CONFINING POTENTIAL IN COULOMB-GAUGE LATTICE QCD*

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In the Coulomb-gauge QCD, there exists an instantaneous chromoelectric interaction between static quark–antiquark pairs. The confining nature of this 'Coulomb potential' was hypothesized as the main contributor to the large-distance behavior of the Wilson loop in the non-gauge fixed QCD. We examine the existing definitions of this interaction in the SU(2) and SU(3) Yang–Mills theory on anisotropic lattices by performing the Hamiltonian limit. We find an artificial enhancement of the corresponding string tension and suggest a corrected definition of the potential.

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1. Introduction

Since the discovery of hadron resonances in the 1950s, hadron spectroscopy has developed into a rich field of study. More recent discoveries, such as the discovery of the $X(3872)$ by the Belle Collaboration, reinvigorated the field, as exotic hadrons with quantum numbers not allowed by quark models were measured. Significant progress in improving experimental apparatus and analysis of the data has allowed for the discovery of dozens of new hadrons; organizing and classifying them into a system that makes

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sense physically is now an incredibly daunting task [1]. Determination of the resonant- and bound-state properties directly from QCD is difficult due to its non-perturbative nature, and none of the known analytic methods have yet offered a systematic solution to that problem. This task can be accomplished numerically using lattice QCD (LQCD), which led to many fruitful results; for a review of recent progress, see [2]. However, LQCD alone cannot provide a complete picture of the internal dynamics of hadrons, including the quark confinement phenomenon.

The precise mechanism of confinement is not well understood. Moreover, *confinement* can take on different meanings in different contexts [3]. For static heavy quarks, we define it as the existence of the linear potential, as computed in a gauge-invariant way from the large Wilson loops in LQCD [4]. It is parameterized as

$$V(r) = A + \frac{B}{r} + \sigma r\,, \tag{1}$$

where σ is the string tension. To understand the emergence of this potential in QCD and gain some measure of physical intuition, we explore the Hamiltonian formulation of the theory in the Coulomb gauge [5].

In this approach, one finds a useful analogy between the Hamiltonian of QCD and QED due to the existence of the term describing instantaneous interaction between charged objects, H_{C} [1]. In QED, H_{C} gives rise to diagrams such as the one depicted in Fig. 1 (a), which correspond to classical $1/r$ Coulomb potential in electrodynamics. In the non-Abelian theory, H_{C} includes additional coupling to gluons, which can lead to diagrams like the one in Fig. 1 (b). These gluon interactions change the potential's large-distance behavior which, in addition to the $1/r$ dependence, acquires a term linear in r [8]. This resembles the confining force exhibited in Eq. (1).

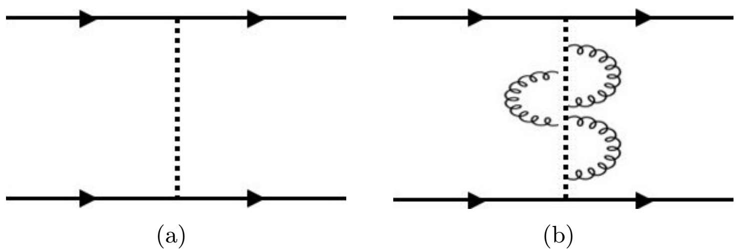

Fig. 1. Diagrams contributing to the Coulomb potential from H_{C} in the Coulomb-gauge QED (diagram (a)) and QCD (diagram (b)) Hamiltonian. Time flows in the horizontal direction and the dashed line represents the instantaneous interaction.

[1] It is worth noting that the Coulomb gauge also has a unique feature, not shared by covariant gauges, that the combination $g\,A^0$ is a renormalization-group invariant [6, 7].

The instantaneous Coulomb potential, $V_{\mathrm{C}}(r)$, is defined as the expectation value of H_{C} on the state of a bare, static $q\bar{q}$ pair with a spatial separation r [2]. It describes an interaction energy between a static quark–antiquark pair annihilated immediately after its creation, before the dynamical quark–gluon interactions antipolarize the QCD vacuum around the pair.

The Coulomb potential and related chromo-electric energy density profile have been studied extensively, both analytically and on the lattice [8–22], but no precise measurement of the related (Coulomb) string tension, σ_{C}, has been performed. Precise calculation is difficult due to the definition of near-simultaneity for the creation and annihilation process of static sources. To approach the continuum value of an observable in LQCD, the finite lattice spacing a forces one to take a limit $a \to 0$ that is free of discretization artifacts. This technical step becomes a fundamental requirement in the study of V_{C} since a sets a lower bound on the minimal (Euclidean) times that can be studied. To define the Coulomb potential, and hence σ_{C}, one must investigate the limit in which the lattice spacing shrinks to zero in order to ensure that only the instantaneous part of the QCD Hamiltonian is probed.

Previous measurements on the lattice found that the Coulomb potential in SU(2) and SU(3) Yang–Mills theory has a string tension two to three times larger than the Wilson string tension [13, 19, 23]. However, these studies were performed with a limited consideration of the continuum limit. In this contribution, we report on a precise study of the Hamiltonian limit (time spacing $a_{\mathrm{t}} \to 0$) of the Coulomb potential in LQCD by using anisotropic lattices of large sizes in the time-like direction. Using the widely accepted lattice definition of the observable, we extract the trajectory of the Coulomb string tension in this limit and find it is ill-defined. Surprisingly, we find that the available definition leads to a superficial enhancement of the string tension that grows indefinitely as the lattice spacing shrinks.

We describe this finding in the following sections. First, we define the observable of interest and describe our lattice set-up. We present a preliminary result of our calculation and suggest a hypothetical resolution to the problem.

2. Coulomb potential in LQCD

In our study, we consider the Euclidean lattice formulation of the SU(N) Yang–Mills theory (for $N = 2, 3$) with no dynamical fermions. The basic

[2] As such, it is not an eigenvalue of the full theory. For comparison, the true ground state defines the Wilson potential, $V(r)$, between the "dressed" static quark and an antiquark. Thus, the Coulomb potential provides an upper bound on $V(r)$, and the lack of confining term in $V_{\mathrm{C}}(r)$ would imply no confinement in Eq. (1), see [9].

degree of freedom, the gauge link $U_\mu(n) = \exp ia A_\mu(n)$, is an SU($N$) matrix defined at a discrete position $n = (n_\mathrm{t}, \vec{n})$ on the lattice that points in a direction $\mu = 0, \ldots, 4$. The theory is defined via the Wilson action for anisotropic lattices

$$S = \sum_x \left[\frac{\beta_\mathrm{s}}{N} \sum_{j>i=1}^{3} \mathrm{Re}\,\mathrm{Tr}\,(1 - U_{ij}(x)) + \frac{\beta_\mathrm{t}}{N} \sum_{i=1}^{3} \mathrm{Re}\,\mathrm{Tr}\,(1 - U_{i0}(x)) \right], \tag{2}$$

where

$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n) \tag{3}$$

is a plaquette oriented in the $(\mu\nu)$ plane and placed at position n. Here, $\beta_\mathrm{s} = \beta/\xi_0$ and $\beta_\mathrm{t} = \beta\xi_0$ are coupling constants introduced to alter the shape of the lattice in physical units, in the spatial and temporal directions, respectively. The quantity ξ_0 is the bare anisotropy and is related to the renormalized anisotropy, $\xi = a_\mathrm{s}/a_\mathrm{t}$, via a non-perturbative relation $\xi_0 \equiv \xi_0(\xi, \beta)$ [23–26].

The lattice definition of the Coulomb potential was derived by Greensite and Olejnik in Ref. [13]. It is given by the expectation value of the correlator of very short time-like Wilson lines $L(T, \vec{r})^\dagger$ and $L(T, \vec{0})$. These are lines of length $a_\mathrm{t} T$ (*i.e.*, formed by the product of T link variables) oriented in the temporal direction and located at positions $(t = 0, \vec{r})$ and $(t = 0, \vec{0})$, as shown in Fig. 2. Since they must be short, on the lattice they become just single time-like links, and the potential is given simply as

$$V_C(r) = -\lim_{a_\mathrm{t} \to 0} \frac{1}{a_\mathrm{t}} \log \left\langle \frac{1}{N} \mathrm{Tr} \left[U_0\left(0, \vec{0}\right) U_0^\dagger(0, \vec{r}) \right] \right\rangle . \tag{4}$$

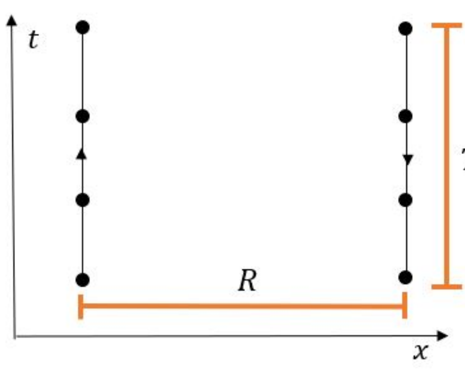

Fig. 2. Time-like Wilson lines $L(T, \vec{0})$ and $L(T, \vec{x})^\dagger$ on the lattice.

We can interpret this physically as an interaction energy corresponding to a $q\bar{q}$ pair popping into the QCD vacuum at some distance $r = a_{\mathrm{s}} R$ away from each other, propagating for one unit of lattice time, a_{t}, and then annihilating. In the limit where the length of these time-like Wilson lines, $a_{\mathrm{t}} T$, becomes infinite, we should recover the Wilson potential as we are giving time for the $q\bar{q}$ pair to exchange gluons with each other and with the vacuum, allowing it to reach the ground state. This behavior was indeed verified in Ref. [13].

On the other hand, if we pop these quark–antiquark pairs into existence and allow no time for them to propagate, they have no time to exchange gluons and the observable we measure corresponds to an instantaneous interaction between the charges.

3. Lattice framework and results

Simulations are still in progress at the time of this publication. The limited set of data presented here was obtained for $\beta = 2.25$ and lattice size $V = 24^3 \times 96$, with anisotropy $\xi = 1, \ldots, 7$ and periodic boundary conditions. This corresponds to spatial lattice spacing $a_{\mathrm{s}} \in [0.206\ \mathrm{fm}, 0.236\ \mathrm{fm}]$ and temporal lattice spacing $a_{\mathrm{t}} \in [0.206\ \mathrm{fm}, 0.0295\ \mathrm{fm}]$. Lattice configurations were generated using the heat-bath algorithm, and were considered to be thermalized after 1000 initial sweeps for each β and ξ. The data presented here was taken from 2000 independent configurations. In measuring the Coulomb potential, we must perform one additional step after thermalizing our lattice, *i.e.*, fix the gauge by enforcing the Coulomb gauge condition, $\partial_i A_i = 0$ [27–30]. The Coulomb gauge was considered fixed when the quality $\Delta F = |F_i - F_{i+1}| < 10^{-7}$ was reached, where

$$F_i = \frac{1}{4V} \sum_{\mu=1}^{3} \sum_{x} \mathrm{Tr}\ U_\mu(x) \tag{5}$$

is the value of the gauge-fixing functional after the i^{th} sweep of the gauge-fixing algorithm over the lattice.

After computing the potential in Eq. (4), we perform a fit with the form given in Eq. (1) and extract the corresponding Coulomb string tension. In Fig. 3, we present σ_{C} as a function of the temporal lattice spacing a_{t}. Contrary to expectations, we observe that the Coulomb string tension becomes infinite as we decrease a_{t}. The divergence as $a_{\mathrm{t}} \to 0$ follows a power law, going roughly as $a_{\mathrm{t}}^{-\frac{1}{2}}$. This behavior was observed for multiple coupling constants β, lattice volumes V, and for all anisotropies measured, both in the SU(2) and SU(3) simulations. It was cross-checked using different lattice codes. These results will be presented in full detail in the upcoming article.

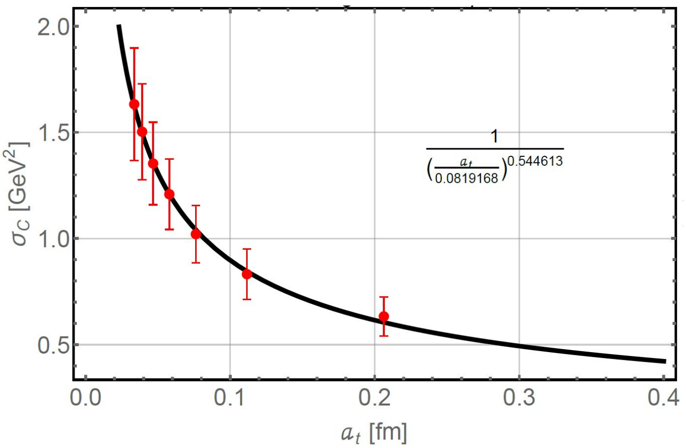

Fig. 3. (Color online) Coulomb string tension, σ_{C}, as a function of a_{t} for the SU(2) Yang–Mills lattice with $\beta = 2.25$ and $\xi = 1$–7. Lattice data is shown as red/gray points, while the best power-law fit is shown as the black curve.

4. Conclusions

We performed the most precise calculation of the SU(2) and SU(3) Coulomb string tensions to date using a wide range of anisotropic lattice ensembles. As far as the available literature is concerned, we performed the first determination of the Hamiltonian limit of the Coulomb potential on the lattice. We discovered that σ_{C}, as currently introduced in the literature, is ill-defined. More work is needed to determine whether the issue lies with the formalism used to define $V_{\mathrm{C}}(r)$ analytically, or if a suitable alternative equation for $V_{\mathrm{C}}(r)$ on the lattice is needed. We hypothesize that the problem lies in the definition of Eq. (4) itself. Namely, the original formula relies on the specific order in which two defining limits, of small T and a_{t}, are taken. The alternative limiting procedure must likely be considered, leading to a modified definition of the observable. We leave details of this discussion to the subsequent paper.

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