BIANCHI TYPE-I COSMOLOGICAL MODELS WITH TIME DEPENDENT GRAVITATIONAL AND COSMOLOGICAL CONSTANTS: AN ALTERNATIVE APPROACH

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The present study deals with the exact solutions of the Einstein's field equations with variable gravitational and cosmological "constants" for a spatially homogeneous and anisotropic Bianchi type-I space-time. We assume that the conservation law for the matter is fulfilled. Hence giving rise to separate equations one for the perfect fluid and other connecting gravitational and cosmological constants. Assuming that G be a function of volume scale V, the metric functions, Λ-term, energy density and pressure are found to be functions of V. The equation for V is found through Einstein's field equations and solved both analytically and numerically. The present study also allows a time dependent deceleration parameter (DP). It is found that for empty universe, the derived model is accelerating whereas for radiating dominated and stiff fluid universes, we obtain models that depict a transition of the universe from the early decelerated phase to the recent accelerating phase. The cosmological constant Λ is obtained as a decreasing function of time and approaching a small positive value at present epoch which is corroborated by consequences from recent supernovae Ia observations. The physical significances of the cosmological models have also been discussed.

Key words: Cosmology, Variable gravitational & cosmological constants, Variable deceleration parameter.

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1. INTRODUCTION

Einstein's theory of gravity contains two fundamental constants: Newton's gravitational constant G and the cosmological constant Λ. Here G plays the role of coupling constant between geometry and matter while, Λ was introduced by Einstein [1] as the universal repulsion to make the universe static in accordance with generally accepted picture of that time. But a general expansion of the universe was observed by Hubble [2] subsequently. The variability of G and Λ is also one of the most striking and unsettled problems in cosmology. A time variation of G has been first suggested by Dirac [3–5] and extensively described in literature [6–8].

The recent observations by Perlmutter *et al.* [9, 10] and Riess *et al.* [11, 12] suggested a positive value of Λ, which causes the acceleration in the expansion of universe. The status of cosmological constant has been reviewed by Carroll *et al.* [13], Sahni and Starobinsky [14], Peebles and Ratra [15], Padmanabhan [16, 17]. Among the various solutions proposed is the phenomenologically simple one of endowing the effective (Λ) with a variable dynamic degree of freedom which allows it to relax to its present value in an expanding universe. Berman [18, 19], Chen and Wu [20] have argued in favour of the dependence $\Lambda \sim t$. Recently, several authors [21–45] have studied variable cosmological constant $\Lambda(t)$ in different context.

Anisotropic Bianchi type-I universe, which is more general than FRW universe, plays a significant role to understand the phenomenon like formation of galaxies in early universe. Theoretical arguments as well as the recent observations of cosmic microwave background radiation (CMBR) support the existence of anisotropic phase that approaches an isotropic one. Motivated by the above discussions, in this paper, we propose to study homogeneous and anisotropic Bianchi type-I cosmological models with time dependent gravitational and cosmological "constants". The paper is organized as follows. In Sect. 2, the metric and basic equations have been presented. Section 3 deals with results and discussions. Finally, conclusions are summarized in the last Sect. 4.

2. THE METRIC AND BASIC EQUATIONS

We consider the space-time metric of the spatially homogeneous and anisotropic Bianchi-I of the form

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2. \tag{1}$$

where A(t), B(t) and C(t) are the metric functions of cosmic time t.

Einstein field equations with time-dependent G and Λ are given by

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G\, T_{ij} + \Lambda\, g_{ij}, \tag{2}$$

where the symbols have their usual meaning.

For a perfect fluid, the stress-energy-momentum tensor T_{ij} is given by

$$T_{ij} = (\rho + p)u_i u_j - p\, g_{ij}, \tag{3}$$

where ρ is the matter density, p is the thermodynamics pressure and u^i is the fluid four-velocity vector of the fluid satisfying the condition

$$u^i u_i = 1. \tag{4}$$

In the field equations (2), Λ accounts for vacuum energy with its energy density ρ_v

and pressure p_v satisfying the equation of state

$$p_v = -\rho_v = -\frac{\Lambda}{8\pi G} \tag{5}$$

The critical density and the density parameters for matter and cosmological constant are, respectively, defined as

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{6}$$

$$\Omega_M = \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}, \tag{7}$$

$$\Omega_\Lambda = \frac{\rho_v}{\rho_c} = \frac{\Lambda}{3H^2}. \tag{8}$$

In a comoving system of coordinates, the field Eqs. (2) for the metric (1) with (3) read as

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -8\pi G p + \Lambda, \tag{9}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -8\pi G p + \Lambda, \tag{10}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi G p + \Lambda, \tag{11}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = 8\pi G\rho + \Lambda. \tag{12}$$

The covariant divergence of Eqs. (2) yield

$$\dot{\rho} + 3(\rho + p)H + \rho\frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{13}$$

The spatial volume for the model (1) is given by

$$V = ABC \tag{14}$$

We define the average scale factor a of anisotropic model as

$$a = (ABC)^{\frac{1}{3}} = V^{\frac{1}{3}} \tag{15}$$

So that the generalized mean Hubble parameter H is given by

$$H = \frac{1}{3}(H_x + H_y + H_z), \tag{16}$$

where $H_x = \frac{\dot{A}}{A}, H_y = \frac{\dot{B}}{B}, H_z = \frac{\dot{C}}{C}$ are the directional Hubble parameters in the direction of x, y and z respectively and a dot denotes differentiation with respect to cosmic time t.

From Eqs. (15) and (16), we obtain an important relation

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \frac{1}{3}\frac{\dot{V}}{V}. \tag{17}$$

The expression for the dynamical scalars such as the expansion scalar (θ), anisotropy parameter (A_m) and the shear scalar (σ) are defined as usual:

$$\theta = u^i_{;i} = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \frac{\dot{V}}{V}, \tag{18}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \tag{19}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right] - \frac{\theta^2}{6}. \tag{20}$$

We define the DP (deceleration parameter) q as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{H} + H^2}{H^2}\right) = 2 - 3\frac{\ddot{V}V}{\dot{V}^2}. \tag{21}$$

3. RESULTS AND DISCUSSION

The field Eqs. (9−12) are a system of four equations with seven unknown parameters A, B, C, G, p, ρ and Λ. Hence, three additional constraints relating these parameters are required to obtain explicit solution of the system.

From Eqs. (9)−(11), following Saha [46], one can derive the metric functions in terms of V as

$$A(t) = l_1 V^{1/3}\exp\left(m_1\int V^{-1}dt\right), \tag{22}$$

$$B(t) = l_2 V^{1/3}\exp\left(m_2\int V^{-1}dt\right), \tag{23}$$

$$C(t) = l_3 V^{1/3}\exp\left(m_3\int V^{-1}dt\right), \tag{24}$$

where constants m_1, m_2, m_3 and l_1, l_2, l_3 satisfy the following two relations:

$$m_1 + m_2 + m_3 = 0, \quad l_1 l_2 l_3 = 1. \tag{25}$$

Note that in a previous paper [47], we assumed some concrete form for a or V. In this case we take another approach, by assuming that conservation law for the material field fulfils. In this case Eq. (13) can be separated to write

$$\dot{\rho}+3(\rho+p)H=0, \tag{26}$$

and

$$8\pi\rho\dot{G}+\dot{\Lambda}=0. \tag{27}$$

First we assume equation of state for perfect fluid as

$$p=\gamma\rho, \tag{28}$$

where γ $(0\leq\gamma\leq1)$ is constant.

On account of Eqs. (28) and (17), Eq. (26) can be solved to get

$$\rho=\frac{\rho_0}{V^{1+\gamma}}, \tag{29}$$

where ρ_0 is an arbitrary constant.

Second, we assume a power-law form of the gravitational constant (G) with scale factor as proposed by Singh and Kumar [31], Chawla *et al.* [44] and Pradhan *et al.* [45]

$$G\propto a^m, \tag{30}$$

where m is a constant. For sake of mathematical simplicity, Eq. (30) may be written as

$$G=G_0a^m=G_0V^{\frac{m}{3}}, \tag{31}$$

where G_0 is a positive constant.

Now inserting Eqs. (29) and (31) into Eq. (27) one finds

$$\Lambda=\Lambda_0-\left[\frac{8\pi\rho_0G_0m}{m-3(1+\gamma)}\right]V^{\frac{m}{3}-(1+\gamma)}, \tag{32}$$

where Λ_0 is an arbitrary constant. From above equation we see that Λ is a decreasing function of time and it is always positive when

$$V<\left[\Lambda_0\frac{m-3(1+\gamma)}{8\pi\rho_0G_0m}\right]^{\frac{3}{m-3(1+\gamma)}}.$$

Thus we see that the metric functions, energy density, pressure, Newton's gravitational constant and Einstein's cosmological constant can be expressed as a function of volume scale V. In what follows we find the equation for determining V and solve it both analytically and numerically. Now summation of Eqs. (9), (10), (11) and three times Eq. (12) gives

$$\ddot{V}=12\pi(1-\gamma)G\rho V+3\Lambda V. \tag{33}$$

Inserting ρ, G and Λ into (33) one finds

$$\ddot{V}=XV^{\frac{m}{3}-\gamma}+3\Lambda_0V, \tag{34}$$

with

$$X = 12\pi\rho_0 G_0 (1+\gamma)\left[\frac{3(1-\gamma)+m}{3(1+\gamma)-m}\right]. \tag{35}$$

The foregoing equation allows the first integral

$$\dot{V}^2 = X_1 V^{1+\frac{m}{3}-\gamma} + 3\Lambda_0 V^2 + V_0, \tag{36}$$

where we denote $X_1 = \frac{6X}{[3(1-\gamma)+m]}$. Here V_0 is the constant of integration. In view of Eq. (36) the solution for V can be written in quadrature as

$$\int \frac{dV}{\sqrt{\left[X_1 V^{1+\frac{m}{3}-\gamma} + 3\Lambda_0 V^2 + V_0\right]}} = t + t_0, \tag{37}$$

where t_0 is an integrating constant. Thus, we have the solution to the corresponding equation in quadrature.

In Fig. 1, we plot the evolution of V in time. As one see, it is an expanding function of time, with V expanding as a power-law function at early stage and exponentially at late. Here and in the figures following the solid line (red) corresponds to $\gamma = 0$ (dust Universe), dot line (magenta) corresponds to $\gamma = 1/3$ (radiation dominated Universe) and dash line (blue) corresponds to $\gamma = 1$ (stiff or Zel'dovich Universe). Here and in the figures following we have considered the values of different constants as $\rho_0 = \Lambda_0 = m = 1$, $\Lambda_0 = 0.1$, $m_1 = 0.25$, $m = 0.75$ and $m_3 = -1$ as a representative case. We have solved Eq. (33) numerically with initials $V_0 = 0.001$ and $\dot{V}_0 = 0.003$. In all figures, we have not mentioned the parameter, as we give a qualitative picture of the evolution.

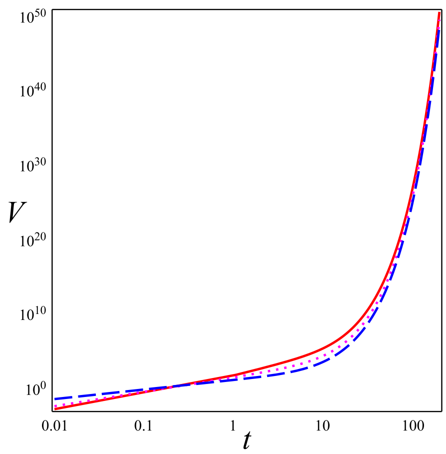

Fig. 1 – Evolution of the volume scale V *vs.* time t.

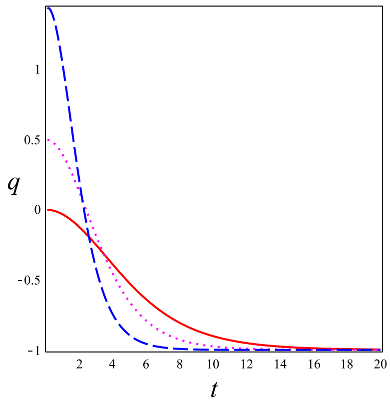

Fig. 2 – Evolution of deceleration parameter q *vs.* time t.

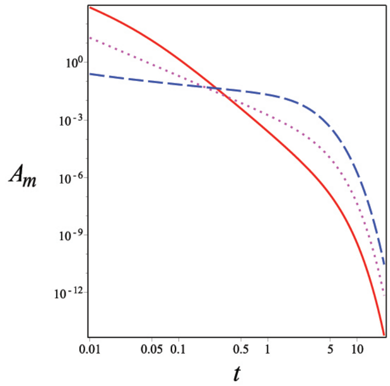

Fig. 3 – Evolution of anisotropic parameter A_m *vs.* time t.

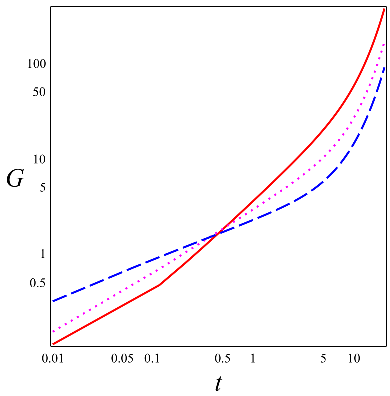

Fig. 4 – Evolution of gravitational constant G *vs.* time t.

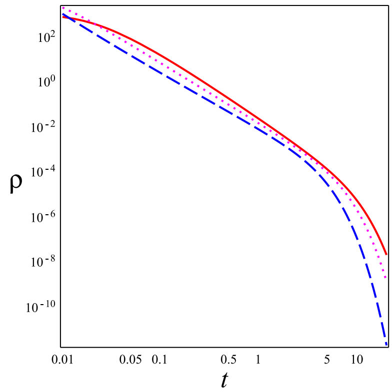

Fig. 5 – Evolution of energy density ρ *vs.* time t.

In this case, expressions for physical parameters such as expansion scalar (θ), Hubble's parameter (H), deceleration parameter, shear scalar (σ) and anisotropy parameter (A_m) are given by

$$\theta = 3H = \sqrt{\left[X_1 V^{\frac{m}{3}-1-\gamma} + 3\Lambda_0 + \frac{V_0}{V^2}\right]}, \tag{38}$$

$$q = 2 - \frac{3\ddot{V}V}{\dot{V}^2} = 2 - 3\left[\frac{XV^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2}{X_1 V^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2 + V_0}\right], \tag{39}$$

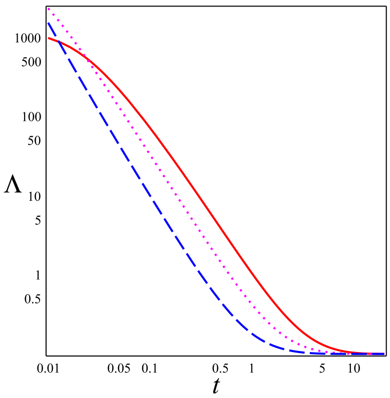

Fig. 6 – Evolution of cosmological constant Λ *vs.* time t.

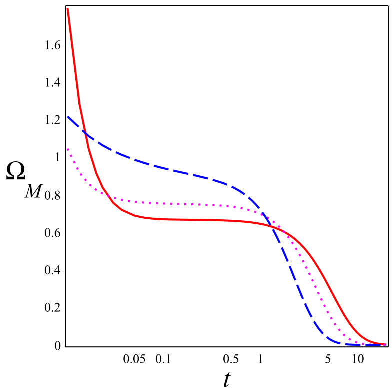

Fig. 7 – Evolution of density parameter Ω_M *vs.* time t.

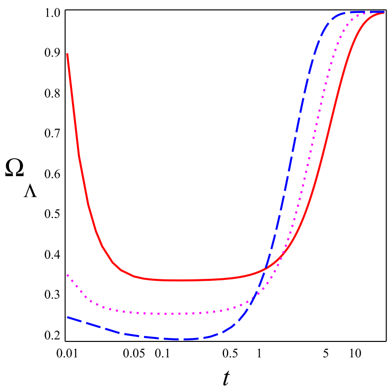

Fig. 8 – Evolution of density parameter Ω_Λ *vs.* time t.

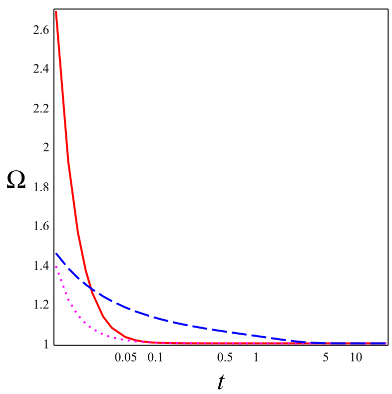

Fig. 9 – Evolution of total density parameter Ω *vs.* time t.

$$\sigma^2 = \frac{m_1^2+m_2^2+m_3^2}{V^2} + \frac{3}{2}H^2 = \frac{m_1^2+m_2^2+m_3^2}{V^2} + \frac{1}{6}\left[X_1 V^{\frac{m}{3}-1-\gamma} + 3\Lambda_0 + \frac{V_0}{V^2}\right], \quad (40)$$

$$A_m = \frac{m_1^2+m_2^2+m_3^2}{3V^2H^2} = \frac{3(m_1^2+m_2^2+m_3^2)}{\left[X_1 V^{1+\frac{m}{3}-\gamma} + 3\Lambda_0 V^2 + V_0\right]}. \quad (41)$$

From above relations (29), (31), (32) and (38)–(41), we can obtain the expressions for different physical parameters for four types of models:

- When $\gamma = 0$, we obtain empty model.
- When $\gamma = \frac{1}{3}$, we obtain radiating dominated model.
- When $\gamma = -1$, we have the degenerate vacuum or false vacuum or ρ vacuum model [48].
- When $\gamma = 1$, the fluid distribution corresponds with the equation of state $\rho = p$ which is known as Zeldovich fluid or stiff fluid model [49, 50].

Therefore, we study the variation of these parameter with respect to cosmic time t for three values of $\gamma = 0$, $\frac{1}{3}$ and 1.

From Eq. (38), we observe that the Hubble parameter tends to infinity where $V \to 0$. From Eq. (39), we observe that

$$q > 0 \ \text{ if } \ \frac{XV^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2}{X_1 V^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2 + V_0} < \frac{2}{3},$$

and

$$q < 0 \ \text{ if } \ \frac{XV^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2}{X_1 V^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2 + V_0} > \frac{2}{3}.$$

Figure 2 depicts the variation of q *versus* cosmic time t. From this figure, we observe that for empty model ($\gamma = 0$), the universe is accelerating whereas for radiating dominated ($\gamma = \frac{1}{3}$) and stiff fluid model ($\gamma = 1$), the universe has transition from very early decelerated phase to the present accelerating phase. In such type of universe, the DP must show signature flipping [51−53].

From Eq. (41), one observes that anisotropy is constant as $V \to 0$, but vanishes away with increase of V. The variation of anisotropic parameter with t is shown in Fig. 3 for $\gamma = 0$, $\frac{1}{3}$ and 1. From this figure we observe that A_m decreases with time and approaches to zero as $t \to \infty$. Thus, the observed isotropy of the universe can be achieved in our three types of models at present epoch.

From Eq. (31), we observe that G is an increasing function of V if $m > 0$ whereas G is a decreasing function of V if $m < 0$. The nature of variation of G with cosmic time is shown in Fig. 4 for three values of $\gamma = 0, \frac{1}{3}$ and 1 by considering $m = 1$. We observe that for all three types of models (empty, radiating dominated and stiff fluid), G is found to be increasing function of time. The possibility of an increasing G has also been suggested by several authors [32−34, 54].

The energy density has been graphed versus time in Fig. 5 for $\gamma = 0, \frac{1}{3}$ and 1. It is evident that the energy density remains positive in all three types of models under appropriate condition. However, it decreases more sharply with the cosmic time in empty universe compare to radiating dominated and Zeldovich Universes.

Figure 6 is the plot of cosmological term Λ *versus* time for $\gamma = 0, \frac{1}{3}$ and 1. In all three types of models, we observe that Λ is decreasing function of time t and it approaches a small positive value at late time (*i.e.* at present epoch). However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and stiff fluid universes. Recent cosmological observations [9−12] suggest the existence of a positive and small cosmological constant Λ at present epoch. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus, the nature of Λ in our derived models are supported by recent observations.

The vacuum energy density (ρ_ν), critical density (ρ_c) and the density parameter (Ω_M, Ω_Λ) are obtained as

$$\rho_\nu = \frac{\Lambda_0}{8\pi G_0 V^{\frac{m}{3}}} + \frac{m\rho_0}{[3(1+\gamma)-m]V^{1+\gamma}}, \tag{42}$$

$$\rho_c = \frac{1}{24\pi G_0}\left[X_1 V^{-(1+\gamma)} + 3\Lambda_0 V^{-\frac{m}{3}} + V_0 V^{-(2+\frac{m}{3})}\right], \tag{43}$$

$$\Omega_M = \frac{24\pi G_0 \rho_0 V^{\frac{m}{3}+1-\gamma}}{X_1 V^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2 + V_0}, \tag{44}$$

$$\Omega_\Lambda = \frac{3}{(3+3\gamma-m)}\left[\frac{(3+3\gamma-m)\Lambda_0 V^2 + 8\pi\rho_0 G_0 m V^{\frac{m}{3}+1-\gamma}}{X_1 V^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2 + V_0}\right], \tag{45}$$

$$\Omega = \frac{3}{(3+3\gamma-m)}\left[\frac{24\pi G_0 \rho_0 (1+\gamma) V^{\frac{m}{3}+1-\gamma} + (3+3\gamma-m)\Lambda_0 V^2}{X_1 V^{\frac{m}{3}+1-\gamma} + 3\Lambda_0 V^2 + V_0}\right]. \tag{46}$$

From Eq. (43) we see that when $V \to 0$, $\rho_c \to \infty$. From Eqs. (44) and (45), we observe that for $V \to 0$, both Ω_M and Ω_Λ vanish but for $V \to \infty$, both Ω_M and Ω_Λ approach to some constant. Figures 7 and 8 plot the variation of density parameters for matter (Ω_M) and cosmological constant (Ω_Λ) *versus* t respectively. From these figures it is clear that the universe is dominated by matter in early stage of evolution whereas the universe is dominated by dark energy (cosmological constant Λ) at present epoch. Figure 9 plots the variation of total energy parameter (Ω) *versus* cosmic time t. From the Fig. 9, we observe that $\Omega \to 1$ at late time which is in good agreement with the observational results [55].

4. CONCLUSION

In this report we have studied the exact solutions of the Einstein's field equations with variable gravitational and cosmological "constants" for a spatially homogeneous and anisotropic Bianchi type-I space-time by taking an alternative approach. Our solution has been given in quadrature and it is regarded as exact solution [56]. We assume that the conservation law for the matter is fulfilled. Hence giving rise to separate equations one for the perfect fluid and other connecting gravitational and cosmological constants. The equation for volume scale V is found by using Einstein's field equations and solved both analytically and numerically. The present study also permit a time dependent deceleration parameter. It is found that for empty universe, the derived model is accelerating whereas for radiating dominated and stiff fluid universes, we obtain models that depict a transition of the universe from the early decelerated phase to the recent accelerating phase. The observations suggest that the universe was previously decelerating which entered into an accelerating phase. It was decelerating when dominant was matter, but afterwards when dark energy became dominant the phase transition took place. A real model should describe both phases, that is why we introduce both usual matter and DE. The cosmological constant Λ is obtained as a decreasing function of time and approaching a small positive value at present epoch which is corroborated by consequences from recent supernovae Ia observations. The gravitation constant G is found to be an increasing function of time. Our models are expanding, shearing and non-rotating and they all are accelerating at present epoch.

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