and that is the Chern-Simons term, these terms need to be considered since they account for the missing axions when one takes the duals of the field strengths thus giving a enhancement of E_7 group.

A consistent Kaluza-Klein reduction on S^n that retains all the gauge fields of $SO(n+1)$ will be possible only if the coset of the scalar manifold $GL(n, \mathbb{R})/O(n)$ will become large enough to contain the $SO(n+1)$ group of the n-sphere. The global symmetry can be written as $GL(n, \mathbb{R}) \sim \mathbb{R} \times SL(n, \mathbb{R})$, where the $\mathbb{R}$ factor is the trombone symmetry for the equations of motion and $SL(n, \mathbb{R})$ is the invariant group for our Lagrangian. The scalar coset $GL(n, \mathbb{R})/O(n)$ describes the diffeomorphism and the general coordinate transformations of the compactified torus which makes our theory Lorentz invariant. We can achieve this by introducing dilatonic scalar together with gravity and the p-form field strength, two cases for the consistency of the theory will arise when we have a 2-form or a 3-form.

The key reference for this chapter will be [10].

4.1 Scalar coset Lagrangians

We will now establish scalar coset Lagrangians and show that they are invariant under the group enhancements. First we establish the hidden symmetry of toroidal compactifications, in order to see that the lower dimensional theories of this type of compactification has a larger group structure requires some work. As we will show one has to

dualise the field strengths and introducing a Lagrange multiplier which can be viewed as a field on its own giving us the correct amount of axions to support the enhancement. Toroidal compactifications gives us ungauged supergravities, as we described above one can do compactifications on spheres where we now obtain gauged supergravities. These gauged supergravities will give us scalar potentials which is a result from more complicated structures these scalar potentials describes effective theories. We will consider the S^3 case later on where one promotes T^3 to a gauge theory where the six gauge potentials from the Kaluza-Klein reduction where $SO(4) \subset SL(4,\mathbb{R})$ global symmetry can be gauged with the six vector potentials becoming Yang-Mills fields of $SO(4)$. These conjectures are only valid for a p-form taking the value $p = 2, 3$. Let us evaluate the toroidal compactifications in lower dimensions and see how group structure is enhanced and later on discuss the conjecture of gauging these compactifications to describe sphere compactifications. In the appendix the reader can find an example of coset representatives and Iwasawa decomposition which will be useful in order to understand the enhancement.

We consider higher toroidal reductions in particular $n \geq 6$, yielding us the following exceptional groups E_6, E_7 and E_8 for $D = 5$, $D = 4$ and $D = 3$ respectively. Showing the invariance of these groups follow the same principles as above, however there is a inconsistency in the amount of axions and the positive roots of the exceptional groups. Counting the degree's of freedom for the axions on an n-torus are given as $\frac{1}{2}n(n-1)$ for the vielbein reductions and $\frac{1}{6}n(n-1)(n-2)$ for the axion fields coming from the 4-form for $n \geq 6$ we have the following number of axions $(35, 56, 84)$ meanwhile the positive roots for the exceptional groups are given as $(36, 63, 120)$. Constructing the coset representative in this case would fail due to the discrepancies of the number of fields. This problem can be fixed by taking the hodge dual of the 4-form field strength which will yield us the missing axions. Let us consider the $D = 5$ case we want to dualise the 4-form we get a 1-form field strength where one has a 0-form gauge potential. From previous discussion on the Lagrangian for the scalar manifold one must now introduce the 3-form potential the Lagrangian for the field strength it given as

$$\mathcal{L} = -\frac{1}{2}e^{\vec{a}\cdot\vec{\phi}} * F_{(4)} \wedge F_{(4)} - \frac{1}{72}A_{(0)ijk} \wedge dA_{(0)\ell mn} \wedge F_{(4)}\epsilon^{ijk\ell mn} \ . \tag{4.1}$$

One can see from the toroidal reduction Lagrangian given in chapter (3.28) that these are the relevant terms where the last term is the Chern-Simons term for $D = 5$. The dualisation requires us to add a Lagrange multiplier to enforce the Bianchi identity $dF_{(4)} = 0$ this also implies that we cannot have bare potentials in our Lagrangian. We can view the Lagrange multiplier as an auxiliary field which can artificially enlarge the symmetry, integrating the auxiliary field then reduces the symmetry [31]. The following Lagrangian is given as

$$\mathcal{L} = -\frac{1}{2}e^{\vec{a}\cdot\vec{\phi}} * F_{(4)} \wedge F_{(4)} - \frac{1}{72}A_{(0)ijk} \wedge dA_{(0)\ell mn} \wedge F_{(4)}\epsilon^{ijk\ell mn} - \chi dF_{(4)} \ . \tag{4.2}$$

Taking the variation w.r.t χ indeed gives us the Bianchi identity, if we vary w.r.t 4-form field strength we obtain the following equation of motion

$$e^{\vec{a}\cdot\vec{\phi}} * F_{(4)} = d\chi - \frac{1}{72}A_{(0)ijk} \wedge dA_{(0)\ell mn}\epsilon^{ijk\ell mn} \ . \tag{4.3}$$

The 4-form field strength is no longer a field strength instead it is a auxiliary field that we then integrate over to obtain the dualisation given as

$$G_{(1)} = d\chi - \frac{1}{72}A_{(0)ijk} \wedge dA_{(0)\ell mn}e^{ijk\ell mn} \ , \tag{4.4}$$

substituting this back into the Lagrangian yields us

$$\mathcal{L} = -\frac{1}{2}e^{-\vec{a}\cdot\vec{\phi}} * G_{(1)} \wedge G_{(1)} \ . \tag{4.5}$$

The dualisation of the field strength has the effect of reversing the sign of the dilaton vector $\vec{a}$, the effect of the dualisation has turned the Chern-Simons term to its fully modified term *i.e* the non-linear Kaluza-Klein modifications these terms are expressed in the vielbein basis where the basis is given as

$$h^i = dz^i + \mathcal{A}^i_{(1)} + \mathcal{A}^{ij}_{(0)} \wedge dz^j \ . \tag{4.6}$$

The coset representative will contain these modified field strengths.

We have now found the missing axions and we need to introduce more generators for the positive roots in order to construct a coset representative. The extra positive root generator we denote as J together with the commutation rules in (C.16) the algebra is extended to

$$[\vec{H}, J] = -\vec{a}J, \quad [E_i{}^j, J] = 0, \quad [E^{ijk}, J] = 0 \ , \tag{4.7}$$

$$[E^{ijk}, E^{\ell mn}] = -\epsilon^{ijk\ell mn} J \ . \tag{4.8}$$

The dilaton vectors are associated with the E_6 algebra and the dilatonic scalars are associated with the Cartan generators, notice that we have an relation now coming from the extra generator namely

$$\vec{a}_{ijk} + \vec{a}_{\ell mn} = -\vec{a} \ , \tag{4.9}$$

when i, j, k, ℓ, m, n are all different. The negative dilaton vector is indeed the one coming from χ, the new axions will generate an extra term in the coset representative. The coset representative obtained from (C.18) together with the extra generator and taking the exterior derivative yields us

$$d\mathcal{V}\mathcal{V}^{-1} = \frac{1}{2} d\vec{\phi} \cdot \vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}} \mathcal{F}^i{}_{(1)j} E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}} F_{(1)ijk} E^{ijk} + e^{-\vec{a}\cdot\vec{\phi}} G_{(1)} J \ . \tag{4.10}$$

This coset representative will indeed make the Lagrangian of the scalars in $D = 5$ E_6 invariant as we will sketch soon. Using the Iwasawa decomposition one can find the field-dependent transformation $\mathcal{O}$ such that $\mathcal{V}' = \mathcal{O}\mathcal{V}\Lambda$ is of upper triangular form. The upper triangular gauge is usually called Borel gauge, the Borel subgroup of any Lie group is generated by the positive roots generators and the Cartan generators. We obtain our coset representative by exponentiating the Borel subalgebra including the Cartan subalgebra. The subgroup of E_6 is $USp(8)$ therefore the field-dependent transformation $\mathcal{O}$ is a $USp(8)$ matrix. Notice now that the invariance of (C.9) we used the transpose of the matrix $\lambda \in SL(2, \mathbb{R})$, since $USp(8)$ is unitary one needs to modify the invariance of the coset representative. Due to E_6 not being orthogonal, we therefore define *Cartan involution* which introduces a generalised *transpose* #. We denote the positive root generators, negative root generators and the Cartan generators by $\{E_{\vec{\alpha}}, \{E_{-\vec{\alpha}}, \vec{H}\}\}$ where $\vec{\alpha}$ ranges over all positive roots, our algebra τ effects the mapping

$$\tau : \quad (E_{\vec{\alpha}}, E_{-\vec{\alpha}}, \vec{H}) \to (-E_{-\vec{\alpha}}, E_{\vec{\alpha}}, -\vec{H}) \ . \tag{4.11}$$

We can construct a generalisation of $\mathcal{M} = \mathcal{V}^T\mathcal{V}$ now using the generalised transpose and the mapping, we define

$$X^{\#} \equiv \tau(X^{-1}) \ , \tag{4.12}$$

thus for compact lie groups with an orthogonal group this simply is the regular transpose as we have seen in the lower dimensional reductions. If the generator is compact but has a unitary group then we obtain the matrix $X^\dagger$, using this we can rewrite the Lagrangian

(C.9) as a generalised form giving us

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\,\mathrm{Tr}\left(\partial(\mathcal{V}^{\#}\mathcal{V})^{-1}\partial(\mathcal{V}^{\#}\mathcal{V})\right)\\ &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\mathcal{V}^{-1}\partial(\mathcal{V}^{\#})^{-1}+\partial\mathcal{V}^{-1}(\mathcal{V}^{\#})^{-1}\right)\left(\mathcal{V}^{\#}\partial\mathcal{V}+\partial\mathcal{V}^{\#}\mathcal{V}\right)\right)\\ &= \frac{1}{4}\,\mathrm{Tr}\left(\partial\mathcal{V}\mathcal{V}^{-1}(\partial\mathcal{V}\mathcal{V}^{-1})^{\#}+\partial\mathcal{V}\mathcal{V}^{-1}\partial\mathcal{V}\mathcal{V}^{-1}\right)\ .\end{aligned} \tag{4.13}$$

Where we have used the following relation $\mathcal{V}\partial\mathcal{V}^{-1} = -\partial\mathcal{V}\mathcal{V}^{-1}$. We can now insert the coset representative in the equation above and one can obtain the scalar Lagrangian for $D = 5$ supergravity. Using the normalisation conditions

$$\mathrm{Tr}(H_iH_j) = 2\delta_{ij},\quad \mathrm{Tr}(E^j_iE^k_\ell) = 0,\quad \mathrm{Tr}(E^i_j(E^k_\ell)^{\#}) = \delta^{ik}\delta_{j\ell},\quad \mathrm{Tr}(HE^i_j) = 0\ , \tag{4.14}$$

and

$$\mathrm{Tr}(E^{ijk}E^{k\ell mn}) = 0,\quad \mathrm{Tr}(E^{ijk}(E^{\ell mn})^{\#}) = 6\delta^i_{[\ell}\delta^h_m\delta^k_{n]}\ , \tag{4.15}$$

we obtain the following scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}(\mathcal{F}^i_{(1)j})^2 + \frac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}(F_{(0)ijk})^2 + \frac{1}{2}e^{-\vec{a}\cdot\vec{\phi}}(G_{(1)})^2\ . \tag{4.16}$$

The last term is precisely the Lagrangian we wrote down in (4.5), where the extra axions came from. This concludes the coset scalar Lagrangians we can apply the same reasoning to $D = 4$ and $D = 3$, we summarise the coset groups in the table below

	G	K
$D = 10$	$O(1,1)$	-
$D = 9$	$GL(2,\mathbb{R})$	$O(2)$
$D = 8$	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
$D = 7$	$SL(5,\mathbb{R})$	$SO(3) \times SO(5)$
$D = 6$	$O(5,5)$	$O(5) \times O(5)$
$D = 5$	$E_{6(+6)}$	$USp(8)$
$D = 4$	$E_{7(+7)}$	$SU(8)$
$D = 3$	$E_{8(+8)}$	$SO(16)$

For $D = 4$ and $D = 3$ our extra generators are defined as χ^i which comes from taking the dual of the 2-form potentials to get 0-forms for $D = 4$ case and for $D = 3$ case one has to take the dual of the 1-form potentials. The coset representative for the extra axions coming from the given potentials is then given as

$$\mathcal{V}_{extra} = e^{\chi^i J_i}\ . \tag{4.17}$$

Our discussion so far have been about toroidal compactifications, the goal is to obtain consistent sphere reductions [10]. We are interested in S^n reductions that have a $SO(n+1)$ local gauge symmetry, this is essentially the ungauged theory that would result from performing a reduction on a n-torus which has a scalar coset that is at least $SO(n+1)$. For a reduction on S^n to be consistent, a reduction on a T^n must contain a global symmetry group that has a maximal compact subgroup that is large enough to contain $SO(n+1)$. The consistency referred to here is that the gauge bosons should live on the n-spheres corresponding isometry group and be retained in the truncation together with the massless scalars, this will be discussed in later chapters. Let us establish why we need to have simple-laced enhanced symmetry meaning that the simple roots of the dilaton vectors $\vec{b}$ and $\vec{a}$ have the same length. The dilaton vectors $\vec{a}$ associated with the axions coming from the p-form field strength form the weights of some representation

of $SL(n,\mathbb{R})$, if an enhancement of global symmetry is to occur the dilaton vectors must become additional simple positive roots.

The lengths for the dilaton vectors are given as

$$|\vec{a}|^2 = \delta - \frac{2(p-1)(D-p-1)}{D-2} \tag{4.18}$$

δ will take the value of $\delta = 4$ the reason is because the dilaton vectors $\vec{a}_i$ associated with the axions are to have the same length as $\vec{b}_i$ then the coupling of the field strength must take this value. This is a requirement because we are restricted to simple-laced enhancements. An extensive analysis of different values of δ can be found in [32], they found that the dilaton vectors $\vec{b}_i$ associated with the axions coming from the metric reduction always have $\delta = 4$.

Notice that the rank of the p-form can not exceed the dimension, since we are including a dilaton field our only possible ranks of p-forms will be of $p = 2$ and $p = 3$, this gives us the following cases

$$p = 3: \quad a^2 = \frac{8}{D-2}, \tag{4.19}$$

$$p = 2: \quad a^2 = \frac{2(D-1)}{D-2}\ . \tag{4.20}$$

Our Lagrangian for these sphere reductions will take the following form

$$\mathcal{L} = \hat{R}\hat{*}\mathbb{1} - \frac{1}{2}\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \frac{1}{2}e^{-a\hat{\phi}}\hat{*}\hat{F}_{(p)}\wedge\hat{F}_{(p)} \tag{4.21}$$

doing a T^n reduction of this using $p = 3$, the global symmetry will be enhanced to

$$D-n>3: \qquad \mathbb{R}\times\frac{O(n,n)}{O(n)\times O(n)}\ , \tag{4.22}$$

$$D-n=3: \qquad \frac{O(D-2,D-2)}{O(D-2)\times O(D-2)} \tag{4.23}$$

the coset scalars above can be obtained when doing toroidal compactifications of bosonic string theory this corresponds to the T-duality. Using the $p = 3$ we can either to a compactification on S^3 or S^{D-3}, if we consider the $n = 3$ we obtain the following coset group

$$\mathbb{R}\times\frac{O(3,3)}{O(3)\times O(3)} \sim \mathbb{R}\times\frac{SL(4,\mathbb{R})}{SO(4)}\ , \tag{4.24}$$

we can then see that the toroidal reduction will indeed contain a large enough local symmetry group to contain the Yang-Mills gauge fields. We will have six gauge potentials coming from the T^3 reduction these gauge potentials can be gauged such that the global symmetry will be $SL(4,\mathbb{R})$ and the subgroup is $SO(4)$ these will indeed be the Yang-Mills gauge fields of the isometry group of the sphere S^3.

Let us conclude this chapter with a table summarising the possible consistent reductions of these types

p-form	Dilaton	Higher-Dim	Lower-Dim	Sphere	Gauge Group	Extra fields
$F_{(2)}$	Yes	Any D	$D-2$	S^2	$SO(3)$	None
$F_{(3)}$	Yes	Any D	$D-3$	S^3	$SO(4)$	$A_{(2)}$
$F_{(3)}$	Yes	Any D	3	S^{D-3}	$SO(D-2)$	None
$F_{(4)}$	No	11	7	S^4	$SO(5)$	$A^i_{(3)}$
$F_{(4)}$	No	11	4	S^7	$SO(8)$	$\phi_{[ijk\ell]+}$
$F_{(5)} = *F_{(5)}$	No	10	5	S^5	$SO(6)$	None

notice that $e.g$ in S^7 we have to deal with extra pseudoscalars coming from the metric ansatz the number of pseudoscalars are 35 which are in a different representation of the isometry group $SO(8)$. The table above summarises the consistent reductions for eleven dimensional supergravity and Type-IIA supergravity, recently in [19] they found a consistent truncation of Type-IIA reduction on a S^6 with the gauging $ISO(7)$ as the isometry group for the sphere. These are all maximally supersymmetric compactifications of supergravity as they preserve all the 32 supercharges.

4.2 Consistency checks

We now move on to look at the reduction of S^n let us first establish what consistency means and how one achieves it. The references that has done the consistency checks for the symmetric potentials coming from the spherical reductions are discussed in detail [15, 16].

We start with the scalar fields which are parametrised by the coset $SL(N,\mathbb{R})/SO(N)$ submanifold of the full scalar coset of the gauged supergravity, and they are described by a symmetric potential. We have seen from before that the maximal supergravity can be parametrised by the coset E_{11-D}/K where E_{11-D} is the exceptional group and K is the maximally compact subgroup. Our focus will be on the subgroup of the exceptional group namely $SL(N,\mathbb{R})$, we can use the local $SO(4)$ transformation in order to diagonalise the scalar potential as we have seen in the previous chapter using the Iwasawa decomposition. The Lagrangian is given as

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\varphi})^2 + \frac{1}{2}g^2\left(\left(\sum_{i=1}^{N} X_i\right)^2 - 2\sum_{i=1}^{N} X_i^2\right) \tag{4.25}$$

where the last term is the potential. The N quantities X_i are restricted to the constraint

$$\prod_{i=1}^{N} X_i = 1 \ , \tag{4.26}$$

which can be parameterised in terms of $(N-1)$ independent dilatonic scalars $\vec{\varphi}$ described as

$$X_i = e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\varphi}} \ , \tag{4.27}$$

where $\vec{b}_i$ are the weight representations of $SL(N,\mathbb{R})$ satisfying the following relations

$$\vec{b}_i\cdot\vec{b}_j = 8\delta_{ij} - \frac{8}{N}, \quad \sum_i \vec{b}_i = 0, \quad (\vec{u}\cdot\vec{b}_i)\vec{b}_i = 8\vec{u}_i \ . \tag{4.28}$$

The last relation allows us to write the scalar tensors in terms of the dilaton $i.e$

$$\vec{\varphi} = -\frac{1}{4}\sum_i \vec{b}_i \log X_i \ , \tag{4.29}$$

we can now find the equations of motion for the scalar where we derive the potential w.r.t the dilatonic fields the equation of motion after doing this yields us

$$\Box \log X_i = 2g^2\left(2X_i^2 - X_i\sum_j X_j - \frac{2}{N}\sum_j X_j^2 + \frac{1}{N}\left(\sum_j X_j\right)^2\right) \ . \tag{4.30}$$

Using the potential one can find consistent truncations for $D = 4, 5, 7$ these have been found in [**?**], where they found embeddings of lower dimensional theories will be solutions

of the higher dimensional theories. Namely one can find consistent truncations of *e.g* $D = 4$ where we have eight supercharges, we set the scalar potentials to be pairwise and one finds the potential to be

$$V = -4g^2 \sum_{a<b} \tilde{X}_a \tilde{X}_b \ , \tag{4.31}$$

this example gives us a $U(1)^4$ truncation, the discussion on consistent truncations will be considered later on.

The known results for consistent truncation where the lower dimensional equations of motion also satisfy the higher dimensional equations of motion, can be described by the same formulas. The relevant parts of the higher dimensional supergravity theories include the metric and a antisymmetric tensor field $\hat{F}_{(D)}$, from now on the hatted objects are the higher dimensional components. For the given dimensions the field strengths for $D = 4$ and $D = 7$ are duals of each other namely $\hat{F}_{(7)} = \hat{*}\hat{F}_{(4)}$ and vice versa meanwhile for the $D = 5$ case we have a self-dual field strength given as $\hat{H}_{(5)} = \hat{F}_{(5)} + \hat{*}\hat{F}_{(5)}$, the expression for the metric and the field strengths will be given as

$$d\hat{s}^2 = \tilde{\Delta}^{\frac{2}{D-1}} ds_D^2 + \frac{1}{2} g^2 \tilde{\Delta}^{-\frac{D-3}{D-1}} \sum_i X_i^{-1} d\mu_i^2 \ , \tag{4.32}$$

$$\hat{F}_{(D)} = g \sum_i \left(2X_i^2 \mu_i^2 - \tilde{\Delta} X_i \right) \epsilon_{(D)} - \frac{1}{2g} \sum_i X_i^{-1} * dX_i \wedge d(\mu_i^2) \ . \tag{4.33}$$

Where $\epsilon_{(D)}$ is the volume form of the metric ds_D^2 and

$$\tilde{\Delta} = \sum_i X_i \mu_i^2 \ , \tag{4.34}$$

and the μ_i are the embedding coordinates on the sphere which satisfies the constraint $\mu_i \mu_i = 1$. If we set all the scalar fields $\vec{\varphi}$ to vanish we obtain the metric for the round-sphere assuming we have $X_i = 1$. When the scalars are non-zero we which parametrised inhomogeneous deformations of the round sphere metric. The scalars also appear in the field strength so we have to check the consistency of the reduction for the field strength. The Bianchi identity and the field equation are given as

$$d\hat{F} = 0, \quad d\hat{*}\hat{F} = 0 \tag{4.35}$$

the ansatz for our field strength should satisfy both these equations. One can show that the ansatz indeed satisfy the Bianchi identity by substituting the lower dimensional equations of motion for the scalars. The exterior derivative only acts on the μ_i coordinates which gives us

$$\sum_i \left(\Box \log X_i - 4g^2 X_i^2 + 2g^2 X_i \sum_j X_j \right) d(\mu_i^2) = 0 \tag{4.36}$$

we can use the constraint $\mu_i \mu_i = 1$ which gives us the following equation

$$\Box \log X_i = 4g^2 X_i^2 - 2g^2 X_i \sum_j X_j + Q \ . \tag{4.37}$$

The Q term is undetermined i-independent quantity, we also have that the scalar fields are satisfied by (3.49) it then follows that $\sum_i \log X_i = 0$. This implies that the left hand side must vanish and we get an equation for Q, doing the substitution for Q into the equation above we retrieve the equation of motion for the scalars where again the last two terms are assumed to be zero when summed over i. Checking that the field equation is satisfied requires slightly more work.

We consider the metric

$$ds^2 = \sum_i X_i^{-1} d\mu_i^2 \tag{4.38}$$

where we now split the sphere coordinates into $\mu_i = (\mu_\alpha, \mu_0)$, we can thus write the metric in terms of μ_α giving us

$$g_{\alpha\beta} = X_\alpha^{-1}\delta_{\alpha\beta} + \frac{1}{X_0\mu_0^2}\mu_\alpha\mu_\beta \ . \tag{4.39}$$

The inverse metric is given as

$$g^{\alpha\beta} = X_\alpha\delta_{\alpha\beta} - \delta^{-1}\mu_\alpha\mu_\beta X_\alpha X_\beta \ . \tag{4.40}$$

Let us go back to the field strengths now, we introduce a tensor density $\epsilon_{M_1\ldots M_D}$ and using the metric to find the upper index components of the field strength, we can then establish the field strengths given as

$$\sqrt{-g}\hat{F}^{\nu_1\ldots\nu_D} = \frac{U}{g^{N-2}\mu_0\Delta^2}\epsilon^{\nu_1\ldots\nu_D} \ , \tag{4.41}$$

$$\sqrt{-g}\hat{F}^{\nu_1\ldots\nu_{D-1}\alpha} = \frac{1}{g^{N-2}\mu_0}\epsilon^{\nu_1\ldots\nu_{D-1}\sigma}\partial_\sigma\left(\frac{X_\alpha\mu_\alpha}{\Delta}\right) \ , \tag{4.42}$$

where

$$U \equiv \sum_i \left(2X_i^2\mu_i^2 - \Delta X_i\right) \ . \tag{4.43}$$

It is now convenient to define the Hodge duals for the field strengths we just written, there is several relation given [16] that one needs to define the Hodge duals. We will just present the results for the dual field strengths and referred to the article for more details, thus Hodge duals of the equations above is given as

$$\hat{*}\hat{F}_{\alpha_1\ldots\alpha_n} = \frac{U}{g^{N-2}\mu_0\Delta^2}\epsilon_{\alpha_1\ldots\alpha_n}, \quad \hat{*}\hat{F}_{\alpha_1\ldots\alpha_{n-1}\nu} = -\frac{1}{g^{N-2}\mu_0}\epsilon_{\alpha_1\ldots\alpha_{n-1}\nu}\partial_\nu\left(\frac{X_\beta\mu_\beta}{\Delta}\right) \ . \tag{4.44}$$

We can rewrite this as

$$\hat{*}\hat{F} = \frac{1}{g^{N-2}}\left(\frac{U}{\Delta^2}W + \partial_\nu\left(\frac{X_i\mu_i}{\Delta}\right)dx^\nu \wedge Z_i\right) \ , \tag{4.45}$$

it is now straightforward to show that the field equation is indeed satisfied, the relation that are required to show this is given as

$$d\mu_j \wedge Z_j = -(\delta_{ij} - \mu_i\mu_j)W, \quad dZ_i = n\mu_i W \ . \tag{4.46}$$

Hence the field equation is given as

$$g^{N-2}d\hat{*}\hat{F} = \partial_\nu\left(\frac{U}{\Delta^2}\right)dx^\nu\wedge W - \partial_\nu\left(\frac{X_i\mu_i}{\Delta}\right)dx^\nu\wedge dZ_i - \partial_{\mu_j}\partial_\nu\left(\frac{X_i\mu_i}{\Delta}\right)dx^\nu\wedge d\mu_j\wedge Z_i = 0 \ , \tag{4.47}$$

where we have used the relations above. This concludes the consistency check for the field strength equations of motion, we are now left with the most computational consistency check namely the Einstein field equations. We will write the rewrite the metric to

$$d\hat{s}^2 = \Delta^a ds_D^2 + \Delta^{-b}\sum_i X_i^{-1}d\mu_i^2 \ , \tag{4.48}$$

where

$$a = \frac{2}{D-1}, \quad b = \frac{D-3}{D-1} \ . \tag{4.49}$$

We will denote the lower dimensional spacetime indices as $\mu, \nu, \ldots$ and the cosine directions *i.e* the parametersation of the sphere coordinates as $\alpha\,\beta, \gamma, \ldots$. Keep in mind that we split our cosine directions.

The Ricci tensor for the given metric is given as

$$\begin{aligned}
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4}X_i^{-2}\partial_\mu X_i \partial_\nu X_i + \frac{1}{2}\Delta^{-1}X_i^{-1}\mu_i^2 \partial_\mu X_i \partial_\nu X_i - \frac{1}{2}\Delta^{-2}\partial_\mu\Delta\partial_\nu\Delta \\
&\quad + \frac{1}{2}a\left(\Delta^{-2}\partial_\lambda\Delta\partial^\lambda\Delta - \Delta^{-1}\Box\Delta\right)g_{\mu\nu} \\
&\quad - a\left(\sum_i X_i^2 - \Delta^{-1}X_i^2\mu_i^2\sum_j X_j - 2\Delta^{-1}X_i^3\mu_i^2 + 2\Delta^{-2}(X_i^2\mu_i^2)^2\right)g_{\mu\nu} \\
\hat{R}_{\alpha\beta} &= R_{\alpha\beta} + \frac{1}{2}bg_{\alpha\beta}\Delta^{-2}\Box\Delta - \frac{1}{2}bg_{\alpha\beta}\Delta^{-3}\partial_\lambda\Delta\partial^\lambda\Delta - \frac{1}{2}\Delta^{-1}\Box g_{\alpha\beta} \\
&\quad + \frac{1}{2}\Delta^{-1}g^{\gamma\delta}\partial_\lambda g_{\alpha\gamma}\partial^\lambda g_{\beta\delta} - \frac{1}{4}\Delta^{-2}\partial_\alpha\Delta\partial_\beta\Delta - \frac{1}{2}\Delta^{-1}\nabla_\alpha\partial_\beta\Delta \\
&\quad - \frac{1}{4}bg_{\alpha\beta}\Delta^{-2}\partial_\gamma\Delta\partial^\gamma\Delta + \frac{1}{2}bg_{\alpha\beta}\Delta^{-1}\nabla_\gamma\partial^\gamma\Delta \\
\hat{R}_{\alpha\mu} &= -\frac{1}{2}\Delta^{-2}U(X_\alpha^{-1}\partial_\mu X_\alpha - X_0^{-1}\partial_\mu X_0)\mu_\alpha
\end{aligned} \tag{4.50}$$

notice that $R_{\mu\nu}$ and $R_{\alpha\beta}$ are the lower dimensional Ricci tensors, where $R_{\mu\nu}$ is obtained from the Lagrangian (3.48). Having obtained the Ricci tensors we can check that the equations of motion are satisfied where the field equation is given as

$$\hat{R}_{MN} = \hat{S}_{MN} \tag{4.51}$$

where

$$\hat{S}_{MN} = \frac{1}{2(D-1)!}\left(\hat{F}^2_{MN} - \frac{D-3}{D(D-1)}\hat{F}^2 g_{MN}\right)\ . \tag{4.52}$$

One can now find the $\hat{S}_{MN}$ tensors using the field strengths given above the result is given as

$$\begin{aligned}
\hat{S}_{\mu\nu} &= \frac{1}{2}\Delta^{-1}X_i^{-1}\mu_i^2\partial_\mu X_i\partial_\nu X_i - \frac{1}{2}\Delta^{-2}\partial_\mu\Delta\partial_\nu\Delta \\
&\quad - \frac{1}{D-1}\Delta^{-2}\left(U^2 - \partial_\lambda\Delta\partial^\lambda\Delta + \Delta X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i\right)g_{\mu\nu} \\
\hat{S}_{\alpha\beta} &= \frac{1}{2}b\Delta^{-3}U^2 g_{\alpha\beta} + \frac{1}{2}\Delta^{-2}g_{\alpha\beta}X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i - \frac{1}{2}b\Delta^{-3}\partial_\lambda\Delta\partial^\lambda\Delta \\
&\quad - \frac{1}{2}\Delta^{-2}(X_\alpha^{-1}\partial_\lambda X_\alpha - X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta - X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta \\
\hat{S}_{\alpha\mu} &= -\frac{1}{2}\Delta^{-2}U\left(X_\alpha^{-1}\partial_\mu X_\alpha - X_0^{-1}\partial_\mu X_0\right)\mu_\alpha\ .
\end{aligned} \tag{4.53}$$

One now needs to use different relations given in [16] to show that the left hand side is equal to the right hand side, the calculations for simplifying the Ricci tensors can be found in the appendix C. We see that the Ricci tensor $\hat{R}_{\alpha\beta}$ is given as

$$\begin{aligned}
\hat{R}_{\alpha\beta} &= \frac{1}{2}b\Delta^{-3}U^2 g_{\alpha\beta} + \frac{1}{2}\Delta^{-2}g_{\alpha\beta}X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i - \frac{1}{2}b\Delta^{-3}\partial_\lambda\Delta\partial^\lambda\Delta \\
&\quad -\frac{1}{2}\Delta^{-2}(X_\alpha^{-1}\partial_\lambda X_\alpha - X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta - X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta\ ,
\end{aligned} \tag{4.54}$$

thus it is true that $\hat{R}_{\alpha\beta} = \hat{S}_{\alpha\beta}$ is indeed satisfied, this is true for all the equations. Hence we have checked the consistency of spherical reductions described by the scalar coset $SL(N,\mathbb{R})/SO(N)$ where we have scalar fields with gravity and a p-form field strength.

4.3 Consistent S^3 reduction

We start off with the Lagrangian

$$\mathcal{L} = \hat{R}\hat{*}\mathbb{1} - \frac{1}{2}\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \frac{1}{2}e^{-a\hat{\phi}}\hat{*}\hat{F}_{(p)}\wedge\hat{F}_{(p)}\ , \tag{4.55}$$

we have seen that toroidal reductions where one includes the dilaton field will yield a consistent reduction of certain sphere, our main focus here will be S^3. Due to the enhancement of the global symmetry which results in a subgroup being large enough to contain the Yang-mills gauge fields, we can thus consider a consistent reduction where one keeps all the $SO(4)$ Yang-Mills fields $A^{ij}_{(1)}$ together with the scalar fields described by a symmetric tensor T_{ij}, where the index are vectors under the isometry group $SO(4)$.

The metric is given and the field strength is given as

$$d\hat{s}^2_D = Y^{\frac{1}{D-2}}\left(\Delta^{\frac{2}{D-2}}ds^2_{D-3} + g^{-2}\Delta^{-\frac{D-4}{D-2}}T^{-1}_{ij}\mathcal{D}\mu_i\mathcal{D}\mu_j\right)\ , \tag{4.56}$$

$$\begin{aligned}\hat{F}_{(3)} &= F_{(3)} + \frac{1}{6}\epsilon_{i_1i_2i_3i_4}\Big(g^{-2}U\Delta^{-2}\mathcal{D}\mu^{i_1}\wedge\mathcal{D}\mu^{i_2}\wedge\mathcal{D}\mu^{i_3}\mu^{i_4}\\ &\quad - 3g^{-2}\Delta^{-2}\mathcal{D}\mu^{i_1}\wedge\mathcal{D}\mu^{i_2}\wedge\mathcal{D}T_{i_3j}T_{i_4k}\mu^j\mu^k - 3g^{-1}\Delta^{-1}F^{i_1i_2}_{(2)}\wedge\mathcal{D}\mu^{i_3}T_{i_4j}\mu^j\Big)\ ,\\ e^{\sqrt{(D-2)/2\hat{\phi}}} &= \Delta^{-1}Y^{\frac{(D-4)}{4}}\ ,\end{aligned} \tag{4.57}$$

where

$$\mu^i\mu^i = 1,\quad \Delta = T_{ij}\mu^i\mu^j,\quad U = 2T_{ik}T_{jk}\mu^i\mu^j - \Delta T_{ii},\quad Y = \det(T_{ij})\ . \tag{4.58}$$

The $i,j,\ldots$ have the range of 4 values, the gauge covariant derivatives are defined as

$$\mathcal{D}\mu^i = d\mu^i + gA^{ij}_{(1)}\mu^j,\quad \mathcal{D}T_{ij} = dT_{ij} + gA^{ik}_{(1)}T_{kj} + gA^{jk}_{(1)}T_{ik} \tag{4.59}$$

and the field strength for the 1-form is given as

$$F^{ij}_{(2)} = dA^{ij}_{(1)} + gA^{ik}_{(1)}\wedge A^{kj}_{(1)}\ , \tag{4.60}$$

where the gauge potentials denotes the $SO(4)$ gauge potentials coming from the isometry group of the sphere S^3. The lower dimensional fields in the Kaluza-Klein ansatz involve the metric ds^2_{D-3}, the six gauge potentials $A^{ij}_{(1)}$ of $SO(4)$ and the ten scalars fields described by the symmetric tensor T_{ij}, together with the 2-form potential coming from the modified Chern-Simons fields where the field strength is given as

$$F_{(3)} = dA_{(2)} + \frac{1}{8}\epsilon_{ijk\ell}\left(F^{ij}_{(2)}\wedge A^{k\ell}_{(1)} - \frac{1}{3}gA^{ij}_{(1)}\wedge A^{km}_{(1)}\wedge A^{m\ell}_{(1)}\right)\ . \tag{4.61}$$

We will now find the lower dimensional equations of motion by plugging in the ansatz above in the higher dimensional equations of motion given as

$$\begin{aligned}d\hat{*}d\hat{\phi} &= -\frac{1}{2}ae^{-a\phi}\hat{F}_{(3)}\wedge *F_{(3)}\ ,\\ d(e^{-a\phi}\hat{*}\hat{F}_{(3)}) &= 0\ ,\\ \hat{R}_{MN} &= \frac{1}{2}\partial_M\hat{\phi}\partial_N\hat{\phi} + \frac{1}{4}\left(\hat{F}^2_{MN} - \frac{2}{3(D-2)}\hat{F}^2_{(3)}\hat{g}_{MN}\right)\ .\end{aligned} \tag{4.62}$$

The consistency check for the Einstein field equation is very complicated however if the dilaton equation of motion and the field strength field equation is satisfied then the Einstein field equation is satisfied automatically. Like we saw in the last chapter we also have the Bianchi identity $d\hat{F}_{(3)} = 0$ however this time since we have a modified Chern-Simons term we have an additional constraint that the lower dimensional field equation must satisfy which is given as

$$dF_{(3)} = \frac{1}{8}\epsilon_{i_1 i_2 i_3 i_4} F^{i_1 i_2}_{(2)} \wedge F^{i_3 i_4}_{(2)} \ . \tag{4.63}$$

In order to show that the embedded field equations are satisfied we have to calculate the hodge dual for the field strength, using the definitions in appendix we obtain

$$\begin{aligned} e^{-\sqrt{8/(D-2)}\hat{\phi}} \hat{*} \hat{F}_{(3)} = \frac{1}{6g^3}\epsilon_{ijk\ell} * F_{(3)} \wedge \mu^i \mathcal{D}\mu^j \wedge \mathcal{D}\mu^k \wedge \mathcal{D}\mu^\ell - gU\epsilon_{D-3} \\ + g^{-1} T^{-1}_{ij} * \mathcal{D}T_{jk} \wedge (\mu^k \mathcal{D}^i) - \frac{1}{2g^2} T^{-1}_{ik} T^{-1}_{j\ell} * F^{ij}_{(2)} \wedge \mathcal{D}\mu^k \wedge \mathcal{D}\mu^\ell \ . \end{aligned} \tag{4.64}$$

Plugging in the dual field strength into the field equation gives us the following lower dimensional equations of motion

$$\begin{aligned} (-1)^D \mathcal{D}(T^{-1}_{ik} T^{-1}_{j\ell} * F^{k\ell}_{(2)}) &= -2g T^{-1}_{k[i} * \mathcal{D}T_{j]k} - \frac{1}{2}\epsilon_{ijk\ell} * F_{(3)} \wedge F^{k\ell}_{(2)} \ , \\ (-1)^D \mathcal{D}(\tilde{T}^{-1}_{ik} * \mathcal{D}\tilde{T}_{kj}) &= 2g^2 \left(2T_{ik}T_{jk} - T_{ij}T_{kk}\right)\epsilon_{D-3} - T^{-1}_{\ell m} T^{-1}_{ik} * F^{\ell k}_{(2)} \wedge F^{mj}_{(2)} \\ &\quad - \frac{1}{4}\delta_{ij}\left(2g^2\left(2T_{nk}T_{nk} - (T_{kk})^2\right)\epsilon_{D-3} - T^{-1}_{\ell m} T^{-1}_{nk} * F^{\ell k}_{(2)} \wedge F^{mn}_{(2)}\right) \ , \\ d(Y^{-1} * F_{(3)}) &= 0 \ . \end{aligned} \tag{4.65}$$

We introduced the unimodular matrix $\tilde{T}_{ij} = Y^{\frac{1}{4}} T_{ij}$, the consistency of the field equation for the field strength is indeed consistent since there is no cosine coordinates of the sphere S^3 in the lower dimensional field equations. Notice that we would not have a consistent compactification if we were to set the gauge potentials to zero, the reason is because of the term $2gT^{-1}_{k[i} * \mathcal{D}T_{j]k}$ since this acts as a source for the equation of motion.

These field equations transform under $SO(4)$ local group. Obtaining these field equations requires one to use over antisymmetrisation identity $\epsilon_{[i_1 i_2 i_3 i_4} V_{i_5]} = 0$ and most terms cancel among themselves.

We can now check the dilaton equations consistency by first checking the equation of motion for the dilaton given in (4.57), this yield us

$$d\hat{\phi} = \sqrt{\frac{2}{D-2}}\left(\frac{1}{4}(D-4)Y^{-1} * dY - \Delta^{-1} d\Delta\right) \ , \tag{4.66}$$

one has to find the hodge dual coming from the term $d\Delta$ which itself is related to the S^3 coordinates. The Hodge dual of this 1-form is given as

$$\hat{*}(T_{ij}\mu^i \mathcal{D}\mu^j) = -\frac{1}{2}\epsilon_{i_1 i_2 i_3 i_4} T_{i\ell}\mu^i \epsilon_{D-3} \wedge (\Delta T_{i_1 \ell} - T_{i_1 j} T_{k\ell} \mu^j \mu^k)\mu^{i_2} \mathcal{D}\mu^{i_3} \wedge \mathcal{D}\mu^{i_4} \tag{4.67}$$

plugging this into the dilaton equation we find the equations of motion are given as

$$\begin{aligned} \frac{D-5}{4}(-1)^D d(Y^{-1} * dY) = \frac{1}{2}g^2(2T_{ij}T_{ij} - (T_{ii})^2)\epsilon_{D-3} - Y^{-1} * F_{(3)} \wedge F_{(3)} \\ - \frac{1}{4} Y^{-1} T^{-1}_{ik} T^{-1}_{j\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} \ . \end{aligned} \tag{4.68}$$

The lower dimensional Lagrangian is given as

$$\begin{aligned}\mathcal{L}_{D-3} =& R*\mathbb{1} - \frac{D-5}{16}Y^{-2}*dY\wedge dY - \frac{1}{4}\tilde{T}^{-1}_{ij}*\mathcal{D}\tilde{T}_{jk}\wedge\tilde{T}^{-1}_{k\ell}\mathcal{D}\tilde{T}_{\ell i} - \frac{1}{2}Y^{-1}*F_{(3)}\wedge F_{(3)}\\ &- \frac{1}{4}Y^{-\frac{1}{2}}\tilde{T}^{-1}_{ik}\tilde{T}^{-1}_{j\ell}*F^{ij}_{(2)}\wedge F^{k\ell}_{(2)} - \frac{1}{2}g^2Y^{\frac{1}{2}}(2\tilde{T}_{ij}\tilde{T}_{ij} - (\tilde{T}_{ii})^2)\end{aligned} \tag{4.69}$$

this Lagrangian gives us the equations of motion we have obtained above, the derivation of the equations of motion can be found in the appendix using this Lagrangian. The last term in the Lagrangian is the scalar potential, and the Lagrangian is invariant under the scalar coset $SL(4,\mathbb{R})/SO(4)$.

4.4 Truncations of reductions

We will now consider two consistent truncations of the S^3 reduction above, we first consider the truncation for $SO(4)\sim SU(2)_L\times SU(2)_R$. From a group theory argument one can see the Lie group of $SU(2)$ being isomorphic to S^3 where one either picks the right-action or the left-action. We achieve this truncation by considering the self-dual or the anti-self-dual of the $SO(4)$ gauge potential $A^{ij}_{(1)}$

$$A^{ij}_{(1)} = \pm\epsilon_{ijk\ell}A^{k\ell}_{(1)}\ . \tag{4.70}$$

The choice of sign determines with group action we consider, we label the isometry group indices as $0,1,2,3$ and write the gauge potentials in term of triplets given as

$$A^{01}_{(1)} = A^{23}_{(1)} \equiv \frac{1}{2}A^1_{(1)},\quad A^{02}_{(1)} = A^{31}_{(1)} \equiv -\frac{1}{2}A^2_{(1)},\quad A^{03}_{(1)} = A^{12}_{(1)} \equiv A^3_{(1)}\ , \tag{4.71}$$

these are the gauge potentials of $SU(2)_L$. In order to be consistent we also have to truncate the scalar fields where the truncation requires us to set all the scalars equal to each other. This will just give us a single degree of freedom where the scalar is a singlet, the scalars can be described as

$$T_{ij} = X\delta_{ij}\ . \tag{4.72}$$

We set the scalar $Y = X^4$, and we will now express the cosine coordinates of the sphere in terms of Euler angles

$$\mu_0 + i\mu_3 = \cos\frac{1}{2}\theta e^{i(\psi+\phi)/2},\qquad \mu_1 + i\mu_2 = \sin\frac{1}{2}\theta e^{i(\psi+\phi)/2}\ , \tag{4.73}$$

calculating the line element one will find that we can write these angles in terms of left invariant one forms given as

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin\theta d\phi),\qquad \sigma_3 = d\psi + \cos\theta d\phi\ . \tag{4.74}$$

These one forms satisfy the $SU(2)$ algebra

$$d\sigma_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma}\sigma_\beta\wedge\sigma_\gamma\ . \tag{4.75}$$

Using the Euler angles now we can rewrite our ansatz where one notices that the covariant derivative of the cosine coordinates can be written as $\mathcal{D}\mu^i = \sum_\alpha(\sigma_\alpha - gA^\alpha_{(1)})$, thus the ansatz we described in (4.57) is now given as

$$\begin{aligned} d\hat{s}^2_D &= X^{\frac{6}{D-2}}ds^2_{D-3} + \frac{1}{4}X^{-\frac{2(D-5)}{D-2}}\sum_\alpha(\sigma_\alpha - gA^\alpha_{(1)})\ ,\\ e^{\sqrt{(D-2)/2}\hat{\phi}} &= X^{D-5}\ ,\\ \hat{F}_{(3)} &= F_{(3)} - \frac{1}{4g^2}\Omega_{(3)} - \frac{1}{12}\epsilon_{\alpha\beta\gamma}F^\alpha_{(2)}\wedge(\sigma_\beta - gA^\beta_{(1)})\wedge(\sigma_\gamma - gA^\gamma_{(1)})\ , \end{aligned} \tag{4.76}$$

where the volume form $\Omega_{(3)}$ is given as

$$\Omega_{(3)} = \frac{1}{6}\epsilon_{\alpha\beta\gamma}(\sigma_\alpha - gA^\alpha_{(1)}) \wedge (\sigma_\beta - gA^\beta_{(1)}) \wedge (\sigma_\gamma - gA^\gamma_{(1)}) \ . \tag{4.77}$$

All fields that transformed non-trivially under $SU(2)_R$ has been set to zero and all the retained fields are singlets under $SU(2)$. The ansatz for $SU(2)$ truncation parameterises homogeneous deformations of the S^3. Since S^3 is isomorphic to $SU(2)$ where the Lie group is a group manifold, these type of dimensional reductions are consistent by default since all the fields are retained as singlets. The non-singlet are discarded thus will not act as sources for the fields that we truncate. Notice that we do not have any warp factors Δ in the metric ansatz thus concluding that this is indeed a homogeneous deformation which tells us that this is a group manifold compactification.

Consistent truncations of $SU(2)$ where one compactified on a S^4 can be found at [17] where they considered a second order formulation, meanwhile a consistent reduction for $AdS_7 \times S^4$ given in [11] where the truncation was of $SU(2)$ but in the first order formalism. As we can see there has been consistent truncation of different compactifications leading to the same consistency, this consistency truncation for S^4 goes back to [33], where they obtain the same Lagrangian as [11] including the modified non-linear Chern-Simons modifications $\Omega_{(3)}$ as well as $\Omega_{(5)}$ these terms will not allow for taking the gauge limit $g \to 0$. Since the Chern-Simons terms are gauge invariant, there should not be any scalars included in these terms otherwise it would ruin the supersymmetry. Taking the gauge limit of certain compactifications has been shown to lead to certain already known consistent truncations [14] where they instead considered truncations of $U(1)$, we will discuss the consistency for S^3 under this truncation and compare to other known results.

We will now discuss truncation for $U(1)\times U(1)$ which is a subgroup of $SU(2)\times SU(2)$ gauge fields of the full $SO(4)$ ansatz. The range of our vectors will now be denoted as $i, j \ldots = 1, 2, 3, 4$, and we will set all the gauge potentials $A^{ij}_{(1)}$ to zero except for $A^{12}_{(1)}$ and $A^{34}_{(1)}$. We write them as

$$A^{12}_{(1)}) = A^1_{(1)}, \qquad A^{34}_{(1)} = A^2_{(1)} \ . \tag{4.78}$$

The parameterisation for the S^3 is now given as

$$\mu_1 + i\mu_2 = \tilde{\mu}_1 e^{i\phi_1}, \qquad \mu_3 + i\mu_4 = \tilde{\mu}_2 e^{i\phi_2} \ . \tag{4.79}$$

We will set certain scalars to zero and only keep two scalars X_1 and X_2, this writing our symmetric scalar tensor as

$$T_{ij} = \mathrm{diag}(X_1, X_1, X_2, X_2) \tag{4.80}$$

where we now have

$$Y = (X_1X_2)^2, \qquad \Delta = X_1\tilde{\mu}_1^2 + X_2\tilde{\mu}_2^2, \qquad U = 2\sum_{i=1}^{2}(X_i^2\tilde{\mu}_i^2 - \Delta X_i) \ . \tag{4.81}$$

The ansatz using this parameterisation gives us

$$\begin{aligned} d\hat{s}_D^2 &= (X_1X_2)^{\frac{2}{D-2}}\left(\Delta^{\frac{2}{D-2}}ds_{D-3}^2 + g^{-2}\Delta^{-\frac{D-4}{D-2}}\sum_{i=1}^{2}X_i^{-1}\left(d\tilde{\mu}_i^2 + \tilde{\mu}_i^2\left(d\phi_i - gA^i_{(1)}\right)^2\right)\right) \ , \\ e^{\sqrt{(D-2)/2}\hat{\phi}} &= \Delta^{-1}(X_1X_2)^{\frac{D-4}{2}} \ . \end{aligned} \tag{4.82}$$

We will express the field strength tensor in terms of its dual $\hat{F}_{(3)}$, using the truncation above we obtain

$$e^{-\sqrt{8/(D-2)}\hat{\phi}}\hat{*}\hat{F}_{(3)} = -2g\sum_{i=1}^{2}\left(X_i^2\tilde{\mu}_i^2 - \Delta X_i\right)\epsilon_{D-3} + \frac{1}{2g}\sum_{i=1}^{2}X_i^{-1} * dX_i \wedge d(\tilde{\mu}_i^2) - \frac{1}{2g^2}\sum_{i=1}^{2}X_i^{-2}d(\tilde{\mu}_i^2 \wedge (d\phi_i - gA_{(1)}^i) \wedge *F_{(2)}^i + g^{-3}Y^{-1} * F_{(3)} \ . \tag{4.83}$$

The truncation for this case is non-trivial since the scalars X_1 and X_2 parameterises the inhomogeneous deformations on the sphere S^3, these type of deformations could be some squashed sphere where we now have warp factors in the metric ansatz. Comparing previous work concerning the truncation of $U(1)^2$ has been studied in detail [14], let us briefly present the results in this discussion.

They considered two limits of S^4 one that goes to a flat 4 dimensional space and the other one where the metric contains a S^3 where the consistency of the S^4 reduction of 11 dimensional supergravity will be a consistent S^3 reduction of ten dimensional reduction. The reduction to a flat four space $\mathbb{R}^4$ will yield us a massive supergravity in $D = 7$ with a topological massive 3-form potential, this limit will not be discussed here although it is worth mentioning.

Our focus will be on the $\mathbb{R}^1 \times S^3$ limit which yields us a consitent type IIA reduction on S^3. We begin by writing the Lagrangian for $D = 7$ $N = 2$ supergravity which contains the metric, the dilaton and the 3-form vector potential and the $SU(2)$ Yang-Mills gauge fields

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2e^{\frac{8}{\sqrt{10}}\phi} + 4gme^{\frac{3}{\sqrt{10}}\phi} + 4g^2e^{-\frac{2}{\sqrt{10}}\phi} - \frac{1}{48}e^{-\frac{4}{\sqrt{10}}\phi}\left(F_{(2)}\right)^2 - \frac{1}{4}e^{\frac{2}{\sqrt{10}}\phi}\left(F_{(2)}^a\right)^2 + \frac{1}{2}F_{(4)} \wedge F_{(2)}^2 \wedge A_{(1)}^a + \frac{1}{2}mF_{(4)} \wedge A_{(3)} \tag{4.84}$$

we will set the mass parameter to $m = 0$, and make the rescalings of the gauge fields and the scalars to be

$$\vec{\phi} = \vec{\phi}' - \frac{1}{2}(\vec{a}_1 + \vec{a}_2)\lambda, \quad A_{(1)}^i = e^{\frac{1}{5}\lambda}A_{(1)}^{i'}, \quad g = e^{-\frac{1}{5}\lambda}g' \ , \tag{4.85}$$

where λ is a constant. The fields can be parameterised by X_1 and X_2 they parameterise two dilatons described as

$$X_i = e^{\frac{1}{5}\lambda}X_i', \qquad X_0 = e^{-\frac{4}{5}\lambda}X_0' \ , \tag{4.86}$$

here we have introduced factors of e^{λ} which are the scaling symmetries for our equations of motion. We consider equations of motion for eleven dimensional supergravity where we now also rescale the cosine coordinates of the S^4 as

$$\mu_0 = e^{-\frac{1}{2}\lambda}\mu_0', \qquad \mu_i = \mu_i' \ , \tag{4.87}$$

where the constraint is now given as

$$e^{-\lambda}\mu_0' + \mu_i'\mu_i' = 1 \ . \tag{4.88}$$

Taking the limit $\lambda \to \infty$ we obtain the constraint $\mu_i'\mu_i' = 1$ where $i = 1, 2$, notice now that since μ_0' is unconstrained our definition of Δ takes the following form once the limit is taken

$$\tilde{\Delta} \to e^{\frac{1}{5}\lambda}\sum_{i=1}^{2}X_i'\mu_i'^2 \ . \tag{4.89}$$

We can now use these rescalings and constraint to obtained the new ansatz for eleven dimensional supergravity which is now given as

$$
\begin{aligned}
ds_{11}^2 &= e^{\frac{1}{15}\lambda}\left\{\tilde{\Delta}^{1/3}ds_7^2 + g^{-2}\tilde{\Delta}^{-2/3}\left(X_0^{-1}d\mu_0^2 + \sum_{i=1}^{2}X_i^{-1}\left(d\mu_i^2 + \mu_i^2\left(d\psi_i + gA_{(1)}^i\right)^2\right)\right)\right\} \\
F_{(4)} &= e^{\frac{1}{5}\lambda}\Bigg\{2g\sum_{\alpha=0}^{2}\left(X_\alpha^2\mu_\alpha^2 - \tilde{\Delta}X_\alpha\right)\epsilon_{(7)} + \frac{1}{2g}\sum_{i=1}^{2}X_i^{-1}\bar{}dX_\alpha\wedge d(\mu_i^2) \\
&\qquad + \frac{1}{2g^2}\sum_{i=1}^{2}X_i^{-2}d(\mu_i^2)\wedge\left(d\psi_i + gA_{(1)}^i\right)\wedge\bar{*}F_{(2)}^i\Bigg\}
\end{aligned}
\tag{4.90}
$$

where $\bar{*}$ denotes the dual of the 7-form. Notice now that the constraint we have above after taking the limit yields us a parameterisation of a 3-sphere, where the coordinate μ_0 now instead parameterises the forth direction being $\mathbb{R}$, which just covers the real line.

We can now go from eleven dimensional supergravity down to Type-IIA supergravity by compactifying the real line into a circle by interpreting it as an angular coordinate, we can also now drop the prescaling factor e^{λ} due to the trombone symmetry of $D = 11$ equations of motion. The trombone symmetry scales the metric and the 3-form potential by constant factors up to the rank of the p-form and the indices of the metric. We can use the same tactic as before when we did toroidal compactification using the standard Kaluza-Klein metric ansatz

$$
d\hat{s}_{11}^2 = e^{-\frac{1}{6}\phi}ds_{10}^2 + e^{\frac{4}{3}\phi}(dz + \mathcal{A}_{(1)})^2 \ , \tag{4.91}
$$

and the 4-form becomes

$$
\hat{*}F_{(4)} = e^{\frac{1}{2}\phi} * F_{(4)}\wedge(dz + \mathcal{A}_{(1)}) + e^{-\phi} * F_{(3)} \ . \tag{4.92}
$$

We can now interpret our limited ansatz and we obtain the Type-IIA reduction ansatz of S^3, which now is given as

$$
\begin{aligned}
ds_{10}^2 &= \tilde{\Delta}^{1/4}(X_1X_2)^{1/4}ds_7^2 + g^{-2}\tilde{\Delta}^{-3/4}(X_1X_2)^{1/4}\sum_{i=1}^{2}X_2^{-1}\left(d\mu_i^2 + \mu_i^2(d\psi_i + gA_{(1)}^i)^2\right) \ , \\
e^{-\phi} * F_{(3)} &= 2g\sum_{i=1}^{2}\left(X_i^2\mu_i^2 - \tilde{\Delta}X_i\right)\epsilon_{(7)} + \frac{1}{2g}\sum_{i=1}^{2}X_i^{-1}\bar{*}dX_i\wedge d(\mu_i^2) \\
&\qquad + \frac{1}{2g^2}\sum_{i=1}^{2}X_i^{-2}d(\mu_i^2)\wedge(d\psi_i + gA_{(1)}^i)\wedge\bar{*}F_{(2)}^i \ , \\
e^{2\phi} &= \frac{(X_1X_2)^3}{\tilde{\Delta}}, \quad F_{(4)} = 0, \quad \mathcal{F}_{(2)} = 0 \ ,
\end{aligned}
\tag{4.93}
$$

where ϕ is the dilaton of this theory. Thus we have sketched how one obtains a reduction from S^4 of eleven dimensional theory down to a Type-IIA theory compactified on a S^3, where the parameterisation of the scalars now describes the inhomogeneous sphere deformations just like the ones we saw when we considered a truncation of $U(1)\times U(1)$ for the case of S^3 reduction of a Lagrangian including gravity, a dilaton and a 3-form field strength.

Chapter 5

Holonomy and the squashed S^7

In this chapter we will define properties and the integrability conditions for surviving supersymmetries. Holonomy[1] has been shown to be a very useful tool for checking how many supersymmetries one has left after compactification or in the case of membrane solutions. The amount of Killing spinors that are invariant under the holonomy group $\mathcal{H}$ is equal to the amount of supersymmetries one has. The key references in this chapter are [34–39].

5.1 Holonomy of M-theory and integrability conditions

The focus here is still supergravity however this discussion can be extended to the entire M-theory, one important feature of supersymmetric theories is that fermions come in naturally. We therefore invoke that the manifold $\mathcal{M}_7$ should have a spin structure such that the second Steifel-Whitney class disappears [23]. We have seen in the previous chapters that the isometries of the groups on the sphere determines the number of massless gauge bosons and we will now see that the holonomy group $\mathcal{H}$ determines how many massless gravitinos there are.

Two questions of interest is what are the symmetries of supergravity (M-theory) and how many supersymmetries can vacua in supergravity preserve. It has been shown that not only does supergravity have hidden internal symmetries as we argued in the previous chapter but also the theory admits a hidden spacetime symmetry. The number of preserved supersymmetries are given by the number of singlets appearing in the decomposition of the 32-dimensional representation of $\mathcal{G}$ under $\mathcal{G} \supset \mathcal{H}$, where $\mathcal{H}$ is the generalised holonomy group [36, 38].

We consider the bosonic sector of supergravity where we set $\Psi_M = 0$, the supersymmetry transformations are simplified to $\delta e^A{}_M = 0$ and $\delta A_{MNP} = 0$. The supersymmetric transformation rule for the gravitino is reduced to purely bosonic part which is given as

$$\delta\Psi_M = \mathcal{D}_M\epsilon = \left(\partial_M + \frac{1}{4}\omega_M{}^{AB}\Gamma_{AB} + \frac{1}{288}\left(\Gamma_M{}^{NPQR} - 8\Gamma^{[MNP}\delta^{Q]}{}_N\right)F_{NPQR}\right)\epsilon = 0 \ , \tag{5.1}$$

where the regular Riemann covariant derivative involves the spin connection with the spin structure group $Spin(10,1)$ which is a double covering of $SO(10,1)$. The equation above is what determines how many Killing spinors are satisfied *i.e* how many supersymmetries are left, the decomposition of the spinors is given as

$$\epsilon(x,y) = \epsilon(x) \otimes \eta(y) \tag{5.2}$$

[1] Holonomy measures how vectors and tensors on the fiber transform under parallel transport around a closed loop

where $\epsilon(x)$ is a anti-commuting component and $\eta(y)$ is a commuting spinor component on the compact manifold.

The supercovariant derivative above is describing the generalised holonomy, let us illustrate a simpler case where we set $F_{NPQR} = 0$. We see that we are just left with the Riemannian covariant derivative where the integrability condition is given as

$$M_{MN} = [\mathcal{D}_M, \mathcal{D}_n] \equiv \frac{1}{4}\mathcal{R}_{MN}(\omega) \ , \tag{5.3}$$

as one would expect, since this is the definition of the curvature. The Killing spinors are covariantly constant with respect to the Levi-Civita connection, *i.e* they correspond to the singlets under the decomposition **32** of $Spin(10,1)$ under the holonomy group $\mathcal{H}(\mathcal{D})$ satisfying (5.1). The Ambrose-Singer theorem describes the Lie algebra of the holonomy group in terms of the curvature, the dimensions of the Lie algebra spanned by the connection should be in agreement with the dimensions of the Lie algebra spanned by $\mathcal{H}$ [23]. The Killing spinors satisfy the integrability condition (5.3) where the subgroup of $Spin(10,1)$ are generated by linear combinations of $Spin(10,1)$ generators Γ_{AB} corresponding to the holonomy group $\mathcal{H}$ of the spin connection.

When considering a non-vanishing field strength $F_{NPQR} \neq 0$ the Killing spinors satisfy the Killing spinor equation above where we now have the generalised connection. We see that the Clifford algebra is larger where the holonomy group now is spanned by $\{\mathbf{1}_{32}, \Gamma_A, \Gamma_{AB}, \Gamma_{ABC}, \Gamma_{ABCD}, \Gamma_{ABCDE}\}$ which is the Clifford algebra in $D = 11$ where the structure group $\mathcal{G}$ is $GL(32,\mathbb{R})$. The dimensions of the Clifford algebra is given as

$$\mathrm{Dim}\, C\ell(11,5) = \binom{11}{0} + \binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4} + \binom{11}{5} = 1024 \tag{5.4}$$

this coincides with the dimensions of Lie algebra $gl(32,\mathbb{R})$. The Lie algebra of the holonomy group of the generalised connection is spanned by the values of the curvature under the structure group. The supercovariant curvature takes its values in the spinor representations of the Clifford algebra, however it does not contain any term that is proportional to the identity $\mathbf{1}_{32}$ and thus the trace of the curvature is

$$\mathrm{Tr}(\mathcal{R}_{MN}(\omega)) = 0 \tag{5.5}$$

hence the spinor values on the trace elements vanishes. The curvature takes its values in the subset $M^0_{32}(\mathbb{R}) \subset M_{32}(\mathbb{R})$ where the corresponding lie algebra is given as $sl(32,\mathbb{R}) = M^0_{32}$ of $SL(32,\mathbb{R})$ [40]. The holonomy group of the generalised connection must be contained in $SL(32,\mathbb{R})$. The Lie algebra for the generalised connection takes values of the 1023-dimensional Lie subalgebra which is spanned by its traceless generators $\{\Gamma_A, \Gamma_{AB}, \Gamma_{ABC}, \Gamma_{ABCD}, \Gamma_{ABCDE}\}$. The first order integrability condition (5.3) changes when we now consider the generalised connection when $F_{ABCD} \neq 0$, one can see that the curvature takes on larger values spanned by a larger Lie algebra of the holonomy group $\mathcal{H}$. We will see that when we consider compactifications this does is not necessarily true since the holonomy group might be smaller.

The generalised holonomy group of $D = 11$ supergravity is given as

$$\mathcal{H}(\omega) \supseteq SL(32-n,\mathbb{R}) \ltimes (\mathbb{R}^{32-n} \otimes \overset{n}{\ldots} \otimes \mathbb{R}^{32-k}) \equiv SL(32-n,\mathbb{R}) \ltimes \mathbb{R}^{n(32-n)} \tag{5.6}$$

a similar expression can be written for the Lie algebra. We thus see that the amount of n supersymmetries on preserves is classified by the generalised holonomy group $\mathcal{H}(\omega)$.

The integrability condition is an algebraic equation compare to the Killing spinor equation which is a differential equation, in general it is easier to determine the surviving supersymmetries by considering the integrability conditions. One thing to notice

however is that if one considers the Levi-Civita connection one immediately see's that taking higher order integrability conditions *i.e* $[\mathcal{D}_P, M_{MN}]$ yield no new information. This is not the case when we consider $F_{ABCD} \neq 0$, since it takes its values in the Clifford algebra. Therefore once we introduce a non vanishing field strength we need to check higher order integrability conditions in order to have all the generators generating the generalised holonomy group.

In most cases the first integrability condition does not give the complete set of generators so one needs to close the algebra, however higher integrability conditions completes the entire set of generators corresponding to a structure group under the holonomy group.

Higher order integrability conditions can be thought of as corrections to the curvature $\mathcal{R}_{MN}$ away from the original base point, therefore we obtain more information about the connection between the holonomy group $\mathcal{H}(\omega)$ and the generalised connection. We will now go further into these examples and let us consider a example of the $M5$-brane and then go to a squashed S^7 with the fibration $S^3 \hookrightarrow S^7 \to S^4$.

5.2 generalised holonomy of the M5-brane

We will review an example of how higher order integrability conditions give rise to the entire span of generators for $M5$-brane. We will just present the result from [34, 37] to show the significance of holonomy groups and integrability conditions.

The first integrability condition for $D = 11$ supergravity is given as

$$M_{MN} = \mathcal{R}_{MN}(\Omega) = \frac{1}{4}R_{MNAB}\Gamma^{AB} + \frac{1}{2}\nabla_{[M}\tilde{\Omega}_{N]} + \frac{1}{9}\tilde{\Omega}_{[M}\tilde{\Omega}_{N]} \ , \tag{5.7}$$

where $\tilde{\Omega}$ is the extra term from the generalised connection. We are now working the entire Clifford algebra of $D = 11$ supergravity such that the holonomy group takes its values in the Clifford algebra $C\ell(\mathbb{R}^{10,1})$. Higher integrability conditions are just the covariant derivatives of the first order integrability condition, *i.e* higher order integrability conditions is just further away from the base point, they are given as follows

$$\begin{aligned} M_{AMN} &= \mathcal{D}_A M_{MN} = \nabla_A M_{MN} + \frac{1}{4}[\tilde{\Omega}_A, M_{MN}] \ , \\ M_{ABMN} &= \mathcal{D}_A M_{BMN} = \nabla_A M_{BMN} + \frac{1}{4}[\tilde{\Omega}, M_{MN}] \quad \text{etc} \ . \end{aligned} \tag{5.8}$$

The integrability condition for $M5$-brane was first considered in [37] where the authors closed the algebra by checking the commutators of the generators coming from the first order. They found that the spinor representation after doing so is $SO(5)_+$ and the generalised holonomy group was given as

$$\mathcal{H}(M5) = SO(5)_+ \ltimes 6\mathbb{R}^{4(4)} \ , \tag{5.9}$$

this coincides with the results from [36], where the $M5$-brane is split in 6/5 with the holonomy group given as $SO(5)_+ \supset SO(5,1) \times SO(5)_+ \times SO(5)_-$ with the spinor decomposition $(4,4,1) + (\bar{4},1,4) = 4(4) + 16(1)$. Thus we have 16 singlets under this holonomy group which preserves half the supersymmetry of the $M5$-brane.

Closing the algebra by hand might be inefficient if one deals with larger Lie algebras, hence it was shown that the missing generators shows up in higher order integrability conditions [34]. The $M5$-brane solutions is given as

$$\begin{aligned} ds^2_{11} &= H_5^{-1/3} dx^2_\mu + H_5^{2/3} d\vec{y}^2 \\ F_{ijkl} &= \epsilon_{ijklm}\partial_m H_5 \end{aligned} \tag{5.10}$$

where x_μ are the longitudinal direction and y are in the transverse direction spanned by $\{y^i\}$ and $H_5(\vec{y})$ is a harmonic function. The supercovariant connection is given as

$$\Omega_\mu = -\frac{2}{3}H^{-1/2}\partial_i \ln H\mathcal{K}_{\mu i}, \quad \Omega_i = -\frac{1}{3}\partial_i \ln H\Gamma^5 + \frac{4}{3}\delta_{i[j}\partial_{k]} \ln H\hat{\mathcal{T}}_{jk} \ , \tag{5.11}$$

where Γ^5, $\mathcal{K}_{\mu i}$ and $\hat{\mathcal{T}}_{jk}$ are generators of the Lie algebra[2]. They are defined as

$$\Gamma^5 = \frac{1}{5!}\epsilon_{ijklm}\Gamma^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}}, \quad \mathcal{K}_{\mu i} = \Gamma_{\bar{\mu}\bar{i}}P_5^+, \quad \hat{\mathcal{T}}_{jk} = \Gamma_{\bar{i}\bar{j}}P_5^+ \ , \tag{5.12}$$

where $P_5^+ = \frac{1}{2}(1+\Gamma^5)$ are the BPS projection operator, one thing to stress is that Γ^5 does not appear in the generalised curvature and hence will not contribute to the generalised holonomy group.

The first order integrability for the longitudinal direction is zero $M_{\mu\nu} = 0$ since its trivial along the brane, the non zero components are given as

$$\begin{aligned} M_{\mu i} &= \mathcal{R}_{\mu i} = H^{-1/2}\left(\frac{1}{6}\left(\partial_i\partial_j \ln H - \frac{2}{3}\partial_i \ln H\partial_j \ln H\right) + \frac{1}{18}\delta_{ij}(\partial H)^2\right)\mathcal{K}_{\mu j} \\ M_{ij} &= \mathcal{R}_{ij} = \left(\frac{2}{3}\left(\partial_\ell\partial_{[i} \ln H - \frac{2}{3}\partial_\ell \ln H\partial_{[i} \ln H\right)\delta_{j]k} - \frac{2}{9}(\partial H)^2\delta^k_{[i}\delta^\ell_{j]}\right)\hat{\mathcal{T}}_{k\ell} \ . \end{aligned} \tag{5.13}$$

The generators found in the first order integrability does not close the algebra, since we are missing generators of $\mathcal{K}_\mu$ and $\mathcal{K}_{\mu ij}$ which were found in [37] where they closed the algebra finding the linear combination of the generators forming the complete set of generators. The missing generators will come from higher order integrability conditions, the second order is given as

$$\begin{aligned} &M_{\mu\nu\lambda} = M^{\rho i}_{\mu\nu\lambda}\mathcal{K}_{\rho i}, \quad M_{\mu\nu i} = \frac{1}{2}M^{jk}_{\mu\nu i}\hat{\mathcal{T}}_{jk}, \quad M_{\mu ij} = M^{\nu k}_{\mu ij}\mathcal{K}_{\nu k} + \frac{1}{2}M^{\nu k\ell}_{\mu ij}\mathcal{K}_{\nu k\ell} \ , \\ &M_{i\mu\nu} = 0, \quad M_{i\mu j} = M^{\nu k}_{i\mu j}\mathcal{K}_{\nu k} + \frac{1}{2}M^{\nu k\ell}_{i\mu j}\mathcal{K}_{\nu k\ell}, \quad M_{ijk} = \frac{1}{2}M^{\ell m}_{ijk}\hat{\mathcal{T}}_{\ell m} \ , \end{aligned} \tag{5.14}$$

where the components of M_{AMN} are functions of H and its derivatives where two components are given as

$$\begin{aligned} M^{\rho i}_{\mu\nu\lambda} &= \frac{1}{36}H^{-3/2}\left(\partial_j \ln H\partial_j\partial_i \ln H - \frac{1}{3}\partial_i \ln H(\partial H)^2\right)\eta_{\mu[\nu}\delta^\rho_{\lambda]} \ , \\ M^{jk}_{\mu\nu i} &= \frac{4}{9}H^{-1}\left(\partial_{[j} \ln H\partial^i\partial_{k]} \ln H - \delta_{i[j}\delta^\ell \ln H\partial^\ell\partial_{k]} \ln H\right)\eta_{\mu\nu} \ . \end{aligned} \tag{5.15}$$

The second order gave us the generator $\mathcal{K}_{\mu ij}$ and in similar fashion the third order integrability condition $M_{ki\mu j} \equiv [\mathcal{D}_k, [\mathcal{D}_i, \mathcal{R}_{\mu j}]]$ gives the last generator $\mathcal{K}_\mu$. The 10 generators from $\mathcal{T}_{jk}$ correspond to $SO(5)$ and $\{\mathcal{K}_\mu, \mathcal{K}_{\mu i}, \mathcal{K}_{\mu ij}\}$ span the 96-dimensional Lie algebra $\mathbb{R}^{96}$ where they transform as spinor representations of $SO(5)_+$. The generalised holonomy is then given as

$$\mathcal{H}_{M5} = SO(5)_+ \ltimes 6\mathbb{R}^{4(4)} \ , \tag{5.16}$$

this is the same result as they got in [36, 37][3]. Similar calculation can be performed on the $M2$-brane with a 3/8 split where the generalised holonomy group is given as

$$\mathcal{H}_{M2} = SO(8)_+ \ltimes 12\mathbb{R}^{2(8_s)} \tag{5.17}$$

where the spinor representations are under transformed under $SO(8)_+$ and yield 16 singlets *i.e* $(2, 16) = 2(8) + 16(1)$. This result was first obtained in [41] and by using the generalised holonomy approach one could find that the results coincide as they considered in [34, 37].

[2]Notice from before that the generators of the connection should take values in the Clifford algebra therefore we have generators in terms of Γ matrices

[3]The 96 dimensional Lie algebra $\mathbb{R}^{96}$ is decomposed as four dimensional spinors representation with 6 copies of the same reducible representation *i.e* $\mathcal{H}_{M5} = SO(5) \ltimes (\mathbb{R}^{4(4)} \oplus \overset{6}{\ldots} \oplus \mathbb{R}^{4(4)})$.

5.3 Squashed S^7

We will now consider the squashed S^7 sphere which admits the following isometry group $SO(5) \otimes SU(2)$, the generalised holonomy group is G_2. We consider the case of spontaneous compactification where one yields two different solutions depending on the orientation one chooses.

The supersymmetry $\mathcal{N} = 8$ is broken down to $\mathcal{N} = 1$ or $\mathcal{N} = 0$, however it is not clear from the level of first order integrability condition that these two different solutions are in fact different. The Freund-Rubin ansatz is given as

$$F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma}, \quad \mu = 0, 1, 2, 3 \ , \tag{5.18}$$

where m is a constant and the rest of the components vanish. This leads to a spontaneous compactification of the form $AdS_4 \times X^7$, where X^7 is a compact Einstein manifold, plugging in the Freund-Rubin ansatz in the equations of motion for $D = 11$ supergravity yields us the following solutions

$$R_{\mu\nu} = -12m^2 g_{\mu\nu}, \quad R_{mn} = 6m^2 g_{mn} \ . \tag{5.19}$$

Decomposition of the Clifford algebra is given as

$$\Gamma_M = (\gamma_\mu \otimes \mathbf{1}, \gamma_5 \otimes \Gamma_m), \quad \mu = 0, 1, 2, 3, \quad m = 1, \ldots, 7 \tag{5.20}$$

with the usual direct product split of the spinors $\epsilon(x^\mu) \otimes \eta(y^m)$ gives us the following Killing spinor equations

$$\begin{aligned} \mathcal{D}_\mu \epsilon &= \left(\partial_\mu + \frac{1}{4} \Omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta} + m\gamma_\mu \gamma_5 \right) \epsilon = 0 \\ \mathcal{D}_m \eta &= \left(\partial_m + \frac{1}{4} \Omega_m{}^{ab} \Gamma_{ab} - \frac{i}{2} m \Gamma_m \right) \eta = 0 \ . \end{aligned} \tag{5.21}$$

The structure group of these two connections are $Spin(3,2)$ and $Spin(8)$, notice that the when $m = 0$ we have the regular Levi-Civita connection with the holonomy group $Spin(3,1) \otimes Spin(7)$. Introducing the generalised connection we enlarge the holonomy group to be of the structure group $Spin(3,2) \otimes Spin(8)$. The equation of motion for $R_{\mu\nu}$ admits a maximally symmetric AdS space which has four Killing spinors, hence we are only interested in the number of Killing spinors satisfied by the Killing spinor equation for the internal space X^7. The first integrability condition for the internal space X^7 is given as

$$[\mathcal{D}_m, \mathcal{D}_n]\eta \equiv -\frac{1}{4} \mathcal{R}_{mn}{}^{ab} \Gamma_{ab} \eta + \frac{1}{2} m^2 \Gamma_{mn} \eta \ , \tag{5.22}$$

if the Killing spinor equations admits eight covariantly constant spinors then the space is maximally symmetric *i.e* the curvature tensor is given as

$$R_{mnpq} = m^2 \left(g_{mp} g_{nq} - g_{mq} g_{np} \right) \ . \tag{5.23}$$

This corresponds to the round S^7, where the holonomy is trivial $\mathcal{H} = 1$ hence preserves $\mathcal{N} = 8$ supersymmetries in $D = 4$ with a local $SO(8)$ invariance. We are interested in different solutions of the Einstein field equations namely squashed S^7 which has the same topology as the round S^7 yields different vacua which can be thought of as a spontaneously broken theory of the original theory. We will consider the squashed S^7 with the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$, which admits two orientation depending on the choice of the sign of the Freund-Rubin ansatz. We denote $F_4 = m\epsilon_4$ as left orientation and $F_4 = -m\epsilon_4$ as right orientation.

The metric for the squashed S^7 is given as

$$ds^2 = d\mu^2 + \frac{1}{4}\sin^2\mu\omega_i^2 + \frac{1}{4}\lambda^2(\nu_i + \cos\mu\omega_i)^2 \tag{5.24}$$

where ω_i and ν_i are linear combinations of the 1-forms forming the $SU(2)$ algebra which satisfy the condition

$$d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k, \quad d\Sigma_i = -\frac{1}{2}\epsilon_{ijk}\Sigma_j \wedge \Sigma_k \ . \tag{5.25}$$

The λ is the squashing parameter describing the deformations of the sphere, for the round sphere we set $\lambda^2 = 1$ and for the squashed sphere we have that $\lambda^2 = 1/5$. The values of the parameters are obtained from using the Einstein condition $R_{ab} \propto g_{ab}$ where the Ricci tensor is defined as $R_{ab} = \mathrm{diag}(\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta)$, α is the S^4 coordinates and β are the S^3 fibers.

The spin connection for the squashed S^7 with the splitting of the coordinates $a = (0, i, \hat{i})$ where $i = 1, 2, 3$ and $\hat{i} = 4, 5, 6$ is given as

$$\begin{aligned}
&\Omega_{0i} = -\cot\mu e^i + \frac{1}{2}\lambda e^{\hat{i}}, \quad \Omega_{0\hat{i}} = \frac{1}{2}\lambda e^i, \quad \Omega_{ij} = \epsilon_{ijk}\left(\cot\mu e^k + \left(\frac{\lambda}{2} - \frac{1}{\lambda}\right)e^{\hat{k}}\right) \\
&\Omega_{\hat{i}\hat{j}} = -\frac{1}{2\lambda}\epsilon_{ijk}e^{\hat{k}}, \quad \Omega_{i\hat{j}} = -\frac{1}{2}\lambda\delta_{ij}e^0 - \frac{1}{2}\lambda\epsilon_{ijk}e^k \ ,
\end{aligned}$$

and the curvature tensors are given as

$$\begin{aligned}
\mathcal{R}_{0i} &= \left(1 - \frac{3}{4}\lambda^2\right)e^0 \wedge e^i + \frac{1}{4}\left(1 - \lambda^2\right)\epsilon_{ijk}e^{\hat{j}} \wedge e^{\hat{k}} \ , \quad \mathcal{R}_{0\hat{i}} = \frac{1}{4}\lambda^2 e^0 \wedge e^{\hat{i}} - \frac{1}{4}(1-\lambda^2)\epsilon_{ijk}e^j \wedge e^{\hat{k}} \ , \\
\mathcal{R}_{ij} &= \left(1 - \frac{3}{4}\lambda^2\right)e^i \wedge e^j + \frac{1}{2}(1-\lambda^2)e^{\hat{i}} \wedge e^{\hat{j}} \ , \quad \mathcal{R}_{\hat{i}\hat{j}} = \frac{1}{2\lambda^2}e^{\hat{i}} \wedge e^{\hat{j}} + \frac{1}{2}(1-\lambda^2)\left(\epsilon_{ijk}e^0 \wedge e^k + e^i \wedge e^j\right) \\
\mathcal{R}_{i\hat{j}} &= \frac{1}{4}\lambda^2 e^i \wedge e^{\hat{j}} - \frac{1}{4}(1-\lambda^2)e^j \wedge e^{\hat{i}} - \frac{1}{4}(1-\lambda^2)\epsilon_{ijk}e^0 \wedge e^{\hat{k}} + \frac{1}{4}(1-\lambda^2)\delta_{ij}e^k \wedge e^{\hat{k}} \ .
\end{aligned} \tag{5.26}$$

We can now discuss the first order integrability condition for both the orientations, the right orientation Killing spinor is different from the (5.21) up to a sign given as

$$\mathcal{D}_m\eta = \left(\partial_m + \frac{1}{4}\Omega_m{}^{ab}\Gamma_{ab} + \frac{i}{2}m\Gamma_m\right)\eta = 0 \ . \tag{5.27}$$

The generalised connection takes its values in the algebra spanned by $\{\Gamma_{ab}, \Gamma_a\}$, the corresponding structure group is $SO(8)$. The integrability condition for both orientations are the same *i.e*

$$[\mathcal{D}_m, \mathcal{D}_n]\eta = -\frac{1}{4}\mathcal{R}_{mn}{}^{ab}\Gamma_{ab}\eta + \frac{1}{2}m^2\Gamma_{mn}\eta = \frac{1}{4}\mathcal{C}_{mn}{}^{ab}\Gamma_{ab}\eta = 0 \ , \tag{5.28}$$

hence the first order is not sufficient enough to distinguish the orientations, since the Killing spinor solving the integrability condition might not solve the Killing spinor equation for one of the choices of orientation. The skew-whiffing theorem [42] states that at most one orientation can have $\mathcal{N} > 0$, an exception is the round S^7 which admits $\mathcal{N} = 8$ for both orientations since its a symmetric space that admits orientation-reversing isometry. In order to determine the spinors that are invariant under the holonomy group for the squashed S^7 one needs to go beyond the first order to obtain the number of singlets in the decomposition of spinors under $SO(8)$.

Using the curvature terms and plugging in $\lambda^2 = 1/5$ we can analyze the Weyl tensor $\mathcal{C}_{mn}$. It turns out that we obtain 21 components of the Weyl tensor which are linear

combinations of Γ_{ab} given as

$$\begin{aligned}
\mathcal{C}_{0i} &= \frac{4}{5}\left(\Gamma_{0i} + \frac{1}{2}\epsilon_{ijk}\Gamma_{\hat{j}\hat{k}}\right), \quad \mathcal{C}_{ij} = \frac{4}{5}\left(\Gamma_{ij} + \Gamma_{\hat{i}\hat{j}}\right), \quad \mathcal{C}_{0\hat{i}} = -\frac{4}{5}\left(\Gamma_{0\hat{i}} + \frac{1}{2}\epsilon_{ijk}\Gamma_{j\hat{k}}\right) \\
\mathcal{C}_{\hat{i}j} &= \frac{4}{5}\left(-\Gamma_{i\hat{j}} - \frac{1}{2}\Gamma_{j\hat{i}} + \frac{1}{2}\delta_{ij}\Gamma_{k\hat{k}} - \frac{1}{2}\epsilon_{ijk}\Gamma_{0\hat{k}}\right), \quad \mathcal{C}_{\hat{i}\hat{j}} = \frac{4}{5}\left(2\Gamma_{\hat{i}\hat{j}} + \Gamma_{ij} + \epsilon_{ijk}\Gamma_{0k}\right)
\end{aligned} \tag{5.29}$$

not all of these are independent, it turns out that 7 of these are redundant and we end up with

$$\mathcal{C}_{0i} = \frac{4}{5}\left(\Gamma_{0i} + \frac{1}{2}\epsilon_{ijk}\Gamma_{\hat{j}\hat{k}}\right) \tag{5.30}$$

$$\mathcal{C}_{ij} = \frac{4}{5}\left(\Gamma_{ij} + \Gamma_{\hat{i}\hat{j}}\right) \tag{5.31}$$

$$\mathcal{C}_{\hat{i}j} = \frac{4}{5}\left(-\Gamma_{i\hat{j}} - \frac{1}{2}\Gamma_{j\hat{i}} + \frac{1}{2}\delta_{ij}\Gamma_{k\hat{k}} - \frac{1}{2}\epsilon_{ijk}\Gamma_{0\hat{k}}\right) . \tag{5.32}$$

We have 14 linearly independent combinations which correspond to the generators of the exceptional group G_2, the structure group is $SO(7)$ and the generalised holonomy group would decompose as $\mathbf{8}_s \to \mathbf{8} \to \mathbf{7} + \mathbf{1}$ under $SO(8) \supset SO(7) \supset G_2$. This is not the correct generalised holonomy group since both orientations would decompose in the same way leading to both orientations having $\mathcal{N} = 1$ this is not true due to the skew-whiffing theorem. The first order integrability condition is not sufficient enough to determine the generalised holonomy.

The second order integrability condition for a general Freund-Rubin compactification [43] is given as

$$M_{\ell mn}\eta = \frac{1}{4}\left(\nabla_\ell \mathcal{C}_{mn}{}^{ab}\Gamma_{ab} \mp \mathcal{C}_{mn\ell}{}^{a}\Gamma_a\right)\eta = 0 , \tag{5.33}$$

the $\mp$ sign correspond to left and right orientation respectively. The complete linear combinations for the second order integrability condition was given in [34], they found 21 $M_{\ell mn}$ that are linearly independent combinations of the Dirac matrices. The independent generators from the second order can be written as

$$M_{ij} = \Gamma_{\hat{i}\hat{j}} \mp \frac{2}{3}\sqrt{5}im\epsilon_{ijk}\Gamma^{\hat{k}}, \quad M_i = \Gamma_{0\hat{i}} \mp \frac{2}{3}\sqrt{5}im\Gamma_i, \quad M = \delta^{k\ell}\Gamma_{k\hat{\ell}} \pm 2\sqrt{5}im\Gamma_0 , \tag{5.34}$$

together with the generators above *i.e* $\{\mathcal{C}_{0i}, \mathcal{C}_{ij}, \mathcal{C}_{i\hat{j}}\}$ which span the G_2 and the extra generators from the second order integrability condition one finds that they generate the 21 dimensional algebra of $SO(7)$. The embedding of $SO(7)$ into $SO(8)$ is however different, since the decomposition of the spinors representations are different. Taking $SO(7)_-$ to be left orientation the decomposition is $\mathbf{8}_s \to \mathbf{7}+\mathbf{1}$ which has one singlet hence preserves $\mathcal{N} = 1$ supersymmetries. The right orientation turns out to have no singlets where the spinor decomposition is given as $\mathbf{8}_s \to \mathbf{8}$. The conclusion is that both higher order integrability and the embedding of the algebra is important in evaluating the surviving supersymmetries. The generators that came from the first order integrability condition is enough to close the algebra but it would just be the subalgebra of the generalised holonomy algebra.

Chapter 6

Discussion

The consistency of sphere reductions have been well studied, where one shows that the conjecture of gauging ungauged supergravity is possible if one can retain all the gauge field singlets under the isometry group of the sphere $SO(n+1)$. This can be done by introducing a dilatonic scalar together with gravity and a p-form field strength, where the dilaton vectors are associated with the simple roots of the given Lie algebra, forming the proper coset manifold that retains all the gauge fields. The simple roots should have equal length as the dilaton vectors due to needing simple-laced enhanced symmetry. The further down we go in dimensions there will be a discrepancy in the degree's of freedom of the p-form and the simple roots, this is however fixed if we take the hodge dual of the field strength which yields us the correct amount of axions. The lower dimensional equations of motion will be embedded into the $D = 11$ equations of motion which then guarantees the consistency. The scalar potential comes from the duality transformation of the field strength and the dilaton field in the lower dimensional equations of motion.

We have seen how spontaneous compactification of parallelizable manifolds yields us four dimensional Minkowski background. Parallelizable manifolds allows us to write down the Weitzenböck connection on the manifold explicitly in terms of the fermionic bilinears, this flattens the geometry where the curvature gets canceled by the torsion.

The ansatz where we set the supercovariant field strength $\langle \hat{F}_{ABCD} \rangle = 0$ however breaks all of the supersymmetry for $\mathcal{M}_4 \times S^7$ and $\mathcal{M}_4 \times S^3 \times T^4$ since there is no spinor parameter that allow us to shift the fields to become zero *i.e* satisfying the susy transformations.

The solution of $\mathcal{M}_4 \times T^7$ preserves two of the supersymmetries but breaks the Lorentz invariance due to the non-vanishing fermionic fields, in contrast to the bosonic sector of supergravity where the compactification of T^7 preserves all supersymmetries. These can be seen from the holonomy group being trivial *i.e* $\mathcal{H} = 1$ resulting in having eight Killing spinors satisfied by the gravitino transformation which tells us that all 32 supercharges are preserved. We want to point out that the scalar potential for this configuration is unknown in our case. One sees that the equations of motion are satisfied allowing us to perhaps use consistent truncation in order to evaluate the scalar potential. However since it breaks Lorentz symmetry one would like to find a Freund-Rubin ansatz with a non-vanishing field strength on the compact manifold $F_{abcd} \neq 0$.

The squashed S^7 with the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$ considered in chapter 5 has a vanishing field strength on the compact manifold X^7. This solution turns out to have a $\mathcal{N} = 1$ supersymmetries surviving after compactification for left orientation and breaks down to $\mathcal{N} = 0$ for the right orientation.

The Freund-Rubin compactifications in the cases we have discussed, either breaks the Lorentz invariance or the gravitino condition $\Psi_M \neq 0$. One has to choose which one to break since there is no decomposition of the gravitino field $\Psi_\alpha(x)$ and $\Psi_m(y)$ that satisfies both the gravitino condition and does not break the Lorentz invariance.

An interesting case would be to consider domain walls in bosonic supergravity, where one considers non-Killing spinor $\nabla_M \epsilon \neq 0$. Under this construction one obtains that all the fields $\{\Psi_M, e^A{}_M, A_{MNP}\}$ are non-zero compare to if ϵ was a Killing spinor where we set fermionic field to zero $\Psi_M = 0$. However if one introduces bilinears such that after one acts with the local supersymmetries gives rise to a fermionic field with a mass i.e $\Psi_M \neq 0$. In a sense the non-Killing spinors give rise to a non-vanishing fermionic field that allows us to introduce bilinears that have a non-vanishing expectation value.

The domain wall solutions in lower dimensions would have the fermionic VEV in only one direction and not the entire space, essentially we would only break Lorentz invariance in one direction. Hence one could perhaps construct domain wall solutions of $\mathcal{M}_3 \times \mathbb{R} \times X^7$ or membrane solutions with non vanishing fermionic bilinears.

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Appendix A

Clifford Algebra

The representation of the Clifford algebra can be written in terms of the Pauli matrices σ_i and are constructed in the following way

$$\begin{aligned}\gamma_{2n-1} &= \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_1 \\ \gamma_{2n} &= \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_2 \\ \gamma_{2n+1} &= \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_3 \ .\end{aligned}$$

The decomposition of gamma matrices in Kaluza-Klein reductions where K_n is the internal space and the spacetime is M_m, can then be written as Dirac matrices $\hat{\gamma}$ of $M_m \times K_n$

$$\begin{aligned}\text{even, odd}: &\quad \hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbf{1}, \quad \hat{\Gamma}_i = \gamma \otimes \gamma_i \\ \text{odd, even}: &\quad \hat{\Gamma}_\mu = \gamma_\mu \otimes \gamma, \quad \hat{\Gamma}_i = \mathbf{1} \otimes \gamma_i \\ \text{even, even}: &\quad \hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbf{1}, \quad \hat{\Gamma}_i = \gamma \otimes \gamma_i \\ \text{odd, odd}: &\quad \hat{\Gamma}_\mu = \sigma_1 \otimes \gamma_\mu \otimes \mathbf{1}, \quad \hat{\Gamma}_i = \sigma_2 \otimes \mathbf{1} \otimes \gamma_i\end{aligned}$$

Gamma matrix manipulations are given by the general formula

$$\gamma^{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s} \gamma_{\nu_s \ldots \nu_1} = \frac{(D-r)!}{(D-r-s)!} \gamma^{\mu_1 \ldots \mu_r} \ . \tag{A.1}$$

General order reversal of gamma matrices is given as

$$\gamma^{\nu_1 \ldots \nu_r} = (-)^{r(r-1)/2} \gamma^{\nu_r \ldots \nu_1} \tag{A.2}$$

a example of how to use gamma matrix manipulation

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} \ . \tag{A.3}$$

The Clifford algebra of $2^m \times 2^m$ matrices for both even and odd representations one can distinguish the antisymmetric and the symmetric gamma matrices by a charge conjugation matrix. There exist unitary matrix C such that each matrix $C\gamma^A$ is either antisymmetric or symmetric. The symmetry depends on the rank of the γ^A in other words

$$(C\gamma^{(r)})^T = -t_r C\gamma^{(r)}, \quad t_r = \pm 1 \ . \tag{A.4}$$

Appendix B

Hodge duality

The r-form $\Omega^r(M)$ is isomorphic to the $(m-r)$-form $\Omega m - r(M)$ on a M dimensional manifold. If M is equipped with a metric g we can define a natural isomorphism between them called Hodge * operation. We define the totally antisymmetric tensor ϵ by

$$\epsilon_{\mu_1 \dots \mu_m} = \begin{cases} +1, & \text{if} \quad (\mu_1 \dots \mu_m) \quad \text{is an even permutation} \\ -1, & \text{if} \quad (\mu_1 \dots \mu_m) \quad \text{is an odd permutation} \\ 0, & \text{otherwise} \end{cases} . \tag{B.1}$$

To raise or lower the indices we use the metric

$$\epsilon^{\mu_1 \dots \mu_m} = g^{-1} \epsilon_{\mu_1 \dots \mu_m} .$$

The Hodge star is a linear map taking $\Omega^r(M) \to \Omega^{m-r}(M)$ whose action on the basis vector of $\Omega^r(M)$ is defined as

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(m-r)!} \epsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_m} dx^{r+1} \wedge \dots \wedge dx^{\nu_m} . \tag{B.2}$$

Defining the one form as

$$\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} , \tag{B.3}$$

acting with the star operation we obtain

$$*\omega = \frac{\sqrt{|g|}}{r!(m-r)!} \omega_{\mu_1 \dots \mu_r} \epsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_m} dx^{\nu_{r+1}} \wedge \dots \wedge dx^{\nu_m} . \tag{B.4}$$

Acting with the hodge star twice should give back the same r-form as we started with up to a minus sign $i.e$

$$** \omega = (-1)^{r(m-r)} \omega$$

for a Riemannian manifold and for a Lorentzian we have

$$** \omega = (-1)^{1+r(m-r)} \omega .$$

Appendix C

Iwasawa decomposition and coset group

For simplicity we will now consider the $SL(2,\mathbb{R})$ this would be the coset group for a T^2 reduction, the $SL(2,\mathbb{R})$ is the non-compact group of $SU(2)$ and its associated Lie algebra is essentially the same as that of $SU(2)$. The generators for $SU(2)$ satisfy the following Lie algebra

$$[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = H \ . \tag{C.1}$$

H is the Cartan subalgebra generator and $E_\pm$ are the raising/lowering operators, the representation for the generators can be written as a 2×2 matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{C.2}$$

where $H = \tau_3$ and $E_\pm = \frac{1}{2}(\tau_1 \pm \tau_2)$ where τ_i are the Pauli matrices. We now consider the exponentiation of the H and E_+ and define

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{\chi E_+} \tag{C.3}$$

where ϕ and χ are fields depending on the coordinates of a D-dimensional spacetime. We can rewrite the exponent as

$$\mathcal{V} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} \ . \tag{C.4}$$

Computing the exterior derivative one finds

$$d\mathcal{V}\mathcal{V}^{-1} = \frac{1}{2} d\phi H + e^{\phi} d\chi E_+ \ . \tag{C.5}$$

We can now define the following matrix $\mathcal{M} = \mathcal{V}^T\mathcal{V}$, using the above matrix we can obtain the following matrices

$$\mathcal{M} = \begin{pmatrix} e^{\phi} & \chi e^{\phi} \\ \chi e^{\phi} & e^{-\phi} + e^{\phi}\chi^2 \end{pmatrix}, \quad \mathcal{M}^{-1} = \begin{pmatrix} e^{\phi} + e^{\phi}\chi^2 & -\chi e^{\phi} \\ -\chi e^{\phi} & e^{\phi} \end{pmatrix} \ , \tag{C.6}$$

using these matrices one can now define the kinetic Lagrangian for the scalar fields

$$\mathcal{L} = \frac{1}{4} \operatorname{Tr}(\partial \mathcal{M}^{-1} \partial \mathcal{M}) = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2} e^{2\phi} (\partial\chi)^2 \ . \tag{C.7}$$

This Lagrangian is $SL(2,\mathbb{R})$ invariant and we can show it by considering a matrix $\Lambda \in SL(2,\mathbb{R})$ which is given as

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - cb = 1 \ , \tag{C.8}$$

we then shift the matrix $\mathcal{V}$ to $\mathcal{V}'' = \Lambda\mathcal{V}$ which yields us $\mathcal{M} \to (\mathcal{V}'')^T\mathcal{V}'' = \Lambda^T\mathcal{M}\Lambda$ this gives us that our Lagrangian is manifestly invariant *i.e*

$$\mathcal{L} \to \frac{1}{4}\,\mathrm{Tr}\left(\Lambda^{-1}\partial\mathcal{M}(\Lambda^T)^{-1}\Lambda^T\mathcal{M}\Lambda\right) = \frac{1}{4}\,\mathrm{Tr}\left(\partial\mathcal{M}^{-1}\partial\mathcal{M}\right)\ . \tag{C.9}$$

Notice however that the shift we used makes the $\mathcal{V}''$ matrix is not of the upper triangular form, which is not correct since acting with Λ ruins the transformation of $SL(2,\mathbb{R})$ on the fields ϕ and χ. We can however fix this by introducing a local transformation $\mathcal{O}$ that acts on the $\mathcal{V}$ on the left and we also multiply by the constant $SL(2,\mathbb{R})$ matrix on the right. Defining the transformation as

$$\mathcal{V}' = \mathcal{O}\mathcal{V}\Lambda\ , \tag{C.10}$$

where the matrix $\mathcal{O}$ will restore the upper triangular form of $\mathcal{V}'$. One can find the unique orthogonal matrix $\mathcal{O}$ that does the job of restoring the upper triangular form, and it is given by

$$\mathcal{O} = \left(c^2 + e^{2\phi}(c\chi + a)^2\right)^{-1/2}\begin{pmatrix} e^{\phi}(c\chi + a) & c \\ -c & e^{\phi}(c\chi + a)\ . \end{pmatrix} \tag{C.11}$$

This matrix restores the upper triangular form of the $SL(2,\mathbb{R})$ matrix $\mathcal{V}$ and we can now interpret the action of $SL(2,\mathbb{R})$ in terms of transformations on ϕ and χ. The matrix $\mathcal{O}$ also leaves the Lagrangian invariant since it compensates for the $SL(2,\mathbb{R})$ transformations with the constant parameters of Λ. Since $\mathcal{O}$ is local it depends on both the constant parameters and the fields leaving the Lagrangian invariant, we can see this by doing the following transformation

$$\mathcal{M} \to \mathcal{M}' = (\mathcal{V}')^T\mathcal{V}' = \Lambda^T\mathcal{V}^T\mathcal{O}^T\mathcal{O}\mathcal{V}\Lambda = \Lambda^T\mathcal{M}\Lambda\ . \tag{C.12}$$

We have parameterised points on the scalar manifold in terms of upper triangular gauge matrix, where acting with a matrix Λ allows us to get to any other point on the scalar manifold. We can do this if we compensate with a $O(2)$ transformation that makes us stay in our original parameterisation scheme in terms of our upper triangular matrices $\mathcal{V}$. We can specify points on the scalar by the coset $SL(2,\mathbb{R})/O(2)$ where $SL(2,\mathbb{R})$ is the global symmetry and $O(2)$ is the local symmetry.

In general one does never need to find the orthogonal matrix $\mathcal{O}$, since one can use Iwasawa decomposition that states that every element g in the Lie group G obtained by exponentiating the Lie algebra $\mathcal{G}$ can be uniquely expressed as the following product [44]

$$g = g_K g_H g_N\ . \tag{C.13}$$

Here g_K is the maximal compact subgroup K of G, g_H is the Cartan subalgebra of G and g_N is the exponentiation of the positive root part of the algebra $\mathcal{G}$. Our coset representative $\mathcal{V}$ is constructed by exponentiating the Cartan generators and the full set of positive root generators. We can write our coset representative as $\mathcal{V} = g_H g_N$ acting with the general group element of $\Lambda \in G$ which means that $\mathcal{V}\Lambda$ is some group element of G. Using the Iwasawa decomposition tells us that we can write our group element in the form $g_K\mathcal{V}'$ where $\mathcal{V}'$ is of the form $g'_H g'_N$. This assures that there exist a way of pulling out an element $\mathcal{O}$ of the maximal compact subgroup K of G such that we can write $\mathcal{V}\Lambda$ as $\mathcal{O}\mathcal{V}'$.

Let us now continue with the analysis of roots as we have mentioned above. The generators of a Lie algebra $\mathcal{G}$ can be organised into Cartan generators $\vec{H}$ which commute with each other and also with the raising and lowering operators $E_{\vec{\alpha}}$, we can generalised the commutation relation we mentioned in the start of the chapter as

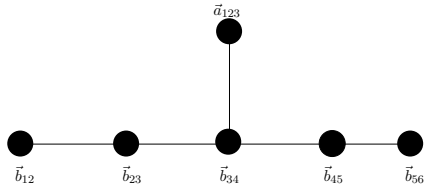
$$[\vec{H}, E_{\vec{\alpha}}] = \vec{\alpha}E_{\vec{\alpha}}\ . \tag{C.14}$$

The $\vec{\alpha}$ are called the root vectors associated with the generators $E_{\vec{\alpha}}$ the root vectors can be negative or positive, positive roots are associated with the raising operator and the negative roots are associated with lowering operator. The commutator of two non-zero root generators $E_{\vec{\alpha}}$ are given as

$$[E_{\vec{\alpha}}, E_{\vec{\beta}}] = N(\alpha, \beta) E_{\vec{\alpha}+\vec{\beta}} \ , \tag{C.15}$$

for some constant $N(\alpha, \beta)$. Lie algebras can be classified by classifying all possible root systems, these root systems can be characterised in terms of simple roots. Simple roots are subset of positive roots that allows one to write linear combinations of the simple roots. There are as many simple roots as there are root vectors which tells us that the rank of the algebra is equal to the number of simple roots.

We can now proceed to lower dimensions of toroidal compactifications we encountered in chapter 3.1 using these scalar cosets to obtain the enhanced symmetries of lower dimensional bosonic supergravity. We will consider compactifications when n-torus is $n \leq 5$ where our dimensions will be $D = 11 - n \geq 6$. The axionic scalars will be $\mathcal{A}^{i}_{(0)j}$ and $\mathcal{A}_{(0)ijk}$ in each dimensions and the dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ for these axions form the positive root generators. The dilaton vector $\vec{b}_{ij}$ comes from the reduction of the vielbein where the linear combination of the simpe roots are given as $\vec{b}_{ij} = -\vec{f}_i + \vec{f}_j$ and the $\vec{a}_{ijk}$ comes from the 4-form field strength reduction with the linear combination $\vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}$. We can construct a coset representative $\mathcal{V}$ for G/K by exponentiating the associated positive roots with the axions as coefficients and exponentiating the Cartan generators with the dilatons as coefficients. If one can identify the subset of the dilaton vectors corresponding to the simple roots of the Lie algebra then one can identify the group G for each dimension. All Lie algebras are classified in terms of their Dynkin diagrams *e.g* the exceptional group E_6 has the following Dynkin diagram



for higher exceptional groups we just add more roots to the Dynkin diagram. For $n = 2$ we just have one dilaton vector $\vec{b}_{12}$ where the algebra is $SL(2, \mathbb{R})$ for $n = 4$ we have the following dilaton vectors $(\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}, \vec{a}_{123})$ and the Dynkin diagram is of $SL(5, \mathbb{R})$, last but not least for $n = 5$ our dilaton vectors are $(\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}, \vec{b}_{45}, \vec{a}_{123})$ and the corresponding Dynkin diagram is of D_5 or $O(5, 5)$. We can see the dilaton vectors as the positive roots of the given algebra, we therefore introduce commutation relations for the Cartan generators and the generators $E_i{}^j$ associated with the dilaton vectors $\vec{b}_{ij}$ and the generator E^{ijk} associated with the dilaton vector $\vec{a}_{ijk}$. The commutation relations are given as

$$\begin{aligned}
[\vec{H}, E_i{}^j] &= \vec{b}_{ij} E_i{}^j, \quad [\vec{H}, E^{ijk}] = \vec{a}_{ijk} E^{ijk} \\
[E_i{}^j, E_k{}^\ell] &= \delta^i_k E_i{}^\ell - \delta^\ell_i E_k{}^j \ , \\
[E_\ell{}^m, E^{ijk}] &= -3\delta^{[i}_\ell E^{|m|jk]} \ , \\
[E^{ijk}, E^{\ell mn}] &= 0 \ .
\end{aligned} \tag{C.16}$$

Notice that these commutation rules for the generators above are the ones for $SL(n, \mathbb{R})$, however we have an extra commutation rule for the dilaton vector $\vec{a}_{ijk}$ which extends the algebra from $SL(2, \mathbb{R})$ to larger ones which precisely corresponds to $n \geq 3$. If

we remove the dilaton vector associated with the generator E^{ijk} we indeed obtain the Dynkin diagram for $SL(2,\mathbb{R})$ Lie group.

We can write the coset representative for all the cases of $n \leq 5$ as

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \left(\prod_{i<j} e^{\mathcal{A}^{i}_{(0)j} E_i{}^j} \right) \exp\left(\sum_{i<j<k} A_{(0)ijk} E^{ijk} \right) \ , \tag{C.17}$$

we order the terms in a anti-lexigraphical[1] where we invert the lexigraphical order such that we order in the following way $(24)(23)\ldots(14)(13)(12)$. Computing the exterior derivative yields us

$$d\mathcal{V}\mathcal{V}^{-1} = \frac{1}{2} d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}} \mathcal{F}^{i}{}_{(1)j} E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}} F_{(1)ijk} E^{ijk} \ , \tag{C.18}$$

where one needs to use the commutation relations together with the two identities

$$de^{X}e^{-X} = dX + \frac{1}{2}[dX, X] + \frac{1}{6}[X,[X,dX]] + \ldots \tag{C.19}$$

$$e^{X}Ye^{-X} = Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \ldots \ . \tag{C.20}$$

The terms $\mathcal{F}^{i}{}_{(1)j}$ are coming from the vielbein reduction and the $F_{(0)ijk}$ coming from the 4-form, the scalar Lagrangian for these cases where $n \leq 5$ can be written as $\mathcal{L} = \frac{1}{4}\,\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M})$.

[1] Given two partially ordered sets A and B then the lexigraphical order of the Cartesian product $A \times B$ is defined as

$$(a,b) \leq (a',b'), \quad \text{iff} \quad a > a' \quad \text{or} \quad a = a', b \leq b'$$

the result is a partial order.

Appendix D

Calculations

D.1 Equations of motion for S^{D-3}

The Lagrangian is given as

$$\begin{aligned}\mathcal{L}_{D-3} =& R * \mathbb{1} - \frac{D-5}{16} Y^{-2} * dY \wedge dY - \frac{1}{4}\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk} \wedge \tilde{T}^{-1}_{k\ell} \mathcal{D}\tilde{T}_{\ell i} - \frac{1}{2} Y^{-1} * F_{(3)} \wedge F_{(3)} \\ & - \frac{1}{4} Y^{-\frac{1}{2}} \tilde{T}^{-1}_{ik} \tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} - \frac{1}{2} g^2 Y^{\frac{1}{2}} (2\tilde{T}_{ij}\tilde{T}_{ij} - (\tilde{T}_{ii})^2)\end{aligned} \tag{D.1}$$

we also define the field strengths as

$$F^{ij}_{(2)} = dA^{ij}_1 + gA^{ik} \wedge A^{kj}_{(1)} \tag{D.2}$$

$$F_{(3)} = dA_{(2)} + \frac{1}{8}\epsilon_{ijk\ell}\left(F^{ij}_{(2)} \wedge A^{k\ell}_{(1)} - \frac{1}{3} g A^{ij}_{(1)} \wedge A^{km}_{(1)} \wedge A^{m\ell}_{(1)}\right) . \tag{D.3}$$

Ignoring the variation of metric which would give us the Einstein field equations and only consider the variation of the scalar field, the gauge potential and the vector Y. We begin with the variation of the vector Y which is given as

$$\begin{aligned}\delta_Y S = & -\frac{D-5}{16}\delta_Y (Y^{-2} * dY \wedge dY) - \frac{1}{2}\delta_Y Y^{-1} F_{(3)} \wedge F_{(3)} \\ & - \frac{1}{4}\delta_Y Y^{-\frac{1}{2}} \tilde{T}^{-1}_{ik} \tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} - \frac{1}{2} g^2 \delta_Y Y^{\frac{1}{2}} \left(2\tilde{T}^{-1}_{ij}\tilde{T}^{-1}_{ij} - (\tilde{T}^{-1}_{ii})^2\right) ,\end{aligned} \tag{D.4}$$

using the relation $\tilde{T}_{ij} = Y^{\frac{1}{4}} T_{ij}$ and after partial integration where all the boundary terms are set to zero we obtain the following equation of motion of the dilaton given as

$$\begin{aligned}\frac{D-5}{4} d(Y^{-1} * dY) =& \frac{1}{2} g^2 (2T_{ij}T_{ij} - (T_{ii})^2)_{\epsilon_{D-3}} - Y^{-1} * F_{(3)} \wedge F_{(3)} \\ & - \frac{1}{4} Y^{-1} \tilde{T}^{-1}_{ik} \tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} .\end{aligned} \tag{D.5}$$

The variation of the 2-form gauge potential and the 1-form gauge potential is given by

$$\begin{aligned}\delta_A S = & -\frac{1}{4}\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk} \wedge \tilde{T}^{-1}_{k\ell} \delta_A (d\tilde{T}_{\ell i} + gA^{\ell m}\tilde{T}_{mi} + gA^{im}\tilde{T}_{\ell m}) - \frac{1}{4} Y^{\frac{1}{2}} \tilde{T}^{-1}_{ik} \tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge \delta_A F^{k\ell}_{(2)} \\ & - \frac{1}{2} Y^{-1} * F_{(3)} \wedge \delta_A \left(dA_{(2)} + \frac{1}{8}\epsilon_{ijkl}\left(F^{ij}_{(2)} \wedge A^{kl}_{(1)} - \frac{1}{3} g A^{ij}_{(1)} \wedge A^{km}_{(1)} \wedge A^{m\ell}_{(1)}\right)\right) .\end{aligned} \tag{D.6}$$

Neglecting the total derivatives and and using the relation $\tilde{T}^{-1}_{k\ell}\tilde{T}_{\ell m} = 1$ we end up with the following terms

$$\begin{aligned}\delta_A S = &-\frac{1}{4}g\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk} \wedge \tilde{T}^{-1}_{k\ell}\delta_A A^{\ell m}_{(1)}\tilde{T}_{mi} - \frac{1}{4}g\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}^{-1}_{jk} \wedge \tilde{T}^{-1}_{k\ell}\delta_A A^{im}_{(1)}T_{\ell m} \\ &-\frac{1}{2}Y^{-1} * F_{(3)} \wedge d\delta_A A_{(2)} - \frac{1}{8}Y^{\frac{1}{2}}\tilde{T}^{-1}_{ik}\tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge \mathcal{D}\delta_A A^{k\ell}_{(1)} \\ &-\frac{1}{16}Y^{-1}\epsilon_{ijk\ell} * F_{(3)} \wedge F^{ij}_{(2)} \wedge \delta_A A^{k\ell}_{(1)} \ ,\end{aligned} \tag{D.7}$$

doing the partial integration and using the scalar relation above we obtain the following equations of motion

$$d(Y^{-1} * F_{(3)}) = 0 \tag{D.8}$$

$$(-1)^D\mathcal{D}(T^{-1}_{ik}T^{-1}_{j\ell} * F^{k\ell}_{(2)}) = -2gT^{-1}_{k[i} * \mathcal{D}T_{j]k} - \frac{1}{2}\epsilon_{ijk\ell} * F_{(3)} \wedge F^{k\ell}_{(2)} \ . \tag{D.9}$$

Notice that we relabeled some terms and the $(-1)^D$ term comes from doing a partial integration on differential forms where D is the degree of the p-form.

The last variation is w.r.t $\delta_T T$ the terms that will contribute to the equations of motion will be

$$\delta_T S = -\frac{1}{4}\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk} \wedge \tilde{T}^{-1}_{k\ell}\mathcal{D}\delta_T\tilde{T}_{\ell i} - \frac{1}{4}Y^{-\frac{1}{2}}\tilde{T}^{-1}_{ik}\tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} - \delta_T V * \mathbb{1} \ , \tag{D.10}$$

we will deal with the potential later on. Using the following relation $\delta_T\tilde{T}^{-1}_{ij} = -\tilde{T}^{-2}_{ik}\delta_T\tilde{T}_{kj}$ and rewriting the terms above we obtain

$$\begin{aligned}\delta_T S = &-\frac{1}{4}\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk} \wedge \tilde{T}^{-1}_{k\ell}\mathcal{D}\delta_T\tilde{T}_{\ell i} + \frac{1}{4}Y^{-\frac{1}{2}}\tilde{T}^{-1}_{ik}\tilde{T}^{-2}_{jn}\delta_T\tilde{T}_{n\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} \\ &+\frac{1}{4}Y^{-\frac{1}{2}}\tilde{T}^{-2}_{im}\delta_T\tilde{T}_{mk}\tilde{T}^{-1}_{j\ell} * F^{ij}_{(2)} \wedge F^{k\ell}_{(2)} \ .\end{aligned} \tag{D.11}$$

Relabeling the second and third term and doing partial integration on the first term yields us

$$\begin{aligned}\delta_T S = &\frac{1}{4}(-1)^D\mathcal{D}\left(\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk}\right)\tilde{T}^{-1}_{k\ell}\delta_T\tilde{T}_{\ell i} + \frac{1}{4}Y^{-\frac{1}{2}}\tilde{T}^{-1}_{k\ell}\tilde{T}^{-1}_{k\ell}\delta_T\tilde{T}_{\ell i}\tilde{T}^{-1}_{mn} * F^{nk}_{(2)} \wedge F^{mi}_{(2)} \\ &+ \frac{1}{4}Y^{-\frac{1}{2}}\tilde{T}^{-1}_{k\ell}\tilde{T}^{-1}_{k\ell}\delta_T\tilde{T}^{-1}_{\ell i}\tilde{T}^{-1}_{jm} * F^{kj}_{(2)} \wedge F^{m\ell}_{(2)} \ ,\end{aligned} \tag{D.12}$$

gathering all the terms we obtain

$$\delta_T S = \frac{1}{2}\tilde{T}^{-1}_{k\ell}\delta_T\tilde{T}_{\ell i}\left((-1)^D\mathcal{D}\left(\tilde{T}^{-1}_{ij} * \mathcal{D}\tilde{T}_{jk}\right) + T^{-1}_{k\ell}T^{-1}_{mn} * F^{nk}_{(2)} \wedge F^{mi}_{(2)} + T^{-1}_{k\ell}T^{-1}_{jm} * F^{kj}_{(2)} \wedge F^{m\ell}_{(2)}\right) \ . \tag{D.13}$$

We now have to deal with the potential, which is given as

$$\delta_T V = \frac{1}{2}g^2Y^{\frac{1}{2}}\delta_T\left(\tilde{T}^{-1}_{kj}\tilde{T}_{jm}\left(2\tilde{T}_{ij}\tilde{T}_{ij} - (\tilde{T}_{ii})^2\right)\right) \tag{D.14}$$

taking the variation we obtain

$$\begin{aligned}\delta_T V &= \frac{1}{2}g^2Y^{\frac{1}{2}}\left(\tilde{T}^{-1}_{kj}\delta_T\tilde{T}_{jm}\left(2\tilde{T}_{ij}\tilde{T}_{ij} - 2(\tilde{T}_{ii})^2\right) + \tilde{T}^{-1}_{kj}\tilde{T}_{jm}\left(2\delta_T\tilde{T}_{ij}\tilde{T}_{ij} - 2\tilde{T}_{ii}\delta_T\tilde{T}_{np}\delta^{np}\right)\right) \\ &= \frac{1}{2}g^2\tilde{T}^{-1}_{k\ell}\delta_T\tilde{T}_{\ell i}\left(2\left(2T_{\ell i}T_{\ell i} - (T_{ii})^2\right) + (2T_{jm}T_{\ell i} - 2T_{pp}T_{\ell m})\right) \ .\end{aligned} \tag{D.15}$$

We thus see that the variations are all the same now and we obtain the following equations of motion

$$(-1)^D\mathcal{D}\left(\tilde{T}^{-1}_{ij}*\mathcal{D}\tilde{T}_{jk}\right)=2g^2\left(2T_{ik}T_{jk}-T_{ij}T_{kk}\right)\epsilon_{D-3}-\tilde{T}^{-1}_{\ell m}\tilde{T}^{-1}_{ik}*F^{\ell k}_{(2)}\wedge F^{mj}_{(2)}$$
$$-\frac{1}{4}\delta_{ij}\left(2g^2\left(2T_{nk}T_{nk}-(T_{kk})^2\right)\epsilon_{D-3}-\tilde{T}^{-1}_{\ell m}\tilde{T}^{-1}_{nk}*F^{\ell k}_{(2)}\wedge F^{mn}_{(2)}\right) \tag{D.16}$$

D.2 Einstein field equation consistency check

The Ricci tensor is given as

$$\begin{aligned}
\hat{R}_{\mu\nu}=&R_{\mu\nu}-\frac{1}{4}X_i^{-2}\partial_\mu X_i\partial_\nu X_i+\frac{1}{2}\Delta^{-1}X_i^{-1}\mu_i^2\partial_\mu X_i\partial_\nu X_i-\frac{1}{2}\Delta^{-2}\partial_\mu\Delta\partial_\nu\Delta\\
&+\frac{1}{2}a\left(\Delta^{-2}\partial_\lambda\Delta\partial^\lambda\Delta-\Delta^{-1}\Box\Delta\right)g_{\mu\nu}\\
&-a\left(\sum_i X_i^2-\Delta^{-1}X_i^2\mu_i^2\sum_j X_j-2\Delta^{-1}X_i^3\mu_i^2+2\Delta^{-2}(X_i^2\mu_i^2)^2\right)g_{\mu\nu}\\
\hat{R}_{\alpha\beta}=&R_{\alpha\beta}+\frac{1}{2}bg_{\alpha\beta}\Delta^{-2}\Box\Delta-\frac{1}{2}bg_{\alpha\beta}\Delta^{-3}\partial_\lambda\Delta\partial^\lambda\Delta-\frac{1}{2}\Delta^{-1}\Box g_{\alpha\beta}\\
&+\frac{1}{2}\Delta^{-1}g^{\gamma\delta}\partial_\lambda g_{\alpha\gamma}\partial^\lambda g_{\beta\delta}-\frac{1}{4}\Delta^{-2}\partial_\alpha\Delta\partial_\beta\Delta-\frac{1}{2}\Delta^{-1}\nabla_\alpha\partial_\beta\Delta\\
&-\frac{1}{4}bg_{\alpha\beta}\Delta^{-2}\partial_\gamma\Delta\partial^\gamma\Delta+\frac{1}{2}bg_{\alpha\beta}\Delta^{-1}\nabla_\gamma\partial^\gamma\Delta\\
\hat{R}_{\alpha\mu}=&-\frac{1}{2}\Delta^{-2}U(X_\alpha^{-1}\partial_\mu X_\alpha-X_0^{-1}\partial_\mu X_0)\mu_\alpha
\end{aligned} \tag{D.17}$$

and useful properties are given as

$$\begin{aligned}
\partial^\alpha\Delta\partial_\alpha\Delta&=4X_i^3\mu_i^2-4\Delta^{-1}(X_i^2\mu_i^2)^2\\
\Gamma^\alpha{}_{\alpha\beta}&=\frac{1}{2}\Delta^{-1}\partial_\beta\Delta+\frac{1}{\mu_0^2}\mu_\beta\\
\nabla_\alpha\partial^\alpha\Delta&=2\sum_i X_i^2-2\Delta^{-1}X_i^2\mu_i^2\sum_j X_j+4\Delta^{-2}(X_i^2\mu_i^2)^2-4\Delta^{-1}X_i^3\mu_i^2+\frac{1}{2}\Delta^{-1}\partial^\alpha\Delta\partial_\alpha\Delta\\
R_{\alpha\beta}&=\Delta^{-1}\bar{g}_{\alpha\beta}\sum_\gamma X_\gamma-\Delta^{-2}(X_i^2\mu_i^2)\bar{g}_{\alpha\beta}+\Delta^{-2}(X_\alpha-X_0)(X_\beta-X_0)\mu_\alpha\mu_\beta-\Delta^{-1}(X_\alpha-X_0)\delta_{\alpha\beta}\\
\Box g_{\alpha\beta}&=X_\alpha^{-3}\partial_\lambda X_\alpha\partial^\lambda X^\alpha\delta_{\alpha\beta}+X_0^{-3}\partial_\lambda X_0\partial^\lambda X_0\hat{\mu}_\alpha\hat{\mu}_\beta-4(X_\alpha\delta_{\alpha\beta}+X_0\hat{\mu}_\alpha\hat{\mu}_\beta)+2\bar{g}_{\alpha\beta}\sum_j X_j+\frac{4}{N}Vg_{\alpha\beta}\\
g^{\gamma\delta}\partial_\lambda g_{\alpha\gamma}\partial^\lambda g_{\beta\delta}&=X_\alpha^{-3}\partial_\lambda X_\alpha\partial^\lambda X^\alpha\delta_{\alpha\beta}+X_0^{-3}\partial_\lambda X_0\partial^\lambda X_0\hat{\mu}_\alpha\hat{\mu}_\beta\\
&\quad-\Delta^{-1}(X_\alpha^{-1}\partial_\lambda X_\alpha-X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta-X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta\ .
\end{aligned}$$

We can use these relations to simplify the Ricci tensors which will be needed for checking that the equations of motion are satisfied for the Einstein field equations. The scalar equation of motion can be written as

$$\Box\Delta=X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i+4X_i^3\mu_i^2-2X_i^2\mu_i^2\sum_j X_j-\frac{4}{N}\Delta V\ . \tag{D.18}$$

Let us first consider $\hat{R}_{\alpha\beta}$ using the relations above we can expand the Ricci tensor to

$$\begin{aligned}
\hat{R}_{\alpha\beta} = &\Delta^{-1}\bar{g}_{\alpha\beta}\sum_{\gamma}X_\gamma - \Delta^{-2}(X_i^2\mu_i^2)\bar{g}_{\alpha\beta} + \Delta^{-2}(X_\alpha - X_0)(X_\beta - X_0)\mu_\alpha\mu_\beta - \Delta^{-1}(X_\alpha - X_0)\delta_{\alpha\beta} \\
&+ \frac{1}{2}bg_{\alpha\beta}\Delta^{-2}X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i + 2bg_{\alpha\beta}\Delta^{-2}X_i^3\mu_i^2 - bg_{\alpha\beta}\Delta^{-2}X_i^2\mu_i^2\sum_j X_j - \frac{2}{N}bg_{\alpha\beta}\Delta^{-1}V \\
&- 2g_{\alpha\beta}\Delta^{-3}X_i^3\mu_i^2 - 2g_{\alpha\beta}\Delta^{-4}(X_i^2\mu_i^2)^2 - \frac{1}{2}\Delta^{-1}X_\alpha^{-3}\partial_\lambda X_\alpha\partial^\lambda X^\alpha\delta_{\alpha\beta} - \frac{1}{2}\Delta^{-1}X_0^{-3}\partial_\lambda X_0\partial^\lambda X_0\hat{\mu}_\alpha\hat{\mu}_\beta \\
&- 2\Delta^{-1}(X_\alpha\delta_{\alpha\beta} + X_0\hat{\mu}_\alpha\hat{\mu}_\beta) - \Delta^{-1}\bar{g}_{\alpha\beta}\sum_j X_j - \Delta^{-1}\frac{2}{N}Vg_{\alpha\beta} + \frac{1}{2}\Delta^{-1}X_\alpha^{-3}\partial_\lambda X_\alpha\partial^\lambda X^\alpha\delta_{\alpha\beta} \\
&+ \frac{1}{2}\Delta^{-1}X_0^{-3}\partial_\lambda X_0\partial^\lambda X_0\hat{\mu}_\alpha\hat{\mu}_\beta - \frac{1}{2}\Delta^{-2}(X_\alpha^{-1}\partial_\lambda X_\alpha - X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta - X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta \\
&- \frac{1}{4}\Delta^{-2}\partial_\alpha\Delta\partial_\beta\Delta - \frac{1}{2}\Delta^{-1}\nabla_\alpha\partial_\beta\Delta - bg_{\alpha\beta}\Delta^{-2}X_i^3\mu_i^2 - bg_{\alpha\beta}\Delta^{-3}(X_i^2\mu_i^2)^2 \\
&+ \frac{1}{2}bg_{\alpha\beta}\Delta^{-1}\left(2\sum_i X_i^2 - 2\Delta^{-1}X_i^2\mu_i^2\sum_j X_j + 4\Delta^{-2}(X_i^2\mu_i^2)^2 - 4\Delta^{-1}X_i^3\mu_i^2 + \frac{1}{2}\Delta^{-1}\partial^\gamma\Delta\partial_\gamma\Delta\right)
\end{aligned}$$

we can simplify this using the following relations

$$\hat{\mu}_\alpha = \mu_\alpha/\mu_0, \quad \bar{g}_{\alpha\beta} = \delta_{\alpha\beta} + \hat{\mu}_\alpha\hat{\mu}_\beta \ . \tag{D.19}$$

First we gather some terms that will not disappear

$$\begin{aligned}
\hat{R}_{\alpha\beta} \supset &- 2bg_{\alpha\beta}\Delta^{-2}X_i^2\mu_i^2\sum_j X_j + bg_{\alpha\beta}\Delta^{-1}\sum_j X_j^2 + bg_{\alpha\beta}\Delta^{-3}(X_i^2\mu_i^2)^2 = \frac{1}{2}b\Delta^{-3}U^2g_{\alpha\beta} \\
\hat{R}_{\alpha\beta} \supset &\frac{1}{2}bg_{\alpha\beta}\Delta^{-2}X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i \\
\hat{R}_{\alpha\beta} \supset &\frac{1}{2}\Delta^{-2}(X_\alpha^{-1}\partial_\lambda X_\alpha - X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta - X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta \ ,
\end{aligned}$$

notice that some terms cancel among them self and we end up with the following terms

$$\begin{aligned}
\hat{R}_{\alpha\beta} = &-\Delta^{-2}(X_i^2\mu_i^2)\bar{g}_{\alpha\beta} + \Delta^{-2}(X_\alpha - X_0)(X_\beta - X_0)\mu_\alpha\mu_\beta - \Delta^{-1}(X_\alpha - X_0)\delta_{\alpha\beta} \\
&+ \frac{1}{2}bg_{\alpha\beta}\Delta^{-2}X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i + \frac{1}{2}b\Delta^{-3}U^2g_{\alpha\beta} - \frac{1}{2}bg_{\alpha\beta}\Delta^{-3}\partial_\lambda\Delta\partial^\lambda\Delta \\
&- \frac{2}{N}bg_{\alpha\beta}\Delta^{-1}V - 2\Delta^{-1}(X_\alpha\delta_{\alpha\beta} + X_0\hat{\mu}_\alpha\hat{\mu}_\beta) - \frac{2}{N}g_{\alpha\beta}\Delta^{-1}V - \frac{1}{4}\Delta^{-2}\partial_\alpha\Delta\partial_\beta\Delta \\
&- \frac{1}{2}\Delta^{-1}\nabla_\alpha\partial_\beta\Delta - bg_{\alpha\beta}\Delta^{-2}X_i^3\mu_i^2 + \frac{1}{4}bg_{\alpha\beta}\Delta^{-2}\left(4X_i^3\mu_i^2 - 4\Delta^{-1}(X_i^2\mu_i^2)^2\right) \\
&- \frac{1}{2}\Delta^{-2}(X_\alpha^{-1}\partial_\lambda X_\alpha - X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta - X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta \ ,
\end{aligned}$$

after more algebra manipulations making the rest of the non surviving terms canceling among them self, we can finally simplify the Ricci tensor to

$$\begin{aligned}
\hat{R}_{\alpha\beta} = &\frac{1}{2}b\Delta^{-3}U^2g_{\alpha\beta} + \frac{1}{2}\Delta^{-2}g_{\alpha\beta}X_i^{-1}\mu_i^2\partial_\lambda X_i\partial^\lambda X_i - \frac{1}{2}b\Delta^{-3}\partial_\lambda\Delta\partial^\lambda\Delta \\
&- \frac{1}{2}\Delta^{-2}(X_\alpha^{-1}\partial_\lambda X_\alpha - X_0^{-1}\partial_\lambda X_0)(X_\beta^{-1}\partial^\lambda X_\beta - X_0^{-1}\partial^\lambda X_0)\mu_\alpha\mu_\beta \ .
\end{aligned} \tag{D.20}$$

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