Entanglement in systems of indistinguishable fermions

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Abstract. The characterization of entanglement is a fundamental issue for Quantum Information Theory. But the definition of entanglement depends on the notion of locality, and thus on the tensor product structure of the state space of the composite system. This notion is affected by the presence of superselection rules that restrict the accessible Hilbert space to a direct sum of subspaces.

Indistinguishability of particles imposes one such restriction, namely to totally symmetric or totally antisymmetric states. The entanglement can in this case be defined with respect to partitions of modes in the second quantization formalism. For fermionic systems the Fock space of m modes is isomorphic to the space of m qubits, but the action of creation and annihilation operators is not local, due to their anticommutation.

Conservation of the parity of fermion number imposes another relevant superselection rule. It requires that local physical observables commute with the local parity operator.

Taking into account the considerations above, it is possible to define the set of separable states or equivalently the concept of entanglement for fermionic systems in a number of ways. Here we analyze systematically these possibilities and the relation among the various sets of separable states. We also discuss the behavior of the different classes when taking several copies of the state, as well as the characterization of the sets in terms of the usual criteria regarding the tensor product.

1. Introduction

Besides being one of the characteristic properties of quantum mechanics, entanglement is a valuable resource for quantum information processing. Entangled states can for instance be used to achieve cryptographic tasks that would not be allowed classically. The goal of entanglement theory is to characterize this resource and to quantify it, as well as to identify how it can be transformed and manipulated.

The concept of entanglement [1] is very much related to the definition of locality. In a composite quantum system, the notion of locality typically relies on the tensor product structure of the total Hilbert space. In the case of indistinguishable particles such structure disappears due to the restriction of the physical states to the completely symmetric or antisymmetric part of the Hilbert space. Something similar happens in the presence of other superselection rules, which affect the concept of locality and thus also that of entanglement [2–6]. Since a superselection rule restricts physical local operations to the ones compatible with the conserved quantity, there exist states that cannot be prepared locally, although they can be written as convex combinations of product states.

In this work we study entanglement in a system of indistinguishable fermions. We consider systems in second quantization, and analyze entanglement between distinguishable sets of modes, or regions in space. Entanglement between particles was studied in [7–9] in first quantization. Other works have dealt with entanglement between modes [10–16].

The goal of the present work is to systematically study the possible definitions of entanglement in this system, given the indistinguishability of fermions, the anticommutation relations and the parity superselection rule, i.e. the conservation of the parity of the number of fermions. We find that the various possible mathematical definitions carry different physical menaings, related to the ability to prepare, use or measure the entanglement.

In the rest of this paper, we review the main results from our study. We discuss how the main separability definitions arise, and emphasize the different meaning of the identified sets. A more complete mathematical treatment, including the proofs of all the relations stated here, can be found in [17].

In section 2 we introduce the basic ingredients for the study, namely the different representations of the system (in terms of fermionic operators or in the Fock space) and the parity superselection rule. The definition of product state, first step in the process of determining what separable states are, is discussed in section 3. From the various sets of product states, separability sets are constructed in section 4. The different sets can be characterized in terms of the usual tensor product criteria, as discussed in section 5, what turns out to be useful in order to determine when a given state belongs to any of the separability sets. An interesting consideration regards the behavior of the various definitions when we analyse the separability of several copies, and in particular, the asymptotic limit. As described in section 6, in this limit the differences among the various definitions seem to vanish. Finally, in section 7 we summarize the results of the work.

2. Basic concepts

We will be studying systems formed by a finite number, m, of fermionic modes. There, we define a bipartition in two subsets $A = 1, \dots m_A$ and $B = m_A + 1, \dots m$. The mathematical objects describing this system are creation and annihilation operators, $\{a_1, a_1^\dagger, \dots a_m, a_m^\dagger\}$, satisfying canonical anticommutation relations, $\{a_k, a_l^\dagger\} = \delta_{kl}$. They generate the algebra of all observables. Correspondingly, we define the subalgebras $\mathcal{A}$ and $\mathcal{B}$ as the ones generated by the subsets of modes A and B.

The fermionic system can also be described in the Fock representation, defined by

$$|n_1, \dots n_m\rangle = (a_1^\dagger)^{n_1} \dots (a_m^\dagger)^{n_m} |0\rangle, \tag{1}$$

where n_k is the occupation number of mode number k. The Jordan-Wigner transformation,

$$a_k^\dagger = \prod_{i=1}^{k-1} \sigma_z^{(i)} \sigma_+^{(k)}, \quad a_k = \prod_{i=1}^{k-1} \sigma_z^{(i)} \sigma_-^{(k)}, \tag{2}$$

maps fermionic creation and annihilation operators onto Pauli spin operators. The Fock space corresponding to m fermionic modes is then isomorphic to a m-qubit space. However, the action of fermionic operators in this space is not local, due to the anticommutation relations. Therefore, the operators in the subalgebras $\mathcal{A}$ and $\mathcal{B}$ cannot be considered local to the corresponding partitions.

One fundamental ingredient in the systems of fermions that we considered is the conservation of the parity of the fermionic number,

$$\hat{P} = \prod_k (1 - 2a_k^\dagger a_k).$$

Table 1. Relations among sets of physical product states.

Set	Definition	Relation	Example in 1×1
$\mathcal{P}1$	$\rho(A_\pi B_\pi) = \rho(A_\pi)\rho(B_\pi)$	$\mathcal{P}2_\pi \subset \mathcal{P}1_\pi$	$\rho_{\mathcal{P}1} = \frac{1}{16}\begin{pmatrix} 9 & 0 & 0 & -i \\ 0 & 3 & -i & 0 \\ 0 & i & 3 & 0 \\ i & 0 & 0 & 1 \end{pmatrix} \in \mathcal{P}1_\pi \setminus \mathcal{P}2_\pi$
$\mathcal{P}2$	$\rho = \rho_A \otimes \rho_B$		
$\mathcal{P}3$	$\rho(A\,B) = \rho(A)\rho(B)$	$\mathcal{P}3_\pi = \mathcal{P}2_\pi$	$\rho_{\mathcal{P}2} = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix} \otimes \begin{pmatrix} b & 0 \\ 0 & 1-b \end{pmatrix} \in \mathcal{P}2_\pi$

The conservation of $\hat{P}$ requires that physical states have either an even or an odd number of fermions, forbidding any coherent superposition of both. Therefore, the physical Hilbert space is the direct sum of the even and odd subspaces, and all physical observables must commute with the operator $\hat{P}$. The set of physical states is the given by

$$\Pi := \{\rho : [\rho, \hat{P}] = 0\},$$

while $\mathcal{A}_\pi$ and $\mathcal{B}_\pi$ will designate the sets of local physical observables, commuting with the local parity operators $\hat{P}_A$ and $\hat{P}_B$, respectively.

We will say that an observable is *even* (*odd*) if it commutes (anticommutes) with $\hat{P}$, and can then be decomposed as a sum of products of an even (odd) number of fermionic creation and annihilation operators. Notice that even (odd) operators do not correspond to the even (odd) eigenspace of $\hat{P}$. We can define also the projectors onto the eigenspaces of well-defined parity, $\mathbb{P}_{e(o)}$. In terms of them, any physical state can be written as $\rho = \mathbb{P}_e \rho \mathbb{P}_e + \mathbb{P}_o \rho \mathbb{P}_o$, with a block-diagonal structure, and the same is true, in the local subspaces, for parity conserving observables, $A_\pi = \mathbb{P}_e^A A_\pi \mathbb{P}_e^A + \mathbb{P}_o^A A_\pi \mathbb{P}_o^A$.

3. Product states

Since separable states are defined as convex combinations of product states [1], the first step in our analysis must be the definition of product states. Given the above considerations, we could study the entanglement of our system both in the Fock space representation or at the level of the operator subalgebras. In the first case, exploiting the isomorphism to a qubit system, separability can be studied with respect to the tensor product $\mathbb{C}^{2m_A} \otimes \mathbb{C}^{2m_B}$. In terms of the operator subalgebras, there are two possibilities. We could study entanglement between subalgebras $\mathcal{A}$ and $\mathcal{B}$, but they do not commute, and the corresponding observables act in general non-locally in the Fock space. From this argument, it is logical to consider entanglement between $\mathcal{A}_\pi$ and $\mathcal{B}_\pi$, i.e. the restriction of both subalgebras to parity conserving operators. These subalgebras are mutually commuting and can be considered local to each partition.

Using all the criteria above, we may define three fundamental sets of product states. Here we focus only on the relations among physical sets, which are summarized in Table 1 (for the whole treatment of physical and non-physical states, see [17]).

- We may say that a state is a product if the expectation value for all products of local observables factorizes. We call this set $\mathcal{P}1$. Formally,

$$\mathcal{P}1 \;:=\; \{\rho : \rho(A_\pi\, B_\pi) = \rho(A_\pi)\rho(B_\pi) \quad \forall A_\pi \in \mathcal{A}_\pi,\, B_\pi \in \mathcal{B}_\pi\}\,. \tag{3}$$

- A product state can also be defined at the level of the Fock representation. As for the isomorphic m-qubit system, we may then say that a product state is that which can be written as a tensor product.

$$\mathcal{P}2 := \{\rho : \rho = \rho_A \otimes \rho_B\}\,. \tag{4}$$

- If we want to look at the entanglement between $\mathcal{A}$ and $\mathcal{B}$, we may ignore the restriction to parity conserving operators, and define product states $\mathcal{P}3$ as those for which any product of observables factorize [18], namely

$$\mathcal{P}3 := \{\rho : \rho(A\,B) = \rho(A)\rho(B) \quad \forall A \in \mathcal{A},\, B \in \mathcal{B}\}\,. \tag{5}$$

The three sets above are strictly different, but if we look only at physical states, i.e. those commuting with the operator $\hat{P}$, we find that

$$\mathcal{P}3_\pi = \mathcal{P}2_\pi \subset \mathcal{P}1_\pi, \tag{6}$$

where the subindex π indicates the interesection of the sets defined above with the set of physical states, Π.

It is also worth noticing that the difference between $\mathcal{P}2_\pi$ and $\mathcal{P}1_\pi$ vanishes in the case of pure states.

4. Separable states

Separable states are those that can be written as a convex combination of product states. Therefore we may construct different separability sets by simply taking the convex hull of each one of the sets defined above,

- $\mathcal{S}1_\pi := \mathrm{co}\,(\mathcal{P}1_\pi) = \{\rho \in \Pi,\, \rho = \sum \lambda_k \rho_k,\, \rho_k(A_\pi\, B_\pi) = \rho_k(A_\pi)\rho_k(B_\pi),\, \forall A_\pi \in \mathcal{A}_\pi,\, B_\pi \in \mathcal{B}_\pi\}$,
- $\mathcal{S}2_\pi := \mathrm{co}\,(\mathcal{P}2_\pi) = \{\rho \in \Pi,\, \rho = \sum \lambda_k \rho_k^A \otimes \rho_k^B,\, [\rho_k^{A(B)}, \hat{P}_{A(B)}] = 0\}$.

None of these definitions corresponds to the usual definition of separability in the Fock space, as we would get by just importing the definition for a system of m-qubits. This indeed gives a third separability set,

- $\mathcal{S}2'_\pi := \mathrm{co}\,(\mathcal{P}2_\pi) = \{\rho \in \Pi,\, \rho = \sum \lambda_k \rho_k^A \otimes \rho_k^B\}$.

Notice that in the definition of $\mathcal{S}2'_\pi$, no restriction from the conservation of $\hat{P}$ is imposed to the factors in the convex decomposition, whereas $\mathcal{S}2_\pi$ includes such a restriction. Therefore, $\mathcal{S}2_\pi$ contains all states preparable by means of local operations and classical communication in the presence of the superselection rule (LOCC_S), whereas in $\mathcal{S}2'_\pi$ there will be states that cannot be locally prepared.

We may think of a setting in which the only accessible information about the state is the result of local measurements. In such situation, it is not possible to distinguish states which produce the same expectation values for all physical local operators. It makes then sense to say that such states are equivalent, and define thus an equivalence relation between states,

$$\rho_1 \sim \rho_2 \quad \text{if} \quad \rho_1(A_\pi B_\pi) = \rho_2(A_\pi B_\pi) \;\; \forall A_\pi \in \mathcal{A}_\pi,\, \mathcal{B}_\pi \in \mathcal{B}_\pi.$$

As discussed above, when parity conservation is in play, the only states that can be prepared locally are those in $\mathcal{S}2_\pi$. On the other hand, locally accessible observables are quantities of the form $\rho(A_\pi B_\pi)$. It is then natural to define a state as separable if it is equivalent to a state that can be locally prepared. This would define the set of separable states as the equivalence class of $\mathcal{S}2_\pi$ with respect to the relation $\sim$,

Table 2. The different sets of separable states and their relations.

Set	Characterization	Relations	Example in 1×1
$[\mathcal{S}2_\pi]$	$\sum_{\alpha,\,\beta=e,\,o} \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \rho \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \in \mathcal{S}2'_\pi$	$\mathcal{S}1_\pi \subset [\mathcal{S}2_\pi]$	$\rho_{[\mathcal{S}2_\pi]} = \frac{1}{15}\begin{pmatrix} 5 & 0 & 0 & 2\sqrt{5} \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 2\sqrt{5} & 0 & 0 & 4 \end{pmatrix}$ $\in [\mathcal{S}2_\pi] \setminus \mathcal{S}1_\pi$
$\mathcal{S}1_\pi$	$\rho = \sum_k \lambda_k \rho_k, \quad s.t.$ $\sum_{\alpha,\,\beta=e,\,o} \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \rho_k \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \in \mathcal{S}2'_\pi$	$\mathcal{S}2'_\pi \subset \mathcal{S}1_\pi$	For the 1×1 case, $\mathcal{S}1_\pi = \mathcal{S}2'_\pi$. Therefore examples of $\mathcal{S}1_\pi \setminus \mathcal{S}2'_\pi$ can only be found in bigger systems, f.i. 2×2 modes.
$\mathcal{S}2'_\pi$	$\rho = \sum_k \lambda_k \rho^k_A \otimes \rho^k_B$	$\mathcal{S}2_\pi \subset \mathcal{S}2'_\pi$	$\rho_{\mathcal{S}2'_\pi} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\in \mathcal{S}2'_\pi \setminus \mathcal{S}2_\pi$
$\mathcal{S}2_\pi$	$\mathbb{P}_e \rho \mathbb{P}_e \in \mathcal{S}2'_\pi$, $\mathbb{P}_o \rho \mathbb{P}_o \in \mathcal{S}2'_\pi$		$\rho \in \mathcal{S}2_\pi \Leftrightarrow \rho$ diagonal

- $[\mathcal{S}2_\pi] := \{\rho \in \Pi,\, \exists \tilde{\rho} \in \mathcal{S}2_\pi,\, \rho \sim \tilde{\rho}\}$.

It is also possible to define the equivalence classes corresponding to the other sets of separable states ($[\mathcal{S}1_\pi]$, $[\mathcal{S}2'_\pi]$), but it is easy to show that they are all equivalent to $[\mathcal{S}2_\pi]$ [17].

Therefore, we have defined four sets of separable states. It turns out that they are strictly different and

$$\mathcal{S}2_\pi \subset \mathcal{S}2'_\pi \subset \mathcal{S}1_\pi \subset [\mathcal{S}2_\pi]. \tag{7}$$

The non-strict inclusions are easy to show, and table 2 shows some examples of the strict character of some of the relations (see [17] for the complete proofs).

These four sets correspond to four classes of states, each one strictly including the preceding class.

(i) $\mathcal{S}2_\pi$ contains states that can be prepared by means of local operations, when these are restricted by parity conservations, plus classical communication.

(ii) $\mathcal{S}2'_\pi$ contains states that are writable as convex combinations of product states in the Fock representation.

(iii) $\mathcal{S}1_\pi$ includes convex combinations of states for which all products of locally measurable observables factorize.

(iv) $[\mathcal{S}2_\pi]$ contains states such that all locally measurable correlations can be reproduced by a separable state from any of the classes above.

5. Characterization

The separability sets defined in the previous section can be characterized in terms of the usual mathematical criteria, i.e. with respect to the tensor product. This allows one to make use of standard separability conditions [19] in order to decide the membership of a given state with respect to each of these classes. Table 2 shows these characterizations.

The standard separability concept [1] corresponds to set $\mathcal{S}2'_\pi$, applied to parity preserving states only. The set $\mathcal{S}2_\pi$, on the other hand, is formed by states which can be decomposed in terms of tensor products, where every factor commutes with the local version of the parity operator. Since any physical state has a block diagonal structure $\rho = \mathbb{P}_e\rho\mathbb{P}_e + \mathbb{P}_o\rho\mathbb{P}_o$, we can make use of this expression to conclude that each block must have an independent decomposition in the sense of the tensor product. Therefore a state will be in $\mathcal{S}2_\pi$ iff both $\mathbb{P}_e\rho\mathbb{P}_e$ and $\mathbb{P}_o\rho\mathbb{P}_o$ are in $\mathcal{S}2'_\pi$.

It can be seen that for a state to be in $\mathcal{P}1_\pi$, all of its diagonal blocks must be a tensor product [17],

$$\sum_{\alpha,\,\beta=e,\,o} \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \rho \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta = \tilde{\rho}_A \otimes \tilde{\rho}_B \in \mathcal{P}2_\pi.$$

The set $\mathcal{S}1_\pi$ is then characterized as the convex hull of $\mathcal{P}1_\pi$, i.e. it is formed by convex combinations of states that can be written as the sum of a parity preserving tensor product plus some off–diagonal terms.

Finally, the equivalence class $[\mathcal{S}2_\pi]$ is defined in terms of the expectation values of observable products $A_\pi B_\pi$. These have no contribution from off-diagonal blocks in ρ, so that the class can be characterized in terms of the diagonal blocks alone. A state ρ is then in $[\mathcal{S}2_\pi]$ if and only if

$$\sum_{\alpha,\,\beta=e,\,o} \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \rho \mathbb{P}^A_\alpha \otimes \mathbb{P}^B_\beta \in \mathcal{S}2'_\pi. \tag{8}$$

The condition involves only the block diagonal part of the state, and is thus equivalent to the individual separability (with respect to the tensor product) of each of the blocks.

6. Asymptotic behavior

All the definitions introduced in the previous sections apply only to a single copy of the fermionic state. In this section we are instead concerned about the stability of the various criteria when several copies are taken into account. In particular, it is relevant to understand their behavior in the asymptotic limit $N \to \infty$.

The conditions for $\mathcal{S}2'_\pi$ and $\mathcal{S}2_\pi$ are stable when taking several copies of the state.

$$\rho^{\otimes 2} \in \mathcal{S}2'_\pi \iff \rho \in \mathcal{S}2'_\pi,$$
$$\rho^{\otimes 2} \in \mathcal{S}2_\pi \iff \rho \in \mathcal{S}2_\pi.$$

Moreover, in [4] it was shown that both definitions are asymptotically equivalent, as the entanglement cost of $\mathcal{S}2_\pi$ converges to that of $\mathcal{S}2'_\pi$ in the limit $N \to \infty$.

On the contrary, $\mathcal{S}1_\pi$ and $[\mathcal{S}2_\pi]$ are not stable when we take two copies of the state. It is possible to have a state $\rho \in \mathcal{S}1_\pi$ ($[\mathcal{S}2_\pi]$) such that $\rho^{\otimes 2}$ is not separable according to the same criterion. However, the opposite sense of the implication holds.

$$\rho^{\otimes 2} \in \mathcal{S}1_\pi \Rightarrow \rho \in \mathcal{S}1_\pi,$$
$$\rho^{\otimes 2} \in [\mathcal{S}2_\pi] \Rightarrow \rho \in [\mathcal{S}2_\pi].$$

Moreover, it is also possible to prove that separability of two copies according to $[\mathcal{S}2_\pi]$ implies positivity of the partial transpose (PPT) of ρ,

$$\rho^{\otimes 2} \in [\mathcal{S}2_\pi] \Rightarrow \rho \text{ PPT}.$$

Therefore, an non-positive partial transpose (NPPT) of ρ implies that the state is also non-separable according to the broadest definition $[\mathcal{S}2_\pi]$ when one takes several copies. In particular, this is true for distillable states [20, 21]. This suggests that the differences between the various definitions of separability may vanish in the asymptotic regime. The strict equivalence of the classes in this limit, however, is proved only for the simplest case of only 1×1 fermionic modes, described in detail in [17].

7. Conclusions

We have discussed how, in a system of indistinguishable fermions, the anticommutation rules of the fermionic operators and the presence of the parity superselection rule give rise to several reasonable definitions of separability. In general, we proceed by defining first the product states and taking then convex hulls of them to construct the separability sets.

The various definitions of product states reduce in the case of physical systems to only two different sets, $\mathcal{P}2_\pi$, or products in the Fock representation, and $\mathcal{P}1_\pi$, or states whose locally measurable correlations can be reproduced by some state in $\mathcal{P}2_\pi$. For the definition of separability we are left with four reasonable classes of separable states, each one associated with different physical capabilities of preparing or measuring the states. $\mathcal{S}2_\pi$ represents states that can be prepared by $\mathrm{LOCC_S}$. The usual definition of separability in the Fock representation is represented by $\mathcal{S}2'_\pi$, which contains states writable as convex combinations of tensor products but not necessarily preparable in a local way. The set $\mathcal{S}1_\pi$ contains the convex combinations of states such that the expectation values of all local observables factorize. The last class, $[\mathcal{S}2_\pi]$, is characterized by locally measurable correlations reproducible by a state that can be prepared locally. In order to be able to apply the usual separability criteria to determine whether a state is or not within each of these classes, we have also characterized each set in terms of the tensor product.

If we analyse the separability of several copies of the state, several features are to be noticed. First of all, while the definitions of $\mathcal{S}2_\pi$ and $\mathcal{S}2'_\pi$ are stable under taking several copies, the same does not hold for $\mathcal{S}1_\pi$ and $[\mathcal{S}2_\pi]$. On the other hand, in the asymptotic limit the differences among the various definitions seem to vanish. In particular, it is known that $\mathcal{S}2_\pi$ and $\mathcal{S}2'_\pi$ become equivalent for a large number of copies and, in the case of 1×1 modes, i.e. the smallest possible system, the equivalence is true for all the four classes. However whether all the classes of separable states collapse to a single one in the asymptotic regime for a general system remains an open question.

References

[1] Werner R 1989 *Phys. Rev. A* **40** 4277
[2] Verstraete F and Cirac J I 2003 *Phys. Rev. Lett.* **91** 010404 (*Preprint* arXiv:quant-ph/0302039)
[3] Schuch N, Verstraete F and Cirac J I 2004 *Phys. Rev. Lett.* **92** 087904 (*Preprint* arXiv:quant-ph/0310124)
[4] Schuch N, Verstraete F and Cirac J I 2004 *Phys. Rev. A* **70** 042310 (*Preprint* arXiv:quant-ph/0404079)
[5] Bartlett S D and Wiseman H M 2003 *Phys. Rev. Lett.* **91** 097903 (*Preprint* arXiv:quant-ph/0303140)
[6] Caban P, Podlaski K, Rembielinski J, Smolinski K A and Walczak Z 2005 *J. Phys. A: Math. Gen.* **38** L79–L86 (*Preprint* arXiv:quant-ph/0405108)
[7] Schliemann J, Cirac J I, Kuś M, Lewenstein M and Loss D 2001 *Phys. Rev. A* **64** 022303 (*Preprint* arXiv:quant-ph/0012094)
[8] Li Y S, Zeng B, Liu X S and Long G L 2001 *Phys. Rev. A* **64** 054302 (*Preprint* arXiv:quant-ph/0104101)
[9] Lévay P, Nagy S and Pipek J 2005 *Phys. Rev. A* **72** 022302 (*Preprint* arXiv:quant-ph/0501145)
[10] Zanardi P 2002 *Phys. Rev. A* **65** 042101 (*Preprint* arXiv:quant-ph/0104114)
[11] Zanardi P and Wang X 2002 *J. Phys. A: Math. Gen.* **35** 7947–7959 (*Preprint* arXiv:quant-ph/0201028)
[12] Anfossi A, Giorda P, Montorsi A and Traversa F 2005 *Phys. Rev. Lett.* **95** 056402 (*Preprint* arXiv:cond-mat/0502500)
[13] Larsson D and Johannesson H 2006 *Phys. Rev. A* **73** 042320 (*Preprint* arXiv:quant-ph/0602048)
[14] Wolf M M 2006 *Phys. Rev. Lett.* **96** 010404 (*Preprint* arXiv:quant-ph/0503219)
[15] Gioev D and Klich I 2006 *Phys. Rev. Lett.* **96** 100503 (*Preprint* arXiv:quant-ph/0504151)
[16] Cramer M, Eisert J and Plenio M B Statistics dependence of the entanglement entropy (*Preprint* arXiv:quant-ph/0611264)
[17] Bañuls M C, Cirac J I and Wolf M M 2007 *Physical Review A (Atomic, Molecular, and Optical Physics)* **76** 022311 (pages 13) URL http://link.aps.org/abstract/PRA/v76/e022311
[18] Moriya H On separable states for composite systems of distinguishable fermions (*Preprint* arXiv:quant-ph/0405166)
[19] Horodecki R, Horodecki P, Horodecki M and Horodecki K Quantum entanglement (*Preprint* arXiv:quant-ph/0702225)

[20] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev. A* **54** 3824
[21] Horodecki M, Horodecki P and Horodecki R 1998 *Phys. Rev. Lett.* **80** 5239–5242 (*Preprint* quant-ph/9801069)