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APPLICATION OF CANONICAL EFFECTIVE METHODS TO BACKGROUND-INDEPENDENT THEORIES

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Abstract

Effective formalisms play an important role in analyzing phenomena above some given length scale when complete theories are not accessible. In diverse exotic but physically important cases, the usual path-integral techniques used in a standard Quantum Field Theory approach seldom serve as adequate tools.

This thesis exposes a new effective method for quantum systems, called the Canonical Effective Method, which owns particularly wide applicability in background-independent theories as in the case of gravitational phenomena. The central purpose of this work is to employ these techniques to obtain semi-classical dynamics from canonical quantum gravity theories. Application to non-associative quantum mechanics is developed and testable results are obtained. Types of non-associative algebras relevant for magnetic-monopole systems are discussed.

Possible modifications of hypersurface deformation algebra and the emergence of effective space-times are presented.

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Anişkom'a

Chapter 1 Introduction

The "Holy Grail" of contemporary theoretical physics has been to construct a meaningful quantum theory of gravity. More explicitly, it is the quest to reconcile General Relativity (GR) and Quantum Mechanics (QM): On one hand, Einstein's theory governs the macroscopic, classical dynamics of space-time and how it reacts to the presence of matter in a deterministic way while quantum mechanics aims at explaining the microscopic realm dominated by uncertainties and probabilistic distributions all of which in turn deeply altered our notions of causality. However, difficulties arise when trying to apply a mathematically consistent quantization procedure for the gravitational interaction and putting it in a physical context that has predictive power in a way similar to what has been achieved by the quantum field theories (QFTs) of electroweak and strong forces under the Standard Model of particle physics. The incompatibility of the two descriptions of nature exhibits itself when one naïvely tries a perturbative approach: Namely, at each order increasingly higher curvature counter-terms need to be introduced which results in the non-renormalizability of the theory.

Stepping back for a moment, we realize that there are various reasons to motivate the search for quantum gravity (QG). To begin with, at a fundamental level, quantum effects cannot be ignored if, for example, we wanted to examine the space-time structure at the Planck length. Basically, trying to measure the geometry beyond this so-called fundamental length scale would result in the formation of micro-singularities as a result of concentrating extreme energies in such small a space. This, in effect, would simply remove the region which we are actually trying to observe from the accessible space-time. Furthermore, a similar logic can be applied to the whole universe: Our observation of the curretly expanding cosmos immediately leads to the standard cosmological model which asserts that we should come across an infinitely dense and curved singular point that is commonly called as the Big Bang[1]. However, it is at such occasions where QG effects need to be taken into account which are absent from the above mentioned classical model of cosmology. Last but not the least, QG is expected to explain different facets of black hole physics, from the true nature of the Hawking radiation to the root causes of the huge entropy a black hole possesses.

Hence with the need for a consistent QG theory and with the above mentioned failure of standard methods, a number of different approaches have been suggested. String Theory differs from the rest of QG research in the sense that it aims at unifying all known fundamental forces into a common description where the whole spectrum of physical bodies are encoded in the different vibration modes of strings. However, the arena in which these strings live is at least ten dimensional and when one attempts to make contact with the physical four dimensional space-time the practically infinitude of ways to compactify the extra dimensions leads to what is known as the landscape problem: by a rough estimate there are 10^{500} possible inequivalent vacua. The more widely studied and more realistic modern version which incorporates fermionic fields into the same picture as the bosonic ones (superstring theory) uses the idea of supersymmetry: Despite being a mathematically elegant idea, the absence of supersymmetric particles in the mass ranges set by earlier experiments during the latest run of the Large Hadron Collider (LHC) has been a major discouragement for some researchers in the field. It is noteworthy that the type of infinities that present obstacles to the innocent quantization of GR does not appear in a perturbative expansion of String Theory around a fixed background; however, as finite as each term in such an expansion seems, the full series is evidently divergent. This fact motivated string theorists to work on the still ongoing programme of a non-perturbative formulation.

Another popular approach, loop quantum gravity (LQG), is less ambitious and has simply as its sole objective to describe the (quantum) states of the physical four-

[1] As far as the two competing models discussed below are considered, Loop Quantum Cosmology (LQC) - a finite, symmetry reduced model of LQG - involves a quantum bridge, called the Big Bounce, between contracting and expanding universes. String cosmologies, on the other hand, claim a replacement of the singularity by a saddle-point in the evolution of curvature: in this pre-big-bang universe the curvature is increasing. It has always amazed me to compare and contrast the ways these models look at the universe with ancient philosophies; particularly LQC and cyclic universes of the Hindu/Buddhist temporal cosmologies.

dimensional space-time in terms of labelled graphs called spin networks. This model has the (aesthetically) desirable property of manifest background-independence, i.e. the requirement the geometrical structure itself be a local degree of freedom in the sense that it allows us to obtain different space-time configurations as different solutions of the underlying equations. However, this seemingly appealing fact that the equations of LQG are not dependent on the properties of space-time but only on the given topology may also be a shortcoming for the theory in the following sense: Study of three dimensional QG models has suggested that there exist non-zero transition amplitudes between two distinct topologies. But, some derivations of LQG fixes a choice of topology in the first place which may be interpreted as a flaw of the theory not being able to meet the expectation that it should include topology change as a dynamical process. In any case, this model is plagued by an even more important problem: It is not known whether GR can be recovered as a classical limit of LQG. [2]

Indeed, it is this last point that brings us to the main theme of this thesis. As successul as String Theory might be in accomodating a spin-2 boson in its spectrum or LQG in computing the precise size of the granular structure of space-time, neither theory haa suggested any predictions that could be directly verified experimentally. It could be said that we are undeniably at that stage of high energy physics research where the success of precision cosmology experiments in gathering useful data precedes that of predictive theoretical descriptions of nature: Missions like COBE from as early as 1989, and its successors WMAP (2001-2010) and Planck (2009-2013) whose purposes were mainly the measurement of the anisotropies in the Cosmic Microwave Background Radiation (CMB) with high sensitivity helped constrain the Lambda-CDM parametrization of the classical Big Bang cosmology to a high degree, which gave a good explanation of the large-scale structure and the accelerated expansion of the universe.

The dates mentioned above are not to be seen as details but rather as a reference to the timeline of the theoretical attempts. It was 1974 when Schwarz and Scherk [2] proposed bosonic string theory as a theory of QG and it was 1986 when Ashtekar [3] introduced a set of new variables to write down GR in terms of SU(2)-gauge fields

[2]Although some of the difficulties arising in reproducing this limit have been improved or overcome by the introduction of the so-called master constraint [1], the discovery of appropriate semi-classical tools for graph-changing operators such as Hamiltonian constraint does not seem to be within reach in the near future.

which brought the subject closer to the field-theoretic setup of other fundamental interactions. The position theoretical high energy physics is now in demonstrates no resemblance to the historical progression it has witnessed, take for example the case of Gell-Mann's coming up with (what I would call the semi-phenomenological) Eight-Fold Way to describe the myriad of subatomic particles discovered in the newly established accelerators of 60's: His mathematically beautiful model not only organized the existing particles into various multiplets but also predicted one missing component of the decuplet, the Ω^- meson. Two years later, in 1964, it was observed in Brookhaven with the same mass, charge, and quantum numbers as foreseen by Gell-Mann. The current situation carries even a deeper contrast to that of the beginning of last century: There was no experimental contradiction challenging the validity of Newtonian gravity when Einstein came up with his ingenious theory whose consecutive predictions such as gravitational waves, orbital effects and light deflection in the form of gravitational lensing have all been confirmed.

It is this immediate necessity to re-establish the contact of current models with reality and to obtain verifiable predictions that motivates the new effective technique introduced and employed in this thesis. It is the hope that this novel formalism can shed light onto the semi-classical analysis of existing theories that has inspired my research. It is the fear that the common trend in high energy physics is challenging its status as an experimental science and slowly turning it into an undistinguished philosophy [4, 5, 6] which prompts the investigation of effective theories in the next section.

1.1 Effective Theories

Effective theories in physics portray physical phenomena at or below a given energy scale by a kind of averaging over the unknown degrees of freedom of the underlying theory beyond that certain scale. This procedure works well as long as the fundamental energy scale of the full theory is broadly separated from that of this chosen-by-hand cutoff. In the context of QFTs, this formalism is encoded in the concept of an effective action, $\Gamma[q]$, which coincides with the classical action, $S[q]$, for a free field theory, but for which contributions higher order in $\hbar$ can be interpreted as coming from the purely quantum mechanical process of virtual particle exchanges. However, this particle physics driven, loop-correction type of justification of $\Gamma[q]$ does not always go smoothly for various other quantum mechanical systems and presents some technical, as well as heuristic, problems.

First, the solutions of equations of motion coming from the variation of the effective action with respect to q are known to be complex and this fact is related to the q variable lacking any classical attribute and it being associated with the off-diagonal elements of the matrix representation of the position operator $\hat{q}$. Moreover, $\Gamma[q]$ not necessarily being a local functional of q requires, in most cases, a derivative expansion; however, many solutions of the higher derivative effective action, for various technical reasons, destroy the validity of this perturbative scenario in the case where the leading order contribution may be overshadowed by the higher orders in the expansion. In order to keep the legitimacy of this programme, only solutions analytic in its parameter are to be kept which in turn raises questions as to what significance could be attributed to the apparently (spurious) quantum degrees of freedom that had to be discarded.

When such methods are applied particularly to a theory of QG various additional conceptual complications emerge. For example, the unavailability of a standard Fock vacuum for the quantization of a dynamical background space-time presses one to consider more general states. Also, the problem of time in QG, namely that there is none in GR due to the Hamiltonian being a vanishing constraint to ensure general covariance but that time evolution of states are generated by the Hamiltonian operator in QM, obscures the possibility of writing down relevant effective actions. Quantum effects in some cases modify the diffeomorphism covariance of GR which forms the ground for its symmetry algebra; as a result the study of the gauge structure of such theories demand more care. Lastly, on the more technical side, proper inner-products on the physical Hilbert spaces of any back-ground independent theory must be established in a way that allows a meaningful extraction of semi-classical effects.

With the need to overcome the aforementioned obstacles, an effective method for quantum systems resting on their geometrical formulations was established first in [7]. The applications considered in Chaper-2 of this thesis rely on this method. For this reason, in the next subsection an overview of this Canonical Effective Method will be presented.

1.1.1 Canonical effective methods

It is of no surprise that the approach studied in this thesis is based on a Hamiltonian formulation granted that the primary field to which it was originally planned to be applied was in the context of canonical quantum gravity theories (for the time being, LQG to be specific). Expectation value functionals, which are simply positive linear functionals defined on an abstract algebra, are understood as states in this kind of an algebraic method. In this manner, there is no reference needed to a wavefunction on an underlying Hilbert space which, as we have briefly touched upon in the previous section, might not always turn out to be well-defined. Practically, this implies that we are indeed representing expectation values, $(\langle\hat{q}_i\rangle, \langle\hat{p}_i\rangle)$, of canonical variables and their moments ($\Delta(pq)$ for a single canonical pair for example) on a space of states, as demonstrated in Section 2.2.3 below (see e.g. Eqn.(2.23)).

Once this step ensures that a single point in this infinite-dimensional space is fully characterized by these variables, it is endowed with a Poisson structure taking advantage of the commutator of two (generic) elements, $\hat{A}$ and $\hat{B}$, of the abstract algebra under consideration

$$\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} := \frac{1}{i\hbar}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle \tag{1.1}$$

extended by the Leibniz rule for the product of expectation values on the left hand side. Defining an effective (a.k.a quantum) Hamiltonian, $H_Q(\langle.\rangle, \langle.\rangle, \Delta(.)) := \langle\hat{H}\rangle$ where the dots in arguments stand for the various canonical pairs relevant to the particular case and the arbitrary order of moments, from the expectation value of the Hamiltonian operator on the space of states, consequently allows us to describe the dynamics of the system, and determine the equations of motion for various entities as follows:

$$\dot{\langle\hat{A}\rangle} := \left\{\langle\hat{A}\rangle, H_Q\right\}\ . \tag{1.2}$$

Clearly, this prescription above has succeeded in replicating the behavior of a wavefunction subject to partial differential equations equivalently by infinitely many coupled yet ordinary differential equations. However, the central mathematical difficulty in this procedure lies in bringing this infinite set of equations to a finite subset in a consistent way. This is first attacked by utilizing a semi-classical expansion in $\hbar$ which is enabled by the fact that a moment of a given order has a definite $\hbar$ dependence in a semi-classical state (by definition). This means the expansion of a quantity up to a finite $\hbar$-order will only involve finite number of moments. [3] This way the infinite set of coupled differential equations are reduced to a finite coupled sytem of differential equations. However, since the coefficients in these equations can be functions of the position and momentum variables another expansion, an adiabatic expansion, is used to convert these hard-to-solve differential equations into algebraic ones. This procedure in practice boils down to setting the left hand side of (1.2) to zero at the zeroth adiabatic order; once the zeroth order adiabatic contribution to a moment of given $\hbar$ order is computed, one can proceed recursively and compute higher order adiabatic contributions from the already known lower ones and their time derivatives. Thus, all in all, we end up with a finitely coupled system of algebraic equations which is easier to solve.

This method has been shown in [7] to yield the low-energy effective action for the case of a generic anharmonic oscillator

$$\begin{aligned}\Gamma_{\text{eff}}[q(t)] \;&=\; \int \mathrm{d}t \Bigg(\frac{1}{2} \left(m + \frac{\hbar U'''(q)^2}{32 m^2 \omega^5 \left(1 + U''(q)/m\omega^2\right)^{5/2}} \right) \dot{q}^2 \\ &\quad - \frac{1}{2} m \omega^2 q^2 - U(q) - \frac{\hbar\omega}{2} \left(1 + \frac{U''(q)}{m\omega^2} \right)^{1/2} \Bigg) \end{aligned} \tag{1.3}$$

which is identical to the one obtained by standard path-integration [8] up to the first order in $\hbar$ and second order in the adiabatic expansion. Nonetheless, the path integral approach has some disadvantages. The above expression is reached by necessarily applying a derivative expansion around the harmonic oscillator ground state. In the canonical formalism, on the contrary, the moments are only subject to Schwarz inequality type bounds and the quantum corrections do not have to be only associated to a minimal uncertainty state. Moreover, a closer look at (1.3) reveals an infrared problem in the massless limit. In our language, this obstacle is mainly due to the adiabatic approximation and in principle could be surpassed. Additionally, the present method has the capability to deal with Hamiltonians which are non-quadratic in momenta, which is a relevant situation in quantum

[3] This expansion might have half integer $\hbar$ orders because a moment of order n has the dependence $\mathcal{O}(\hbar^{n/2})$. Even though we can impose that an initial state consists of only even order moments, quantum back-reactions can lead to non-zero moments of odd order.

cosmology whereas standard techniques would face problems not being able to employ Gaussian path integrations.

To these orders the above expression only involves corrections to the particle mass and by an effective quantum potential. The analysis was carried out for higher $\hbar$ and adiabatic orders in [9], and it was found there that terms involving higher order time derivatives appear. This situation is naturally understandable in terms of path integrals where observable quantities are identified by the full sum-over-histories which reflect the nature of quantum effects that are non-local in time. The consequences of having higher order derivative corrections are even more far-reaching in a gravitational context; namely that equations of motion corresponding to a gravitational theory extended by higher powers of curvature often includes time derivatives higher than second order.

Such already well-established successes and its advantages mentioned above over the standard methods motivated the usage of Canonical Effective Methods as the main tool in this thesis. The difficulties sometimes arising from the representations of algebras on Hilbert spaces are simply bypassed thanks to the basic characteristic of the formalism that the existence of a Hilbert space is not a prerequisite for extracting the dynamics and that the calculation of the expectation values does not require fixing a vacuum state. In this fashion, being able to relax some of the technical ingredients of QM opens up the possibility to explore more generalized settings.

One such example is non-associative algebras. A simple case, first suggested by Jackiw [10], where such constructs can appear is to consider a generic magnetic field in three dimensions causing a Lorentz force on an electrically charged particle. In this magnetic background translation invariance is lost, i.e. finite translations by a vector a, $U(a) = e^{i\pi/\hbar a}$, fail to commute due to a phase-factor which is proportional to the magnetic flux through a triangle defined by two such vectors. This factor has roots in the non-zero commutation relations of two gauge invariant momenta. Even more significantly, three finite translations have an associator, i.e. the difference between the only two possible bracketings of the three objects, proportional to the flux Φ through a tetrahedron defined by three such vectors. This effect comes from the fact that the Jacobi identity of three physical momenta, based on their non-vanishing commutators, goes as the divergence of the magnetic field. In the case where magnetic monopoles exist ($\nabla.B \neq 0$) in the system, unless the phase factor satisfies $e^{e\Phi/\hbar} \in \pi\mathbb{Z}$, which reflects the Dirac quantization condition for a magnetic charge, the non-associativity of the global translations is non-vanishing. The fact that such non-associative algebras cannot be represented in the standard way on a Hilbert space has been our driving force for employing the representation-independent Canonical Effective Methods described in this section.

At first, this endeavor might seem like a purely mathematical curiousity; but there are a couple reasons to physically motivate the investigation of non-associative algebras. Firstly, research in flux compactifications of string theory has witnessed the emergence of non-associative deformations of closed string background geometries where the gravity sector lives [11]. But even before this observation, non-associative gauge theories arose within the context of D-branes in a non-constant B-field background [12,13,14]. On the other hand, the motivation from the other candidate theory for QG, Loop Quantum Gravity, is not necessarily related to a non-associative structure explicitly but the appearance of modified brackets in some models of LQG points to a possible non-Riemannian (non-)geometry. Also the fact that the area (and volume) operator has a discrete spectrum [15] might signal an underlying non-commutative space-time. Indeed some connection between this fuzzy space-time of canonical QGs for point-like fields and the smooth geometry of String theory on which its finitely extended objects evolve was suggested making use of non-geometric flux compactifications and the T-duality in [16] where the *derived* non-associativity follows from the violation of Jacobi identity. This in turn raises the possibility of a non-commutative (or non-associative?) theory of gravity [17,18,19].

This last point will be discussed in more detail in Chapter 5, but for now we suffice to say part of this thesis is dedicated to exploring various possibilities of deforming the constraint algebra of GR. In the context of canonical QG theories, covariance is implemented by brackets of hypersurface-deformation generators forming a Lie algebroid. Lie algebroid morphisms therefore allow one to relate different versions of the brackets that correspond to the same space-time structure. An application to examples of modified brackets found mainly in models of loop quantum gravity can in some cases map the space-time structure back to the classical Riemannian form after a field redefinition. However, our construction in Chapter 4 reveals that this is not always possible. After all, the central part of this work has been advertised to be establishing an effective method that is relevant to such canonical theories and we tried to lay out the necessary ingredients for this mission.

1.2 Organization of the thesis

State vectors and operators by construction require an associative product of observables in standard quantum mechanics, but some exotic systems such as magnetic monopoles, as briefly demonstrated in Chapter 1, have long been known to lead to non-associative products. In Chapter 2, we employ algebraic methods in order to derive uncertainty relations and semiclassical equations, based on general properties of quantum moments. New results about effective potentials in non-associative quantum mechanics and related observable effects are also derived. This chapter is based on [20, 21], and and further explanation of the physical relevance of such systems are given. Chapter 3, based on [22], elaborates on the type of non-associative algebras that can emerge in these quantum mechanical contexts where such magnetic monopoles are present in the system. It is shown here by using methods of deformation quantization that algebras for such systems cannot be alternative, i.e. their associator cannot be completely anti-symmetric. Chapter 4 returns to the essential question of this thesis: For one type of quantum corrections (holonomies), signature change appears to be a generic feature of effective space-time, and is shown here to be a new quantum space-time phenomenon which cannot be mapped to an equivalent classical structure. Based on [23] we prove that in low-curvature regimes, our constructions prove the existence of classical space-time structures assumed elsewhere in models of loop quantum cosmology, but also shows the existence of additional quantum corrections that have not always been included. Chapter 5 is devoted to a survey of what has been achieved in this formalism for fully constrained systems and a discussion of implications of the present work for quantum gravity. A review of current open projects and how they are suited in the bigger picture will also be given.

Chapter 2
Non-associative quantum theories

2.1 Introduction

Quantum mechanics represents the classical Poisson algebra of basic variables q_j and p_k, $\{q_j, p_k\} = \delta_{jk}$, as an operator algebra acting on a Hilbert space, so that the Poisson bracket is turned into the commutator $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$ of basic operators.[1] The Jacobi identity satisfied by a Poisson bracket has an analog in the associativity of the operator product: A simple calculation shows that

$$\epsilon^{ijk}[[\hat{O}_i, \hat{O}_j], \hat{O}_k] = 2\epsilon^{ijk}[\hat{O}_i, \hat{O}_j, \hat{O}_k] \tag{2.1}$$

where the 3-bracket on the right-hand side is used to denote the associator of the product of the quantum observables, $[\hat{O}_1, \hat{O}_2, \hat{O}_3] := (\hat{O}_1\hat{O}_2)\hat{O}_3 - \hat{O}_1(\hat{O}_2\hat{O}_3)$. If the Jacobiator of the classical bracket — or the Jacobiator of the operator product introduced on the left-hand side of (2.1) — vanishes for all triples of operators $\hat{O}_i$, an associative operator algebra is consistent with Dirac's basic quantization rule relating Poisson brackets to commutators. (These concepts have been formalized mathematically in different ways, for instance in the frameworks of group-theoretical quantization [24], geometric quantization [25], and deformation quantization.)

[1] This latter equation, as usual, holds on a dense subspace only. From a mathematical perspective, it is convenient to consider bounded operators obtained after exponentiating $\hat{q}_i$ and $\hat{p}_j$, resulting in the Weyl algebra. In this chapter, however, we focus on conceptual questions of the construction of non-associative quantum mechanics and related consequences in possible physical applications, postponing more mathematical issues to later work.

For classical systems with modified brackets, such as twisted Poisson structures [26, 27, 28], the Jacobi identity may no longer hold true and be replaced by a non-zero Jacobiator $\epsilon^{ijk}\{\{O_i, O_j\}, O_k\} \neq 0$. As the quantum analog, there must be a non-zero associator $[\hat{O}_1, \hat{O}_2, \hat{O}_3] \neq 0$ of a non-associative operator algebra. Such an algebra cannot be represented on a Hilbert space in the standard way, and alternatives making use, for instance, of non-associative $*$-products must be developed. In this chapter, we focus on the general aspects of states on a non-associative operator algebra and see how the basic notions familiar from quantum mechanics can be derived in representation-independent terms. In some respects (and unless extra conditions on states are imposed) the results seem to differ from existing constructions using non-associative $*$-products [12, 29, 30, 31, 32, 11].

Non-associative structures have recently gained interest in the context of certain flux compactifications of string theory and double-field theory [16, 33, 29, 34, 35]. They have played a role in the understanding of gauge anomalies, and also appear in "standard" quantum mechanics if a charged particle is coupled to a density of magnetic monopoles [36]. These monopoles need not be fundamental, and therefore the systems may describe realistic physics in some analog models of condensed-matter systems (see for instance [37]). A related version is realized in chiral gauge theories [38, 39, 40]. We briefly review how non-Poisson brackets or non-associative algebras appear, which will present the main example to keep in mind throughout this chapter.

In the presence of a magnetic field with vector potential A_i, the canonical momentum of a particle with mass m and charge e is $\pi_i = mv_i + eA_i$ in terms of the velocity $v_i = \dot{q}_i$. While the momentum components obey canonical Poisson brackets with the position variables and have zero brackets with one another, the velocity components or the kinematical momentum components $p_i = mv_i$ have brackets related to the magnetic field:

$$\{p_i, p_j\} = e\epsilon_{ijk}B^k \,. \tag{2.2}$$

(We use the convention that repeated indices are summed over.) These brackets define a Poisson structure provided the magnetic field is divergence free.

If the divergence is non-zero, the magnetic field no longer has a vector potential, but one may still use (2.2) as the definition of a bracket on phase space (together with $\{q_i, p_j\} = \delta_{ij}$ and antisymmetry). One then computes a non-zero Jacobiator

$$\epsilon^{ijk}\{\{p_i, p_j\}, p_k\} = -2e\partial_l B^l \,. \tag{2.3}$$

The corresponding quantum mechanics cannot be represented by an associative operator algebra acting on a Hilbert space.

For a constant monopole density, the bracket is twisted Poisson [26, 27, 28] and can be realized as a Malcev algebra [41, 42]. The $*$-product constructions of [29, 30, 31, 32, 11] simplify for a constant density and allow several explicit results to be derived, but they hold more generally. Our calculations here are complementary and allow for $\partial_l B^l$ to be non-constant even in some explicit results. The existence of relevant algebras and states based on our relations alone is more difficult to show, but if they are assumed to exist, several properties can be derived efficiently by considering expectation-value functions $\omega\colon \mathcal{A} \to \mathbb{C}$.

From the physical perspective, this example is of interest because the existence of a magnetic monopole density, fundamental or effective, *somewhere* within the system necessitates a modification of very fundamental aspects of quantum mechanics. The notion of a Hilbert space is a non-local one, for instance in the sense that wave functions in the Schrödinger representation are normalized by an integration over all of space. Nevertheless, for meaningful experiments it must be possible to construct a local description of quantum physics outside the magnetic monopole density, where it has to reproduce the established and experimentally verified quantum properties (at least to a very high precision). This (perhaps hypothetical) physical system thus provides an interesting playground and a test for the development of non-associative quantum mechanics.

2.2 Properties of states

As briefly derived in the example of a magnetic monopole density, we assume that we have an algebra $\mathcal{A}$ of observables, which includes elements $\hat{q}_i$ and $\hat{p}_j$ (as well as a unit $\mathbb{I}$) and obeys the relations

$$[\hat{q}_i, \hat{q}_j] = 0 \tag{2.4}$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{2.5}$$

$$[\hat{p}_i, \hat{p}_j] = i\hbar e\epsilon_{ijk}\hat{B}^k \quad (2.6)$$
$$[\hat{p}_i, \hat{p}_j, \hat{p}_k] = -\hbar^2 e\epsilon_{ijk}\widehat{\partial_l B^l} \quad (2.7)$$

where $i, j, k \in \{1, 2, 3\}$. Summation over double indices is assumed, and, as before, the 3-bracket denotes the associator. We further assume that, other than (2.7), all associators between the fundamental variables $\hat{q}_i$ and $\hat{p}_j$ vanish. Among basic operators, only the associator (2.7) is non-zero because of the non-vanishing Jacobiator (2.3).

In (2.6) and (2.7), $\hat{B}^k \in \mathcal{A}$ (and similarly $\widehat{\partial_l B^l}$) is obtained by inserting $\hat{q}_i$ in the classical function $B^k(q_i)$. Since the $\hat{q}_i$ commute and associate with one another, $\hat{B}^k$ is well-defined for polynomial B^k. For non-polynomial magnetic fields, we assume that $\hat{B}^k$ can be defined by a formal power series. With the magnetic field and also its divergence allowed to be functions of q_i, the relations may be non-linear. If the magnetic-field components are assumed to be analytic functions, then also their derivatives are well-defined, which we will use in some semiclassical expansions. The assumption of analyticity may have to be weakened in some physical situations because it is not consistent with a monopole density of compact support. For the algebra, we need the first derivatives of B^i, so that these functions should at least be differentiable.

At this point, we encounter the first existence question. In associative cases, it is known that $\hat{q}_i$ and $\hat{p}_j$ are not bounded in Hilbert-space representations. It is then more convenient to use exponentiated (Weyl) operators for explicit constructions of algebra representations. In the present case, Hilbert-space representations cannot exist at all, and existence questions are more complicated. In this chapter, we take a pragmatic view and assume that an algebra with the relations (2.4)–(2.7) (as well as a $*$-relation introduced below) exists. Our aim is to derive properties of states which are of interest for physical questions and can be obtained using only the given relations. This view is akin to the one taken in particle physics, where it is difficult to show that interacting quantum field theories do indeed exist, but powerful computational methods are still available and can be compared with observations.

The relations (2.4)–(2.6) are direct translations of basic brackets, the first two of standard form and (2.6) derived from (2.2). The non-zero Jacobiator (2.3) implies that there must be a non-zero associator. However, (2.1) shows that only the totally antisymmetric part of the associator is determined by the correspondence between classical brackets and commutators. Contributions to the associator which are not totally antisymmetric can be considered as quantization choices, which one may be able to choose so as to realize certain simplifications. For now, we will assume simplifications which appear to be consistent with the equations (2.4)–(2.7), postponing a more precise construction of $\mathcal{A}$ to later work.

We could assume the associator between any three elements of the algebra $\mathcal{A}$ to be completely antisymmetric, or equivalently

$$A(BB) = (AB)B \tag{2.8}$$

$$(AA)B = A(AB) \tag{2.9}$$

$$(AB)A = A(BA) \tag{2.10}$$

for all $A, B, C \in \mathcal{A}$. Any algebra satisfying these conditions (two of which imply the third one) is called an *alternative algebra*. For such an algebra, we have additional relations between algebra elements which are not as strong as assocativity but will turn out to be useful: An alternative algebra satisfies the Moufang identities [43]

$$C(A(CB)) = (CAC)B \tag{2.11}$$

$$((AC)B)C = A(CBC) \tag{2.12}$$

$$(CA)(BC) = C(AB)C\,. \tag{2.13}$$

(If (2.8) holds, we do not need to set further paranthesis in (2.13).) These identities are also useful for an extension of some of the measurement axioms of quantum mechanics to non-associative versions [44]. The algebras constructed by $*$-products in [30, 31, 32, 11], with the same basic associator (2.7), are *not* alternative.[2] Our explicit results derived in the rest of this chapter only require (2.7) to be totally antisymmetric, and the corresponding Moufang identity for A, B and C linear in the $\hat{p}_i$. They will therefore also hold for the known $*$-algebra realizations of (2.7), but there may be deviations at higher moments or $\hbar$-orders.

We turn $\mathcal{A}$ into a $*$-algebra by requiring $\hat{q}_i$ and $\hat{p}_j$ to be self-adjoint. (We then have the usual relations, such as $(\lambda\hat{p}_1)^* = \lambda^*\hat{p}_1$ for all $\lambda \in \mathbb{C}$, and $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$.) This requirement is consistent with (2.7) thanks to the alternative

[2] We are grateful to Peter Schupp and Richard Szabo for pointing this out to us.

nature of the algebra: for self-adjoint $\hat{p}_i^* = \hat{p}_i$, we then have

$$[\hat{p}_1, \hat{p}_2, \hat{p}_3]^* = \hat{p}_3(\hat{p}_2\hat{p}_1) - (\hat{p}_3\hat{p}_2)\hat{p}_1 = -[\hat{p}_3, \hat{p}_2, \hat{p}_1] = [\hat{p}_1, \hat{p}_2, \hat{p}_3] \tag{2.14}$$

so that both sides of (2.7) are self-adjoint.

2.2.1 The Cauchy–Schwarz inequality and uncertainty relations

In treatments of algebra theory relevant for quantum mechanics it is often assumed that one is dealing only with associative algebras. Several important results no longer apply in the non-associative case. However, a notable exception is the Cauchy–Schwarz inequality. It is important in quantum mechanics because it leads to the uncertainty relation, and fortunately, this result is still available for non-associative algebras. Even the standard proof can be used without modifications, which we sketch here for completeness.

For any complex-valued, positive linear functional ω on the algebra $\mathcal{A}$ (that is, $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$), we would like to prove that

$$\omega(A^*A)\omega(B^*B) \geq |\omega(B^*A)|^2 \tag{2.15}$$

for any two elements A and B in $\mathcal{A}$. We define a new element $A' := A\exp(-i\arg\omega(B^*A))$, so that $|\omega(B^*A)| = \omega(B^*A')$, and compute

$$\begin{aligned} 0 &\leq \omega\left((\sqrt{\omega(B^*B)}A' - \sqrt{\omega(A'^*A')}B)^*(\sqrt{\omega(B^*B)}A' - \sqrt{\omega(A'^*A')}B)\right) \\ &= 2\omega(B^*B)\omega(A'^*A') - \sqrt{\omega(B^*B)\omega(A'^*A')}\,(\omega(A'^*B) + \omega(B^*A')) \\ &= 2\omega(B^*B)\omega(A'^*A') - 2\sqrt{\omega(B^*B)\omega(A'^*A')}|\omega(B^*A)|\,. \end{aligned} \tag{2.16}$$

Therefore,

$$|\omega(B^*A)| \leq \sqrt{\omega(B^*B)}\sqrt{\omega(A'^*A')} = \sqrt{\omega(B^*B)\omega(A^*A)}\,.$$

One can then derive the standard uncertainty relation for basic operators by applying the Cauchy–Schwarz inequality to $A = \hat{q}_i - \omega(\hat{q}_i)\mathbb{I}$ and $B = \hat{p}_j - \omega(\hat{p}_j)\mathbb{I}$: We have $\omega(A^*A) = (\Delta q_i)^2$, $\omega(B^*B) = (\Delta p_j)^2$, and $\omega(B^*A)$ can be split into its real part, which equals the covariance $C_{q_ip_j} := \frac{1}{2}\omega(\hat{q}_j\hat{p}_i + \hat{p}_i\hat{q}_j) - \omega(\hat{q}_j)\omega(\hat{p}_i)$, and its imaginary part proportional to the commutator $[\hat{q}_i, \hat{p}_j]$. The inequality (2.15) then implies

$$(\Delta q_j)^2(\Delta p_i)^2 \geq \frac{\hbar^2}{4}\delta_{ij} + C^2_{q_j p_i} \geq \frac{\hbar^2}{4}\delta_{ij}\,. \tag{2.17}$$

In the present case, there is a new uncertainty relation for different components of p^i thanks to the non-zero commutator (2.6):

$$(\Delta p_i)^2(\Delta p_j)^2 \geq \frac{\hbar^2 e^2}{4}(\epsilon_{ijk}\omega(\hat{B}^k))^2\delta_{ij} + C^2_{p_i p_j} \geq \frac{\hbar^2 e^2}{4}(\epsilon_{ijk}\omega(\hat{B}^k))^2\delta_{ij}\,. \tag{2.18}$$

These relations depend only on commutators, and therefore are equivalent to those given in [30, 31, 32, 11] based on $*$-products.

At this stage, we note a difference with the $*$-product treatment of non-associative algebras. When one constructs an analog of a Hilbert-space representation of wave functions ψ acted on by $\mathcal{A}$ using a non-associative $*$-product, one assigns to any $A \in \mathcal{A}$ a map $\psi \mapsto A * \psi$ on a set of wave functions ψ instead of the usual associative action of operators. (The constructions in [32] are more general and consider also density states.) In deriving the uncertainty relation, one applies two such multiplications of the form $A*(B*\psi)$. This product is sensitive to non-associativity, and indeed the derivation of an uncertainty relation is non-trivial. In [32], the problem has been solved by introducing modified (and associative) composition maps derived but different from the original algebra product:[3] $\circ$ is obtained from $(A\circ B)*C = A*(B*C)$ for all A, B, C, and $\bar{\circ}$ from $C*(A\bar{\circ}B) = (C*A)*B$. For the $*$-product action on states, a Cauchy–Schwarz inequality holds for $\circ$ but not for the original $*$. However, as derived in detail in [32], the $\circ$-commutator acts by $(\hat{p}_i \circ \hat{p}_j - \hat{p}_j \circ \hat{p}_i) * \psi = \psi * \hat{K}$, with $\hat{K}$ corresponding to the right-hand side of the commutator (2.6), but now acting from the right. Accordingly, the resulting uncertainty relation is not of the standard form, unless an additional "symmetry" condition is imposed on wave functions, or $\rho * C = C * \rho$ on density states ρ. The general derivation of the Cauchy–Schwarz inequality, on the other hand, makes use of products of at most two operators and is not sensitive to non-associativity. It implies an uncertainty relation that is equivalent to the one obtained using $*$-products only if the symmetry condition is imposed. We view this observation as an additional argument that wave functions should indeed obey the symmetry

[3] These composition maps are important for the construction of states obeying the positivity condition [32].

condition (as already suggested in [32]).

2.2.2 Failure of the GNS construction

Given an *associative* $*$-algebra $\mathcal{A}$ and a positive linear functional ω on it, one can construct a Hilbert-space representation by making use of the GNS construction. (See for instance [45, 46].) It is clear that the construction must fail in the non-associative case because such an algebra cannot act by standard operator multiplication on a Hilbert space. Nevertheless, it is interesting to see where exactly the construction breaks down.

In the GNS construction, one starts with the algebra $\mathcal{A}$ as a linear space and constructs a Hilbert space from it. Multiplication in the algebra then implies an action of the algebra on the Hilbert space. In order to derive the Hilbert space, one introduces a (degenerate) scalar product on $\mathcal{A}$ by $\langle A|B\rangle := \omega(A^*B)$ for all $A, B \in \mathcal{A}$. The scalar product is positive semidefinite because ω is assumed to be a positive linear functional, but it has a kernel spanned by all $C \in \mathcal{A}$ for which $\omega(C^*C) = 0$. Assuming the algebra to be associative, the kernel is a left-ideal in $\mathcal{A}$ and can be factored out, leaving a linear space with a positive definite scalar product which can be completed to a Hilbert space.

In this last step, associativity is important. In order to show that the kernel is a left-ideal, one makes use of the Cauchy–Schwarz inequality and computes (using associativity only at this place in the present chapter)

$$|\omega((AC)^*(AC))|^2 = |\omega(C^*A^*AC)|^2 \leq \omega(C^*C)\omega((A^*AC)^*A^*AC) = 0\,, \tag{2.19}$$

so that AC is in the kernel for any $A \in \mathcal{A}$ and C in the kernel. For a non-associative algebra, (2.19) is not available and it is in general impossible to factor out the kernel consistently in order to obtain a Hilbert space.

It would be possible to obtain a left ideal from the kernel of ω if all C in the kernel would be self-adjoint (or anti-selfadjoint). For an alternative algebra, we could then proceed as in (2.19) thanks to the Moufang identity

$$C(AB)C = (CA)(BC) \tag{2.20}$$

which allows us to write

$$|\omega((AC)^*(AC))|^2 = |\omega((CA^*)(AC))|^2 = |\omega(C(A^*A)C)|^2 \qquad (2.21)$$

$$\leq \omega(C^*C)\omega(((A^*A)C)^*((A^*A)C)) = 0 \qquad (2.22)$$

if $C^* = \pm C$. A real Hilbert space would follow from the GNS construction if the algebra could be restricted to only (anti-)self-adjoint elements. Unfortunately, however, a closed algebra of (anti-)self-adjoint elements can be obtained only with (anti-)commutative multiplication.

The GNS construction plays an important role in algebraic approaches to quantum mechanics and quantum field theory because it shows that Hilbert-space representations do exist. In particular, using all states in a Hilbert-space representation, one is assured that sufficiently many positive linear functionals exist on the algebra, allowing one to derive potential measurement results. A quantum system would not be considered meaningful if it does not allow sufficiently many states, for instance when $\omega(\hat{q}_1) = \omega(\hat{q}_2)$ for all states ω. For every point $(\bar{q}_1, \bar{q}_2, \bar{q}_3; \bar{p}_1, \bar{p}_2, \bar{p}_3)$ in the classical phase space in which we expect a semiclassical quantum description to be available, we should require that there is a state ω such that $\omega(\hat{q}_i) = \bar{q}_i$ and $\omega(\hat{p}_j) = \bar{p}_j$. The classical freedom of choosing initial values then remains unrestricted after quantization.

If sufficiently many states exist, general features of expectation-value functionals can be employed to derive generic properties which are independent of which specific representation is used. For non-associative algebras, we cannot have standard Hilbert-space representations, and we are not aware of an alternative version of the GNS construction that could guarantee the existence of sufficiently many positive linear functionals on the algebra. The methods of [32] show that states can be constructed with an action of the algebra given by a $*$-product, and positivity properties have been demonstrated. However, as shown by the discussion of uncertainty relations in the preceding section, the general algebraic results we make use of here agree with those found by non-associative $*$-products only when the class of states is restricted by an additional symmetry condition. To the best of our knowledge, it is not clear whether sufficiently many positive linear functionals obeying the symmetry condition do exist. In what follows, we will have to assume that there are such states, some of whose properties we will be able to derive.

2.2.3 Moments

Without a Hilbert space, we cannot describe states by wave functions. However, we can use an alternative set of variables which describes a positive linear functional ω on $\mathcal{A}$ in terms of expectation values $\omega(O)$ and moments of the form $\omega((O-\omega(O)\mathbb{I})^n)$ for O one of the basic operators. More generally, covariance parameters in which ω is applied to products of $O_i - \omega(O_i)\mathbb{I}$ for different values of i are also required. We introduce these variables and determine some of their algebraic relations after switching to a physics-oriented notation in which $\omega(A)$ is written as the expectation value $\omega(A) = \langle A \rangle$ of an operator A. These expectation values, by definition, refer to a state as a positive linear functional on the algebra; they do not require wave functions or a Hilbert space. Moreover, we will omit explicit insertions of the unit operator $\mathbb{I}$ and assume that it is understood in expressions such as $\hat{A} - \langle \hat{A} \rangle$.

In the associative case, the definition of the moment variables is as follows:

$$\begin{aligned} \Delta(p_x^{a_1} q_x^{a_2} p_y^{b_1} q_y^{b_2} p_z^{c_1} q_z^{c_2}) \quad := \quad & \langle (\hat{p}_x - \langle \hat{p}_x \rangle)^{a_1} (\hat{q}_x - \langle \hat{q}_x \rangle)^{a_2} \\ & \times (\hat{p}_y - \langle \hat{p}_y \rangle)^{b_1} (\hat{q}_y - \langle \hat{q}_y \rangle)^{b_2} \\ & \times (\hat{p}_z - \langle \hat{p}_z \rangle)^{c_1} (\hat{q}_z - \langle \hat{q}_z \rangle)^{c_2} \rangle_{\text{Weyl}} \end{aligned} \tag{2.23}$$

with totally symmetric or Weyl ordering indicated by the subscript "Weyl." Weyl ordering makes sure that we do not count as different moments which can be obtained from each other by simple applications of the commutator. Moreover, the moments of Weyl ordered products in an associative algebra are defined as real numbers. It turns out to be useful to define them as expectation values of products of the differences $\hat{A} - \langle \hat{A} \rangle$ as opposed to products just of basic operators because a semiclassical state can then be defined as one in which moments of order $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 =: n$ are of the order $O(\hbar^{n/2})$. In this way, one generalizes the family of Gaussian states in which this order relationship can be confirmed by an explicit calculation. We make use of the $\hbar$-orders in our semiclassical equations derived in Sections 2.3 and 2.4.

For a non-associative algebra, we have to be careful with the order in which the products are performed. We define the moments by declaring that products of operators in them are to be evaluated from the left, that is

$$\Delta(p_x p_y p_z) = \langle \{ (\hat{p}_x - \langle \hat{p}_x \rangle)(\hat{p}_y - \langle \hat{p}_y \rangle) \} (\hat{p}_z - \langle \hat{p}_z \rangle) \rangle_{\text{Weyl}} \,. \tag{2.24}$$

A bracket on the space of expectation values and moments is defined via the commutator

$$\{\langle\hat{O}_1\rangle,\langle\hat{O}_2\rangle\}=\frac{\langle[\hat{O}_1,\hat{O}_2]\rangle}{i\hbar} \tag{2.25}$$

combined with the Leibniz rule for products of expectation values. For an associative algebra, this definition gives rise to a Poisson bracket;[4] for a non-associative one, the associator is turned into a non-zero Jacobiator of (2.25). Evaluating the bracket on basic variables gives

$$\{\langle\hat{q}_i\rangle,\langle\hat{q}_j\rangle\} = \frac{1}{i\hbar}\langle[\hat{q}_i,\hat{q}_j]\rangle=0 \tag{2.26}$$

$$\{\langle\hat{q}_i\rangle,\langle\hat{p}_j\rangle\} = \frac{1}{i\hbar}\langle[\hat{q}_i,\hat{p}_j]\rangle=\delta_{ij} \tag{2.27}$$

$$\{\langle\hat{p}_i\rangle,\langle\hat{p}_j\rangle\} = \frac{1}{i\hbar}\langle[\hat{p}_i,\hat{p}_j]\rangle=e\epsilon_{ijk}\langle\hat{B}^k\rangle\,. \tag{2.28}$$

For a magnetic field B^k linear in the q_i, the right-hand side of the last relation is a function of basic expectation values, which from now on we will abbreviate as $q_i=\langle\hat{q}_i\rangle$. For a quadratic function, such as $B^k(q_i)=C(q_i)^2$ with a constant C, we have $\langle\hat{B}^k\rangle=C\langle\hat{q}_i^2\rangle=C(q_i)^2+C\Delta(q_i^2)$ with a moment contribution. In general, if the magnetic field is non-linear, we may further expand

$$\langle\hat{B}^k\rangle=\langle B^k(q_i+(\hat{q}_i-q_i))\rangle=B^k(q_i)+\sum_{a,b,c}\frac{1}{a!b!c!}\frac{\partial^{a+b+c}B^k}{\partial q_x^a\partial q_y^b\partial q_z^c}\Delta(q_x^aq_y^bq_z^c) \tag{2.29}$$

with a series of moment contributions. There will be an infinite number of terms if B is non-polynomial. Such an expansion is usually asymptotic and gives rise to semiclassical or effective equations following the methods of [7, 47].

2.2.4 Volume uncertainty and uncertainty volume

Moments are subject to uncertainty relations and cannot be assigned arbitrary values. For covariances and fluctuations (2.23) with $a_1+a_2+b_1+b_2+c_1+c_2=2$, the standard uncertainty relation follows from the Cauchy–Schwarz inequality with $\hat{A}=\widehat{\Delta O}_1=\hat{O}_1-\langle\hat{O}_1\rangle$ and $\hat{B}=\widehat{\Delta O}_2=\hat{O}_2-\langle\hat{O}_2\rangle$ linear in basic operators $\hat{O}_1$ and $\hat{O}_2$. Moments of higher order are restricted by uncertainty relations that follow

[4] The resulting Poisson manifold is much larger than the classical phase space, and in fact infinite-dimensional owing to infinitely many independent moments.

from the Cauchy–Schwarz inequality with $\hat{A}$ and $\hat{B}$ polynomial in $\widehat{\Delta O}_i$. (See for instace [48, 49].)

A non-zero commutator between two observables $\hat{O}_1$ and $\hat{O}_2$ provides a lower bound for the product of their fluctuations ΔO_1 and ΔO_2. For a set of n canonical pairs $(\hat{q}_i, \hat{p}_i)$, the lower bound of $\prod_{i=1}^{n}(\Delta q_i \Delta p_i) \geq (\hbar/2)^n$ may then be interpreted as an elementary chunk of phase-space volume. For non-canonical commutators, different lower bounds may be realized for a subset of the phase-space variables or even the configuration (or momentum) variables among themselves. For instance, (2.6) suggests that areas in momentum space have lower bounds given by, for instance, $\Delta p_x \Delta p_y \geq \frac{1}{2}\hbar e\langle \hat{B}^z\rangle$, depending on the magnetic field and therefore, possibly, on the position. A new suggestion, going back to [16] and further analyzed in [32], is that a non-zero associator may provide an independent lower bound for triple products of fluctuations, such as $\Delta p_x \Delta p_y \Delta p_z$ for (2.7). Non-associativity in position space may then, intriguingly, imply spatial discreteness. (However, even if there is a lower bound for quantum fluctuations, the relation to discrete structures is not obvious: In [50], uncertainty relations have been computed for a discrete system, given by the cotangent space of a circle, and no lower bound for fluctuations of the discrete momentum was obtained. As discussed there, such lower bounds could rather be taken as an indication for extended fundamental objects, as would be appropriate for lower bounds found in models of string theory.)

It is not obvious how such uncertainty relations may be derived in a general way. The Cauchy–Schwarz inequality quite naturally leads to commutators by expressing the expectation value $\langle A^* B\rangle$ in terms of symmetric and antisymmetric combinations of A and B. It is more difficult to see how the associator might appear in uncertainty relations as an intrinsic quantity (as opposed to a quantity derived from the commutator which happens to resemble the associator). For instance, given a non-trivial uncertainty relation between momentum components, such as

$$(\Delta p_x)^2(\Delta p_y)^2 \geq \frac{1}{4}\hbar^2 e^2 \langle \hat{B}^z\rangle^2\,,$$

and the standard uncertainty relation between Δq_z and Δp_z, a magnetic field with $\partial B^z/\partial q_z \neq 0$ would imply a non-trivial lower bound for the triple product

$$(\Delta p_x)^2(\Delta p_y)^2(\Delta p_z)^2 \;\; \geq \;\; \frac{1}{4}\hbar^2 e^2 \langle \hat{B}^z\rangle^2 (\Delta p_z)^2 \tag{2.30}$$

$$
\begin{aligned}
&= \frac{1}{4}\hbar^2 e^2 \left(B^z(\langle \hat{q}_j\rangle)^2 + B^z(\langle \hat{q}_j\rangle) \frac{\partial^2 B^z(\langle \hat{q}_j\rangle)}{\partial \langle \hat{q}_z\rangle^2} (\Delta q_z)^2 + \cdots \right) (\Delta p_z)^2 \\
&\geq \frac{1}{16}\hbar^4 e^2 B^z \frac{\partial^2 B^z}{\partial q_z^2} + \frac{1}{4}\hbar^2 e^2 (B^z)^2 (\Delta p_z)^2 + \cdots \qquad (2.31)
\end{aligned}
$$

However, such an uncertainty relation is neither simple enough to suggest a universal and state-independent lower bound, nor does it follow directly from the associator. Moreover, there would be no lower bound for a linear magnetic field, or a constant associator. (For a semiclassical state, the second term in (2.31) would be dominant, so that the inequality would just amount to the momentum uncertainty relation for Δp_x and Δp_y, multiplied with an additional factor of Δp_z on both sides.)

A direct definition of volume uncertainty would be the uncertainty $\Delta V = \sqrt{\langle \hat{V}^2\rangle - \langle \hat{V}\rangle}$ of the volume operator $\hat{V} := ((\hat{p}_x\hat{p}_y)\hat{p}_z)_{\rm Weyl}$. (The definition $\hat{V} := (\hat{p}_x(\hat{p}_y\hat{p}_z))_{\rm Weyl}$ would result in the same operator for an alternative algebra.) However, an uncertainty relation follows from the Cauchy-Schwarz inequality only when ΔV is combined with the fluctuation of another observable not commuting with $\hat{V}$. No universal lower bound for ΔV itself would be implied.

One can introduce different quantities which may capture some of the intuition that may be associated with the notion of "volume uncertainty." For instance, the quantity $(\widehat{\Delta p}_x\widehat{\Delta p}_y)\widehat{\Delta p}_z$ could be related to the associator. In what follows, we call this triple product of uncertainties the uncertainty volume, in order to distinguish it from the uncertainty of the volume operator. (As noted in [32], the antisymmetrized uncertainty volume is related to the associator, but it is not clear to us how this quantity may appear as an upper or lower bound.)

Although the uncertainty volume does appear in some uncertainty relations, it turns out that it is subject to an upper rather than lower bound by higher-order uncertainty relations. Choosing $\hat{A} = \frac{1}{2}(\widehat{\Delta p}_x\widehat{\Delta p}_y + \widehat{\Delta p}_y\widehat{\Delta p}_x)$ and $\hat{B} = \widehat{\Delta p}_z$, one can compute

$$
\begin{aligned}
\langle \hat{A}^*\hat{A}\rangle &= \Delta(p_x^2p_y^2) - \frac{1}{4}\langle[\widehat{\Delta p}_x, \widehat{\Delta p}_y]^2\rangle + \frac{1}{6}\langle \widehat{\Delta p}_y[[\widehat{\Delta p}_x, \widehat{\Delta p}_y], \widehat{\Delta p}_x] - \widehat{\Delta p}_x[[\widehat{\Delta p}_x, \widehat{\Delta p}_y], \widehat{\Delta p}_y]\rangle \\
&= \Delta(p_x^2p_y^2) + \frac{e^2\hbar^2}{4}\langle \hat{B}^z\rangle + \frac{e^2\hbar^2}{6}\left\langle \widehat{\Delta p}_x\widehat{\partial_y B^z} - \widehat{\Delta p}_y\widehat{\partial_x B^z}\right\rangle . \qquad (2.32)
\end{aligned}
$$

(For a linear magnetic field, the last term is zero.) We obtain $\langle \hat{B}^*\hat{B}\rangle = (\Delta p_z)^2$, as usual, and $\langle \hat{A}^* \hat{B} \rangle$ contains in its real part the fluctuation volume:

$$\langle \hat{A}^* \hat{B} \rangle = \langle (\widehat{\Delta p_x} \widehat{\Delta p_y}) \widehat{\Delta p_z} \rangle + \frac{1}{2} \hbar^2 e \left\langle \widehat{\partial_z B^z} \right\rangle - \frac{1}{4} i\hbar e \langle \hat{B}^z \widehat{\Delta p_z} + \widehat{\Delta p_z} \hat{B}^z \rangle \,. \qquad (2.33)$$

Therefore, the uncertainty relation for the fluctuation volume $f := \langle (\widehat{\Delta p_x} \widehat{\Delta p_y}) \widehat{\Delta p_z} \rangle$ is of the form

$$\left(f + \frac{1}{2} \hbar^2 e \left\langle \widehat{\partial_z B^z} \right\rangle \right)^2 \leq \Delta(p_x^2 p_y^2) \Delta(p_z^2) + \cdots \qquad (2.34)$$

However, the associator again does not play a direct role in the derivation.

2.3 Algebra of second-order moments

We now calculate some of the brackets between second-order moments, providing characteristic examples in which different features of alternative algebras appear. These brackets are useful for Hamiltonian equations of motion once the dynamics is specified, which we will explore in the next section.

2.3.1 Application of Moufang identities

We begin with an example in which the identity (2.20) (for A, B and C linear in the $\hat{p}_i$) plays an important role. For the bracket of two covariances of different momentum components, we have

$$\begin{aligned} P_1 &:= \{\Delta(p_x p_y), \Delta(p_y p_z)\} \qquad (2.35) \\ &= \frac{1}{4i\hbar} \langle [(\hat{p}_x - p_x)(\hat{p}_y - p_y) + (\hat{p}_y - p_y)(\hat{p}_x - p_x), \\ &\qquad (\hat{p}_y - p_y)(\hat{p}_z - p_z) + (\hat{p}_z - p_z)(\hat{p}_y - p_y)] \rangle \\ &= \frac{1}{4i\hbar} \langle [i\hbar e \hat{B}^z + 2(\hat{p}_y - p_y)(\hat{p}_x - p_x), i\hbar e \hat{B}^x + 2(\hat{p}_z - p_z)(\hat{p}_y - p_y)] \rangle \end{aligned}$$

using the non-zero commutator (2.6). We continue and write out the commutator explicitly,

$$\begin{aligned} P_1 &= \frac{1}{i\hbar} \langle ((\hat{p}_y - p_y)(\hat{p}_x - p_x))((\hat{p}_z - p_z)(\hat{p}_y - p_y)) \\ &\quad - ((\hat{p}_z - p_z)(\hat{p}_y - p_y))((\hat{p}_y - p_y)(\hat{p}_x - p_x)) \qquad (2.36) \\ &\quad + \frac{1}{2} i\hbar e (\hat{B}^z (\hat{p}_z - p_z)(\hat{p}_y - p_y) - (\hat{p}_z - p_z)(\hat{p}_y - p_y) \hat{B}^z) \end{aligned}$$

$$+\frac{1}{2}i\hbar e(-\hat{B}^x(\hat{p}_y-p_y)(\hat{p}_x-p_x)+(\hat{p}_y-p_y)(\hat{p}_x-p_x)\hat{B}^x)\rangle\,.$$

The Moufang identity can be used in the first line, but not in the second line in its present form. Two additional applications of commutators bring the momentum factors of the second line into the form of (2.13):

$$\begin{aligned}P_1 \;=\; & \frac{1}{i\hbar}\langle((\hat{p}_y-p_y)(\hat{p}_x-p_x))((\hat{p}_z-p_z)(\hat{p}_y-p_y))\\ & -((\hat{p}_y-p_y)(\hat{p}_z-p_z)-i\hbar\hat{B}^x)((\hat{p}_x-p_x)(\hat{p}_y-p_y)-i\hbar\hat{B}^z)\\ & +\frac{1}{2}i\hbar e(\hat{B}^z(\hat{p}_z-p_z)(\hat{p}_y-p_y)-(\hat{p}_z-p_z)(\hat{p}_y-p_y)\hat{B}^z)\\ & +\frac{1}{2}i\hbar e(-\hat{B}^x(\hat{p}_y-p_y)(\hat{p}_x-p_x)+(\hat{p}_y-p_y)(\hat{p}_x-p_x)\hat{B}^x)\rangle\,. \end{aligned} \tag{2.37}$$

Now distributing the second term and using (2.13), and collecting the middle terms of the first and second terms into a commutator, we obtain

$$\begin{aligned}P_1 \;=\; & e\left\langle-(\hat{p}_y-p_y)\hat{B}^y(\hat{p}_y-p_y)\right\rangle\\ & +\frac{e}{2}\left\langle\hat{B}^x(\hat{p}_x-p_x)(\hat{p}_y-p_y)+(\hat{p}_y-p_y)(\hat{p}_x-p_x)\hat{B}^x\right\rangle\\ & +\frac{e}{2}\left\langle\hat{B}^z(\hat{p}_z-p_z)(\hat{p}_y-p_y)+(\hat{p}_y-p_y)(\hat{p}_z-p_z)\hat{B}^z\right\rangle\,.\end{aligned}$$

We can now expand $\langle\hat{B}^i\rangle$ as in (2.29) in order to express this expectation value in terms of moments. If we keep up to second-order moments for semiclassical equations, we obtain

$$\{\Delta(p_xp_y),\Delta(p_yp_z)\} \;=\; -eB^y\Delta(p_y^2)+eB^x\Delta(p_xp_y)+eB^z\Delta(p_yp_z)\,.$$

2.3.2 Application of the associator

Another example in which a combination of commutators and the associator can be used directly is

$$\begin{aligned}P_2 \;:=\; & \{\Delta(p_xq_z),\Delta(p_yp_z)\} \qquad (2.38)\\ =\; & \frac{1}{2i\hbar}\langle[(\hat{p}_x-p_x)(\hat{q}_z-q_z),(\hat{p}_y-p_y)(\hat{p}_z-p_z)+(\hat{p}_z-p_z)(\hat{p}_y-p_y)]\rangle\\ =\; & \frac{1}{2i\hbar}\langle((\hat{q}_z-q_z)(\hat{p}_x-p_x))((\hat{p}_y-p_y)(\hat{p}_z-p_z))\end{aligned}$$

$$-((\hat{p}_y - p_y)(\hat{p}_z - p_z))((\hat{p}_x - p_x)(\hat{q}_z - q_z))\rangle + (y \leftrightarrow z \text{ only in } p\text{-terms})\,.$$

The goal here is to bring the triple product of $\hat{p}_i$ in the first term to the form of the second term; for this reason we use the associator first, and concentrate only on the first term (omitting the $(\hat{q}_z - q_z)$ term for now):

$$(\hat{p}_x - p_x)((\hat{p}_y - p_y)(\hat{p}_z - p_z)) = ((\hat{p}_x - p_x)(\hat{p}_y - p_y))(\hat{p}_z - p_z) + \hbar^2 e\widehat{\partial_i B^i}\,. \quad (2.39)$$

After using the commutator in the paranthesis of the first term on the right-hand side we arrive at

$$(\hat{p}_x - p_x)((\hat{p}_y - p_y)(\hat{p}_z - p_z)) = ((\hat{p}_y - p_y)(\hat{p}_x - p_x))(\hat{p}_z - p_z) + i\hbar e\hat{B}^z(\hat{p}_z - p_z) + \hbar^2 e\widehat{\partial_i B^i}\,. \quad (2.40)$$

Once again, we use the associator followed by the commutator, writing

$$\begin{aligned}(\hat{p}_x - p_x)((\hat{p}_y - p_y)(\hat{p}_z - p_z)) \;=\;& (\hat{p}_y - p_y)((\hat{p}_z - p_z)(\hat{p}_x - p_x)) \quad (2.41)\\ & - i\hbar e(\hat{p}_y - p_y)\hat{B}^y + i\hbar e\hat{B}^z(\hat{p}_z - p_z) + 2\hbar^2 e\widehat{\partial_i B^i}\,.\end{aligned}$$

Applying this procedure one last time on the very first term in (2.41) yields

$$((\hat{p}_y - p_y)(\hat{p}_z - p_z))(\hat{p}_x - p_x) - i\hbar e(\hat{p}_y - p_y)\hat{B}^y + i\hbar e\hat{B}^z(\hat{p}_z - p_z) + 3\hbar^2 e\widehat{\partial_i B^i}\,. \quad (2.42)$$

So far we have looked only at the first term in (2.38). Observe that we brought it to the same form as the second term in (2.38) up to the position of $(\hat{q}_z - q_z)$ to the left and right, respectively, which can be combined into a commutator with $(\hat{p}_z - p_z)$ to yield an $i\hbar$. Now doing the same calculation with the third and fourth term in (2.38), which we did not spell out explicitly, yields the bracket wherein the contributions from the associator terms drop out

$$\begin{aligned}\{\Delta(p_x q_z), \Delta(p_y p_z)\} \;=\;& \frac{1}{2}\langle(\hat{p}_y - p_y)(\hat{p}_x - p_x) + (\hat{p}_x - p_x)(\hat{p}_y - p_y)\rangle\\ & + \frac{e}{2}\left\langle \hat{B}^z(\hat{q}_z - q_z)(\hat{p}_z - p_z) + (\hat{q}_z - q_z)(\hat{p}_z - p_z)\hat{B}^z \right\rangle\\ & - \frac{e}{2}\left\langle \hat{B}^y(\hat{q}_z - q_z)(\hat{p}_y - p_y) + (\hat{q}_z - q_z)(\hat{p}_y - p_y)\hat{B}^y \right\rangle\\ & - \frac{i\hbar e}{2}\langle\hat{B}^z\rangle\,. \quad (2.43)\end{aligned}$$

Expanding to second order in moments, we obtain

$$\begin{aligned}\{\Delta(p_xq_z),\Delta(p_yp_z)\} &= \Delta(p_xp_y)+eB^z\Delta(p_zq_z)-eB^y\Delta(p_yq_z)\\ &\quad -\frac{i\hbar e}{2}\left(B^z+\frac{1}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j)\right)+\cdots\,. \end{aligned} \tag{2.44}$$

Here and in what follows, the dots indicate terms having moments higher than second order, or terms of order larger than $\hbar$ in a semiclassical state.

2.3.3 Application of commutator identities

In our third example of the brackets it is sufficient to use standard commutator identities:

$$\begin{aligned}\{\Delta(p_xq_y),\Delta(p_yq_x)\} &= \frac{1}{i\hbar}\langle[(\hat{p}_x-p_x)(\hat{q}_y-q_y),(\hat{p}_y-p_y)(\hat{q}_x-q_x)]\rangle \\ &= \langle(\hat{p}_x-p_x)(\hat{q}_x-q_x)-(\hat{p}_y-p_y)(\hat{q}_y-q_y)+e\hat{B}^z(\hat{q}_y-q_y)(\hat{q}_x-q_x)\rangle \\ &= \Delta(p_xq_x)-\Delta(p_yq_y)+eB^z\Delta(q_xq_y)\end{aligned} \tag{2.45}$$

expanded up to second order in moments.

In general, however, one should be careful with the usual identity $[\hat{A},\hat{B}\hat{C}]=[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]$ when the algebra is not associative, as has already been pointed out in [32] in the context of Heisenberg equations of motion, no longer given by a derivation $[\cdot,\hat{H}]$. In fact, the equation is not valid in general: We have

$$[\hat{A},\hat{B}\hat{C}]=\hat{A}(\hat{B}\hat{C})-(\hat{B}\hat{C})\hat{A}$$

and

$$[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]=(\hat{A}\hat{B})\hat{C}-(\hat{B}\hat{A})\hat{C}+\hat{B}(\hat{A}\hat{C})-\hat{B}(\hat{C}\hat{A})\,.$$

The two terms in the middle of the last equation cancel out only when the multiplication of three given operators is associative. In our example, we have at most two momentum components, so that this requirement is satisfied. In general, one can write the difference of the usual two expressions as a combination of associators:

$$[\hat{A},\hat{B}\hat{C}]-[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]=-[\hat{A},\hat{B},\hat{C}]-[\hat{B},\hat{C},\hat{A}]-[\hat{B},\hat{A},\hat{C}]\,. \tag{2.46}$$

For an alternative algebra, the last two terms cancel out and the difference is just the negative associator $[\hat{A},\hat{B},\hat{C}]$.

2.3.4 Brackets

Having shown a few explicit calculations, we give here a list of some more brackets of generic type, including their expansions up to second-order moments:

$$\begin{aligned}
\{\Delta(p_yp_x),\Delta(q_yq_x)\} &= -\Delta(p_xq_x)-\Delta(p_yq_y) && (2.47)\\
\{\Delta(p_xq_y),\Delta(q_xq_y)\} &= -\Delta(q_y^2) && (2.48)\\
\{\Delta(p_xq_y),\Delta(p_yq_x)\} &= \Delta(p_xq_x)-\Delta(p_yq_y)+e\langle\hat{B}^z(\hat{q}_y-q_y)(\hat{q}_x-q_x)\rangle \\
&= \Delta(p_xq_x)-\Delta(p_yq_y)+eB^z\Delta(q_xq_y)+\cdots && (2.49)\\
\{\Delta(p_xq_y),\Delta(p_zq_z)\} &= -e\langle\hat{B}^y(\hat{q}_y-q_y)(\hat{q}_z-q_z)\rangle \\
&= -eB^y\Delta(q_yq_z)+\cdots && (2.50)\\
\{\Delta(p_xq_x),\Delta(p_yq_y)\} &= e\langle\hat{B}^z(\hat{q}_x-q_x)(\hat{q}_y-q_y)\rangle \\
&= eB^z\Delta(q_xq_y)+\cdots && (2.51)\\
\{\Delta(p_xq_z),\Delta(p_yp_z)\} &= \Delta(p_xp_y)+\frac{e}{2}\langle\hat{B}^z(\hat{q}_z-q_z)(\hat{p}_z-p_z)+(\hat{q}_z-q_z)(\hat{p}_z-p_z)\hat{B}^z\rangle \\
&\quad -\frac{e}{2}\langle(\hat{q}_z-q_z)\hat{B}^y(\hat{p}_y-p_y)+(\hat{q}_z-q_z)(\hat{p}_y-p_y)\hat{B}^y\rangle-\frac{i\hbar e}{2}\langle\hat{B}^z\rangle \\
&= \Delta(p_xp_y)+eB^z\Delta(p_zq_z)-eB^y\Delta(p_yq_z) \\
&\quad -\frac{i\hbar e}{2}\left(B^z+\frac{1}{2}\frac{\partial^2B^z}{\partial q_i\partial q_j}\Delta(q_iq_j)\right)+\cdots && (2.52)\\
\{\Delta(p_xq_x),\Delta(p_yp_z)\} &= \frac{e}{2}\langle\hat{B}^z(\hat{q}_x-q_x)(\hat{p}_z-p_z)+(\hat{q}_x-q_x)(\hat{p}_z-p_z)\hat{B}^z\rangle \\
&\quad -\frac{e}{2}\langle(\hat{q}_x-q_x)\hat{B}^y(\hat{p}_y-p_y)+(\hat{q}_x-q_x)(\hat{p}_y-p_y)\hat{B}^y\rangle \\
&= eB^z\Delta(p_zq_x)-eB^y\Delta(p_yq_x)+\cdots && (2.53)
\end{aligned}$$

For a non-constant magnetic field, some of the brackets of basic expectation values and moments are non-zero as well:

$$\begin{aligned}
\{p_x,\Delta(p_y^2)\} &= e\langle\hat{B}^z(\hat{p}_y-p_y)+(\hat{p}_y-p_y)\hat{B}^z\rangle && (2.54)\\
&= 2e\frac{\partial B^z}{\partial q_i}\Delta(p_yq_i)+\cdots\,. && (2.55)
\end{aligned}$$

2.4 Semiclassical dynamics of a charged particle in a magnetic monopole density

For algebraic states, the dynamics is defined in terms of a flow of positive linear functionals ω_t, $t \in \mathbb{R}$ on $\mathcal{A}$ with respect to a Hamiltonian $H \in \mathcal{A}$:

$$\frac{\mathrm{d}\omega_t(O)}{\mathrm{d}t} := \frac{1}{i\hbar}\omega_t([O,H]) = \{\omega_t(O), \omega_t(H)\} \tag{2.56}$$

in terms of the bracket (2.25). This definition agrees with the standard Schrödinger or Heisenberg flow in the case of an associative algebra of operators represented on a Hilbert space, but it does not require this additional structure. (It is also insensitive to the commutator $[\cdot, \hat{H}]$ no longer being a derivation, which had been noted in [32].) We can therefore apply it to the example of a non-associative algebra studied here.

2.4.1 General magnetic field

We first choose the "free-particle" Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)\,, \tag{2.57}$$

so that we will be considering the motion of a charged particle in a background magnetic field without additional forces. Interactions between the charged particle and the magnetic field are represented by the non-associativity of the algebra or the Jacobiator of the bracket of expectation values and moments, rather than terms in the Hamiltonian. We obtain the quantum Hamiltonian as

$$H_Q := \langle\hat{H}\rangle = p_x^2 + p_y^2 + p_z^2 + \Delta(p_x^2) + \Delta(p_y^2) + \Delta(p_z^2) \tag{2.58}$$

which generates Hamiltonian equations of motion as per (2.56), now writing $\omega(\hat{H}) = \langle\hat{H}\rangle$.

As an example we look at the x-components of the equations of motion,

$$\dot{q}_x = \frac{1}{m}p_x \tag{2.59}$$

$$\begin{aligned}\dot{p}_x &= \frac{1}{2m}\{p_x, p_y^2+p_z^2\} + \frac{1}{2m}\{p_x, \langle\hat{p}_y^2\rangle - p_y^2 + \langle\hat{p}_z^2\rangle - p_z^2\} \\ &= \frac{e}{2m}(\langle\hat{p}_y\hat{B}_z + \hat{B}_z\hat{p}_y\rangle - \langle\hat{p}_z\hat{B}_y + \hat{B}_y\hat{p}_z\rangle)\,. \end{aligned} \tag{2.60}$$

In this expression we expand the magnetic field as

$$\hat{B}^z(\hat{q}) = B^z(q) + (\hat{q}_i - q_i)\frac{\partial B^z}{\partial q_i} + \frac{1}{2}(\hat{q}_j - q_j)(\hat{q}_i - q_i)\frac{\partial^2 B^z}{\partial q_i \partial q_j} + \cdots \tag{2.61}$$

and insert it in the first term of (2.60):

$$\langle\hat{p}_y\hat{B}^z + \hat{B}^z\hat{p}_y\rangle = 2B^z p_y + 2\frac{\partial B^z}{\partial q_i}\Delta(p_y q_i) + p_y\frac{\partial^2 B^z}{\partial q_i \partial q_j}\Delta(q_i q_j) + \cdots\,. \tag{2.62}$$

Here we just added the (vanishing) contribution $\langle p_y(\hat{q}_i - q_i)\rangle$. Similarly expanding the second term in (2.60) and using the definitions of moments we get

$$\begin{aligned} m\ddot{q}_x &= e(B^z v_y - B^y v_z) \qquad (2.63)\\ &+\frac{e}{m}\frac{\partial B^z}{\partial q_i}\Delta(p_y q_i) - \frac{e}{m}\frac{\partial B^y}{\partial q_i}\Delta(p_z q_i) + \frac{e}{2m}\left(p_y\frac{\partial^2 B^z}{\partial q_i\partial q_j} - p_z\frac{\partial^2 B^y}{\partial q_i \partial q_j}\right)\Delta(q_i q_j) + \cdots\,. \end{aligned}$$

The first term is the classical Lorentz force. The additional terms are quantum corrections to the equation of motion, which vanish for a constant magnetic field. Indeed, it is well-known that a charged particle in a constant magnetic field can be described by a harmonic-oscillator Hamiltonian, and the harmonic oscillator does not give rise to quantum corrections in Ehrenfest equations.

The moments that appear in (2.63) are themselves subject to dynamical equations of motion with respect to the effective Hamiltonian. We have

$$\begin{aligned} \dot{\Delta}(p_y q_i) &= 2\Delta(p_y p_x)\delta_{ix} + 2\Delta(p_y^2)\delta_{iy} + 2\Delta(p_y p_z)\delta_{iz} - 2eB^z\Delta(p_x q_i) \\ &+2eB^x\Delta(p_z q_i) + 2eq_i\frac{\partial B^z}{\partial q_j}\Delta(p_x q_j) + eq_i p_x\frac{\partial^2 B^z}{\partial q_j \partial q_k}\Delta(q_j q_k) \\ &-2eq_i\frac{\partial B^x}{\partial q_j}\Delta(p_z q_j) - eq_i p_z\frac{\partial^2 B^x}{\partial q_j\partial q_k}\Delta(q_j q_k) + \cdots\,. \end{aligned} \tag{2.64}$$

The equation for $\Delta(p_z q_i)$ is analogous to the equation above. The remaining moment in (2.63) has an equation of motion of the form

$$\begin{aligned}\dot{\Delta}(q_iq_j) &= 2\Delta(p_xq_i)\delta_{jx} + 2\Delta(p_xq_j)\delta_{ix} + 2\Delta(p_yq_i)\delta_{jx} + 2\Delta(p_yq_j)\delta_{ix} \\ &\quad +2\Delta(p_zq_i)\delta_{jx} + 2\Delta(p_zq_j)\delta_{ix} + \cdots .\end{aligned} \tag{2.65}$$

For a closed set of equations, we need an equation of motion for moments of the form $\Delta(p_yp_x)$, which appears in (2.64). This calculation turns out to be more challenging, but it can be handled by using the associator as well as the defining identities (2.8) for an alternative algebra

$$\begin{aligned}\dot{\Delta}(p_yp_x) &= 2e\Bigg[-B^z\Delta(p_x^2) + B^z\Delta(p_y^2) + p_x\frac{\partial B^z}{\partial q_i}\Delta(p_xq_i) + \frac{p_x^2}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j) \\ &\quad -p_y\frac{\partial B^z}{\partial q_i}\Delta(p_yq_i) - \frac{p_y^2}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j) - B^y\Delta(p_yp_z) + B^x\Delta(p_xp_z) \\ &\quad -p_z\frac{\partial B^y}{\partial q_i}\Delta(p_yq_i) + p_z\frac{\partial B^x}{\partial q_i}\Delta(p_xq_i) \\ &\quad -i\hbar\frac{\partial^2 B^j}{\partial q_i\partial q_j}\Delta(p_zq_i) - \frac{i\hbar p_z}{2}\left(\frac{\partial B^j}{\partial q_j} + \frac{1}{2}\frac{\partial^3 B^j}{\partial q_jq_iq_k}\Delta(q_iq_k)\right)\Bigg] + \end{aligned} \tag{2.66}$$

We now have a closed system of equations for the moments up to second order.

2.4.2 Canonical variables in the absence of a magnetic charge density

In order to test the quantum corrections for a non-constant magnetic field, we use moment expansions in a derivation of semiclassical equations for the canonical variables q_i and $\pi_j = m\dot{q}_j + eA_j$ (with $\{q_i, \pi_j\} = i\hbar$). These variables can be used only in the absence of a magnetic charge density, in which case we can compare their dynamics with (2.63).

In canonical variables, the Hamiltonian operator (2.57) is

$$\hat{H} = \frac{1}{2m}(\hat{\pi} - eA)^2 = \frac{1}{2m}\delta^{ij}(\pi_i - eA_i)(\pi_j - eA_j)\,. \tag{2.67}$$

To second order in moments, it implies a quantum Hamiltonian

$$
\begin{aligned}
H_Q = \langle \hat{H} \rangle &= \frac{1}{2m}\delta^{ij}\pi_i\pi_j - \frac{e}{m}\delta^{ij}\pi_i A_j + \frac{e^2}{2m}\delta^{ij}A_iA_j \\
&+\frac{1}{2m}\delta^{ij}\Delta(\pi_i\pi_j) - \frac{e}{m}\delta^{ik}\frac{\partial A_i}{\partial q_j}\Delta(\pi_k q_j) \\
&-\frac{e}{2m}\delta^{il}\left((\pi_i - eA_i)\frac{\partial^2 A_l}{\partial q_j \partial q_k} - e\frac{\partial A_i}{\partial q_j}\frac{\partial A_l}{\partial q_k}\right)\Delta(q_jq_k)
\end{aligned}
\tag{2.68}
$$

where A_i is understood as the classical function $A_i(\langle \hat{q}_j \rangle)$ evaluated at expectation values.

We compute Hamiltonian equations of motion

$$
\begin{aligned}
\dot{q}_i &= \frac{1}{m}\pi_i - \frac{e}{m}A_i - \frac{e}{2m}\frac{\partial^2 A_i}{\partial q_k \partial q_l}\Delta(q_kq_l) \\
&= \frac{1}{m}(\pi_i - e\langle \hat{A}_i \rangle)
\end{aligned}
\tag{2.69}
$$

and

$$
\begin{aligned}
\dot{\pi}_i &= \frac{e}{m}\delta^{jk}\pi_j\frac{\partial A_k}{\partial q_i} - \frac{e^2}{m}\delta^{jk}A_j\frac{\partial A_k}{\partial q_i} \\
&+\frac{e}{m}\delta^{jk}\frac{\partial^2 A_j}{\partial q_i \partial q_l}\delta(\pi_k q_l) \\
&+\frac{q}{2m}\delta^{jk}\left((\pi_j - eA_j)\frac{\partial^3 A_k}{\partial q_i \partial q_m \partial q_n}\right. \\
&\left.-e\left(\frac{\partial A_j}{\partial q_i}\frac{\partial^2 A_k}{\partial q_m \partial q_n} + \frac{\partial \dot{\Delta}(p_yp_x)A_j}{\partial q_m}\frac{\partial^2 A_k}{\partial q_i \partial q_n} + \frac{\partial A_j}{\partial q_n}\frac{\partial^2 A_k}{\partial q_i \partial q_m}\right)\right)\Delta(q_mq_n) + \cdots .
\end{aligned}
\tag{2.70}
$$

We will also need the equations of motion for some moments:

$$
\dot{\Delta}(q_mq_n) = \frac{1}{m}\left(\Delta(\pi_m q_n) + \Delta(\pi_n q_m)\right) - \frac{e}{m}\left(\frac{\partial A_m}{\partial q_l}\Delta(q_lq_n) + \frac{\partial A_n}{\partial q_l}\Delta(q_lq_m)\right) + \cdots .
\tag{2.71}
$$

With these results, we can rewrite the Hamiltonian equations of motion as second-order differential equations for the components q_i:

$$
m\ddot{q}_i = \frac{e}{m}\pi_j\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right) - \frac{e^2}{m}A_j\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)
\tag{2.72}
$$

$$+\frac{e}{m}\frac{\partial}{\partial q_m}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)\Delta(\pi_j q_m)$$
$$+\frac{e}{2m}\left((\pi_j-eA_j)\frac{\partial^2}{\partial q_m\partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)-e\frac{\partial^2 A_j}{\partial q_m\partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)\right.$$
$$\left.-2e\frac{\partial A_j}{\partial q_m}\frac{\partial}{\partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)\right)\Delta(q_m q_n)+\cdots .$$

After several simplifications, we can bring this equation into the form

$$m\ddot{q}_i = \frac{e}{m}(\pi_j-e\langle\hat{A}_j\rangle)\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right) \tag{2.73}$$
$$+\frac{e}{m}\frac{\partial}{\partial q_m}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)\Delta((\pi_j-eA_j)q_m)$$
$$+\frac{e}{2m}(\pi_j-eA_j)\frac{\partial^2}{\partial q_m\partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)\Delta(q_m q_n)+\cdots$$
$$= \frac{e}{m}(\pi_j-e\langle\hat{A}_j\rangle)\left\langle\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right\rangle \tag{2.74}$$
$$+\frac{e}{m}\frac{\partial}{\partial q_m}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l}-\frac{\partial A_l}{\partial q_j}\right)\Delta((\pi_j-eA_j)q_m)+\cdots .$$

This equation agrees with (2.63), but is valid only in the absence of a magnetic charge density.

2.4.3 Potential and magnetic charge density

If there is a position-dependent potential in addition to the magnetic field, the effective Hamiltonian is

$$H_Q = \frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V(q_i) \tag{2.75}$$
$$+\frac{1}{2m}(\Delta(p_x^2)+\Delta(p_y^2)+\Delta(p_z^2))+\frac{1}{2}\frac{\partial^2 V}{\partial q_i\partial q_j}\Delta(q_iq_j)+\cdots .$$

The potential implies the usual additional terms $-\partial V/\partial q_i$ and $-\frac{1}{2}(\partial^3 V/\partial q_i\partial q_j\partial q_k)\Delta(q_jq_k)$ in the equation of motion for $m\ddot{q}_i$.

$$m\ddot{q}_x = e(B^z v_y - B^y v_z)-\frac{\partial V}{\partial q_x} \tag{2.76}$$

$$+\frac{e}{m}\frac{\partial B^z}{\partial q_i}\Delta(p_y q_i) - \frac{e}{m}\frac{\partial B^y}{\partial q_i}\Delta(p_z q_i) + \frac{e}{2m}\left(p_y\frac{\partial^2 B^z}{\partial q_i \partial q_j} - p_z\frac{\partial^2 B^y}{\partial q_i \partial q_j}\right)\Delta(q_i q_j)$$
$$-\frac{1}{2}\frac{\partial^3 V}{\partial q_i \partial q_j \partial q_k}\Delta(q_j q_k) + \cdots .$$

Equations of motion for moments (which appear in the above equation) in this case are modified as follows:

$$\begin{aligned}
\dot{\Delta}(p_y q_i) &= 2\Delta(p_y p_x)\delta_{ix} + 2\Delta(p_y^2)\delta_{iy} + 2\Delta(p_y p_z)\delta_{iz} - 2eB^z\Delta(p_x q_i) \\
&\quad +2eB^x\Delta(p_z q_i) + 2eq_i\frac{\partial B^z}{\partial q_j}\Delta(p_x q_j) + eq_i p_x\frac{\partial^2 B^z}{\partial q_j \partial q_k}\Delta(q_j q_k) \\
&\quad -2eq_i\frac{\partial B^x}{\partial q_j}\Delta(p_z q_j) - eq_i p_z\frac{\partial^2 B^x}{\partial q_j \partial q_k}\Delta(q_j q_k) \\
&\quad -\frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k}\left[\Delta(q_i q_j)\delta_{ky} + \Delta(q_i q_k)\delta_{jy}\right] + \cdots \qquad (2.77) \\
\dot{\Delta}(q_i q_j) &= 2\Delta(p_x q_i)\delta_{jx} + 2\Delta(p_x q_j)\delta_{ix} + 2\Delta(p_y q_i)\delta_{jx} + 2\Delta(p_y q_j)\delta_{ix} \\
&\quad +2\Delta(p_z q_i)\delta_{jx} + 2\Delta(p_z q_j)\delta_{ix} + \cdots . \qquad (2.78)
\end{aligned}$$

For completeness, we also note

$$\begin{aligned}
\dot{\Delta}(p_y p_x) &= 2e\Bigg[-B^z\Delta(p_x^2) + B^z\Delta(p_y^2) + p_x\frac{\partial B^z}{\partial q_i}\Delta(p_x q_i) + \frac{p_x^2}{2}\frac{\partial^2 B^z}{\partial q_i \partial q_j}\Delta(q_i q_j) \\
&\quad -p_y\frac{\partial B^z}{\partial q_i}\Delta(p_y q_i) - \frac{p_y^2}{2}\frac{\partial^2 B^z}{\partial q_i \partial q_j}\Delta(q_i q_j) - B^y\Delta(p_y p_z) + B^x\Delta(p_x p_z) \\
&\quad -p_z\frac{\partial B^y}{\partial q_i}\Delta(p_y q_i) + p_z\frac{\partial B^x}{\partial q_i}\Delta(p_x q_i) \\
&\quad -i\hbar\frac{\partial^2 B^j}{\partial q_i \partial q_j}\Delta(p_z q_i) - \frac{i\hbar p_z}{2}\left(\frac{\partial B^j}{\partial q_j} + \frac{1}{2}\frac{\partial^3 B^j}{\partial q_j q_i q_k}\Delta(q_i q_k)\right)\Bigg] \\
&\quad -\frac{1}{2}\frac{\partial^2 V}{\partial q_i \partial q_j}\left(\Delta(p_y q_i)\delta_{jx} + \Delta(p_y q_j)\delta_{ix} + \Delta(p_x q_i)\delta_{jy} + \Delta(p_x q_j)\delta_{iy}\right) + \cdots \qquad (2.79)
\end{aligned}$$

As is evident from (2.64), (2.65), (2.66), (2.77) and (2.79) the equations of motion for $\Delta(p_y q_i)$ (and $\Delta(p_z q_i)$) and $\Delta(p_y p_x)$ get some additional terms due to the potential , whereas that for $\Delta(q_i q_j)$ remains the same.

2.5 Application: Testing non-associative quantum mechanics

Quantum mechanics is being tested ever more precisely by experiments, even while conceptual questions remain. We suggest a new kind of test in an extended setting in which the usual concepts of wave functions or state vectors and operators do not exist. Therefore, the standard axioms about outcomes of individual measurements are unavailable, or at least not known yet, and even at a practical level, no computational methods have been available so far. Here we show how one can derive semiclassical corrections to the motion of a particle and associated new phenomena.

The formalism of state vectors and operators implies that the action of the latter on the former is associative:

$$(\hat{A}\hat{B})\hat{C}\psi = \hat{A}\hat{B}\psi' = \hat{A}(\hat{B}\psi') = \hat{A}(\hat{B}\hat{C})\psi \tag{2.80}$$

if $\psi' = \hat{C}\psi$, for an arbitrary ψ. However, some exotic ingredients, such as magnetic monopoles, require an underlying non-associative algebra in order to quantize such systems. Quantum observables then no longer obey $(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$. Not much has been known about physical effects when this basic identity is not available. In this section, we develop and utilize a novel method in order to reveal testable quantum effects in such a system.

A non-associative algebra cannot be represented by operators on a Hilbert space. Instead, one has to work with an abstract non-associative algebra, which can be constructed by methods of deformation quantization as applied in [29,30,31,32,11]. States in this setting are not defined as normalized vectors in a Hilbert space, but as suitable linear functionals $\hat{A} \mapsto \langle\hat{A}\rangle$ from the algebra of observables to the complex numbers. (For associative observable algebras, this definition of a state is equivalent to the Hilbert-space picture thanks to the Gelfand–Naimark–Segal theorem; see for instance [45].) The primary object is therefore not a wave function but the set of expectation values assigned by a single state to all possible observables. We will demonstrate, for the first time in the context of non-associative quantum mechanics, that algebraic properties of such expectation-value functionals can be used to derive new semiclassical effects.

The best-known example of a non-associative quantum system is a charged particle in a magnetic monopole density [36]. Even if fundamental magnetic monopoles may not exist, such systems are gaining interest from a physical perspective, following recent constructions of analog systems of magnetic monopoles in condensed-matter physics [51, 37, 52, 53, 54, 55]. (Related models also play a role in string theory [16, 33, 29, 34, 35].) As is well known, the canonical momentum of a charged particle in a magnetic field $\vec{B}$ without monopoles, $\mathrm{div}\vec{B} = 0$, is a combination of the particle velocity and the vector potential. For the kinematical momentum $\vec{p} = m\dot{\vec{q}}$, one has non-canonical commutators of momentum components,

$$[\hat{p}_j, \hat{p}_k] = ie\hbar \sum_{l=1}^{3} \epsilon_{jkl} \hat{B}^l \qquad (2.81)$$

where e is the particle's electric charge.

This relation depends only on the magnetic field and does not require a vector potential. Therefore, it can be used to define the basic commutators of a charged particle moving in a magnetic field with $\mathrm{div}\vec{B} \neq 0$, where no vector potential exists. However, the resulting algebra does not fulfill the Jacobi identity of commutators:

$$\begin{aligned} & [[\hat{p}_x, \hat{p}_y], \hat{p}_z] + [[\hat{p}_y, \hat{p}_z], \hat{p}_x] + [[\hat{p}_z, \hat{p}_x], \hat{p}_y] \\ = \; & ie\hbar \sum_{j=1}^{3} [\hat{B}^j, \hat{p}_j] = -e\hbar^2 \widehat{\mathrm{div}\vec{B}} \,. \end{aligned} \qquad (2.82)$$

As the name suggests, the Jacobi *identity* normally follows without further assumptions, provided the algebra is associative. The non-zero result just obtained can therefore be consistent only if the multiplication of momentum components is not associative. (Finite translations generated by momentum operators are not associative [56]. The classical analog is a twisted Poisson bracket [26, 27, 28].)

A physical version of the property of non-associativity is a "triple" uncertainty relation, just as the usual uncertainty relation is a consequence of non-commuting operators. As usual, (2.6) implies that $\Delta p_x \Delta p_y \geq \frac{1}{2} e\hbar \langle \hat{B}^z \rangle$: a large magnetic field in the z-direction deflects the particle from a straight line, making it harder to measure momentum components. A characteristic of monopole fields is that they change along the direction in which they point, for instance if $B^z = \mu z$ with a constant μ. The commutator of $\hat{p}_x$ and $\hat{p}_y$ then depends on the measurement of $\hat{z}$, which itself is subject to the standard uncertainty relation $\Delta z \Delta p_z \geq \frac{1}{2}\hbar$. Therefore, all three fluctuations, Δp_x, Δp_y and Δp_z, together determine how small the momentum fluctuations can be.

As already mentioned, states then cannot be defined as vectors in a Hilbert space, but their physical properties can be analyzed by treating them as linear expectation-value functionals on the algebra. Any such functional must be normalized, $\langle\hat{\mathbb{I}}\rangle = 1$ for the identity $\hat{\mathbb{I}}$ in the algebra, and obey a positivity condition which implies uncertainty relations. Having identified basic operators as the components of position and kinematical momentum, we can parameterize a state by the basic expectation values $\langle\hat{q}_i\rangle$ and $\langle\hat{p}_j\rangle$ as well as fluctuations, correlations and higher moments. The latter are defined to be of the form

$$\Delta(p_i p_j) := \frac{1}{2}\langle\hat{p}_i\hat{p}_j + \hat{p}_j\hat{p}_i\rangle - \langle\hat{p}_i\rangle\langle\hat{p}_j\rangle \tag{2.83}$$

for the example of two momentum components. (With this notation, we slightly modify the usual denotation of quantum fluctuations, identifying $\Delta(p_i^2) = (\Delta p_i)^2$.) The symmetrization in (2.83) takes into account the non-commuting nature of kinematical momentum components in a magnetic field. For higher moments with more than two factors of momentum or position components, different symmetrizations are possible, of which we choose, following [7], totally symmetric (or Weyl) ordering by summing with equal weights over all permutations of factors. Moreover, non-associativity requires a fixed choice for the bracketing of products of observables, which we choose to be done from the left as was done in the earlier sections of this chapter.

A Hamiltonian leads to equations of motion for the basic expectation values coupled to moments, giving an infinite-dimensional dynamical system. In a semi-classical approximation, only a finite number of moments need be considered, corresponding to a specific order in $\hbar$. The Hamiltonian we use in this section is of the standard form,

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{3}\hat{p}_j^2 + V(\hat{x},\hat{y},\hat{z})\,, \tag{2.84}$$

where interactions of a charged particle with the magnetic field are implemented not by a term in the potential but by the non-trivial commutators of momentum components. (The potential $V(x, y, z)$ for an additional, non-magnetic force will be specified below.) Given a Hamiltonian, the equations of motion for expectation values and moments follow by using

$$\frac{\mathrm{d}\langle\hat{O}\rangle}{\mathrm{d}t} = \frac{\langle[\hat{O},\hat{H}]\rangle}{i\hbar} \tag{2.85}$$

which is still available in the non-associative case. However, the non-associative nature requires great care when evaluating commutators of products of the basic observables, for which we refer to section 2.2.3 above.

The specific effect we will derive, related to stable motion in an effective potential, requires the particle to be completely confined. A magnetic field in the z-direction, $B^x = 0 = B^y$, confines the particle motion to a plane normal to the magnetic field. For complete confinement, we combine the magnetic force with a harmonic force in the same direction, choosing the potential to be $V(x,y,z) = \frac{1}{2}m\omega^2 z^2$. This force could be generated by an electric field. For simplicity, we consider a linear z-component $B^z = \mu z$, so that μ is the magnetic charge density. The resulting Hamiltonian is

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{3}\hat{p}_j^2 + \frac{1}{2}m\omega^2\hat{z}^2\,, \tag{2.86}$$

and the magnetic field enters via $[\hat{p}_x,\hat{p}_y] = ie\mu\hbar\hat{z}$ while the other pairs of momentum components commute.

We are interested in deriving an effective potential for the motion of such a particle. If one knows a suitable state of the system, the effective potential can be obtained from the expectation value of the Hamiltonian in which one sets all momentum expectation values to zero in order to remove the kinetic term, $V_{\mathrm{eff}} = \langle\hat{H}\rangle_{\langle\hat{p}_i\rangle=0}$. (We do not require solutions for momentum expectation values to be zero at all times.) In the given case with a quadratic Hamiltonian, the effective potential is the classical potential plus a sum of fluctuations:

$$\begin{aligned} V_{\mathrm{eff}}(\langle\hat{z}\rangle) &= \frac{1}{2}m\omega^2\langle\hat{z}\rangle^2 \\ &\quad+\frac{1}{2m}\left(\Delta(p_x^2)+\Delta(p_y^2)+\Delta(p_z^2)\right)+\frac{1}{2}m\omega^2\Delta(z^2)\,. \end{aligned} \tag{2.87}$$

In order to express this potential as a function of the coordinates, we have to compute the values of quantum fluctuations. Following the methods of [57], we can compute the relevant state properties without using a wave function. Instead, we solve equations of motion for fluctuations in an adiabatic approximation (giving stationary moments in a near-coherent state) and saturating uncertainty relations (minimizing fluctuations). For well-understood (associative) systems such as anharmonic oscillators [8] or Coleman–Weinberg potentials in self-interacting scalar field theories [58], the correct results are obtained in this way [7, 57]. In our new situation, we minimize fluctuations by saturating the uncertainty relations, in the standard form for q_i and p_i and for non-commuting momentum components.

We can derive Ehrenfest-type equations of motion by using (2.85) for the moments. Expanded up to first order in $\hbar$ for semiclassical states, thus including no moments of higher than second order, they are

$$\begin{aligned} m\dot{\Delta}(q_iq_j) &= \Delta(p_xq_i)\delta_{jx} + \Delta(p_xq_j)\delta_{ix} + \Delta(p_yq_i)\delta_{jx} \\ &\quad + \Delta(p_yq_j)\delta_{ix} + \Delta(p_zq_i)\delta_{jx} + \Delta(p_zq_j)\delta_{ix} \end{aligned} \tag{2.88}$$

for all position moments,

$$\begin{aligned} m\dot{\Delta}(p_xq_i) &= \Delta(p_x^2)\delta_{ix} + \Delta(p_xp_y)\delta_{iy} + \Delta(p_xp_z)\delta_{iz} \\ &\quad + e\mu\left(\langle\hat{z}\rangle\Delta(p_yq_i) - \langle\hat{q}_i\rangle\Delta(p_yz)\right) , \end{aligned} \tag{2.89}$$

$$\begin{aligned} m\dot{\Delta}(p_yq_i) &= \Delta(p_xp_y)\delta_{ix} + \Delta(p_y^2)\delta_{iy} + \Delta(p_yp_z)\delta_{iz} \\ &\quad - e\mu\left(\langle\hat{z}\rangle\Delta(p_xq_i) - \langle\hat{q}_i\rangle\Delta(p_xz)\right) , \end{aligned} \tag{2.90}$$

$$\begin{aligned} m\dot{\Delta}(p_zq_i) &= \Delta(p_xp_z)\delta_{ix} + \Delta(p_yp_z)\delta_{iy} + \Delta(p_z^2)\delta_{iz} \\ &\quad - m^2\omega^2\Delta(q_iz) \end{aligned} \tag{2.91}$$

for the position-momentum covariances,

$$\begin{aligned} m\dot{\Delta}(p_xp_y) &= -e\mu\Big(\langle\hat{z}\rangle\Delta(p_x^2) - \langle\hat{z}\rangle\Delta(p_y^2) \\ &\quad - \langle\hat{p}_x\rangle\Delta(p_xz) + \langle\hat{p}_y\rangle\Delta(p_yz)\Big) , \end{aligned} \tag{2.92}$$

$$\begin{aligned} m\dot{\Delta}(p_yp_z) &= -e\mu\left(\langle\hat{z}\rangle\Delta(p_xp_z) + \langle\hat{p}_x\rangle\Delta(p_zz)\right) \\ &\quad - m^2\omega^2\Delta(p_yz) , \end{aligned} \tag{2.93}$$

$$\begin{aligned} m\dot{\Delta}(p_xp_z) &= e\mu\left(\langle\hat{z}\rangle\Delta(p_yp_z) + \langle\hat{p}_y\rangle\Delta(p_zz)\right) \\ &\quad - m^2\omega^2\Delta(p_xz) \end{aligned} \tag{2.94}$$

for momentum covariances, and

$$m\dot{\Delta}(p_x^2) = 2e\mu\left(\langle\hat{z}\rangle\Delta(p_xp_y) + 2\langle\hat{p}_y\rangle\Delta(p_xz) + \langle\hat{p}_x\rangle\Delta(p_yz)\right) \tag{2.95}$$
$$m\dot{\Delta}(p_y^2) = -2e\mu\left(\langle\hat{z}\rangle\Delta(p_xp_y) + \langle\hat{p}_y\rangle\Delta(p_xz) + 2\langle\hat{p}_x\rangle\Delta(p_yz)\right) \tag{2.96}$$
$$m\dot{\Delta}(p_z^2) = -2m^2\omega^2\Delta(zp_z) \tag{2.97}$$

for momentum fluctuations.

We solve these equations to zeroth adiabatic order in the moments, so that all time derivatives on the left-hand sides can be set to zero. The moments are then subject to linear algebraic equations. In order to solve the set of coupled equations, we use (2.88) for all possible index combinations to conclude that

$$\Delta(xp_x) = \Delta(yp_y) = \Delta(zp_z) = 0 \tag{2.98}$$
$$\Delta(yp_x) + \Delta(xp_y) = 0 \tag{2.99}$$
$$\Delta(zp_x) + \Delta(xp_z) = 0 \tag{2.100}$$
$$\Delta(zp_y) + \Delta(yp_z) = 0\,. \tag{2.101}$$

Using equations for mixed position-momentum moments, we obtain

$$\Delta(p_x^2) = -e\mu\left(\langle\hat{z}\rangle\Delta(xp_y) - \langle\hat{x}\rangle\Delta(zp_y)\right)\,, \tag{2.102}$$
$$\Delta(p_xp_y) = -e\mu\left(\langle\hat{z}\rangle\Delta(yp_y) - \langle\hat{y}\rangle\Delta(zp_y)\right)\,, \tag{2.103}$$
$$\Delta(p_xp_z) = 0 \tag{2.104}$$

from (2.89),

$$\Delta(p_xp_y) = e\mu\left(\langle\hat{z}\rangle\Delta(xp_x) - \langle\hat{x}\rangle\Delta(zp_x)\right)\,, \tag{2.105}$$
$$\Delta(p_y^2) = e\mu\left(\langle\hat{z}\rangle\Delta(yp_x) - \langle\hat{y}\rangle\Delta(zp_x)\right)\,, \tag{2.106}$$
$$\Delta(p_yp_z) = 0 \tag{2.107}$$

from (2.90), and

$$\Delta(p_xp_z) = m^2\omega^2\Delta(xz)\,, \tag{2.108}$$
$$\Delta(p_yp_z) = m^2\omega^2\Delta(yz)\,, \tag{2.109}$$
$$\Delta(p_z^2) = m^2\omega^2\Delta(z^2) \tag{2.110}$$

from (2.91). The equations of motion (2.92), (2.93) and (2.94) for momentum covariances provide

$$\begin{aligned}\Delta(p_x^2)-\Delta(p_y^2) &= \frac{\langle\hat{p}_x\rangle\Delta(zp_x)-\langle\hat{p}_y\rangle\Delta(zp_y)}{\langle\hat{z}\rangle}\,, &(2.111)\\ \Delta(zp_z) &= -\frac{m^2\omega^2}{e\mu\langle\hat{p}_x\rangle}\Delta(zp_y)-\frac{\langle\hat{z}\rangle}{\langle\hat{p}_x\rangle}\Delta(p_xp_z)\,, &(2.112)\\ \Delta(zp_z) &= \frac{m^2\omega^2}{e\mu\langle\hat{p}_y\rangle}\Delta(zp_x)-\frac{\langle\hat{z}\rangle}{\langle\hat{p}_y\rangle}\Delta(p_yp_z)\,. &(2.113)\end{aligned}$$

Since $\Delta(zp_z)=0$ from (2.98) and $\Delta(p_xp_z)=0=\Delta(p_yp_z)$ from (2.104) and (2.107), (2.112) and (2.113) imply $\Delta(zp_x)=0=\Delta(zp_y)$. From (2.111) and (2.103) or (2.105), we immediately conclude that

$$\Delta(p_x^2)=\Delta(p_y^2) \quad \text{and} \quad \Delta(p_xp_y)=0\,, \tag{2.114}$$

also using (2.98). These values are consistent with (2.102) and (2.106), in which the same fluctuations appear. All equations are then solved and the adiabatic approximation is self-consistent, showing that an effective potential exists.

We now consider states saturating the uncertainty relations. For the pair (z,p_z), we have the standard one,

$$\Delta(z^2)\Delta(p_z^2)-\Delta(zp_z)^2\geq\frac{\hbar^2}{4}\,, \tag{2.115}$$

while (2.6) implies an uncertainty relation

$$\Delta(p_x^2)\Delta(p_y^2)-\Delta(p_xp_y)^2\geq\frac{1}{4}e^2\hbar^2\langle\hat{B}\rangle^2\,. \tag{2.116}$$

If both inequalities are saturated, we obtain

$$\Delta(p_x^2)=\Delta(p_y^2)=\frac{1}{2}e\hbar\langle\hat{B}\rangle \tag{2.117}$$

and

$$\Delta(z^2)=\frac{\hbar}{2m\omega} \quad , \quad \Delta(p_z^2)=\frac{1}{2}\hbar\omega\,. \tag{2.118}$$

Finally, inserting these values in (2.87), we obtain

$$V_{\rm eff}(\langle\hat{z}\rangle) = \frac{1}{2}m\omega^2\langle\hat{z}\rangle^2 + \frac{1}{2}\hbar\frac{e|B(\langle\hat{z}\rangle)|}{m} + \frac{1}{2}\hbar\omega\,. \qquad (2.119)$$

If the magnetic field is constant, the fraction $eB/m = \omega_{\rm c}$ in (2.119) is the cyclotron frequency. It is well known that the Hamiltonian of a charged particle in a constant magnetic field can be transformed to one of a harmonic oscillator with the cyclotron frequency, so that our derivation provides the correct result of a constant $\hbar$-term in the effective potential given by the sum of zero-point energies $\frac{1}{2}\hbar\omega_{\rm c}$ and $\frac{1}{2}\hbar\omega$ of two uncoupled oscillators.

With the absolute value in (2.119), the effective potential differs from the classical potential by a kink around $\langle\hat{z}\rangle = 0$, accompanied by a modification of the potential in a neighborhood around $\langle\hat{z}\rangle = 0$ where the linear contribution to the potential is dominant. (At the kink, where the $\hbar$-correction implied by the magnetic field is zero, it is likely that higher-order corrections are relevant.) The motion of a charged particle in a magnetic monopole density therefore differs from the classical motion by anharmonic behavior around the minimum of an additional quadratic potential.

This effect cannot be mimicked by magnetic fields without monopole densities. Such a magnetic field could produce a z-dependent potential only if there are non-vanishing components B^x or B^y, either by cancelling the z-derivative of B^z in the divergence or by having the z-dependence come only from B^x or B^y. However, the motion would then be more complicated than circular motion in the $x-y$-plane at some fixed value of z.

Our results have important conceptual and potentially observable consequences. They demonstrate that physical effects can be derived in quantum mechanics even when the usual and widely used notions of state vectors and operators are unavailable. Non-associative quantum mechanics is thereby shown to be meaningful physically, which, despite its exotic appearance, can be applied in diverse ways, including some versions of string theory and analog magnetic monopoles.

Regarding the latter, we have specialized our general methods to a system in which closed-form solutions can be obtained, providing a model system with clear new effects. Such models always play important roles in situations like the present one: not much is known about testable quantum effects of analog condensed matter monopoles, even while experimental realizations seem to be within reach [59]. Our model amounts to an idealized example which brings out new effects clearly.

In practice, although it seems hard to have a constant monopole density, for sufficiently large amplitude of the oscillating motions of a charged particle, it is conceivable that a fine lattice of magnetic monopoles could be used to test the new effect found here. Specifically, one should arrange the lattice in cylinder shape, so as to impose a preferred direction identified here with the z-direction. On scales larger than the lattice spacing (but well within the entire lattice), the complicated dynamics of electric charges moving around monopoles can be approximated by electric charges moving through a uniform monopole density to which our methods apply. Analog monopoles do have Dirac strings [59], which may still have an effect after averaging to a continuous density, making the magnetic field non-linear. For more accurate derivations of the effective potential, applying our methods to non-linear magnetic fields, the same equations for moments are available, but they are coupled in more complicated ways which are likely to require numerical input and further research. Similarly, the equations can be extended to higher orders in $\hbar$ by including higher moments, but again we are not aware of closed analytic solutions.

Chapter 3
Alternative algebras and monopole star products

Deformation quantization [60,61] has been explored much in the associative setting. If one drops the condition that the star product be associative, some of the usual methods are no longer available. The classification of such star products therefore remains open. In this chapter, we present one general result in this direction, motivated by a recent resurgence of interest in magnetic-monopole systems [29, 30, 31, 32, 11, 62], where standard quantization methods show that associative algebras cannot constitute consistent quantizations of the relevant observables [10, 41].

In the original version of deformation quantization, associativity of the star product represents an important condition on the coefficients in the formal power series of the product. If one works with star products without the condition of associativity, at first sight it may seem easier to find acceptable versions because they may appear to be subject to fewer consistency requirements. However, if one is forced to use a non-associative star product for physical reasons, one is not fully liberated from imposing conditions on the associator

$$[a,b,c] = a*(b*c)-(a*b)*c\,. \tag{3.1}$$

For a specific set of basic observables, the associator, like the usual commutator

$$[a,b] = a*b-b*a\,, \tag{3.2}$$

is prescribed based on physical arguments.

Formulated for position and momentum components as basic observables, the commutator of an acceptable star product should be $[q_i, p_j] = i\hbar\{q_i, p_j\} = i\hbar\delta_{ij}$, mimicking the Poisson bracket. If these are coordinates of a charged particle (with electric charge e) moving in the magnetic field $B^l(q_i)$ of a magnetic monopole distribution, so that $\mathrm{div}B = \partial_l B^l \neq 0$, the classical brackets are modified: They are twisted Poisson brackets for which the Jacobi identity does not hold [26, 27, 28]. An algebra that quantizes the bracket endows phase-space functions with a new product $\star$ and the associated commutator (3.2) and associator (3.1). The Jacobiator of the commutator is proportional to the totally antisymmetric part of the associator and can be non-zero for non-associative $\star$-products. In the present context, one is led to the relations [10, 41]

$$
\begin{aligned}
[q_i, q_j] &= 0 && (3.3)\\
[q_i, p_j] &= i\hbar\delta_{ij} && (3.4)\\
[p_i, p_j] &= i\hbar e\epsilon_{ijk}B^k && (3.5)\\
\left[q_i, x^I, x^J\right] &= 0 && (3.6)\\
[p_i, p_j, p_k] &= -\hbar^2 e\epsilon_{ijk}\partial_l B^l && (3.7)
\end{aligned}
$$

to be realized by a star product. Here $(x^I)_{I=1}^6$ is a collective notation for the Cartesian coordinates $(q_i, p_i)_{i=1}^3$. In the absence of a magnetic charge density, one can introduce a canonical momentum π_i with zero brackets for its components. However, the definition, $\pi_i := p_i + A_i$, makes use of a vector potential A through $B = \mathrm{rot}A$, which does not exist if $\mathrm{div}B$ does not vanish. Instead of a zero associator in standard star products, the specific form of (3.7) imposes restrictions on acceptable star products for magnetic-monopole systems.

Most of the usual properties of quantum mechanics are no longer valid and must be modified when observables cannot be represented as associative operators on a Hilbert space. In some studies, a weaker condition given by an alternative algebra has been found advantageous [44]—if it can be realized. This is also the case explored in Chapter 2. An alternative algebra is one where the associator (3.1) is completely antisymmetric, or, equivalently, where the $*$-product obeys

$$a * (a * b) = (a * a) * b$$

$$(a * b) * b \;\; = \;\; a * (b * b) \tag{3.8}$$

for any a, b in the algebra. Many well-known non-associative algebras are of this form, such as the octonionic ones. Requiring an algebra to be alternative, provides a priori a tempting option for the case of a charged particle in the background of magnetic monopoles, in particular in view of the total anti-symmetry of the basic relation (3.7).

However, in this chapter we demonstrate the impossibility of such an algebra as a set of quantized observables of a charged particle in the presence of magnetic monopole densities, obtained by deformation quantization. While (3.7) implies a totally antisymmetric associator for linear functions of the basic observables, the associator of general algebra elements is not guaranteed to be totally antisymmetric. Different examples for algebras consistent with the relations (3.3)–(3.7) have been constructed using star products [29,30,31,32,11,62], one of which has explicitly been shown to be non-alternative [63]. In what follows, we will analyze the possibility of alternative monopole star products in general terms, using deformation theory, the basics of which we first recall in the next section.

3.1 Deformation quantization with non-associativity

The classical theory is described by the commutative algebra of smooth functions on $T^*\mathbb{R}^3$, equipped with the bivector field[1]

$$\Pi = \left(\frac{\partial}{\partial q_i} + \epsilon_{jik} B^k(q) \frac{\partial}{\partial p_j} \right) \wedge \frac{\partial}{\partial p_i} \, , \tag{3.9}$$

in the canonical linear coordinates $(x^I)_{I=1}^6 \equiv (q_1, q_2, q_3, p_1, p_2, p_3)$. For a vector field B with non-vanishing divergence, this is only a twisted Poisson bivector: Its Schouten bracket with itself does not vanish but is given by

$$\frac{1}{2}[\Pi, \Pi] = \Pi^\sharp(H) \tag{3.10}$$

[1] We set the electric charge to $e = 1$ from now on.

where the 3-form H takes the form

$$H = \pi^* \mathrm{d}B \,. \tag{3.11}$$

Here the magnetic field B is considered a 2-form on $\mathbb{R}^3$ by means of $B = \epsilon_{ijk} B^i \mathrm{d}q^j \wedge \mathrm{d}q^k$ and $\pi\colon T^*\mathbb{R}^3 \to \mathbb{R}^3$ is the canonical projection. Maxwell's equations link $\mathrm{d}B$ directly to the magnetic monopole density: $\mathrm{d}B = *\rho_{\text{magnetic}}$.

The bivector field Π then induces the following bracket on the functions $f, g \in C^\infty(T^*\mathbb{R}^3)$,

$$\{f, g\} = \frac{1}{2} \Pi^{IJ}(x) \frac{\partial f}{\partial x^I} \frac{\partial g}{\partial x^J} \,. \tag{3.12}$$

This bracket is an antisymmetric bi-derivation, but no longer a Lie bracket and thus not a Poisson bracket: the r.h.s. of (3.10) provides precisely the non-zero Jacobiator.

3.1.1 Star product

Deformation quantization turns the classical commutative algebra $(C^\infty(T^*\mathbb{R}^3), \cdot)$ into the quantum algebra $\mathcal{A} := (C^\infty(T^*\mathbb{R}^3)[[\lambda]], \star)$, where $\lambda = \frac{1}{2} i\hbar$ is considered as a formal deformation or expansion parameter:

$$f \star g = \sum_{j=0}^{\infty} \lambda^j B_j(f, g) \,. \tag{3.13}$$

Here $B_j\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ are bilinear maps on $\mathcal{A}$.[2] To zeroth order in λ, we have the classical product given by pointwise multiplication, $B_0(f, g) = f \cdot g \equiv fg$. Following [60], we will assume that B_j is a bi-differential operator of maximum degree j which is zero on constants for strictly positive j:

$$B_j(f, g) = \sum_{k,l=1}^{j} B_j^{k,l}(f, g) \qquad \text{for} \quad j \geq 1 \tag{3.14}$$

$$B_j^{k,l}(f, g) = \sum_{I_1,\ldots,I_k,J_1,\ldots,J_l=1}^{6} B^{k,l}_{j;I_1,\ldots,I_k,J_1,\ldots,J_l}(q) \frac{\partial^k f}{\partial x^{I_1} \cdots \partial x^{I_k}} \frac{\partial^l g}{\partial x^{J_1} \cdots \partial x^{J_l}} \tag{3.15}$$

[2] Using the same letter for these bilinear maps and the magnetic field should not cause confusion.

The property implies in particular that the star product defines a unital algebra, with the unit function as unit.

Let us for a moment assume that $\star$ would be associative. In this case, we would have that the commutator (3.2) evidently satisfies the Jacobi identity and also that $[f, g \star h] = [f, g] \star h + g \star [f, h]$. Both equations together, evaluated at lowest non-vanishing order in λ, imply that the antisymmetric part $B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f))$ of $B_1(f,g)$ is a Poisson bivector. On the other hand, for physical reasons, we want that the antisymmetric part of the first order deformation is determined by the classical bracket:

$$B_1^-(f,g) = \{f,g\}\,. \tag{3.16}$$

This then shows that the $\star$-product cannot be associative for the deformation quantization of the above classical system, cf., in particular, Eq. (3.10)—as anticipated already in the Introduction.

In fact, in the present chapter, we want to strengthen eq. (3.16) in a two-fold way: First, we require in addition that B_1 is antisymmetric itself already, so that

$$B_1(f,g) = \{f,g\}\,. \tag{3.17}$$

This, in fact, is not really a restriction: it can be shown that every star product either satisfies this condition or has an equivalent deformation for which (3.17) is fulfilled. We will come back to this below and assume it for now in any case. Second, we want that for linear coordinate functions on $T^*\mathbb{R}^3$ the bracket determines the commutator even to next-to-leading order, i.e. we require

$$\frac{x^I \star x^J - x^J \star x^I}{i\hbar} = \{x^I, x^J\} + O(\hbar^2)\,. \tag{3.18}$$

The first condition is equivalent to requiring $B_1^+(f,g) = 0$ for all functions f, g, the second one to demanding

$$B_2^-(x^I, x^J) = 0\,. \tag{3.19}$$

We remark in parenthesis that the equation (3.18) is implied if the x^I are implemented as distinguished observables in the sense of [61].

3.1.2 Monopole star products

Since we found above that the associator of the monopole star product cannot be zero, we also expand it into a formal power series in λ:

$$A(f,g,h) = f \star (g \star h) - (f \star g) \star h := \sum_{j=0}^{\infty} \lambda^j A_j(f,g,h)\,. \tag{3.20}$$

The maps B_i and A_j are not independent; in fact, A_j is determined by the B_i with $i \leq j$. It is easy to evaluate the low orders: We always have $A_0 = 0$, because the point-wise multiplication of phase-space functions is associative. At first order, we have

$$A_1(f,g,h) = fB_1(g,h) - B_1(f,g)h + B_1(f,gh) - B_1(fg,h) = 0 \tag{3.21}$$

simply since B_1 is bi-differential of order $(1,1)$.

At second order, one finds

$$\begin{aligned} A_2(f,g,h) &= fB_2(g,h) - B_2(f,g)h + B_2(f,gh) - B_2(fg,h) \\ &\quad + B_1(f,B_1(g,h)) - B_1(B_1(f,g),h)\,. \end{aligned} \tag{3.22}$$

For a non-associative star product, the coefficient A_2, as the first non-zero one in the expansion (3.20), plays a role similar to the coefficient B_1 in specifying conditions on the star product as a quantization of the classical bracket. The totally antisymmetric contribution

$$\begin{aligned} A_2^-(f,g,h) &:= \frac{1}{6}\big(A_2(f,g,h) + A_2(h,f,g) + A_2(g,h,f) \\ &\quad - A_2(f,h,g) - A_2(g,f,h) - A_2(h,g,f)\big) \end{aligned}$$

to A_2, in view of (3.19), only depends on B_1 *if* it is evaluated on linear functions of the basic variables x^I: We have

$$A_2^-(x^I,x^J,x^K) = \frac{1}{2} J(x^I,x^J,x^K) \tag{3.23}$$

where $J(f,g,h)$ is the Jacobiator of B_1, i.e. of the classical bracket $\{\cdot,\cdot\}$. In particular, $A_2^-(p_1,p_2,p_3) = 4\pi^*\mathrm{d}B$ for a star product that quantizes a twisted

Poisson bivector obeying (3.10). It is then consistent to assume that $A_2(p_1, p_2, p_3) = A_2^-(p_1, p_2, p_3)$ is totally antisymmetric, as written in the basic relation (3.7). The basic relations do not give us direct statements about A_2 evaluated on functions not linear in the global coordinates x^I. We will assume that $A_2(f, g, h)$ can be chosen totally antisymmetric even in this case — since our aim is to prove that monopole star products cannot be alternative, there would be nothing to show if this assumption were violated. However, this condition does not already imply that the star product is alternative, since non-linear functions generically lead to contributions to $A(f, g, h)$ of higher order in λ, which do not directly follow from simple combinations of the basic relations (3.7).

We summarize our conditions on A_2 in

Definition 1. *A* monopole star product *is a non-associative star product $\star$ on $C^\infty(T^*\mathbb{R}^3)[[\lambda]]$ such that (3.18) holds, its associator to second order in λ is totally antisymmetric and further obeys the following conditions:*

1. $A_2(p_1, p_2, p_3) \neq 0$,
2. $A_2(q_i, x^I, x^J) = 0$ *for all* $i = 1, 2, 3$ *and* $I, J = 1, \dots, 6$, *and*
3. $B_1(q_i, A_2(p_1, p_2, p_3)) = 0$ *for* $i = 1, 2, 3$.

where $(x^I)_{I=1}^6 = (q_1, q_2, q_2, p_1, p_2, p_3)$ *are the canonical linear coordinates on* $T^*\mathbb{R}^3$.

3.1.3 Hochschild cohomology

For an associative algebra $\mathcal{A}$, the space of multilinear maps from $\mathcal{A}$ to itself can be equipped with a coboundary operator d, used in Hochschild cohomology. For a multilinear map $\phi\colon \mathcal{A}^{\otimes n} \to \mathcal{A}$ of n arguments, $\mathrm{d}\phi$ is a multilinear function of $n+1$ arguments given by

$$\mathrm{d}\phi(a_0, a_1, \dots, a_n) = a_0 \cdot \phi(a_1, \dots, a_n) + \sum_{j=0}^{n-1} (-1)^j \phi(a_0, \dots, a_{j-1}, a_j \cdot a_{j+1}, a_{j+2}, \dots, a_n) + (-1)^n \phi(a_0, \dots, a_{n-1}) \cdot a_n \,. \tag{3.24}$$

Hochschild cohomology plays an important role in classifying equivalent star products with respect to a redefinition of higher orders in a λ-expansion: If

$$D(f) = \sum_{j=0}^{\infty} D_j(f)\lambda^j \tag{3.25}$$

with linear differential operators D_j starting with $D_0 = \mathrm{id}$, for any given star product $\star$ a new product $\star'$ can be defined by means of

$$D(f) \star' D(g) = D(f \star g)\,. \tag{3.26}$$

The condition on D_0 ensures that D is invertible as a map on formal power series. If functions in $C^\infty(M)$ are written as symbols of operators, for instance by a Weyl correspondence, a non-trivial map D changes the factor-ordering choice in the correspondence. To first order, $B_1' = B_1 - \mathrm{d}D_1$ while $\mathrm{d}B_1 = 0$; see (3.21). The first Hochschild cohomology therefore classifies inequivalent choices of B_1 which cannot be related by a different choice of factor ordering. For a given bracket $\{\cdot,\cdot\}$, all star products quantizing it respect the condition (3.16), but not necessarily (3.17).

If $\mathcal{A}$ is not associative, $\not{\mathrm{d}}$, defined just like d for an associative algebra, is not a coboundary operator: For a linear function $\phi\colon \mathcal{A} \to \mathcal{A}$, we have

$$\not{\mathrm{d}}\phi(a_0, a_1) = a_0 \star \phi(a_1) - \phi(a_0 \star a_1) + \phi(a_0) \star a_1 \tag{3.27}$$

and

$$\not{\mathrm{d}}^2\phi(a_0, a_1, a_2) = A(a_0, a_1, \phi(a_2)) + A(a_0, \phi_0(a_1), a_2) + A(\phi(a_0), a_1, a_2) - \phi_0(A(a_0, a_1, a_2)) \tag{3.28}$$

with the associator A. Therefore, Hochschild cohomology is not available for non-associative algebras. However, the coboundary operator d of the classical associative commutative algebra of smooth functions may still be used in constructing non-associative deformations, as we will do below. For instance, the product in (3.21) refers to $\cdot$, not to $\star$. Moreover, we can refer to the standard argument [64] for changing the star product within its equivalence class to show that the symmetric part in B_1 can always be set to zero and (3.17) be achieved. Thus, up to operator ordering, we can always assume that B_1 is given by the classical bracket, even if it is not Poisson, but for example twisted Poisson as here.

3.2 The main result

Our main result is

Theorem 1. *Let $\star$ be a monopole star product as defined above, cf. Definition 1. Then the associator $A(f,g,h) \equiv f \star (g \star h) - (f \star g) \star h$ cannot be totally antisymmetric in its arguments.*

We will prove this result by making use of three lemmas:

Lemma 1. *Let $\star$ be a star product obeying (3.18). If $\star$ is flexible at second order, that is $A_2(f,g,h) = -A_2(h,g,f)$, then B_2 is symmetric.*

Proof. We evaluate A_2 in (3.22) on functions with $f = h$, writing the result as

$$\begin{aligned} A_2(f,g,f) &= fB_2(g,f) - B_2(f,g)f + B_2(f,gf) - B_2(fg,f) \\ &= -2fB_2^-(f,g) + 2B_2^-(f,fg) \end{aligned} \quad (3.29)$$

using the antisymmetric part $B_2^-(f,g) := \frac{1}{2}(B_2(f,g) - B_2(g,f))$ of B_2. If $A_2(f,g,h) = -A_2(h,g,f)$ holds, $A_2(f,g,f) = 0$, and we obtain

$$B_2^-(f,fg) = fB_2^-(f,g)\,. \quad (3.30)$$

For an antisymmetric bi-differential form, this equation can hold only if the degree is $(1,1)$. However, if B_2^- has a contribution of degree $(1,1)$, (3.19) cannot hold. Therefore, $B_2^- = 0$ and B_2 is symmetric. □

In particular, the conclusion holds for a monopole star product (3.13). All explicit star products that have been constructed for monopole systems indeed have a symmetric B_2. For associative star products, Kontsevich's formula [65] has the same property. If symmetry of B_j holds at all even orders j, the star product gives rise to a formal deformation of the twisted Poisson bracket by powers of λ^2, or a Vey deformation as defined in [60].

Lemma 2. *If (3.13) is a star product with symmetric B_2, then the totally antisymmetric part of A_3 is equal to zero.*

Proof. Using the definition of the associator and the star product, we derive

$$\begin{aligned} A_3(f,g,h) &= \mathrm{d}B_3(f,g,h) + B_2(f,B_1(g,h)) \\ &\quad - B_2(B_1(f,g),h) + B_1(f,B_2(g,h)) - B_1(B_2(f,g),h)\,, \end{aligned} \tag{3.31}$$

where d is the coboundary operator of Hochschild cohomology, cf. eq. (3.24). In particular, $\mathrm{d}B_3(f,g,h) \equiv fB_3(g,h) + B_3(f,gh) - hB_3(f,g) - B_3(fg,h)$. The totally anti-symmetric part A_3^- of A_3, defined as in (3.23), is given by

$$\begin{aligned} 3A_3^-(f,g,h) &= B_2^-(f,2B_1^-(g,h)) + B_2^-(h,2B_1^-(f,g)) + B_2^-(g,2B_1^-(h,f)) \\ &\quad + B_1^-(f,2B_2^-(g,h)) + B_1^-(f,2B_2^-(g,h)) + B_1^-(f,2B_2^-(g,h)) \end{aligned} \tag{3.32}$$

where, as before, $B_j^-(f,g) = \frac{1}{2}(B_j(f,g) - B_j(g,f))$ is the antisymmetric part of B_j.[3] Since all terms on the right-hand side of (3.32) contain a B_2^-, $B_2^- = 0$ implies $A_3^- = 0$. □

We remark that for the last conclusion it is important that the antisymmetric part of A_3, unlike the full A_3, does not depend on B_3.

Lemma 3. *Let $\star$ be a star product such that*

$$\begin{aligned} O(f,g,h,k) &:= A_2(f,g,B_1(h,k)) - A_2(f,B_1(g,h),k) + A_2(B_1(f,g),h,k) \\ &\quad + B_1(A_2(g,h,k),f) - B_1(A_2(f,g,h),k) \end{aligned} \tag{3.33}$$

is not identically zero. Then the third-order contribution A_3 to the associator is non-zero.

Proof. Again, we use the Hochschild coboundary operator and consider

$$\mathrm{d}A_3(f,g,h,k) = fA_3(g,h,k) - A_3(fg,h,k) + A_3(f,gh,k) - A_3(f,g,hk) + kA_3(f,g,h) \tag{3.34}$$

Our goal is to show that $\mathrm{d}A_3$ is non-zero for algebras with non-zero O, which implies immediately also that $A_3 \neq 0$. The Pentagon identity

$$f \star A(g,h,k) + A(f,g,h) \star k = A(f \star g,h,k) - A(f,g \star h,k) + A(f,g,h \star k) \tag{3.35}$$

for non-associative algebras can be used for a compact proof of this statement.

[3] See App. A for a detailed derivation of (3.32).

Expanding it to third order in λ, we obtain

$$
\begin{aligned}
& fA_3(g,h,k) + B_1(f, A_2(g,h,k)) + kA_3(f,g,h) + B_1(A_2(f,g,h),k) \\
= \; & A_3(fg,h,k) - A_3(f,gh,k) + A_3(f,g,hk) \\
& + A_2(B_1(f,g),h,k) - A_2(f,B_1(g,h),k) + A_2(f,g,B_1(h,k)) \qquad (3.36)
\end{aligned}
$$

where we used $A_1 = 0$, cf. eq. (3.21). These terms can be organized to obtain

$$
\begin{aligned}
\mathrm{d}A_3(f,g,h,k) \; = \; & A_2(f,g,B_1(h,k)) - A_2(f,B_1(g,h),k) + A_2(B_1(f,g),h,k) \\
& + B_1(A_2(g,h,k),f) - B_1(A_2(f,g,h),k)\,. \qquad (3.37)
\end{aligned}
$$

Alternatively, one can prove directly that $\mathrm{d}A_3$ is of this form without invoking the Pentagon identity, as shown in App. B. The right-hand side of this equation is equal to $O(f,g,h,k)$. If it is not identically zero, A_3 is non-zero. □

We are now ready to prove our main result:

Proof (of Theorem 1)**:** By Lemmas 1 and 2, a monopole star product has an A_3 with zero totally antisymmetric part. If the star product is alternative, we must then have $A_3 = 0$. If the obstruction O provided by Lemma 3 is not identically zero, however, it is not possible that $A_3 = 0$. We now show that $O \neq 0$ for a monopole star product, discussing two cases separately depending on whether the associator (the monopole density) is constant or a function of the position.

For a constant associator, we may choose $f = p_1$, $g = p_2$, $h = p_3$ and $k = q_3 p_3$. Using the twisted Poisson bracket for B_1, all but the first term in $O(f,g,h,k)$ are zero, while $A_2(f,g,B_1(h,k))$ is proportional to the monopole density and therefore non-zero.

If the monopole density is not constant, we specialize $O(f,g,h,k)$ to

$$
O(f,g,h,g) = A_2(B_1(f,g),h,g) - B_1(A_2(f,g,h),g)\,. \qquad (3.38)
$$

Since the associator is not constant, it depends on at least one position coordinate, say q_1 without loss of generality. If we then choose $f = p_2$, $g = p_1$ and $h = p_3$ we have $B_1(A_2(f,g,h),g) \neq 0$ while $A_2(B_1(f,g),h,g) = 0$. □

The conclusion is independent of the choice of the star product within an equivalence class, with [30] or [62] as concrete examples, because alternativity is independent of the choice of the ordering (the "gauge") [66].

More generally, Lemma 3 gives us an obstruction to alternativity which only depends on B_1 and A_2, and therefore can be tested for general non-associative star products more easily than the full associator.

3.3 Monopole Weyl star product

Two different star products have been proposed recently for the magnetic-monopole system, one by using the Kontsevich formula [29, 30, 31, 32, 11], and one from Weyl products [62]. The former is known to be non-alternative [63]. Since it satisfies our assumptions, it provides an explicit example for our general result. We now discuss the star product of [62] in more detail.

Example (Weyl star product): The star product of [62] has the first coefficient $B_1(f,g) = \frac{1}{2}\{f,g\}$ with an atisymmetric bracket $\{f,g\} = \frac{1}{2}\Pi^{IJ}\partial_I f \partial_J g$ given by an arbitrary bivector Π^{IJ}. It can therefore be applied to monopole star products. The second coefficient is

$$B_2(f,g) = -\frac{1}{2}\Pi^{IJ}\Pi^{KL}(\partial_I\partial_K f)(\partial_J\partial_L g) - \frac{1}{3}\Pi^{IJ}\partial_J\Pi^{KL}\left((\partial_I\partial_K f)(\partial_L g) - (\partial_K f)(\partial_I\partial_L g)\right), \tag{3.39}$$

transferred to our notation. It obeys our assumptions. In particular, B_2 has no contribution of bi-differential degree $(1,1)$, and it is symmetric thanks to the antisymmetry of the twisted Poisson tensor Π^{IJ}. Therefore, our conditions on monopole star products are satisfied and the algebra cannot be alternative.[4] □

In [67, 66], an explicit expression for B_3 is given as well. It is therefore possible to compute A_3 in specific examples and show that it is not totally antisymmetric. In

[4]This star product has been conjectured to be alternative in [67], with a proof suggested in [66]. However, the arguments given are not complete: They are based on a computation of the associator $A_{\xi,\eta,\zeta} := A(\exp(i\xi\cdot z), \exp(i\eta\cdot z), \exp(i\zeta\cdot z))$ with phase-space variables z, together with a Fourier representation $f(z) = \int \mathrm{d}\mu f(\xi)\exp(i\xi\cdot z)$ of smooth functions. The direct calculation of $A_{\xi,\eta,\zeta}$ shows that it is zero whenever two of its arguments are equal. If $A_{\xi,\eta,\zeta}$ were tri-linear in (ξ,η,ζ), this fact would imply that it is antisymmetric, which would imply alternativity. However, $A_{\xi,\eta,\zeta}$ is not tri-linear in (ξ,η,ζ) but rather in $(\exp(i\xi\cdot z), \exp(i\eta\cdot z), \exp(i\zeta\cdot z))$, and antisymmetry is not implied. In fact, direct inspection of the result given in [66] shows that $A_{\xi,\eta,\zeta}$ is not antisymmetric in (ξ,η,ζ), even though it is zero whenever two of its arguments are equal.

particular, for monopole star products, it is not difficult to find functions $f(p_1, p_2, p_3)$ such that $A_3(f, f, f) \neq 0$.

Lemma 4. *Let $\star$ be a Weyl star product on $C^\infty(T^*\mathbb{R}^3)[[\lambda]]$ according to [62] which quantizes a twisted Poisson tensor (3.9), and let $f(p_1, p_2, p_3)$ be a function of the fiber coordinates of $T^*\mathbb{R}^3$ such that $\partial_{p_i}\partial_{p_j} f = 0$ whenever $i \neq j$. The third coefficient of the associator of $\star$ then obeys*

$$A_3(f, f, f) = \frac{4}{3} i \left(\partial_{q_1}\Pi^{p_2p_3} + \partial_{q_2}\Pi^{p_3p_1} + \partial_{q_3}\Pi^{p_1p_2}\right) \sum_{\sigma\in Z_3} \Pi^{p_{\sigma(1)}p_{\sigma(2)}}\partial_{p_{\sigma(3)}} f \partial^2_{p_{\sigma(1)}} f \partial^2_{p_{\sigma(2)}} f \,, \tag{3.40}$$

summing over elements of the alternating group $A_3 = Z_3$ of cyclic permutations.

Proof. We have explicitly computed $A_3(f, f, f)$ for arbitrary f using Cadabra software [68, 69]:

$$\begin{aligned} A_3(f,f,f) \;\; = \;\; & \frac{2i}{3}\Big(\Pi^{LM}\partial_L\Pi^{NO}\,\partial_N\Pi^{PQ}\,\partial_M f\,\partial_P f\,\partial_O\partial_Q f \\ & -\Pi^{LM}\partial_L\Pi^{NO}\,\partial_N\Pi^{PQ}\,\partial_O f\,\partial_P f\,\partial_M\partial_Q f \\ & -2\,\Pi^{LM}\Pi^{NO}\partial_L\Pi^{PQ}\,\partial_P f\,\partial_M\partial_N f\,\partial_O\partial_Q f \\ & +\Pi^{LM}\Pi^{NO}\partial_L\Pi^{PQ}\,\partial_M f\,\partial_{NP} f\,\partial_O\partial_Q f\Big)\,. \end{aligned} \tag{3.41}$$

For a monopole star product, the bivector Π is a function only of the position coordinates q_i via the magnetic field. Therefore, L and N must be position indices for non-zero contributions in the first two terms of (3.41). These terms are then identically zero because each contains a factor of $\partial_L\Pi^{NO}$, which is zero for a bivector of the form (3.9).

In the third term, only L is required to be a position index, while M, N, O, P, and Q are momentum indices if f depends only on momenta. The components Π^{LM} then equal δ^{LM} since they contain one position and one momentum index. The remaining terms in (3.41) yield

$$\begin{aligned} \frac{3}{2i}A_3(f,f,f) \;\; = \;\; & -2\,\Pi^{NO}(\partial_{q_1}\Pi^{UQ}\,\partial_U f\,\partial_{p_1}\partial_N f\,\partial_O\partial_Q f\; + \partial_{q_2}\Pi^{UQ}\,\partial_U f\,\partial_{p_2}\partial_N f\,\partial_O\partial_Q f \\ & \qquad +\partial_{q_3}\Pi^{UQ}\,\partial_U f\,\partial_{p_3}\partial_N f\,\partial_O\partial_Q f\,) \\ & +\Pi^{NO}(\partial_{q_1}\Pi^{UQ}\,\partial_{p_1} f\,\partial_N\partial_U f\,\partial_O\partial_Q f + \partial_{q_2}\Pi^{UQ}\,\partial_{p_2} f\,\partial_N\partial_U f\,\partial_O\partial_Q f \\ & \qquad +\partial_{q_3}\Pi^{UQ}\,\partial_{p_3} f\,\partial_N\partial_U f\,\partial_O\partial_Q f) \end{aligned}$$

We collect terms with the same factor of $\partial_{q_i}\Pi^{IJ}$ from derivatives of the bivector. Such a contribution with $\partial_{q_1}\Pi^{IJ}$ is of the form

$$\begin{aligned} & \Pi^{NO}\left(-2\partial_{q_1}\Pi^{UQ}\,\partial_U f\,\partial_{p_1}\partial_N f\,\partial_O\partial_Q f+\partial_{q_1}\Pi^{UQ}\,\partial_{p_1}f\,\partial_N\partial_U f\,\partial_O\partial_Q f\right)\\ = \; & \Pi^{NO}\big(\partial_{p_1}f\left(-\partial_{q_1}\Pi^{p_1Q}\partial_{p_1}\partial_N f+\partial_{q_1}\Pi^{p_2Q}\,\partial_N\partial_{p_2}f+\partial_{q_1}\Pi^{p_3Q}\,\partial_N\partial_{p_3}f\right)\\ & -2\partial_{p_2}f\partial_{q_1}\Pi^{p_2Q}\,\partial_{p_1}\partial_N f-2\partial_{p_3}f\partial_{q_1}\Pi^{p_3Q}\,\partial_{p_1}\partial_N f\big)\partial_O\partial_Q f\,, \end{aligned}$$

arranging by factors of first-order derivatives $\partial_{p_i}f$. By our assumptions on f, the index N is determined in all terms for non-zero contributions and we obtain

$$\begin{aligned} & \Big(\partial_{p_1}f\left(-\Pi^{p_1O}\partial_{q_1}\Pi^{p_1Q}\partial^2_{p_1}f+\Pi^{p_2O}\partial_{q_1}\Pi^{p_2Q}\,\partial^2_{p_2}f+\Pi^{p_3O}\partial_{q_1}\Pi^{p_3Q}\,\partial^2_{p_3}f\right)\\ & -2\partial_{p_2}f\Pi^{p_1O}\partial_{q_1}\Pi^{p_2Q}\,\partial^2_{p_1}f-2\partial_{p_3}f\Pi^{p_1O}\partial_{q_1}\Pi^{p_3Q}\,\partial^2_{p_1}f)\Big)\partial_O\partial_Q f\\ = \; & \sum_O\big(\partial_{p_1}f\left(-\Pi^{p_1O}\partial_{q_1}\Pi^{p_1O}\partial^2_{p_1}f+\Pi^{p_2O}\partial_{q_1}\Pi^{p_2O}\,\partial^2_{p_2}f+\Pi^{p_3O}\partial_{q_1}\Pi^{p_3O}\,\partial^2_{p_3}f\right)\\ & -2\partial_{p_2}f\Pi^{p_1O}\partial_{q_1}\Pi^{p_2O}\,\partial^2_{p_1}f-2\partial_{p_3}f\Pi^{p_1O}\partial_{q_1}\Pi^{p_3O}\,\partial^2_{p_1}f\big)\partial^2_O f\big) \end{aligned}$$

setting $O=Q$ in the last step, again by our assumptions on f. We now go through all remaining choices of the only free index O. All contributions to terms containing $\partial_{q_1}\Pi^{p_1O}$ cancel out. We arrive at

$$\begin{aligned} & 2\partial_{p_1}f\Pi^{p_2p_3}\partial_{q_1}\Pi^{p_2p_3}\partial^2_{p_2}f\partial^2_{p_3}f-2\partial_{p_2}f\Pi^{p_1p_3}\partial_{q_1}\Pi^{p_2p_3}\partial^2_{p_1}f\partial^2_{p_3}f-2\partial_{p_3}f\Pi^{p_1p_2}\partial_{q_1}\Pi^{p_3p_2}\partial^2_{p_1}f\partial^2_{p_2}f\\ = \; & 2\partial_{q_1}\Pi^{p_2p_3}\sum_{\sigma\in Z_3}\Pi^{p_{\sigma(1)}p_{\sigma(2)}}\partial_{p_{\sigma(3)}}f\partial^2_{p_{\sigma(1)}}f\partial^2_{p_{\sigma(2)}}f\,. \end{aligned}$$

Bringing back contributions with the remaining $\partial_{q_i}\Pi^{IJ}$, we have (3.40). □

For specific choices of f obeying the condition stated in the Lemma, we can compute $A_3(f,f,f)$ more explicitly. The first parenthesis in (3.40) is half the Jacobiator of the bivector, which is non-zero for a monopole star product. The sum over cyclic permutations depends on the specific f.

Example: Let Π be a bivector as stated in the conditions on a monopole star product.

1. Let $f = |p|^2 = p_1^2 + p_2^2 + p_3^2$. We have

$$\sum_{\sigma \in Z_3} \Pi^{p_{\sigma(1)} p_{\sigma(2)}} \partial_{p_{\sigma(3)}} f \partial^2_{p_{\sigma(1)}} f \partial^2_{p_{\sigma(2)}} f = 8 \sum_{\sigma \in Z_3} \Pi^{p_{\sigma(1)} p_{\sigma(2)}} p_{\sigma(3)} \,.$$

   With a bivector as implied by (3.5),

$$A_3(|p|^2, |p|^2, |p|^2) = \frac{32}{3} i(p \cdot B)\, \mathrm{div} B \,. \tag{3.42}$$

   For a monopole star product, $\mathrm{div} B \neq 0$, and $p \cdot B$ is generically non-zero for a charged particle with momentum p moving in the magnetic field B. Therefore, the a monopole star product obtained from a Weyl star product cannot be alternative to third order in λ.

2. Another example in which (3.40) can be used is $f = e^{i\alpha_1 p_1} + e^{i\alpha_2 p_2} + e^{i\alpha^3 p_3}$ for $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, a family of bounded functions. The sum over cyclic permutations then equals

$$\sum_{\sigma \in Z_3} \Pi^{p_{\sigma(1)} p_{\sigma(2)}} \partial_{p_{\sigma(3)}} f \partial^2_{p_{\sigma(1)}} f \partial^2_{p_{\sigma(2)}} f = i\alpha_1^2 \alpha_2^2 \alpha_3^2 \left(\frac{\Pi^{p_1 p_2}}{\alpha_3} + \frac{\Pi^{p_2 p_3}}{\alpha_1} + \frac{\Pi^{p_3 p_1}}{\alpha_2} \right) e^{i(p_1 + p_2 + p_3)} \,.$$

   For a bivector as in (3.5), we have

$$\begin{aligned} & A_3(e^{ip_1} + e^{ip_2} + e^{ip_3}, e^{ip_1} + e^{ip_2} + e^{ip_3}, e^{ip_1} + e^{ip_2} + e^{ip_3}) \\ = \; & -\frac{4}{3} \alpha_1^2 \alpha_2^2 \alpha_3^2 e^{i(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3)} \left(\frac{B^1}{\alpha_1} + \frac{B^2}{\alpha_2} + \frac{B^3}{\alpha_3} \right) \mathrm{div} B \,. \end{aligned} \tag{3.43}$$

   For any non-zero B, there is a triple $(\alpha_1, \alpha_2, \alpha_3)$ such that $B^1/\alpha_1 + B^2/\alpha_2 + B^3/\alpha_3$ is not identically zero. Therefore, every magnetic field with non-zero divergence gives rise to an f with $A_3(f, f, f) \neq 0$.

□

The Lemma implies non-alternativity of monopole star products obtained from a Weyl star product quantizing (3.9), but this already follows from Theorem 1. Having explicit examples with $A_3(f, f, f) \neq 0$ implies further results.

A property weaker than alternativity is *flexibility*, for which, by definition, only anti-symmetry with respect to the first and third entry is required:

$$A(f,g,h) = -A(h,g,f)\,. \tag{3.44}$$

Flexibility is important for quantum mechanics because it is a necessary and sufficient condition [70] for the commutator

$$[f,g] = f\star g - g\star f \tag{3.45}$$

to be a derivation of the Jordan product

$$f\circ g := \frac{1}{2}(f\star g + g\star f)\,. \tag{3.46}$$

Heisenberg equations of motion

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{[f,H]}{i\hbar} \tag{3.47}$$

with a Hamiltonian H then obey a product rule of the form

$$\frac{\mathrm{d}(f\circ g)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}\circ g + f\circ\frac{\mathrm{d}g}{\mathrm{d}t}\,. \tag{3.48}$$

To second order in λ, flexibility of the associator follows from (3.22) for any star product with symmetric B_2. However, as with alternativity, this fact does not guarantee that flexibility is realized at higher orders.

Another condition weaker than alternativity is *power-associativity:* A power-associative algebra is defined as an algebra $\mathcal{A}$ such that the subalgebra generated by any single element $a\in\mathcal{A}$ is associative. For any positive integer n, the n-th power a^n is then uniquely defined even though the algebra product may be non-associative. For Weyl star products of monopole systems, we have

Theorem 2. *A Weyl star product which quantizes (3.9) with* $\mathrm{div}B\neq 0$ *cannot be flexible or power associative.*

Proof. Since there is an f such that $A_3(f,f,f)\neq 0$, the associator cannot be antisymmetric in its first and last arguments. Moreover, we have $f\star(f\star f) - (f\star f)\star f = A_3(f,f,f)\lambda^3 + O(\lambda^4)$ and the subalgebra generated by f cannot be associative. □

3.4 Conclusions

We have shown that, under rather weak conditions, star products that quantize the phase space of a charged particle in the presence of a magnetic monopole density cannot be alternative. More generally, we have provided obstructions for a non-associative star product with symmetric B_2 being alternative. By the non-associative Gelfand–Naimark theorem [71], this result, together with the fact that the algebra is unital, implies that there is no norm that would turn the quantum algebra into a C^*-algebra, even if the algebra can be restricted to bounded functions; see (3.43). This version of our result strengthens the usual statement that non-associative systems cannot be quantized in the standard way by representing observables on a Hilbert space. One way to circumvent the use of Hilbert spaces in associative systems is to take an algebraic view point and define quantum states as positive linear functionals on the C^*-algebra of bounded observables; see for instance [45]. For non-associative systems of the kind studied here, this route must be generalized because the star-product algebra cannot be turned into a C^*-algebra. One can still use positive linear functionals, but only on a $*$-algebra.

Non-alternativity rules out the use of octonions as realizations of observable algebras of the relevant physical systems. Recently, in [72], octonions have been used to realize the relations (3.5) and (3.7) for linear functions of the momentum components. An extension to non-linear functions would encounter the same obstructions found here for star products, and a purely octonionic construction would no longer suffice.

Chapter 4
Effective space-time models

4.1 Introduction

Several independent examples of modified gauge transformations have been found in different models of canonical quantum gravity, using effective [73,74,75,76,77,78,79] and operator calculations [80, 81, 82, 83, 84]. In classical canonical formulations, space-time structure is encoded not in the usual form of general covariance of tensors, but by the equivalent version of gauge covariance under hypersurface deformations in space-time [85, 86]. The new structures found as a direct consequence of key ingredients of the quantization process using holonomies instead of connections therefore confirm a general expectation: Quantum geometry may lead to modified space-time structures [87, 88].

Although these modified gauge structures have been found within a variety of models of loop quantum gravity and by virtue of different computational methods, they all share some important properties. There is a phase-space function β modifying only the Poisson bracket of two smeared Hamiltonian constraints (or normal deformations of hypersurfaces). Denoting the constraints by $H[N]$ with the lapse function N that specifies the magnitude of the normal deformation at every point on a spatial hypersurface, we have

$$\{H[N], H[M]\} = -H_a[\beta q^{ab}(N\partial_b M - M\partial_b N)]\,. \tag{4.1}$$

On the right-hand side, H_a are the components of the diffeomorphism constraint (generating tangential deformations) and q^{ab} is the inverse metric on a spatial hypersurface. Brackets involving $H_a[M^a]$ retain the classical form

$$\{H_a[M_1^a], H_b[M_2^b]\} = -H_c[\mathcal{L}_{M_2} M_1^c] \quad (4.2)$$

$$\{H[N], H_a[M^a]\} = -H[\mathcal{L}_M N]\,. \quad (4.3)$$

There have been attempts to modify the brackets involving not only the Hamiltonian constraint as in (4.1) but also the diffeomorphism constraint [89, 90]. Other such examples are given by fractional space-time models, in which the modification functions can, however, be absorbed [91]. A discrete version of the brackets has been defined in [92], which differs from (4.2) and (4.3). In the present chapter, we focus on continuum effective theories in which space (but not necessarily space-time) has the classical structure. Accordingly, (4.2) will not be modified. We will derive a new form of brackets in which (4.3) is modified, but (4.2) is not. Nevertheless, our main focus will be on brackets with modifications as in (4.1).

The correction function $\beta \neq 1$ depends on the phase-space variables, and transforms as a spatial scalar. In the classical case, the hypersurface-deformation brackets are (on shell) related to the Lie algebra of space-time diffeomorphims, reflecting the coordinate invariance of general relativity. Brackets with $\beta \neq 1$ modify general covariance of the effective theory, but in such a way that no gauge transformations are violated. (Obeying the condition of anomaly freedom, gauge transformations are allowed to be modified by quantum corrections but not to be destroyed.)

With modified brackets, the effective metric q_{ab} appearing in (4.1) cannot be part of a space-time line element of classical form: Modified gauge transformations of q_{ab}, generated by $H[\epsilon]$ and $H_a[\epsilon^a]$, do not complement coordinate transformations of $\mathrm{d}x^a$ to form an invariant space-time line element

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t) \quad (4.4)$$

in canonical form. Nevertheless, there may be field redefinitions of different kinds which allow one to find a classical space-time picture for some function β of the phase-space variables. For instance, in some cases β can be absorbed in the lapse function by $N' := \sqrt{|\beta|}N$, with classical brackets in terms of N'. Or, a combination of the original spatial metric and extrinsic curvature could determine the spatial geometry of an effective space-time of classical type. The question has been investigated in certain spherically symmetric models in [93], with some encouraging results: Gauge transformations of the original canonical fields of the effective theory (including q_{ab}) are deformed, but by applying canonical transformations it is possible, in some cases, to recover the classical hypersurface-deformation brackets and hence to restore general covariance. Specifically, a canonical transformation with this effect has been found in [93] when β depends only on metric components. Absorbing β in q^{ab} then provides a simple canonical transformation. If β depends on the momentum (extrinsic curvature) as well, it is more difficult to see whether it can be removed from the brackets.

In this chapter, we analyze the same question from a different perspective which is insensitive to the availability of canonical transformations. Our discussion makes use of the general setting of Lie algebroids, of which a suitable fiber-bundle formulation of (4.1) provides an example [94]. More generally, the language of Lie algebroids is a well-defined mathematical structure that allows one to formalize theories with structure functions. Our results are independent of details of any specific form of quantum gravity in the sense that we will not use equations or methods characteristic of a specific approach. Instead, we use the general form (4.1) of the modified bracket of two normal deformations as a guiding principle and study possible Lie-algebroid realizations. Modifications of the classical brackets can be understood as a generic form of quantum corrections, introduced by some effective quantum gravity theory.

We will be able to classify different inequivalent space-time structures corresponding to modified brackets of the type (4.1) that cannot be related by morphisms. While there appears to be an arbitrary modification function β in (4.1) with virtually unrestricted quantum corrections, only $\mathrm{sgn}\beta$ remains as the single choice left after equivalence classes of brackets up to morphisms are considered. This result helps to clarify the implications of modified brackets (4.1) for space-time structures. In particular, they can be related to the classical brackets by Lie algebroid morphisms as long as β has a definite sign and is non-zero. The existence of effective Riemannian space-time structures is confirmed in this case, which so far has only been assumed, for instance in [95, 96, 97, 98]. Such modifications therefore do not imply radical changes of the space-time structure, even though they may still lead to a modified dynamics on and of the effective space-time. If β does not have a fixed sign, a new version of quantum space-time is obtained which exhibits signature change as a new physical effect.

In some cases, concrete morphisms can be formulated with simple interpretations of their implications on canonical variables and the dynamics. For instance, with spatially constant $\beta \neq 1$, as in cosmological models with first-order perturbative inhomogeneity, a suitable morphism is obtained by changing the usual conventions in setting up the canonical formulation based on space-time foliations into spatial slices. Somewhat akin to absorbing β in the lapse function, one can make use of a generalized canonical formulation which is a hybrid version of, on one side, Dirac's [85] and the ADM [99] formulation with variables adapted to directions normal and tangent to a spatial hypersurface, and on the other Rosenfeld's [101] earlier derivation of canonical gravity without reference to a foliation or preferred directions. We will use a foliation, but do not require the timelike vector n^μ to be normalized or orthogonal to the spatial tangent plane. The normalization function $n^\mu n_\mu$ can be related to β. Therefore, non-standard normalizations present a more-general way of relating modified brackets to classical space-time structures than absorbing β in the lapse function would do. The angles between n^μ and the spatial tangent plane give rise to new modifications of the brackets not yet encountered elsewhere. At the same time, we make use of a concise derivation of the hypersurface-deformation brackets and use the example to introduce Lie algebroids in this context. Morphisms of Lie algebroids will lead to further transformations that can be used to relate modified brackets of different types, still with the classical signature as the only parameter that characterizes inequivalent space-time structures of brackets of the form (4.1) via $\mathrm{sgn}\beta$. This result allows us to draw rather general conclusions about implications of the modified dynamics according to (4.1).

4.2 Canonical gravity and Lie algebroids

In order to set up the canonical formalism, we assume, as usual, space-time $\mathcal{M}$ to be globally hyperbolic and introduce a foliation by constant-level surfaces of a parameter $t \in \mathbb{R}$, such that the hypersurfaces are all spacelike. Each spatial slice is homeomorphic to a 3-manifold σ, on which we may choose local coordinates x^a, $a \in \{1, 2, 3\}$. We realize σ as a spatial hypersurface $\Sigma_t := X_t(\sigma)$ at constant t by an embedding $X\colon \mathbb{R} \times \sigma \hookrightarrow \mathcal{M}$ with $(t, x) \mapsto X(t, x) =: X_t(x)$.

We choose a foliation $X_t = X(t,\cdot)$ and define a time-evolution vector field τ^μ by

$$\tau(X) := \partial_t X^\mu(t,x)\partial_\mu \,. \tag{4.5}$$

This vector field is, in general, not normal to Σ_t. Following ADM [99], it is convenient to introduce vector fields tangential to Σ_t, given by

$$X_a(X) := \partial_a X^\mu(t,x)\partial_\mu \,, \tag{4.6}$$

and to define a time-like vector field normal to the time slice Σ_t by

$$g_{\mu\nu}n^\mu X_a^\nu = 0 \quad , \quad g_{\mu\nu}n^\mu n^\nu = -1 \,. \tag{4.7}$$

If we further require that n^μ point toward the future, that is, $n^\mu\partial_\mu t > 0$, it is uniquely defined. By introducing the lapse function $N(X)$ and the shift vector field $M^a(X)$ the time-evolution vector field τ^μ is decomposed into its components normal and tangential to Σ_t:

$$\tau^\mu(X) = N(X)n^\mu(X) + N^a(X)X_a^\mu(X) \,. \tag{4.8}$$

Since the choice of the embedding X is arbitrary, the components of lapse and shift are free functions as long as they give rise to a timelike τ^μ.

So far, we have used only well-known and basic ingredients of the canonical formulation. (See [102] for further details.) The decomposition (4.8) and the normalization condition of n^μ in (4.7) play a key role in our considerations of modified space-time structures. In order to exhibit the full freedom of the formalism, we will not follow the common convention of normalizing n^μ by $g_{\mu\nu}n^\mu n^\nu = -1$. We may fix any other negative constant, or even a phase-space function, for Lorentzian space-time signature, or a positive constant (or phase-space function) for Euclidean signature. We may therefore require that $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$, where $\epsilon = -1$ in the Lorentzian case and $\epsilon = +1$ in the Euclidean case. If the signature is constant, $\beta > 0$ is a positive phase-space function. But in anticipation of applying these methods to some of the models found in the context of loop quantum gravity, we allow for β to change its sign, so that $\mathrm{sgn}\beta =: \epsilon_\beta$ may not be constant. The overall signature is then locally given by the product $\epsilon\epsilon_\beta$.

In order to compare dynamical results obtained with different normalizations, we should demand that $\tau^\mu(X)$ remain the same and be independent of β:

$$\tau^\mu(X) = \frac{1}{\sqrt{|\beta|}} N(X) n^\mu(X) + M^a(X) X^\mu_a(X) =: N_\beta(X) n^\mu(X) + M^a(X) X^\mu_a(X) \tag{4.9}$$

where now $n^\mu/\sqrt{|\beta|}$ is normalized to $\pm 1 = \epsilon\epsilon_\beta$. This condition ensures that equations of motion for evolution along τ^μ exist independently of the canonical decomposition in terms of hypersurfaces. At this stage, we see the simple result that the lapse function has to absorb any non-standard normalization factor β, but later on we will be able to draw more benefit from these simple-looking considerations. The only requirement for (4.9) to be used is that n^μ and $M^\mu = M^a X^\mu_a$ form a basis of the tangent space to $\mathcal{M}$ at each point. We may therefore drop normalization conditions as well as orthogonality of n^μ and M^μ.

4.2.1 A concise derivation of the hypersurface-deformation brackets

We derive the brackets of hypersurface deformations with non-standard normalization by repurposing a derivation of the usual result given in [94]. The main aim of this chapter was to analyze the Lie-algebroid structure of the brackets, which we will describe in the following subsection. Some part of the mathematical analysis of [94] amounts to a brief derivation of the brackets which we formulate here in abstract index notation and, at the same time, use it to derive the brackets with non-standard normalization. As a further generalization, we will also assume a non-orthogonality relation between n^μ and X^μ_a. More traditional derivations using ADM-style evolution equations or geometrodynamics are given in App. C for the case of a non-unit normal n^μ, with equivalent results.

The explicit derivation of hypersurface deformations depends on choices of coordinates or embedding functions, but the brackets must be covariant under changes of these auxiliary structures. As in [94], one can exploit the coordinate freedom by working with embeddings such that the space-time metric, from which the spatial metric q_{ab} in the structure functions is induced, is Gaussian with respect to the hypersurfaces:

$$\mathrm{d}s^2 = \epsilon \mathrm{d}t^2 + q_{ab} \mathrm{d}x^a \mathrm{d}x^b \,. \tag{4.10}$$

In this way, one fixes a representative in each equivalence class of hypersurface embeddings. The remaining coordinate freedom is given by diffeomorphisms generated by so-called g-Gaussian vector fields v^μ which preserve the Gaussian form of the metric and therefore satisfy

$$n^\mu \mathcal{L}_v g_{\mu\nu} = 0 \tag{4.11}$$

with some vector field n^μ normal to $t =$ constant, but not necessarily normalized. This condition ensures that an infinitesimal diffeomorphism along v^μ, changing $g_{\mu\nu}$ to $g'_{\mu\nu} := g_{\mu\nu} + \mathcal{L}_v g_{\mu\nu}$, respects the relations $n^\mu n^\nu g'_{\mu\nu} = n^\mu n^\nu g_{\mu\nu} = \epsilon$ and $n^\mu w^\nu g'_{\mu\nu} = 0$ if $n^\mu w^\nu g_{\mu\nu} = 0$ of the Gaussian system. Because they generate diffeomorphisms preserving the Gaussian form of the metric, g-Gaussian vector fields form a subalgebra of the Lie algebra of all vector fields with bracket the usual Lie bracket. As found in [94], one can derive the hypersurface-deformation brackets by rewriting the Lie bracket using properties of vector fields v^μ satisfying (4.11).

Some restriction on the form of vector fields is necessary because the hypersurface deformations as gauge transformations are known to be equal to infinitesimal space-time diffeomorphisms only on-shell [86], that is, when some of the generators H and H_a and the equations of motion they generate are set to zero as phase-space functions. The restriction is implemented here by using g-Gaussian vector fields, which turn out to have Lie brackets directly related to the hypersurface-deformation brackets. Such a restriction cannot be chosen arbitrarily but must fulfill three conditions: (i) The vector fields considered must provide a unique extension from spatial (lapse) functions N and spatial (shift) vector fields M^a to a space-time vector field v^μ which equals $Nn^\mu + M^a X^\mu_a$ on the spatial slice. If this condition is fulfilled, it is possible to compute space-time Lie brackets. (ii) The vector fields considered must form a subalgebra of the Lie algebra of all space-time vector fields. And (iii), the Lie bracket of space-time extensions of two pairs (N_1, M^a_1) and (N_2, M^a_2) should not depend on the extensions but only on spatial derivatives of N_i and M^a_i in addition to the functions and vector fields themselves. With conditions (ii) and (iii) fulfilled, it is then possible to interpret the Lie bracket of extensions of two pairs (N_1, M^a_1) and (N_2, M^a_2) as the unique extension of a third pair (N_3, M^a_3), and to define a new bracket $[(N_1, M^a_1), (N_2, M^a_2)] := (N_3, M^a_3)$. All three conditions can be shown to be true for g-Gaussian vector fields [94], recovered as a special case

($\beta = 1$ and $\alpha^a = 0$) of the following calculations. To the best of our knowledge, it is not known whether g-Gaussian vector fields are the only choice fulfilling all three conditions, but having one such choice is sufficient for a derivation of the brackets.

We first derive properties (i), (ii) and (iii), found in [94], using abstract index notation. We write (4.11) as

$$0 = n^\mu \mathcal{L}_v g_{\mu\nu} = n^\mu v^\rho \partial_\rho g_{\mu\nu} + n^\mu g_{\nu\rho} \partial_\mu v^\rho + n^\mu g_{\mu\rho} \partial_\nu v^\rho \,. \tag{4.12}$$

The first two terms can be expressed by the Lie bracket of n^μ and v^ν if we write $n^\mu v^\rho \partial_\rho g_{\mu\nu} = v^\rho \partial_\rho n_\nu - g_{\mu\nu} v^\rho \partial_\rho n^\mu$. The last term in $n^\mu \mathcal{L}_v g_{\mu\nu}$ can be replaced by a total derivative using $n^\mu g_{\mu\rho} \partial_\nu v^\rho = \partial_\nu (n^\mu v^\rho g_{\mu\rho}) - v^\rho \partial_\nu n_\rho$. In addition to the Lie bracket and the total derivative, there remain two extra terms related to the 2-form dn:

$$0 = n^\mu \mathcal{L}_v g_{\mu\nu} = [n, v]^\mu g_{\mu\nu} + \partial_\nu (n^\mu v^\rho g_{\mu\rho}) + v^\rho (\mathrm{d}n)_{\rho\nu} \,. \tag{4.13}$$

If n^μ is hypersurface orthogonal, by the Frobenius theorem we have $\mathrm{d}n = n \wedge w$ with some 1-form w which can, without loss of generality, be assumed to be orthogonal to n^μ. For $n^\mu n_\mu = \epsilon$ and the metric in Gaussian form, $w = 0$ because $n = \epsilon \mathrm{d}t$ is closed. In this case, n^μ is hypersurface orthogonal in a neighborhood of the initial slice by construction of the Gaussian system. If $n^\mu n_\mu = \epsilon\beta$, the analog of the Gaussian system has n^μ hypersurface orthogonal only if β is spatially constant. In order to allow for spatially non-constant β, we use a Gaussian system constructed from a unit normal, which would be $\tilde{n}^\mu := n^\mu / \sqrt{|\beta|}$ if $n^\mu n_\mu = \epsilon\beta$. This rescaled normal is extended to a closed 1-form in its Gaussian system. We can compute $\mathrm{d}n = n \wedge w$ from the equation $\mathrm{d}\tilde{n} = 0$, resulting in $w = -\frac{1}{2}\beta^{-1}(\mathrm{d}\beta - \epsilon|\beta|^{-1/2}\dot{\beta} n)$. The second term, with $\dot{\beta} = \partial\beta/\partial t$, is chosen such that $n^\mu w_\mu = 0$.

We include one further generalization by relaxing the usual orthogonality relation to $g_{\mu\nu} n^\mu X^\nu_a = \alpha_a$ with fixed phase-space functions α_a allowed to be non-zero. The components of α_a are related to the direction cosines (hyperbolicus) of n^μ with respect to the spatial basis X^ν_a. The new condition can equivalently be written as an orthogonality relation $g_{\mu\nu} n'^\mu X^\nu_a = 0$ with a redefined $n'^\mu := n^\mu - \alpha^a X^\mu_a$. With the non-standard normalization of n^μ, the redefined vector satisfies $n'^\mu n'_\mu = \epsilon\beta - \alpha_a \alpha^a =: \epsilon\gamma$. In the Euclidean case, $\epsilon = 1$, we must have $\gamma > 0$ and therefore $\alpha^a \alpha_a < \beta$. The same condition ensures that n^μ and X^μ_a form a basis because the angle between the direction n^μ and the spatial tangent plane spanned by X^μ_a is less than ninety degrees. In the Lorentzian case, $\alpha^a\alpha_a$ is unrestricted.

We construct a Gaussian system as before. The hypersurface orthogonal vector is now given by $\tilde{n}'^\mu := n'^\mu/\sqrt{|\beta-\epsilon\alpha^a\alpha_a|} = n'^\mu/\sqrt{|\gamma|}$. Computing $\mathrm{d}n' = n'\wedge w$ from the equation $\mathrm{d}\tilde{n}'=0$ now results in $w = -\frac{1}{2}\gamma^{-1}(\mathrm{d}\gamma - \epsilon|\gamma|^{-1/2}\dot{\gamma}n')$. With the redefined normal, (4.13) takes the form

$$0 = n'^\mu \mathcal{L}_v g_{\mu\nu} = [n',v]^\mu g_{\mu\nu} + \partial_\nu(n'^\mu v^\rho g_{\mu\rho}) + v^\rho(\mathrm{d}n')_{\rho\nu}\,. \tag{4.14}$$

We use n'^μ because we need a normal vector for the condition of a g-Gaussian vector field. However, we may decompose a g-Gaussian vector field v^μ according to our original basis (n^μ, X^ν_a) or according to the redefined basis using n'^μ instead of n^μ:

$$v^\mu = Nn^\mu + M^\mu = Nn'^\mu + M'^\mu \tag{4.15}$$

with $M^\mu = M^a X^\mu_a$ and $M'^\mu = M^\mu + N\alpha^a X^\mu_a$, or $M'^a = M^a + N\alpha^a$. The latter choice simplifies some derivations and is therefore employed below, but for full generality we will transform the final result to a decomposition with respect to (n^μ, X^ν_a).

We will need the following ingredients in order to rewrite (4.14) with a decomposed vector field v^μ. In contrast to the standard case, $n'^\mu n'_\mu = \epsilon\gamma$ is not a constant because α^a and β may depend on space and time via phase-space variables. Therefore, for spatial M^μ (or M'^μ), $[n',M]^\mu$ has a normal component given by

$$\begin{aligned}
\frac{n'^\mu n'_\nu[n',M]^\nu}{n'^\kappa n'_\kappa} &= \frac{1}{\epsilon\gamma}n'^\mu\left(n'_\nu n'^\rho\nabla_\rho M^\nu - n'_\nu M^\rho\nabla_\rho n'^\nu\right)\\
&= -\frac{1}{\epsilon\gamma}n'^\mu\left(M^\nu n'^\rho\nabla_\rho n'_\nu + n'_\nu M^\rho\nabla_\rho n'^\nu\right)\\
&= -\frac{1}{\epsilon\gamma}n'^\mu\left(2M^\nu n'^\rho\nabla_\rho n'_\nu + n'^\nu M^\rho(\mathrm{d}n')_{\rho\nu}\right)\\
&= -\frac{1}{\epsilon\gamma}n'^\mu\left(2M^\nu\sqrt{|\gamma|}\tilde{n}'^\rho\nabla_\rho(\sqrt{|\gamma|}\tilde{n}'_\nu) + 2n'^\nu M^\rho n'_{[\rho}w_{\nu]}\right)\\
&= \frac{1}{\epsilon\gamma}n'^\mu n'^\nu n'_\nu M^\rho w_\rho = n'^\mu M^\rho w_\rho
\end{aligned} \tag{4.16}$$

using $M^\nu\tilde{n}'_\nu = 0$ and the geodesic property $\tilde{n}'^\rho\nabla_\rho\tilde{n}'_\nu = 0$ of the normal in a Gaussian system. With

$$v^\rho(\mathrm{d}n')_{\rho\nu} = 2(Nn'^\rho + M'^\rho)n'_{[\rho}w_{\nu]} = \epsilon\gamma N w_\nu - M'^\rho w_\rho n'_\nu\,, \tag{4.17}$$

we can write (4.14) as

$$0 = n'^\mu g_{\mu\nu} n'^\rho \partial_\rho N + [n', M']^\mu g_{\mu\nu} + \partial_\nu(Nn'^\mu n'^\rho g_{\mu\rho}) + \epsilon\gamma N w_\nu - M'^\rho w_\rho n'_\nu \tag{4.18}$$

or

$$0 = [n', M']^\mu + n'^\mu n'^\rho \partial_\rho N + \epsilon\partial^\mu(\gamma N) + \epsilon\gamma N w^\mu - M'^\rho w_\rho n'^\mu\,. \tag{4.19}$$

The equation can now be split into components parallel and orthogonal to n'^μ: The normal component implies

$$n'^\rho \partial_\rho N = -\frac{1}{2}\frac{N}{\gamma} n'^\nu \partial_\nu \gamma \tag{4.20}$$

(the contribution from $\mathrm{d}n'$ cancelling out with the normal contribution from $[n', M']$) while the spatial component gives

$$[n', M']^a = q^a_\mu [n', M']^\mu = -\epsilon q^{ab}\partial_b(\gamma N) - \epsilon\gamma N w^a = -\epsilon(\mathrm{grad}_q(\gamma N))^a - \epsilon\gamma N w^a\,. \tag{4.21}$$

The full space-time commutator is

$$[n', M']^\mu = q^\mu_a [n', M']^a + M'^\rho w_\rho n'^\mu\,, \tag{4.22}$$

combining (4.21) with (4.16).

With these relations, the hypersurface-deformation brackets follow immediately from the Lie brackets of g-Gaussian vector fields. First, in the Gaussian system, (4.20) and (4.22) provide first-order partial differential equations for N and M^μ or M'^μ to be extended into a neighborhood of the initial slice. (Importantly, all M^μ-dependent terms cancel out in (4.20) even with non-standard normalization. The equation for N is therefore decoupled from the equation for M^μ.) We can then compute space-time Lie brackets of two g-Gaussian vector fields

$$\begin{aligned}[v_1, v_2] &= [N_1 n' + M'_1, N_2 n' + M'_2] \\ &= [N_1 n', N_2 n'] + [N_1 n', M'_2] + [M'_1, N_2 n'] + [M'_1, M'_2]\end{aligned} \tag{4.23}$$

$$= (N_1\mathcal{L}_{n'}N_2 - N_2\mathcal{L}_{n'}N_1)n' + (\mathcal{L}_{M_1'}N_2 - \mathcal{L}_{M_2'}N_1)n' + N_1[n', M_2'] - N_2[n', M_1'] + [M_1', M_2']$$

The first term, $N_1\mathcal{L}_{n'}N_2 - N_2\mathcal{L}_{n'}N_1$, is zero even with the new contributions in (4.20) for non-constant γ. Similarly, the w^a-term in (4.21) does not contribute to $N_1[n', M_2'] - N_2[n', M_1']$. However, the normal contribution $M'^\rho w_\rho n'^\mu = -\frac{1}{2}\gamma^{-1}n'^\mu M'^\rho\partial_\rho\gamma$ in (4.22) does not cancel out and provides a new normal term in

$$\begin{aligned} [v_1, v_2] &= \left(\mathcal{L}_{M_1'}N_2 - \mathcal{L}_{M_2'}N_1 - \frac{1}{2\gamma}\left(N_1\mathcal{L}_{M_2'}\gamma - N_2\mathcal{L}_{M_1'}\gamma\right)\right)n' \\ &\quad -\epsilon N_1 \mathrm{grad}_q(N_2\gamma) + \epsilon N_2 \mathrm{grad}_q(N\gamma) + [M_1', M_2'] \\ &= \frac{1}{\sqrt{|\gamma|}}\left(\mathcal{L}_{M_1'}(\sqrt{|\gamma|}N_2) - \mathcal{L}_{M_2'}(\sqrt{|\gamma|}N_1)\right)n' \\ &\quad -\epsilon\gamma(N_1\mathrm{grad}_q N_2 - N_2\mathrm{grad}_q N_1) + [M_1', M_2']\,. \end{aligned} \tag{4.24}$$

The last line can now be transformed from $n'^\mu = n^\mu - \alpha^\mu$ and $M'^\mu = M^\mu + N\alpha^\mu$ to n^μ and M^μ. Inserting the expressions for the primed vectors leads to several extra terms, most of which cancel out. However, two new contributions remain:

$$\begin{aligned} [v_1, v_2] &= \left(\frac{1}{\sqrt{|\gamma|}}\left(\mathcal{L}_{M_1}(\sqrt{|\gamma|}N_2) - \mathcal{L}_{M_2}(\sqrt{|\gamma|}N_1)\right) + N_1\mathcal{L}_\alpha N_2 - N_2\mathcal{L}_\alpha N_1\right)n \\ &\quad -\epsilon\gamma(N_1\mathrm{grad}_q N_2 - N_2\mathrm{grad}_q N_1) - \sqrt{|\gamma|}\left(N_1\mathcal{L}_{M_2}\frac{\alpha}{\sqrt{|\gamma|}} - N_2\mathcal{L}_{M_1}\frac{\alpha}{\sqrt{|\gamma|}}\right) + [M_1, M_2]\,. \end{aligned} \tag{4.25}$$

By extracting terms parallel to n or the tangent plane, we write this Lie bracket as bracket relationships between pairs (N, M^a):

$$[(0, M_1^a), (0, M_2^b)] = (0, [M_1, M_2]^c) \tag{4.26}$$

$$[(N, 0), (0, M^a)] = \left(-|\gamma^{-1/2}|\mathcal{L}_M(|\gamma|^{1/2}N), -|\gamma|^{1/2}N\mathcal{L}_M(|\gamma|^{-1/2}\alpha^a)\right) \tag{4.27}$$

$$[(N_1, 0), (N_2, 0)] = \left(N_1\mathcal{L}_\alpha N_2 - N_2\mathcal{L}_\alpha N_1, -\epsilon\gamma(N_1\mathrm{grad}_q^a N_2 - N_2\mathrm{grad}_q^a N_1)\right) \tag{4.28}$$

The following special cases are of interest:

- If $\alpha^a \neq 0$, there is a new class of modified brackets which have not been derived explicitly in models of loop quantum gravity. New features are a transversal deformation (along a non-normal n^μ) contributing to the bracket of two transversal deformations, and a spatial diffeomorphism contributing to the bracket of a transversal deformation and a spatial diffeomorphism. If this example is realized by quantum-gravity effects, it would require the existence of a preferred spatial direction α^a.

- If $\alpha^a = 0$, the bracket of two normal deformations is a spatial diffeomorphism, as in the classical version, but with a multiplicative correction function $\gamma = \beta$. One can obtain the modified brackets (4.28) by replacing N_i with $\sqrt{|\gamma|}N_i$ and n' with $n'/\sqrt{|\gamma|}$ in the standard brackets, in accordance with the rescaling transformations of the normal keeping Nn' invariant for (4.9) to be preserved. However, our calculation shows more than this because it ensures that the three conditions required for a meaningful relation between hypersurface-deformation brackets and space-time Lie brackets are still satisfied for g-Gaussian vector fields with a non-standard normal.

- If $\alpha^a = 0$ and $\gamma = \beta$ is spatially constant, all derivatives of γ cancel out and the bracket of a normal deformation and a spatial diffeomorphism is unmodified. A time-dependent γ therefore leads only to a multiplicative modification of the standard brackets, and it appears only in the bracket of two normal deformations. This is the example (4.1) found in models of loop quantum cosmology with first-order perturbative inhomogeneity.

4.2.2 Lie algebroids

The hypersurface-deformation generators do not form a Lie algebra, owing to the appearance of structure functions. Structure functions can be elegantly described by the notion of Lie algebroids, which may be motivated as follows: Assume that we have a finite number of constraints C_I, $I = 1, \ldots, n$, on a Poisson manifold B, which satisfy an algebra $\{C_I, C_J\} = c^K_{IJ}(x)C_K$ with structure functions $c^K_{IJ}(x)$ depending on $x \in B$. We can formally rewrite brackets with structure functions in terms of structure constants by defining an extended system of infinitely many constraints

$$C_I \quad , \qquad C_{IJ} := \{C_I, C_J\} = c^K_{IJ}C_K \quad ,$$
$$C_{HIJ} := \{C_H, C_{IJ}\} = (\{C_H, c^K_{IJ}\} + c^L_{IJ}c^K_{HL})C_K \quad , \quad \ldots \tag{4.29}$$

The brackets $\{C_I, C_J\} = C_{IJ}$, $\{C_H, C_{IJ}\} = C_{HIJ}$, $\ldots$ of the extended system then have structure constants.

These constraints span a certain linear subspace of the space $\Gamma(A)$ of sections $\alpha = \alpha(x)^I C_I$ of a vector bundle A over the base manifold B (phase space) with fiber $\pi^{-1}(x) \approx \mathbb{R}^n \ni \{\alpha(x)^1, \ldots, \alpha(x)^n\}$. The sections of this bundle form a Lie algebra by taking Poisson brackets $[\alpha_1, \alpha_2] = \{\alpha_1(x)^I C_I, \alpha_2(x)^J C_J\}$. Moreover, we can define a linear map $\rho\colon \Gamma(A) \to \Gamma(TB), \alpha = \alpha^I(x) C_I \mapsto \{\alpha(x)^I C_I, \cdot\}$ which appears in a Leibniz rule

$$\begin{aligned}
[\alpha, g\beta] &= \{\alpha(x)^I C_I, g(x)\beta(x)^J C_J\} = g(x)\{\alpha(x)^I C_I, \beta(x)^J C_J\} + \{\alpha(x)^I C_I, g(x)\}\beta(x)^J C_J \\
&= g(x)\{\alpha(x)^I C_I, \beta(x)^J C_J\} + \left(\rho(\alpha(x)^I C_I) g(x)\right) \beta(x)^J C_J \\
&= g[\alpha, \beta] + (\rho(\alpha) g)\beta \,, \qquad (4.30)
\end{aligned}$$

and ρ is a homomorphism of Lie algebras:

$$\begin{aligned}
\rho([\alpha, \beta]) &= \{\{\alpha(x)^I C_I, \beta(x)^J C_J\}, \cdot\} \\
&= \{\alpha(x)^I C_I, \{\beta(x)^J C_J, \cdot\}\} - \{\beta(x)^J C_J, \{\alpha(x)^I C_I, \cdot\}\} \\
&= \rho(\alpha)\rho(\beta) - \rho(\beta)\rho(\alpha) = [\rho(\alpha), \rho(\beta)] \qquad (4.31)
\end{aligned}$$

using the Jacobi identity. The Lie bracket on sections together with a homomorphism ρ characterize A as a Lie algebroid [103].

Definition 2. *A* Lie algebroid *is a vector bundle A over a smooth base manifold B together with a Lie bracket $[\cdot, \cdot]_A$ on the set $\Gamma(A)$ of sections of A and a bundle map $\rho\colon \Gamma(A) \to \Gamma(TB)$, called the* anchor, *provided that*

- $\rho\colon (\Gamma(A), [\cdot, \cdot]_A) \to (\Gamma(TB), [\cdot, \cdot])$ *is a homomorphism of Lie algebras, that is*

  $$\rho\left([\xi, \eta]_A\right) = [\rho(\xi), \rho(\eta)] \,,$$

  where $[\cdot, \cdot]$ is the commutator of vector fields in $\Gamma(TB)$.

- *For any ξ, $\eta \in \Gamma(A)$ and for any $f \in C^\infty(B)$ the Leibniz identity*

  $$[\xi, f\eta]_A = f[\xi, \eta]_A + (\rho(\xi) f)\, \eta$$

  holds.

If the base manifold B is a point, the Lie algebroid is a Lie algebra. Another example for a Lie algebroid is the tangent bundle TB of a manifold B with $\rho\colon \Gamma(TB) \to \Gamma(TB)$ the identity map and the Lie bracket of vector fields as the bracket on sections. The hypersurface-deformation brackets have been shown in [94] to be captured by a certain Lie algebroid more specific than the construction based on (4.29). This notion can therefore provide useful methods in an analysis of different versions of hypersurface deformations. In order to identify classes of equivalent Lie algebroids, one may generalize the notion of a Lie algebra morphism to the Lie algebroid case.

Definition 3. *A* base-preserving morphism *between Lie algebroids* $(A, [\cdot,\cdot]_A, \rho)$ *and* $(A', [\cdot,\cdot]_{A'}, \rho')$ *is a bundle map* $\Phi\colon A \to A'$ *over* $\mathrm{id}_B\colon B \to B' = B$ *such that* Φ *induces a Lie algebra homomorphism* $\Phi\colon\ (\Gamma(A), [\cdot,\cdot]_A) \to (\Gamma(A'), [\cdot,\cdot]_{A'})$ *and satisfies* $\rho' \circ \Phi = \rho$*.*

If the induced base map ϕ_0 is a diffeomorphism, the definition can still be used. In such cases, which will be of interest to us, the bundle map induces a map on sections via $\Phi(\xi)(y) = \xi(\phi_0^{-1}(y))$ for $\xi \in \Gamma(A)$ and $y \in B'$. For completeness, we mention that a Lie algebroid morphism which does not preserve the base manifold can be defined as follows; see for instance [104]:

Definition 4. *A* Lie algebroid morphism *from* $A \to B$ *to* $A' \to B'$ *is a bundle map* $\phi\colon A'^* \to A^*$ *with induced base map* $\phi_0\colon B' \to B$*, such that:*

1. *The induced map* $\Phi\colon \Gamma(A) \to \Gamma(A')$*, defined by* $\Phi(\xi)(y) = \phi^*\xi(\phi_0(y))$ *for* $y \in B'$*, preserves the Lie bracket on sections:* $[\Phi(\xi), \Phi(\eta)] = \Phi([\xi, \eta])$ *for all* $\xi, \eta \in \Gamma(A)$*.*

2. *We have* $\rho = \phi_{0*} \circ \rho' \circ \Phi$*.*

We will not use general morphisms in this thesis, but note that an example of a morphism as in the preceding definition could be used to relate the space-time structures underlying general relativity and higher-curvature actions, respectively. The latter are higher-derivative theories and have additional canonical degrees of freedom compared with general relativity; therefore, the base manifolds are not diffeomorphic. Nevertheless, the hypersurface-deformation brackets are the same in both settings [105] and could be used to construct a Lie algebroid morphism.

From now on, we focus on the specific example of the algebroid underlying general relativity. We quote useful definitions and one central result from [94]:

- A connected Lorentzian manifold (or space-time) $(\mathcal{M}, g)$ is called Σ-*adapted* if it admits an embedding of Σ as a spacelike hypersurface. Such an embedding is called a Σ-*space* in $\mathcal{M}$, and a pair consisting of a space-time and a Σ-space in it is called a Σ-*space-time*. On every Σ-space we have an induced, or spatial, metric $q = i^*g$ using the embedding $i\colon (\Sigma, q) \hookrightarrow (\mathcal{M}, g)$.

- Coordinate independence leads to the concept of a Σ-*universe*, an equivalence class $[i]$ of Σ-space-times where $i\colon\ (\Sigma, q) \hookrightarrow (\mathcal{M}, g)$ and $i'\colon\ (\Sigma, q) \hookrightarrow (\mathcal{M}', g')$ are equivalent if there is an isometry $\Psi\colon\ (\mathcal{M}, g) \to (\mathcal{M}', g')$ which preserves the coorientation of Σ and satisfies $\Psi \circ i = i'$. The set of all Σ-universes is denoted by $\mathcal{U}\Sigma$. In order to confirm that this definition is consistent, we pull back g' along i' and obtain the same result as before applying the isometry: $(i')^* g' = (\Psi \circ i)^* g' = i^* (\Psi^* g') = i^* g = q$.

- So far, the relations between a Cauchy hypersurface Σ and space-time $\mathcal{M}$ have been formalized. The next step is to look at the evolutions of one time slice into another time slice. A time slice is defined to be an embedding i_t for a fixed time parameter $t =$ constant within a 1-parameter family. Different time slices are related by Σ-*evolutions*, equivalence classes $[i_1, i_0]$ of pairs (i_1, i_0) of Σ-spaces in the same space-time, where a pair (i_1, i_0) in $\mathcal{M}$ is equivalent to (i_1', i_0') in $\mathcal{M}'$ if there is a single isometry $\Psi\colon \mathcal{M} \to \mathcal{M}'$ which is consistent with the coorientations of time slices and which satisfies both $\Psi \circ i_1 = i_1'$ and $\Psi \circ i_0 = i_0'$. The set of all Σ-evolutions is denoted by $\mathcal{E}\Sigma$.

The set of Σ-evolutions, $\mathcal{E}\Sigma$, forms a Lie groupoid [94] with elements in $\mathcal{U}\Sigma$, source map $s([i_1, i_0]) = [i_0]$ and target map $t([i_1, i_0]) = [i_1]$, multiplication given by $[i_2, i_1][i_1, i_0] = [i_2, i_0]$ and inversion by $[i_1, i_0]^{-1} = [i_0, i_1]$. The definition therefore gives rise to an evolution picture in terms of groupoid multiplication. The Lie algebroid $A\mathcal{E}\Sigma$ belonging to the Lie groupoid $\mathcal{E}\Sigma$ provides the link between this formulation and the infinitesimal one used for instance in [86]. According to [94],

Proposition 1. *The Lie algebroid $A\mathcal{E}\Sigma$ of $\mathcal{E}\Sigma$ is isomorphic as a vector bundle to the trivial bundle $\mathcal{U}\Sigma \times (\Gamma(T\Sigma) \oplus C^\infty(\Sigma))$ over the base manifold $\mathcal{U}\Sigma$.*

Proposition 1 tells us that infinitesimal evolutions of an equivalence class in $\mathcal{U}\Sigma$ are described by (shift) vector fields in $\Gamma(T\Sigma)$ and (lapse) C^∞-functions on Σ. The base manifold of the Lie algebroid is the space of equivalence classes of spatial embeddings. Structure functions of the classical hypersurface-deformation brackets depend on the spatial metric, which in turn depends only on the equivalence class of embeddings $\Sigma \hookrightarrow \mathcal{M}$ for a given space-time metric. Similarly, extrinsic curvature on Σ depends on the embedding in $(\mathcal{M}, g)$, but it is not invariant under space-time isometries fixing (Σ, q). Since the modification function β may depend on all phase-space variables, we should refine the equivalence classes to those transformations that keep both q_{ab} and K_{cd} fixed on Σ. However, if the hypersurface-deformation brackets are modified, it is not clear whether a space-time metric structure exists which can induce a spatial metric. It is then more appropriate to formulate the Lie algebroid directly over a base manifold of spatial metrics and extrinsic-curvature tensors on Σ (or the classical phase space). In fact, [94] indicates the way to such a formulation using Gaussian representatives.

For an explicit construction of Lie algebroid brackets and the anchor, [94] chooses as a representative for a Σ-universe a slicing which is locally of Gaussian form, as in the derivation of Sec. 4.2.1. A representative of a class in $\mathcal{U}\Sigma$ can then be fixed by specifying the induced metric q instead of the embedding. The tangent space of the resulting base manifold of spatial metrics is, at a point q, given by $T_q\mathcal{U}\Sigma = S^2T^*\Sigma$, the space of symmetric tensors identified with Lie derivatives of the space-time metric by g-Gaussian vector fields $v^\mu = Nn^\mu + M^\mu$: Since such vector fields preserve the Gaussian form, $\mathcal{L}_v g$ is equivalent to a change $\delta_v q := \mathcal{L}_M q + N\dot{q}$ of just the spatial metric, where $\dot{q} = \mathcal{L}_n q = 2K$ is proportional to the extrinsic-curvature tensor. The latter changes by $\delta_v K = \mathcal{L}_M K + N\dot{K}(q, K)$ where $\dot{K} = \mathcal{L}_n K$ is a function of q_{ab} and K_{cd} via the field equations. (The field equations had been bypassed on [94] by working with equivalence classes of entire neighborhood of embeddings of Σ in M.) Notice that the anchor ρ depends on the field equations of the theory, while the brackets do not.

The anchor map of the Lie algebroid with the gravitational phase space as base manifold is given by $(N, M) \mapsto (\delta_{Nn+M}q, \delta_{Nn+M}K)$. This base manifold and anchor have been extended to the space of induced metrics and extrinsic-curvature tensors, which is necessary if one works with modified brackets where β depends on q_{ab} and K_{ab}. The same calculations as in Sec. 4.2.1 imply that the Lie algebra of g-Gaussian vector fields v leads to a Lie-algebroid bracket

$$\begin{aligned} & [(N_1, M_1), (N_2, M_2)] \\ = & \left(\frac{1}{\sqrt{|\beta|}} \left(\mathcal{L}_{M_1}(\sqrt{|\beta|} N_2) - \mathcal{L}_{M_2}(\sqrt{|\beta|} N_1) \right), \epsilon\beta(N_1 \mathrm{grad}_q N_2 - N_2 \mathrm{grad}_q N_1) + [M_1, M_2] \right) \end{aligned} \quad (4.32)$$

(if $\alpha^a = 0$) once the decomposition $v^\mu = N n^\mu + M^\mu$ is introduced.

4.3 Physics from hypersurface-deformation algebroids

Using the Lie-algebroid structure of hypersurface deformations, we can now look at possible modified versions and their relations to the classical brackets. In some cases, they turn out to be related by algebroid morphisms. We begin with a review of existing examples for deformed brackets.

4.3.1 Modified brackets

The classical hypersurface-deformation brackets have been derived from the usual space-time structure, using for instance infinitesimal space-time diffeomorphisms in (4.24). They are independent of specific solutions to Einstein's or modified field equations as long as the theory is based on Riemannian geometry. For instance, the same brackets are obtained for higher-curvature actions [105]. In several effective models of loop quantum gravity, however, modified versions of the brackets have been found, and it has not been clear what space-time structure or what effective actions they may correspond to. In this subsection, we discuss several relevant conceptual details of such models, leaving aside technical features.

Modified brackets have been derived canonically, by including possible quantum corrections in the classical constraints and checking under which conditions they still give rise to a closed set of Poisson brackets. Generically, quantum corrections suggested by loop quantum gravity, based on real connection variables, could be implemented consistently only when the brackets were modified as in (4.1). For complex connections, the derivative structure of the Hamiltonian constraint is different, in that there are no second-order derivatives of the triad unlike in real formulations which have the generic pattern responsible for signature-change type deformations [106]. At least in spherically symmetric models, it is then possible to have undeformed brackets even in the presence of holonomy modifications [107]. Such models are less restrictive than the full theory, and therefore it is not clear whether the full brackets can be undeformed.

Two main classes of models in which deformed brackets have been derived are: (i) cosmological perturbations [73,108] where, to linear order, β is a function only of time (via the background spatial metric and extrinsic curvature) and (ii) spherically symmetric models [74,76,75,79] where β may also depend on the radial coordinate. With so-called holonomy modifications of the classical dynamics, β depends on K_{ab} as some kind of higher-curvature correction, but only in spatial terms so that the modification is not necessarily space-time covariant. Detailed calculations have shown that it is possible to have such spatial-curvature modifications and still maintain closed brackets of correspondingly modified hypersurface-deformation generators, but only when β and the way it appears in the equations of motion are restricted. This is the condition of anomaly-freedom. Generically, whenever β depends on K_{ab}, it changes sign at large curvature if quantum effects lead to bounded curvature or densities (so-called bounce models). The same observations have been found in cosmological and spherically symmetric models, with agreement also in the specific functional form of β [109]. There are, however, obstructions in models with local physical degrees of freedom [110,111], in which no anomaly-free holonomy-modified versions have been found yet. (There are also obstructions in some operator versions of spherically symmetric models that implement spatial discreteness [112].)

In these two classes of models, two kinds of methods have been used to provide complementary insights: Effective calculations proceed by computing Poisson brackets of classical hypersurface-deformation generators modified by potential quantum corrections, following a systematic canonical version of effective-action techniques [7,9,113,57]. Operator methods compute commutators of quantized generators. Also here, there is full agreement between results from these two different methods: The operator calculations of [84] in spherically symmetric models provide the same restrictions on modifications and the function β as found by effective methods [74]. It is not known how to implement cosmological perturbations at the operator level, but there is a set of $2+1$-dimensional models which provide complementary insights. In [80], a modification function for holonomies has been found that shows the same features related to the change of sign of β; see also [114].

Other operator calculations in $2+1$-dimensional models [81, 82, 83] are only partially off-shell so far, and therefore are not able to show the full brackets. In particular, since they amount to factoring out spatial diffeomorphisms everywhere except at a finite number of isolated points, they cannot exhibit holonomy modifications which are spatially non-local. The interesting conclusion of β changing sign therefore cannot yet be tested in this setting. Nevertheless, these models have confirmed the presence of modified brackets for metric-dependent modifications. For instance, Eq. (9.27) in [81] gives a definition of the right-hand side of the operator equivalent of (4.1), which contains an inverse-metric operator with a factor of $(\det q)^{-1/4}$ modified by so-called inverse-triad corrections [115, 116]. We note that reading off modified brackets from commutators is not straightforward because in addition to the commutator, an effective bracket contains information about semiclassical states. Defining such states and computing expectation values in them is notoriously difficult in background-independent quantum-gravity theories. Nevertheless, it is clear that the naive classical limit of the equation just cited shows a modification of the classical bracket. (In the naive classical limit, one replaces operator factors in the quantized constraints and structure functions with their expectation values in simple states, thus ignoring fluctuations and higher moments.)

Some quantization schemes of constrained gravitational systems represent hypersurface deformations in an indirect way, after reformulating the classical constraints so as to make them easier to quantize. In the present context, two examples are relevant in which one can use reformulations in order to eliminate structure functions from the constraint brackets. In [83], $2+1$-dimensional gravity is quantized by writing the bracket of two Hamiltonian constraints in the schematic form $\{H[N], H[M]\} = \{D[N'^a], D[M'^b]\}$ where N'^a and M'^b are shift vector fields related to N and M, respectively. There are no structure functions on the right-hand side, and it is possible to represent the bracket relation without modifications. However, this result does not imply that the hypersurface-deformation brackets are undeformed; in fact, one can check that $\{H[N], H[M]\}$ written as a single diffeomorphism constraint has quantum-corrected structure functions. (The vector fields N'^a and M'^b mentioned above depend on the spatial metric and give rise to new terms in structure functions when $\{D[N'^a], D[M'^b]\}$ is expressed as a term linear in D.)

Similarly, spherically symmetric systems can be reformulated in a way that partially Abelianizes the constraint algebra [117,118]. The Hamiltonian constraint is here replaced by a linear combination $C[L] := H[L'] + D[L'']$ with L' and L'' suitably related to L such that $\{C[L_1], C[L_2]\} = 0$. Structure functions are thus eliminated from the constrained system (C, D), and the brackets can be represented without quantum corrections in their coefficients. However, if one tries to find hypersurface-deformation generators of quantum constraints with the correct classical limit, it turns out that this is possible only if the hypersurface-deformation brackets are deformed [110, 111].

Since all these examples are obtained after quantizing generators of normal deformations with respect to n^μ such that $g_{\mu\nu}n^\mu n^\nu = \epsilon$ and the vector field n^μ is not subject to quantum corrections, the deformed algebra refers to a unit normal vector. With such modified brackets but standard normalization, the space-time considerations of [86] no longer apply, and therefore a non-classical space-time structure seems to be realized.

The new brackets, in general, cannot be viewed as describing deformations of hypersurfaces in a Riemannian space-time with metric $g_{\mu\nu}$. They do, however, determine a well-defined canonical theory, in which one can, in principle, solve the constraints and compute gauge-invariant observables, which is all that is needed for physical predictions. Importantly, the brackets are still closed, which is the challenging part of their constructions. If the brackets were not closed, the models would be anomalous and inconsistent because gauge transformations would be violated and results would depend on choices of coordinates.

Modified brackets can be formulated as a Lie algebroid over the space of pairs of symmetric tensor fields (q_{ab}, K_{cd}) with positive-definite q_{ab}. The inverse of q_{ab}, as well as K_{ab} through possible modifications in β, appear in the structure functions of the constraint brackets, but they play the role only of phase-space functions which need not have a geometrical interpretation as spatial metric and extrinsic curvature associated with a slice Σ in space-time $(\mathcal{M}, g)$. Instead of defining these spatial tensors in terms of the embedding functions $X(x)$ and a space-time metric $g_{\mu\nu}$, the only option is to view q_{ab} and K_{ab} as independent phase-space degrees of freedom on which the constraints and the structure functions depend. The modification function must be covariant under transformations with brackets (4.2), (4.3) and (4.1). In particular, since these brackets contain infinitesimal spatial diffeomorphisms as a subalgebra, β must be a spatial scalar. In the modified case, the theory is not necessarily standard space-time covariant, but if the brackets close, β and the resulting theory are covariant under transformations generated by Poisson brackets with the modified constraints. In the absence of a space-time picture, the physical meaning of q_{ab} and K_{cd} is supplied by how they appear in canonical observables. The latter have a known interpretation in the classical limit of $\beta \to 1$ (low curvature), which is extended to non-classical regimes in an anomaly-free deformed theory. Alternatively, one may employ field redefinitions such that a relation of Lie algebroid elements to space-time metrics becomes possible. We discuss two possible types in the following subsections.

4.3.2 Base transformations

In (4.1), β always appears in combination with the inverse of q^{ab}, whose components can be used as coordinates on the base manifold along with the components of K_{ab}. We can define a transformation of the base manifold by mapping (q_{ab}, K_{cd}) to $(|\beta|^{-1}q_{ab}, K_{cd})$ and extend it to a fiber map $(q_{ab}, K_{cd}, N, M^e) \mapsto (|\beta|^{-1}q_{ab}, K_{cd}, N, M^e)$. Here, the fiber coordinates N and M^e as well as K_{cd} are unchanged, while q_{ab} absorbs $|\beta|$. As long as $\beta \neq 0$, the base map is a diffeomorphism and a well-defined Lie algebroid morphism is obtained, eliminating $|\beta|$ from the brackets. The only parameter that cannot be absorbed is $\mathrm{sgn}\beta$ because q_{ab} is required to be positive definite and, in particular, invertible.

We may then consider $|\beta|^{-1}q_{ab}$ as the spatial metric on a spatial slice in a space-time with line element

$$\mathrm{d}s^2 = \epsilon\epsilon_\beta N^2 \mathrm{d}t^2 + |\beta|^{-1} q_{ab}(\mathrm{d}x^a + M^a \mathrm{d}t)(\mathrm{d}x^b + M^b \mathrm{d}t) \tag{4.33}$$

which generically cannot be obtained by a coordinate transformation from (4.4). (If this were possible, one could eliminate the scale factor $a = |\beta|^{-1/2}$ of a Friedmann–Robertson–Walker metric by a coordinate transformation.) The extrinsic curvature of a $t =$ constant slice in (4.33) is not equal to K_{ab}. However, we can use the field equations of the modified theory in order to relate K_{ab} to $\dot{q}_{ab} = \mathcal{L}_n q_{ab}$. Using the standard equation for extrinsic curvature computed from (4.33), a relationship between K_{ab} and extrinsic curvature is obtained, which may not be the identity.

The new variables $(|\beta|^{-1}q_{ab}, K_{cd})$ are no longer canonical coordinates on the base manifold. Non-canonical base coordinates do not make a difference for a Lie algebroid, which in general does not even have a Poisson structure on its base. However, we need a Poisson structure on the base manifold in order to derive the dynamics generated by the constraints, and for this it is useful to have a canonical set of variables. Modifying the map $(q_{ab}, K_{cd}) \mapsto (|\beta|^{-1} q_{ab}, K_{cd})$ such that it becomes canonical is possible in some models [93], but may be complicated in general.

While base transformations can map modified brackets to the classical version, as long as β does not change sign, it is not easy to derive general, theory-independent effects because the interpretation of K_{ab} depends on the dynamics, and there may be no simple canonical sets of variables. It turns out that general aspects of physical implications of the absorption are easier to discern if one uses morphisms that originate from fiber maps. We will be able to do so by absorbing $|\beta|$ in the normalization condition, at least partially, allowing us to discuss possible physical implications in general terms.

4.3.3 Change of normalization as algebroid morphism

One usually expects that the classical theory can be recovered when β approaches one in some regime, such as low curvature. However, as already mentioned, the classical theory can be described with a more general β if one uses non-standard normalizations $g_{\mu\nu} n^\mu n^\nu = \epsilon\beta$ of normal vectors to hypersurfaces. Even the classical brackets can therefore be modified without changing the implied physics. Although it is customary to assume the normal vector n^μ to be normalized to $\epsilon = \pm 1$, depending on the signature, this choice is a mere convention and one may as well introduce a different normalization. Thus, the requirement of having the correct classical limit does not restrict β much, except that β should not be identically zero.

Since we know from Sec. 4.2.1 that, for spatially constant β, the hypersurface-deformation brackets belong to a Lie algebroid, irrespective of how the normal is normalized, there are no further conditions on β from the Jacobi identity. As in our explicit derivation of the brackets, we may obtain a deformation by using a non-standard normalization of the normal vector field in classical general relativity.

We introduce a bundle map Φ with fiber map $(N, M^a) \mapsto (\sqrt{|\beta|} N, M^a)$ and the identity as base map. It obeys

$$\begin{aligned}[\Phi((N_1,0)),\Phi((N_2,0))] &= [(\sqrt{|\beta|}N_1,0),(\sqrt{|\beta|}N_2,0)] = (0,|\beta|M^a_{12}) \\ &= \Phi((0,|\beta|M^a_{12})) = \Phi([(N_1,0),(N_2,0)]_\beta) \qquad (4.34)\end{aligned}$$

where $M^a_{12} = q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)$ and we have more specifically denoted the modified bracket by $[\cdot,\cdot]_\beta$ while $[\cdot,\cdot]$ is the classical bracket. The anchor is preserved because $Nn^\mu = \sqrt{|\beta|}\tilde{n}^\mu$ with a non-standard normal $\tilde{n}^\mu$ such that $g_{\mu\nu}\tilde{n}^\mu\tilde{n}^\nu = 1/|\beta|$. If β is spatially constant, as in models of first-order cosmological perturbations, modified brackets of sections in the Lie algebroid A are mapped to the classical brackets on A', with the required anchor because $Nn^\mu \mapsto (N/\sqrt{|\beta|})n^\mu = N\tilde{n}^\mu$. With spatially dependent β, the existence of a morphism is less clear because $\{H[N], H_a[M^a]\}$ is not modified in effective models of loop quantum gravity, while it would change in (4.24). Fiber transformations are therefore less general than base transformations in mapping modified brackets to the classical ones.

The fiber map just introduced is valid only if β has constant sign. When β is of indefinite sign, no β-absorbing morphism can exist: For opposite signs of β, the corresponding groupoids are inequivalent because their compositions are concatenations of slices in Lorentzian space-time and 4-dimensional space of Euclidean signature, respectively.

For spatially constant $\beta > 0$, we have a Lie algebroid morphism between modified and unmodified brackets irrespective of where the deformation function β originates. In the modified case, we then have the classical space-time structure after applying the morphism that absorbs β in the normalization. But the classical structure is obtained after a field redefinition: The space-time metric obtained from q_{ab} is not of the standard canonical form but reads

$$\mathrm{d}s^2 = \epsilon\beta N^2\mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t) \qquad (4.35)$$

depending, in general, on q_{ab} and K_{ab}. This line element is conformally related to (4.33).

4.3.4 Equations of motion

When we interpret hypersurface deformations as actual moves in space-time, we refer to time-evolution vector fields, and therefore to coordinate structures. Space-time coordinates are not quantized in canonical quantum gravity, and therefore the vector field should not receive quantum corrections if there is a classical manifold picture for the effective theory. Deformed brackets with $\beta > 0$ can sometimes be mapped to the classical space-time structure in terms of hypersurface deformations, but this does not necessarily lead to the same physics in terms of time evolution.

For a classical deformation with standard normalization, we use

$$\tau^\mu = \delta X^\mu = \delta N \tilde{n}^\mu + \delta N^a X^\mu_a \tag{4.36}$$

in order to identify time deformations, while in the classical case with non-standard normalization, we have

$$\delta X^\mu = \delta N_\beta n^\mu + \delta N^a X^\mu_a \tag{4.37}$$

with $n^\mu = \sqrt{|\beta|}\tilde{n}^\mu$. These vector fields must be the same: Changing the normalization of the normal vector should not affect the relative position of two hypersurfaces X^μ and $X^\mu + \delta X^\mu$ embedded in space-time. Thus, the two time-evolution vector fields have to be the same, and it follows that the infinitesimal lapse function δN_β of the modified theory must be given by

$$\delta N_\beta = \frac{1}{\sqrt{|\beta|}} \delta N \,. \tag{4.38}$$

4.3.4.1 Classical theory with non-standard normalization

Classically, we have standard hypersurface-deformation brackets with the normalization condition $g_{\mu\nu} n^\mu n^\nu = \epsilon$ and we know, by [86], that second-order equations of motion for the metric are the classical field equations of general relativity. However, we may change the normalization condition to $g_{\mu\nu} n^\mu n^\nu = \epsilon|\beta|$. The theory is still classical, but the generator of normal deformations is rescaled. Accordingly, the hypersurface-deformation brackets are modified. Since the physics is insensitive to our choice of normalization, we should be able to recover Einstein's field equations from the new brackets.

In [86, 119] the Lie derivative with respect to the normal vector field plays an important role in the derivation of possible Hamiltonian constraints consistent with the brackets and hence in the derivation of the equations of motion. One obtains a partial differential equation which the Hamiltonian constraint as the generator of normal deformations must obey [86], and similarly there is a related partial differential equation for the Lagrangian [119]. If the brackets are modified, the differential equation is changed by a new coefficient β. For instance, a metric-dependent Lagrangian $L[q_{ab}(x), K_{ab}(x)]$ consistent with constraints satisfying (4.1) must satisfy the functional equation [87]

$$\frac{\delta L(x)}{\delta q_{ab}(x')}K_{ab}(x')+2(\partial_b\beta)(x)\frac{\partial L(x)}{K_{ab}(x)}\partial_a\delta(x,x')+2\beta(x)\frac{\partial L(x)}{\partial K_{ab}(x)}\partial_a\partial_b\delta(x,x')-(x\leftrightarrow x')=0 \tag{4.39}$$

where $K_{ab}=\frac{1}{2}\mathcal{L}_n q_{ab}$ is taken with a non-standard normal n^μ. The normal derivative is subsequently written as a Lie derivative along τ^μ in order to arrive at equations of motion with respect to the time-evolution vector field. For the classical equations to result in this second case, in which the algebroid and the normalization are modified in such a way that we are still dealing with the classical theory, the function β appearing in n^μ with non-standard normalization (and therefore in the Lie derivative $\mathcal{L}_n$ as well) must cancel the function β appearing in the modified brackets. We will make use of the presence of such cancellations in our discussion of the modified case.

4.3.4.2 Modified theory

In models of loop quantum gravity, the hypersurface-deformation brackets are modified. However, since one sets up the models in the standard canonical formulation, the normalization $g_{\mu\nu}n^\mu n^\nu=\epsilon$ is preserved. Since the normal does not depend on phase-space variables and is not quantized, the normalization convention does not change. And yet, the brackets are modified. This case is therefore different from simply rescaling the normal vector. Nevertheless, one can understand the resulting structures by rescaling the normal after new brackets have been obtained from quantum effects. For spatially constant β, a morphism to the classical brackets is obtained. By applying the preceding arguments, we nevertheless expect non-classical equations of motion: There is a function β from the modified brackets appearing in the Hamiltonian constraint or Lagrangian regained from the brackets, but now there is no compensating β in the normal Lie derivative in relation to the

τ^μ-derivative because it is defined with respect to the standard normal vector n^μ.

The dynamics is therefore modified, which is consistent with the results of several detailed investigations of cosmological [120, 121, 122, 123, 124, 78, 125, 126, 127] and black-hole consequences [128,76,75,129] in terms of physical, coordinate-independent effects. An open question has been whether one can introduce a modified effective space-time metric which is generally covariant in the standard sense, or whether the deformed algebroid modifies this symmetry and leads to an entirely new space-time structure.

For spatially constant β, we know that deformed brackets can be mapped to classical brackets by a Lie-algebroid morphism so long as β does not change sign. In terms of space-time geometries, rescaling the normal vector n^μ to $\tilde{n}^\mu = |\beta|^{-1/2}\, n^\mu$ then leads us back to the unmodified brackets. We already know that this algebroid implements standard space-time covariance in the canonical formalism. We therefore see, in qualitative agreement with [93], that a field redefinition allows us to restore the undeformed brackets, and consequently general covariance in the classical form. The equations of motion are nevertheless different from the classical ones because we moved the β appearing in the modified brackets into the new normal vector, which is not cancelled out when we finally switch to equations of motion with respect to τ^μ.

4.4 Consequences

Hypersurface-deformation brackets can be modified by replacing the usual normalization of the normal vector by $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$, while the time-evolution vector field must be the same for the modified as well as the unmodified theory. These two facts raise the question of whether it is possible to distinguish between classical modifications from non-standard normalizations and modifications induced by quantum gravity theories. We have answered this question in the affirmative because equations of motion with respect to a fixed time-evolution vector field do change.

4.4.1 Field equations and matter couplings

If β has definite sign and is spatially constant, one can absorb the bracket modifications in a non-standard normalization. Gauge transformations generated by the algebroid then amount to the standard symmetries of covariance. Accordingly, regained constraints or Lagrangians must belong to the canonical theory of some higher-curvature action, assuming that a local effective action exists.

We expect higher-curvature effective actions when a local derivative expansion exists. In canonical terms, a non-local quantum effective action is obtained by coupling expectation values to independent quantum moments [7, 9], which formally play the role of auxiliary fields in a non-local theory. Only when moments behave adiabatically can they be eliminated from the equations of motion, and a local effective action results. As shown in [113], moments do not appear in structure functions such as β here, but they lead to higher-order constraints which restrict the moments as independent variables. For a local, higher-curvature version of the effective theory one would have to solve for almost all the higher-order constraints, which may not always be possible. A canonical effective theory still exists.

However, even if we have a standard higher-curvature effective action after a field redefinition, there are additional effects from modified brackets. The Hamiltonian constraint in such a system generates deformations along a non-standard normal vector. Therefore, when equations of motion are written with respect to a time coordinate, they belong to an effective action in which time derivatives are multiplied by a factor of β. The main consequence of modified algebroids is therefore a non-classical propagation speed, which is in agreement with the specific results obtained in [120, 121, 77, 124, 78, 108] for cosmological scalar and tensor modes. From (4.35), we have the kinetic term $\ddot{\phi}/\beta - \Delta\phi$ in an equation of motion for a scalar field on the effective Riemannian space-time. This result is in agreement with a related one derived in [87] for metric-dependent β, following [119, 87]. At the same time, we have generalized the result of [87] by extending it to functions β that may depend on extrinsic curvature as in cases of interest for signature change.

One can turn these arguments around and try to generate explicit consistent models with modified brackets by introducing non-standard normalizations in different classical actions or constraints. More generally, we could relax the orthogonality condition between n^μ and X^μ_a in order to find models with the new modified brackets (4.27) and (4.28) with $\alpha^a \neq 0$. The recent analysis of [130] suggests that such modified versions of constraints will have to be of higher than second order in extrinsic curvature.

4.4.2 (Non-)existence of an effective Riemannian structure

Sometimes, the classical space-time structure is *assumed* in toy models of quantum gravity, without checking closure of modified constraints. In fact, one should not consider such constructions as models of quantum gravity but rather of quantum-field theory on (modified) curved space-times because quantum gravity is usually understood as including a derivation of non-classical space-time structures in addition to a modified dynamics. For instance, some constructions [96, 97, 98] use perturbation equations on a modified background $\bar{q}_{ab}$ subject to evolution equations with quantum corrections. Perturbations are gauge-fixed or combined into gauge-invariant expressions before quantization, and therefore one assumes the classical space-time structure. As confirmed here, an effective formulation with the classical space-time structure does exist as long as $\beta > 0$, but only after a field redefinition using either base transformations or, in the case of a spatially constant β as it is realized in first-order cosmological perturbation theory, fiber transformations of the hypersurface-deformation algebroid.

There are therefore two important caveats regarding assumptions as in [96,97,98]: First, if the evolution of $\bar{q}_{ab}$ is modified, a consistent description of space-time transformations for inhomogeneous modes requires a modified N which can only be computed if one knows a consistent set of β-modified brackets. (The lapse function of the postulated space-time metrics in [96, 97, 98] do have quantum corrections, but in an incomplete way that ignores the field redefinition required for a consistent space-time structure.) The modified N, as opposed to the classical N, then implies further quantum corrections not directly present in the evolving $\bar{q}_{ab}$. One can, of course, partially absorb $\sqrt{|\beta|}$ in $N' = \sqrt{|\beta|}N$ by introducing a new time coordinate t' with $\mathrm{d}t' = \sqrt{|\beta|}\mathrm{d}t$. But the dependence of $\bar{q}_{ab}$ on this new t' is different from the original dependence on t, so that additional quantum corrections are present.

4.4.3 Signature change

In particular, as the second caveat, the signature of the effective space-time metric can be determined only if one knows the sign $\epsilon\epsilon_\beta$ by which $\beta N^2 \mathrm{d}t^2$ enters the metric (4.35) in the equivalent Riemannian space-time structure, which can differ from the classical value if β does not have definite sign. The sign, in turn, affects the form of well-posed partial differential equations on the background; see for instance [106,131]. In the presence of signature change, there is no deterministic evolution through large curvature. And even if one tries to ignore this conclusion for a formal analysis of the resulting phenomenology, no viable results are obtained [132].

If β is of indefinite sign, it can no longer be absorbed globally. The classical space-time structure can be used only to model disjoint pieces of a solution in which β has definite sign, corresponding to Lorentzian space-time patches when β is positive and Euclidean spatial patches when it is negative. We then have non-isomorphic Lie algebroids. A non-constant sign of β therefore triggers signature change [87, 133, 131] with the effective signature locally given by $\epsilon\epsilon_\beta$. Globally, such a solution of an effective quantum-gravity model can be described consistently only with a modified algebroid, in which all structure functions are continuous and well-defined even when β goes through zero. It is no longer possible to absorb β globally, and therefore a new version of quantum space-time is obtained.

Chapter 5
Conclusions and Future Directions

Gravity, from the point of view of canonical formulations, can be described by slicing the full space-time into three dimensional hypersurfaces that evolve in time. Not possessing a particular notion of time and being a fully constrained system implies for General Relativity that a combination of first class constraints is what constitutes its total Hamiltonian which is trivial on the constraint surface. Classically speaking, in general, the Poisson brackets of two constraints is required to close into another first class constraint for any generally covariant theory. Being one such theory, in GR this condition is satisfied by its Hamiltonian and diffeomorphism constraints. The same algebra is still obeyed by the constraints obtained from higher curvature gravity theories. This type of theories are significant as these are the only actions consistent with local covariance. Perturbative corrections that generically come about on quantizing the classical theory will basically be identified with such higher curvature terms of a corresponding effective theory of gravity. In [113] these higher curvature effects have been shown to correspond to constraints themselves getting corrections due to quantum back-reaction by moments on expectation values, but that, under a anomaly-free quantization scheme, constraint algebra stays undeformed. This observation also supports the view that if quantization essentially persists all the way down to Planck length, then semi-classical effects might take forms other than higher curvature terms. In turn, one is moved to look for non-perturbative corrections within a given theory.

In LQG there are two known such non-perturbative contributions. The lack of a well defined curvature operator acting on the Hilbert space of this theory resulted in these expressions being written in terms of holonomies of connection components. Other than this so-called (nonlocal) holonomy corrections, another non-perturbative effect that goes under the name of inverse-triad corrections is present due to the constraint of the theory including an operator inverse for the volume. Since constraint algebra characterizes the local diffeomorphism symmetry of the particular theory at hand, it is necessary that it still closes even after any such non-perturbative corrections are included in the constraints simply because it should still reflect some version of general covariance.

The well-established feature of Quantum Mechanics that the order of the measurement of two of its dynamical quantities actually matters is mathematically encoded in the fact that the operators corresponding to these variables have a non-vanishing commutator. On the other hand, Einstein's GR teaches us that gravity is equivalently the geometry of space-time. These simple statements hint at us that the tool for explaining the underlying space-time structure for a quantum gravity theory could very well be non-commutative geometry (NCG). There are various approaches, tailored for different purposes, in this direction [134, 135, 136]. For example, claims in [137] about the role NCG plays in M-Theory (a unification of all five consistent versions of superstring theory) sparked the recent interest in the subject, which eventually led to what is known as Noncommutative Standard Model [138, 139], a model that extends Standard Model by a modified version of GR. [1]

Such encouraging physical results along with the mathematical indications coming from String theory and LQG which we talked about in Chapter 1 provoked us to consider non-commutativity as another possible cause for obtaining modifications of hypersurface deformation algebra. For this purpose, endowed with the differential geometry of deformed diffeomorphisms of [141, 142] which is formulated in an explicitly covariant and coordinate-independent way, our plan is to apply the mechanism of Gaussian vector fields from Section 4.2.1 above to this non-commutative setup. In particular, making use of the $\star$-Lie derivative defined as in Eqn. (4.12) and the deformed Leibniz rule it obeys written as in Eqn.(4.13) of [141], it would be interesting to start from our Eqn. (4.11) and see whether we would still be able to get a closed yet deformed algebra of constraints if any

[1] One of the impressive achievements of this model is that the Higgs mass predicted within this model is in agreement with the 125-127 GeV range set by recent runs at CERN. [140]

correction terms appear in this process. After all, at least in a (Minkowskian) field theory context, non-vanishing commutators between space-time coordinates for example as in the case of Weyl-Moyal star product are known to lead to verifiable predictions for Lorentz violating effects [143].

Our method has proven successful in obtaining observable predictions which cannot be replicated in the absence of monopole densities as we have detailed in 2.5. Fur the purpose of extracting even more such verifiable results in more complicated systems, we attempt a new definition of spectra of elements in some unital algebra $\mathcal{A}$ which does not make use of a Hilbert-space representation. In order to achieve this we generalize the definition of a state to be positive linear functionals, ω, obeying Cauchy-Schwarz inequalities on a (not necessarily associative) *-algebra. Given such a unital algebra and a Hamiltonian constructed from the product structure on this algebra, an eigenvalue λ was defined with respect to the following equation

$$\omega(a(H - \lambda 1)) = 0 \tag{5.1}$$

if there exists a state $\omega : \mathcal{A} \to \mathcal{C}$ for all $a \in \mathcal{A}$. If further $\mathcal{A}$ happens to be a unital *-algebra, imposing the self-adjointness condition on the Hamiltonian results in its eigenvalues defined as above to be real, as expected from the usual quantum mechanical calculations but this time formulated in the language of such linear functionals. One remark about Eqn. (5.1) is that it can only yield what is called the discrete spectrum of an operator acting on a Hilbert space. This above definition of an eigenvalue has an advantage over the more standard definition of the spectrum of let's say $A \in \mathcal{A}$ as the set of all $\lambda \in \mathcal{C}$ such that $A - \lambda 1$ does not have an inverse in $\mathcal{A}$: It is not as hard to use in practice when one does not have a Hilbert-space representation and the associated topologies on state spaces.

In order to define moments in this formalism, we make use of a set of distinguished observables, $\mathcal{O}$, in an algebra $\mathcal{A}$ as follows: Given a classical phase space $\mathcal{M}$ and a classical Poisson algebra defined on it, the global coordinates $x_i \in C^\infty(\mathcal{M})$ are required to form a linear subalgebra of the the full Poisson algebra and hence generate a Lie algebra, g, which is finite dimensional if $\mathcal{M}$ is finite dimensional. Then $\mathcal{O} \ni d_i$ is a subalgebra of the Lie algebra associated to $\mathcal{A}$ through the commutator such that $\mathcal{A}$ is a quantization, for example with respect to the deformation quantization scheme, of the classical Poisson algebra $(C^\infty(\mathcal{M}, \{,\})$

with a Lie algebra isomorphism $g \to \mathcal{O}, x_i \to d_i$. Then moments of such observables are defined with the Weyl ordering as before making use of the expectation values $\omega(d_i)$.

It turns out that these definitions are very functional, and in the case of hydrogen atom, with Hamiltonian $H = \frac{1}{2}|p|^2 - \alpha r^{-1}$, the choice of distinguished observables as $r, P := r|p|^2, Q := xp_x + yp_y + zp_z$ which is enough to reproduce the known Hydrogen eigenvalues

$$\lambda_{l+1} = -\frac{m\alpha^2}{2\hbar^2(l+1)^2} \tag{5.2}$$

for maximal angular-momentum quantum number l from the standard uncertainty relation. However, even more surprisingly, following this same derivation in the case where we introduce a non-zero $[p,p]$ commutator taken to be proportional to a magnetic field which is, a priori, chosen to be generic, we obtain the *very* same algebra

$$[r, Q] = ir, \qquad [r, P] = 2iQ, \qquad [Q, P] = iP \tag{5.3}$$

of distinguished observables as in the standard Hydrogen atom case if the magnetic field components satisfy the conditions:

$$B_x = \frac{xB_z}{z}, \qquad B_y = \frac{yB_z}{z} \tag{5.4}$$

where B_z is a free parameter.[2] It should be noted here that $\nabla.B \neq 0$ if $B_z \neq 0$. Under these conditions the usual Virial Theorem relating the expectation value of the kinetic term and the potential term of the standard Hydrogen hamiltonian and their dependence on the energy eigenvalue remain unaltered, hence we recover Eqn. (5.2). The remaining step in this non-associative Hydrogen problem is to check if the transition rates from time correlation functions get modified at all. If we get some new contributions, as minute as they might be, which are not present in the transition rates for the associative Hydrogen atom, this could give us an indication as to whether non-associativity, at least at this energy scale, is a physical entity or not once a proper experiment to distinguish such effects for this already extremely well-tested system is devised.

Last but not the least, we would like to touch upon what consequences our results from Chapter 3 might have. There we concluded that star products used

[2]As a special case the linear magnetic field is covered with the choice $B_z = z$

for mganetic monopole systems cannot be alternative, that is the associator cannot be completely anti-symmetric. This means a totally symmetric and two mixed type $2 \times 2 \times 2$ tensors can contribute to the Jacobiator. A totally symmetric $2 \times 2 \times 2$-tensor T_{ijk} has components

$$T_{ij1} = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_3 \end{pmatrix} \quad , \quad T_{ij2} = \begin{pmatrix} C_2 & C_3 \\ C_3 & C_4 \end{pmatrix} . \tag{5.5}$$

There are two mixed types obtained from a generic matrix S_{ijk} by a product of one symmetrizer and one antisymmetrizer in different pairs of indices. We first apply a symmetrizer in the first pair, followed by an antisymmetrizer in the first and last index. A matrix with these symmetries has the form

$$T_{ijk} = S_{ijk} + S_{jik} - S_{kji} - S_{kij} \tag{5.6}$$

in terms of the generic matrix S_{ijk}. Its components are

$$T_{ij1} = \begin{pmatrix} 0 & A_1 \\ A_1 & -2A_2 \end{pmatrix} \quad , \quad T_{ij2} = \begin{pmatrix} -2A_1 & A_2 \\ A_2 & 0 \end{pmatrix} . \tag{5.7}$$

The last type can be obtained by symmetrizing in the first and last index, followed by antisymmetrizing in the first index pair:

$$T_{ijk} = S_{ijk} + S_{kji} - S_{jik} - S_{jki} \tag{5.8}$$

with components

$$T_{ij1} = \begin{pmatrix} 0 & -2B_1 \\ B_1 & B_2 \end{pmatrix} \quad , \quad T_{ij2} = \begin{pmatrix} B_1 & B_2 \\ -2B_2 & 0 \end{pmatrix} . \tag{5.9}$$

As appropriate for a decomposition of a $2 \times 2 \times 2$-tensor, we have eight independent components.

If we use S_{ijk}, written in terms of the contributions A_1, A_2, B_1, B_2 and C_1 through C_4, as the associator for a single pair of degrees of freedom, we obtain

$$[x, x, x] = C_1 \tag{5.10}$$

$$[x, x, p] = -2A_1 + B_1 + C_2 \tag{5.11}$$

$$\begin{aligned} [x,p,x] &= A_1 - 2B_1 + C_2 && (5.12) \\ [p,x,x] &= A_1 + B_1 + C_2 && (5.13) \\ [x,p,p] &= A_2 + B_2 + C_3 && (5.14) \\ [p,x,p] &= A_2 - 2B_2 + C_3 && (5.15) \\ [p,p,x] &= -2A_2 + B_2 + C_3 && (5.16) \\ [p,p,p] &= C_4\,. && (5.17) \end{aligned}$$

The associators $[x,p,p]$ and $[p,x,x]$, together with $[p,x,p]$ and $[x,p,x]$, are particularly relevant for equations of motion of expectation values in a non-alternative harmonic oscillator. With $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$, we compute

$$\begin{aligned} [x,H] &= \frac{1}{2m}\left(x(p^2) - (p^2)x\right) + \frac{1}{2}m\omega^2\left(x(x^2) - (x^2)x\right) \\ &= \frac{1}{2m}\left((xp)p + [x,p,p] - p(px) - [p,p,x]\right) + \frac{1}{2}m\omega^2[x,x,x] \\ &= \frac{1}{2m}\left((px + i\hbar)p - p(xp - i\hbar) + [x,p,p] - [p,p,x]\right) + \frac{1}{2}m\omega^2[x,x,x] \\ &= \frac{i\hbar}{m}p + \frac{1}{2m}\left([x,p,p] - [p,p,x] - [p,x,p]\right) + \frac{1}{2}m\omega^2[x,x,x]\,. \qquad (5.18) \end{aligned}$$

In terms of the decomposition coefficients, we have

$$[x,H] = \frac{i\hbar}{m}p + \frac{1}{2m}(2A_2 + 2B_2 - C_3) + \frac{1}{2}m\omega^2 C_4\,. \qquad (5.19)$$

An additional drift term is then obtained in the equation of motion for $\langle x\rangle$, which implies that $\langle x\rangle$ can change even for vanishing momentum. Similarly, the equation of motion for $\langle p\rangle$ would have an extra shift term corresponding to a constant force or a linear potential, which shifts the minimum of the harmonic potential. Even such preliminary results, again, illustrate the possibility of deriving experimentally testable effects in the context of such exotic system.

Appendix A
Details on the derivation of eq. (3.32)

Starting from (3.31), and using all its cyclic permutations, we can write the fully anti-symmetric part of A_3 as

$$\begin{aligned}6A_3(f,g,h)^- \;=\;& B_2(f,B_1(g,h)) - B_2(B_1(f,g),h) + B_2(h,B_1(f,g)) \\ &-B_2(B_1(h,f)g) + B_2(g,B_1(h,f)) - B_2(B_1(g,h),f) \\ &-B_2(f,B_1(h,g)) + B_2(B_1(f,h),g) - B_2(g,B_1(f,h)) \\ &+B_2(B_1(g,f),h) - B_2(h,B_1(g,f)) + B_2(B_1(h,g),f) + (B_1 \leftrightarrow B_2)\,.\end{aligned} \tag{A.1}$$

Using the definition of the anti-symmetric parts of the B_i, we have

$$\begin{aligned}6A_3(f,g,h)^- \;=\;& 2B_2^-(f,B_1(g,h)) + 2B_2^-(h,B_1(f,g)) + 2B_2^-(g,B_1(h,f)) \\ &-2B_2^-(f,B_1(h,g)) - 2B_2(g,B_1(f,h)) - 2B_2^-(h,B_1(g,f)) + (B_1 \leftrightarrow B_2)\,.\end{aligned} \tag{A.2}$$

Finally, using the fact that the B_i are linear in their arguments, we obtain the required form for the fully anti-symmetric part of A_3 as in (3.32).

Appendix B
Proof of Lemma 3 without Pentagon identity

To begin with, let us write the third-order associator as before:

$$\begin{aligned} A_3(f,g,h) &= \mathrm{d}B_3(f,g,h) + B_2(f,B_1(g,h)) \\ &\quad - B_2(B_1(f,g),h) + B_1(f,B_2(g,h)) - B_1(B_2(f,g),h)\,, \end{aligned} \tag{B.1}$$

where $\mathrm{d}B_n = fB_n(g,h) + B_n(f,gh) - hB_n(f,g) - B_n(fg,h)$.If we apply the Hochschild coboundary operator to A_3, the first term in (B.1) should give zero because $\mathrm{d}^2 = 0$. (Again, when applied to coefficients in an λ-expansion of a non-assocative star product, only the associative multiplication of smooth functions is used in the definition of d.) However, for completeness we will explicitly show this. The part in $\mathrm{d}A_3(f,g,h,k)$ involving contributions only from the B_3 terms has the form

$$f\,\mathrm{d}B_3(g,h,k) - \mathrm{d}B_3(fg,h,k) + \mathrm{d}B_3(f,gh,k) - \mathrm{d}B_3(f,g,hk) + k\,\mathrm{d}B_3(f,g,h) \tag{B.2}$$

Using the definition of $\mathrm{d}B_n$ for $n=3$ gives

$$\begin{aligned} & f\Big(g\,B_3(h,k) + B_3(g,hk) - k\,B_3(g,h) - B_3(gh,k)\Big) \\ & -\Big(f\,g\,B_3(h,k) + B_3(fg,hk) - k\,B_3(fg,h) - B_3(fgh,k)\Big) \\ & +\Big(f\,B_3(gh,k) + B_3(f,ghk) - k\,B_3(f,gh) - B_3(fgh,k)\Big) \end{aligned}$$

$$-\Big(f\, B_3(g,hk) + B_3(f,ghk) - h\,k\, B_3(f,g) - B_3(fg,hk) \Big)$$
$$+k\Big(f\, B_3(g,h) + B_3(f,gh) - h\, B_3(f,g) - B_3(fg,h) \Big)\,.$$

Upon a close inspection of this expression, we see that there is a counterterm for each term, and thus it is zero. We are left with the action of the coboundary operator on the last four terms in (B.1). Concentrating, for now, on its action on the B_2 terms, using the generic definition of $\mathrm{d}B_n$ for $n = 2$, we obtain a part in $\mathrm{d}A_3(f,g,h,k)$ that is of the form:

$$\begin{aligned}
&-f\Big(B_2(g, B_1(h,k)) - B_2(B_1(g,h),k) \Big) \\
&-B_2(fg, B_1(h,k)) + B_2(B_1(fg,h),k) \\
&+B_2(f, B_1(gh,k)) - B_2(B_1(f,gh),k) \\
&-B_2(f, B_1(g,hk)) + B_2(B_1(f,g),hk) \\
&+k\Big(B_2(f, B_1(g,h)) - B_2(B_1(f,g),h) \Big)\,.
\end{aligned} \tag{B.3}$$

Using the Leibniz property of B_1, and removing terms that identically cancel out, we are left with

$$\begin{aligned}
&-f\, B_2(g, B_1(h,k)) - f\, B_2(B_1(g,h),k) - B_2(fg, B_1(h,k)) \\
&+B_2(fB_1(g,h),k) + B_2(f, gB_1(h,k)) - B_2(hB_1(f,g),k) \\
&-B_2(f, kB_1(g,h)) + B_2(B_1(f,g),hk) + k\, B_2(f, B_1(g,h)) - k\, B_2(B_1(f,g),h)\,.
\end{aligned}$$

This expression can be cast into a more succinct form in terms of $\mathrm{d}A_2$, by adding and subracting a few terms as follows:

$$\begin{aligned}
&\mathrm{d}B_2(f,g,B_1(h,k)) - \mathrm{d}B_2(f,B_1(g,h),k) + \mathrm{d}B_2(B_1(f,g),h,k) \\
&+B_1(h,k)B_2(f,g) - B_2(h,k)B_1(f,g)\,.
\end{aligned} \tag{B.4}$$

The action of the differential on the B_1 terms in (B.1) gives an expression similar to (B.3), with the roles of B_1 and B_2 exchanged. Again upon using the Leibniz property of B_1 and cancelling terms, we have the contribution to $\mathrm{d}A_3$ as

$$-f\,B_1(B_2(g,h),k) - g\,B_1(f,B_2(h,k)) + B_1(B_2(fg,h),k) + B_1(f,B_2(gh,k))$$
$$-B_1(B_2(f,gh),k) - B_1(f,B_2(g,hk)) + h\,B_1(B_2(f,g),k) + k\,B_1(f,B_2(g,h))\,.$$

Using anti-symmetry and linearity in either of the arguments of B_1, and again adding and subtracting a few terms, we introduce $\mathrm{d}B_2$ as

$$B_1(\mathrm{d}B_2(g,h,k),f) - B_1(\mathrm{d}B_2(f,g,h),k) - B_2(f,g)B_1(h,k) + B_2(h,k)B_1(f,g) \tag{B.5}$$

As the final result, (B.4) and (B.5) give

$$\begin{aligned}\mathrm{d}A_3(f,g,h,k) \;&=\; \mathrm{d}B_2(f,g,B_1(h,k)) - \mathrm{d}B_2(f,B_1(g,h),k) + \mathrm{d}B_2(B_1(f,g),h,k) \\ &\quad +B_1(\mathrm{d}B_2(g,h,k),f) - B_1(\mathrm{d}B_2(f,g,h),k)\,.\end{aligned} \tag{B.6}$$

To get the same result as in (3.37), which was obtained using the Pentagon identity, we just use the definition of $\mathrm{d}B_2$ in terms of the second-order associator as $\mathrm{d}B_2(f,g,h) = A_2(f,g,h) - B_1(f,B_1(g,h)) + B_1(B_1(f,g),h)$, and use the linearity of B_1 in its first argument in the last two terms.

Appendix C
ADM and geometrodynamics derivation of non-standard classical constraints

We derive the results of Sec. 4.2.1 for $\alpha^a = 0$ using more familiar methods.

C.1 ADM

Given a space-time metric $g_{\mu\nu}$ and a time-evolution vector field of the form (4.9) with respect to a foliation, we obtain the canonical form of the metric by expanding $g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu$ using

$$\mathrm{d}X^\mu = \partial_t X^\mu \mathrm{d}t + \partial_a X^\mu \mathrm{d}x^a = \left(N_\beta n^\mu + N^a X^\mu_a\right)\mathrm{d}t + X^\mu_a \mathrm{d}x^a \tag{C.1}$$

with $N_\beta = N/\sqrt{|\beta|}$. If n^μ has non-standard normalization $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$, the metric components are

$$g_{tt} = N^a N_a + \epsilon\beta N_\beta^2 = N^a N_a + \epsilon\epsilon_\beta N^2 \quad , \quad g_{at} = N_a \quad , \quad g_{tb} = N_b \quad , \quad g_{ab} = q_{ab}\,. \tag{C.2}$$

With respect to a non-standard normal, we define the tensor

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n q_{\mu\nu}\,. \tag{C.3}$$

It differs from the extrinsic-curvature tensor bu a factor of $\sqrt{|\beta|}$, as can be seen from the alternative version

$$K_{\mu\nu} = \frac{1}{2N_\beta}\mathcal{L}_{\tau-\vec{N}}q_{\mu\nu} \tag{C.4}$$

derived from (C.3) using (4.9). The relationship between $K_{ab} = K_{\mu\nu}X^\mu_a X^\nu_b$ and the τ-derivative $\dot{q}_{ab} = \mathcal{L}_\tau q_{ab}$ is therefore

$$K_{ab} = \frac{1}{2N_\beta}\left(\dot{q}_{ab} - \mathcal{L}_{\vec{N}}q_{ab}\right) . \tag{C.5}$$

In order to relate K_{ab} to the momentum of q_{ab}, we need the gravitational action $S = \int \mathrm{d}y^4\sqrt{|\det g|}R$ in new variables defined with respect to a non-standard normal. (We set $16\pi G = 1$.) The standard derivation from Gauss–Codazzi equations gives us the space-time Ricci scalar

$$R = \mathcal{R} - \frac{\epsilon}{\beta}\left(K_{ab}K^{ab} - K^2\right) \tag{C.6}$$

expressed as a combination of the spatial Ricci scalar $\mathcal{R}$ and K_{ab}. (See also [144], where a time-dependent β has been assumed to study classical signature change.) Together with

$$\sqrt{|\det(X^*g)|} = N\sqrt{\det(g_{ab})} = N_\beta\sqrt{|\beta|}\sqrt{\det(q_{ab})}\,, \tag{C.7}$$

all contributions to the Einstein–Hilbert action appear are written in terms of new variables. The momentum of q_{ab} is

$$P^{ab}(t,x) = \frac{\delta\mathcal{S}}{\delta\dot{q}_{ab}} = -\frac{\epsilon\epsilon_\beta}{\sqrt{|\beta|}}\sqrt{\det(q_{ab})}\left(K^{ab} - q^{ab}K^c_c\right) , \tag{C.8}$$

while the momenta P of N and P_a of N^a vanish as usual. (The factor of $\epsilon\epsilon_\beta/\sqrt{|\beta|} = (\epsilon/\beta)(N/N_\beta)$ in (C.8) is a result of combining ϵ/β in (C.6) with N in (C.7) and one of the N_β obtained after converting K_{ab} to $\dot{q}_{ab}$ using (C.5).) For the primary constraints $P = 0$ and $P_a = 0$ to be preserved in time, we obtain as secondary constraints the diffeomorphism and Hamiltonian constraints

$$\mathcal{H}_a \;:=\; -2q_{ab}\nabla_b P^{bc} \tag{C.9}$$

$$\mathcal{H} \;:=\; -\frac{\epsilon\epsilon_\beta\sqrt{|\beta|}}{\sqrt{\det(q_{ab})}}\left(q_{ac}q_{bd}-\frac{1}{2}q_{ab}q_{cd}\right)P^{ab}P^{cd}-\sqrt{|\beta|}\sqrt{\det(q_{ab})}\mathcal{R}\,. \tag{C.10}$$

These constraints have closed Poisson brackets corresponding to (4.24). In terms of extrinsic curvature instead of the momentum, the first term of (C.10) has a factor of $\epsilon_\beta/\sqrt{|\beta|}$, in agreement with expressions regained from modified brackets [87] following the methods of [86, 119].

C.2 Geometrodynamics

Using the formalism of hyperspace [145, 146, 147], the hypersurface-deformation brackets can be derived from infinitesimal deformations, irrespective of the dynamics. An infinitesimal deformation δX^μ may be decomposed as

$$\delta X^\mu = \delta N_\beta n^\mu + \delta N^a X^\mu_a\,. \tag{C.11}$$

The (non-standard) normalization and orthogonality relations $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$ and $g_{\mu\nu}n^\mu X^\nu_a = 0$ allow us to compute δN_β and δN^a from δX^μ:

$$\delta N_\beta = \frac{\epsilon}{\beta}n_\mu\,\delta X^\mu \quad , \quad \delta N^a = X^a_\mu\,\delta X^\mu \tag{C.12}$$

Here we do not refer to τ^μ or δN because the present geometrical considerations refer to what is considered as the normal vector with a non-standard normalization.

An arbitrary functional $F = F[X^\mu(x^a)]$ on hyperspace changes if we deform the hypersurface by $\delta N_\beta(x)$ along a normal geodesic and stretch it by $\delta N^a(x)$. Using (C.11), we write the infinitesimal change of F as

$$\delta F = \int_\sigma \mathrm{d}^3x\delta X^\mu(x)\frac{\delta}{\delta X^\mu(x)}F = \int_\sigma \mathrm{d}^3x\left(\delta N_\beta(x)\rho_0(x)+\delta N^a(x)\rho_a(x)\right)F \tag{C.13}$$

with the generators of pure deformations and pure stretchings given by

$$\rho_0(x) := n^\mu(X(x))\frac{\delta}{\delta X^\mu(x)} \quad , \quad \rho_a(x) := X^\mu_a(x)\frac{\delta}{\delta X^\mu(x)}\,. \tag{C.14}$$

These generators can be interpreted as the Lie-algebroid anchor $\rho\colon \Gamma(A)\to\Gamma(TB)$, with base manifold B the space of embeddings $X\colon \sigma\to\mathcal{M}$, expressed in a local basis: In a neighborhood $U \subset B$, we introduce a smooth chart $(U, \{x^a\})$ of the manifold B and a local frame $\{e_i\}$ for sections of the Lie algebroid $\pi^{-1}(U) \subset A$. Then there exist smooth functions $c_{ij}^k, \rho_i^a \colon B \to \mathbb{R}$, such that

$$[e_i, e_j]_A = c_{ij}^k e_k \quad , \quad \rho(e_i) = \rho_i^a \frac{\partial}{\partial x^a} \,. \tag{C.15}$$

These functions are called the structure functions of the Lie algebroid with respect to the local frame $\{e_i\}$ and local coordinates $\{x^a\}$. For the hypersurface-deformation algebroid, $\rho_0 = \rho(e_0)$ and $\rho_a = \rho(e_a)$.

There are infinitely many generators $\rho_0(x)$ and $\rho_a(x)$ which span the tangent space to hyperspace at each hypersurface. Compared with the coordinate basis $\delta/\delta X^\mu$, an important advantage of this basis is its independence of the choice of space-time coordinates X^μ. We can therefore describe the kinematics in terms intrinsic to the hypersurfaces. However, the basis is non-holonomic: commutators of the generators $\rho_0(x)$ and $\rho_a(x)$ do not vanish in general.

In order to establish the commutators of deformation generators (C.14) we have to know how the normal vector changes under an infinitesimal deformation. To this end, the formula

$$\delta n^\mu = -\epsilon X^{\mu a} \delta N_{,a} + K_{ab} X^{\mu a} \delta N^b - \Gamma^\mu_{\rho\sigma} X^\rho_c n^\sigma \delta N^c - \Gamma^\mu_{\rho\sigma} n^\rho n^\sigma \delta N \tag{C.16}$$

has been used in [86, 145] in order to compute the commutator of normal deformations $\rho_0(x)$ in which $\delta n^\mu(x)/\delta X^\nu(x')$ appear. Only the first term in (C.16) contributes to this commutator, while all other terms are irrelevant for this purpose because they present variations proportional to delta functions. Since delta functions are symmetric in their arguments they will cancel out thanks to the anti-symmetry of a commutator. The variation given by the first term in (C.16), on the other hand, is proportional to $\delta_{,a}(x, x') = -\delta_{,a'}(x', x)$, which is anti-symmetric and does contribute.

The first term in (C.16) follows from a simple consideration that can easily be extended to non-standard normalizations of n^μ. One can compute the full (C.16) in terms of its normal and tangential components by varying $g_{\mu\nu} n^\mu n^\nu = \epsilon$ and $g_{\mu\nu} X^\mu_a n^\nu = 0$. Since the first term in (C.16) does not contribute to the normal component $n_\mu \delta n^\mu$, it must result from $\delta(g_{\mu\nu} X^\mu_a n^\nu) = 0$. This variation has three terms, so that the equation can be solved for

$$X_{a\mu}\delta n^{\mu} = -n^{\mu}\delta X_{\mu,a} - X^{\mu}_{a} n^{\nu}\delta g_{\mu\nu} = -(n^{\mu}\delta X_{\mu})_{,a} - n^{\mu}_{,a}\delta X_{\mu} - X^{\mu}_{a} n^{\nu}\delta g_{\mu\nu}\,.$$

The metric variations in the last term as well as $n^{\mu}_{,a}$ in the second term can be written in terms of extrinsic curvature and the Christoffel symbol, while the first term provides the first part of (C.16) upon using (C.12) with $\beta = 1$. For $\beta \neq 1$, the first term in (C.16) is replaced by $-\epsilon(\beta\delta N_{\beta})_{,a}$, or $-\epsilon\beta(\delta N_{\beta})_{,a}$ if the derivative of β is combined with the last term in (C.16) which drops out of commutators. As a result, there is a factor of β in the commutator

$$[\rho_0(x), \rho_0(x')] = \epsilon\beta \left(q^{ab}(x)\,\delta_{,a}(x,x')\,\rho_b(x) - q^{ab}(x')\,\delta_{,a}(x',x)\,\rho_b(x')\right)\,. \qquad \text{(C.17)}$$

This result agrees with (4.24).

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Vita

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- M. Bojowald, S. Brahma, U. Büyükçam, and F. D'Ambrosio, *Hypersurface-deformation algebroids and effective spacetime models*, Phys. Rev. D 94 (2016) 104032. [arXiv:1610.08355]

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