E. L. Chu
TN-63-9
January 21, 1963

THE SIMPLE BEAM-LOADING THEORY FOR CONSTANT-GRADIENT LINACS

1. Design Philosophy of Constant-Gradient Linacs

This has been described by Neal.* No further discussion is needed here.

2. Conditions and Assumptions

(a) The linac waveguide is almost a periodic structure. Waveguide dimensions vary slowly from cell to cell in a suitable manner, so that the forward E_z field excited by an external source located at $z = 0$ is constant over the whole length of the waveguide from $z = 0$ to $z = \ell$.

(b) Both the attenuation constant α and the group velocity v_g are functions of z. We assume that α and v_g are independent of frequency ω. The group velocity is much smaller than the velocity of light c.

(c) The phase velocity of wave propagation at the operating frequency is equal to c.

(d) The rf bunching of the electron beam is infinitely sharp. Electron velocity is equal to c.

(e) Backward wave propagation in the waveguide may be neglected.

(f) The shunt impedance r per unit length of the waveguide is assumed to be a constant, independent of both ω and z.

3. Differential Equation

Corresponding to Eq. (1) in TN-62-69 we have, for a linac waveguide of slowly-varying geometrical dimensions,

$$\left[\frac{\partial}{\partial z} + \frac{1}{v_g}\left\{\frac{\partial}{\partial t} + i\beta v_g - i\omega + \alpha v_g\left(1 + r\frac{\partial}{\partial z}\frac{1}{2\alpha r}\right)\right\}\right] E_{z,t}(z,t) = - F(z,t). \tag{1}$$

*R. B. Neal, "Theory of the Constant Gradient Linear Electron Accelerator," M.L. Report No. 513, Stanford University, May 1958.

Here $E_{z,+}$, F, and β are as defined in TN-62-69. In order that $E_{z,+}$ may have a constant amplitude in the steady state, i.e., $E_{z,+} = \text{const.} \times \exp(i\omega t - i\beta z)$, the waveguide must be so designed as to satisfy

$$1 + r \frac{\partial}{\partial z} \frac{1}{2\alpha r} = 0 \tag{2}$$

Under this condition Eq. (1) becomes

$$\left\{\left(\frac{\partial}{\partial z} + i\beta\right) + \frac{1}{v_g}\left(\frac{\partial}{\partial t} - i\omega\right)\right\} E_{z,+}(z,t) = - F(z,t). \tag{3}$$

Since r is assumed to be a constant, $(1/\alpha)$ must vary linearly with z. Let α_o be the value of α at $z = 0$, then

$$\alpha = \alpha_o/(1 - 2\alpha_o z). \tag{4}$$

Since $v_g = \omega/2\alpha Q = (\omega/2\alpha_o Q) \cdot (1 - 2\alpha_o z)$,

$$v_g = v_{go}(1 - 2\alpha_o z). \tag{5}$$

Here, $v_{go} = \omega/2\alpha_o Q$ is the value of v_g at $z = 0$.

Substituting Eq. (5) into Eq. (3) we obtain the differential equation for a constant-gradient linac waveguide as follows:

$$\left\{(1 - 2\alpha_o z)\left(\frac{\partial}{\partial z} + i\beta\right) + \frac{1}{v_{go}}\left(\frac{\partial}{\partial t} - i\omega\right)\right\} E_{z,t}(z,t) = - (1 - 2\alpha_o z)F(z,t). \tag{6}$$

4. Green's Function

$$\left\{(1 - 2\alpha_o z)\left(\frac{\partial}{\partial z} + i\beta\right) + \frac{1}{v_{go}}\left(\frac{\partial}{\partial t} - i\omega\right)\right\} g_+(z,t) = - \delta(z)\delta(t). \tag{7}$$

$$g_+(z,t) = - U(z)\delta\left(t - \int_0^z \frac{dz}{v_g}\right) e^{i(\omega t - \beta z)}. \tag{8}$$

As a contrast, we re-write the Green function given by Eq. (3) in TN-62-69 for constant-α linacs as

$$g_+(z,t) = - U(z)\delta\left(t - \frac{z}{v_g}\right) e^{-\alpha z+i(\omega t-\beta z)}.$$

The differences in characteristics between these two kinds of linac waveguides may be seen most clearly by comparing these two forms of Green's functions.

5. The Formal Solution

$$E_{z,t}(z,t) = \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dt' \; g_+(z - z', t - t') \cdot (1 - 2\alpha_0 z')F(z',t'). \tag{9}$$

6. Source Terms

As in TN-62-69 we denote the external source by $(-F_a)$ and the internal source accounting for the sharply-bunched electron beam by $(-F_b)$.

$$-F_a(z,t) = E_z(0)U(t + \tau)\delta(z)f(t + \tau) \cdot \exp(i\omega t + i\theta). \tag{10}$$

$$-F_b(z,t) = -\alpha r\bar{J} \cdot U(z)U(t - z/c) \cdot \exp(i\omega t - i\beta z). \tag{11}$$

In Eq. (10) $f(t + \tau)$ is some suitable function not yet defined. The right-hand side of Eq. (11) is obtained from $\left\{-\alpha rJ(z,t)\right\}$ by omitting all the harmonics $\exp\left\{in(\omega t - \beta z)\right\}$ of $J(z,t)$ with $n \neq 1$. This approximation is justifiable in the simple beam-loading theory as discussed in TN-63-6.

7. An Auxiliary Function

For subsequent analyses we find it convenient to introduce the auxiliary function $\psi(w)$ such that, if

$$w = \int_0^u \frac{dz}{v_g'}, \tag{12a}$$

$v_g' = v_g/(1 - v_g/c)$, $0 \leq u \leq \ell$, then

$$u = \psi(w). \tag{12b}$$

Using Eq. (5) we obtain

$$(1 - 2\alpha_o u) \exp(2\alpha_o v_{go} u/c) = \exp(- 2\alpha_o v_{go} w).$$

Since $2\alpha_o u \leq 1$ and $v_{go}/c \ll 1$, the foregoing equation gives, on omitting the higher order terms of Lagrange's series expansion,

$$\psi(w) \cong \frac{1}{2\alpha_o} \left\{ \left(1 - e^{-2\alpha_o v_{go} w}\right) + \frac{v_{go}}{c} \left(e^{-2\alpha_o v_{go} w} - e^{-4\alpha_o v_{go} w}\right) \right\} \tag{13}$$

and

$$\frac{d\psi}{dw} \cong v_{go} \left\{ e^{-2\alpha_o v_{go} w} - \frac{v_{go}}{c} \left(e^{-2\alpha_o v_{go} w} - 2e^{-4\alpha_o v_{go} w}\right) \right\} . \tag{14}$$

8. The Forward E_z Field

Assuming that the phase angle θ in Eq. (10) is zero, we obtain by integrating Eq. (9)

$$E^+_{za}(z,t) = E_z(0)U(z)U\left(t + \tau - \int_0^z \frac{dz}{v_g}\right) f\left(t + \tau - \int_0^z \frac{dz}{v_g}\right) e^{i(\omega t - \beta z)} \tag{15}$$

and

$$E^+_{zb}(z,t) = - \alpha_o r\bar{J}\, e^{i(\omega t-\beta z)} \cdot U(z)U\left(t - \frac{z}{c}\right) \cdot \left[z - U\left(\int_0^z \frac{dz}{v_g} - t\right) \cdot \left\{ z - \psi\left(t - \frac{z}{c}\right)\right\}\right]. \tag{16}$$

The resultant field $E_{z,+}(z,t)$ is the sum of the two fields given above.

To facilitate comparison we reproduce here the corresponding field expressions for constant-α linacs:

$$E^+_{za}(z,t) = E_z(0)U(z)U\left(t + \tau - \frac{z}{v_g}\right) f\left(t + \tau - \frac{z}{v_g}\right) \cdot e^{-\alpha z + i(\omega t - \beta z)}$$

$$E^+_{zb}(z,t) = - r\bar{J}\, e^{i(\omega t - \beta z)} \cdot U(z)U\left(t - \frac{z}{c}\right)$$

$$\cdot \left[\left(1 - e^{-\alpha z}\right) - U\left(\frac{z}{v_g} - t\right) \cdot \left\{ e^{-\alpha v_g'(t - z/c)} - e^{-\alpha z}\right\}\right].$$

9. Electron Energy Gain

The electron energy gain $V_\ell(t)$ in length ℓ of the linac waveguide is given by Eq. (11) in TN-62-69. This consists of two parts; $V_\ell(t) = V_{\ell a}(t) + V_{\ell b}(t)$. For the sake of convenience, we again define the quantities $L_a(t) = V_{\ell a}(t)/E_z(0)$, $L_b(t) = V_{\ell b}(t)/E_z(0)$, and $L(t) = V_\ell(t)/E_z(0)$ as in TN-62-78 and consider the dimensionless quantity

$$\alpha_o L(t) = \alpha_o L_a(t) + \alpha_o L_b(t). \tag{17}$$

Substituting Eqs. (15) and (16) given above into Eq. (11) of TN-62-69 and simplifying the resulting equation we obtain

$$\alpha_o L_a(t) = \int_{t+\tau-t_F'}^{t+\tau} U(\xi) f(\xi) \alpha_o \psi'(t + \tau - \xi) d\xi \tag{18}$$

and

$$\alpha_o L_b(t) = -\chi U(t) \cdot \left[\int_o^{\alpha_o \ell} \alpha_o z d(\alpha_o z) - U(t_F' - t) \int_o^{t_F'-t} \alpha_o \psi'(t + \eta) \left\{ \alpha_o \psi(t + \eta) - \alpha_o \psi(t) \right\} d\eta \right]. \tag{19}$$

In these two equations, $\chi = r\bar{J}/E_z(0)$, $t_F' = \int_o^\ell \frac{dz}{v_g'}$, and $\psi'(w) = d\psi(w)/dw$. The quantity t_F' may be called the effective filling time to distinguish it from the filling time $t_F = \int_o^\ell \frac{dz}{v_g}$. Evidently, $\ell = \psi(t_F')$. Eq. (19) can easily be integrated. Thus,

$$\alpha_o L_b(t) = -U(t) \frac{\chi}{2} \cdot \left[(\alpha_o \ell)^2 - U(t_F' - t) \cdot \left\{ \alpha_o \ell - \alpha_o \psi(t) \right\}^2 \right]. \tag{20}$$

For comparison, we may note the following corresponding equations for the case of constant α:

$$\alpha L_a(t) = \int_{t+\tau-t_F'}^{t+\tau} U(\xi)f(\xi)\alpha v_g' \, e^{-\alpha v_g'(t+\tau-\xi)} d\xi .$$

$$\alpha L_b(t) = -\chi U(t)\left[\int_0^{\alpha \ell} \left(1 - e^{-\alpha z}\right) d(\alpha z) - U(t_F' - t) \int_0^{t_F'-t} \alpha v_g' \left\{e^{-\alpha v_g' t} - e^{-\alpha v_g'(t+\eta)}\right\} d\eta \right] .$$

10. Time Derivative of $\alpha_o L(t)$

It is to be understood that $f(\xi)$ is a continuous function of ξ and $f(0) = 0$. By differentiating Eqs. (18) and (20) with respect to t we obtain

$$\frac{d}{dt}\left\{\alpha_o L_a(t)\right\} = \int_{t+\tau-t_F'}^{t+\tau} U(\xi)f'(\xi)\alpha_o \psi'(t + \tau - \xi)d\xi \tag{21}$$

and

$$\frac{d}{dt}\left\{\alpha_o L_b(t)\right\} = - U(t)U(t_F' - t) \cdot \chi \alpha_o \psi'(t)\left\{\alpha_o \ell - \alpha_o \psi(t)\right\} . \tag{22}$$

11. The Condition for Attaining Constant Electron Energy

Let us restrict the time variable to $t \geq 0$. From $(d/dt)\left\{\alpha_o L(t)\right\} = 0$, we obtain this condition as

$$\int_{t+\tau-t_F'}^{t+\tau} U(\xi)f'(\xi)\alpha_o \psi'(t + \tau - \xi)d\xi = \begin{cases} \chi \alpha_o \psi'(t)\left[\alpha_o \psi(t_F') - \alpha_o \psi(t)\right] , & t \leq t_F' ; \quad (23a) \\ 0, & t \geq t_F' . \quad (23b) \end{cases}$$

From Eq. (23b) it is clear that, when $\xi \geq \tau$, either $f'(\xi)$ must vanish or $\alpha_o \psi'(t + \tau - \xi) f'(\xi)$ must be an oscillating function of period t_F'. In the latter case, the integral of $\alpha_o \psi'(t + \tau - \xi) \cdot f'(\xi)$ over any whole period must vanish. Here we may recall that $f'(\xi)$ satisfies similar conditions for constant-α linacs.

12. The Required Function $f(t + \tau)$ for Constant Energy Gain

For the sake of simplicity, we consider $\tau \geq t_F'$. Under this condition Eq. (23) is not difficult to solve. Let us denote

$$\psi_o(w) = (1/2\alpha_o) \left(1 - e^{-2\alpha_o v_{go} w}\right) \tag{24a}$$

and

$$\psi_o'(w) = v_{go} \; e^{-2\alpha_o v_{go} w} \tag{24b}$$

so that, according to Eqs. (13) and (14),

$$\psi(w) \cong \psi_o(w) + (v_{go}/c) \left\{\psi_o(2w) - \psi_o(w)\right\} \tag{25}$$

and

$$\psi'(w) \cong \psi_o'(w) + (v_{go}/c) \left\{2\psi_o'(2w) - \psi_o'(w)\right\}. \tag{26}$$

Evidently, $f(\xi)$ may also be expanded in powers of the small quantity v_{go}/c. We thus take

$$f(\xi) \cong f_o(\xi) + (v_{go}/c) f_1(\xi) \tag{27a}$$

and

$$f'(\xi) \cong f_o'(\xi) + (v_{go}/c) f_1'(\xi). \tag{27b}$$

In terms of these new functions Eq. (23) may be written as

$$\int_{t+\tau-t_F'}^{t+\tau} d\xi \cdot \Big[f_o'(\xi)\alpha_o\psi_o'(t+\tau-\xi) + \frac{v_{go}}{c}\Big\{ f_1'(\xi)\alpha_o\psi_o'(t+\tau-\xi)$$

$$- f_o'(\xi)\alpha_o\psi_o'(t+\tau-\xi) + 2f_o'(\xi)\alpha_o\psi_o'(2t+2\tau-2\xi)\Big\}\Big]$$

$$= U(t_F'-t)\cdot \chi\alpha_o^2 \cdot \Big[\psi_o'(t)\Big\{\psi_o(t_F') - \psi_o(t)\Big\} - \frac{v_{go}}{c}2\psi_o'(t)\Big\{\psi_o(t_F') - \psi_o(t)\Big\}$$

$$+ \frac{v_{go}}{c}\psi_o'(t)\Big\{\psi_o(2t_F') - \psi_o(2t)\Big\} + \frac{v_{go}}{c}2\psi_o'(2t)\Big\{\psi_o(t_F') - \psi_o(t)\Big\}\Big] . \qquad (28)$$

Since $\psi_o(t_F') - \psi_o(t) = -(1/2\alpha_o v_{go})\Big\{\psi_o'(t_F') - \psi_o'(t)\Big\}$, $\psi_o'(2t) = (1/v_{go})\psi_o'(t)\cdot\psi_o'($

and $\psi_o'(t)\cdot\psi_o'(t_F') = v_{go}\psi_o'(t_F'+t)$, we have

$$\psi_o'(2t)\Big\{\psi_o(t_F') - \psi_o(t)\Big\} = \psi_o'(t)\Big\{\psi_o(t_F'+t) - \psi_o(2t)\Big\} .$$

Thus, every term inside the square brackets on the right-hand side of Eq. (28) contains the common factor $\psi_o'(t)$. Similarly, every term inside the square brackets on the left-hand side of Eq. (28) contains the common factor $\psi_o'(t+\tau-\xi)$, which is equal to $\exp\Big\{-2\alpha_o v_{go}(\tau-\xi)\Big\}\cdot\psi_o'(t)$. Therefore, we can cancel the factor $\alpha_o\psi_o'(t)$ on both sides of Eq. (28). We then get

$$\int_{t+\tau-t_F'}^{t+\tau} d\xi\, e^{-2\alpha_o v_{go}(\tau-\xi)} \cdot \Big[f_o'(\xi) + \frac{v_{go}}{c}\Big\{f_1'(\xi) - f_o'(\xi)$$

$$+ f_o'(\xi)(2/v_{go})\,\psi_o'(t+\tau-\xi)\Big\}\Big] .$$

$$= U(t_F'-t)\cdot\chi\alpha_o \cdot \Big[\Big\{\psi_o(t_F') - \psi_o(t)\Big\} + \frac{v_{go}}{c}\Big\{\psi_o(2t_F') - 3\psi_o(2t)$$

$$+ 2\psi_o(t_F'+t) - 2\psi_o(t_F') + 2\psi_o(t)\Big\}\Big] . \qquad (29)$$

The zeroth-order solution is obtained by neglecting the small terms having the factor v_{go}/c. Noting that

$$\psi_o(t_F') - \psi_o(t) = \int_{t+\tau-t_F'}^{\tau} \psi_o'(t_F' - \tau + \xi)d\xi \ ,$$

we easily find

$$f_o'(\xi) = U(\tau - \xi) \cdot \chi\alpha_o \ e^{2\alpha_o v_{go} t_F'} \cdot \psi_o'(2t_F' - 2\tau + 2\xi). \tag{30}$$

Having obtained $f_o'(\xi)$ we may then solve Eq. (29) for $f_1'(\xi)$ in a straightforward though somewhat tedious manner. By using the zeroth-order solution, Eq. (29) may be reduced to

$$\int_{t+\tau-t_F'}^{t+\tau} e^{-2\alpha_o v_{go}(\tau-\xi)} \cdot f_1'(\xi)d\xi = U(t_F' - t) \cdot \chi\alpha_o \cdot \Big[3\Big\{\psi_o(2t_F') - \psi_o(2t)\Big\} - 2\Big\{\psi_o(2t_F') - \psi_o(t + t_F')\Big\} - \Big\{\psi_o(t_F') - \psi_o(t)\Big\} - 2(t_F' - t)\psi_o'(t + t_F')\Big]. \tag{31}$$

The solution of this equation is as follows:

$$f_1'(\xi) = U(\tau - \xi) \cdot \chi\alpha_o \ e^{2\alpha_o v_{go} t_F'} \cdot \Big[6\psi_o'(3t_F' - 3\tau + 3\xi) - 2\psi_o'(3t_F' - 2\tau + 2\xi) - \psi_o'(2t_F' - 2\tau + 2\xi) - 2\Big\{1 + 2\alpha_o v_{go}(\tau - \xi)\Big\} \psi_o'(3t_F' - 2\tau + 2\xi)\Big]. \tag{32}$$

Noting that $f(\xi)$ is continuous and $f(0) = 0$, we obtain by integrating Eqs. (30) and (32)

$$f_o(\xi) = \frac{\chi\alpha_o}{2} e^{2\alpha_o v_{go}(2\tau - t_F')} \cdot \left[\left\{U(\xi) - U(\xi - \tau)\right\} \cdot \psi_o(2\xi) + U(\xi - \tau)\psi_o(2\tau)\right] \qquad (33)$$

and

$$f_1(\xi) = \frac{\chi\alpha_o}{2} e^{2\alpha_o v_{go}(3\tau - 2t_F')} \cdot \left\{U(\xi) - U(\xi - \tau)\right\} \cdot \left[4\psi_o(3\xi) - 3\left\{\psi_o(2\xi + \tau) - \psi_o(\tau)\right\}\right.$$
$$\left. - \left\{\psi_o(2\xi + \tau - t_F') - \psi_o(\tau - t_F')\right\} + \left\{(2\tau - 2\xi)\psi_o'(2\xi + \tau) - 2\tau\psi_o'(\tau)\right\}\right]$$
$$+ \frac{\chi\alpha_o}{2} e^{2\alpha_o v_{go}(3\tau - 2t_F')} \cdot U(\xi - \tau) \cdot \left[4\psi_o(3\tau) - 3\left\{\psi_o(3\tau) - \psi_o(\tau)\right\}\right.$$
$$\left. - \left\{\psi_o(3\tau - t_F') - \psi_o(\tau - t_F')\right\} - 2\tau\psi_o'(\tau)\right]. \qquad (34)$$

The functional form of $f(\xi) = f_o(\xi) + (v_{go}/c)f_1(\xi)$ is more complicated than the corresponding form in the case of constant-α. For comparison, Eq. (15) in TN-62-78 is rewritten as follows:

$$f(\xi) = \chi e^{\alpha v_g' \tau} \cdot \left[\left\{U(\xi) - U(\xi - \tau)\right\} \cdot \left(1 - e^{-\alpha v_g' \xi}\right) + U(\xi - \tau) \cdot \left(1 - e^{-\alpha v_g' \tau}\right)\right].$$

Now we substitute the foregoing expressions $f_o(\xi)$ and $f_1(\xi)$ into Eq. (18) and carry out the integration to obtain $\alpha_o L_a(t)$. The result is

$$\alpha_o L_a(t) = \alpha_o \ell \cdot f(\tau) - U(t_F' - t) \cdot \frac{\chi}{2}\left\{\alpha_o \ell - \alpha_o \psi(t)\right\}^2. \qquad (35)$$

Thus

$$\alpha_o L(t) = \alpha_o \ell \cdot f(\tau) - \chi \int_0^{\alpha_o \ell} (\alpha_o z) d(\alpha_o z). \qquad (36)$$

This is, as expected, independent of t.

The corresponding expression in the case of constant-α is, according to Eq. (16) in TN-62-78,

$$\alpha L(t) = \left(1 - e^{-\alpha \ell}\right) \cdot f(\tau) - \varkappa \int_0^{\alpha \ell} \left(1 - e^{-\alpha z}\right) d(\alpha z).$$